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# PREFACE

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## BACKGROUND AND PURPOSE

As in the previous three editions, the primary objective of the fourth edition of *Basic Econometrics* is to provide an elementary but comprehensive introduction to econometrics without resorting to matrix algebra, calculus, or statistics beyond the elementary level.

In this edition I have attempted to incorporate some of the developments in the theory and practice of econometrics that have taken place since the publication of the third edition in 1995. With the availability of sophisticated and user-friendly statistical packages, such as Eviews, Limdep, Microfit, Minitab, PcGive, SAS, Shazam, and Stata, it is now possible to discuss several econometric techniques that could not be included in the previous editions of the book. I have taken full advantage of these statistical packages in illustrating several examples and exercises in this edition.

I was pleasantly surprised to find that my book is used not only by economics and business students but also by students and researchers in several other disciplines, such as politics, international relations, agriculture, and health sciences. Students in these disciplines will find the expanded discussion of several topics very useful.

## THE FOURTH EDITION

The major changes in this edition are as follows:

1. In the introductory chapter, after discussing the steps involved in traditional econometric methodology, I discuss the very important question of how one chooses among competing econometric models.
2. In Chapter 1, I discuss very briefly the measurement scale of economic variables. It is important to know whether the variables are *ratio*

*scale, interval scale, ordinal scale, or nominal scale*, for that will determine the econometric technique that is appropriate in a given situation.

3. The appendices to Chapter 3 now include the large-sample properties of OLS estimators, particularly the property of consistency.

4. The appendix to Chapter 5 now brings into one place the properties and interrelationships among the four important probability distributions that are heavily used in this book, namely, the *normal*, *t*, *chi square*, and *F*.

5. Chapter 6, on functional forms of regression models, now includes a discussion of regression on standardized variables.

6. To make the book more accessible to the nonspecialist, I have moved the discussion of the matrix approach to linear regression from old Chapter 9 to Appendix C. Appendix C is slightly expanded to include some advanced material for the benefit of the more mathematically inclined students. The new Chapter 9 now discusses dummy variable regression models.

7. Chapter 10, on multicollinearity, includes an extended discussion of the famous Longley data, which shed considerable light on the nature and scope of multicollinearity.

8. Chapter 11, on heteroscedasticity, now includes in the appendix an intuitive discussion of White's robust standard errors.

9. Chapter 12, on autocorrelation, now includes a discussion of the Newey–West method of correcting the OLS standard errors to take into account likely autocorrelation in the error term. The corrected standard errors are known as HAC standard errors. This chapter also discusses briefly the topic of forecasting with autocorrelated error terms.

10. Chapter 13, on econometric modeling, replaces old Chapters 13 and 14. This chapter has several new topics that the applied researcher will find particularly useful. They include a compact discussion of model selection criteria, such as the *Akaike information criterion*, the *Schwarz information criterion*, *Mallows's  $C_p$  criterion*, and *forecast chi square*. The chapter also discusses topics such as *outliers*, *leverage*, *influence*, *recursive least squares*, and *Chow's prediction failure test*. This chapter concludes with some cautionary advice to the practitioner about econometric theory and econometric practice.

11. Chapter 14, on nonlinear regression models, is new. Because of the easy availability of statistical software, it is no longer difficult to estimate regression models that are nonlinear in the parameters. Some econometric models are intrinsically nonlinear in the parameters and need to be estimated by iterative methods. This chapter discusses and illustrates some comparatively simple methods of estimating nonlinear-in-parameter regression models.

12. Chapter 15, on qualitative response regression models, which replaces old Chapter 16, on dummy dependent variable regression models, provides a fairly extensive discussion of regression models that involve a dependent variable that is qualitative in nature. The main focus is on logit

and probit models and their variations. The chapter also discusses the *Poisson regression model*, which is used for modeling count data, such as the number of patents received by a firm in a year; the number of telephone calls received in a span of, say, 5 minutes; etc. This chapter has a brief discussion of multinomial logit and probit models and duration models.

**13.** Chapter 16, on panel data regression models, is new. A panel data combines features of both time series and cross-section data. Because of increasing availability of panel data in the social sciences, panel data regression models are being increasingly used by researchers in many fields. This chapter provides a nontechnical discussion of the *fixed effects* and *random effects* models that are commonly used in estimating regression models based on panel data.

**14.** Chapter 17, on dynamic econometric models, has now a rather extended discussion of the Granger causality test, which is routinely used (and misused) in applied research. The Granger causality test is sensitive to the number of lagged terms used in the model. It also assumes that the underlying time series is stationary.

**15.** Except for new problems and minor extensions of the existing estimation techniques, Chapters 18, 19, and 20 on simultaneous equation models are basically unchanged. This reflects the fact that interest in such models has dwindled over the years for a variety of reasons, including their poor forecasting performance after the OPEC oil shocks of the 1970s.

**16.** Chapter 21 is a substantial revision of old Chapter 21. Several concepts of time series econometrics are developed and illustrated in this chapter. The main thrust of the chapter is on the nature and importance of stationary time series. The chapter discusses several methods of finding out if a given time series is stationary. Stationarity of a time series is crucial for the application of various econometric techniques discussed in this book.

**17.** Chapter 22 is also a substantial revision of old Chapter 22. It discusses the topic of economic forecasting based on the *Box-Jenkins (ARIMA)* and *vector autoregression (VAR)* methodologies. It also discusses the topic of measuring volatility in financial time series by the techniques of *autoregressive conditional heteroscedasticity (ARCH)* and *generalized autoregressive conditional heteroscedasticity (GARCH)*.

**18.** Appendix A, on statistical concepts, has been slightly expanded. Appendix C discusses the linear regression model using matrix algebra. This is for the benefit of the more advanced students.

As in the previous editions, all the econometric techniques discussed in this book are illustrated by examples, several of which are based on concrete data from various disciplines. The end-of-chapter questions and problems have several new examples and data sets. For the advanced reader, there are several technical appendices to the various chapters that give proofs of the various theorems and or formulas developed in the text.

## ORGANIZATION AND OPTIONS

Changes in this edition have considerably expanded the scope of the text. I hope this gives the instructor substantial flexibility in choosing topics that are appropriate to the intended audience. Here are suggestions about how this book may be used.

**One-semester course for the nonspecialist:** Appendix A, Chapters 1 through 9, an overview of Chapters 10, 11, 12 (omitting all the proofs).

**One-semester course for economics majors:** Appendix A, Chapters 1 through 13.

**Two-semester course for economics majors:** Appendices A, B, C, Chapters 1 to 22. Chapters 14 and 16 may be covered on an optional basis. Some of the technical appendices may be omitted.

**Graduate and postgraduate students and researchers:** This book is a handy reference book on the major themes in econometrics.

## SUPPLEMENTS

### Data CD

Every text is packaged with a CD that contains the data from the text in ASCII or text format and can be read by most software packages.

### Student Solutions Manual

Free to instructors and salable to students is a Student Solutions Manual (ISBN 0072427922) that contains detailed solutions to the 475 questions and problems in the text.

### EViews

With this fourth edition we are pleased to provide Eviews Student Version 3.1 on a CD along with all of the data from the text. This software is available from the publisher packaged with the text (ISBN: 0072565705). Eviews Student Version is available separately from QMS. Go to <http://www.eviews.com> for further information.

### Web Site

A comprehensive web site provides additional material to support the study of econometrics. Go to [www.mhhe.com/econometrics/gujarati4](http://www.mhhe.com/econometrics/gujarati4).

## ACKNOWLEDGMENTS

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*Damodar N. Gujarati*

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# INTRODUCTION

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## I.1 WHAT IS ECONOMETRICS?

Literally interpreted, *econometrics* means “economic measurement.” Although measurement is an important part of econometrics, the scope of econometrics is much broader, as can be seen from the following quotations:

Econometrics, the result of a certain outlook on the role of economics, consists of the application of mathematical statistics to economic data to lend empirical support to the models constructed by mathematical economics and to obtain numerical results.<sup>1</sup>

... econometrics may be defined as the quantitative analysis of actual economic phenomena based on the concurrent development of theory and observation, related by appropriate methods of inference.<sup>2</sup>

Econometrics may be defined as the social science in which the tools of economic theory, mathematics, and statistical inference are applied to the analysis of economic phenomena.<sup>3</sup>

Econometrics is concerned with the empirical determination of economic laws.<sup>4</sup>

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<sup>1</sup>Gerhard Tintner, *Methodology of Mathematical Economics and Econometrics*, The University of Chicago Press, Chicago, 1968, p. 74.

<sup>2</sup>P. A. Samuelson, T. C. Koopmans, and J. R. N. Stone, “Report of the Evaluative Committee for *Econometrica*,” *Econometrica*, vol. 22, no. 2, April 1954, pp. 141–146.

<sup>3</sup>Arthur S. Goldberger, *Econometric Theory*, John Wiley & Sons, New York, 1964, p. 1.

<sup>4</sup>H. Theil, *Principles of Econometrics*, John Wiley & Sons, New York, 1971, p. 1.

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The art of the econometrician consists in finding the set of assumptions that are both sufficiently specific and sufficiently realistic to allow him to take the best possible advantage of the data available to him.<sup>5</sup>

Econometricians . . . are a positive help in trying to dispel the poor public image of economics (quantitative or otherwise) as a subject in which empty boxes are opened by assuming the existence of can-openers to reveal contents which any ten economists will interpret in 11 ways.<sup>6</sup>

The method of econometric research aims, essentially, at a conjunction of economic theory and actual measurements, using the theory and technique of statistical inference as a bridge pier.<sup>7</sup>

**I.2 WHY A SEPARATE DISCIPLINE?**

As the preceding definitions suggest, econometrics is an amalgam of economic theory, mathematical economics, economic statistics, and mathematical statistics. Yet the subject deserves to be studied in its own right for the following reasons.

Economic theory makes statements or hypotheses that are mostly qualitative in nature. For example, microeconomic theory states that, other things remaining the same, a reduction in the price of a commodity is expected to increase the quantity demanded of that commodity. Thus, economic theory postulates a negative or inverse relationship between the price and quantity demanded of a commodity. But the theory itself does not provide any numerical measure of the relationship between the two; that is, it does not tell by how much the quantity will go up or down as a result of a certain change in the price of the commodity. It is the job of the econometrician to provide such numerical estimates. Stated differently, econometrics gives empirical content to most economic theory.

The main concern of mathematical economics is to express economic theory in mathematical form (equations) without regard to measurability or empirical verification of the theory. Econometrics, as noted previously, is mainly interested in the empirical verification of economic theory. As we shall see, the econometrician often uses the mathematical equations proposed by the mathematical economist but puts these equations in such a form that they lend themselves to empirical testing. And this conversion of mathematical into econometric equations requires a great deal of ingenuity and practical skill.

Economic statistics is mainly concerned with collecting, processing, and presenting economic data in the form of charts and tables. These are the

<sup>5</sup>E. Malinvaud, *Statistical Methods of Econometrics*, Rand McNally, Chicago, 1966, p. 514.

<sup>6</sup>Adrian C. Darnell and J. Lynne Evans, *The Limits of Econometrics*, Edward Elgar Publishing, Hants, England, 1990, p. 54.

<sup>7</sup>T. Haavelmo, "The Probability Approach in Econometrics," Supplement to *Econometrica*, vol. 12, 1944, preface p. iii.



jobs of the economic statistician. It is he or she who is primarily responsible for collecting data on gross national product (GNP), employment, unemployment, prices, etc. The data thus collected constitute the raw data for econometric work. But the economic statistician does not go any further, not being concerned with using the collected data to test economic theories. Of course, one who does that becomes an econometrician.

Although mathematical statistics provides many tools used in the trade, the econometrician often needs special methods in view of the unique nature of most economic data, namely, that the data are not generated as the result of a controlled experiment. The econometrician, like the meteorologist, generally depends on data that cannot be controlled directly. As Spanos correctly observes:

In econometrics the modeler is often faced with **observational** as opposed to **experimental** data. This has two important implications for empirical modeling in econometrics. First, the modeler is required to master very different skills than those needed for analyzing experimental data. . . . Second, the separation of the data collector and the data analyst requires the modeler to familiarize himself/herself thoroughly with the nature and structure of data in question.<sup>8</sup>

### I.3 METHODOLOGY OF ECONOMETRICS

How do econometricians proceed in their analysis of an economic problem? That is, what is their methodology? Although there are several schools of thought on econometric methodology, we present here the **traditional** or **classical** methodology, which still dominates empirical research in economics and other social and behavioral sciences.<sup>9</sup>

Broadly speaking, traditional econometric methodology proceeds along the following lines:

1. Statement of theory or hypothesis.
2. Specification of the mathematical model of the theory
3. Specification of the statistical, or econometric, model
4. Obtaining the data
5. Estimation of the parameters of the econometric model
6. Hypothesis testing
7. Forecasting or prediction
8. Using the model for control or policy purposes.

To illustrate the preceding steps, let us consider the well-known Keynesian theory of consumption.

<sup>8</sup>Aris Spanos, *Probability Theory and Statistical Inference: Econometric Modeling with Observational Data*, Cambridge University Press, United Kingdom, 1999, p. 21.

<sup>9</sup>For an enlightening, if advanced, discussion on econometric methodology, see David F. Hendry, *Dynamic Econometrics*, Oxford University Press, New York, 1995. See also Aris Spanos, *op. cit.*



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**1. Statement of Theory or Hypothesis**

Keynes stated:

The fundamental psychological law . . . is that men [women] are disposed, as a rule and on average, to increase their consumption as their income increases, but not as much as the increase in their income.<sup>10</sup>

In short, Keynes postulated that the **marginal propensity to consume (MPC)**, the rate of change of consumption for a unit (say, a dollar) change in income, is greater than zero but less than 1.

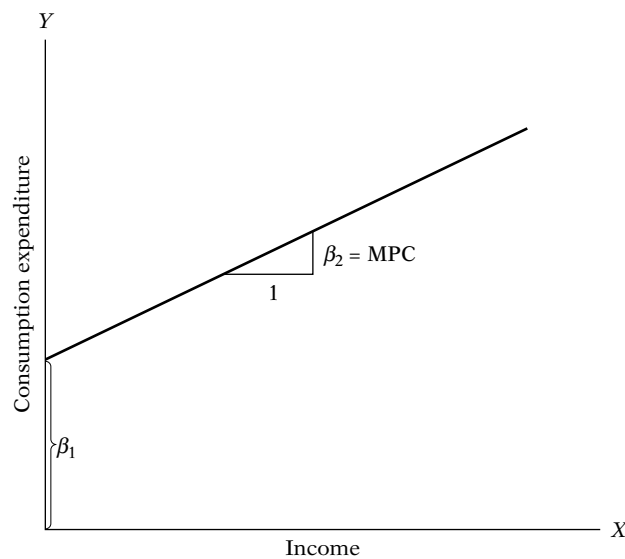
**2. Specification of the Mathematical Model of Consumption**

Although Keynes postulated a positive relationship between consumption and income, he did not specify the precise form of the functional relationship between the two. For simplicity, a mathematical economist might suggest the following form of the Keynesian consumption function:

$$Y = \beta_1 + \beta_2 X \quad 0 < \beta_2 < 1 \quad (\text{I.3.1})$$

where  $Y$  = consumption expenditure and  $X$  = income, and where  $\beta_1$  and  $\beta_2$ , known as the **parameters** of the model, are, respectively, the **intercept** and **slope** coefficients.

The slope coefficient  $\beta_2$  measures the MPC. Geometrically, Eq. (I.3.1) is as shown in Figure I.1. This equation, which states that consumption is lin-



**FIGURE I.1** Keynesian consumption function.

<sup>10</sup>John Maynard Keynes, *The General Theory of Employment, Interest and Money*, Harcourt Brace Jovanovich, New York, 1936, p. 96.

early related to income, is an example of a mathematical model of the relationship between consumption and income that is called the **consumption function** in economics. A model is simply a set of mathematical equations. If the model has only one equation, as in the preceding example, it is called a **single-equation model**, whereas if it has more than one equation, it is known as a **multiple-equation model** (the latter will be considered later in the book).

In Eq. (I.3.1) the variable appearing on the left side of the equality sign is called the **dependent variable** and the variable(s) on the right side are called the **independent, or explanatory, variable(s)**. Thus, in the Keynesian consumption function, Eq. (I.3.1), consumption (expenditure) is the dependent variable and income is the explanatory variable.

### 3. Specification of the Econometric Model of Consumption

The purely mathematical model of the consumption function given in Eq. (I.3.1) is of limited interest to the econometrician, for it assumes that there is an *exact* or *deterministic* relationship between consumption and income. But relationships between economic variables are generally inexact. Thus, if we were to obtain data on consumption expenditure and disposable (i.e., aftertax) income of a sample of, say, 500 American families and plot these data on a graph paper with consumption expenditure on the vertical axis and disposable income on the horizontal axis, we would not expect all 500 observations to lie exactly on the straight line of Eq. (I.3.1) because, in addition to income, other variables affect consumption expenditure. For example, size of family, ages of the members in the family, family religion, etc., are likely to exert some influence on consumption.

To allow for the inexact relationships between economic variables, the econometrician would modify the deterministic consumption function (I.3.1) as follows:

$$Y = \beta_1 + \beta_2 X + u \quad (\text{I.3.2})$$

where  $u$ , known as the **disturbance, or error, term**, is a **random (stochastic) variable** that has well-defined probabilistic properties. The disturbance term  $u$  may well represent all those factors that affect consumption but are not taken into account explicitly.

Equation (I.3.2) is an example of an **econometric model**. More technically, it is an example of a **linear regression model**, which is the major concern of this book. The econometric consumption function hypothesizes that the dependent variable  $Y$  (consumption) is linearly related to the explanatory variable  $X$  (income) but that the relationship between the two is not exact; it is subject to individual variation.

The econometric model of the consumption function can be depicted as shown in Figure I.2.

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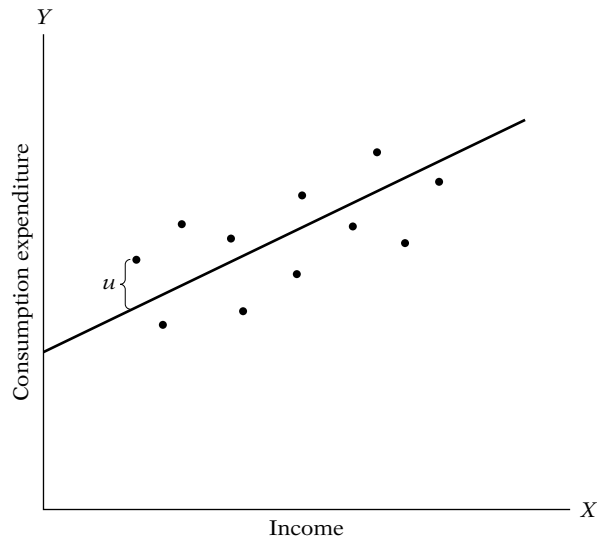


FIGURE I.2 Econometric model of the Keynesian consumption function.

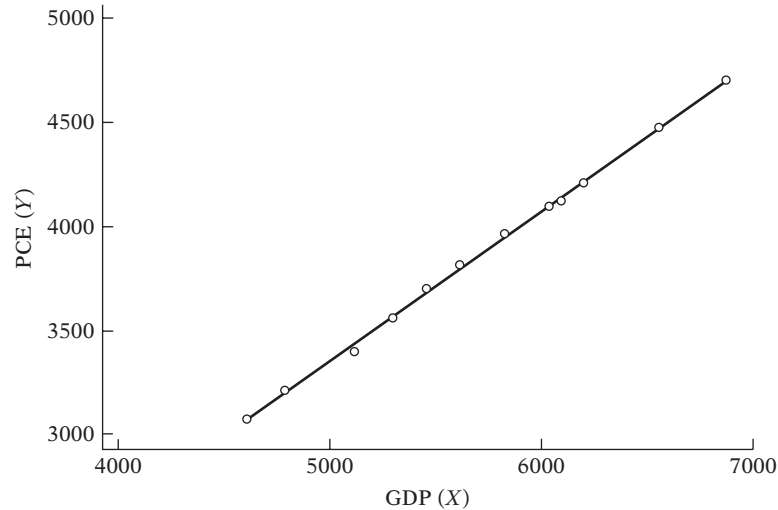
#### 4. Obtaining Data

To estimate the econometric model given in (I.3.2), that is, to obtain the numerical values of  $\beta_1$  and  $\beta_2$ , we need data. Although we will have more to say about the crucial importance of data for economic analysis in the next chapter, for now let us look at the data given in Table I.1, which relate to

**TABLE I.1** DATA ON Y (PERSONAL CONSUMPTION EXPENDITURE) AND X (GROSS DOMESTIC PRODUCT, 1982–1996), BOTH IN 1992 BILLIONS OF DOLLARS

Year	Y	X
1982	3081.5	4620.3
1983	3240.6	4803.7
1984	3407.6	5140.1
1985	3566.5	5323.5
1986	3708.7	5487.7
1987	3822.3	5649.5
1988	3972.7	5865.2
1989	4064.6	6062.0
1990	4132.2	6136.3
1991	4105.8	6079.4
1992	4219.8	6244.4
1993	4343.6	6389.6
1994	4486.0	6610.7
1995	4595.3	6742.1
1996	4714.1	6928.4

Source: *Economic Report of the President*, 1998, Table B–2, p. 282.



**FIGURE I.3** Personal consumption expenditure ( $Y$ ) in relation to GDP ( $X$ ), 1982–1996, both in billions of 1992 dollars.

the U.S. economy for the period 1981–1996. The  $Y$  variable in this table is the *aggregate* (for the economy as a whole) personal consumption expenditure (PCE) and the  $X$  variable is gross domestic product (GDP), a measure of aggregate income, both measured in billions of 1992 dollars. Therefore, the data are in “real” terms; that is, they are measured in constant (1992) prices. The data are plotted in Figure I.3 (cf. Figure I.2). For the time being neglect the line drawn in the figure.

## 5. Estimation of the Econometric Model

Now that we have the data, our next task is to estimate the parameters of the consumption function. The numerical estimates of the parameters give empirical content to the consumption function. The actual mechanics of estimating the parameters will be discussed in Chapter 3. For now, note that the statistical technique of **regression analysis** is the main tool used to obtain the estimates. Using this technique and the data given in Table I.1, we obtain the following estimates of  $\beta_1$  and  $\beta_2$ , namely,  $-184.08$  and  $0.7064$ . Thus, the estimated consumption function is:

$$\hat{Y} = -184.08 + 0.7064X_i \quad (\text{I.3.3})$$

The hat on the  $Y$  indicates that it is an estimate.<sup>11</sup> The estimated consumption function (i.e., regression line) is shown in Figure I.3.

<sup>11</sup>As a matter of convention, a hat over a variable or parameter indicates that it is an estimated value.

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As Figure I.3 shows, the regression line fits the data quite well in that the data points are very close to the regression line. From this figure we see that for the period 1982–1996 the slope coefficient (i.e., the **MPC**) was about 0.70, suggesting that for the sample period an increase in real income of 1 dollar led, *on average*, to an increase of about 70 cents in real consumption expenditure.<sup>12</sup> We say *on average* because the relationship between consumption and income is inexact; as is clear from Figure I.3; not all the data points lie exactly on the regression line. In simple terms we can say that, according to our data, the *average*, or *mean*, consumption expenditure went up by about 70 cents for a dollar's increase in real income.

## 6. Hypothesis Testing

Assuming that the fitted model is a reasonably good approximation of reality, we have to develop suitable criteria to find out whether the estimates obtained in, say, Eq. (I.3.3) are in accord with the expectations of the theory that is being tested. According to “positive” economists like Milton Friedman, a theory or hypothesis that is not verifiable by appeal to empirical evidence may not be admissible as a part of scientific enquiry.<sup>13</sup>

As noted earlier, Keynes expected the MPC to be positive but less than 1. In our example we found the MPC to be about 0.70. But before we accept this finding as confirmation of Keynesian consumption theory, we must enquire whether this estimate is sufficiently below unity to convince us that this is not a chance occurrence or peculiarity of the particular data we have used. In other words, is 0.70 *statistically less than 1*? If it is, it may support Keynes' theory.

Such confirmation or refutation of economic theories on the basis of sample evidence is based on a branch of statistical theory known as **statistical inference (hypothesis testing)**. Throughout this book we shall see how this inference process is actually conducted.

## 7. Forecasting or Prediction

If the chosen model does not refute the hypothesis or theory under consideration, we may use it to predict the future value(s) of the dependent, or **forecast, variable**  $Y$  on the basis of known or expected future value(s) of the explanatory, or **predictor, variable**  $X$ .

To illustrate, suppose we want to predict the mean consumption expenditure for 1997. The GDP value for 1997 was 7269.8 billion dollars.<sup>14</sup> Putting

<sup>12</sup>Do not worry now about how these values were obtained. As we show in Chap. 3, the statistical method of **least squares** has produced these estimates. Also, for now do not worry about the negative value of the intercept.

<sup>13</sup>See Milton Friedman, “The Methodology of Positive Economics,” *Essays in Positive Economics*, University of Chicago Press, Chicago, 1953.

<sup>14</sup>Data on PCE and GDP were available for 1997 but we purposely left them out to illustrate the topic discussed in this section. As we will discuss in subsequent chapters, it is a good idea to save a portion of the data to find out how well the fitted model predicts the out-of-sample observations.

this GDP figure on the right-hand side of (I.3.3), we obtain:

$$\begin{aligned}\hat{Y}_{1997} &= -184.0779 + 0.7064(7269.8) \\ &= 4951.3167\end{aligned}\tag{I.3.4}$$

or about 4951 billion dollars. Thus, given the value of the GDP, the mean, or average, forecast consumption expenditure is about 4951 billion dollars. The actual value of the consumption expenditure reported in 1997 was 4913.5 billion dollars. The estimated model (I.3.3) thus **overpredicted** the actual consumption expenditure by about 37.82 billion dollars. We could say the **forecast error** is about 37.82 billion dollars, which is about 0.76 percent of the actual GDP value for 1997. When we fully discuss the linear regression model in subsequent chapters, we will try to find out if such an error is “small” or “large.” But what is important for now is to note that such forecast errors are inevitable given the statistical nature of our analysis.

There is another use of the estimated model (I.3.3). Suppose the President decides to propose a reduction in the income tax. What will be the effect of such a policy on income and thereby on consumption expenditure and ultimately on employment?

Suppose that, as a result of the proposed policy change, investment expenditure increases. What will be the effect on the economy? As macroeconomic theory shows, the change in income following, say, a dollar’s worth of change in investment expenditure is given by the **income multiplier  $M$** , which is defined as

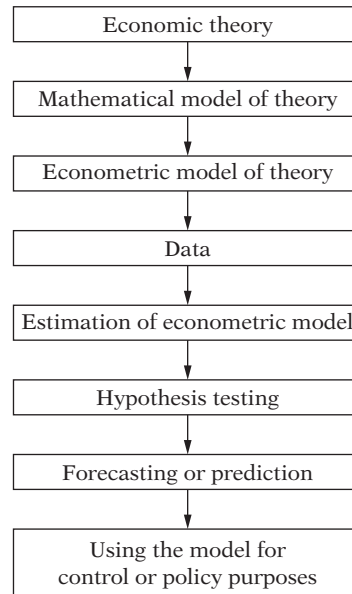
$$M = \frac{1}{1 - \text{MPC}}\tag{I.3.5}$$

If we use the MPC of 0.70 obtained in (I.3.3), this multiplier becomes about  $M = 3.33$ . That is, an increase (decrease) of a dollar in investment will *eventually* lead to more than a threefold increase (decrease) in income; note that it takes time for the multiplier to work.

The critical value in this computation is MPC, for the multiplier depends on it. And this estimate of the MPC can be obtained from regression models such as (I.3.3). Thus, a quantitative estimate of MPC provides valuable information for policy purposes. Knowing MPC, one can predict the future course of income, consumption expenditure, and employment following a change in the government’s fiscal policies.

## 8. Use of the Model for Control or Policy Purposes

Suppose we have the estimated consumption function given in (I.3.3). Suppose further the government believes that consumer expenditure of about 4900 (billions of 1992 dollars) will keep the unemployment rate at its



**FIGURE I.4** Anatomy of econometric modeling.

current level of about 4.2 percent (early 2000). What level of income will guarantee the target amount of consumption expenditure?

If the regression results given in (I.3.3) seem reasonable, simple arithmetic will show that

$$4900 = -184.0779 + 0.7064X \quad (\text{I.3.6})$$

which gives  $X = 7197$ , approximately. That is, an income level of about 7197 (billion) dollars, given an MPC of about 0.70, will produce an expenditure of about 4900 billion dollars.

As these calculations suggest, an estimated model may be used for control, or policy, purposes. By appropriate fiscal and monetary policy mix, the government can manipulate the **control variable**  $X$  to produce the desired level of the **target variable**  $Y$ .

Figure I.4 summarizes the anatomy of classical econometric modeling.

### Choosing among Competing Models

When a governmental agency (e.g., the U.S. Department of Commerce) collects economic data, such as that shown in Table I.1, it does not necessarily have any economic theory in mind. How then does one know that the data really support the Keynesian theory of consumption? Is it because the Keynesian consumption function (i.e., the regression line) shown in Figure I.3 is extremely close to the actual data points? Is it possible that an-



other consumption model (theory) might equally fit the data as well? For example, Milton Friedman has developed a model of consumption, called the *permanent income hypothesis*.<sup>15</sup> Robert Hall has also developed a model of consumption, called the *life-cycle permanent income hypothesis*.<sup>16</sup> Could one or both of these models also fit the data in Table I.1?

In short, the question facing a researcher in practice is how to choose among competing hypotheses or models of a given phenomenon, such as the consumption–income relationship. As Miller contends:

No encounter with data is step towards genuine confirmation unless the hypothesis does a better job of coping with the data than some natural rival. . . . What strengthens a hypothesis, here, is a victory that is, at the same time, a defeat for a plausible rival.<sup>17</sup>

How then does one choose among competing models or hypotheses? Here the advice given by Clive Granger is worth keeping in mind:<sup>18</sup>

I would like to suggest that in the future, when you are presented with a new piece of theory or empirical model, you ask these questions:

- (i) What purpose does it have? What economic decisions does it help with? and;
- (ii) Is there any evidence being presented that allows me to evaluate its quality compared to alternative theories or models?

I think attention to such questions will strengthen economic research and discussion.

As we progress through this book, we will come across several competing hypotheses trying to explain various economic phenomena. For example, students of economics are familiar with the concept of the production function, which is basically a relationship between output and inputs (say, capital and labor). In the literature, two of the best known are the *Cobb–Douglas* and the *constant elasticity of substitution* production functions. Given the data on output and inputs, we will have to find out which of the two production functions, if any, fits the data well.

The eight-step classical econometric methodology discussed above is neutral in the sense that it can be used to test any of these rival hypotheses.

Is it possible to develop a methodology that is comprehensive enough to include competing hypotheses? This is an involved and controversial topic.

<sup>15</sup>Milton Friedman, *A Theory of Consumption Function*, Princeton University Press, Princeton, N.J., 1957.

<sup>16</sup>R. Hall, “Stochastic Implications of the Life Cycle Permanent Income Hypothesis: Theory and Evidence,” *Journal of Political Economy*, 1978, vol. 86, pp. 971–987.

<sup>17</sup>R. W. Miller, *Fact and Method: Explanation, Confirmation, and Reality in the Natural and Social Sciences*, Princeton University Press, Princeton, N.J., 1978, p. 176.

<sup>18</sup>Clive W. J. Granger, *Empirical Modeling in Economics*, Cambridge University Press, U.K., 1999, p. 58.

## 12 BASIC ECONOMETRICS

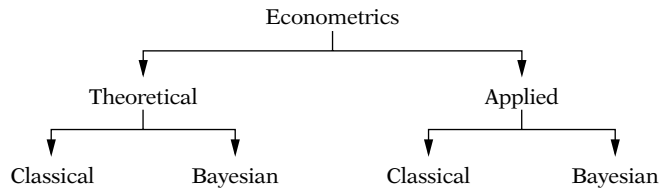


FIGURE I.5 Categories of econometrics.

We will discuss it in Chapter 13, after we have acquired the necessary econometric theory.

#### I.4 TYPES OF ECONOMETRICS

As the classificatory scheme in Figure I.5 suggests, econometrics may be divided into two broad categories: **theoretical econometrics** and **applied econometrics**. In each category, one can approach the subject in the **classical** or **Bayesian** tradition. In this book the emphasis is on the classical approach. For the Bayesian approach, the reader may consult the references given at the end of the chapter.

Theoretical econometrics is concerned with the development of appropriate methods for measuring economic relationships specified by econometric models. In this aspect, econometrics leans heavily on mathematical statistics. For example, one of the methods used extensively in this book is **least squares**. Theoretical econometrics must spell out the assumptions of this method, its properties, and what happens to these properties when one or more of the assumptions of the method are not fulfilled.

In applied econometrics we use the tools of theoretical econometrics to study some special field(s) of economics and business, such as the production function, investment function, demand and supply functions, portfolio theory, etc.

This book is concerned largely with the development of econometric methods, their assumptions, their uses, their limitations. These methods are illustrated with examples from various areas of economics and business. But this is *not* a book of applied econometrics in the sense that it delves deeply into any particular field of economic application. That job is best left to books written specifically for this purpose. References to some of these books are provided at the end of this book.

#### I.5 MATHEMATICAL AND STATISTICAL PREREQUISITES

Although this book is written at an elementary level, the author assumes that the reader is familiar with the basic concepts of statistical estimation and hypothesis testing. However, a broad but nontechnical overview of the basic statistical concepts used in this book is provided in **Appendix A** for

the benefit of those who want to refresh their knowledge. Insofar as mathematics is concerned, a nodding acquaintance with the notions of differential calculus is desirable, although not essential. Although most graduate level books in econometrics make heavy use of matrix algebra, I want to make it clear that it is not needed to study this book. It is my strong belief that the fundamental ideas of econometrics can be conveyed without the use of matrix algebra. However, for the benefit of the mathematically inclined student, **Appendix C** gives the summary of basic regression theory in matrix notation. For these students, **Appendix B** provides a succinct summary of the main results from matrix algebra.

## I.6 THE ROLE OF THE COMPUTER

Regression analysis, the bread-and-butter tool of econometrics, these days is unthinkable without the computer and some access to statistical software. (Believe me, I grew up in the generation of the slide rule!) Fortunately, several excellent regression packages are commercially available, both for the mainframe and the microcomputer, and the list is growing by the day. Regression software packages, such as **ET, LIMDEP, SHAZAM, MICRO TSP, MINITAB, EVIEWS, SAS, SPSS, STATA, Microfit, PcGive**, and **BMD** have most of the econometric techniques and tests discussed in this book.

In this book, from time to time, the reader will be asked to conduct **Monte Carlo** experiments using one or more of the statistical packages. Monte Carlo experiments are “fun” exercises that will enable the reader to appreciate the properties of several statistical methods discussed in this book. The details of the Monte Carlo experiments will be discussed at appropriate places.

## I.7 SUGGESTIONS FOR FURTHER READING

The topic of econometric methodology is vast and controversial. For those interested in this topic, I suggest the following books:

Neil de Marchi and Christopher Gilbert, eds., *History and Methodology of Econometrics*, Oxford University Press, New York, 1989. This collection of readings discusses some early work on econometric methodology and has an extended discussion of the British approach to econometrics relating to time series data, that is, data collected over a period of time.

Wojciech W. Charemza and Derek F. Deadman, *New Directions in Econometric Practice: General to Specific Modelling, Cointegration and Vector Autoregression*, 2d ed., Edward Elgar Publishing Ltd., Hants, England, 1997. The authors of this book critique the traditional approach to econometrics and give a detailed exposition of new approaches to econometric methodology.

Adrian C. Darnell and J. Lynne Evans, *The Limits of Econometrics*, Edward Elgar Publishers Ltd., Hants, England, 1990. The book provides a somewhat

balanced discussion of the various methodological approaches to econometrics, with renewed allegiance to traditional econometric methodology.

Mary S. Morgan, *The History of Econometric Ideas*, Cambridge University Press, New York, 1990. The author provides an excellent historical perspective on the theory and practice of econometrics, with an in-depth discussion of the early contributions of Haavelmo (1990 Nobel Laureate in Economics) to econometrics. In the same spirit, David F. Hendry and Mary S. Morgan, *The Foundation of Econometric Analysis*, Cambridge University Press, U.K., 1995, have collected seminal writings in econometrics to show the evolution of econometric ideas over time.

David Colander and Reuven Brenner, eds., *Educating Economists*, University of Michigan Press, Ann Arbor, Michigan, 1992, present a critical, at times agnostic, view of economic teaching and practice.

For Bayesian statistics and econometrics, the following books are very useful: John H. Dey, *Data in Doubt*, Basic Blackwell Ltd., Oxford University Press, England, 1985. Peter M. Lee, *Bayesian Statistics: An Introduction*, Oxford University Press, England, 1989. Dale J. Porier, *Intermediate Statistics and Econometrics: A Comparative Approach*, MIT Press, Cambridge, Massachusetts, 1995. Arnold Zeller, *An Introduction to Bayesian Inference in Econometrics*, John Wiley & Sons, New York, 1971, is an advanced reference book.

# PART ONE

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## SINGLE-EQUATION REGRESSION MODELS

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Part I of this text introduces single-equation regression models. In these models, one variable, called the *dependent variable*, is expressed as a linear function of one or more other variables, called the *explanatory variables*. In such models it is assumed implicitly that causal relationships, if any, between the dependent and explanatory variables flow in one direction only, namely, from the explanatory variables to the dependent variable.

In Chapter 1, we discuss the historical as well as the modern interpretation of the term *regression* and illustrate the difference between the two interpretations with several examples drawn from economics and other fields.

In Chapter 2, we introduce some fundamental concepts of regression analysis with the aid of the two-variable linear regression model, a model in which the dependent variable is expressed as a linear function of only a single explanatory variable.

In Chapter 3, we continue to deal with the two-variable model and introduce what is known as the *classical linear regression model*, a model that makes several simplifying assumptions. With these assumptions, we introduce the method of *ordinary least squares* (OLS) to estimate the parameters of the two-variable regression model. The method of OLS is simple to apply, yet it has some very desirable statistical properties.

In Chapter 4, we introduce the (two-variable) classical *normal* linear regression model, a model that assumes that the random dependent variable follows the normal probability distribution. With this assumption, the OLS estimators obtained in Chapter 3 possess some stronger statistical properties than the nonnormal classical linear regression model—properties that enable us to engage in statistical inference, namely, hypothesis testing.

Chapter 5 is devoted to the topic of hypothesis testing. In this chapter, we try to find out whether the estimated regression coefficients are compatible with the hypothesized values of such coefficients, the hypothesized values being suggested by theory and/or prior empirical work.

Chapter 6 considers some extensions of the two-variable regression model. In particular, it discusses topics such as (1) regression through the origin, (2) scaling and units of measurement, and (3) functional forms of regression models such as double-log, semilog, and reciprocal models.

In Chapter 7, we consider the multiple regression model, a model in which there is more than one explanatory variable, and show how the method of OLS can be extended to estimate the parameters of such models.

In Chapter 8, we extend the concepts introduced in Chapter 5 to the multiple regression model and point out some of the complications arising from the introduction of several explanatory variables.

Chapter 9 on dummy, or qualitative, explanatory variables concludes Part I of the text. This chapter emphasizes that not all explanatory variables need to be quantitative (i.e., ratio scale). Variables, such as gender, race, religion, nationality, and region of residence, cannot be readily quantified, yet they play a valuable role in explaining many an economic phenomenon.

# 1

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## THE NATURE OF REGRESSION ANALYSIS

---

As mentioned in the Introduction, regression is a main tool of econometrics, and in this chapter we consider very briefly the nature of this tool.

### 1.1 HISTORICAL ORIGIN OF THE TERM *REGRESSION*

The term *regression* was introduced by Francis Galton. In a famous paper, Galton found that, although there was a tendency for tall parents to have tall children and for short parents to have short children, the average height of children born of parents of a given height tended to move or “regress” toward the average height in the population as a whole.<sup>1</sup> In other words, the height of the children of unusually tall or unusually short parents tends to move toward the average height of the population. Galton’s *law of universal regression* was confirmed by his friend Karl Pearson, who collected more than a thousand records of heights of members of family groups.<sup>2</sup> He found that the average height of sons of a group of tall fathers was less than their fathers’ height and the average height of sons of a group of short fathers was greater than their fathers’ height, thus “regressing” tall and short sons alike toward the average height of all men. In the words of Galton, this was “regression to mediocrity.”

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<sup>1</sup>Francis Galton, “Family Likeness in Stature,” *Proceedings of Royal Society, London*, vol. 40, 1886, pp. 42–72.

<sup>2</sup>K. Pearson and A. Lee, “On the Laws of Inheritance,” *Biometrika*, vol. 2, Nov. 1903, pp. 357–462.



## 1.2 THE MODERN INTERPRETATION OF REGRESSION

The modern interpretation of regression is, however, quite different. Broadly speaking, we may say

Regression analysis is concerned with the study of the dependence of one variable, the *dependent variable*, on one or more other variables, the *explanatory variables*, with a view to estimating and/or predicting the (population) mean or average value of the former in terms of the known or fixed (in repeated sampling) values of the latter.

The full import of this view of regression analysis will become clearer as we progress, but a few simple examples will make the basic concept quite clear.

### Examples

1. Reconsider Galton's law of universal regression. Galton was interested in finding out why there was a stability in the distribution of heights in a population. But in the modern view our concern is not with this explanation but rather with finding out how the *average* height of sons changes, given the fathers' height. In other words, our concern is with predicting the average height of sons knowing the height of their fathers. To see how this can be done, consider Figure 1.1, which is a **scatter diagram**, or **scatter-**

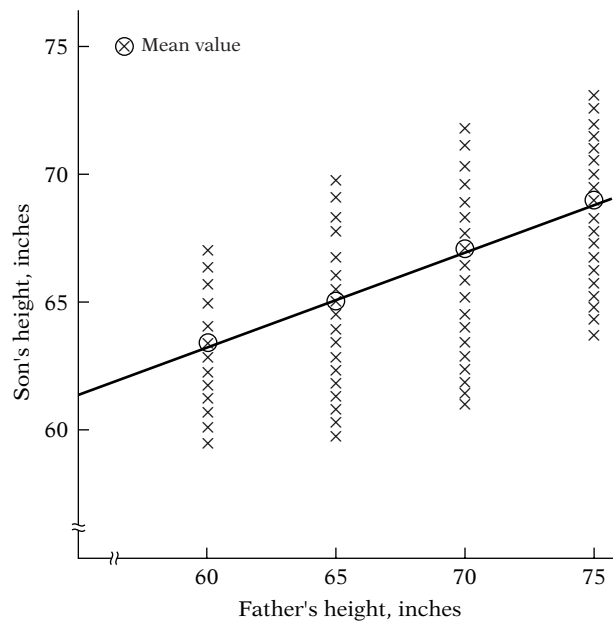


FIGURE 1.1 Hypothetical distribution of sons' heights corresponding to given heights of fathers.

**gram.** This figure shows the distribution of heights of sons in a hypothetical population corresponding to the given or *fixed* values of the father's height. Notice that corresponding to any given height of a father is a *range* or distribution of the heights of the sons. However, notice that despite the variability of the height of sons for a given value of father's height, the average height of sons generally increases as the height of the father increases. To show this clearly, the circled crosses in the figure indicate the *average* height of sons corresponding to a given height of the father. Connecting these averages, we obtain the line shown in the figure. This line, as we shall see, is known as the **regression line**. It shows how the *average* height of sons increases with the father's height.<sup>3</sup>

2. Consider the scattergram in Figure 1.2, which gives the distribution in a hypothetical population of heights of boys measured at *fixed* ages. Corresponding to any given age, we have a range, or distribution, of heights. Obviously, not all boys of a given age are likely to have identical heights. But height *on the average* increases with age (of course, up to a certain age), which can be seen clearly if we draw a line (the regression line) through the

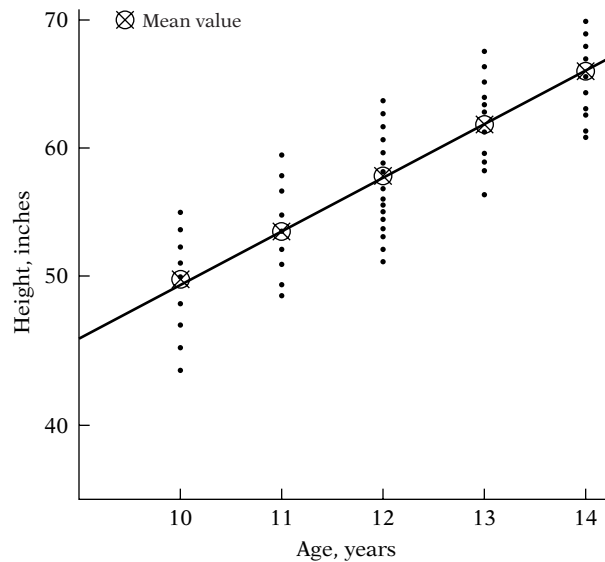


FIGURE 1.2 Hypothetical distribution of heights corresponding to selected ages.

<sup>3</sup>At this stage of the development of the subject matter, we shall call this regression line simply the *line connecting the mean, or average, value of the dependent variable (son's height) corresponding to the given value of the explanatory variable (father's height)*. Note that this line has a positive slope but the slope is less than 1, which is in conformity with Galton's regression to mediocrity. (Why?)

circled points that represent the average height at the given ages. Thus, knowing the age, we may be able to predict from the regression line the average height corresponding to that age.

3. Turning to economic examples, an economist may be interested in studying the dependence of personal consumption expenditure on after-tax or disposable real personal income. Such an analysis may be helpful in estimating the marginal propensity to consume (MPC), that is, average change in consumption expenditure for, say, a dollar's worth of change in real income (see Figure I.3).

4. A monopolist who can fix the price or output (but not both) may want to find out the response of the demand for a product to changes in price. Such an experiment may enable the estimation of the **price elasticity** (i.e., price responsiveness) of the demand for the product and may help determine the most profitable price.

5. A labor economist may want to study the rate of change of money wages in relation to the unemployment rate. The historical data are shown in the scattergram given in Figure 1.3. The curve in Figure 1.3 is an example of the celebrated *Phillips curve* relating changes in the money wages to the unemployment rate. Such a scattergram may enable the labor economist to predict the average change in money wages given a certain unemployment rate. Such knowledge may be helpful in stating something about the inflationary process in an economy, for increases in money wages are likely to be reflected in increased prices.

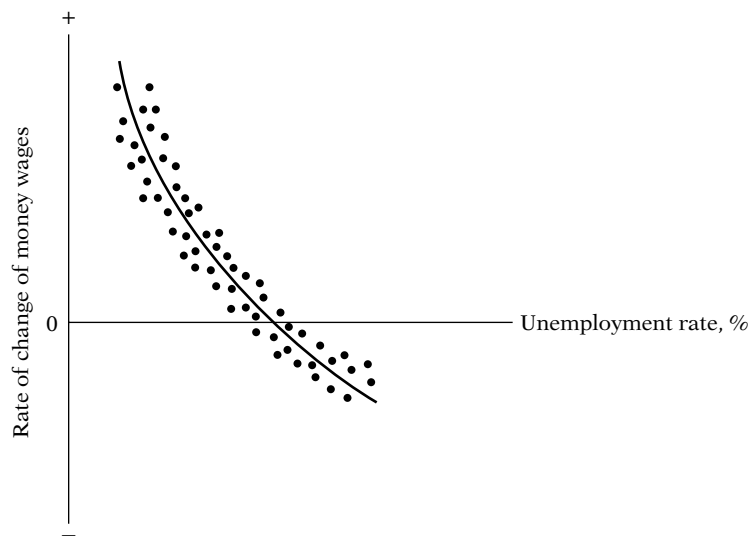


FIGURE 1.3 Hypothetical Phillips curve.

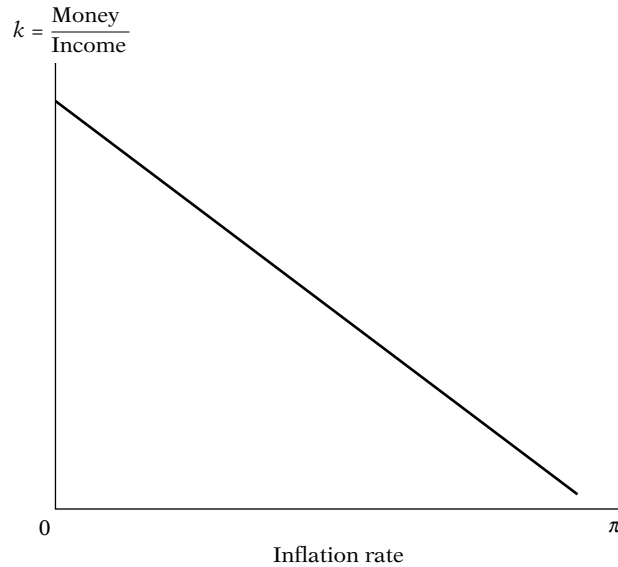


FIGURE 1.4 Money holding in relation to the inflation rate  $\pi$ .

6. From monetary economics it is known that, other things remaining the same, the higher the rate of inflation  $\pi$ , the lower the proportion  $k$  of their income that people would want to hold in the form of money, as depicted in Figure 1.4. A quantitative analysis of this relationship will enable the monetary economist to predict the amount of money, as a proportion of their income, that people would want to hold at various rates of inflation.

7. The marketing director of a company may want to know how the demand for the company's product is related to, say, advertising expenditure. Such a study will be of considerable help in finding out the **elasticity of demand** with respect to advertising expenditure, that is, the percent change in demand in response to, say, a 1 percent change in the advertising budget. This knowledge may be helpful in determining the "optimum" advertising budget.

8. Finally, an agronomist may be interested in studying the dependence of crop yield, say, of wheat, on temperature, rainfall, amount of sunshine, and fertilizer. Such a dependence analysis may enable the prediction or forecasting of the average crop yield, given information about the explanatory variables.

The reader can supply scores of such examples of the dependence of one variable on one or more other variables. The techniques of regression analysis discussed in this text are specially designed to study such dependence among variables.

### 1.3 STATISTICAL VERSUS DETERMINISTIC RELATIONSHIPS

From the examples cited in Section 1.2, the reader will notice that in regression analysis we are concerned with what is known as the *statistical*, not *functional* or *deterministic*, dependence among variables, such as those of classical physics. In statistical relationships among variables we essentially deal with **random** or **stochastic**<sup>4</sup> variables, that is, variables that have probability distributions. In functional or deterministic dependency, on the other hand, we also deal with variables, but these variables are not random or stochastic.

The dependence of crop yield on temperature, rainfall, sunshine, and fertilizer, for example, is statistical in nature in the sense that the explanatory variables, although certainly important, will not enable the agronomist to predict crop yield exactly because of errors involved in measuring these variables as well as a host of other factors (variables) that collectively affect the yield but may be difficult to identify individually. Thus, there is bound to be some “intrinsic” or random variability in the dependent-variable crop yield that cannot be fully explained no matter how many explanatory variables we consider.

In deterministic phenomena, on the other hand, we deal with relationships of the type, say, exhibited by Newton’s law of gravity, which states: Every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. Symbolically,  $F = k(m_1m_2/r^2)$ , where  $F$  = force,  $m_1$  and  $m_2$  are the masses of the two particles,  $r$  = distance, and  $k$  = constant of proportionality. Another example is Ohm’s law, which states: For metallic conductors over a limited range of temperature the current  $C$  is proportional to the voltage  $V$ ; that is,  $C = (\frac{1}{k})V$  where  $\frac{1}{k}$  is the constant of proportionality. Other examples of such deterministic relationships are Boyle’s gas law, Kirchhoff’s law of electricity, and Newton’s law of motion.

In this text we are not concerned with such deterministic relationships. Of course, if there are errors of measurement, say, in the  $k$  of Newton’s law of gravity, the otherwise deterministic relationship becomes a statistical relationship. In this situation, force can be predicted only approximately from the given value of  $k$  (and  $m_1$ ,  $m_2$ , and  $r$ ), which contains errors. The variable  $F$  in this case becomes a random variable.

### 1.4 REGRESSION VERSUS CAUSATION

Although regression analysis deals with the dependence of one variable on other variables, it does not necessarily imply causation. In the words of Kendall and Stuart, “A statistical relationship, however strong and however

<sup>4</sup>The word *stochastic* comes from the Greek word *stokhos* meaning “a bull’s eye.” The outcome of throwing darts on a dart board is a stochastic process, that is, a process fraught with misses.

suggestive, can never establish causal connection: our ideas of causation must come from outside statistics, ultimately from some theory or other.”<sup>5</sup>

In the crop-yield example cited previously, there is no *statistical reason* to assume that rainfall does not depend on crop yield. The fact that we treat crop yield as dependent on rainfall (among other things) is due to nonstatistical considerations: Common sense suggests that the relationship cannot be reversed, for we cannot control rainfall by varying crop yield.

In all the examples cited in Section 1.2 the point to note is that **a statistical relationship in itself cannot logically imply causation**. To ascribe causality, one must appeal to a priori or theoretical considerations. Thus, in the third example cited, one can invoke economic theory in saying that consumption expenditure depends on real income.<sup>6</sup>

## 1.5 REGRESSION VERSUS CORRELATION

Closely related to but conceptually very much different from regression analysis is **correlation analysis**, where the primary objective is to measure the *strength* or *degree of linear association* between two variables. The **correlation coefficient**, which we shall study in detail in Chapter 3, measures this strength of (linear) association. For example, we may be interested in finding the correlation (coefficient) between smoking and lung cancer, between scores on statistics and mathematics examinations, between high school grades and college grades, and so on. In regression analysis, as already noted, we are not primarily interested in such a measure. Instead, we try to estimate or predict the average value of one variable on the basis of the fixed values of other variables. Thus, we may want to know whether we can predict the average score on a statistics examination by knowing a student’s score on a mathematics examination.

Regression and correlation have some fundamental differences that are worth mentioning. In regression analysis there is an asymmetry in the way the dependent and explanatory variables are treated. The dependent variable is assumed to be statistical, random, or stochastic, that is, to have a probability distribution. The explanatory variables, on the other hand, are assumed to have fixed values (in repeated sampling),<sup>7</sup> which was made explicit in the definition of regression given in Section 1.2. Thus, in Figure 1.2 we assumed that the variable age was fixed at given levels and height measurements were obtained at these levels. In correlation analysis, on the

<sup>5</sup>M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, Charles Griffin Publishers, New York, 1961, vol. 2, chap. 26, p. 279.

<sup>6</sup>But as we shall see in Chap. 3, classical regression analysis is based on the assumption that the model used in the analysis is the correct model. Therefore, the direction of causality may be implicit in the model postulated.

<sup>7</sup>It is crucial to note that the explanatory variables may be intrinsically stochastic, but for the purpose of regression analysis we assume that their values are fixed in repeated sampling (that is,  $X$  assumes the same values in various samples), thus rendering them in effect non-random or nonstochastic. But more on this in Chap. 3, Sec. 3.2.

other hand, we treat any (two) variables symmetrically; there is no distinction between the dependent and explanatory variables. After all, the correlation between scores on mathematics and statistics examinations is the same as that between scores on statistics and mathematics examinations. Moreover, both variables are assumed to be random. As we shall see, most of the correlation theory is based on the assumption of randomness of variables, whereas most of the regression theory to be expounded in this book is conditional upon the assumption that the dependent variable is stochastic but the explanatory variables are fixed or nonstochastic.<sup>8</sup>

## 1.6 TERMINOLOGY AND NOTATION

Before we proceed to a formal analysis of regression theory, let us dwell briefly on the matter of terminology and notation. In the literature the terms *dependent variable* and *explanatory variable* are described variously. A representative list is:

Dependent variable	Explanatory variable
⇕	⇕
Explained variable	Independent variable
⇕	⇕
Predictand	Predictor
⇕	⇕
<b>Regressand</b>	<b>Regressor</b>
⇕	⇕
Response	Stimulus
⇕	⇕
Endogenous	Exogenous
⇕	⇕
Outcome	Covariate
⇕	⇕
Controlled variable	Control variable

Although it is a matter of personal taste and tradition, in this text we will use the dependent variable/explanatory variable or the more neutral, regressand and regressor terminology.

If we are studying the dependence of a variable on only a single explanatory variable, such as that of consumption expenditure on real income, such a study is known as *simple*, or **two-variable, regression analysis**. However, if we are studying the dependence of one variable on more than

<sup>8</sup>In advanced treatment of econometrics, one can relax the assumption that the explanatory variables are nonstochastic (see introduction to Part II).



one explanatory variable, as in the crop-yield, rainfall, temperature, sunshine, and fertilizer examples, it is known as **multiple regression analysis**. In other words, in two-variable regression there is only one explanatory variable, whereas in multiple regression there is more than one explanatory variable.

The term **random** is a synonym for the term **stochastic**. As noted earlier, a random or stochastic variable is a variable that can take on any set of values, positive or negative, with a given probability.<sup>9</sup>

Unless stated otherwise, the letter  $Y$  will denote the dependent variable and the  $X$ 's ( $X_1, X_2, \dots, X_k$ ) will denote the explanatory variables,  $X_k$  being the  $k$ th explanatory variable. The subscript  $i$  or  $t$  will denote the  $i$ th or the  $t$ th observation or value.  $X_{ki}$  (or  $X_{kt}$ ) will denote the  $i$ th (or  $t$ th) observation on variable  $X_k$ .  $N$  (or  $T$ ) will denote the total number of observations or values in the population, and  $n$  (or  $t$ ) the total number of observations in a sample. As a matter of convention, the observation subscript  $i$  will be used for **cross-sectional data** (i.e., data collected at one point in time) and the subscript  $t$  will be used for **time series data** (i.e., data collected over a period of time). The nature of cross-sectional and time series data, as well as the important topic of the nature and sources of data for empirical analysis, is discussed in the following section.

## 1.7 THE NATURE AND SOURCES OF DATA FOR ECONOMIC ANALYSIS<sup>10</sup>

The success of any econometric analysis ultimately depends on the availability of the appropriate data. It is therefore essential that we spend some time discussing the nature, sources, and limitations of the data that one may encounter in empirical analysis.

### Types of Data

Three types of data may be available for empirical analysis: **time series**, **cross-section**, and **pooled** (i.e., combination of time series and cross-section) data.

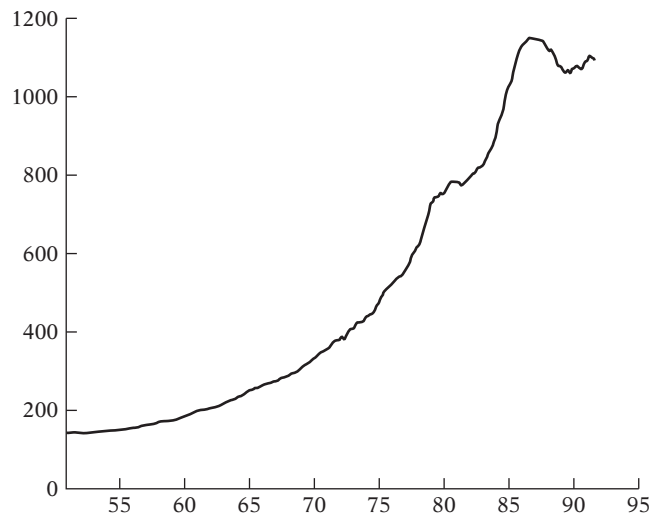
**Time Series Data** The data shown in Table I.1 of the Introduction are an example of time series data. A *time series* is a set of observations on the values that a variable takes at different times. Such data may be collected at regular time intervals, such as **daily** (e.g., stock prices, weather reports), **weekly** (e.g., money supply figures), **monthly** [e.g., the unemployment rate, the Consumer Price Index (CPI)], **quarterly** (e.g., GDP), **annually** (e.g.,

<sup>9</sup>See **App. A** for formal definition and further details.

<sup>10</sup>For an informative account, see Michael D. Intriligator, *Econometric Models, Techniques, and Applications*, Prentice Hall, Englewood Cliffs, N.J., 1978, chap. 3.

government budgets), **quinquennially**, that is, every 5 years (e.g., the census of manufactures), or **decennially** (e.g., the census of population). Sometime data are available both quarterly as well as annually, as in the case of the data on GDP and consumer expenditure. With the advent of high-speed computers, data can now be collected over an extremely short interval of time, such as the data on stock prices, which can be obtained literally continuously (the so-called *real-time quote*).

Although time series data are used heavily in econometric studies, they present special problems for econometricians. As we will show in chapters on **time series econometrics** later on, most empirical work based on time series data assumes that the underlying time series is **stationary**. Although it is too early to introduce the precise technical meaning of stationarity at this juncture, *loosely speaking a time series is stationary if its mean and variance do not vary systematically over time*. To see what this means, consider Figure 1.5, which depicts the behavior of the M1 money supply in the United States from January 1, 1959, to July 31, 1999. (The actual data are given in exercise 1.4.) As you can see from this figure, the M1 money supply shows a steady upward **trend** as well as variability over the years, suggesting that the M1 time series is not stationary.<sup>11</sup> We will explore this topic fully in Chapter 21.



**FIGURE 1.5** M1 money supply: United States, 1951:01–1999:09.

<sup>11</sup>To see this more clearly, we divided the data into four time periods: 1951:01 to 1962:12; 1963:01 to 1974:12; 1975:01 to 1986:12, and 1987:01 to 1999:09. For these subperiods the mean values of the money supply (with corresponding standard deviations in parentheses) were, respectively, 165.88 (23.27), 323.20 (72.66), 788.12 (195.43), and 1099 (27.84), all figures in billions of dollars. This is a rough indication of the fact that the money supply over the entire period was not stationary.

**Cross-Section Data** Cross-section data are data on one or more variables collected *at the same point in time*, such as the census of population conducted by the Census Bureau every 10 years (the latest being in year 2000), the surveys of consumer expenditures conducted by the University of Michigan, and, of course, the opinion polls by Gallup and umpteen other organizations. A concrete example of cross-sectional data is given in Table 1.1 This table gives data on egg production and egg prices for the 50 states in the union for 1990 and 1991. For each year the data on the 50 states are cross-sectional data. Thus, in Table 1.1 we have two cross-sectional samples.

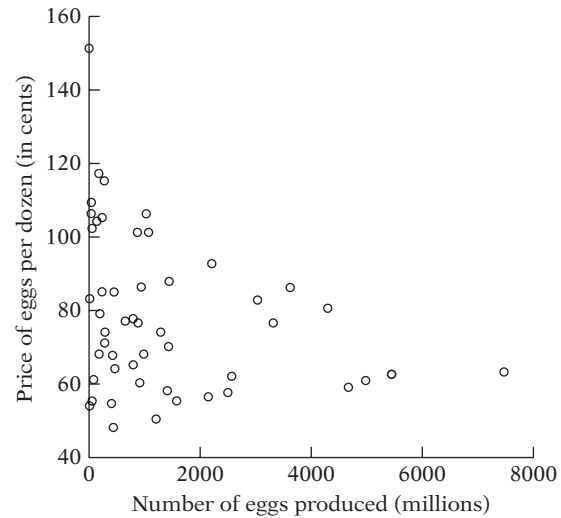
Just as time series data create their own special problems (because of the stationarity issue), cross-sectional data too have their own problems, specifically the problem of *heterogeneity*. From the data given in Table 1.1 we see that we have some states that produce huge amounts of eggs (e.g., Pennsylvania) and some that produce very little (e.g., Alaska). When we

**TABLE 1.1** U.S. EGG PRODUCTION

State	Y <sub>1</sub>	Y <sub>2</sub>	X <sub>1</sub>	X <sub>2</sub>	State	Y <sub>1</sub>	Y <sub>2</sub>	X <sub>1</sub>	X <sub>2</sub>
AL	2,206	2,186	92.7	91.4	MT	172	164	68.0	66.0
AK	0.7	0.7	151.0	149.0	NE	1,202	1,400	50.3	48.9
AZ	73	74	61.0	56.0	NV	2.2	1.8	53.9	52.7
AR	3,620	3,737	86.3	91.8	NH	43	49	109.0	104.0
CA	7,472	7,444	63.4	58.4	NJ	442	491	85.0	83.0
CO	788	873	77.8	73.0	NM	283	302	74.0	70.0
CT	1,029	948	106.0	104.0	NY	975	987	68.1	64.0
DE	168	164	117.0	113.0	NC	3,033	3,045	82.8	78.7
FL	2,586	2,537	62.0	57.2	ND	51	45	55.2	48.0
GA	4,302	4,301	80.6	80.8	OH	4,667	4,637	59.1	54.7
HI	227.5	224.5	85.0	85.5	OK	869	830	101.0	100.0
ID	187	203	79.1	72.9	OR	652	686	77.0	74.6
IL	793	809	65.0	70.5	PA	4,976	5,130	61.0	52.0
IN	5,445	5,290	62.7	60.1	RI	53	50	102.0	99.0
IA	2,151	2,247	56.5	53.0	SC	1,422	1,420	70.1	65.9
KS	404	389	54.5	47.8	SD	435	602	48.0	45.8
KY	412	483	67.7	73.5	TN	277	279	71.0	80.7
LA	273	254	115.0	115.0	TX	3,317	3,356	76.7	72.6
ME	1,069	1,070	101.0	97.0	UT	456	486	64.0	59.0
MD	885	898	76.6	75.4	VT	31	30	106.0	102.0
MA	235	237	105.0	102.0	VA	943	988	86.3	81.2
MI	1,406	1,396	58.0	53.8	WA	1,287	1,313	74.1	71.5
MN	2,499	2,697	57.7	54.0	WV	136	174	104.0	109.0
MS	1,434	1,468	87.8	86.7	WI	910	873	60.1	54.0
MO	1,580	1,622	55.4	51.5	WY	1.7	1.7	83.0	83.0

Note: Y<sub>1</sub> = eggs produced in 1990 (millions)  
Y<sub>2</sub> = eggs produced in 1991 (millions)  
X<sub>1</sub> = price per dozen (cents) in 1990  
X<sub>2</sub> = price per dozen (cents) in 1991

Source: *World Almanac*, 1993, p. 119. The data are from the Economic Research Service, U.S. Department of Agriculture.



**FIGURE 1.6** Relationship between eggs produced and prices, 1990.

include such heterogeneous units in a statistical analysis, the **size** or **scale effect** must be taken into account so as not to mix apples with oranges. To see this clearly, we plot in Figure 1.6 the data on eggs produced and their prices in 50 states for the year 1990. This figure shows how widely scattered the observations are. In Chapter 11 we will see how the scale effect can be an important factor in assessing relationships among economic variables.

**Pooled Data** In pooled, or combined, data are elements of both time series and cross-section data. The data in Table 1.1 are an example of pooled data. For each year we have 50 cross-sectional observations and for each state we have two time series observations on prices and output of eggs, a total of 100 pooled (or combined) observations. Likewise, the data given in exercise 1.1 are pooled data in that the Consumer Price Index (CPI) for each country for 1973–1997 is time series data, whereas the data on the CPI for the seven countries for a single year are cross-sectional data. In the pooled data we have 175 observations—25 annual observations for each of the seven countries.

**Panel, Longitudinal, or Micropanel Data** This is a special type of pooled data in which the *same* cross-sectional unit (say, a family or a firm) is surveyed over time. For example, the U.S. Department of Commerce carries out a census of housing at periodic intervals. At each periodic survey the same household (or the people living at the same address) is interviewed to find out if there has been any change in the housing and financial conditions of that household since the last survey. By interviewing the same household periodically, the panel data provides very useful information on the dynamics of household behavior, as we shall see in Chapter 16.

### The Sources of Data<sup>12</sup>

The data used in empirical analysis may be collected by a governmental agency (e.g., the Department of Commerce), an international agency (e.g., the International Monetary Fund (IMF) or the World Bank), a private organization (e.g., the Standard & Poor's Corporation), or an individual. Literally, there are thousands of such agencies collecting data for one purpose or another.

**The Internet** The Internet has literally revolutionized data gathering. If you just “surf the net” with a keyword (e.g., exchange rates), you will be swamped with all kinds of data sources. In **Appendix E** we provide some of the frequently visited web sites that provide economic and financial data of all sorts. Most of the data can be downloaded without much cost. You may want to bookmark the various web sites that might provide you with useful economic data.

The data collected by various agencies may be **experimental** or **nonexperimental**. In experimental data, often collected in the natural sciences, the investigator may want to collect data while holding certain factors constant in order to assess the impact of some factors on a given phenomenon. For instance, in assessing the impact of obesity on blood pressure, the researcher would want to collect data while holding constant the eating, smoking, and drinking habits of the people in order to minimize the influence of these variables on blood pressure.

In the social sciences, the data that one generally encounters are nonexperimental in nature, that is, not subject to the control of the researcher.<sup>13</sup> For example, the data on GNP, unemployment, stock prices, etc., are not directly under the control of the investigator. As we shall see, this lack of control often creates special problems for the researcher in pinning down the exact cause or causes affecting a particular situation. For example, is it the money supply that determines the (nominal) GDP or is it the other way round?

### The Accuracy of Data<sup>14</sup>

Although plenty of data are available for economic research, the quality of the data is often not that good. There are several reasons for that. First, as noted, most social science data are nonexperimental in nature. Therefore, there is the possibility of observational errors, either of omission or commission. Second, even in experimentally collected data errors of measurement arise from approximations and roundoffs. Third, in questionnaire-type surveys, the problem of nonresponse can be serious; a researcher is lucky to

<sup>12</sup>For an illuminating account, see Albert T. Somers, *The U.S. Economy Demystified: What the Major Economic Statistics Mean and their Significance for Business*, D.C. Heath, Lexington, Mass., 1985.

<sup>13</sup>In the social sciences too sometimes one can have a controlled experiment. An example is given in exercise 1.6.

<sup>14</sup>For a critical review, see O. Morgenstern, *The Accuracy of Economic Observations*, 2d ed., Princeton University Press, Princeton, N.J., 1963.

get a 40 percent response to a questionnaire. Analysis based on such partial response may not truly reflect the behavior of the 60 percent who did not respond, thereby leading to what is known as (sample) **selectivity bias**. Then there is the further problem that those who respond to the questionnaire may not answer all the questions, especially questions of financially sensitive nature, thus leading to additional selectivity bias. Fourth, the sampling methods used in obtaining the data may vary so widely that it is often difficult to compare the results obtained from the various samples. Fifth, economic data are generally available at a highly aggregate level. For example, most macrodata (e.g., GNP, employment, inflation, unemployment) are available for the economy as a whole or at the most for some broad geographical regions. Such highly aggregated data may not tell us much about the individual or microunits that may be the ultimate object of study. Sixth, because of confidentiality, certain data can be published only in highly aggregate form. The IRS, for example, is not allowed by law to disclose data on individual tax returns; it can only release some broad summary data. Therefore, if one wants to find out how much individuals with a certain level of income spent on health care, one cannot do that analysis except at a very highly aggregate level. But such macroanalysis often fails to reveal the dynamics of the behavior of the microunits. Similarly, the Department of Commerce, which conducts the census of business every 5 years, is not allowed to disclose information on production, employment, energy consumption, research and development expenditure, etc., at the firm level. It is therefore difficult to study the interfirm differences on these items.

Because of all these and many other problems, **the researcher should always keep in mind that the results of research are only as good as the quality of the data**. Therefore, if in given situations researchers find that the results of the research are “unsatisfactory,” the cause may be not that they used the wrong model but that the quality of the data was poor. Unfortunately, because of the nonexperimental nature of the data used in most social science studies, researchers very often have no choice but to depend on the available data. But they should always keep in mind that the data used may not be the best and should try not to be too dogmatic about the results obtained from a given study, especially when the quality of the data is suspect.

#### A Note on the Measurement Scales of Variables<sup>15</sup>

The variables that we will generally encounter fall into four broad categories: *ratio scale*, *interval scale*, *ordinal scale*, and *nominal scale*. It is important that we understand each.

**Ratio Scale** For a variable  $X$ , taking two values,  $X_1$  and  $X_2$ , the ratio  $X_1/X_2$  and the distance  $(X_2 - X_1)$  are meaningful quantities. Also, there is a

<sup>15</sup>The following discussion relies heavily on Aris Spanos, *Probability Theory and Statistical Inference: Econometric Modeling with Observational Data*, Cambridge University Press, New York, 1999, p. 24.

natural ordering (ascending or descending) of the values along the scale. Therefore, comparisons such as  $X_2 \leq X_1$  or  $X_2 \geq X_1$  are meaningful. Most economic variables belong to this category. Thus, it is meaningful to ask how big is this year's GDP compared with the previous year's GDP.

**Interval Scale** An interval scale variable satisfies the last two properties of the ratio scale variable but not the first. Thus, the distance between two time periods, say (2000–1995) is meaningful, but not the ratio of two time periods (2000/1995).

**Ordinal Scale** A variable belongs to this category only if it satisfies the third property of the ratio scale (i.e., natural ordering). Examples are grading systems (A, B, C grades) or income class (upper, middle, lower). For these variables the ordering exists but the distances between the categories cannot be quantified. Students of economics will recall the *indifference curves* between two goods, each higher indifference curve indicating higher level of utility, but one cannot quantify by how much one indifference curve is higher than the others.

**Nominal Scale** Variables in this category have none of the features of the ratio scale variables. Variables such as gender (male, female) and marital status (married, unmarried, divorced, separated) simply denote categories. *Question:* What is the reason why such variables cannot be expressed on the ratio, interval, or ordinal scales?

As we shall see, econometric techniques that may be suitable for ratio scale variables may not be suitable for nominal scale variables. Therefore, it is important to bear in mind the distinctions among the four types of measurement scales discussed above.

## 1.8 SUMMARY AND CONCLUSIONS

1. The key idea behind regression analysis is the statistical dependence of one variable, the dependent variable, on one or more other variables, the explanatory variables.

2. The objective of such analysis is to estimate and/or predict the mean or average value of the dependent variable on the basis of the known or fixed values of the explanatory variables.

3. In practice the success of regression analysis depends on the availability of the appropriate data. This chapter discussed the nature, sources, and limitations of the data that are generally available for research, especially in the social sciences.

4. In any research, the researcher should clearly state the sources of the data used in the analysis, their definitions, their methods of collection, and any gaps or omissions in the data as well as any revisions in the data. Keep in mind that the macroeconomic data published by the government are often revised.



5. Since the reader may not have the time, energy, or resources to track down the data, the reader has the right to presume that the data used by the researcher are properly gathered and that the computations and analysis are correct.

## EXERCISES

- 1.1. Table 1.2 gives data on the Consumer Price Index (CPI) for seven industrialized countries with 1982–1984 = 100 as the base of the index.
- From the given data, compute the inflation rate for each country.<sup>16</sup>
  - Plot the inflation rate for each country against time (i.e., use the horizontal axis for time and the vertical axis for the inflation rate.)
  - What broad conclusions can you draw about the inflation experience in the seven countries?
  - Which country's inflation rate seems to be most variable? Can you offer any explanation?

TABLE 1.2 CPI IN SEVEN INDUSTRIAL COUNTRIES, 1973–1997 (1982–1984 = 100)

Year	Canada	France	Germany	Italy	Japan	U.K.	U.S.
1973	40.80000	34.60000	62.80000	20.60000	47.90000	27.90000	44.40000
1974	45.20000	39.30000	67.10000	24.60000	59.00000	32.30000	49.30000
1975	50.10000	43.90000	71.10000	28.80000	65.90000	40.20000	53.80000
1976	53.90000	48.10000	74.20000	33.60000	72.20000	46.80000	56.90000
1977	58.10000	52.70000	76.90000	40.10000	78.10000	54.20000	60.60000
1978	63.30000	57.50000	79.00000	45.10000	81.40000	58.70000	65.20000
1979	69.20000	63.60000	82.20000	52.10000	84.40000	66.60000	72.60000
1980	76.10000	72.30000	86.70000	63.20000	90.90000	78.50000	82.40000
1981	85.60000	81.90000	92.20000	75.40000	95.30000	87.90000	90.90000
1982	94.90000	91.70000	97.10000	87.70000	98.10000	95.40000	96.50000
1983	100.4000	100.4000	100.3000	100.8000	99.80000	99.80000	99.60000
1984	104.7000	108.1000	102.7000	111.5000	102.1000	104.8000	103.9000
1985	109.0000	114.4000	104.8000	121.1000	104.1000	111.1000	107.6000
1986	113.5000	117.3000	104.7000	128.5000	104.8000	114.9000	109.6000
1987	118.4000	121.1000	104.9000	134.4000	104.8000	119.7000	113.6000
1988	123.2000	124.4000	106.3000	141.1000	105.6000	125.6000	118.3000
1989	129.3000	128.7000	109.2000	150.4000	108.1000	135.3000	124.0000
1990	135.5000	133.0000	112.2000	159.6000	111.4000	148.2000	130.7000
1991	143.1000	137.2000	116.3000	169.8000	115.0000	156.9000	136.2000
1992	145.3000	140.5000	122.1000	178.8000	116.9000	162.7000	140.3000
1993	147.9000	143.5000	127.6000	186.4000	118.4000	165.3000	144.5000
1994	148.2000	145.8000	131.1000	193.7000	119.3000	169.4000	148.2000
1995	151.4000	148.4000	133.5000	204.1000	119.1000	175.1000	152.4000
1996	153.8000	151.4000	135.5000	212.0000	119.3000	179.4000	156.9000
1997	156.3000	153.2000	137.8000	215.7000	121.3000	185.0000	160.5000

<sup>16</sup>Subtract from the current year's CPI the CPI from the previous year, divide the difference by the previous year's CPI, and multiply the result by 100. Thus, the inflation rate for Canada for 1974 is  $[(45.2 - 40.8)/40.8] \times 100 = 10.78\%$  (approx.).

- 1.2.
  - a. Plot the inflation rate of Canada, France, Germany, Italy, Japan, and the United Kingdom against the United States inflation rate.
  - b. Comment generally about the behavior of the inflation rate in the six countries vis-à-vis the U.S. inflation rate.
  - c. If you find that the six countries' inflation rates move in the same direction as the U.S. inflation rate, would that suggest that U.S. inflation "causes" inflation in the other countries? Why or why not?
- 1.3. Table 1.3 gives the foreign exchange rates for seven industrialized countries for years 1977–1998. Except for the United Kingdom, the exchange rate is defined as the units of foreign currency for one U.S. dollar; for the United Kingdom, it is defined as the number of U.S. dollars for one U.K. pound.
  - a. Plot these exchange rates against time and comment on the general behavior of the exchange rates over the given time period.
  - b. The dollar is said to *appreciate* if it can buy more units of a foreign currency. Contrarily, it is said to *depreciate* if it buys fewer units of a foreign currency. Over the time period 1977–1998, what has been the general behavior of the U.S. dollar? Incidentally, look up any textbook on macroeconomics or international economics to find out what factors determine the appreciation or depreciation of a currency.
- 1.4. The data behind the M1 money supply in Figure 1.5 are given in Table 1.4. Can you give reasons why the money supply has been increasing over the time period shown in the table?

**TABLE 1.3** EXCHANGE RATES FOR SEVEN COUNTRIES: 1977–1998

Year	Canada	France	Germany	Japan	Sweden	Switzerland	U.K.
1977	1.063300	4.916100	2.323600	268.6200	4.480200	2.406500	1.744900
1978	1.140500	4.509100	2.009700	210.3900	4.520700	1.790700	1.918400
1979	1.171300	4.256700	1.834300	219.0200	4.289300	1.664400	2.122400
1980	1.169300	4.225100	1.817500	226.6300	4.231000	1.677200	2.324600
1981	1.199000	5.439700	2.263200	220.6300	5.066000	1.967500	2.024300
1982	1.234400	6.579400	2.428100	249.0600	6.283900	2.032700	1.748000
1983	1.232500	7.620400	2.553900	237.5500	7.671800	2.100700	1.515900
1984	1.295200	8.735600	2.845500	237.4600	8.270800	2.350000	1.336800
1985	1.365900	8.980000	2.942000	238.4700	8.603200	2.455200	1.297400
1986	1.389600	6.925700	2.170500	168.3500	7.127300	1.797900	1.467700
1987	1.325900	6.012200	1.798100	144.6000	6.346900	1.491800	1.639800
1988	1.230600	5.959500	1.757000	128.1700	6.137000	1.464300	1.781300
1989	1.184200	6.380200	1.880800	138.0700	6.455900	1.636900	1.638200
1990	1.166800	5.446700	1.616600	145.0000	5.923100	1.390100	1.784100
1991	1.146000	5.646800	1.661000	134.5900	6.052100	1.435600	1.767400
1992	1.208500	5.293500	1.561800	126.7800	5.825800	1.406400	1.766300
1993	1.290200	5.666900	1.654500	111.0800	7.795600	1.478100	1.501600
1994	1.366400	5.545900	1.621600	102.1800	7.716100	1.366700	1.531900
1995	1.372500	4.986400	1.432100	93.96000	7.140600	1.181200	1.578500
1996	1.363800	5.115800	1.504900	108.7800	6.708200	1.236100	1.560700
1997	1.384900	5.839300	1.734800	121.0600	7.644600	1.451400	1.637600
1998	1.483600	5.899500	1.759700	130.9900	7.952200	1.450600	1.657300

Source: *Economic Report of the President*, January 2000 and January 2001.

**TABLE 1.4** SEASONALLY ADJUSTED M1 SUPPLY: 1959:01–1999:09 (BILLIONS OF DOLLARS)

1959:01	138.8900	139.3900	139.7400	139.6900	140.6800	141.1700
1959:07	141.7000	141.9000	141.0100	140.4700	140.3800	139.9500
1960:01	139.9800	139.8700	139.7500	139.5600	139.6100	139.5800
1960:07	140.1800	141.3100	141.1800	140.9200	140.8600	140.6900
1961:01	141.0600	141.6000	141.8700	142.1300	142.6600	142.8800
1961:07	142.9200	143.4900	143.7800	144.1400	144.7600	145.2000
1962:01	145.2400	145.6600	145.9600	146.4000	146.8400	146.5800
1962:07	146.4600	146.5700	146.3000	146.7100	147.2900	147.8200
1963:01	148.2600	148.9000	149.1700	149.7000	150.3900	150.4300
1963:07	151.3400	151.7800	151.9800	152.5500	153.6500	153.2900
1964:01	153.7400	154.3100	154.4800	154.7700	155.3300	155.6200
1964:07	156.8000	157.8200	158.7500	159.2400	159.9600	160.3000
1965:01	160.7100	160.9400	161.4700	162.0300	161.7000	162.1900
1965:07	163.0500	163.6800	164.8500	165.9700	166.7100	167.8500
1966:01	169.0800	169.6200	170.5100	171.8100	171.3300	171.5700
1966:07	170.3100	170.8100	171.9700	171.1600	171.3800	172.0300
1967:01	171.8600	172.9900	174.8100	174.1700	175.6800	177.0200
1967:07	178.1300	179.7100	180.6800	181.6400	182.3800	183.2600
1968:01	184.3300	184.7100	185.4700	186.6000	187.9900	189.4200
1968:07	190.4900	191.8400	192.7400	194.0200	196.0200	197.4100
1969:01	198.6900	199.3500	200.0200	200.7100	200.8100	201.2700
1969:07	201.6600	201.7300	202.1000	202.9000	203.5700	203.8800
1970:01	206.2200	205.0000	205.7500	206.7200	207.2200	207.5400
1970:07	207.9800	209.9300	211.8000	212.8800	213.6600	214.4100
1971:01	215.5400	217.4200	218.7700	220.0000	222.0200	223.4500
1971:07	224.8500	225.5800	226.4700	227.1600	227.7600	228.3200
1972:01	230.0900	232.3200	234.3000	235.5800	235.8900	236.6200
1972:07	238.7900	240.9300	243.1800	245.0200	246.4100	249.2500
1973:01	251.4700	252.1500	251.6700	252.7400	254.8900	256.6900
1973:07	257.5400	257.7600	257.8600	259.0400	260.9800	262.8800
1974:01	263.7600	265.3100	266.6800	267.2000	267.5600	268.4400
1974:07	269.2700	270.1200	271.0500	272.3500	273.7100	274.2000
1975:01	273.9000	275.0000	276.4200	276.1700	279.2000	282.4300
1975:07	283.6800	284.1500	285.6900	285.3900	286.8300	287.0700
1976:01	288.4200	290.7600	292.7000	294.6600	295.9300	296.1600
1976:07	297.2000	299.0500	299.6700	302.0400	303.5900	306.2500
1977:01	308.2600	311.5400	313.9400	316.0200	317.1900	318.7100
1977:07	320.1900	322.2700	324.4800	326.4000	328.6400	330.8700
1978:01	334.4000	335.3000	336.9600	339.9200	344.8600	346.8000
1978:07	347.6300	349.6600	352.2600	353.3500	355.4100	357.2800
1979:01	358.6000	359.9100	362.4500	368.0500	369.5900	373.3400
1979:07	377.2100	378.8200	379.2800	380.8700	380.8100	381.7700
1980:01	385.8500	389.7000	388.1300	383.4400	384.6000	389.4600
1980:07	394.9100	400.0600	405.3600	409.0600	410.3700	408.0600
1981:01	410.8300	414.3800	418.6900	427.0600	424.4300	425.5000
1981:07	427.9000	427.8500	427.4600	428.4500	430.8800	436.1700
1982:01	442.1300	441.4900	442.3700	446.7800	446.5300	447.8900
1982:07	449.0900	452.4900	457.5000	464.5700	471.1200	474.3000
1983:01	476.6800	483.8500	490.1800	492.7700	499.7800	504.3500
1983:07	508.9600	511.6000	513.4100	517.2100	518.5300	520.7900

(Continued)

TABLE 1.4 (Continued)

1984:01	524.4000	526.9900	530.7800	534.0300	536.5900	540.5400
1984:07	542.1300	542.3900	543.8600	543.8700	547.3200	551.1900
1985:01	555.6600	562.4800	565.7400	569.5500	575.0700	583.1700
1985:07	590.8200	598.0600	604.4700	607.9100	611.8300	619.3600
1986:01	620.4000	624.1400	632.8100	640.3500	652.0100	661.5200
1986:07	672.2000	680.7700	688.5100	695.2600	705.2400	724.2800
1987:01	729.3400	729.8400	733.0100	743.3900	746.0000	743.7200
1987:07	744.9600	746.9600	748.6600	756.5000	752.8300	749.6800
1988:01	755.5500	757.0700	761.1800	767.5700	771.6800	779.1000
1988:07	783.4000	785.0800	784.8200	783.6300	784.4600	786.2600
1989:01	784.9200	783.4000	782.7400	778.8200	774.7900	774.2200
1989:07	779.7100	781.1400	782.2000	787.0500	787.9500	792.5700
1990:01	794.9300	797.6500	801.2500	806.2400	804.3600	810.3300
1990:07	811.8000	817.8500	821.8300	820.3000	822.0600	824.5600
1991:01	826.7300	832.4000	838.6200	842.7300	848.9600	858.3300
1991:07	862.9500	868.6500	871.5600	878.4000	887.9500	896.7000
1992:01	910.4900	925.1300	936.0000	943.8900	950.7800	954.7100
1992:07	964.6000	975.7100	988.8400	1004.340	1016.040	1024.450
1993:01	1030.900	1033.150	1037.990	1047.470	1066.220	1075.610
1993:07	1085.880	1095.560	1105.430	1113.800	1123.900	1129.310
1994:01	1132.200	1136.130	1139.910	1141.420	1142.850	1145.650
1994:07	1151.490	1151.390	1152.440	1150.410	1150.440	1149.750
1995:01	1150.640	1146.740	1146.520	1149.480	1144.650	1144.240
1995:07	1146.500	1146.100	1142.270	1136.430	1133.550	1126.730
1996:01	1122.580	1117.530	1122.590	1124.520	1116.300	1115.470
1996:07	1112.340	1102.180	1095.610	1082.560	1080.490	1081.340
1997:01	1080.520	1076.200	1072.420	1067.450	1063.370	1065.990
1997:07	1067.570	1072.080	1064.820	1062.060	1067.530	1074.870
1998:01	1073.810	1076.020	1080.650	1082.090	1078.170	1077.780
1998:07	1075.370	1072.210	1074.650	1080.400	1088.960	1093.350
1999:01	1091.000	1092.650	1102.010	1108.400	1104.750	1101.110
1999:07	1099.530	1102.400	1093.460			

Source: Board of Governors, Federal Reserve Bank, USA.

- 1.5. Suppose you were to develop an economic model of criminal activities, say, the hours spent in criminal activities (e.g., selling illegal drugs). What variables would you consider in developing such a model? See if your model matches the one developed by the Nobel laureate economist Gary Becker.<sup>17</sup>
- 1.6. *Controlled experiments in economics:* On April 7, 2000, President Clinton signed into law a bill passed by both Houses of the U.S. Congress that lifted earnings limitations on Social Security recipients. Until then, recipients between the ages of 65 and 69 who earned more than \$17,000 a year would lose 1 dollar's worth of Social Security benefit for every 3 dollars of income earned in excess of \$17,000. How would you devise a study to assess the impact of this change in the law? *Note:* There was no income limitation for recipients over the age of 70 under the old law.

<sup>17</sup>G. S. Becker, "Crime and Punishment: An Economic Approach," *Journal of Political Economy*, vol. 76, 1968, pp. 169–217.

**TABLE 1.5** IMPACT OF ADVERTISING EXPENDITURE

Firm	Impressions, millions	Expenditure, millions of 1983 dollars
1. Miller Lite	32.1	50.1
2. Pepsi	99.6	74.1
3. Stroh's	11.7	19.3
4. Fed'l Express	21.9	22.9
5. Burger King	60.8	82.4
6. Coca Cola	78.6	40.1
7. McDonald's	92.4	185.9
8. MCI	50.7	26.9
9. Diet Cola	21.4	20.4
10. Ford	40.1	166.2
11. Levi's	40.8	27.0
12. Bud Lite	10.4	45.6
13. ATT/Bell	88.9	154.9
14. Calvin Klein	12.0	5.0
15. Wendy's	29.2	49.7
16. Polaroid	38.0	26.9
17. Shasta	10.0	5.7
18. Meow Mix	12.3	7.6
19. Oscar Meyer	23.4	9.2
20. Crest	71.1	32.4
21. Kibbles 'N Bits	4.4	6.1

Source: <http://lib.stat.cmu.edu/DASL/Datafiles/tvadsdat.html>

**1.7.** The data presented in Table 1.5 was published in the March 1, 1984 issue of the *Wall Street Journal*. It relates to the advertising budget (in millions of dollars) of 21 firms for 1983 and millions of impressions retained per week by the viewers of the products of these firms. The data are based on a survey of 4000 adults in which users of the products were asked to cite a commercial they had seen for the product category in the past week.

- a. Plot impressions on the vertical axis and advertising expenditure on the horizontal axis.
- b. What can you say about the nature of the relationship between the two variables?
- c. Looking at your graph, do you think it pays to advertise? Think about all those commercials shown on Super Bowl Sunday or during the World Series.

*Note:* We will explore further the data given in Table 1.5 in subsequent chapters.

# 2

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## TWO-VARIABLE REGRESSION ANALYSIS: SOME BASIC IDEAS

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In Chapter 1 we discussed the concept of regression in broad terms. In this chapter we approach the subject somewhat formally. Specifically, this and the following two chapters introduce the reader to the theory underlying the simplest possible regression analysis, namely, the **bivariate**, or **two-variable**, regression in which the dependent variable (the regressand) is related to a single explanatory variable (the regressor). This case is considered first, not because of its practical adequacy, but because it presents the fundamental ideas of regression analysis as simply as possible and some of these ideas can be illustrated with the aid of two-dimensional graphs. Moreover, as we shall see, the more general **multiple** regression analysis in which the regressand is related to one or more regressors is in many ways a logical extension of the two-variable case.

### 2.1 A HYPOTHETICAL EXAMPLE<sup>1</sup>

As noted in Section 1.2, regression analysis is largely concerned with estimating and/or predicting the (population) mean value of the dependent variable on the basis of the known or fixed values of the explanatory variable(s).<sup>2</sup> To understand this, consider the data given in Table 2.1. The data

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<sup>1</sup>The reader whose statistical knowledge has become somewhat rusty may want to freshen it up by reading the statistical appendix, **App. A**, before reading this chapter.

<sup>2</sup>The *expected value*, or *expectation*, or *population mean of a random variable*  $Y$  is denoted by the symbol  $E(Y)$ . On the other hand, the mean value computed from a sample of values from the  $Y$  population is denoted as  $\bar{Y}$ , read as  $Y$  bar.

TABLE 2.1 WEEKLY FAMILY INCOME  $X$ , \$

$Y \downarrow$ / $X \rightarrow$	80	100	120	140	160	180	200	220	240	260
Weekly family consumption expenditure $Y$ , \$	55	65	79	80	102	110	120	135	137	150
	60	70	84	93	107	115	136	137	145	152
	65	74	90	95	110	120	140	140	155	175
	70	80	94	103	116	130	144	152	165	178
	75	85	98	108	118	135	145	157	175	180
	–	88	–	113	125	140	–	160	189	185
	–	–	–	115	–	–	–	162	–	191
Total	325	462	445	707	678	750	685	1043	966	1211
Conditional means of $Y$ , $E(Y X)$	65	77	89	101	113	125	137	149	161	173

in the table refer to a total **population** of 60 families in a hypothetical community and their weekly income ( $X$ ) and weekly consumption expenditure ( $Y$ ), both in dollars. The 60 families are divided into 10 income groups (from \$80 to \$260) and the weekly expenditures of each family in the various groups are as shown in the table. Therefore, we have 10 *fixed* values of  $X$  and the corresponding  $Y$  values against each of the  $X$  values; so to speak, there are 10  $Y$  subpopulations.

There is considerable variation in weekly consumption expenditure in each income group, which can be seen clearly from Figure 2.1. But the general picture that one gets is that, despite the variability of weekly consump-

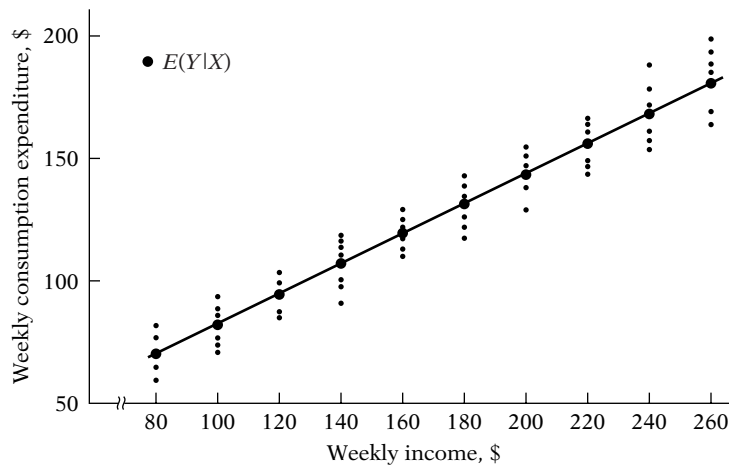


FIGURE 2.1 Conditional distribution of expenditure for various levels of income (data of Table 2.1).

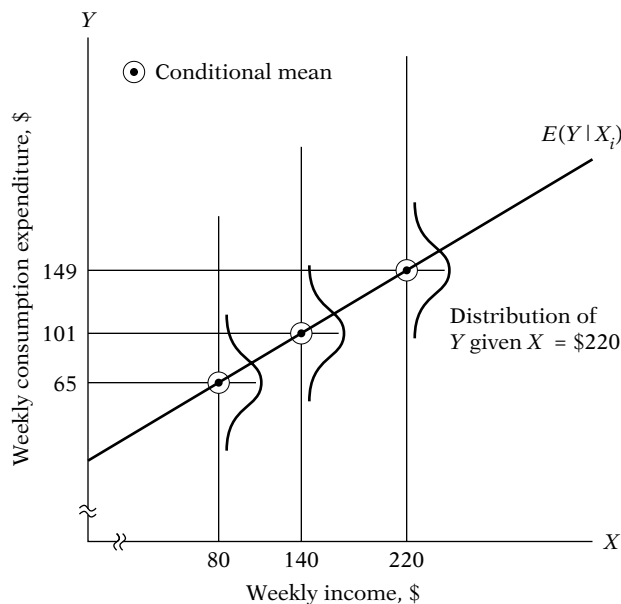




say, \$140,” we get the answer \$101 (the conditional mean). To put it differently, if we ask the question, “What is the best (mean) prediction of weekly expenditure of families with a weekly income of \$140,” the answer would be \$101. Thus the knowledge of the income level may enable us to better predict the mean value of consumption expenditure than if we do not have that knowledge.<sup>4</sup> This probably is the essence of regression analysis, as we shall discover throughout this text.

The dark circled points in Figure 2.1 show the conditional mean values of  $Y$  against the various  $X$  values. If we join these conditional mean values, we obtain what is known as the **population regression line (PRL)**, or more generally, the **population regression curve**.<sup>5</sup> More simply, it is the **regression of  $Y$  on  $X$** . The adjective “population” comes from the fact that we are dealing in this example with the entire population of 60 families. Of course, in reality a population may have many families.

*Geometrically, then, a population regression curve is simply the locus of the conditional means of the dependent variable for the fixed values of the explanatory variable(s). More simply, it is the curve connecting the means of the subpopulations of  $Y$  corresponding to the given values of the regressor  $X$ . It can be depicted as in Figure 2.2.*



**FIGURE 2.2** Population regression line (data of Table 2.1).

<sup>4</sup>I am indebted to James Davidson on this perspective. See James Davidson, *Econometric Theory*, Blackwell Publishers, Oxford, U.K., 2000, p. 11.

<sup>5</sup>In the present example the PRL is a straight line, but it could be a curve (see Figure 2.3).

This figure shows that for each  $X$  (i.e., income level) there is a population of  $Y$  values (weekly consumption expenditures) that are spread around the (conditional) mean of those  $Y$  values. For simplicity, we are assuming that these  $Y$  values are distributed symmetrically around their respective (conditional) mean values. And the regression line (or curve) passes through these (conditional) mean values.

With this background, the reader may find it instructive to reread the definition of regression given in Section 1.2.

## 2.2 THE CONCEPT OF POPULATION REGRESSION FUNCTION (PRF)

From the preceding discussion and Figures 2.1 and 2.2, it is clear that each conditional mean  $E(Y | X_i)$  is a function of  $X_i$ , where  $X_i$  is a given value of  $X$ . Symbolically,

$$E(Y | X_i) = f(X_i) \quad (2.2.1)$$

where  $f(X_i)$  denotes some function of the explanatory variable  $X$ . In our example,  $E(Y | X_i)$  is a linear function of  $X_i$ . Equation (2.2.1) is known as the **conditional expectation function (CEF)** or **population regression function (PRF)** or **population regression (PR)** for short. It states merely that the *expected value* of the distribution of  $Y$  given  $X_i$  is functionally related to  $X_i$ . In simple terms, it tells how the mean or average response of  $Y$  varies with  $X$ .

What form does the function  $f(X_i)$  assume? This is an important question because in real situations we do not have the entire population available for examination. The functional form of the PRF is therefore an empirical question, although in specific cases theory may have something to say. For example, an economist might posit that consumption expenditure is linearly related to income. Therefore, as a first approximation or a working hypothesis, we may assume that the PRF  $E(Y | X_i)$  is a linear function of  $X_i$ , say, of the type

$$E(Y | X_i) = \beta_1 + \beta_2 X_i \quad (2.2.2)$$

where  $\beta_1$  and  $\beta_2$  are unknown but fixed parameters known as the **regression coefficients**;  $\beta_1$  and  $\beta_2$  are also known as **intercept** and **slope coefficients**, respectively. Equation (2.2.1) itself is known as the **linear population regression function**. Some alternative expressions used in the literature are *linear population regression model* or simply *linear population regression*. In the sequel, the terms **regression**, **regression equation**, and **regression model** will be used synonymously.

In regression analysis our interest is in estimating the PRFs like (2.2.2), that is, estimating the values of the unknowns  $\beta_1$  and  $\beta_2$  on the basis of observations on  $Y$  and  $X$ . This topic will be studied in detail in Chapter 3.

### 2.3 THE MEANING OF THE TERM *LINEAR*

Since this text is concerned primarily with linear models like (2.2.2), it is essential to know what the term *linear* really means, for it can be interpreted in two different ways.

#### Linearity in the Variables

The first and perhaps more “natural” meaning of linearity is that the conditional expectation of  $Y$  is a linear function of  $X_i$ , such as, for example, (2.2.2).<sup>6</sup> Geometrically, the regression curve in this case is a straight line. In this interpretation, a regression function such as  $E(Y|X_i) = \beta_1 + \beta_2 X_i^2$  is not a linear function because the variable  $X$  appears with a power or index of 2.

#### Linearity in the Parameters

The second interpretation of linearity is that the conditional expectation of  $Y$ ,  $E(Y|X_i)$ , is a linear function of the parameters, the  $\beta$ 's; it may or may not be linear in the variable  $X$ .<sup>7</sup> In this interpretation  $E(Y|X_i) = \beta_1 + \beta_2 X_i^2$  is a linear (in the parameter) regression model. To see this, let us suppose  $X$  takes the value 3. Therefore,  $E(Y|X = 3) = \beta_1 + 9\beta_2$ , which is obviously linear in  $\beta_1$  and  $\beta_2$ . All the models shown in Figure 2.3 are thus linear regression models, that is, models linear in the parameters.

Now consider the model  $E(Y|X_i) = \beta_1 + \beta_2^2 X_i$ . Now suppose  $X = 3$ ; then we obtain  $E(Y|X_i) = \beta_1 + 3\beta_2^2$ , which is nonlinear in the parameter  $\beta_2$ . The preceding model is an example of a **nonlinear (in the parameter) regression model**. We will discuss such models in Chapter 14.

Of the two interpretations of linearity, linearity in the parameters is relevant for the development of the regression theory to be presented shortly. Therefore, *from now on the term “linear” regression will always mean a regression that is linear in the parameters; the  $\beta$ 's (that is, the parameters are raised to the first power only). It may or may not be linear in the explanatory variables, the  $X$ 's*. Schematically, we have Table 2.3. Thus,  $E(Y|X_i) = \beta_1 + \beta_2 X_i$ , which is linear both in the parameters and variable, is a LRM, and so is  $E(Y|X_i) = \beta_1 + \beta_2 X_i^2$ , which is linear in the parameters but nonlinear in variable  $X$ .

<sup>6</sup>A function  $Y = f(X)$  is said to be linear in  $X$  if  $X$  appears with a power or index of 1 only (that is, terms such as  $X^2$ ,  $\sqrt{X}$ , and so on, are excluded) and is not multiplied or divided by any other variable (for example,  $X \cdot Z$  or  $X/Z$ , where  $Z$  is another variable). If  $Y$  depends on  $X$  alone, another way to state that  $Y$  is linearly related to  $X$  is that the rate of change of  $Y$  with respect to  $X$  (i.e., the slope, or derivative, of  $Y$  with respect to  $X$ ,  $dY/dX$ ) is independent of the value of  $X$ . Thus, if  $Y = 4X$ ,  $dY/dX = 4$ , which is independent of the value of  $X$ . But if  $Y = 4X^2$ ,  $dY/dX = 8X$ , which is not independent of the value taken by  $X$ . Hence this function is not linear in  $X$ .

<sup>7</sup>A function is said to be linear in the parameter, say,  $\beta_1$ , if  $\beta_1$  appears with a power of 1 only and is not multiplied or divided by any other parameter (for example,  $\beta_1\beta_2$ ,  $\beta_2/\beta_1$ , and so on).

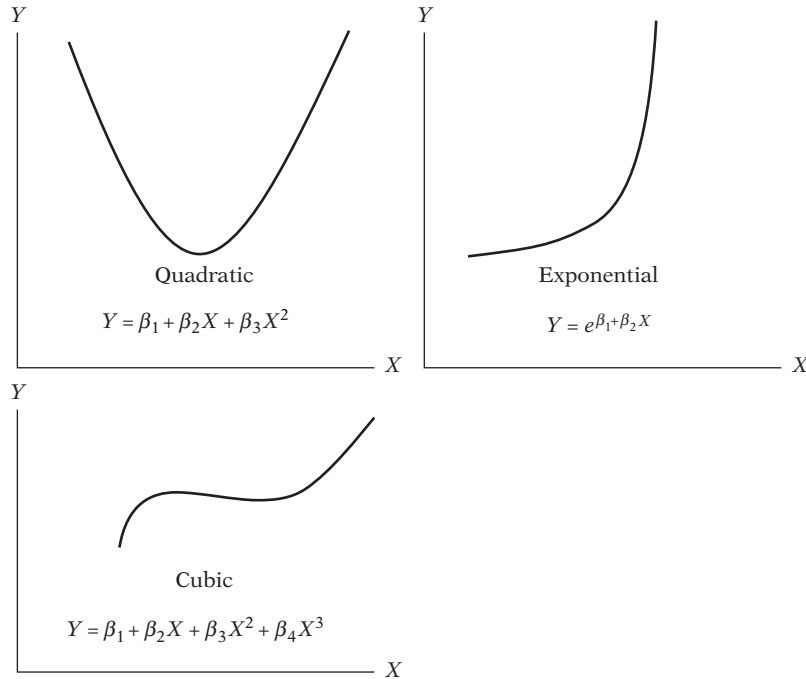


FIGURE 2.3 Linear-in-parameter functions.

TABLE 2.3 LINEAR REGRESSION MODELS

Model linear in parameters?	Model linear in variables?	
	Yes	No
Yes	LRM	LRM
No	NLRM	NLRM

Note: LRM = linear regression model  
NLRM = nonlinear regression model

## 2.4 STOCHASTIC SPECIFICATION OF PRF

It is clear from Figure 2.1 that, as family income increases, family consumption expenditure on the average increases, too. But what about the consumption expenditure of an individual family in relation to its (fixed) level of income? It is obvious from Table 2.1 and Figure 2.1 that an individual family's consumption expenditure does not necessarily increase as the income level increases. For example, from Table 2.1 we observe that corresponding to the income level of \$100 there is one family whose consumption expenditure of \$65 is less than the consumption expenditures of two families whose weekly income is only \$80. But notice that the *average* consumption

expenditure of families with a weekly income of \$100 is greater than the average consumption expenditure of families with a weekly income of \$80 (\$77 versus \$65).

What, then, can we say about the relationship between an individual family's consumption expenditure and a given level of income? We see from Figure 2.1 that, given the income level of  $X_i$ , an individual family's consumption expenditure is clustered around the average consumption of all families at that  $X_i$ , that is, around its conditional expectation. Therefore, we can express the *deviation* of an individual  $Y_i$  around its expected value as follows:

$$u_i = Y_i - E(Y | X_i)$$

or

$$Y_i = E(Y | X_i) + u_i \quad (2.4.1)$$

where the deviation  $u_i$  is an unobservable random variable taking positive or negative values. Technically,  $u_i$  is known as the **stochastic disturbance** or **stochastic error term**.

How do we interpret (2.4.1)? We can say that the expenditure of an individual family, given its income level, can be expressed as the sum of two components: (1)  $E(Y | X_i)$ , which is simply the mean consumption expenditure of all the families with the same level of income. This component is known as the **systematic**, or **deterministic**, component, and (2)  $u_i$ , which is the random, or **nonsystematic**, component. We shall examine shortly the nature of the stochastic disturbance term, but for the moment assume that it is a *surrogate or proxy* for all the omitted or neglected variables that may affect  $Y$  but are not (or cannot be) included in the regression model.

If  $E(Y | X_i)$  is assumed to be linear in  $X_i$ , as in (2.2.2), Eq. (2.4.1) may be written as

$$\begin{aligned} Y_i &= E(Y | X_i) + u_i \\ &= \beta_1 + \beta_2 X_i + u_i \end{aligned} \quad (2.4.2)$$

Equation (2.4.2) posits that the consumption expenditure of a family is linearly related to its income plus the disturbance term. Thus, the individual consumption expenditures, given  $X = \$80$  (see Table 2.1), can be expressed as

$$\begin{aligned} Y_1 &= 55 = \beta_1 + \beta_2(80) + u_1 \\ Y_2 &= 60 = \beta_1 + \beta_2(80) + u_2 \\ Y_3 &= 65 = \beta_1 + \beta_2(80) + u_3 \\ Y_4 &= 70 = \beta_1 + \beta_2(80) + u_4 \\ Y_5 &= 75 = \beta_1 + \beta_2(80) + u_5 \end{aligned} \quad (2.4.3)$$

Now if we take the expected value of (2.4.1) on both sides, we obtain

$$\begin{aligned} E(Y_i | X_i) &= E[E(Y | X_i)] + E(u_i | X_i) \\ &= E(Y | X_i) + E(u_i | X_i) \end{aligned} \quad (2.4.4)$$

where use is made of the fact that the expected value of a constant is that constant itself.<sup>8</sup> Notice carefully that in (2.4.4) we have taken the conditional expectation, conditional upon the given  $X$ 's.

Since  $E(Y_i | X_i)$  is the same thing as  $E(Y | X_i)$ , Eq. (2.4.4) implies that

$$E(u_i | X_i) = 0 \quad (2.4.5)$$

Thus, the assumption that the regression line passes through the conditional means of  $Y$  (see Figure 2.2) implies that the conditional mean values of  $u_i$  (conditional upon the given  $X$ 's) are zero.

From the previous discussion, it is clear (2.2.2) and (2.4.2) are equivalent forms if  $E(u_i | X_i) = 0$ .<sup>9</sup> But the stochastic specification (2.4.2) has the advantage that it clearly shows that there are other variables besides income that affect consumption expenditure and that an individual family's consumption expenditure cannot be fully explained only by the variable(s) included in the regression model.

## 2.5 THE SIGNIFICANCE OF THE STOCHASTIC DISTURBANCE TERM

As noted in Section 2.4, the disturbance term  $u_i$  is a surrogate for all those variables that are omitted from the model but that collectively affect  $Y$ . The obvious question is: Why not introduce these variables into the model explicitly? Stated otherwise, why not develop a multiple regression model with as many variables as possible? The reasons are many.

**1. Vagueness of theory:** The theory, if any, determining the behavior of  $Y$  may be, and often is, incomplete. We might know for certain that weekly income  $X$  influences weekly consumption expenditure  $Y$ , but we might be ignorant or unsure about the other variables affecting  $Y$ . Therefore,  $u_i$  may be used as a substitute for all the excluded or omitted variables from the model.

**2. Unavailability of data:** Even if we know what some of the excluded variables are and therefore consider a multiple regression rather than a simple regression, we may not have quantitative information about these

<sup>8</sup>See **App. A** for a brief discussion of the properties of the expectation operator  $E$ . Note that  $E(Y | X_i)$ , once the value of  $X_i$  is fixed, is a constant.

<sup>9</sup>As a matter of fact, in the method of least squares to be developed in Chap. 3, it is assumed explicitly that  $E(u_i | X_i) = 0$ . See Sec. 3.2.

variables. It is a common experience in empirical analysis that the data we would ideally like to have often are not available. For example, in principle we could introduce family wealth as an explanatory variable in addition to the income variable to explain family consumption expenditure. But unfortunately, information on family wealth generally is not available. Therefore, we may be forced to omit the wealth variable from our model despite its great theoretical relevance in explaining consumption expenditure.

**3. Core variables versus peripheral variables:** Assume in our consumption-income example that besides income  $X_1$ , the number of children per family  $X_2$ , sex  $X_3$ , religion  $X_4$ , education  $X_5$ , and geographical region  $X_6$  also affect consumption expenditure. But it is quite possible that the joint influence of all or some of these variables may be so small and at best nonsystematic or random that as a practical matter and for cost considerations it does not pay to introduce them into the model explicitly. One hopes that their combined effect can be treated as a random variable  $u_i$ .<sup>10</sup>

**4. Intrinsic randomness in human behavior:** Even if we succeed in introducing all the relevant variables into the model, there is bound to be some “intrinsic” randomness in individual  $Y$ 's that cannot be explained no matter how hard we try. The disturbances, the  $u$ 's, may very well reflect this intrinsic randomness.

**5. Poor proxy variables:** Although the classical regression model (to be developed in Chapter 3) assumes that the variables  $Y$  and  $X$  are measured accurately, in practice the data may be plagued by errors of measurement. Consider, for example, Milton Friedman's well-known theory of the consumption function.<sup>11</sup> He regards *permanent consumption* ( $Y^p$ ) as a function of *permanent income* ( $X^p$ ). But since data on these variables are not directly observable, in practice we use proxy variables, such as current consumption ( $Y$ ) and current income ( $X$ ), which can be observable. Since the observed  $Y$  and  $X$  may not equal  $Y^p$  and  $X^p$ , there is the problem of errors of measurement. The disturbance term  $u$  may in this case then also represent the errors of measurement. As we will see in a later chapter, if there are such errors of measurement, they can have serious implications for estimating the regression coefficients, the  $\beta$ 's.

**6. Principle of parsimony:** Following Occam's razor,<sup>12</sup> we would like to keep our regression model as simple as possible. If we can explain the behavior of  $Y$  “substantially” with two or three explanatory variables and if

<sup>10</sup>A further difficulty is that variables such as sex, education, and religion are difficult to quantify.

<sup>11</sup>Milton Friedman, *A Theory of the Consumption Function*, Princeton University Press, Princeton, N.J., 1957.

<sup>12</sup>“That descriptions be kept as simple as possible until proved inadequate,” *The World of Mathematics*, vol. 2, J. R. Newman (ed.), Simon & Schuster, New York, 1956, p. 1247, or, “Entities should not be multiplied beyond necessity,” Donald F. Morrison, *Applied Linear Statistical Methods*, Prentice Hall, Englewood Cliffs, N.J., 1983, p. 58.



our theory is not strong enough to suggest what other variables might be included, why introduce more variables? Let  $u_i$  represent all other variables. Of course, we should not exclude relevant and important variables just to keep the regression model simple.

7. *Wrong functional form:* Even if we have theoretically correct variables explaining a phenomenon and even if we can obtain data on these variables, very often we do not know the form of the functional relationship between the regressand and the regressors. Is consumption expenditure a linear (invariable) function of income or a nonlinear (invariable) function? If it is the former,  $Y_i = \beta_1 + \beta_2 X_i + u_i$  is the proper functional relationship between  $Y$  and  $X$ , but if it is the latter,  $Y_i = \beta_1 + \beta_2 X_i + \beta_3 X_i^2 + u_i$  may be the correct functional form. In two-variable models the functional form of the relationship can often be judged from the scattergram. But in a multiple regression model, it is not easy to determine the appropriate functional form, for graphically we cannot visualize scattergrams in multiple dimensions.

For all these reasons, the stochastic disturbances  $u_i$  assume an extremely critical role in regression analysis, which we will see as we progress.

## 2.6 THE SAMPLE REGRESSION FUNCTION (SRF)

By confining our discussion so far to the population of  $Y$  values corresponding to the fixed  $X$ 's, we have deliberately avoided sampling considerations (note that the data of Table 2.1 represent the population, not a sample). But it is about time to face up to the sampling problems, for in most practical situations what we have is but a sample of  $Y$  values corresponding to some fixed  $X$ 's. Therefore, our task now is to estimate the PRF on the basis of the sample information.

As an illustration, pretend that the population of Table 2.1 was not known to us and the only information we had was a randomly selected sample of  $Y$  values for the fixed  $X$ 's as given in Table 2.4. Unlike Table 2.1, we now have only one  $Y$  value corresponding to the given  $X$ 's; each  $Y$  (given  $X_i$ ) in Table 2.4 is chosen randomly from similar  $Y$ 's corresponding to the same  $X_i$  from the population of Table 2.1.

The question is: From the sample of Table 2.4 can we predict the average weekly consumption expenditure  $Y$  in the population as a whole corresponding to the chosen  $X$ 's? In other words, can we estimate the PRF from the sample data? As the reader surely suspects, we may not be able to estimate the PRF "accurately" because of sampling fluctuations. To see this, suppose we draw another random sample from the population of Table 2.1, as presented in Table 2.5.

Plotting the data of Tables 2.4 and 2.5, we obtain the scattergram given in Figure 2.4. In the scattergram two sample regression lines are drawn so as

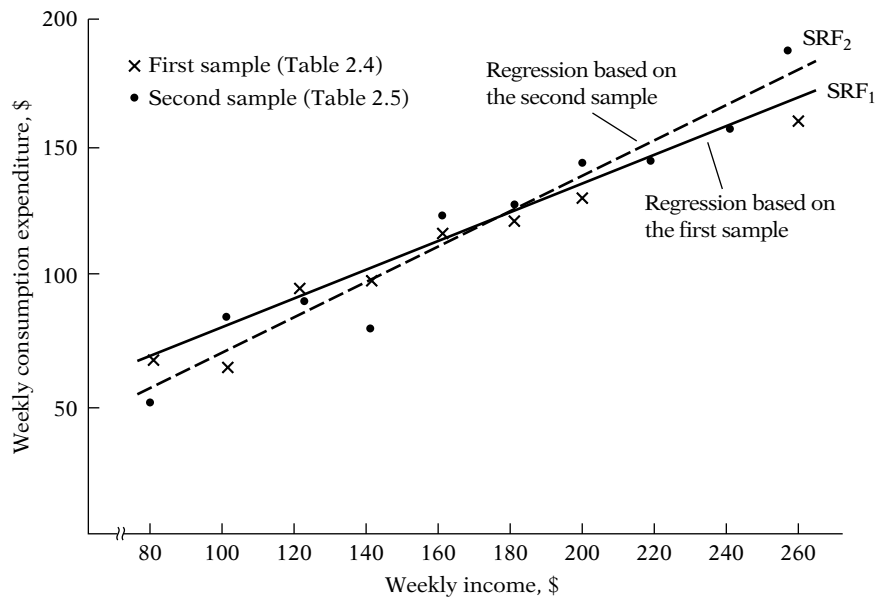


**TABLE 2.4**  
A RANDOM SAMPLE FROM THE  
POPULATION OF TABLE 2.1

Y	X
70	80
65	100
90	120
95	140
110	160
115	180
120	200
140	220
155	240
150	260

**TABLE 2.5**  
ANOTHER RANDOM SAMPLE FROM  
THE POPULATION OF TABLE 2.1

Y	X
55	80
88	100
90	120
80	140
118	160
120	180
145	200
135	220
145	240
175	260



**FIGURE 2.4** Regression lines based on two different samples.

to “fit” the scatters reasonably well:  $SRF_1$  is based on the first sample, and  $SRF_2$  is based on the second sample. Which of the two regression lines represents the “true” population regression line? If we avoid the temptation of looking at Figure 2.1, which purportedly represents the PR, there is no way we can be absolutely sure that either of the regression lines shown in Figure 2.4 represents the true population regression line (or curve). The regression lines in Figure 2.4 are known as the **sample regression lines**. Sup-

posedly they represent the population regression line, but because of sampling fluctuations they are at best an approximation of the true PR. In general, we would get  $N$  different SRFs for  $N$  different samples, and these SRFs are not likely to be the same.

Now, analogously to the PRF that underlies the population regression line, we can develop the concept of the **sample regression function** (SRF) to represent the sample regression line. The sample counterpart of (2.2.2) may be written as

$$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i \quad (2.6.1)$$

where  $\hat{Y}$  is read as “Y-hat” or “Y-cap”

$\hat{Y}_i$  = estimator of  $E(Y | X_i)$

$\hat{\beta}_1$  = estimator of  $\beta_1$

$\hat{\beta}_2$  = estimator of  $\beta_2$

Note that an **estimator**, also known as a (sample) **statistic**, is simply a rule or formula or method that tells how to estimate the population parameter from the information provided by the sample at hand. A particular numerical value obtained by the estimator in an application is known as an **estimate**.<sup>13</sup>

Now just as we expressed the PRF in two equivalent forms, (2.2.2) and (2.4.2), we can express the SRF (2.6.1) in its stochastic form as follows:

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \quad (2.6.2)$$

where, in addition to the symbols already defined,  $\hat{u}_i$  denotes the (sample) **residual** term. Conceptually  $\hat{u}_i$  is analogous to  $u_i$  and can be regarded as an *estimate* of  $u_i$ . It is introduced in the SRF for the same reasons as  $u_i$  was introduced in the PRF.

To sum up, then, we find our primary objective in regression analysis is to estimate the PRF

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (2.4.2)$$

on the basis of the SRF

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 x_i = \hat{u}_i \quad (2.6.2)$$

because more often than not our analysis is based upon a single sample from some population. But because of sampling fluctuations our estimate of

<sup>13</sup>As noted in the Introduction, a hat above a variable will signify an estimator of the relevant population value.

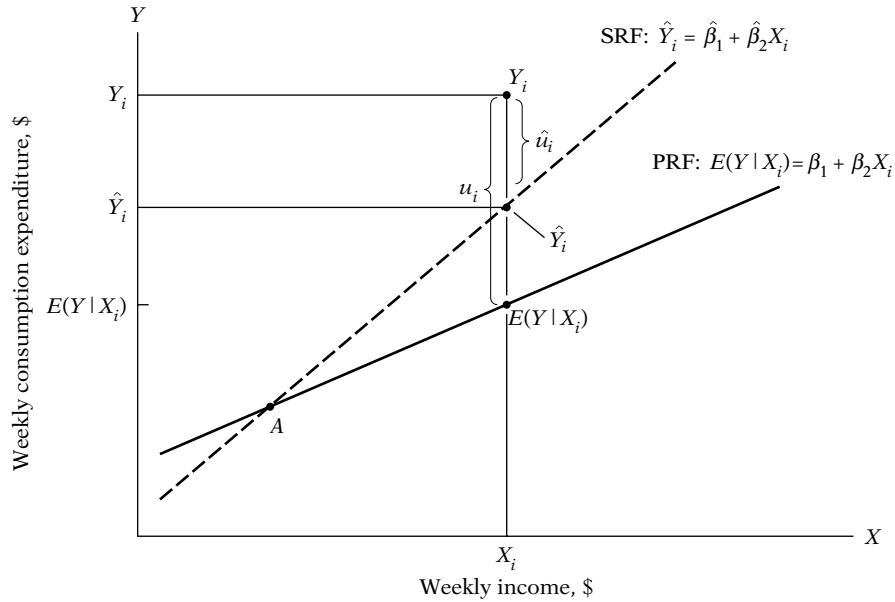


FIGURE 2.5 Sample and population regression lines.

the PRF based on the SRF is at best an approximate one. This approximation is shown diagrammatically in Figure 2.5.

For  $X = X_i$ , we have one (sample) observation  $Y = Y_i$ . In terms of the SRF, the observed  $Y_i$  can be expressed as

$$Y_i = \hat{Y}_i + \hat{u}_i \tag{2.6.3}$$

and in terms of the PRF, it can be expressed as

$$Y_i = E(Y | X_i) + u_i \tag{2.6.4}$$

Now obviously in Figure 2.5  $\hat{Y}_i$  overestimates the true  $E(Y | X_i)$  for the  $X_i$  shown therein. By the same token, for any  $X_i$  to the left of the point A, the SRF will underestimate the true PRF. But the reader can readily see that such over- and underestimation is inevitable because of sampling fluctuations.

The critical question now is: Granted that the SRF is but an approximation of the PRF, can we devise a rule or a method that will make this approximation as “close” as possible? In other words, how should the SRF be constructed so that  $\hat{\beta}_1$  is as “close” as possible to the true  $\beta_1$  and  $\hat{\beta}_2$  is as “close” as possible to the true  $\beta_2$  even though we will never know the true  $\beta_1$  and  $\beta_2$ ?

The answer to this question will occupy much of our attention in Chapter 3. We note here that we can develop procedures that tell us how to construct the SRF to mirror the PRF as faithfully as possible. It is fascinating to consider that this can be done even though we never actually determine the PRF itself.

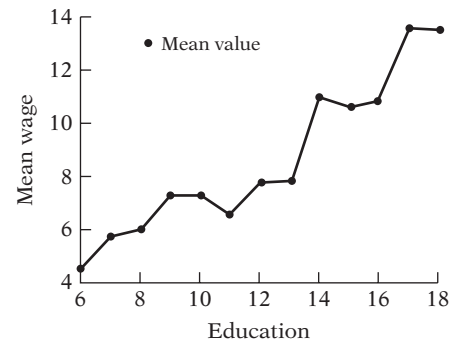
### 2.7 AN ILLUSTRATIVE EXAMPLE

We conclude this chapter with an example. Table 2.6 gives data on the level of education (measured by the number of years of schooling), the mean hourly wages earned by people at each level of education, and the number of people at the stated level of education. Ernst Berndt originally obtained the data presented in the table, and he derived these data from the current population survey conducted in May 1985.<sup>14</sup> We will explore these data (with additional explanatory variables) in Chapter 3 (Example 3.3, p. 91).

Plotting the (conditional) mean wage against education, we obtain the picture in Figure 2.6. The regression curve in the figure shows how mean wages vary with the level of education; they generally increase with the level of education, a finding one should not find surprising. We will study in a later chapter how variables besides education can also affect the mean wage.

**TABLE 2.6**  
MEAN HOURLY WAGE BY EDUCATION

Years of schooling	Mean wage, \$	Number of people
6	4.4567	3
7	5.7700	5
8	5.9787	15
9	7.3317	12
10	7.3182	17
11	6.5844	27
12	7.8182	218
13	7.8351	37
14	11.0223	56
15	10.6738	13
16	10.8361	70
17	13.6150	24
18	13.5310	31
		Total 528



**FIGURE 2.6**  
Relationship between mean wages and education.

Source: Arthur S. Goldberger, *Introductory Econometrics*, Harvard University Press, Cambridge, Mass., 1998, Table 1.1, p. 5 (adapted).

<sup>14</sup>Ernst R. Berndt, *The Practice of Econometrics: Classic and Contemporary*, Addison Wesley, Reading, Mass., 1991. Incidentally, this is an excellent book that the reader may want to read to find out how econometricians go about doing research.

## 2.8 SUMMARY AND CONCLUSIONS

1. The key concept underlying regression analysis is the concept of the **conditional expectation function (CEF), or population regression function (PRF)**. Our objective in regression analysis is to find out how the average value of the dependent variable (or regressand) varies with the given value of the explanatory variable (or regressor).

2. This book largely deals with **linear PRFs**, that is, regressions that are linear in the parameters. They may or may not be linear in the regressand or the regressors.

3. For empirical purposes, it is the **stochastic PRF** that matters. The **stochastic disturbance term**  $u_i$  plays a critical role in estimating the PRF.

4. The PRF is an idealized concept, since in practice one rarely has access to the entire population of interest. Usually, one has a sample of observations from the population. Therefore, one uses the **stochastic sample regression function (SRF)** to estimate the PRF. How this is actually accomplished is discussed in Chapter 3.

## EXERCISES

### Questions

- 2.1. What is the conditional expectation function or the population regression function?
- 2.2. What is the difference between the population and sample regression functions? Is this a distinction without difference?
- 2.3. What is the role of the stochastic error term  $u_i$  in regression analysis? What is the difference between the stochastic error term and the residual,  $\hat{u}_i$ ?
- 2.4. Why do we need regression analysis? Why not simply use the mean value of the regressand as its best value?
- 2.5. What do we mean by a *linear* regression model?
- 2.6. Determine whether the following models are linear in the parameters, or the variables, or both. Which of these models are linear regression models?

#### Model

a.  $Y_i = \beta_1 + \beta_2 \left( \frac{1}{X_i} \right) + u_i$

b.  $Y_i = \beta_1 + \beta_2 \ln X_i + u_i$

c.  $\ln Y_i = \beta_1 + \beta_2 X_i + u_i$

d.  $\ln Y_i = \ln \beta_1 + \beta_2 \ln X_i + u_i$

e.  $\ln Y_i = \beta_1 - \beta_2 \left( \frac{1}{X_i} \right) + u_i$

#### Descriptive title

Reciprocal

Semilogarithmic

Inverse semilogarithmic

Logarithmic or double logarithmic

Logarithmic reciprocal

Note:  $\ln$  = natural log (i.e., log to the base  $e$ );  $u_i$  is the stochastic disturbance term. We will study these models in Chapter 6.

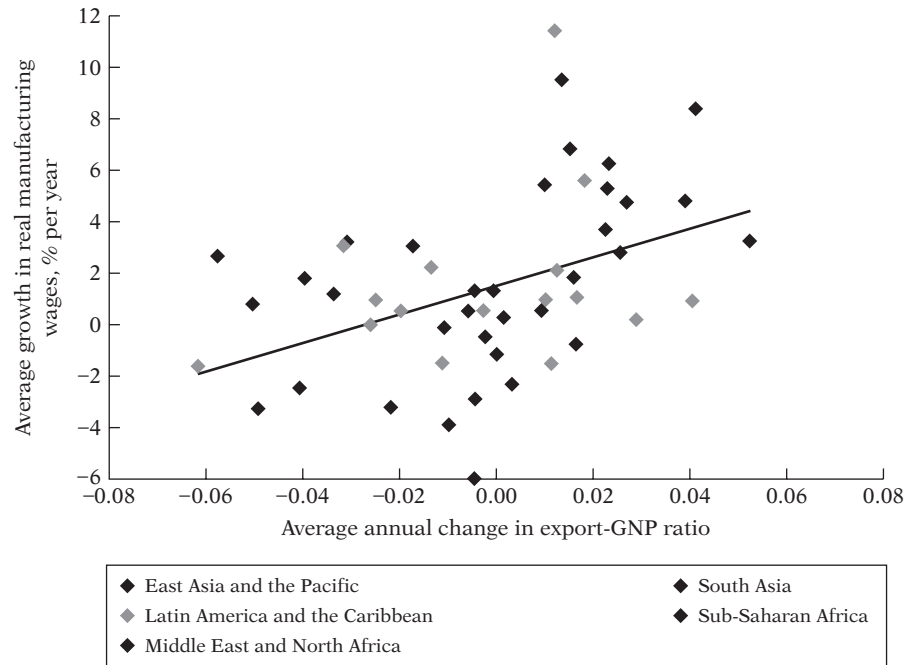
- 2.7. Are the following models linear regression models? Why or why not?

a.  $Y_i = e^{\beta_1 + \beta_2 X_i + u_i}$

b.  $Y_i = \frac{1}{1 + e^{\beta_1 + \beta_2 X_i + u_i}}$

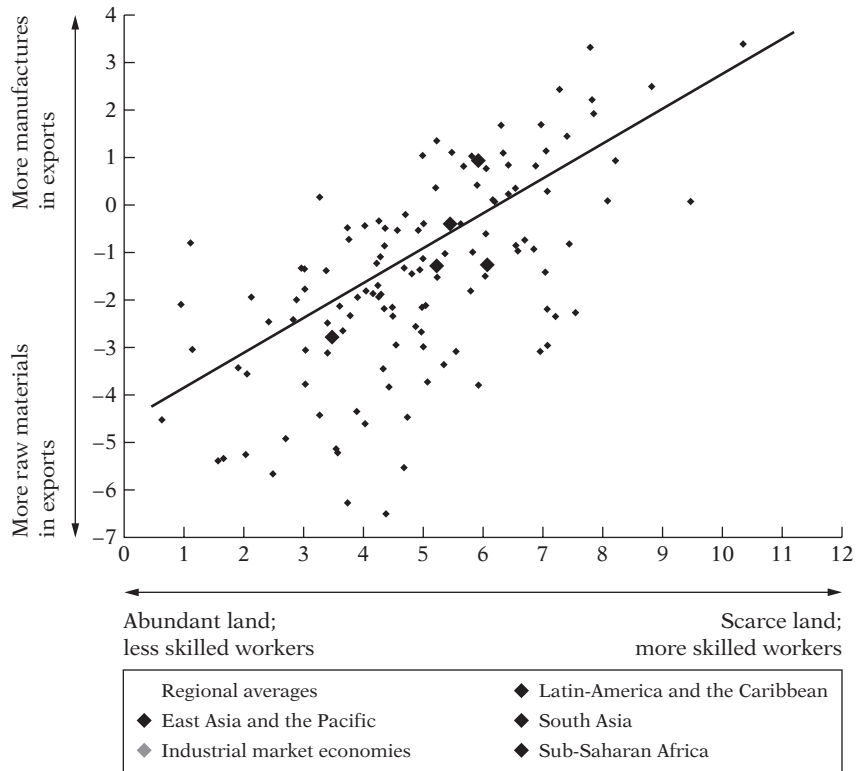
- c.  $\ln Y_i = \beta_1 + \beta_2 \left( \frac{1}{X_i} \right) + u_i$
- d.  $Y_i = \beta_1 + (0.75 - \beta_1)e^{-\beta_2(X_i-2)} + u_i$
- e.  $Y_i = \beta_1 + \beta_2^3 X_i + u_i$

- 2.8. What is meant by an *intrinsically linear* regression model? If  $\beta_2$  in exercise 2.7d were 0.8, would it be a linear or nonlinear regression model?
- \*2.9. Consider the following nonstochastic models (i.e., models without the stochastic error term). Are they linear regression models? If not, is it possible, by suitable algebraic manipulations, to convert them into linear models?
- a.  $Y_i = \frac{1}{\beta_1 + \beta_2 X_i}$
  - b.  $Y_i = \frac{X_i}{\beta_1 + \beta_2 X_i}$
  - c.  $Y_i = \frac{1}{1 + \exp(-\beta_1 - \beta_2 X_i)}$
- 2.10. You are given the scattergram in Figure 2.7 along with the regression line. What general conclusion do you draw from this diagram? Is the regression line sketched in the diagram a population regression line or the sample regression line?



**FIGURE 2.7** Growth rates of real manufacturing wages and exports. Data are for 50 developing countries during 1970–90.

Source: The World Bank, *World Development Report 1995*, p. 55. The original source is UNIDO data, World Bank data.



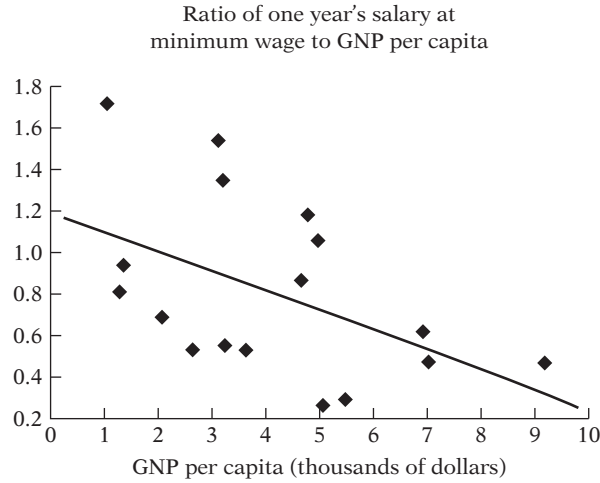
**FIGURE 2.8** Skill intensity of exports and human capital endowment. Data are for 126 industrial and developing countries in 1985. Values along the horizontal axis are logarithms of the ratio of the country's average educational attainment to its land area; vertical axis values are logarithms of the ratio of manufactured to primary-products exports.

Source: World Bank, *World Development Report 1995*, p. 59. Original sources: Export data from United Nations Statistical Office COMTRADE data base; education data from UNDP 1990; land data from the World Bank.

- 2.11. From the scattergram given in Figure 2.8, what general conclusions do you draw? What is the economic theory that underlies this scattergram? (*Hint: Look up any international economics textbook and read up on the Heckscher–Ohlin model of trade.*)
- 2.12. What does the scattergram in Figure 2.9 reveal? On the basis of this diagram, would you argue that minimum wage laws are good for economic well-being?
- 2.13. Is the regression line shown in Figure I.3 of the Introduction the PRF or the SRF? Why? How would you interpret the scatterpoints around the regression line? Besides GDP, what other factors, or variables, might determine personal consumption expenditure?

### Problems

- 2.14. You are given the data in Table 2.7 for the United States for years 1980–1996.



**FIGURE 2.9** The minimum wage and GNP per capita. The sample consists of 17 developing countries. Years vary by country from 1988 to 1992. Data are in international prices.

Source: World Bank, *World Development Report 1995*, p. 75.

**TABLE 2.7** LABOR FORCE PARTICIPATION DATA

Year	CLFPRM <sup>1</sup>	CLFPRF <sup>2</sup>	UNRM <sup>3</sup>	UNRF <sup>4</sup>	AHE82 <sup>5</sup>	AHE <sup>6</sup>
1980	77.4	51.5	6.9	7.4	7.78	6.66
1981	77.0	52.1	7.4	7.9	7.69	7.25
1982	76.6	52.6	9.9	9.4	7.68	7.68
1983	76.4	53.9	9.9	9.2	7.79	8.02
1984	76.4	53.6	7.4	7.6	7.80	8.32
1985	76.3	54.5	7.0	7.4	7.77	8.57
1986	76.3	55.3	6.9	7.1	7.81	8.76
1987	76.2	56.0	6.2	6.2	7.73	8.98
1988	76.2	56.6	5.5	5.6	7.69	9.28
1989	76.4	57.4	5.2	5.4	7.64	9.66
1990	76.4	57.5	5.7	5.5	7.52	10.01
1991	75.8	57.4	7.2	6.4	7.45	10.32
1992	75.8	57.8	7.9	7.0	7.41	10.57
1993	75.4	57.9	7.2	6.6	7.39	10.83
1994	75.1	58.8	6.2	6.0	7.40	11.12
1995	75.0	58.9	5.6	5.6	7.40	11.44
1996	74.9	59.3	5.4	5.4	7.43	11.82

Source: *Economic Report of the President, 1997*. Table citations below refer to the source document.

<sup>1</sup>CLFPRM, Civilian labor force participation rate, male (%), Table B-37, p. 343.

<sup>2</sup>CLFPRF, Civilian labor force participation rate, female (%), Table B-37, p. 343.

<sup>3</sup>UNRM, Civilian unemployment rate, male (%) Table B-40, p. 346.

<sup>4</sup>UNRF, Civilian unemployment rate, female (%) Table B-40, p. 346.

<sup>5</sup>AHE82, Average hourly earnings (1982 dollars), Table B-45, p. 352.

<sup>6</sup>AHE, Average hourly earnings (current dollars), Table B-45, p. 352.



- a. Plot the male civilian labor force participation rate against male civilian unemployment rate. Eyeball a regression line through the scatter points. A priori, what is the expected relationship between the two and what is the underlying economic theory? Does the scattergram support the theory?
  - b. Repeat part a for females.
  - c. Now plot both the male and female labor participation rates against average hourly earnings (in 1982 dollars). (You may use separate diagrams.) Now what do you find? And how would you rationalize your finding?
  - d. Can you plot the labor force participation rate against the unemployment rate and the average hourly earnings simultaneously? If not, how would you verbalize the relationship among the three variables?
- 2.15.** Table 2.8 gives data on expenditure on food and total expenditure, measured in rupees, for a sample of 55 rural households from India. (In early 2000, a U.S. dollar was about 40 Indian rupees.)

**TABLE 2.8** FOOD AND TOTAL EXPENDITURE (RUPEES)

Observation	Food expenditure	Total expenditure	Observation	Food expenditure	Total expenditure
1	217.0000	382.0000	29	390.0000	655.0000
2	196.0000	388.0000	30	385.0000	662.0000
3	303.0000	391.0000	31	470.0000	663.0000
4	270.0000	415.0000	32	322.0000	677.0000
5	325.0000	456.0000	33	540.0000	680.0000
6	260.0000	460.0000	34	433.0000	690.0000
7	300.0000	472.0000	35	295.0000	695.0000
8	325.0000	478.0000	36	340.0000	695.0000
9	336.0000	494.0000	37	500.0000	695.0000
10	345.0000	516.0000	38	450.0000	720.0000
11	325.0000	525.0000	39	415.0000	721.0000
12	362.0000	554.0000	40	540.0000	730.0000
13	315.0000	575.0000	41	360.0000	731.0000
14	355.0000	579.0000	42	450.0000	733.0000
15	325.0000	585.0000	43	395.0000	745.0000
16	370.0000	586.0000	44	430.0000	751.0000
17	390.0000	590.0000	45	332.0000	752.0000
18	420.0000	608.0000	46	397.0000	752.0000
19	410.0000	610.0000	47	446.0000	769.0000
20	383.0000	616.0000	48	480.0000	773.0000
21	315.0000	618.0000	49	352.0000	773.0000
22	267.0000	623.0000	50	410.0000	775.0000
23	420.0000	627.0000	51	380.0000	785.0000
24	300.0000	630.0000	52	610.0000	788.0000
25	410.0000	635.0000	53	530.0000	790.0000
26	220.0000	640.0000	54	360.0000	795.0000
27	403.0000	648.0000	55	305.0000	801.0000
28	350.0000	650.0000			

Source: Chandan Mukherjee, Howard White, and Marc Wuyts, *Econometrics and Data Analysis for Developing Countries*, Routledge, New York, 1998, p. 457.

- a. Plot the data, using the vertical axis for expenditure on food and the horizontal axis for total expenditure, and sketch a regression line through the scatterpoints.
  - b. What broad conclusions can you draw from this example?
  - c. A priori, would you expect expenditure on food to increase linearly as total expenditure increases regardless of the level of total expenditure? Why or why not? You can use total expenditure as a proxy for total income.
- 2.16.** Table 2.9 gives data on mean Scholastic Aptitude Test (SAT) scores for college-bound seniors for 1967–1990.
- a. Use the horizontal axis for years and the vertical axis for SAT scores to plot the verbal and math scores for males and females separately.
  - b. What general conclusions can you draw?
  - c. Knowing the verbal scores of males and females, how would you go about predicting their math scores?
  - d. Plot the female verbal SAT score against the male verbal SAT score. Sketch a regression line through the scatterpoints. What do you observe?

**TABLE 2.9** MEAN SCHOLASTIC APTITUDE TEST SCORES FOR COLLEGE-BOUND SENIORS, 1967–1990\*

Year	Verbal			Math		
	Males	Females	Total	Males	Females	Total
1967	463	468	466	514	467	492
1968	464	466	466	512	470	492
1969	459	466	463	513	470	493
1970	459	461	460	509	465	488
1971	454	457	455	507	466	488
1972	454	452	453	505	461	484
1973	446	443	445	502	460	481
1974	447	442	444	501	459	480
1975	437	431	434	495	449	472
1976	433	430	431	497	446	472
1977	431	427	429	497	445	470
1978	433	425	429	494	444	468
1979	431	423	427	493	443	467
1980	428	420	424	491	443	466
1981	430	418	424	492	443	466
1982	431	421	426	493	443	467
1983	430	420	425	493	445	468
1984	433	420	426	495	449	471
1985	437	425	431	499	452	475
1986	437	426	431	501	451	475
1987	435	425	430	500	453	476
1988	435	422	428	498	455	476
1989	434	421	427	500	454	476
1990	429	419	424	499	455	476

\*Data for 1967–1971 are estimates.  
Source: The College Board. *The New York Times*, Aug. 28, 1990, p. B-5.

# 3

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## TWO-VARIABLE REGRESSION MODEL: THE PROBLEM OF ESTIMATION

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As noted in Chapter 2, our first task is to estimate the population regression function (PRF) on the basis of the sample regression function (SRF) as accurately as possible. In **Appendix A** we have discussed two generally used methods of estimation: (1) **ordinary least squares (OLS)** and (2) **maximum likelihood (ML)**. By and large, it is the method of OLS that is used extensively in regression analysis primarily because it is intuitively appealing and mathematically much simpler than the method of maximum likelihood. Besides, as we will show later, in the linear regression context the two methods generally give similar results.

### 3.1 THE METHOD OF ORDINARY LEAST SQUARES

The method of ordinary least squares is attributed to Carl Friedrich Gauss, a German mathematician. Under certain assumptions (discussed in Section 3.2), the method of least squares has some very attractive statistical properties that have made it one of the most powerful and popular methods of regression analysis. To understand this method, we first explain the least-squares principle.

Recall the two-variable PRF:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (2.4.2)$$

However, as we noted in Chapter 2, the PRF is not directly observable. We

estimate it from the SRF:

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \quad (2.6.2)$$

$$= \hat{Y}_i + \hat{u}_i \quad (2.6.3)$$

where  $\hat{Y}_i$  is the estimated (conditional mean) value of  $Y_i$ .

But how is the SRF itself determined? To see this, let us proceed as follows. First, express (2.6.3) as

$$\begin{aligned} \hat{u}_i &= Y_i - \hat{Y}_i \\ &= Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i \end{aligned} \quad (3.1.1)$$

which shows that the  $\hat{u}_i$  (the residuals) are simply the differences between the actual and estimated  $Y$  values.

Now given  $n$  pairs of observations on  $Y$  and  $X$ , we would like to determine the SRF in such a manner that it is as close as possible to the actual  $Y$ . To this end, we may adopt the following criterion: Choose the SRF in such a way that the sum of the residuals  $\sum \hat{u}_i = \sum (Y_i - \hat{Y}_i)$  is as small as possible. Although intuitively appealing, this is not a very good criterion, as can be seen in the hypothetical scattergram shown in Figure 3.1.

If we adopt the criterion of minimizing  $\sum \hat{u}_i^2$ , Figure 3.1 shows that the residuals  $\hat{u}_2$  and  $\hat{u}_3$  as well as the residuals  $\hat{u}_1$  and  $\hat{u}_4$  receive the same weight in the sum ( $\hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4$ ), although the first two residuals are much closer to the SRF than the latter two. In other words, all the residuals receive

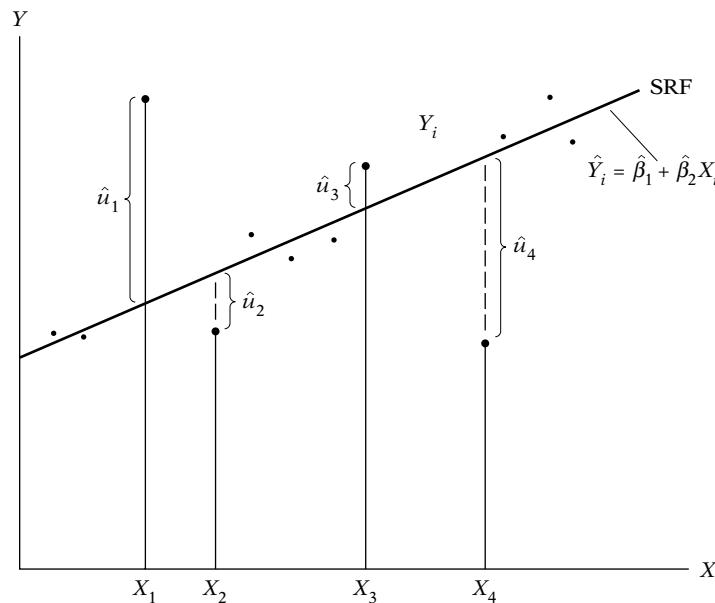


FIGURE 3.1 Least-squares criterion.

equal importance no matter how close or how widely scattered the individual observations are from the SRF. A consequence of this is that it is quite possible that the algebraic sum of the  $\hat{u}_i$  is small (even zero) although the  $\hat{u}_i$  are widely scattered about the SRF. To see this, let  $\hat{u}_1, \hat{u}_2, \hat{u}_3,$  and  $\hat{u}_4$  in Figure 3.1 assume the values of 10, -2, +2, and -10, respectively. The algebraic sum of these residuals is zero although  $\hat{u}_1$  and  $\hat{u}_4$  are scattered more widely around the SRF than  $\hat{u}_2$  and  $\hat{u}_3$ . We can avoid this problem if we adopt the *least-squares criterion*, which states that the SRF can be fixed in such a way that

$$\begin{aligned} \sum \hat{u}_i^2 &= \sum (Y_i - \hat{Y}_i)^2 \\ &= \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2 \end{aligned} \tag{3.1.2}$$

is as small as possible, where  $\hat{u}_i^2$  are the squared residuals. By squaring  $\hat{u}_i$ , this method gives more weight to residuals such as  $\hat{u}_1$  and  $\hat{u}_4$  in Figure 3.1 than the residuals  $\hat{u}_2$  and  $\hat{u}_3$ . As noted previously, under the minimum  $\sum \hat{u}_i$  criterion, the sum can be small even though the  $\hat{u}_i$  are widely spread about the SRF. But this is not possible under the least-squares procedure, for the larger the  $\hat{u}_i$  (in absolute value), the larger the  $\sum \hat{u}_i^2$ . A further justification for the least-squares method lies in the fact that the estimators obtained by it have some very desirable statistical properties, as we shall see shortly.

It is obvious from (3.1.2) that

$$\sum \hat{u}_i^2 = f(\hat{\beta}_1, \hat{\beta}_2) \tag{3.1.3}$$

that is, the sum of the squared residuals is some function of the estimators  $\hat{\beta}_1$  and  $\hat{\beta}_2$ . For any given set of data, choosing different values for  $\hat{\beta}_1$  and  $\hat{\beta}_2$  will give different  $\hat{u}$ 's and hence different values of  $\sum \hat{u}_i^2$ . To see this clearly, consider the hypothetical data on  $Y$  and  $X$  given in the first two columns of Table 3.1. Let us now conduct two experiments. In experiment 1,

**TABLE 3.1** EXPERIMENTAL DETERMINATION OF THE SRF

$Y_i$ (1)	$X_i$ (2)	$\hat{Y}_{1i}$ (3)	$\hat{u}_{1i}$ (4)	$\hat{u}_{1i}^2$ (5)	$\hat{Y}_{2i}$ (6)	$\hat{u}_{2i}$ (7)	$\hat{u}_{2i}^2$ (8)
4	1	2.929	1.071	1.147	4	0	0
5	4	7.000	-2.000	4.000	7	-2	4
7	5	8.357	-1.357	1.841	8	-1	1
12	6	9.714	2.286	5.226	9	3	9
Sum: 28	16		0.0	12.214		0	14

Notes:  $\hat{Y}_{1i} = 1.572 + 1.357X_i$  (i.e.,  $\hat{\beta}_1 = 1.572$  and  $\hat{\beta}_2 = 1.357$ )  
 $\hat{Y}_{2i} = 3.0 + 1.0X_i$  (i.e.,  $\hat{\beta}_1 = 3$  and  $\hat{\beta}_2 = 1.0$ )  
 $\hat{u}_{1i} = (Y_i - \hat{Y}_{1i})$   
 $\hat{u}_{2i} = (Y_i - \hat{Y}_{2i})$

let  $\hat{\beta}_1 = 1.572$  and  $\hat{\beta}_2 = 1.357$  (let us not worry right now about how we got these values; say, it is just a guess).<sup>1</sup> Using these  $\hat{\beta}$  values and the  $X$  values given in column (2) of Table 3.1, we can easily compute the estimated  $Y_i$  given in column (3) of the table as  $\hat{Y}_{1i}$  (the subscript 1 is to denote the first experiment). Now let us conduct another experiment, but this time using the values of  $\hat{\beta}_1 = 3$  and  $\hat{\beta}_2 = 1$ . The estimated values of  $Y_i$  from this experiment are given as  $\hat{Y}_{2i}$  in column (6) of Table 3.1. Since the  $\hat{\beta}$  values in the two experiments are different, we get different values for the estimated residuals, as shown in the table;  $\hat{u}_{1i}$  are the residuals from the first experiment and  $\hat{u}_{2i}$  from the second experiment. The squares of these residuals are given in columns (5) and (8). Obviously, as expected from (3.1.3), these residual sums of squares are different since they are based on different sets of  $\hat{\beta}$  values.

Now which sets of  $\hat{\beta}$  values should we choose? Since the  $\hat{\beta}$  values of the first experiment give us a lower  $\sum \hat{u}_i^2 (= 12.214)$  than that obtained from the  $\hat{\beta}$  values of the second experiment ( $= 14$ ), we might say that the  $\hat{\beta}$ 's of the first experiment are the "best" values. But how do we know? For, if we had infinite time and infinite patience, we could have conducted many more such experiments, choosing different sets of  $\hat{\beta}$ 's each time and comparing the resulting  $\sum \hat{u}_i^2$  and then choosing that set of  $\hat{\beta}$  values that gives us the least possible value of  $\sum \hat{u}_i^2$  assuming of course that we have considered all the conceivable values of  $\beta_1$  and  $\beta_2$ . But since time, and certainly patience, are generally in short supply, we need to consider some shortcuts to this trial-and-error process. Fortunately, the method of least squares provides us such a shortcut. The principle or the method of least squares chooses  $\hat{\beta}_1$  and  $\hat{\beta}_2$  in such a manner that, for a given sample or set of data,  $\sum \hat{u}_i^2$  is as small as possible. In other words, for a given sample, the method of least squares provides us with unique estimates of  $\beta_1$  and  $\beta_2$  that give the smallest possible value of  $\sum \hat{u}_i^2$ . How is this accomplished? This is a straight-forward exercise in differential calculus. As shown in Appendix 3A, Section 3A.1, the process of differentiation yields the following equations for estimating  $\beta_1$  and  $\beta_2$ :

$$\sum Y_i = n\hat{\beta}_1 + \hat{\beta}_2 \sum X_i \quad (3.1.4)$$

$$\sum Y_i X_i = \hat{\beta}_1 \sum X_i + \hat{\beta}_2 \sum X_i^2 \quad (3.1.5)$$

where  $n$  is the sample size. These simultaneous equations are known as the **normal equations**.

<sup>1</sup>For the curious, these values are obtained by the method of least squares, discussed shortly. See Eqs. (3.1.6) and (3.1.7).

Solving the normal equations simultaneously, we obtain

$$\begin{aligned}\hat{\beta}_2 &= \frac{n \sum X_i Y_i - \sum X_i \sum Y_i}{n \sum X_i^2 - (\sum X_i)^2} \\ &= \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} \\ &= \frac{\sum x_i y_i}{\sum x_i^2}\end{aligned}\tag{3.1.6}$$

where  $\bar{X}$  and  $\bar{Y}$  are the sample means of  $X$  and  $Y$  and where we define  $x_i = (X_i - \bar{X})$  and  $y_i = (Y_i - \bar{Y})$ . Henceforth we adopt the convention of letting the lowercase letters denote deviations from mean values.

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum X_i^2 \sum Y_i - \sum X_i \sum X_i Y_i}{n \sum X_i^2 - (\sum X_i)^2} \\ &= \bar{Y} - \hat{\beta}_2 \bar{X}\end{aligned}\tag{3.1.7}$$

The last step in (3.1.7) can be obtained directly from (3.1.4) by simple algebraic manipulations.

Incidentally, note that, by making use of simple algebraic identities, formula (3.1.6) for estimating  $\beta_2$  can be alternatively expressed as

$$\begin{aligned}\hat{\beta}_2 &= \frac{\sum x_i y_i}{\sum x_i^2} \\ &= \frac{\sum x_i Y_i}{\sum X_i^2 - n\bar{X}^2} \\ &= \frac{\sum X_i y_i}{\sum X_i^2 - n\bar{X}^2}\end{aligned}\tag{3.1.8}^2$$

The estimators obtained previously are known as the **least-squares estimators**, for they are derived from the least-squares principle. Note the following **numerical properties** of estimators obtained by the method of OLS: “Numerical properties are those that hold as a consequence of the use

<sup>2</sup>Note 1:  $\sum x_i^2 = \sum (X_i - \bar{X})^2 = \sum X_i^2 - 2 \sum X_i \bar{X} + \sum \bar{X}^2 = \sum X_i^2 - 2\bar{X} \sum X_i + \sum \bar{X}^2$ , since  $\bar{X}$  is a constant. Further noting that  $\sum X_i = n\bar{X}$  and  $\sum \bar{X}^2 = n\bar{X}^2$  since  $\bar{X}$  is a constant, we finally get  $\sum x_i^2 = \sum X_i^2 - n\bar{X}^2$ .

Note 2:  $\sum x_i y_i = \sum x_i (Y_i - \bar{Y}) = \sum x_i Y_i - \bar{Y} \sum x_i = \sum x_i Y_i - \bar{Y} \sum (X_i - \bar{X}) = \sum x_i Y_i$ , since  $\bar{Y}$  is a constant and since the sum of deviations of a variable from its mean value [e.g.,  $\sum (X_i - \bar{X})$ ] is always zero. Likewise,  $\sum y_i = \sum (Y_i - \bar{Y}) = 0$ .

of ordinary least squares, regardless of how the data were generated.”<sup>3</sup> Shortly, we will also consider the **statistical properties** of OLS estimators, that is, properties “that hold only under certain assumptions about the way the data were generated.”<sup>4</sup> (See the classical linear regression model in Section 3.2.)

- I. The OLS estimators are expressed solely in terms of the observable (i.e., sample) quantities (i.e.,  $X$  and  $Y$ ). Therefore, they can be easily computed.
- II. They are **point estimators**; that is, given the sample, each estimator will provide only a single (point) value of the relevant population parameter. (In Chapter 5 we will consider the so-called **interval estimators**, which provide a range of possible values for the unknown population parameters.)
- III. Once the OLS estimates are obtained from the sample data, the sample regression line (Figure 3.1) can be easily obtained. The regression line thus obtained has the following properties:
  1. It passes through the sample means of  $Y$  and  $X$ . This fact is obvious from (3.1.7), for the latter can be written as  $\bar{Y} = \hat{\beta}_1 + \hat{\beta}_2 \bar{X}$ , which is shown diagrammatically in Figure 3.2.

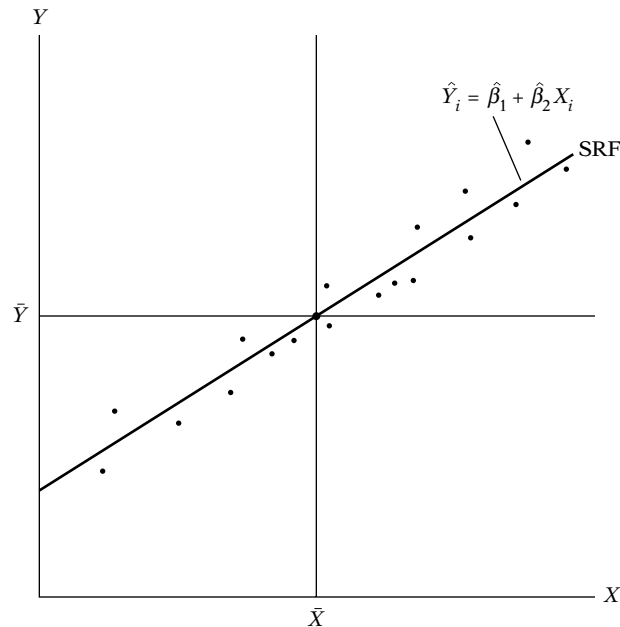


FIGURE 3.2 Diagram showing that the sample regression line passes through the sample mean values of  $Y$  and  $X$ .

<sup>3</sup>Russell Davidson and James G. MacKinnon, *Estimation and Inference in Econometrics*, Oxford University Press, New York, 1993, p. 3.

<sup>4</sup>*Ibid.*



2. The mean value of the estimated  $Y = \hat{Y}_i$  is equal to the mean value of the actual  $Y$  for

$$\begin{aligned}\hat{Y}_i &= \hat{\beta}_1 + \hat{\beta}_2 X_i \\ &= (\bar{Y} - \hat{\beta}_2 \bar{X}) + \hat{\beta}_2 X_i \\ &= \bar{Y} + \hat{\beta}_2 (X_i - \bar{X})\end{aligned}\quad (3.1.9)$$

Summing both sides of this last equality over the sample values and dividing through by the sample size  $n$  gives

$$\bar{\hat{Y}} = \bar{Y} \quad (3.1.10)^5$$

where use is made of the fact that  $\sum (X_i - \bar{X}) = 0$ . (Why?)

3. The mean value of the residuals  $\hat{u}_i$  is zero. From Appendix 3A, Section 3A.1, the first equation is

$$-2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i) = 0$$

But since  $\hat{u}_i = Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i$ , the preceding equation reduces to  $-2 \sum \hat{u}_i = 0$ , whence  $\bar{\hat{u}} = 0$ .<sup>6</sup>

As a result of the preceding property, the sample regression

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \quad (2.6.2)$$

can be expressed in an alternative form where both  $Y$  and  $X$  are expressed as deviations from their mean values. To see this, sum (2.6.2) on both sides to give

$$\begin{aligned}\sum Y_i &= n\hat{\beta}_1 + \hat{\beta}_2 \sum X_i + \sum \hat{u}_i \\ &= n\hat{\beta}_1 + \hat{\beta}_2 \sum X_i \quad \text{since } \sum \hat{u}_i = 0\end{aligned}\quad (3.1.11)$$

Dividing Eq. (3.1.11) through by  $n$ , we obtain

$$\bar{Y} = \hat{\beta}_1 + \hat{\beta}_2 \bar{X} \quad (3.1.12)$$

which is the same as (3.1.7). Subtracting Eq. (3.1.12) from (2.6.2), we obtain

$$Y_i - \bar{Y} = \hat{\beta}_2 (X_i - \bar{X}) + \hat{u}_i$$

<sup>5</sup>Note that this result is true only when the regression model has the intercept term  $\beta_1$  in it. As **App. 6A, Sec. 6A.1** shows, this result need not hold when  $\beta_1$  is absent from the model.

<sup>6</sup>This result also requires that the intercept term  $\beta_1$  be present in the model (see **App. 6A, Sec. 6A.1**).

or

$$y_i = \hat{\beta}_2 x_i + \hat{u}_i \quad (3.1.13)$$

where  $y_i$  and  $x_i$ , following our convention, are deviations from their respective (sample) mean values.

Equation (3.1.13) is known as the **deviation form**. Notice that the intercept term  $\hat{\beta}_1$  is no longer present in it. But the intercept term can always be estimated by (3.1.7), that is, from the fact that the sample regression line passes through the sample means of  $Y$  and  $X$ . An advantage of the deviation form is that it often simplifies computing formulas.

In passing, note that in the deviation form, the SRF can be written as

$$\hat{y}_i = \hat{\beta}_2 x_i \quad (3.1.14)$$

whereas in the original units of measurement it was  $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$ , as shown in (2.6.1).

4. The residuals  $\hat{u}_i$  are uncorrelated with the predicted  $\hat{Y}_i$ . This statement can be verified as follows: using the deviation form, we can write

$$\begin{aligned} \sum \hat{y}_i \hat{u}_i &= \hat{\beta}_2 \sum x_i \hat{u}_i \\ &= \hat{\beta}_2 \sum x_i (y_i - \hat{\beta}_2 x_i) \\ &= \hat{\beta}_2 \sum x_i y_i - \hat{\beta}_2^2 \sum x_i^2 \\ &= \hat{\beta}_2^2 \sum x_i^2 - \hat{\beta}_2^2 \sum x_i^2 \\ &= 0 \end{aligned} \quad (3.1.15)$$

where use is made of the fact that  $\hat{\beta}_2 = \sum x_i y_i / \sum x_i^2$ .

5. The residuals  $\hat{u}_i$  are uncorrelated with  $X_i$ ; that is,  $\sum \hat{u}_i X_i = 0$ . This fact follows from Eq. (2) in Appendix 3A, Section 3A.1.

### 3.2 THE CLASSICAL LINEAR REGRESSION MODEL: THE ASSUMPTIONS UNDERLYING THE METHOD OF LEAST SQUARES

If our objective is to estimate  $\beta_1$  and  $\beta_2$  only, the method of OLS discussed in the preceding section will suffice. But recall from Chapter 2 that in regression analysis our objective is not only to obtain  $\hat{\beta}_1$  and  $\hat{\beta}_2$  but also to draw inferences about the true  $\beta_1$  and  $\beta_2$ . For example, we would like to know how close  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are to their counterparts in the population or how close  $\hat{Y}_i$  is to the true  $E(Y|X_i)$ . To that end, we must not only specify the functional form of the model, as in (2.4.2), but also make certain assumptions about

the manner in which  $Y_i$  are generated. To see why this requirement is needed, look at the PRF:  $Y_i = \beta_1 + \beta_2 X_i + u_i$ . It shows that  $Y_i$  depends on both  $X_i$  and  $u_i$ . Therefore, unless we are specific about how  $X_i$  and  $u_i$  are created or generated, there is no way we can make any statistical inference about the  $Y_i$  and also, as we shall see, about  $\beta_1$  and  $\beta_2$ . Thus, the assumptions made about the  $X_i$  variable(s) and the error term are extremely critical to the valid interpretation of the regression estimates.

**The Gaussian, standard, or classical linear regression model (CLRM)**, which is the cornerstone of most econometric theory, makes 10 assumptions.<sup>7</sup> We first discuss these assumptions in the context of the two-variable regression model; and in Chapter 7 we extend them to multiple regression models, that is, models in which there is more than one regressor.

**Assumption 1: Linear regression model.** The regression model is linear in the parameters, as shown in (2.4.2)

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (2.4.2)$$

We already discussed model (2.4.2) in Chapter 2. Since linear-in-parameter regression models are the starting point of the CLRM, we will maintain this assumption throughout this book. Keep in mind that the regressand  $Y$  and the regressor  $X$  themselves may be nonlinear, as discussed in Chapter 2.<sup>8</sup>

**Assumption 2:  $X$  values are fixed in repeated sampling.** Values taken by the regressor  $X$  are considered fixed in repeated samples. More technically,  $X$  is assumed to be *nonstochastic*.

This assumption is implicit in our discussion of the PRF in Chapter 2. But it is very important to understand the concept of “fixed values in repeated sampling,” which can be explained in terms of our example given in Table 2.1. Consider the various  $Y$  populations corresponding to the levels of income shown in that table. Keeping the value of income  $X$  fixed, say, at level \$80, we draw at random a family and observe its weekly family consumption expenditure  $Y$  as, say, \$60. Still keeping  $X$  at \$80, we draw at random another family and observe its  $Y$  value as \$75. In each of these drawings (i.e., repeated sampling), the value of  $X$  is fixed at \$80. We can repeat this process for all the  $X$  values shown in Table 2.1. As a matter of fact, the sample data shown in Tables 2.4 and 2.5 were drawn in this fashion.

What all this means is that our regression analysis is **conditional regression analysis**, that is, conditional on the given values of the regressor(s)  $X$ .

<sup>7</sup>It is classical in the sense that it was developed first by Gauss in 1821 and since then has served as a norm or a standard against which may be compared the regression models that do not satisfy the Gaussian assumptions.

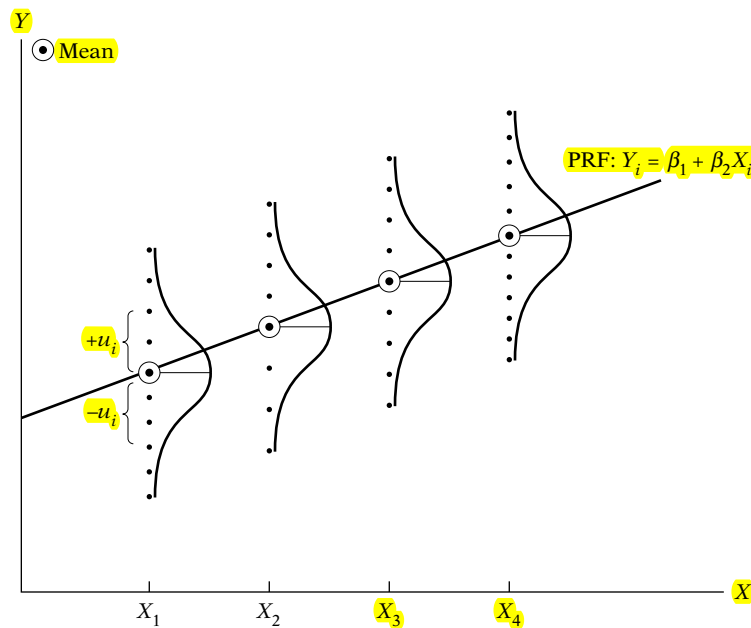
<sup>8</sup>However, a brief discussion of nonlinear-in-the-parameter regression models is given in Chap. 14.

**Assumption 3: Zero mean value of disturbance  $u_i$ .** Given the value of  $X$ , the mean, or expected, value of the random disturbance term  $u_i$  is zero. Technically, the conditional mean value of  $u_i$  is zero. Symbolically, we have

$$E(u_i | X_i) = 0 \tag{3.2.1}$$

Assumption 3 states that the mean value of  $u_i$ , conditional upon the given  $X_i$ , is zero. Geometrically, this assumption can be pictured as in Figure 3.3, which shows a few values of the variable  $X$  and the  $Y$  populations associated with each of them. As shown, each  $Y$  population corresponding to a given  $X$  is distributed around its mean value (shown by the circled points on the PRF) with some  $Y$  values above the mean and some below it. The distances above and below the mean values are nothing but the  $u_i$ , and what (3.2.1) requires is that the average or mean value of these deviations corresponding to any given  $X$  should be zero.<sup>9</sup>

This assumption should not be difficult to comprehend in view of the discussion in Section 2.4 [see Eq. (2.4.5)]. All that this assumption says is that the factors not explicitly included in the model, and therefore subsumed in  $u_i$ , do not systematically affect the mean value of  $Y$ ; so to speak, the positive  $u_i$



**FIGURE 3.3** Conditional distribution of the disturbances  $u_i$ .

<sup>9</sup>For illustration, we are assuming merely that the  $u$ 's are distributed symmetrically as shown in Figure 3.3. But shortly we will assume that the  $u$ 's are distributed normally.

values cancel out the negative  $u_i$  values so that their average or mean effect on  $Y$  is zero.<sup>10</sup>

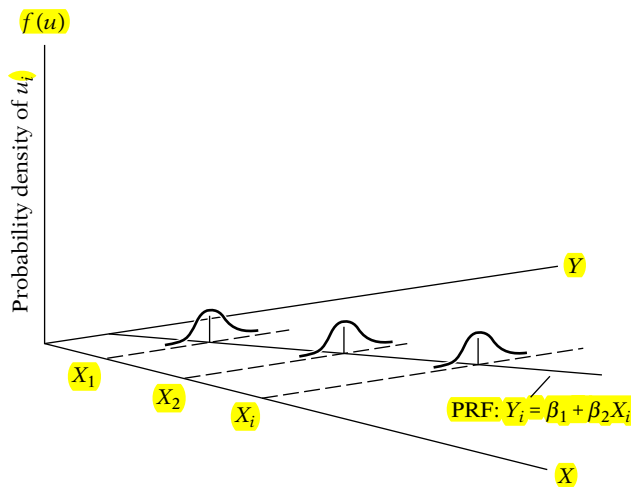
In passing, note that the assumption  $E(u_i | X_i) = 0$  implies that  $E(Y_i | X_i) = \beta_1 + \beta_2 X_i$ . (Why?) Therefore, the two assumptions are equivalent.

**Assumption 4: Homoscedasticity or equal variance of  $u_i$ .** Given the value of  $X$ , the variance of  $u_i$  is the same for all observations. That is, the conditional variances of  $u_i$  are identical. Symbolically, we have

$$\begin{aligned} \text{var}(u_i | X_i) &= E[u_i - E(u_i | X_i)]^2 \\ &= E(u_i^2 | X_i) \text{ because of Assumption 3} \\ &= \sigma^2 \end{aligned} \tag{3.2.2}$$

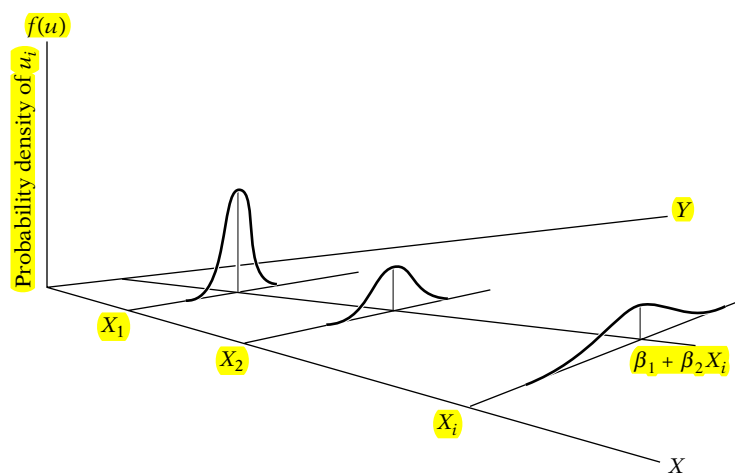
where **var** stands for variance.

Eq. (3.2.2) states that the variance of  $u_i$  for each  $X_i$  (i.e., the conditional variance of  $u_i$ ) is some positive constant number equal to  $\sigma^2$ . Technically, (3.2.2) represents the assumption of **homoscedasticity**, or *equal* (homo) *spread* (scedasticity) or *equal variance*. The word comes from the Greek verb *skedanime*, which means to disperse or scatter. Stated differently, (3.2.2) means that the  $Y$  populations corresponding to various  $X$  values have the same variance. Put simply, the variation around the regression line (which is the line of average relationship between  $Y$  and  $X$ ) is the same across the  $X$  values; it neither increases or decreases as  $X$  varies. Diagrammatically, the situation is as depicted in Figure 3.4.



**FIGURE 3.4** Homoscedasticity.

<sup>10</sup>For a more technical reason why Assumption 3 is necessary see E. Malinvaud, *Statistical Methods of Econometrics*, Rand McNally, Chicago, 1966, p. 75. See also exercise 3.3.



**FIGURE 3.5** Heteroscedasticity.

In contrast, consider Figure 3.5, where the conditional variance of the  $Y$  population varies with  $X$ . This situation is known appropriately as **heteroscedasticity**, or *unequal spread*, or *variance*. Symbolically, in this situation (3.2.2) can be written as

$$\text{var}(u_i | X_i) = \sigma_i^2 \quad (3.2.3)$$

Notice the subscript on  $\sigma^2$  in Eq. (3.2.3), which indicates that the variance of the  $Y$  population is no longer constant.

To make the difference between the two situations clear, let  $Y$  represent weekly consumption expenditure and  $X$  weekly income. Figures 3.4 and 3.5 show that as income increases the average consumption expenditure also increases. But in Figure 3.4 the variance of consumption expenditure remains the same at all levels of income, whereas in Figure 3.5 it increases with increase in income. In other words, richer families on the average consume more than poorer families, but there is also more variability in the consumption expenditure of the former.

To understand the rationale behind this assumption, refer to Figure 3.5. As this figure shows,  $\text{var}(u | X_1) < \text{var}(u | X_2), \dots, < \text{var}(u | X_i)$ . Therefore, the likelihood is that the  $Y$  observations coming from the population with  $X = X_1$  would be closer to the PRF than those coming from populations corresponding to  $X = X_2, X = X_3$ , and so on. In short, not all  $Y$  values corresponding to the various  $X$ 's will be equally reliable, reliability being judged by how closely or distantly the  $Y$  values are distributed around their means, that is, the points on the PRF. If this is in fact the case, would we not prefer to sample from those  $Y$  populations that are closer to their mean than those that are widely spread? But doing so might restrict the variation we obtain across  $X$  values.

By invoking Assumption 4, we are saying that at this stage all  $Y$  values corresponding to the various  $X$ 's are equally important. In Chapter 11 we shall see what happens if this is not the case, that is, where there is heteroscedasticity.

In passing, note that Assumption 4 implies that the conditional variances of  $Y_i$  are also homoscedastic. That is,

$$\text{var}(Y_i | X_i) = \sigma^2 \quad (3.2.4)$$

Of course, the *unconditional variance* of  $Y$  is  $\sigma_Y^2$ . Later we will see the importance of distinguishing between conditional and unconditional variances of  $Y$  (see Appendix A for details of conditional and unconditional variances).

**Assumption 5: No autocorrelation between the disturbances.** Given any two  $X$  values,  $X_i$  and  $X_j$  ( $i \neq j$ ), the correlation between any two  $u_i$  and  $u_j$  ( $i \neq j$ ) is zero. Symbolically,

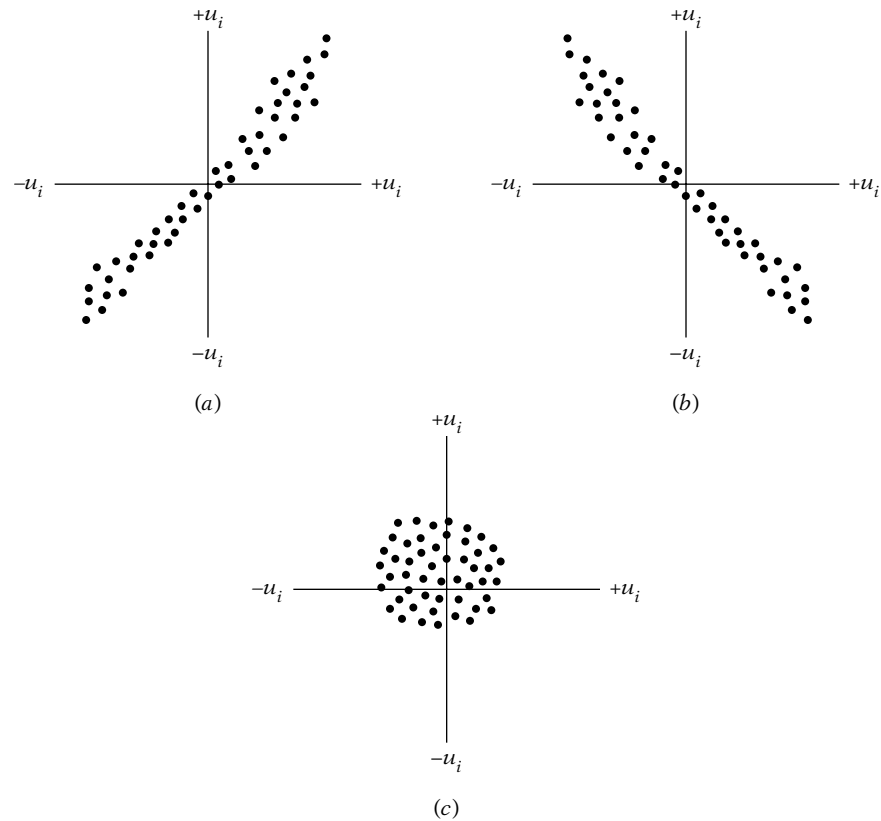
$$\begin{aligned} \text{cov}(u_i, u_j | X_i, X_j) &= E\{[u_i - E(u_i) | X_i]\{[u_j - E(u_j) | X_j]\} \\ &= E(u_i | X_i)(u_j | X_j) \quad (\text{why?}) \\ &= 0 \end{aligned} \quad (3.2.5)$$

where  $i$  and  $j$  are two different observations and where **cov** means **covariance**.

In words, (3.2.5) postulates that the disturbances  $u_i$  and  $u_j$  are uncorrelated. Technically, this is the assumption of **no serial correlation**, or **no autocorrelation**. This means that, given  $X_i$ , the deviations of any two  $Y$  values from their mean value do not exhibit patterns such as those shown in Figure 3.6a and b. In Figure 3.6a, we see that the  $u$ 's are **positively correlated**, a positive  $u$  followed by a positive  $u$  or a negative  $u$  followed by a negative  $u$ . In Figure 3.6b, the  $u$ 's are **negatively correlated**, a positive  $u$  followed by a negative  $u$  and vice versa.

If the disturbances (deviations) follow systematic patterns, such as those shown in Figure 3.6a and b, there is auto- or serial correlation, and what Assumption 5 requires is that such correlations be absent. Figure 3.6c shows that there is no systematic pattern to the  $u$ 's, thus indicating zero correlation.

The full import of this assumption will be explained thoroughly in Chapter 12. But intuitively one can explain this assumption as follows. Suppose in our PRF ( $Y_t = \beta_1 + \beta_2 X_t + u_t$ ) that  $u_t$  and  $u_{t-1}$  are positively correlated. Then  $Y_t$  depends not only on  $X_t$  but also on  $u_{t-1}$  for  $u_{t-1}$  to some extent determines  $u_t$ . At this stage of the development of the subject matter, by invoking Assumption 5, we are saying that we will consider the systematic effect, if any, of  $X_t$  on  $Y_t$  and not worry about the other influences that might act on  $Y$  as a result of the possible intercorrelations among the  $u$ 's. But, as noted in Chapter 12, we will see how intercorrelations among the disturbances can be brought into the analysis and with what consequences.



**FIGURE 3.6** Patterns of correlation among the disturbances. (a) positive serial correlation; (b) negative serial correlation; (c) zero correlation.

**Assumption 6: Zero covariance between  $u_i$  and  $X_i$ , or  $E(u_i X_i) = 0$ .** Formally,

$$\begin{aligned}
 \text{cov}(u_i, X_i) &= E[u_i - E(u_i)][X_i - E(X_i)] \\
 &= E[u_i(X_i - E(X_i))] \quad \text{since } E(u_i) = 0 \\
 &= E(u_i X_i) - E(X_i)E(u_i) \quad \text{since } E(X_i) \text{ is nonstochastic} \\
 &= E(u_i X_i) \quad \text{since } E(u_i) = 0 \\
 &= 0 \quad \text{by assumption}
 \end{aligned}
 \tag{3.2.6}$$

Assumption 6 states that the disturbance  $u$  and explanatory variable  $X$  are uncorrelated. The rationale for this assumption is as follows: When we expressed the PRF as in (2.4.2), we assumed that  $X$  and  $u$  (which may represent the influence of all the omitted variables) have separate (and additive) influence on  $Y$ . But if  $X$  and  $u$  are correlated, it is not possible to assess their individual effects on  $Y$ . Thus, if  $X$  and  $u$  are positively correlated,  $X$  increases



when  $u$  increases and it decreases when  $u$  decreases. Similarly, if  $X$  and  $u$  are negatively correlated,  $X$  increases when  $u$  decreases and it decreases when  $u$  increases. In either case, it is difficult to isolate the influence of  $X$  and  $u$  on  $Y$ .

Assumption 6 is automatically fulfilled if  $X$  variable is nonrandom or nonstochastic and Assumption 3 holds, for in that case,  $\text{cov}(u_i, X_i) = [X_i - E(X_i)]E[u_i - E(u_i)] = 0$ . (Why?) But since we have assumed that our  $X$  variable not only is nonstochastic but also assumes fixed values in repeated samples,<sup>11</sup> Assumption 6 is not very critical for us; it is stated here merely to point out that the regression theory presented in the sequel holds true even if the  $X$ 's are stochastic or random, provided they are independent or at least uncorrelated with the disturbances  $u_i$ .<sup>12</sup> (We shall examine the consequences of relaxing Assumption 6 in Part II.)

**Assumption 7: The number of observations  $n$  must be greater than the number of parameters to be estimated.** Alternatively, the number of observations  $n$  must be greater than the number of explanatory variables.

This assumption is not so innocuous as it seems. In the hypothetical example of Table 3.1, imagine that we had only the first pair of observations on  $Y$  and  $X$  (4 and 1). From this single observation there is no way to estimate the two unknowns,  $\beta_1$  and  $\beta_2$ . We need at least two pairs of observations to estimate the two unknowns. In a later chapter we will see the critical importance of this assumption.

**Assumption 8: Variability in  $X$  values.** The  $X$  values in a given sample must not all be the same. Technically,  $\text{var}(X)$  must be a finite positive number.<sup>13</sup>

This assumption too is not so innocuous as it looks. Look at Eq. (3.1.6). If all the  $X$  values are identical, then  $X_i = \bar{X}$  (Why?) and the denominator of that equation will be zero, making it impossible to estimate  $\beta_2$  and therefore  $\beta_1$ . Intuitively, we readily see why this assumption is important. Looking at

<sup>11</sup>Recall that in obtaining the samples shown in Tables 2.4 and 2.5, we kept the same  $X$  values.

<sup>12</sup>As we will discuss in Part II, if the  $X$ 's are stochastic but distributed independently of  $u_i$ , the properties of least estimators discussed shortly continue to hold, but if the stochastic  $X$ 's are merely uncorrelated with  $u_i$ , the properties of OLS estimators hold true only if the sample size is very large. At this stage, however, there is no need to get bogged down with this theoretical point.

<sup>13</sup>The sample variance of  $X$  is

$$\text{var}(X) = \frac{\sum(X_i - \bar{X})^2}{n - 1}$$

where  $n$  is sample size.

our family consumption expenditure example in Chapter 2, if there is very little variation in family income, we will not be able to explain much of the variation in the consumption expenditure. The reader should keep in mind that variation in both  $Y$  and  $X$  is essential to use regression analysis as a research tool. In short, the variables must vary!

**Assumption 9: The regression model is correctly specified.** Alternatively, there is no specification bias or error in the model used in empirical analysis.

As we discussed in the Introduction, the classical econometric methodology assumes implicitly, if not explicitly, that the model used to test an economic theory is “correctly specified.” This assumption can be explained informally as follows. An econometric investigation begins with the specification of the econometric model underlying the phenomenon of interest. Some important questions that arise in the specification of the model include the following: (1) What variables should be included in the model? (2) What is the functional form of the model? Is it linear in the parameters, the variables, or both? (3) What are the probabilistic assumptions made about the  $Y_i$ , the  $X_i$ , and the  $u_i$  entering the model?

These are extremely important questions, for, as we will show in Chapter 13, by omitting important variables from the model, or by choosing the wrong functional form, or by making wrong stochastic assumptions about the variables of the model, the validity of interpreting the estimated regression will be highly questionable. To get an intuitive feeling about this, refer to the Phillips curve shown in Figure 1.3. Suppose we choose the following two models to depict the underlying relationship between the rate of change of money wages and the unemployment rate:

$$Y_i = \alpha_1 + \alpha_2 X_i + u_i \quad (3.2.7)$$

$$Y_i = \beta_1 + \beta_2 \left( \frac{1}{X_i} \right) + u_i \quad (3.2.8)$$

where  $Y_i$  = the rate of change of money wages, and  $X_i$  = the unemployment rate.

The regression model (3.2.7) is linear both in the parameters and the variables, whereas (3.2.8) is linear in the parameters (hence a linear regression model by our definition) but nonlinear in the variable  $X$ . Now consider Figure 3.7.

If model (3.2.8) is the “correct” or the “true” model, fitting the model (3.2.7) to the scatterpoints shown in Figure 3.7 will give us wrong predictions: Between points  $A$  and  $B$ , for any given  $X_i$  the model (3.2.7) is going to overestimate the true mean value of  $Y$ , whereas to the left of  $A$  (or to the right of  $B$ ) it is going to underestimate (or overestimate, in absolute terms) the true mean value of  $Y$ .

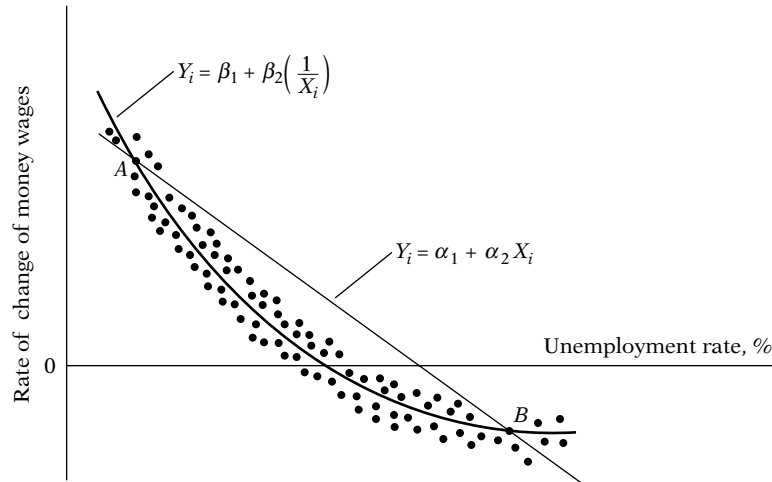


FIGURE 3.7 Linear and nonlinear Phillips curves.

The preceding example is an instance of what is called a **specification bias** or a **specification error**; here the bias consists in choosing the wrong functional form. We will see other types of specification errors in Chapter 13.

Unfortunately, in practice one rarely knows the correct variables to include in the model or the correct functional form of the model or the correct probabilistic assumptions about the variables entering the model for the theory underlying the particular investigation (e.g., the Phillips-type money wage change–unemployment rate tradeoff) may not be strong or robust enough to answer all these questions. Therefore, in practice, the econometrician has to use some judgment in choosing the number of variables entering the model and the functional form of the model and has to make some assumptions about the stochastic nature of the variables included in the model. To some extent, there is some trial and error involved in choosing the “right” model for empirical analysis.<sup>14</sup>

If judgment is required in selecting a model, what is the need for Assumption 9? Without going into details here (see Chapter 13), this assumption is there to remind us that our regression analysis and therefore the results based on that analysis are conditional upon the chosen model and to warn us that we should give very careful thought in formulating econometric

<sup>14</sup>But one should avoid what is known as “**data mining**,” that is, trying every possible model with the hope that at least one will fit the data well. That is why it is essential that there be some economic reasoning underlying the chosen model and that any modifications in the model should have some economic justification. A purely ad hoc model may be difficult to justify on theoretical or a priori grounds. In short, theory should be the basis of estimation. But we will have more to say about data mining in Chap. 13, for there are some who argue that in some situations data mining can serve a useful purpose.

models, especially when there may be several competing theories trying to explain an economic phenomenon, such as the inflation rate, or the demand for money, or the determination of the appropriate or equilibrium value of a stock or a bond. *Thus, econometric model-building, as we shall discover, is more often an art rather than a science.*

Our discussion of the assumptions underlying the classical linear regression model is now completed. It is important to note that all these assumptions pertain to the PRF only and not the SRF. But it is interesting to observe that the method of least squares discussed previously has some properties that are similar to the assumptions we have made about the PRF. For example, the finding that  $\sum \hat{u}_i = 0$ , and, therefore,  $\bar{\hat{u}} = 0$ , is akin to the assumption that  $E(u_i | X_i) = 0$ . Likewise, the finding that  $\sum \hat{u}_i X_i = 0$  is similar to the assumption that  $\text{cov}(u_i, X_i) = 0$ . It is comforting to note that the method of least squares thus tries to “duplicate” some of the assumptions we have imposed on the PRF.

Of course, the SRF does not duplicate all the assumptions of the CLRM. As we will show later, although  $\text{cov}(u_i, u_j) = 0 (i \neq j)$  by assumption, it is *not* true that the *sample*  $\text{cov}(\hat{u}_i, \hat{u}_j) = 0 (i \neq j)$ . As a matter of fact, we will show later that the residuals not only are autocorrelated but also are heteroscedastic (see Chapter 12).

When we go beyond the two-variable model and consider multiple regression models, that is, models containing several regressors, we add the following assumption.

**Assumption 10: There is no perfect multicollinearity.** That is, there are *no perfect linear relationships among the explanatory variables.*

We will discuss this assumption in Chapter 7, where we discuss multiple regression models.

### A Word about These Assumptions

The million-dollar question is: How realistic are all these assumptions? The “reality of assumptions” is an age-old question in the philosophy of science. Some argue that it does not matter whether the assumptions are realistic. What matters are the predictions based on those assumptions. Notable among the “irrelevance-of-assumptions thesis” is Milton Friedman. To him, unreality of assumptions is a positive advantage: “to be important . . . a hypothesis must be descriptively false in its assumptions.”<sup>15</sup>

One may not subscribe to this viewpoint fully, but recall that in any scientific study we make certain assumptions because they facilitate the

<sup>15</sup>Milton Friedman, *Essays in Positive Economics*, University of Chicago Press, Chicago, 1953, p. 14.

development of the subject matter in gradual steps, not because they are necessarily realistic in the sense that they replicate reality exactly. As one author notes, “. . . if simplicity is a desirable criterion of good theory, all good theories idealize and oversimplify outrageously.”<sup>16</sup>

What we plan to do is first study the properties of the CLRM thoroughly, and then in later chapters examine in depth what happens if one or more of the assumptions of CLRM are not fulfilled. At the end of this chapter, we provide in Table 3.4 a guide to where one can find out what happens to the CLRM if a particular assumption is not satisfied.

As a colleague pointed out to me, when we review research done by others, we need to consider whether the assumptions made by the researcher are appropriate to the data and problem. All too often, published research is based on implicit assumptions about problem and data that are likely not correct and that produce estimates based on these assumptions. Clearly, the knowledgeable reader should, realizing these problems, adopt a skeptical attitude toward the research. The assumptions listed in Table 3.4 therefore provide a checklist for guiding our research and for evaluating the research of others.

With this backdrop, we are now ready to study the CLRM. In particular, we want to find out the **statistical properties** of OLS compared with the purely **numerical properties** discussed earlier. The statistical properties of OLS are based on the assumptions of CLRM already discussed and are enshrined in the famous **Gauss–Markov theorem**. But before we turn to this theorem, which provides the theoretical justification for the popularity of OLS, we first need to consider the **precision** or **standard errors** of the least-squares estimates.

### 3.3 PRECISION OR STANDARD ERRORS OF LEAST-SQUARES ESTIMATES

From Eqs. (3.1.6) and (3.1.7), it is evident that least-squares estimates are a function of the sample data. But since the data are likely to change from sample to sample, the estimates will change ipso facto. Therefore, what is needed is some measure of “reliability” or **precision** of the estimators  $\hat{\beta}_1$  and  $\hat{\beta}_2$ . In statistics the precision of an estimate is measured by its standard error (se).<sup>17</sup> Given the Gaussian assumptions, it is shown in Appendix 3A, Section 3A.3 that the standard errors of the OLS estimates can be obtained

<sup>16</sup>Mark Blaug, *The Methodology of Economics: Or How Economists Explain*, 2d ed., Cambridge University Press, New York, 1992, p. 92.

<sup>17</sup>The **standard error** is nothing but the standard deviation of the sampling distribution of the estimator, and the sampling distribution of an estimator is simply a probability or frequency distribution of the estimator; that is, a distribution of the set of values of the estimator obtained from all possible samples of the same size from a given population. Sampling distributions are used to draw inferences about the values of the population parameters on the basis of the values of the estimators calculated from one or more samples. (For details, see **App. A.**)

as follows:

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_i^2} \quad (3.3.1)$$

$$\text{se}(\hat{\beta}_2) = \frac{\sigma}{\sqrt{\sum x_i^2}} \quad (3.3.2)$$

$$\text{var}(\hat{\beta}_1) = \frac{\sum X_i^2}{n \sum x_i^2} \sigma^2 \quad (3.3.3)$$

$$\text{se}(\hat{\beta}_1) = \sqrt{\frac{\sum X_i^2}{n \sum x_i^2}} \sigma \quad (3.3.4)$$

where var = variance and se = standard error and where  $\sigma^2$  is the constant or homoscedastic variance of  $u_i$  of Assumption 4.

All the quantities entering into the preceding equations except  $\sigma^2$  can be estimated from the data. As shown in Appendix 3A, Section 3A.5,  $\sigma^2$  itself is estimated by the following formula:

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2} \quad (3.3.5)$$

where  $\hat{\sigma}^2$  is the OLS estimator of the true but unknown  $\sigma^2$  and where the expression  $n-2$  is known as the **number of degrees of freedom (df)**,  $\sum \hat{u}_i^2$  being the sum of the residuals squared or the **residual sum of squares (RSS)**.<sup>18</sup>

Once  $\sum \hat{u}_i^2$  is known,  $\hat{\sigma}^2$  can be easily computed.  $\sum \hat{u}_i^2$  itself can be computed either from (3.1.2) or from the following expression (see Section 3.5 for the proof):

$$\sum \hat{u}_i^2 = \sum y_i^2 - \hat{\beta}_2^2 \sum x_i^2 \quad (3.3.6)$$

Compared with Eq. (3.1.2), Eq. (3.3.6) is easy to use, for it does not require computing  $\hat{u}_i$  for each observation although such a computation will be useful in its own right (as we shall see in Chapters 11 and 12).

Since

$$\hat{\beta}_2 = \frac{\sum x_i y_i}{\sum x_i^2}$$

<sup>18</sup>The term **number of degrees of freedom** means the total number of observations in the sample ( $= n$ ) less the number of independent (linear) constraints or restrictions put on them. In other words, it is the number of independent observations out of a total of  $n$  observations. For example, before the RSS (3.1.2) can be computed,  $\hat{\beta}_1$  and  $\hat{\beta}_2$  must first be obtained. These two estimates therefore put two restrictions on the RSS. Therefore, there are  $n-2$ , not  $n$ , independent observations to compute the RSS. Following this logic, in the three-variable regression RSS will have  $n-3$  df, and for the  $k$ -variable model it will have  $n-k$  df. **The general rule is this:** df = ( $n$  - number of parameters estimated).

an alternative expression for computing  $\sum \hat{u}_i^2$  is

$$\sum \hat{u}_i^2 = \sum y_i^2 - \frac{(\sum x_i y_i)^2}{\sum x_i^2} \quad (3.3.7)$$

In passing, note that the positive square root of  $\hat{\sigma}^2$

$$\hat{\sigma} = \sqrt{\frac{\sum \hat{u}_i^2}{n-2}} \quad (3.3.8)$$

is known as the **standard error of estimate** or the **standard error of the regression (se)**. It is simply the standard deviation of the  $Y$  values about the estimated regression line and is often used as a summary measure of the “goodness of fit” of the estimated regression line, a topic discussed in Section 3.5.

Earlier we noted that, given  $X_i$ ,  $\sigma^2$  represents the (conditional) variance of both  $u_i$  and  $Y_i$ . Therefore, the standard error of the estimate can also be called the (conditional) standard deviation of  $u_i$  and  $Y_i$ . Of course, as usual,  $\sigma_Y^2$  and  $\sigma_Y$  represent, respectively, the unconditional variance and unconditional standard deviation of  $Y$ .

Note the following features of the variances (and therefore the standard errors) of  $\hat{\beta}_1$  and  $\hat{\beta}_2$ .

1. The variance of  $\hat{\beta}_2$  is directly proportional to  $\sigma^2$  but inversely proportional to  $\sum x_i^2$ . That is, given  $\sigma^2$ , the larger the variation in the  $X$  values, the smaller the variance of  $\hat{\beta}_2$  and hence the greater the precision with which  $\beta_2$  can be estimated. In short, given  $\sigma^2$ , if there is substantial variation in the  $X$  values (recall Assumption 8),  $\beta_2$  can be measured more accurately than when the  $X_i$  do not vary substantially. Also, given  $\sum x_i^2$ , the larger the variance of  $\sigma^2$ , the larger the variance of  $\beta_2$ . Note that as the sample size  $n$  increases, the number of terms in the sum,  $\sum x_i^2$ , will increase. As  $n$  increases, the precision with which  $\beta_2$  can be estimated also increases. (Why?)

2. The variance of  $\hat{\beta}_1$  is directly proportional to  $\sigma^2$  and  $\sum X_i^2$  but inversely proportional to  $\sum x_i^2$  and the sample size  $n$ .

3. Since  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are estimators, they will not only vary from sample to sample but in a given sample they are likely to be dependent on each other, this dependence being measured by the covariance between them. It is shown in Appendix 3A, Section 3A.4 that

$$\begin{aligned} \text{cov}(\hat{\beta}_1, \hat{\beta}_2) &= -\bar{X} \text{var}(\hat{\beta}_2) \\ &= -\bar{X} \left( \frac{\sigma^2}{\sum x_i^2} \right) \end{aligned} \quad (3.3.9)$$



Since  $\text{var}(\hat{\beta}_2)$  is always positive, as is the variance of any variable, the nature of the covariance between  $\hat{\beta}_1$  and  $\hat{\beta}_2$  depends on the sign of  $\bar{X}$ . If  $\bar{X}$  is positive, then as the formula shows, the covariance will be negative. Thus, if the slope coefficient  $\beta_2$  is *overestimated* (i.e., the slope is too steep), the intercept coefficient  $\beta_1$  will be *underestimated* (i.e., the intercept will be too small). Later on (especially in the chapter on multicollinearity, Chapter 10), we will see the utility of studying the covariances between the estimated regression coefficients.

How do the variances and standard errors of the estimated regression coefficients enable one to judge the reliability of these estimates? This is a problem in statistical inference, and it will be pursued in Chapters 4 and 5.

### 3.4 PROPERTIES OF LEAST-SQUARES ESTIMATORS: THE GAUSS-MARKOV THEOREM<sup>19</sup>

As noted earlier, given the assumptions of the classical linear regression model, the least-squares estimates possess some ideal or optimum properties. These properties are contained in the well-known **Gauss-Markov theorem**. To understand this theorem, we need to consider the **best linear unbiasedness property** of an estimator.<sup>20</sup> As explained in Appendix A, an estimator, say the OLS estimator  $\hat{\beta}_2$ , is said to be a best linear unbiased estimator (BLUE) of  $\beta_2$  if the following hold:

1. It is **linear**, that is, a linear function of a random variable, such as the dependent variable  $Y$  in the regression model.
2. It is **unbiased**, that is, its average or expected value,  $E(\hat{\beta}_2)$ , is equal to the true value,  $\beta_2$ .
3. It has minimum variance in the class of all such linear unbiased estimators; an unbiased estimator with the least variance is known as an **efficient estimator**.

In the regression context it can be proved that the OLS estimators are BLUE. This is the gist of the famous Gauss-Markov theorem, which can be stated as follows:

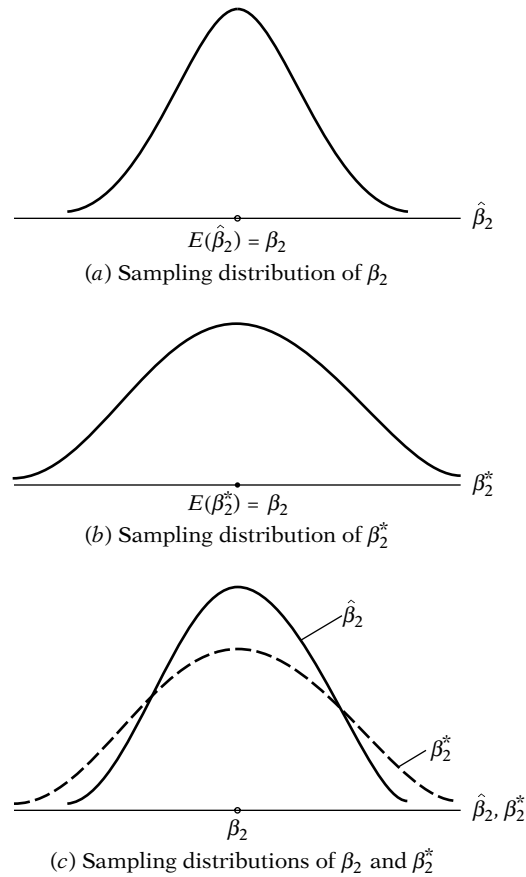
**Gauss-Markov Theorem:** Given the assumptions of the classical linear regression model, the least-squares estimators, in the class of unbiased linear estimators, have minimum variance, that is, they are BLUE.

The proof of this theorem is sketched in **Appendix 3A, Section 3A.6**. The full import of the Gauss-Markov theorem will become clearer as we move

<sup>19</sup>Although known as the *Gauss-Markov theorem*, the least-squares approach of Gauss antedates (1821) the minimum-variance approach of Markov (1900).

<sup>20</sup>The reader should refer to **App. A** for the importance of linear estimators as well as for a general discussion of the desirable properties of statistical estimators.





**FIGURE 3.8** Sampling distribution of OLS estimator  $\hat{\beta}_2$  and alternative estimator  $\hat{\beta}_2^*$

along. It is sufficient to note here that the theorem has theoretical as well as practical importance.<sup>21</sup>

What all this means can be explained with the aid of Figure 3.8.

In Figure 3.8(a) we have shown the **sampling distribution** of the OLS estimator  $\hat{\beta}_2$ , that is, the distribution of the values taken by  $\hat{\beta}_2$  in repeated sampling experiments (recall Table 3.1). For convenience we have assumed  $\hat{\beta}_2$  to be distributed symmetrically (but more on this in Chapter 4). As the figure shows, the mean of the  $\hat{\beta}_2$  values,  $E(\hat{\beta}_2)$ , is equal to the true  $\beta_2$ . In this situation we say that  $\hat{\beta}_2$  is an *unbiased estimator* of  $\beta_2$ . In Figure 3.8(b) we have shown the sampling distribution of  $\hat{\beta}_2^*$ , an alternative estimator of  $\beta_2$

<sup>21</sup>For example, it can be proved that any linear combination of the  $\beta$ 's, such as  $(\beta_1 - 2\beta_2)$ , can be estimated by  $(\hat{\beta}_1 - 2\hat{\beta}_2)$ , and this estimator is BLUE. For details, see Henri Theil, *Introduction to Econometrics*, Prentice-Hall, Englewood Cliffs, N.J., 1978, pp. 401–402. Note a technical point about the Gauss–Markov theorem: It provides only the sufficient (but not necessary) condition for OLS to be efficient. I am indebted to Michael McAleer of the University of Western Australia for bringing this point to my attention.

obtained by using another (i.e., other than OLS) method. For convenience, assume that  $\beta_2^*$ , like  $\hat{\beta}_2$ , is unbiased, that is, its average or expected value is equal to  $\beta_2$ . Assume further that both  $\hat{\beta}_2$  and  $\beta_2^*$  are linear estimators, that is, they are linear functions of  $Y$ . Which estimator,  $\hat{\beta}_2$  or  $\beta_2^*$ , would you choose?

To answer this question, superimpose the two figures, as in Figure 3.8(c). It is obvious that although both  $\hat{\beta}_2$  and  $\beta_2^*$  are unbiased the distribution of  $\beta_2^*$  is more diffused or widespread around the mean value than the distribution of  $\hat{\beta}_2$ . In other words, the variance of  $\beta_2^*$  is larger than the variance of  $\hat{\beta}_2$ . Now given two estimators that are both linear and unbiased, one would choose the estimator with the smaller variance because it is more likely to be close to  $\beta_2$  than the alternative estimator. In short, one would choose the BLUE estimator.

The Gauss–Markov theorem is remarkable in that it makes no assumptions about the probability distribution of the random variable  $u_i$ , and therefore of  $Y_i$  (in the next chapter we will take this up). As long as the assumptions of CLRM are satisfied, the theorem holds. As a result, we need not look for another linear unbiased estimator, for we will not find such an estimator whose variance is smaller than the OLS estimator. Of course, if one or more of these assumptions do not hold, the theorem is invalid. For example, if we consider nonlinear-in-the-parameter regression models (which are discussed in Chapter 14), we may be able to obtain estimators that may perform better than the OLS estimators. Also, as we will show in the chapter on heteroscedasticity, if the assumption of homoscedastic variance is not fulfilled, the OLS estimators, although unbiased and consistent, are no longer minimum variance estimators even in the class of linear estimators.

The statistical properties that we have just discussed are known as **finite sample properties**: These properties hold regardless of the sample size on which the estimators are based. Later we will have occasions to consider the **asymptotic properties**, that is, properties that hold only if the sample size is very large (technically, infinite). A general discussion of finite-sample and large-sample properties of estimators is given in **Appendix A**.

### 3.5 THE COEFFICIENT OF DETERMINATION $r^2$ : A MEASURE OF “GOODNESS OF FIT”

Thus far we were concerned with the problem of estimating regression coefficients, their standard errors, and some of their properties. We now consider the **goodness of fit** of the fitted regression line to a set of data; that is, we shall find out how “well” the sample regression line fits the data. From Figure 3.1 it is clear that if all the observations were to lie on the regression line, we would obtain a “perfect” fit, but this is rarely the case. Generally, there will be some positive  $\hat{u}_i$  and some negative  $\hat{u}_i$ . What we hope for is that these residuals around the regression line are as small as possible. The **coefficient of determination**  $r^2$  (two-variable case) or  $R^2$  (multiple regression) is a summary measure that tells how well the sample regression line fits the data.

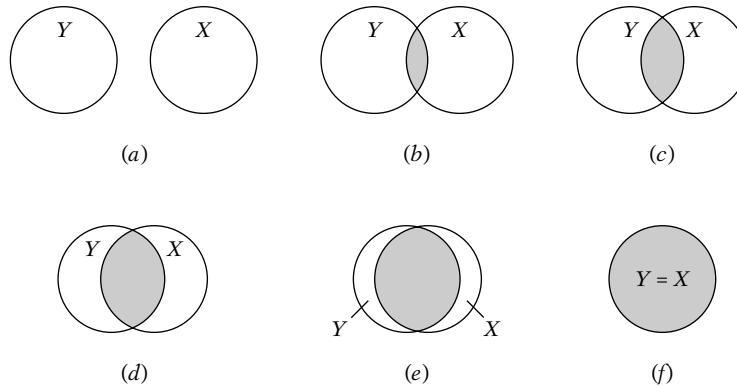


FIGURE 3.9 The Ballentine view of  $r^2$ : (a)  $r^2 = 0$ ; (f)  $r^2 = 1$ .

Before we show how  $r^2$  is computed, let us consider a heuristic explanation of  $r^2$  in terms of a graphical device, known as the **Venn diagram**, or the **Ballentine**, as shown in Figure 3.9.<sup>22</sup>

In this figure the circle  $Y$  represents variation in the dependent variable  $Y$  and the circle  $X$  represents variation in the explanatory variable  $X$ .<sup>23</sup> The overlap of the two circles (the shaded area) indicates the extent to which the variation in  $Y$  is explained by the variation in  $X$  (say, via an OLS regression). The greater the extent of the overlap, the greater the variation in  $Y$  is explained by  $X$ . The  $r^2$  is simply a numerical measure of this overlap. In the figure, as we move from left to right, the area of the overlap increases, that is, successively a greater proportion of the variation in  $Y$  is explained by  $X$ . In short,  $r^2$  increases. When there is no overlap,  $r^2$  is obviously zero, but when the overlap is complete,  $r^2$  is 1, since 100 percent of the variation in  $Y$  is explained by  $X$ . As we shall show shortly,  $r^2$  lies between 0 and 1.

To compute this  $r^2$ , we proceed as follows: Recall that

$$Y_i = \hat{Y}_i + \hat{u}_i \quad (2.6.3)$$

or in the deviation form

$$y_i = \hat{y}_i + \hat{u}_i \quad (3.5.1)$$

where use is made of (3.1.13) and (3.1.14). Squaring (3.5.1) on both sides

<sup>22</sup>See Peter Kennedy, "Ballentine: A Graphical Aid for Econometrics," *Australian Economics Papers*, vol. 20, 1981, pp. 414–416. The name Ballentine is derived from the emblem of the well-known Ballantine beer with its circles.

<sup>23</sup>The term *variation* and *variance* are different. Variation means the sum of squares of the deviations of a variable from its mean value. Variance is this sum of squares divided by the appropriate degrees of freedom. In short, variance = variation/df.

and summing over the sample, we obtain

$$\begin{aligned} \sum y_i^2 &= \sum \hat{y}_i^2 + \sum \hat{u}_i^2 + 2 \sum \hat{y}_i \hat{u}_i \\ &= \sum \hat{y}_i^2 + \sum \hat{u}_i^2 \\ &= \hat{\beta}_2^2 \sum x_i^2 + \sum \hat{u}_i^2 \end{aligned} \tag{3.5.2}$$

since  $\sum \hat{y}_i \hat{u}_i = 0$  (why?) and  $\hat{y}_i = \hat{\beta}_2 x_i$ .

The various sums of squares appearing in (3.5.2) can be described as follows:  $\sum y_i^2 = \sum (Y_i - \bar{Y})^2 =$  total variation of the actual  $Y$  values about their sample mean, which may be called the **total sum of squares (TSS)**.  $\sum \hat{y}_i^2 = \sum (\hat{Y}_i - \bar{Y})^2 = \sum (\hat{Y}_i - \bar{Y})^2 = \hat{\beta}_2^2 \sum x_i^2 =$  variation of the estimated  $Y$  values about their mean ( $\bar{Y} = \bar{Y}$ ), which appropriately may be called the sum of squares due to regression [i.e., due to the explanatory variable(s)], or explained by regression, or simply the **explained sum of squares (ESS)**.  $\sum \hat{u}_i^2 =$  residual or **unexplained** variation of the  $Y$  values about the regression line, or simply the **residual sum of squares (RSS)**. Thus, (3.5.2) is

$$\text{TSS} = \text{ESS} + \text{RSS} \tag{3.5.3}$$

and shows that the total variation in the observed  $Y$  values about their mean value can be partitioned into two parts, one attributable to the regression line and the other to random forces because not all actual  $Y$  observations lie on the fitted line. Geometrically, we have Figure 3.10.

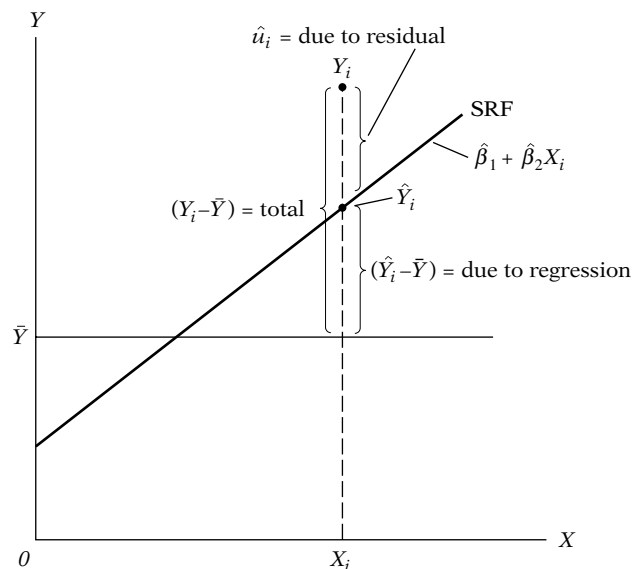


FIGURE 3.10 Breakdown of the variation of  $Y_i$  into two components.

Now dividing (3.5.3) by TSS on both sides, we obtain

$$\begin{aligned} 1 &= \frac{\text{ESS}}{\text{TSS}} + \frac{\text{RSS}}{\text{TSS}} \\ &= \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} + \frac{\sum \hat{u}_i^2}{\sum(Y_i - \bar{Y})^2} \end{aligned} \quad (3.5.4)$$

We now define  $r^2$  as

$$r^2 = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} = \frac{\text{ESS}}{\text{TSS}} \quad (3.5.5)$$

or, alternatively, as

$$\begin{aligned} r^2 &= 1 - \frac{\sum \hat{u}_i^2}{\sum(Y_i - \bar{Y})^2} \\ &= 1 - \frac{\text{RSS}}{\text{TSS}} \end{aligned} \quad (3.5.5a)$$

The quantity  $r^2$  thus defined is known as the (sample) **coefficient of determination** and is the most commonly used measure of the goodness of fit of a regression line. Verbally,  $r^2$  measures the proportion or percentage of the total variation in  $Y$  explained by the regression model.

Two properties of  $r^2$  may be noted:

1. It is a nonnegative quantity. (Why?)
2. Its limits are  $0 \leq r^2 \leq 1$ . An  $r^2$  of 1 means a perfect fit, that is,  $\hat{Y}_i = Y_i$  for each  $i$ . On the other hand, an  $r^2$  of zero means that there is no relationship between the regressand and the regressor whatsoever (i.e.,  $\hat{\beta}_2 = 0$ ). In this case, as (3.1.9) shows,  $\hat{Y}_i = \hat{\beta}_1 = \bar{Y}$ , that is, the best prediction of any  $Y$  value is simply its mean value. In this situation therefore the regression line will be horizontal to the  $X$  axis.

Although  $r^2$  can be computed directly from its definition given in (3.5.5), it can be obtained more quickly from the following formula:

$$\begin{aligned} r^2 &= \frac{\text{ESS}}{\text{TSS}} \\ &= \frac{\sum \hat{y}_i^2}{\sum y_i^2} \\ &= \frac{\hat{\beta}_2^2 \sum x_i^2}{\sum y_i^2} \\ &= \hat{\beta}_2^2 \left( \frac{\sum x_i^2}{\sum y_i^2} \right) \end{aligned} \quad (3.5.6)$$

If we divide the numerator and the denominator of (3.5.6) by the sample size  $n$  (or  $n - 1$  if the sample size is small), we obtain

$$r^2 = \hat{\beta}_2^2 \left( \frac{S_x^2}{S_y^2} \right) \quad (3.5.7)$$

where  $S_y^2$  and  $S_x^2$  are the sample variances of  $Y$  and  $X$ , respectively. Since  $\hat{\beta}_2 = \sum x_i y_i / \sum x_i^2$ , Eq. (3.5.6) can also be expressed as

$$r^2 = \frac{(\sum x_i y_i)^2}{\sum x_i^2 \sum y_i^2} \quad (3.5.8)$$

an expression that may be computationally easy to obtain.

Given the definition of  $r^2$ , we can express ESS and RSS discussed earlier as follows:

$$\begin{aligned} \text{ESS} &= r^2 \cdot \text{TSS} \\ &= r^2 \sum y_i^2 \end{aligned} \quad (3.5.9)$$

$$\begin{aligned} \text{RSS} &= \text{TSS} - \text{ESS} \\ &= \text{TSS}(1 - \text{ESS}/\text{TSS}) \\ &= \sum y_i^2 \cdot (1 - r^2) \end{aligned} \quad (3.5.10)$$

Therefore, we can write

$$\begin{aligned} \text{TSS} &= \text{ESS} + \text{RSS} \\ \sum y_i^2 &= r^2 \sum y_i^2 + (1 - r^2) \sum y_i^2 \end{aligned} \quad (3.5.11)$$

an expression that we will find very useful later.

A quantity closely related to but conceptually very much different from  $r^2$  is the **coefficient of correlation**, which, as noted in Chapter 1, is a measure of the degree of association between two variables. It can be computed either from

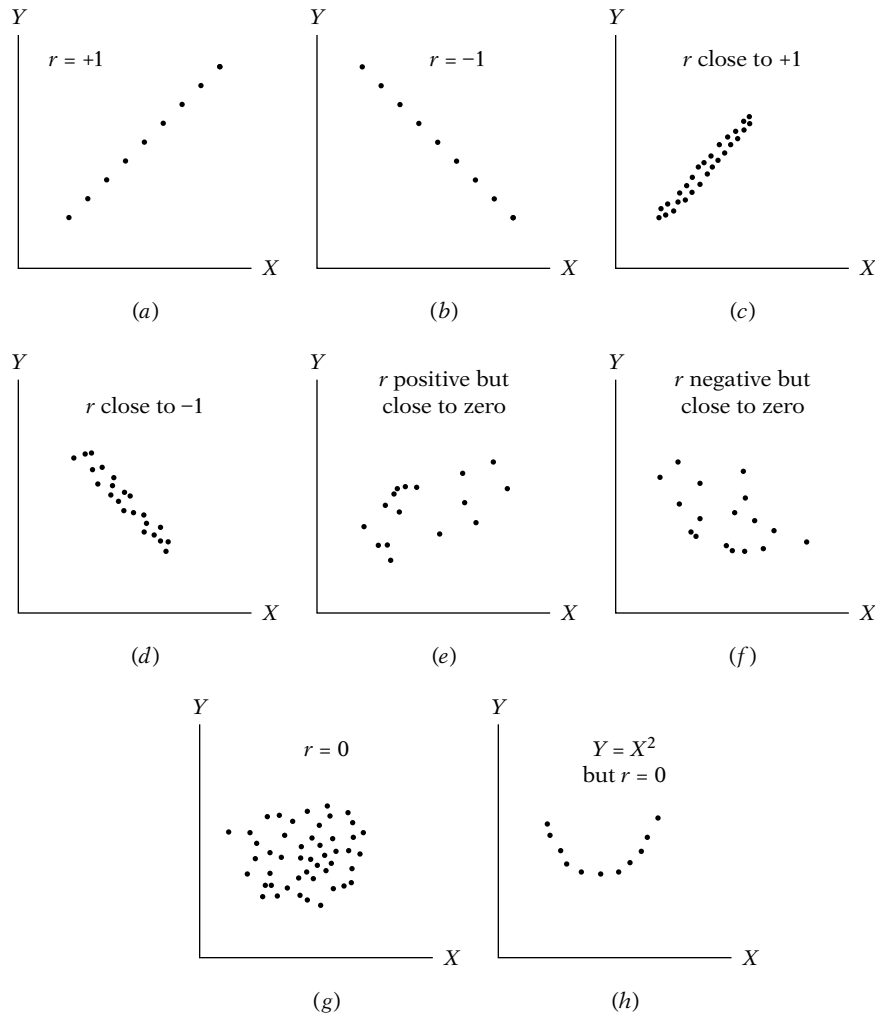
$$r = \pm \sqrt{r^2} \quad (3.5.12)$$

or from its definition

$$r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}} \quad (3.5.13)$$

which is known as the **sample correlation coefficient**.<sup>24</sup>

<sup>24</sup>The population correlation coefficient, denoted by  $\rho$ , is defined in **App. A**.



**FIGURE 3.11** Correlation patterns (adapted from Henri Theil, *Introduction to Econometrics*, Prentice-Hall, Englewood Cliffs, N.J., 1978, p. 86).

Some of the properties of  $r$  are as follows (see Figure 3.11):

1. It can be positive or negative, the sign depending on the sign of the term in the numerator of (3.5.13), which measures the sample *covariation* of two variables.
2. It lies between the limits of  $-1$  and  $+1$ ; that is,  $-1 \leq r \leq 1$ .
3. It is symmetrical in nature; that is, the coefficient of correlation between  $X$  and  $Y$  ( $r_{XY}$ ) is the same as that between  $Y$  and  $X$  ( $r_{YX}$ ).
4. It is independent of the origin and scale; that is, if we define  $X_i^* = aX_i + C$  and  $Y_i^* = bY_i + d$ , where  $a > 0$ ,  $b > 0$ , and  $c$  and  $d$  are constants,

then  $r$  between  $X^*$  and  $Y^*$  is the same as that between the original variables  $X$  and  $Y$ .

5. If  $X$  and  $Y$  are statistically independent (see **Appendix A** for the definition), the correlation coefficient between them is zero; but if  $r = 0$ , it does not mean that two variables are independent. In other words, **zero correlation does not necessarily imply independence**. [See Figure 3.11(*h*).]

6. It is a measure of *linear association* or *linear dependence* only; it has no meaning for describing nonlinear relations. Thus in Figure 3.11(*h*),  $Y = X^2$  is an exact relationship yet  $r$  is zero. (Why?)

7. Although it is a measure of linear association between two variables, it does not necessarily imply any cause-and-effect relationship, as noted in Chapter 1.

In the regression context,  $r^2$  is a more meaningful measure than  $r$ , for the former tells us the proportion of variation in the dependent variable explained by the explanatory variable(s) and therefore provides an overall measure of the extent to which the variation in one variable determines the variation in the other. The latter does not have such value.<sup>25</sup> Moreover, as we shall see, the interpretation of  $r$  ( $= R$ ) in a multiple regression model is of dubious value. However, we will have more to say about  $r^2$  in Chapter 7.

In passing, note that the  $r^2$  defined previously *can also be computed as the squared coefficient of correlation between actual  $Y_i$  and the estimated  $\hat{Y}_i$* , namely,  $\hat{Y}_i$ . That is, using (3.5.13), we can write

$$r^2 = \frac{[\sum(Y_i - \bar{Y})(\hat{Y}_i - \bar{Y})]^2}{\sum(Y_i - \bar{Y})^2 \sum(\hat{Y}_i - \bar{Y})^2}$$

That is,

$$r^2 = \frac{(\sum y_i \hat{y}_i)^2}{(\sum y_i^2)(\sum \hat{y}_i^2)} \quad (3.5.14)$$

where  $Y_i$  = actual  $Y$ ,  $\hat{Y}_i$  = estimated  $Y$ , and  $\bar{Y} = \bar{\hat{Y}}$  = the mean of  $Y$ . For proof, see exercise 3.15. Expression (3.5.14) justifies the description of  $r^2$  as a measure of goodness of fit, for it tells how close the estimated  $Y$  values are to their actual values.

### 3.6 A NUMERICAL EXAMPLE

We illustrate the econometric theory developed so far by considering the Keynesian consumption function discussed in the Introduction. Recall that Keynes stated that “The fundamental psychological law . . . is that men

<sup>25</sup>In regression modeling the underlying theory will indicate the direction of causality between  $Y$  and  $X$ , which, in the context of single-equation models, is generally from  $X$  to  $Y$ .



**TABLE 3.2** HYPOTHETICAL DATA ON  
WEEKLY FAMILY CONSUMPTION  
EXPENDITURE  $Y$  AND  
WEEKLY FAMILY INCOME  $X$

$Y, \$$	$X, \$$
70	80
65	100
90	120
95	140
110	160
115	180
120	200
140	220
155	240
150	260

[women] are disposed, as a rule and on average, to increase their consumption as their income increases, but not by as much as the increase in their income," that is, the marginal propensity to consume (MPC) is greater than zero but less than one. Although Keynes did not specify the exact functional form of the relationship between consumption and income, for simplicity assume that the relationship is linear as in (2.4.2). As a test of the Keynesian consumption function, we use the sample data of Table 2.4, which for convenience is reproduced as Table 3.2. The raw data required to obtain the estimates of the regression coefficients, their standard errors, etc., are given in Table 3.3. From these raw data, the following calculations are obtained, and the reader is advised to check them.

$$\begin{aligned}
 \hat{\beta}_1 &= 24.4545 & \text{var}(\hat{\beta}_1) &= 41.1370 & \text{and} & \text{se}(\hat{\beta}_1) &= 6.4138 \\
 \hat{\beta}_2 &= 0.5091 & \text{var}(\hat{\beta}_2) &= 0.0013 & \text{and} & \text{se}(\hat{\beta}_2) &= 0.0357 \\
 \text{cov}(\hat{\beta}_1, \hat{\beta}_2) &= -0.2172 & \hat{\sigma}^2 &= 42.1591 & & & \\
 r^2 &= 0.9621 & r &= 0.9809 & \text{df} &= 8 & 
 \end{aligned} \tag{3.6.1}$$

The estimated regression line therefore is

$$\hat{Y}_i = 24.4545 + 0.5091X_i \tag{3.6.2}$$

which is shown geometrically as Figure 3.12.

Following Chapter 2, the SRF [Eq. (3.6.2)] and the associated regression line are interpreted as follows: Each point on the regression line gives an *estimate* of the expected or mean value of  $Y$  corresponding to the chosen  $X$  value; that is,  $\hat{Y}_i$  is an estimate of  $E(Y | X_i)$ . The value of  $\hat{\beta}_2 = 0.5091$ , which measures the slope of the line, shows that, within the sample range of  $X$  between \$80 and \$260 per week, as  $X$  increases, say, by \$1, the estimated increase in the mean or average weekly consumption expenditure amounts to about 51 cents. The value of  $\hat{\beta}_1 = 24.4545$ , which is the intercept of the

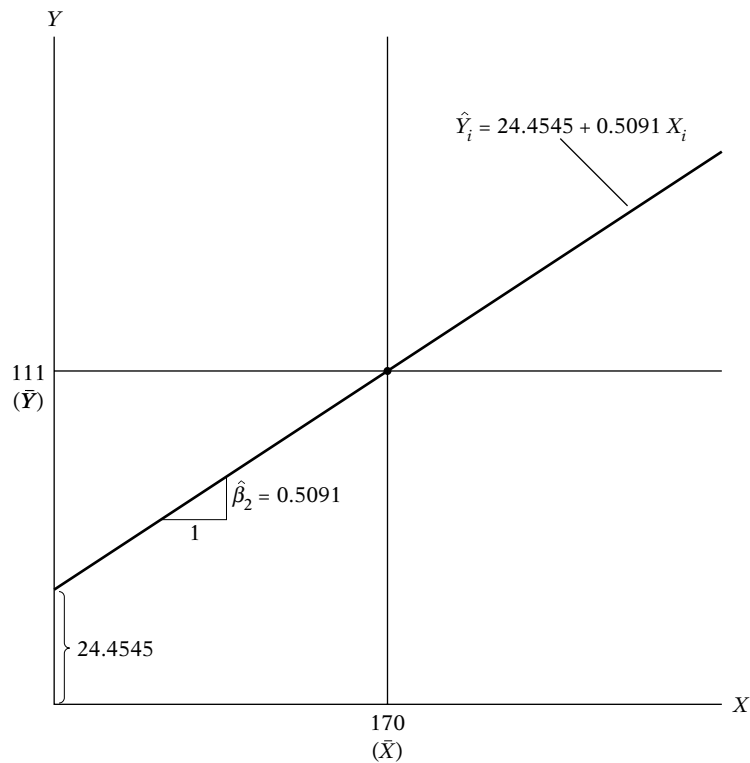
**TABLE 3.3** RAW DATA BASED ON TABLE 3.2

$Y_i$ (1)	$X_i$ (2)	$Y_i X_i$ (3)	$X_i^2$ (4)	$X_i = X_i - \bar{X}$ (5)	$Y_i = Y_i - \bar{Y}$ (6)	$x_i^2$ (7)	$x_i y_i$ (8)	$\hat{Y}_i$ (9)	$\hat{u}_i = Y_i - \hat{Y}_i$ (10)	$\hat{Y}_i \hat{u}_i$ (11)
70	80	5600	6400	-90	-41	8100	3690	65.1818	4.8181	314.0524
65	100	6500	10000	-70	-46	4900	3220	75.3636	-10.3636	-781.0382
90	120	10800	14400	-50	-21	2500	1050	85.5454	4.4545	381.0620
95	140	13300	19600	-30	-16	900	480	95.7272	-0.7272	-69.6128
110	160	17600	25600	-10	-1	100	10	105.9090	4.0909	433.2631
115	180	20700	32400	10	4	100	40	116.0909	-1.0909	-126.6434
120	200	24000	40000	30	9	900	270	125.2727	-6.2727	-792.0708
140	220	30800	48400	50	29	2500	1450	136.4545	3.5454	483.7858
155	240	37200	57600	70	44	4900	3080	145.6363	8.3636	1226.4073
150	260	39000	67600	90	39	8100	3510	156.8181	-6.8181	-1069.2014
Sum 1110	1700	205500	322000	0	0	33000	16800	1109.9995 $\approx 1110.0$	0	0.0040 $\approx 0.0$
Mean 111	170	nc	nc	0	0	nc	nc	110	0	0

$$\hat{\beta}_2 = \frac{\sum X_i Y_i}{\sum X_i^2} = \frac{16,800}{33,000} = 0.5091$$

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = 111 - 0.5091(170) = 24.4545$$

Notes:  $\approx$  symbolizes "approximately equal to"; nc means "not computed."



**FIGURE 3.12** Sample regression line based on the data of Table 3.2.

line, indicates the average level of weekly consumption expenditure when weekly income is zero. However, this is a mechanical interpretation of the intercept term. In regression analysis such literal interpretation of the intercept term may not be always meaningful, although in the present example it can be argued that a family without any income (because of unemployment, layoff, etc.) might maintain some minimum level of consumption expenditure either by borrowing or dissaving. But in general one has to use common sense in interpreting the intercept term, for very often the sample range of  $X$  values may not include zero as one of the observed values.

Perhaps it is best to interpret the intercept term as the mean or average effect on  $Y$  of all the variables omitted from the regression model. The value of  $r^2$  of 0.9621 means that about 96 percent of the variation in the weekly consumption expenditure is explained by income. Since  $r^2$  can at most be 1, the observed  $r^2$  suggests that the sample regression line fits the data very well.<sup>26</sup> The coefficient of correlation of 0.9809 shows that the two variables, consumption expenditure and income, are highly positively correlated. The estimated standard errors of the regression coefficients will be interpreted in Chapter 5.

### 3.7 ILLUSTRATIVE EXAMPLES

#### EXAMPLE 3.1

##### CONSUMPTION–INCOME RELATIONSHIP IN THE UNITED STATES, 1982–1996

Let us return to the consumption income data given in Table I.1 of the Introduction. We have already shown the data in Figure I.3 along with the estimated regression line (I.3.3). Now we provide the underlying OLS regression results. (The results were obtained from the statistical package *Eviews 3*.) *Note:*  $Y$  = personal consumption expenditure (PCE) and  $X$  = gross domestic product (GDP), all measured in 1992 billions of dollars. In this example, our data are *time series* data.

$$\hat{Y}_i = -184.0780 + 0.7064X_i \quad (3.7.1)$$

$$\text{var}(\hat{\beta}_1) = 2140.1707 \quad \text{se}(\hat{\beta}_1) = 46.2619$$

$$\text{var}(\hat{\beta}_2) = 0.000061 \quad \text{se}(\hat{\beta}_2) = 0.007827$$

$$r^2 = 0.998406 \quad \hat{\sigma}^2 = 411.4913$$

Equation (3.7.1) is the aggregate (i.e., for the economy as a whole) Keynesian consumption function. As this equation shows, the **marginal propensity to consume (MPC)** is about 0.71, suggesting that if income goes up by a dollar, the average personal consumption expenditure

(PCE) goes up by about 71 cents. From Keynesian theory, the MPC is less than 1. The intercept value of about  $-184$  tells us that if income were zero, the PCE would be about  $-184$  billion dollars. Of course, such a mechanical interpretation of the intercept term does not make economic sense in the present instance because the zero income value is out of the range of values we are working with and does not represent a likely outcome (see Table I.1). As we will see on many an occasion, very often the intercept term may not make much economic sense. Therefore, in practice the intercept term may not be very meaningful, although on occasions it can be very meaningful, as we will see in some illustrative examples. The more meaningful value is the slope coefficient, MPC in the present case.

The  $r^2$  value of 0.9984 means approximately 99 percent of the variation in the PCE is explained by variation in the GDP. Since  $r^2$  at most can be 1, we can say that the regression line in (3.7.1), which is shown in Figure I.3, fits our data extremely well; as you can see from that figure the actual data points are very tightly clustered around the estimated regression line. As we will see throughout this book, in regressions involving time series data one generally obtains high  $r^2$  values. In the chapter on autocorrelation, we will see the reasons behind this phenomenon.

<sup>26</sup>A formal test of the significance of  $r^2$  will be presented in Chap. 8.

**EXAMPLE 3.2****FOOD EXPENDITURE IN INDIA**

Refer to the data given in Table 2.8 of exercise 2.15. The data relate to a sample of 55 rural households in India. The regressand in this example is expenditure on food and the regressor is total expenditure, a proxy for income, both figures in rupees. The data in this example are thus *cross-sectional* data.

On the basis of the given data, we obtained the following regression:

$$\widehat{\text{FoodExp}}_i = 94.2087 + 0.4368 \text{ TotalExp}_i \quad (3.7.2)$$

$$\text{var}(\hat{\beta}_1) = 2560.9401 \quad \text{se}(\hat{\beta}_1) = 50.8563$$

$$\text{var}(\hat{\beta}_2) = 0.0061 \quad \text{se}(\hat{\beta}_2) = 0.0783$$

$$r^2 = 0.3698 \quad \hat{\sigma}^2 = 4469.6913$$

From (3.7.2) we see that if total expenditure increases by 1 rupee, on average, expenditure on food goes up by about 44 paise (1 rupee = 100 paise). If total expenditure were zero, the average expenditure on food would be about 94 rupees. Again, such a mechanical interpretation of the intercept may not be meaningful. However, in this example one could argue that even if total expenditure is zero (e.g., because of loss of a job), people may still maintain some minimum level of food expenditure by borrowing money or by dissaving.

The  $r^2$  value of about 0.37 means that only 37 percent of the variation in food expenditure is explained by the total expenditure. This might seem a rather low value, but as we will see throughout this text, in cross-sectional data, typically one obtains low  $r^2$  values, possibly because of the diversity of the units in the sample. We will discuss this topic further in the chapter on heteroscedasticity (see Chapter 11).

**EXAMPLE 3.3****THE RELATIONSHIP BETWEEN EARNINGS AND EDUCATION**

In Table 2.6 we looked at the data relating average hourly earnings and education, as measured by years of schooling. Using that data, if we regress<sup>27</sup> average hourly earnings ( $Y$ ) on education ( $X$ ), we obtain the following results.

$$\hat{Y}_i = -0.0144 + 0.7241 X_i \quad (3.7.3)$$

$$\text{var}(\hat{\beta}_1) = 0.7649 \quad \text{se}(\hat{\beta}_1) = 0.8746$$

$$\text{var}(\hat{\beta}_2) = 0.00483 \quad \text{se}(\hat{\beta}_2) = 0.0695$$

$$r^2 = 0.9077 \quad \hat{\sigma}^2 = 0.8816$$

As the regression results show, there is a positive association between education and earnings, an unsurprising finding. For every additional year of schooling, the average hourly earnings go up by about 72 cents an hour. The intercept term is positive but it may have no economic meaning. The  $r^2$  value suggests that about 89 percent of the variation in average hourly earnings is explained by education. For cross-sectional data, such a high  $r^2$  is rather unusual.

**3.8 A NOTE ON MONTE CARLO EXPERIMENTS**

In this chapter we showed that under the assumptions of CLRM the least-squares estimators have certain desirable statistical features summarized in the BLUE property. In the appendix to this chapter we prove this property

<sup>27</sup>Every field of study has its jargon. The expression “regress  $Y$  on  $X$ ” simply means treat  $Y$  as the regressand and  $X$  as the regressor.

more formally. But in practice how does one know that the BLUE property holds? For example, how does one find out if the OLS estimators are unbiased? The answer is provided by the so-called **Monte Carlo** experiments, which are essentially computer simulation, or sampling, experiments.

To introduce the basic ideas, consider our two-variable PRF:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (3.8.1)$$

A Monte Carlo experiment proceeds as follows:

1. Suppose the true values of the parameters are as follows:  $\beta_1 = 20$  and  $\beta_2 = 0.6$ .
2. You choose the sample size, say  $n = 25$ .
3. You fix the values of  $X$  for each observation. In all you will have 25  $X$  values.
4. Suppose you go to a random number table, choose 25 values, and call them  $u_i$  (these days most statistical packages have built-in random number generators).<sup>28</sup>
5. Since you *know*  $\beta_1$ ,  $\beta_2$ ,  $X_i$ , and  $u_i$ , using (3.8.1) you obtain 25  $Y_i$  values.
6. Now using the 25  $Y_i$  values thus generated, you regress these on the 25  $X$  values chosen in step 3, obtaining  $\hat{\beta}_1$  and  $\hat{\beta}_2$ , the least-squares estimators.
7. Suppose you repeat this experiment 99 times, each time using the same  $\beta_1$ ,  $\beta_2$ , and  $X$  values. Of course, the  $u_i$  values will vary from experiment to experiment. Therefore, in all you have 100 experiments, thus generating 100 values each of  $\beta_1$  and  $\beta_2$ . (In practice, many such experiments are conducted, sometimes 1000 to 2000.)
8. You take the averages of these 100 estimates and call them  $\bar{\hat{\beta}}_1$  and  $\bar{\hat{\beta}}_2$ .
9. If these average values are about the same as the true values of  $\beta_1$  and  $\beta_2$  assumed in step 1, this Monte Carlo experiment “establishes” that the least-squares estimators are indeed unbiased. Recall that under CLRM  $E(\hat{\beta}_1) = \beta_1$  and  $E(\hat{\beta}_2) = \beta_2$ .

These steps characterize the general nature of the Monte Carlo experiments. Such experiments are often used to study the statistical properties of various methods of estimating population parameters. They are particularly useful to study the behavior of estimators in small, or finite, samples. These experiments are also an excellent means of driving home the concept of **repeated sampling** that is the basis of most of classical statistical inference, as we shall see in Chapter 5. We shall provide several examples of Monte Carlo experiments by way of exercises for classroom assignment. (See exercise 3.27.)

<sup>28</sup>In practice it is assumed that  $u_i$  follows a certain probability distribution, say, normal, with certain parameters (e.g., the mean and variance). Once the values of the parameters are specified, one can easily generate the  $u_i$  using statistical packages.

### 3.9 SUMMARY AND CONCLUSIONS

The important topics and concepts developed in this chapter can be summarized as follows.

1. The basic framework of regression analysis is the **CLRM**.
2. The CLRM is based on a set of assumptions.
3. Based on these assumptions, the least-squares estimators take on certain properties summarized in the Gauss–Markov theorem, which states that in the class of linear unbiased estimators, the least-squares estimators have minimum variance. In short, they are BLUE.
4. The *precision* of OLS estimators is measured by their **standard errors**. In Chapters 4 and 5 we shall see how the standard errors enable one to draw inferences on the population parameters, the  $\beta$  coefficients.
5. The overall goodness of fit of the regression model is measured by the **coefficient of determination**,  $r^2$ . It tells what proportion of the variation in the dependent variable, or regressand, is explained by the explanatory variable, or regressor. This  $r^2$  lies between 0 and 1; the closer it is to 1, the better is the fit.
6. A concept related to the coefficient of determination is the **coefficient of correlation**,  $r$ . It is a measure of *linear association* between two variables and it lies between  $-1$  and  $+1$ .
7. The CLRM is a theoretical construct or abstraction because it is based on a set of assumptions that may be stringent or “unrealistic.” But such abstraction is often necessary in the initial stages of studying any field of knowledge. Once the CLRM is mastered, one can find out what happens if one or more of its assumptions are not satisfied. The first part of this book is devoted to studying the CLRM. The other parts of the book consider the refinements of the CLRM. Table 3.4 gives the road map ahead.

**TABLE 3.4** WHAT HAPPENS IF THE ASSUMPTIONS OF CLRM ARE VIOLATED?

Assumption number	Type of violation	Where to study?
1	Nonlinearity in parameters	Chapter 14
2	Stochastic regressor(s)	Introduction to Part II
3	Nonzero mean of $u_i$	Introduction to Part II
4	Heteroscedasticity	Chapter 11
5	Autocorrelated disturbances	Chapter 12
6	Nonzero covariance between disturbances and regressor	Introduction to Part II and Part IV
7	Sample observations less than the number of regressors	Chapter 10
8	Insufficient variability in regressors	Chapter 10
9	Specification bias	Chapters 13, 14
10	Multicollinearity	Chapter 10
11*	Nonnormality of disturbances	Introduction to Part II

\*Note: The assumption that the disturbances  $u_i$  are normally distributed is not a part of the CLRM. But more on this in Chapter 4.

## EXERCISES

### Questions

- 3.1. Given the assumptions in column 1 of the table, show that the assumptions in column 2 are equivalent to them.

ASSUMPTIONS OF THE CLASSICAL MODEL

(1)	(2)
$E(u_i   X_i) = 0$	$E(Y_i   X_i) = \beta_2 + \beta_2 X_i$
$\text{cov}(u_i, u_j) = 0 \quad i \neq j$	$\text{cov}(Y_i, Y_j) = 0 \quad i \neq j$
$\text{var}(u_i   X_i) = \sigma^2$	$\text{var}(Y_i   X_i) = \sigma^2$

- 3.2. Show that the estimates  $\hat{\beta}_1 = 1.572$  and  $\hat{\beta}_2 = 1.357$  used in the first experiment of Table 3.1 are in fact the OLS estimators.
- 3.3. According to Malinvaud (see footnote 10), the assumption that  $E(u_i | X_i) = 0$  is quite important. To see this, consider the PRF:  $Y = \beta_1 + \beta_2 X_i + u_i$ . Now consider two situations: (i)  $\beta_1 = 0$ ,  $\beta_2 = 1$ , and  $E(u_i) = 0$ ; and (ii)  $\beta_1 = 1$ ,  $\beta_2 = 0$ , and  $E(u_i) = (X_i - 1)$ . Now take the expectation of the PRF conditional upon  $X$  in the two preceding cases and see if you agree with Malinvaud about the significance of the assumption  $E(u_i | X_i) = 0$ .
- 3.4. Consider the sample regression

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i$$

Imposing the restrictions (i)  $\sum \hat{u}_i = 0$  and (ii)  $\sum \hat{u}_i X_i = 0$ , obtain the estimators  $\hat{\beta}_1$  and  $\hat{\beta}_2$  and show that they are identical with the least-squares estimators given in (3.1.6) and (3.1.7). This method of obtaining estimators is called the **analogy principle**. Give an intuitive justification for imposing restrictions (i) and (ii). (*Hint*: Recall the CLRM assumptions about  $u_i$ .) In passing, note that the analogy principle of estimating unknown parameters is also known as the **method of moments** in which sample moments (e.g., sample mean) are used to estimate population moments (e.g., the population mean). As noted in **Appendix A**, a **moment** is a summary statistic of a probability distribution, such as the expected value and variance.

- 3.5. Show that  $r^2$  defined in (3.5.5) ranges between 0 and 1. You may use the Cauchy-Schwarz inequality, which states that for any random variables  $X$  and  $Y$  the following relationship holds true:

$$[E(XY)]^2 \leq E(X^2)E(Y^2)$$

- 3.6. Let  $\hat{\beta}_{YX}$  and  $\hat{\beta}_{XY}$  represent the slopes in the regression of  $Y$  on  $X$  and  $X$  on  $Y$ , respectively. Show that

$$\hat{\beta}_{YX} \hat{\beta}_{XY} = r^2$$

where  $r$  is the coefficient of correlation between  $X$  and  $Y$ .

- 3.7. Suppose in exercise 3.6 that  $\hat{\beta}_{YX} \hat{\beta}_{XY} = 1$ . Does it matter then if we regress  $Y$  on  $X$  or  $X$  on  $Y$ ? Explain carefully.

3.8. Spearman's rank correlation coefficient  $r_s$  is defined as follows:

$$r_s = 1 - \frac{6 \sum d^2}{n(n^2 - 1)}$$

where  $d$  = difference in the ranks assigned to the same individual or phenomenon and  $n$  = number of individuals or phenomena ranked. Derive  $r_s$  from  $r$  defined in (3.5.13). *Hint:* Rank the  $X$  and  $Y$  values from 1 to  $n$ . Note that the sum of  $X$  and  $Y$  ranks is  $n(n + 1)/2$  each and therefore their means are  $(n + 1)/2$ .

3.9. Consider the following formulations of the two-variable PRF:

$$\text{Model I: } Y_i = \beta_1 + \beta_2 X_i + u_i$$

$$\text{Model II: } Y_i = \alpha_1 + \alpha_2(X_i - \bar{X}) + u_i$$

- Find the estimators of  $\beta_1$  and  $\alpha_1$ . Are they identical? Are their variances identical?
- Find the estimators of  $\beta_2$  and  $\alpha_2$ . Are they identical? Are their variances identical?
- What is the advantage, if any, of model II over model I?

3.10. Suppose you run the following regression:

$$y_i = \hat{\beta}_1 + \hat{\beta}_2 x_i + \hat{u}_i$$

where, as usual,  $y_i$  and  $x_i$  are deviations from their respective mean values. What will be the value of  $\hat{\beta}_1$ ? Why? Will  $\hat{\beta}_2$  be the same as that obtained from Eq. (3.1.6)? Why?

3.11. Let  $r_1$  = coefficient of correlation between  $n$  pairs of values  $(Y_i, X_i)$  and  $r_2$  = coefficient of correlation between  $n$  pairs of values  $(aX_i + b, cY_i + d)$ , where  $a, b, c$ , and  $d$  are constants. Show that  $r_1 = r_2$  and hence *establish the principle that the coefficient of correlation is invariant with respect to the change of scale and the change of origin.*

*Hint:* Apply the definition of  $r$  given in (3.5.13).

*Note:* The operations  $aX_i, X_i + b$ , and  $aX_i + b$  are known, respectively, as the *change of scale, change of origin, and change of both scale and origin.*

3.12. If  $r$ , the coefficient of correlation between  $n$  pairs of values  $(X_i, Y_i)$ , is positive, then determine whether each of the following statements is true or false:

- $r$  between  $(-X_i, -Y_i)$  is also positive.
- $r$  between  $(-X_i, Y_i)$  and that between  $(X_i, -Y_i)$  can be either positive or negative.
- Both the slope coefficients  $\beta_{yx}$  and  $\beta_{xy}$  are positive, where  $\beta_{yx}$  = slope coefficient in the regression of  $Y$  on  $X$  and  $\beta_{xy}$  = slope coefficient in the regression of  $X$  on  $Y$ .

3.13. If  $X_1, X_2$ , and  $X_3$  are uncorrelated variables each having the same standard deviation, show that the coefficient of correlation between  $X_1 + X_2$  and  $X_2 + X_3$  is equal to  $\frac{1}{2}$ . Why is the correlation coefficient not zero?

3.14. In the regression  $Y_i = \beta_1 + \beta_2 X_i + u_i$  suppose we *multiply* each  $X$  value by a constant, say, 2. Will it change the residuals and fitted values of  $Y$ ? Explain. What if we *add* a constant value, say, 2, to each  $X$  value?



- 3.15.** Show that (3.5.14) in fact measures the coefficient of determination.  
*Hint:* Apply the definition of  $r$  given in (3.5.13) and recall that  $\sum y_i \hat{y}_i = \sum (\hat{y}_i + \hat{u}_i) \hat{y}_i = \sum \hat{y}_i^2$ , and remember (3.5.6).
- 3.16.** Explain *with reason* whether the following statements are true, false, or uncertain:
- Since the correlation between two variables,  $Y$  and  $X$ , can range from  $-1$  to  $+1$ , this also means that  $\text{cov}(Y, X)$  also lies between these limits.
  - If the correlation between two variables is zero, it means that there is no relationship between the two variables whatsoever.
  - If you regress  $Y_i$  on  $\hat{Y}_i$  (i.e., actual  $Y$  on estimated  $Y$ ), the intercept and slope values will be 0 and 1, respectively.
- 3.17.** *Regression without any regressor.* Suppose you are given the model:  $Y_i = \beta_1 + u_i$ . Use OLS to find the estimator of  $\beta_1$ . What is its variance and the RSS? Does the estimated  $\beta_1$  make intuitive sense? Now consider the two-variable model  $Y_i = \beta_1 + \beta_2 X_i + u_i$ . Is it worth adding  $X_i$  to the model? If not, why bother with regression analysis?

**Problems**

- 3.18.** In Table 3.5, you are given the ranks of 10 students in midterm and final examinations in statistics. Compute Spearman's coefficient of rank correlation and interpret it.
- 3.19.** *The relationship between nominal exchange rate and relative prices.* From the annual observations from 1980 to 1994, the following regression results were obtained, where  $Y$  = exchange rate of the German mark to the U.S. dollar (GM/\$) and  $X$  = ratio of the U.S. consumer price index to the German consumer price index; that is,  $X$  represents the relative prices in the two countries:

$$\hat{Y}_t = 6.682 - 4.318X_t \quad r^2 = 0.528$$

$$\text{se} = (1.22)(1.333)$$

- Interpret this regression. How would you interpret  $r^2$ ?
  - Does the negative value of  $X_t$  make economic sense? What is the underlying economic theory?
  - Suppose we were to redefine  $X$  as the ratio of German CPI to the U.S. CPI. Would that change the sign of  $X$ ? And why?
- 3.20.** Table 3.6 gives data on indexes of output per hour ( $X$ ) and real compensation per hour ( $Y$ ) for the business and nonfarm business sectors of the U.S. economy for 1959–1997. The base year of the indexes is 1982 = 100 and the indexes are seasonally adjusted.

**TABLE 3.5**

Rank	Student									
	A	B	C	D	E	F	G	H	I	J
Midterm	1	3	7	10	9	5	4	8	2	6
Final	3	2	8	7	9	6	5	10	1	4

**TABLE 3.6** PRODUCTIVITY AND RELATED DATA, BUSINESS SECTOR, 1959–98  
[Index numbers, 1992 = 100; quarterly data seasonally adjusted]

Year or quarter	Output per hour of all persons <sup>1</sup>		Compensation per hour <sup>2</sup>	
	Business sector	Nonfarm business sector	Business sector	Nonfarm business sector
1959 .....	50.5	54.2	13.1	13.7
1960 .....	51.4	54.8	13.7	14.3
1961 .....	53.2	56.6	14.2	14.8
1962 .....	55.7	59.2	14.8	15.4
1963 .....	57.9	61.2	15.4	15.9
1964 .....	60.6	63.8	16.2	16.7
1965 .....	62.7	65.8	16.8	17.2
1966 .....	65.2	68.0	17.9	18.2
1967 .....	66.6	69.2	18.9	19.3
1968 .....	68.9	71.6	20.5	20.8
1969 .....	69.2	71.7	21.9	22.2
1970 .....	70.6	72.7	23.6	23.8
1971 .....	73.6	75.7	25.1	25.4
1972 .....	76.0	78.3	26.7	27.0
1973 .....	78.4	80.7	29.0	29.2
1974 .....	77.1	79.4	31.8	32.1
1975 .....	79.8	81.6	35.1	35.3
1976 .....	82.5	84.5	38.2	38.4
1977 .....	84.0	85.8	41.2	41.5
1978 .....	84.9	87.0	44.9	45.2
1979 .....	84.5	86.3	49.2	49.5
1980 .....	84.2	86.0	54.5	54.8
1981 .....	85.8	87.0	59.6	60.2
1982 .....	85.3	88.3	64.1	64.6
1983 .....	88.0	89.9	66.8	67.3
1984 .....	90.2	91.4	69.7	70.2
1985 .....	91.7	92.3	73.1	73.4
1986 .....	94.1	94.7	76.8	77.2
1987 .....	94.0	94.5	79.8	80.1
1988 .....	94.7	95.3	83.6	83.7
1989 .....	95.5	95.8	85.9	86.0
1990 .....	96.1	96.3	90.8	90.7
1991 .....	96.7	97.0	95.1	95.1
1992 .....	100.0	100.0	100.0	100.0
1993 .....	100.1	100.1	102.5	102.2
1994 .....	100.7	100.6	104.4	104.2
1995 .....	101.0	101.2	106.8	106.7
1996 .....	103.7	103.7	110.7	110.4
1997 .....	105.4	105.1	114.9	114.5

<sup>1</sup>Output refers to real gross domestic product in the sector.

<sup>2</sup>Wages and salaries of employees plus employers' contributions for social insurance and private benefit plans. Also includes an estimate of wages, salaries, and supplemental payments for the self-employed.

Source: *Economic Report of the President*, 1999, Table B-49, p. 384.

- a. Plot  $Y$  against  $X$  for the two sectors separately.
  - b. What is the economic theory behind the relationship between the two variables? Does the scattergram support the theory?
  - c. Estimate the OLS regression of  $Y$  on  $X$ . Save the results for a further look after we study Chapter 5.
- 3.21.** From a sample of 10 observations, the following results were obtained:

$$\sum Y_i = 1110 \quad \sum X_i = 1700 \quad \sum X_i Y_i = 205,500$$

$$\sum X_i^2 = 322,000 \quad \sum Y_i^2 = 132,100$$

with coefficient of correlation  $r = 0.9758$ . But on rechecking these calculations it was found that two pairs of observations were recorded:

Y	X		Y	X
90	120	instead of	80	110
140	220		150	210

What will be the effect of this error on  $r$ ? Obtain the correct  $r$ .

- 3.22.** Table 3.7 gives data on gold prices, the Consumer Price Index (CPI), and the New York Stock Exchange (NYSE) Index for the United States for the period 1977–1991. The NYSE Index includes most of the stocks listed on the NYSE, some 1500 plus.

**TABLE 3.7**

Year	Price of gold at New York, \$ per troy ounce	Consumer Price Index (CPI), 1982–84 = 100	New York Stock Exchange (NYSE) Index, Dec. 31, 1965 = 100
1977	147.98	60.6	53.69
1978	193.44	65.2	53.70
1979	307.62	72.6	58.32
1980	612.51	82.4	68.10
1981	459.61	90.9	74.02
1982	376.01	96.5	68.93
1983	423.83	99.6	92.63
1984	360.29	103.9	92.46
1985	317.30	107.6	108.90
1986	367.87	109.6	136.00
1987	446.50	113.6	161.70
1988	436.93	118.3	149.91
1989	381.28	124.0	180.02
1990	384.08	130.7	183.46
1991	362.04	136.2	206.33

*Source:* Data on CPI and NYSE Index are from the *Economic Report of the President*, January 1993, Tables B-59 and B-91, respectively. Data on gold prices are from U.S. Department of Commerce, Bureau of Economic Analysis, *Business Statistics, 1963–1991*, p. 68.

- a. Plot in the same scattergram gold prices, CPI, and the NYSE Index.
- b. An investment is supposed to be a hedge against inflation if its price and/or rate of return at least keeps pace with inflation. To test this hypothesis, suppose you decide to fit the following model, assuming the scatterplot in **a** suggests that this is appropriate:

$$\begin{aligned} \text{Gold price}_t &= \beta_1 + \beta_2 \text{CPI}_t + u_t \\ \text{NYSE index}_t &= \beta_1 + \beta_2 \text{CPI}_t + u_t \end{aligned}$$

- 3.23.** Table 3.8 gives data on gross domestic product (GDP) for the United States for the years 1959–1997.
- a. Plot the GDP data in current and constant (i.e., 1992) dollars against time.
  - b. Letting  $Y$  denote GDP and  $X$  time (measured chronologically starting with 1 for 1959, 2 for 1960, through 39 for 1997), see if the following model fits the GDP data:

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

Estimate this model for both current and constant-dollar GDP.

- c. How would you interpret  $\beta_2$ ?
- d. If there is a difference between  $\beta_2$  estimated for current-dollar GDP and that estimated for constant-dollar GDP, what explains the difference?

**TABLE 3.8** NOMINAL AND REAL GDP, UNITED STATES, 1959–1997

Year	NGDP	RGDP	Year	NGDP	RGDP
1959	507.2000	2210.200	1979	2557.500	4630.600
1960	526.6000	2262.900	1980	2784.200	4615.000
1961	544.8000	2314.300	1981	3115.900	4720.700
1962	585.2000	2454.800	1982	3242.100	4620.300
1963	617.4000	2559.400	1983	3514.500	4803.700
1964	663.0000	2708.400	1984	3902.400	5140.100
1965	719.1000	2881.100	1985	4180.700	5323.500
1966	787.7000	3069.200	1986	4422.200	5487.700
1967	833.6000	3147.200	1987	4692.300	5649.500
1968	910.6000	3293.900	1988	5049.600	5865.200
1969	982.2000	3393.600	1989	5438.700	6062.000
1970	1035.600	3397.600	1990	5743.800	6136.300
1971	1125.400	3510.000	1991	5916.700	6079.400
1972	1237.300	3702.300	1992	6244.400	6244.400
1973	1382.600	3916.300	1993	6558.100	6389.600
1974	1496.900	3891.200	1994	6947.000	6610.700
1975	1630.600	3873.900	1995	7269.600	6761.700
1976	1819.000	4082.900	1996	7661.600	6994.800
1977	2026.900	4273.600	1997	8110.900	7269.800
1978	2291.400	4503.000			

Note: NGDP = nominal GDP (current dollars in billions).  
 RGDP = real GDP (1992 billions of dollars).

Source: *Economic Report of the President, 1999*, Tables B-1 and B-2, pp. 326–328.

- e. From your results what can you say about the nature of inflation in the United States over the sample period?
- 3.24. Using the data given in Table I.1 of the Introduction, verify Eq. (3.7.1).
- 3.25. For the S.A.T. example given in exercise 2.16 do the following:
- Plot the female verbal score against the male verbal score.
  - If the scatterplot suggests that a linear relationship between the two seems appropriate, obtain the regression of female verbal score on male verbal score.
  - If there is a relationship between the two verbal scores, is the relationship *causal*?
- 3.26. Repeat exercise 3.24, replacing math scores for verbal scores.
- 3.27. Monte Carlo study *classroom assignment*: Refer to the 10  $X$  values given in Table 3.2. Let  $\beta_1 = 25$  and  $\beta_2 = 0.5$ . Assume  $u_i \approx N(0, 9)$ , that is,  $u_i$  are normally distributed with mean 0 and variance 9. Generate 100 samples using these values, obtaining 100 estimates of  $\beta_1$  and  $\beta_2$ . Graph these estimates. What conclusions can you draw from the Monte Carlo study? *Note*: Most statistical packages now can generate random variables from most well-known probability distributions. Ask your instructor for help, in case you have difficulty generating such variables.

## APPENDIX 3A

### 3A.1 DERIVATION OF LEAST-SQUARES ESTIMATES

Differentiating (3.1.2) partially with respect to  $\hat{\beta}_1$  and  $\hat{\beta}_2$ , we obtain

$$\frac{\partial(\sum \hat{u}_i^2)}{\partial \hat{\beta}_1} = -2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i) = -2 \sum \hat{u}_i \quad (1)$$

$$\frac{\partial(\sum \hat{u}_i^2)}{\partial \hat{\beta}_2} = -2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i) X_i = -2 \sum \hat{u}_i X_i \quad (2)$$

Setting these equations to zero, after algebraic simplification and manipulation, gives the estimators given in Eqs. (3.1.6) and (3.1.7).

### 3A.2 LINEARITY AND UNBIASEDNESS PROPERTIES OF LEAST-SQUARES ESTIMATORS

From (3.1.8) we have

$$\hat{\beta}_2 = \frac{\sum x_i Y_i}{\sum x_i^2} = \sum k_i Y_i \quad (3)$$

where

$$k_i = \frac{x_i}{(\sum x_i^2)}$$

which shows that  $\hat{\beta}_2$  is a **linear estimator** because it is a linear function of  $Y_i$ ; actually it is a weighted average of  $Y_i$  with  $k_i$  serving as the weights. It can similarly be shown that  $\hat{\beta}_1$  too is a linear estimator.

Incidentally, note these properties of the weights  $k_i$ :

1. Since the  $X_i$  are assumed to be nonstochastic, the  $k_i$  are nonstochastic too.
2.  $\sum k_i = 0$ .
3.  $\sum k_i^2 = 1/\sum x_i^2$ .
4.  $\sum k_i x_i = \sum k_i X_i = 1$ . These properties can be directly verified from the definition of  $k_i$ .

For example,

$$\begin{aligned}\sum k_i &= \sum \left( \frac{x_i}{\sum x_i^2} \right) = \frac{1}{\sum x_i^2} \sum x_i, & \text{since for a given sample } \sum x_i^2 \text{ is known} \\ &= 0, & \text{since } \sum x_i, \text{ the sum of deviations from} \\ & & \text{the mean value, is always zero}\end{aligned}$$

Now substitute the PRF  $Y_i = \beta_1 + \beta_2 X_i + u_i$  into (3) to obtain

$$\begin{aligned}\hat{\beta}_2 &= \sum k_i (\beta_1 + \beta_2 X_i + u_i) \\ &= \beta_1 \sum k_i + \beta_2 \sum k_i X_i + \sum k_i u_i \\ &= \beta_2 + \sum k_i u_i\end{aligned}\tag{4}$$

where use is made of the properties of  $k_i$  noted earlier.

Now taking expectation of (4) on both sides and noting that  $k_i$ , being nonstochastic, can be treated as constants, we obtain

$$\begin{aligned}E(\hat{\beta}_2) &= \beta_2 + \sum k_i E(u_i) \\ &= \beta_2\end{aligned}\tag{5}$$

since  $E(u_i) = 0$  by assumption. Therefore,  $\hat{\beta}_2$  is an unbiased estimator of  $\beta_2$ . Likewise, it can be proved that  $\hat{\beta}_1$  is also an unbiased estimator of  $\beta_1$ .

### 3A.3 VARIANCES AND STANDARD ERRORS OF LEAST-SQUARES ESTIMATORS

Now by the definition of variance, we can write

$$\begin{aligned}\text{var}(\hat{\beta}_2) &= E[\hat{\beta}_2 - E(\hat{\beta}_2)]^2 \\ &= E(\hat{\beta}_2 - \beta_2)^2 & \text{since } E(\hat{\beta}_2) = \beta_2 \\ &= E\left(\sum k_i u_i\right)^2 & \text{using Eq. (4) above} \\ &= E(k_1^2 u_1^2 + k_2^2 u_2^2 + \cdots + k_n^2 u_n^2 + 2k_1 k_2 u_1 u_2 + \cdots + 2k_{n-1} k_n u_{n-1} u_n)\end{aligned}\tag{6}$$

Since by assumption,  $E(u_i^2) = \sigma^2$  for each  $i$  and  $E(u_i u_j) = 0$ ,  $i \neq j$ , it follows that

$$\begin{aligned}\text{var}(\hat{\beta}_2) &= \sigma^2 \sum k_i^2 \\ &= \frac{\sigma^2}{\sum x_i^2} \quad (\text{using the definition of } k_i^2) \quad (7) \\ &= \text{Eq. (3.3.1)}\end{aligned}$$

The variance of  $\hat{\beta}_1$  can be obtained following the same line of reasoning already given. Once the variances of  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are obtained, their positive square roots give the corresponding standard errors.

### 3A.4 COVARIANCE BETWEEN $\hat{\beta}_1$ AND $\hat{\beta}_2$

By definition,

$$\begin{aligned}\text{cov}(\hat{\beta}_1, \hat{\beta}_2) &= E\{[\hat{\beta}_1 - E(\hat{\beta}_1)][\hat{\beta}_2 - E(\hat{\beta}_2)]\} \\ &= E(\hat{\beta}_1 - \beta_1)(\hat{\beta}_2 - \beta_2) \quad (\text{Why?}) \\ &= -\bar{X}E(\hat{\beta}_2 - \beta_2)^2 \quad (8) \\ &= -\bar{X} \text{var}(\hat{\beta}_2) \\ &= \text{Eq. (3.3.9)}\end{aligned}$$

where use is made of the fact that  $\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$  and  $E(\hat{\beta}_1) = \bar{Y} - \beta_2 \bar{X}$ , giving  $\hat{\beta}_1 - E(\hat{\beta}_1) = -\bar{X}(\hat{\beta}_2 - \beta_2)$ . *Note:*  $\text{var}(\hat{\beta}_2)$  is given in (3.3.1).

### 3A.5 THE LEAST-SQUARES ESTIMATOR OF $\sigma^2$

Recall that

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (9)$$

Therefore,

$$\bar{Y} = \beta_1 + \beta_2 \bar{X} + \bar{u} \quad (10)$$

Subtracting (10) from (9) gives

$$y_i = \beta_2 x_i + (u_i - \bar{u}) \quad (11)$$

Also recall that

$$\hat{u}_i = y_i - \hat{\beta}_2 x_i \quad (12)$$

Therefore, substituting (11) into (12) yields

$$\hat{u}_i = \beta_2 x_i + (u_i - \bar{u}) - \hat{\beta}_2 x_i \quad (13)$$

Collecting terms, squaring, and summing on both sides, we obtain

$$\sum \hat{u}_i^2 = (\hat{\beta}_2 - \beta_2)^2 \sum x_i^2 + \sum (u_i - \bar{u})^2 - 2(\hat{\beta}_2 - \beta_2) \sum x_i(u_i - \bar{u}) \quad (14)$$

Taking expectations on both sides gives

$$\begin{aligned} E\left(\sum \hat{u}_i^2\right) &= \sum x_i^2 E(\hat{\beta}_2 - \beta_2)^2 + E\left[\sum (u_i - \bar{u})^2\right] - 2E\left[(\hat{\beta}_2 - \beta_2) \sum x_i(u_i - \bar{u})\right] \\ &= \sum x_i^2 \text{var}(\hat{\beta}_2) + (n-1) \text{var}(u_i) - 2E\left[\sum k_i u_i(x_i u_i)\right] \\ &= \sigma^2 + (n-1)\sigma^2 - 2E\left[\sum k_i x_i u_i^2\right] \\ &= \sigma^2 + (n-1)\sigma^2 - 2\sigma^2 \\ &= (n-2)\sigma^2 \end{aligned} \quad (15)$$

where, in the last but one step, use is made of the definition of  $k_i$  given in Eq. (3) and the relation given in Eq. (4). Also note that

$$\begin{aligned} E\sum (u_i - \bar{u})^2 &= E\left[\sum u_i^2 - n\bar{u}^2\right] \\ &= E\left[\sum u_i^2 - n\left(\frac{\sum u_i}{n}\right)^2\right] \\ &= E\left[\sum u_i^2 - \frac{1}{n}\sum (u_i^2)\right] \\ &= n\sigma^2 - \frac{n}{n}\sigma^2 = (n-1)\sigma^2 \end{aligned}$$

where use is made of the fact that the  $u_i$  are uncorrelated and the variance of each  $u_i$  is  $\sigma^2$ .

Thus, we obtain

$$E\left(\sum \hat{u}_i^2\right) = (n-2)\sigma^2 \quad (16)$$

Therefore, if we define

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2} \quad (17)$$

its expected value is

$$E(\hat{\sigma}^2) = \frac{1}{n-2} E\left(\sum \hat{u}_i^2\right) = \sigma^2 \quad \text{using (16)} \quad (18)$$

which shows that  $\hat{\sigma}^2$  is an unbiased estimator of true  $\sigma^2$ .



### 3A.6 MINIMUM-VARIANCE PROPERTY OF LEAST-SQUARES ESTIMATORS

It was shown in Appendix 3A, Section 3A.2, that the least-squares estimator  $\hat{\beta}_2$  is linear as well as unbiased (this holds true of  $\hat{\beta}_1$  too). To show that these estimators are also minimum variance in the class of all linear unbiased estimators, consider the least-squares estimator  $\hat{\beta}_2$ :

$$\hat{\beta}_2 = \sum k_i Y_i$$

where

$$k_i = \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} = \frac{x_i}{\sum x_i^2} \quad (\text{see Appendix 3A.2}) \quad (19)$$

which shows that  $\hat{\beta}_2$  is a weighted average of the  $Y$ 's, with  $k_i$  serving as the weights.

Let us define an alternative linear estimator of  $\beta_2$  as follows:

$$\beta_2^* = \sum w_i Y_i \quad (20)$$

where  $w_i$  are also weights, not necessarily equal to  $k_i$ . Now

$$\begin{aligned} E(\beta_2^*) &= \sum w_i E(Y_i) \\ &= \sum w_i (\beta_1 + \beta_2 X_i) \\ &= \beta_1 \sum w_i + \beta_2 \sum w_i X_i \end{aligned} \quad (21)$$

Therefore, for  $\beta_2^*$  to be unbiased, we must have

$$\sum w_i = 0 \quad (22)$$

and

$$\sum w_i X_i = 1 \quad (23)$$

Also, we may write

$$\begin{aligned} \text{var}(\beta_2^*) &= \text{var} \sum w_i Y_i \\ &= \sum w_i^2 \text{var} Y_i \quad [\text{Note: } \text{var} Y_i = \text{var} u_i = \sigma^2] \\ &= \sigma^2 \sum w_i^2 \quad [\text{Note: } \text{cov}(Y_i, Y_j) = 0 (i \neq j)] \\ &= \sigma^2 \sum \left( w_i - \frac{x_i}{\sum x_i^2} + \frac{x_i}{\sum x_i^2} \right)^2 \quad (\text{Note the mathematical trick}) \\ &= \sigma^2 \sum \left( w_i - \frac{x_i}{\sum x_i^2} \right)^2 + \sigma^2 \frac{\sum x_i^2}{(\sum x_i^2)^2} + 2\sigma^2 \sum \left( w_i - \frac{x_i}{\sum x_i^2} \right) \left( \frac{x_i}{\sum x_i^2} \right) \\ &= \sigma^2 \sum \left( w_i - \frac{x_i}{\sum x_i^2} \right)^2 + \sigma^2 \left( \frac{1}{\sum x_i^2} \right) \end{aligned} \quad (24)$$

because the last term in the next to the last step drops out. (Why?)

Since the last term in (24) is constant, the variance of  $(\beta_2^*)$  can be minimized only by manipulating the first term. If we let

$$w_i = \frac{x_i}{\sum x_i^2}$$

Eq. (24) reduces to

$$\begin{aligned} \text{var}(\beta_2^*) &= \frac{\sigma^2}{\sum x_i^2} \\ &= \text{var}(\hat{\beta}_2) \end{aligned} \quad (25)$$

In words, with weights  $w_i = k_i$ , which are the least-squares weights, the variance of the linear estimator  $\beta_2^*$  is equal to the variance of the least-squares estimator  $\hat{\beta}_2$ ; otherwise  $\text{var}(\beta_2^*) > \text{var}(\hat{\beta}_2)$ . To put it differently, if there is a minimum-variance linear unbiased estimator of  $\beta_2$ , it must be the least-squares estimator. Similarly it can be shown that  $\hat{\beta}_1$  is a minimum-variance linear unbiased estimator of  $\beta_1$ .

### 3A.7 CONSISTENCY OF LEAST-SQUARES ESTIMATORS

We have shown that, in the framework of the classical linear regression model, the least-squares estimators are unbiased (and efficient) in any sample size, small or large. But sometimes, as discussed in **Appendix A**, an estimator may not satisfy one or more desirable statistical properties in small samples. But as the sample size increases indefinitely, the estimators possess several desirable statistical properties. These properties are known as the **large sample**, or **asymptotic, properties**. In this appendix, we will discuss one large sample property, namely, the property of **consistency**, which is discussed more fully in **Appendix A**. For the two-variable model we have already shown that the OLS estimator  $\hat{\beta}_2$  is an unbiased estimator of the true  $\beta_2$ . Now we show that  $\hat{\beta}_2$  is also a consistent estimator of  $\beta_2$ . As shown in **Appendix A**, a sufficient condition for consistency is that  $\hat{\beta}_2$  is unbiased and that its variance tends to zero as the sample size  $n$  tends to infinity.

Since we have already proved the unbiasedness property, we need only show that the variance of  $\hat{\beta}_2$  tends to zero as  $n$  increases indefinitely. We know that

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_i^2} = \frac{\sigma^2/n}{\sum x_i^2/n} \quad (26)$$

By dividing the numerator and denominator by  $n$ , we do not change the equality.

Now

$$\lim_{n \rightarrow \infty} \text{var}(\hat{\beta}_2) = \lim_{n \rightarrow \infty} \left( \frac{\sigma^2/n}{\sum x_i^2/n} \right) = 0 \quad (27)$$

where use is made of the facts that (1) the limit of a ratio quantity is the limit of the quantity in the numerator to the limit of the quantity in the denominator (refer to any calculus book); (2) as  $n$  tends to infinity,  $\sigma^2/n$  tends to zero because  $\sigma^2$  is a finite number; and  $[(\sum x_i^2)/n] \neq 0$  because the variance of  $X$  has a finite limit because of Assumption 8 of CLRM.

The upshot of the preceding discussion is that the OLS estimator  $\hat{\beta}_2$  is a consistent estimator of true  $\beta_2$ . In like fashion, we can establish that  $\hat{\beta}_1$  is also a consistent estimator. Thus, in repeated (small) samples, the OLS estimators are unbiased and as the sample size increases indefinitely the OLS estimators are consistent. As we shall see later, even if some of the assumptions of CLRM are not satisfied, we may be able to obtain consistent estimators of the regression coefficients in several situations.

# 4

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## CLASSICAL NORMAL LINEAR REGRESSION MODEL (CNLRM)

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What is known as the **classical theory of statistical inference** consists of two branches, namely, **estimation** and **hypothesis testing**. We have thus far covered the topic of estimation of the parameters of the (two-variable) linear regression model. Using the method of OLS we were able to estimate the parameters  $\beta_1$ ,  $\beta_2$ , and  $\sigma^2$ . Under the assumptions of the *classical linear regression model* (CLRM), we were able to show that the estimators of these parameters,  $\hat{\beta}_1$ ,  $\hat{\beta}_2$ , and  $\hat{\sigma}^2$ , satisfy several desirable statistical properties, such as unbiasedness, minimum variance, etc. (Recall the BLUE property.) Note that, since these are estimators, their values will change from sample to sample. Therefore, these estimators are *random variables*.

But estimation is half the battle. Hypothesis testing is the other half. Recall that in regression analysis our objective is not only to estimate the sample regression function (SRF), but also to use it to draw inferences about the population regression function (PRF), as emphasized in Chapter 2. Thus, we would like to find out how close  $\hat{\beta}_1$  is to the true  $\beta_1$  or how close  $\hat{\sigma}^2$  is to the true  $\sigma^2$ . For instance, in Example 3.2, we estimated the SRF as shown in Eq. (3.7.2). But since this regression is based on a sample of 55 families, how do we know that the estimated MPC of 0.4368 represents the (true) MPC in the population as a whole?

Therefore, since  $\hat{\beta}_1$ ,  $\hat{\beta}_2$ , and  $\hat{\sigma}^2$  are random variables, we need to find out their probability distributions, for without that knowledge we will not be able to relate them to their true values.

#### 4.1 THE PROBABILITY DISTRIBUTION OF DISTURBANCES $u_i$

To find out the probability distributions of the OLS estimators, we proceed as follows. Specifically, consider  $\hat{\beta}_2$ . As we showed in Appendix 3A.2,

$$\hat{\beta}_2 = \sum k_i Y_i \quad (4.1.1)$$

where  $k_i = x_i / \sum x_i^2$ . But since the  $X$ 's are assumed fixed, or nonstochastic, because ours is conditional regression analysis, conditional on the fixed values of  $X_i$ , Eq. (4.1.1) shows that  $\hat{\beta}_2$  is a *linear* function of  $Y_i$ , which is random by assumption. But since  $Y_i = \beta_1 + \beta_2 X_i + u_i$ , we can write (4.1.1) as

$$\hat{\beta}_2 = \sum k_i (\beta_1 + \beta_2 X_i + u_i) \quad (4.1.2)$$

Because  $k_i$ , the betas, and  $X_i$  are all fixed,  $\hat{\beta}_2$  is ultimately a *linear* function of the random variable  $u_i$ , which is random by assumption. Therefore, the probability distribution of  $\hat{\beta}_2$  (and also of  $\hat{\beta}_1$ ) will depend on the assumption made about the probability distribution of  $u_i$ . And since knowledge of the probability distributions of OLS estimators is necessary to draw inferences about their population values, the nature of the probability distribution of  $u_i$  assumes an extremely important role in hypothesis testing.

Since the method of OLS does not make any assumption about the probabilistic nature of  $u_i$ , it is of little help for the purpose of drawing inferences about the PRF from the SRF, the Gauss–Markov theorem notwithstanding. This void can be filled if we are willing to assume that the  $u$ 's follow some probability distribution. For reasons to be explained shortly, in the regression context it is usually assumed that the  $u$ 's follow the normal distribution. Adding the normality assumption for  $u_i$  to the assumptions of the classical linear regression model (CLRM) discussed in Chapter 3, we obtain what is known as the **classical normal linear regression model (CNLRM)**.

#### 4.2 THE NORMALITY ASSUMPTION FOR $u_i$

The classical *normal* linear regression model assumes that each  $u_i$  is distributed *normally* with

$$\text{Mean:} \quad E(u_i) = 0 \quad (4.2.1)$$

$$\text{Variance:} \quad E[u_i - E(u_i)]^2 = E(u_i^2) = \sigma^2 \quad (4.2.2)$$

$$\text{cov}(u_i, u_j): \quad E\{[(u_i - E(u_i))][u_j - E(u_j)]\} = E(u_i u_j) = 0 \quad i \neq j \quad (4.2.3)$$

The assumptions given above can be more compactly stated as

$$u_i \sim N(0, \sigma^2) \quad (4.2.4)$$

where the symbol  $\sim$  means *distributed as* and  $N$  stands for the *normal distribution*, the terms in the parentheses representing the two parameters of the normal distribution, namely, the mean and the variance.

As noted in **Appendix A**, for **two normally distributed variables, zero covariance or correlation means independence of the two variables**. Therefore, with the normality assumption, (4.2.4) means that  $u_i$  and  $u_j$  are not only uncorrelated but are also independently distributed.

Therefore, we can write (4.2.4) as

$$u_i \sim \text{NID}(0, \sigma^2) \quad (4.2.5)$$

where **NID** stands for *normally and independently distributed*.

### Why the Normality Assumption?

Why do we employ the normality assumption? There are several reasons:

1. As pointed out in Section 2.5,  $u_i$  represent the combined influence (on the dependent variable) of a large number of independent variables that are not explicitly introduced in the regression model. As noted, we hope that the influence of these omitted or neglected variables is small and at best random. Now by the celebrated **central limit theorem (CLT)** of statistics (see **Appendix A** for details), it can be shown that if there are a large number of independent and identically distributed random variables, then, with a few exceptions, the distribution of their sum tends to a normal distribution as the number of such variables increase indefinitely.<sup>1</sup> It is the CLT that provides a theoretical justification for the assumption of normality of  $u_i$ .

2. A variant of the CLT states that, even if the number of variables is not very large or if these variables are not strictly independent, their sum may still be normally distributed.<sup>2</sup>

3. With the normality assumption, the probability distributions of OLS estimators can be easily derived because, as noted in **Appendix A**, one property of the normal distribution is that **any linear function of normally distributed variables is itself normally distributed**. As we discussed earlier, OLS estimators  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are linear functions of  $u_i$ . Therefore, if  $u_i$  are normally distributed, so are  $\hat{\beta}_1$  and  $\hat{\beta}_2$ , which makes our task of hypothesis testing very straightforward.

4. The normal distribution is a comparatively simple distribution involving only two parameters (mean and variance); it is very well known and

<sup>1</sup>For a relatively simple and straightforward discussion of this theorem, see Sheldon M. Ross, *Introduction to Probability and Statistics for Engineers and Scientists*, 2d ed., Harcourt Academic Press, New York, 2000, pp. 193–194. One exception to the theorem is the Cauchy distribution, which has no mean or higher moments. See M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, Charles Griffin & Co., London, 1960, vol. 1, pp. 248–249.

<sup>2</sup>For the various forms of the CLT, see Harald Cramer, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, N.J., 1946, Chap. 17.

its theoretical properties have been extensively studied in mathematical statistics. Besides, many phenomena seem to follow the normal distribution.

5. Finally, if we are dealing with a small, or finite, sample size, say data of less than 100 observations, the normality assumption assumes a critical role. It not only helps us to derive the exact probability distributions of OLS estimators but also enables us to use the  $t$ ,  $F$ , and  $\chi^2$  statistical tests for regression models. The statistical properties of  $t$ ,  $F$ , and  $\chi^2$  probability distributions are discussed in **Appendix A**. As we will show subsequently, if the sample size is reasonably large, we may be able to relax the normality assumption.

*A cautionary note:* Since we are “imposing” the normality assumption, it behooves us to find out in practical applications involving small sample size data whether the normality assumption is appropriate. Later, we will develop some tests to do just that. Also, later we will come across situations where the normality assumption may be inappropriate. But until then we will continue with the normality assumption for the reasons discussed previously.

#### 4.3 PROPERTIES OF OLS ESTIMATORS UNDER THE NORMALITY ASSUMPTION

With the assumption that  $u_i$  follow the normal distribution as in (4.2.5), the OLS estimators have the following properties; **Appendix A** provides a general discussion of the desirable statistical properties of estimators.

1. They are unbiased.
2. They have minimum variance. Combined with 1, this means that they are **minimum-variance unbiased, or efficient estimators**.
3. They have **consistency**; that is, as the sample size increases indefinitely, the estimators converge to their true population values.
4.  $\hat{\beta}_1$  (being a linear function of  $u_i$ ) is *normally distributed* with

$$\text{Mean: } E(\hat{\beta}_1) = \beta_1 \quad (4.3.1)$$

$$\text{var}(\hat{\beta}_1): \quad \sigma_{\hat{\beta}_1}^2 = \frac{\sum X_i^2}{n \sum x_i^2} \sigma^2 \quad = (3.3.3) \quad (4.3.2)$$

Or more compactly,

$$\hat{\beta}_1 \sim N(\beta_1, \sigma_{\hat{\beta}_1}^2)$$

Then by the properties of the normal distribution the variable  $Z$ , which is defined as

$$Z = \frac{\hat{\beta}_1 - \beta_1}{\sigma_{\hat{\beta}_1}} \quad (4.3.3)$$

follows the **standard normal distribution**, that is, a normal distribution with zero mean and unit ( $= 1$ ) variance, or

$$Z \sim N(0, 1)$$

5.  $\hat{\beta}_2$  (being a linear function of  $u_i$ ) is *normally* distributed with

$$\text{Mean: } E(\hat{\beta}_2) = \beta_2 \tag{4.3.4}$$

$$\text{var}(\hat{\beta}_2): \sigma_{\hat{\beta}_2}^2 = \frac{\sigma^2}{\sum x_i^2} = (3.3.1) \tag{4.3.5}$$

Or, more compactly,

$$\hat{\beta}_2 \sim N(\beta_2, \sigma_{\hat{\beta}_2}^2)$$

Then, as in (4.3.3),

$$Z = \frac{\hat{\beta}_2 - \beta_2}{\sigma_{\hat{\beta}_2}} \tag{4.3.6}$$

also follows the standard normal distribution.

Geometrically, the probability distributions of  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are shown in Figure 4.1.

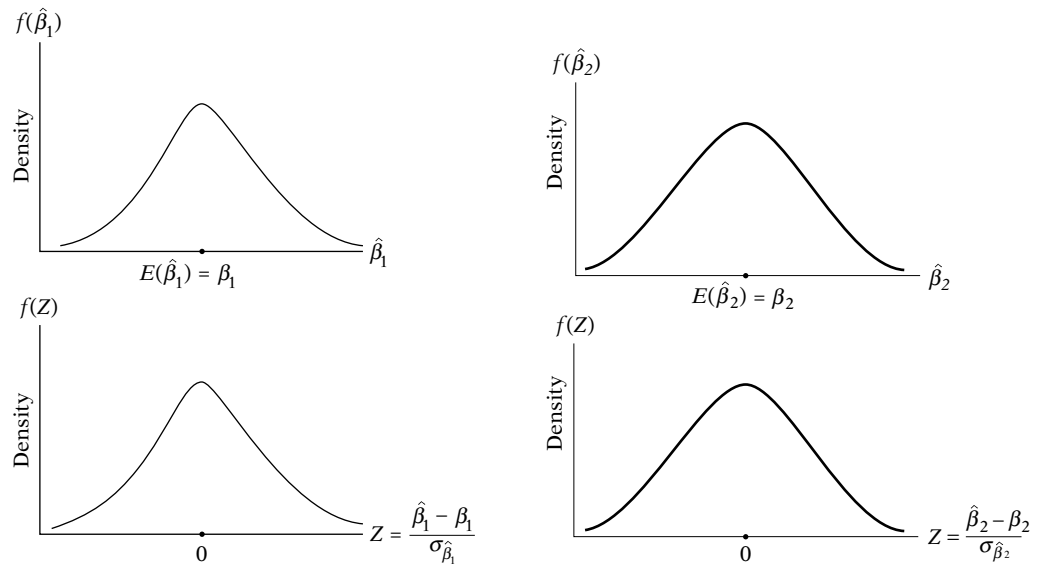


FIGURE 4.1 Probability distributions of  $\hat{\beta}_1$  and  $\hat{\beta}_2$ .



6.  $(n - 2)(\hat{\sigma}^2/\sigma^2)$  is distributed as the  $\chi^2$  (chi-square) distribution with  $(n - 2)$ df.<sup>3</sup> This knowledge will help us to draw inferences about the true  $\sigma^2$  from the estimated  $\sigma^2$ , as we will show in Chapter 5. (The chi-square distribution and its properties are discussed in **Appendix A.**)

7.  $(\hat{\beta}_1, \hat{\beta}_2)$  are distributed independently of  $\hat{\sigma}^2$ . The importance of this will be explained in the next chapter.

8.  $\hat{\beta}_1$  and  $\hat{\beta}_2$  have minimum variance in the entire class of unbiased estimators, whether linear or not. This result, due to Rao, is very powerful because, unlike the Gauss–Markov theorem, it is not restricted to the class of linear estimators only.<sup>4</sup> Therefore, we can say that the least-squares estimators are **best unbiased estimators (BUE)**; that is, they have minimum variance in the entire class of unbiased estimators.

**To sum up:** The important point to note is that the normality assumption enables us to derive the probability, or sampling, distributions of  $\hat{\beta}_1$  and  $\hat{\beta}_2$  (both normal) and  $\hat{\sigma}^2$  (related to the chi square). As we will see in the next chapter, this simplifies the task of establishing confidence intervals and testing (statistical) hypotheses.

In passing, note that, with the assumption that  $u_i \sim N(0, \sigma^2)$ ,  $Y_i$ , being a linear function of  $u_i$ , is itself normally distributed with the mean and variance given by

$$E(Y_i) = \beta_1 + \beta_2 X_i \quad (4.3.7)$$

$$\text{var}(Y_i) = \sigma^2 \quad (4.3.8)$$

More neatly, we can write

$$Y_i \sim N(\beta_1 + \beta_2 X_i, \sigma^2) \quad (4.3.9)$$

#### 4.4 THE METHOD OF MAXIMUM LIKELIHOOD (ML)

A method of point estimation with some stronger theoretical properties than the method of OLS is the method of **maximum likelihood (ML)**. Since this method is slightly involved, it is discussed in the appendix to this chapter. For the general reader, it will suffice to note that if  $u_i$  are assumed to be normally distributed, as we have done for reasons already discussed, the ML and OLS estimators of the regression coefficients, the  $\beta$ 's, are identical, and this is true of simple as well as multiple regressions. The ML estimator of  $\sigma^2$  is  $\sum \hat{u}_i^2/n$ . This estimator is biased, whereas the OLS estimator

<sup>3</sup>The proof of this statement is slightly involved. An accessible source for the proof is Robert V. Hogg and Allen T. Craig, *Introduction to Mathematical Statistics*, 2d ed., Macmillan, New York, 1965, p. 144.

<sup>4</sup>C. R. Rao, *Linear Statistical Inference and Its Applications*, John Wiley & Sons, New York, 1965, p. 258.

of  $\sigma^2 = \sum \hat{u}_i^2 / (n - 2)$ , as we have seen, is unbiased. But comparing these two estimators of  $\sigma^2$ , we see that as the sample size  $n$  gets larger the two estimators of  $\sigma^2$  tend to be equal. Thus, asymptotically (i.e., as  $n$  increases indefinitely), the ML estimator of  $\sigma^2$  is also unbiased.

Since the method of least squares with the added assumption of normality of  $u_i$  provides us with all the tools necessary for both estimation and hypothesis testing of the linear regression models, there is no loss for readers who may not want to pursue the maximum likelihood method because of its slight mathematical complexity.

#### 4.5 SUMMARY AND CONCLUSIONS

1. This chapter discussed the classical *normal* linear regression model (CNLRM).

2. This model differs from the classical linear regression model (CLRM) in that it specifically assumes that the disturbance term  $u_i$  entering the regression model is normally distributed. The CLRM does not require any assumption about the probability distribution of  $u_i$ ; it only requires that the mean value of  $u_i$  is zero and its variance is a finite constant.

3. The theoretical justification for the normality assumption is the **central limit theorem**.

4. Without the normality assumption, under the other assumptions discussed in Chapter 3, the Gauss–Markov theorem showed that the OLS estimators are BLUE.

5. With the additional assumption of normality, the OLS estimators are not only **best unbiased estimators (BUE)** but also follow well-known probability distributions. The OLS estimators of the intercept and slope are themselves normally distributed and the OLS estimator of the variance of  $u_i$  ( $= \hat{\sigma}^2$ ) is related to the chi-square distribution.

6. In Chapters 5 and 8 we show how this knowledge is useful in drawing inferences about the values of the population parameters.

7. An alternative to the least-squares method is the method of **maximum likelihood (ML)**. To use this method, however, one must make an assumption about the probability distribution of the disturbance term  $u_i$ . In the regression context, the assumption most popularly made is that  $u_i$  follows the normal distribution.

8. Under the normality assumption, the ML and OLS estimators of the intercept and slope parameters of the regression model are identical. However, the OLS and ML estimators of the variance of  $u_i$  are different. In large samples, however, these two estimators converge.

9. Thus the ML method is generally called a *large-sample method*. The ML method is of broader application in that it can also be applied to regression models that are nonlinear in the parameters. In the latter case, OLS is generally not used. For more on this, see Chapter 14.

**10.** In this text, we will largely rely on the OLS method for practical reasons: (a) Compared to ML, the OLS is easy to apply; (b) the ML and OLS estimators of  $\beta_1$  and  $\beta_2$  are identical (which is true of multiple regressions too); and (c) even in moderately large samples the OLS and ML estimators of  $\sigma^2$  do not differ vastly.

However, for the benefit of the mathematically inclined reader, a brief introduction to ML is given in the appendix to this chapter and also in **Appendix A**.

## APPENDIX 4A

### 4A.1 MAXIMUM LIKELIHOOD ESTIMATION OF TWO-VARIABLE REGRESSION MODEL

Assume that in the two-variable model  $Y_i = \beta_1 + \beta_2 X_i + u_i$  the  $Y_i$  are normally and independently distributed with mean  $= \beta_1 + \beta_2 X_i$  and variance  $= \sigma^2$ . [See Eq. (4.3.9).] As a result, the joint probability density function of  $Y_1, Y_2, \dots, Y_n$ , given the preceding mean and variance, can be written as

$$f(Y_1, Y_2, \dots, Y_n | \beta_1 + \beta_2 X_i, \sigma^2)$$

But in view of the independence of the  $Y$ 's, this joint probability density function can be written as a product of  $n$  individual density functions as

$$\begin{aligned} f(Y_1, Y_2, \dots, Y_n | \beta_1 + \beta_2 X_i, \sigma^2) \\ = f(Y_1 | \beta_1 + \beta_2 X_i, \sigma^2) f(Y_2 | \beta_1 + \beta_2 X_i, \sigma^2) \cdots f(Y_n | \beta_1 + \beta_2 X_i, \sigma^2) \end{aligned} \quad (1)$$

where

$$f(Y_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \frac{(Y_i - \beta_1 - \beta_2 X_i)^2}{\sigma^2} \right\} \quad (2)$$

which is the density function of a normally distributed variable with the given mean and variance.

(Note: exp means  $e$  to the power of the expression indicated by {}.)

Substituting (2) for each  $Y_i$  into (1) gives

$$f(Y_1, Y_2, \dots, Y_n | \beta_1 + \beta_2 X_i, \sigma^2) = \frac{1}{\sigma^n (\sqrt{2\pi})^n} \exp \left\{ -\frac{1}{2} \sum \frac{(Y_i - \beta_1 - \beta_2 X_i)^2}{\sigma^2} \right\} \quad (3)$$

If  $Y_1, Y_2, \dots, Y_n$  are known or given, but  $\beta_1, \beta_2$ , and  $\sigma^2$  are not known, the function in (3) is called a **likelihood function**, denoted by  $LF(\beta_1, \beta_2, \sigma^2)$ ,

and written as<sup>1</sup>

$$\text{LF}(\beta_1, \beta_2, \sigma^2) = \frac{1}{\sigma^n (\sqrt{2\pi})^n} \exp \left\{ -\frac{1}{2} \sum \frac{(Y_i - \beta_1 - \beta_2 X_i)^2}{\sigma^2} \right\} \quad (4)$$

The **method of maximum likelihood**, as the name indicates, consists in estimating the unknown parameters in such a manner that the probability of observing the given  $Y$ 's is as high (or maximum) as possible. Therefore, we have to find the maximum of the function (4). This is a straightforward exercise in differential calculus. For differentiation it is easier to express (4) in the log term as follows.<sup>2</sup> (*Note:*  $\ln$  = natural log.)

$$\begin{aligned} \ln \text{LF} &= -n \ln \sigma - \frac{n}{2} \ln (2\pi) - \frac{1}{2} \sum \frac{(Y_i - \beta_1 - \beta_2 X_i)^2}{\sigma^2} \\ &= -\frac{n}{2} \ln \sigma^2 - \frac{n}{2} \ln (2\pi) - \frac{1}{2} \sum \frac{(Y_i - \beta_1 - \beta_2 X_i)^2}{\sigma^2} \end{aligned} \quad (5)$$

Differentiating (5) partially with respect to  $\beta_1$ ,  $\beta_2$ , and  $\sigma^2$ , we obtain

$$\frac{\partial \ln \text{LF}}{\partial \beta_1} = -\frac{1}{\sigma^2} \sum (Y_i - \beta_1 - \beta_2 X_i)(-1) \quad (6)$$

$$\frac{\partial \ln \text{LF}}{\partial \beta_2} = -\frac{1}{\sigma^2} \sum (Y_i - \beta_1 - \beta_2 X_i)(-X_i) \quad (7)$$

$$\frac{\partial \ln \text{LF}}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum (Y_i - \beta_1 - \beta_2 X_i)^2 \quad (8)$$

Setting these equations equal to zero (the first-order condition for optimization) and letting  $\tilde{\beta}_1$ ,  $\tilde{\beta}_2$ , and  $\tilde{\sigma}^2$  denote the ML estimators, we obtain<sup>3</sup>

$$\frac{1}{\tilde{\sigma}^2} \sum (Y_i - \tilde{\beta}_1 - \tilde{\beta}_2 X_i) = 0 \quad (9)$$

$$\frac{1}{\tilde{\sigma}^2} \sum (Y_i - \tilde{\beta}_1 - \tilde{\beta}_2 X_i) X_i = 0 \quad (10)$$

$$-\frac{n}{2\tilde{\sigma}^2} + \frac{1}{2\tilde{\sigma}^4} \sum (Y_i - \tilde{\beta}_1 - \tilde{\beta}_2 X_i)^2 = 0 \quad (11)$$

<sup>1</sup>Of course, if  $\beta_1$ ,  $\beta_2$ , and  $\sigma^2$  are known but the  $Y_i$  are not known, (4) represents the joint probability density function—the probability of jointly observing the  $Y_i$ .

<sup>2</sup>Since a log function is a monotonic function,  $\ln \text{LF}$  will attain its maximum value at the same point as  $\text{LF}$ .

<sup>3</sup>We use  $\tilde{\phantom{x}}$  (tilde) for ML estimators and  $\hat{\phantom{x}}$  (cap or hat) for OLS estimators.

After simplifying, Eqs. (9) and (10) yield

$$\sum Y_i = n\tilde{\beta}_1 + \tilde{\beta}_2 \sum X_i \quad (12)$$

$$\sum Y_i X_i = \tilde{\beta}_1 \sum X_i + \tilde{\beta}_2 \sum X_i^2 \quad (13)$$

which are precisely the *normal equations* of the least-squares theory obtained in (3.1.4) and (3.1.5). Therefore, the ML estimators, the  $\tilde{\beta}$ 's, are the same as the OLS estimators, the  $\hat{\beta}$ 's, given in (3.1.6) and (3.1.7). This equality is not accidental. Examining the likelihood (5), we see that the last term enters with a negative sign. Therefore, maximizing (5) amounts to minimizing this term, which is precisely the least-squares approach, as can be seen from (3.1.2).

Substituting the ML (= OLS) estimators into (11) and simplifying, we obtain the ML estimator of  $\tilde{\sigma}^2$  as

$$\begin{aligned} \tilde{\sigma}^2 &= \frac{1}{n} \sum (Y_i - \tilde{\beta}_1 - \tilde{\beta}_2 X_i)^2 \\ &= \frac{1}{n} \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2 \\ &= \frac{1}{n} \sum \hat{u}_i^2 \end{aligned} \quad (14)$$

From (14) it is obvious that the ML estimator  $\tilde{\sigma}^2$  differs from the OLS estimator  $\hat{\sigma}^2 = [1/(n-2)] \sum \hat{u}_i^2$ , which was shown to be an unbiased estimator of  $\sigma^2$  in Appendix 3A, Section 3A.5. Thus, the ML estimator of  $\sigma^2$  is biased. The magnitude of this bias can be easily determined as follows.

Taking the mathematical expectation of (14) on both sides, we obtain

$$\begin{aligned} E(\tilde{\sigma}^2) &= \frac{1}{n} E\left(\sum \hat{u}_i^2\right) \\ &= \left(\frac{n-2}{n}\right) \sigma^2 \quad \text{using Eq. (16) of Appendix 3A,} \\ &= \sigma^2 - \frac{2}{n} \sigma^2 \end{aligned} \quad (15)$$

which shows that  $\tilde{\sigma}^2$  is biased downward (i.e., it underestimates the true  $\sigma^2$ ) in small samples. But notice that as  $n$ , the sample size, increases indefinitely, the second term in (15), the bias factor, tends to be zero. Therefore, *asymptotically* (i.e., in a very large sample),  $\tilde{\sigma}^2$  is *unbiased* too, that is,  $\lim E(\tilde{\sigma}^2) = \sigma^2$  as  $n \rightarrow \infty$ . It can further be proved that  $\tilde{\sigma}^2$  is also a

**consistent** estimator<sup>4</sup>; that is, as  $n$  increases indefinitely  $\tilde{\sigma}^2$  converges to its true value  $\sigma^2$ .

#### 4A.2 MAXIMUM LIKELIHOOD ESTIMATION OF FOOD EXPENDITURE IN INDIA

Return to Example 3.2 and regression (3.7.2), which gives the regression of food expenditure on total expenditure for 55 rural households in India. Since under the normality assumption the OLS and ML estimators of the regression coefficients are the same, we obtain the ML estimators as  $\hat{\beta}_1 = \hat{\beta}_1 = 94.2087$  and  $\hat{\beta}_2 = \hat{\beta}_2 = 0.4386$ . The OLS estimator of  $\sigma^2$  is  $\hat{\sigma}^2 = 4469.6913$ , but the ML estimator is  $\tilde{\sigma}^2 = 4407.1563$ , which is smaller than the OLS estimator. As noted, in small samples the ML estimator is downward biased; that is, on average it underestimates the true variance  $\sigma^2$ . Of course, as you would expect, as the sample size gets bigger, the difference between the two estimators will narrow. Putting the values of the estimators in the log likelihood function, we obtain the value of  $-308.1625$ . If you want the maximum value of the LF, just take the antilog of  $-308.1625$ . No other values of the parameters will give you a higher probability of obtaining the sample that you have used in the analysis.

#### APPENDIX 4A EXERCISES

**4.1.** “If two random variables are statistically independent, the coefficient of correlation between the two is zero. But the converse is not necessarily true; that is, zero correlation does not imply statistical independence. However, if two variables are normally distributed, zero correlation necessarily implies statistical independence.” Verify this statement for the following joint probability density function of two normally distributed variables  $Y_1$  and  $Y_2$  (this joint probability density function is known as the **bivariate normal probability density function**):

$$f(Y_1, Y_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\right. \\ \left.\times \left[\left(\frac{Y_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(Y_1 - \mu_1)(Y_2 - \mu_2)}{\sigma_1\sigma_2} + \left(\frac{Y_2 - \mu_2}{\sigma_2}\right)^2\right]\right\}$$

<sup>4</sup>See **App. A** for a general discussion of the properties of the maximum likelihood estimators as well as for the distinction between asymptotic unbiasedness and consistency. Roughly speaking, in asymptotic unbiasedness we try to find out the  $\lim E(\tilde{\sigma}_n^2)$  as  $n$  tends to infinity, where  $n$  is the sample size on which the estimator is based, whereas in consistency we try to find out how  $\tilde{\sigma}_n^2$  behaves as  $n$  increases indefinitely. Notice that the unbiasedness property is a repeated sampling property of an estimator based on a sample of given size, whereas in consistency we are concerned with the behavior of an estimator as the sample size increases indefinitely.

where  $\mu_1$  = mean of  $Y_1$   
 $\mu_2$  = mean of  $Y_2$   
 $\sigma_1$  = standard deviation of  $Y_1$   
 $\sigma_2$  = standard deviation of  $Y_2$   
 $\rho$  = coefficient of correlation between  $Y_1$  and  $Y_2$

- 4.2.** By applying the second-order conditions for optimization (i.e., second-derivative test), show that the ML estimators of  $\beta_1$ ,  $\beta_2$ , and  $\sigma^2$  obtained by solving Eqs. (9), (10), and (11) do in fact maximize the likelihood function (4).
- 4.3.** A random variable  $X$  follows the **exponential distribution** if it has the following probability density function (PDF):

$$f(X) = (1/\theta)e^{-X/\theta} \quad \text{for } X > 0 \\ = 0 \quad \text{elsewhere}$$

where  $\theta > 0$  is the parameter of the distribution. Using the ML method, show that the ML estimator of  $\theta$  is  $\hat{\theta} = \sum X_i/n$ , where  $n$  is the sample size. That is, show that the ML estimator of  $\theta$  is the sample mean  $\bar{X}$ .

# 5

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## TWO-VARIABLE REGRESSION: INTERVAL ESTIMATION AND HYPOTHESIS TESTING

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Beware of testing too many hypotheses; the more you torture the data, the more likely they are to confess, but confession obtained under duress may not be admissible in the court of scientific opinion.<sup>1</sup>

As pointed out in Chapter 4, estimation and hypothesis testing constitute the two major branches of classical statistics. The theory of estimation consists of two parts: point estimation and interval estimation. We have discussed point estimation thoroughly in the previous two chapters where we introduced the OLS and ML methods of point estimation. In this chapter we first consider interval estimation and then take up the topic of hypothesis testing, a topic intimately related to interval estimation.

### 5.1 STATISTICAL PREREQUISITES

Before we demonstrate the actual mechanics of establishing confidence intervals and testing statistical hypotheses, it is assumed that the reader is familiar with the fundamental concepts of probability and statistics. Although not a substitute for a basic course in statistics, **Appendix A** provides the essentials of statistics with which the reader should be totally familiar. Key concepts such as **probability, probability distributions, Type I and Type II errors, level of significance, power of a statistical test, and confidence interval** are crucial for understanding the material covered in this and the following chapters.

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<sup>1</sup>Stephen M. Stigler, "Testing Hypothesis or Fitting Models? Another Look at Mass Extinctions," in Matthew H. Nitecki and Antoni Hoffman, eds., *Neutral Models in Biology*, Oxford University Press, Oxford, 1987, p. 148.



## 5.2 INTERVAL ESTIMATION: SOME BASIC IDEAS

To fix the ideas, consider the hypothetical consumption-income example of Chapter 3. Equation (3.6.2) shows that the estimated marginal propensity to consume (MPC)  $\beta_2$  is 0.5091, which is a single (point) estimate of the unknown population MPC  $\beta_2$ . How reliable is this estimate? As noted in Chapter 3, because of sampling fluctuations, a single estimate is likely to differ from the true value, although in repeated sampling its mean value is expected to be equal to the true value. [Note:  $E(\hat{\beta}_2) = \beta_2$ .] Now in statistics the reliability of a point estimator is measured by its standard error. Therefore, instead of relying on the point estimate alone, we may construct an interval around the point estimator, say within two or three standard errors on either side of the point estimator, such that this interval has, say, 95 percent probability of including the true parameter value. This is roughly the idea behind **interval estimation**.

To be more specific, assume that we want to find out how “close” is, say,  $\hat{\beta}_2$  to  $\beta_2$ . For this purpose we try to find out two positive numbers  $\delta$  and  $\alpha$ , the latter lying between 0 and 1, such that the probability that the **random interval**  $(\hat{\beta}_2 - \delta, \hat{\beta}_2 + \delta)$  contains the true  $\beta_2$  is  $1 - \alpha$ . Symbolically,

$$\Pr(\hat{\beta}_2 - \delta \leq \beta_2 \leq \hat{\beta}_2 + \delta) = 1 - \alpha \quad (5.2.1)$$

Such an interval, if it exists, is known as a **confidence interval**;  $1 - \alpha$  is known as the **confidence coefficient**; and  $\alpha$  ( $0 < \alpha < 1$ ) is known as the **level of significance**.<sup>2</sup> The endpoints of the confidence interval are known as the **confidence limits** (also known as *critical values*),  $\hat{\beta}_2 - \delta$  being the **lower confidence limit** and  $\hat{\beta}_2 + \delta$  the **upper confidence limit**. In passing, note that in practice  $\alpha$  and  $1 - \alpha$  are often expressed in percentage forms as  $100\alpha$  and  $100(1 - \alpha)$  percent.

Equation (5.2.1) shows that an **interval estimator**, in contrast to a point estimator, is an interval constructed in such a manner that it has a specified probability  $1 - \alpha$  of including within its limits the true value of the parameter. For example, if  $\alpha = 0.05$ , or 5 percent, (5.2.1) would read: The probability that the (random) interval shown there includes the true  $\beta_2$  is 0.95, or 95 percent. The interval estimator thus gives a range of values within which the true  $\beta_2$  may lie.

It is very important to know the following aspects of interval estimation:

1. Equation (5.2.1) does not say that the probability of  $\beta_2$  lying between the given limits is  $1 - \alpha$ . Since  $\beta_2$ , although an unknown, is assumed to be some fixed number, either it lies in the interval or it does not. What (5.2.1)

<sup>2</sup>Also known as the **probability of committing a Type I error**. A Type I error consists in rejecting a true hypothesis, whereas a Type II error consists in accepting a false hypothesis. (This topic is discussed more fully in **App. A**.) The symbol  $\alpha$  is also known as the **size of the (statistical) test**.

states is that, for the method described in this chapter, the probability of constructing an interval that contains  $\beta_2$  is  $1 - \alpha$ .

**2.** The interval (5.2.1) is a **random interval**; that is, it will vary from one sample to the next because it is based on  $\hat{\beta}_2$ , which is random. (Why?)

**3.** Since the confidence interval is random, the probability statements attached to it should be understood in the long-run sense, that is, repeated sampling. More specifically, (5.2.1) means: If in repeated sampling confidence intervals like it are constructed a great many times on the  $1 - \alpha$  probability basis, then, in the long run, on the average, such intervals will enclose in  $1 - \alpha$  of the cases the true value of the parameter.

**4.** As noted in 2, the interval (5.2.1) is random so long as  $\hat{\beta}_2$  is not known. But once we have a specific sample and once we obtain a specific numerical value of  $\hat{\beta}_2$ , the interval (5.2.1) is no longer random; it is fixed. In this case, we **cannot** make the probabilistic statement (5.2.1); that is, we cannot say that the probability is  $1 - \alpha$  that a given *fixed* interval includes the true  $\beta_2$ . In this situation  $\beta_2$  is either in the fixed interval or outside it. Therefore, the probability is either 1 or 0. Thus, for our hypothetical consumption-income example, if the 95% confidence interval were obtained as  $(0.4268 \leq \beta_2 \leq 0.5914)$ , as we do shortly in (5.3.9), we **cannot** say the probability is 95% that this interval includes the true  $\beta_2$ . That probability is either 1 or 0.

How are the confidence intervals constructed? From the preceding discussion one may expect that if the **sampling or probability distributions** of the estimators are known, one can make confidence interval statements such as (5.2.1). In Chapter 4 we saw that under the assumption of normality of the disturbances  $u_i$  the OLS estimators  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are themselves normally distributed and that the OLS estimator  $\hat{\sigma}^2$  is related to the  $\chi^2$  (chi-square) distribution. It would then seem that the task of constructing confidence intervals is a simple one. And it is!

### 5.3 CONFIDENCE INTERVALS FOR REGRESSION COEFFICIENTS $\beta_1$ AND $\beta_2$

#### Confidence Interval for $\beta_2$

It was shown in Chapter 4, Section 4.3, that, with the normality assumption for  $u_i$ , the OLS estimators  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are themselves normally distributed with means and variances given therein. Therefore, for example, the variable

$$\begin{aligned} Z &= \frac{\hat{\beta}_2 - \beta_2}{\text{se}(\hat{\beta}_2)} \\ &= \frac{(\hat{\beta}_2 - \beta_2)\sqrt{\sum x_i^2}}{\sigma} \end{aligned} \tag{5.3.1}$$

as noted in (4.3.6), is a standardized normal variable. It therefore seems that we can use the normal distribution to make probabilistic statements about  $\beta_2$  provided the true population variance  $\sigma^2$  is known. If  $\sigma^2$  is known, an important property of a normally distributed variable with mean  $\mu$  and variance  $\sigma^2$  is that the area under the normal curve between  $\mu \pm \sigma$  is about 68 percent, that between the limits  $\mu \pm 2\sigma$  is about 95 percent, and that between  $\mu \pm 3\sigma$  is about 99.7 percent.

But  $\sigma^2$  is rarely known, and in practice it is determined by the unbiased estimator  $\hat{\sigma}^2$ . If we replace  $\sigma$  by  $\hat{\sigma}$ , (5.3.1) may be written as

$$\begin{aligned} t &= \frac{\hat{\beta}_2 - \beta_2}{\text{se}(\hat{\beta}_2)} = \frac{\text{estimator} - \text{parameter}}{\text{estimated standard error of estimator}} \\ &= \frac{(\hat{\beta}_2 - \beta_2)\sqrt{\sum x_i^2}}{\hat{\sigma}} \end{aligned} \quad (5.3.2)$$

where the  $\text{se}(\hat{\beta}_2)$  now refers to the estimated standard error. It can be shown (see Appendix 5A, Section 5A.2) that the  $t$  variable thus defined follows the  $t$  distribution with  $n - 2$  df. [Note the difference between (5.3.1) and (5.3.2).] Therefore, instead of using the normal distribution, we can use the  $t$  distribution to establish a confidence interval for  $\beta_2$  as follows:

$$\Pr(-t_{\alpha/2} \leq t \leq t_{\alpha/2}) = 1 - \alpha \quad (5.3.3)$$

where the  $t$  value in the middle of this double inequality is the  $t$  value given by (5.3.2) and where  $t_{\alpha/2}$  is the value of the  $t$  variable obtained from the  $t$  distribution for  $\alpha/2$  level of significance and  $n - 2$  df; it is often called the **critical**  $t$  value at  $\alpha/2$  level of significance. Substitution of (5.3.2) into (5.3.3) yields

$$\Pr\left[-t_{\alpha/2} \leq \frac{\hat{\beta}_2 - \beta_2}{\text{se}(\hat{\beta}_2)} \leq t_{\alpha/2}\right] = 1 - \alpha \quad (5.3.4)$$

Rearranging (5.3.4), we obtain

$$\Pr[\hat{\beta}_2 - t_{\alpha/2} \text{se}(\hat{\beta}_2) \leq \beta_2 \leq \hat{\beta}_2 + t_{\alpha/2} \text{se}(\hat{\beta}_2)] = 1 - \alpha \quad (5.3.5)^3$$

<sup>3</sup>Some authors prefer to write (5.3.5) with the df explicitly indicated. Thus, they would write

$$\Pr[\hat{\beta}_2 - t_{(n-2), \alpha/2} \text{se}(\hat{\beta}_2) \leq \beta_2 \leq \hat{\beta}_2 + t_{(n-2), \alpha/2} \text{se}(\hat{\beta}_2)] = 1 - \alpha$$

But for simplicity we will stick to our notation; the context clarifies the appropriate df involved.

Equation (5.3.5) provides a  $100(1 - \alpha)$  percent **confidence interval** for  $\beta_2$ , which can be written more compactly as

100(1 -  $\alpha$ )% confidence interval for  $\beta_2$ :

$$\hat{\beta}_2 \pm t_{\alpha/2} \text{ se}(\hat{\beta}_2) \quad (5.3.6)$$

Arguing analogously, and using (4.3.1) and (4.3.2), we can then write:

$$\Pr [\hat{\beta}_1 - t_{\alpha/2} \text{ se}(\hat{\beta}_1) \leq \beta_1 \leq \hat{\beta}_1 + t_{\alpha/2} \text{ se}(\hat{\beta}_1)] = 1 - \alpha \quad (5.3.7)$$

or, more compactly,

100(1 -  $\alpha$ )% confidence interval for  $\beta_1$ :

$$\hat{\beta}_1 \pm t_{\alpha/2} \text{ se}(\hat{\beta}_1) \quad (5.3.8)$$

Notice an important feature of the confidence intervals given in (5.3.6) and (5.3.8): In both cases *the width of the confidence interval is proportional to the standard error of the estimator*. That is, the larger the standard error, the larger is the width of the confidence interval. Put differently, the larger the standard error of the estimator, the greater is the uncertainty of estimating the true value of the unknown parameter. Thus, the standard error of an estimator is often described as a measure of the **precision** of the estimator, i.e., how precisely the estimator measures the true population value.

Returning to our illustrative consumption–income example, in Chapter 3 (Section 3.6) we found that  $\hat{\beta}_2 = 0.5091$ ,  $\text{se}(\hat{\beta}_2) = 0.0357$ , and  $df = 8$ . If we assume  $\alpha = 5\%$ , that is, 95% confidence coefficient, then the  $t$  table shows that for 8 df the **critical**  $t_{\alpha/2} = t_{0.025} = 2.306$ . Substituting these values in (5.3.5), the reader should verify that the 95% confidence interval for  $\beta_2$  is as follows:

$$0.4268 \leq \beta_2 \leq 0.5914 \quad (5.3.9)$$

Or, using (5.3.6), it is

$$0.5091 \pm 2.306(0.0357)$$

that is,

$$0.5091 \pm 0.0823 \quad (5.3.10)$$

**The interpretation of this confidence interval is:** Given the confidence coefficient of 95%, in the long run, in 95 out of 100 cases intervals like

(0.4268, 0.5914) will contain the true  $\beta_2$ . But, as warned earlier, we cannot say that the probability is 95 percent that the specific interval (0.4268 to 0.5914) contains the true  $\beta_2$  because this interval is now fixed and no longer random; therefore,  $\beta_2$  either lies in it or does not: The probability that the specified fixed interval includes the true  $\beta_2$  is therefore 1 or 0.

### Confidence Interval for $\beta_1$

Following (5.3.7), the reader can easily verify that the 95% confidence interval for  $\beta_1$  of our consumption-income example is

$$9.6643 \leq \beta_1 \leq 39.2448 \quad (5.3.11)$$

Or, using (5.3.8), we find it is

$$24.4545 \pm 2.306(6.4138)$$

that is,

$$24.4545 \pm 14.7902 \quad (5.3.12)$$

Again you should be careful in interpreting this confidence interval. In the long run, in 95 out of 100 cases intervals like (5.3.11) will contain the true  $\beta_1$ ; the probability that this particular fixed interval includes the true  $\beta_1$  is either 1 or 0.

### Confidence Interval for $\beta_1$ and $\beta_2$ Simultaneously

There are occasions when one needs to construct a *joint confidence interval* for  $\beta_1$  and  $\beta_2$  such that with a confidence coefficient  $(1 - \alpha)$ , say, 95%, that interval includes  $\beta_1$  and  $\beta_2$  simultaneously. Since this topic is involved, the interested reader may want to consult appropriate references.<sup>4</sup> We will touch on this topic briefly in Chapters 8 and 10.

## 5.4 CONFIDENCE INTERVAL FOR $\sigma^2$

As pointed out in Chapter 4, Section 4.3, under the normality assumption, the variable

$$\chi^2 = (n - 2) \frac{\hat{\sigma}^2}{\sigma^2} \quad (5.4.1)$$

<sup>4</sup>For an accessible discussion, see John Neter, William Wasserman, and Michael H. Kutner, *Applied Linear Regression Models*, Richard D. Irwin, Homewood, Ill., 1983, Chap. 5.

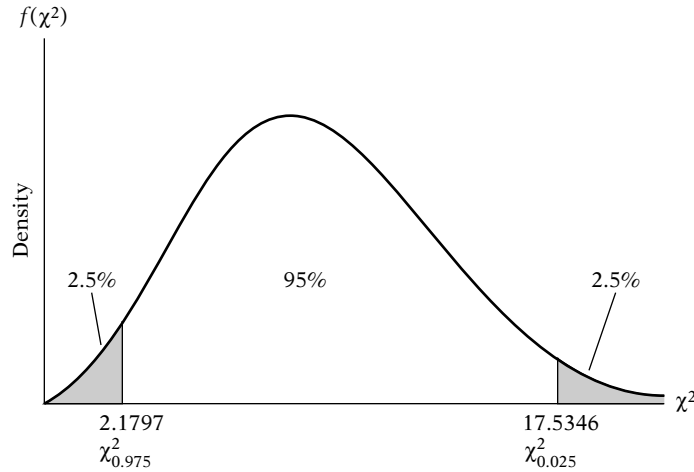


FIGURE 5.1 The 95% confidence interval for  $\chi^2$  (8 df).

follows the  $\chi^2$  distribution with  $n - 2$  df.<sup>5</sup> Therefore, we can use the  $\chi^2$  distribution to establish a confidence interval for  $\sigma^2$

$$\Pr(\chi_{1-\alpha/2}^2 \leq \chi^2 \leq \chi_{\alpha/2}^2) = 1 - \alpha \quad (5.4.2)$$

where the  $\chi^2$  value in the middle of this double inequality is as given by (5.4.1) and where  $\chi_{1-\alpha/2}^2$  and  $\chi_{\alpha/2}^2$  are two values of  $\chi^2$  (the **critical**  $\chi^2$  values) obtained from the chi-square table for  $n - 2$  df in such a manner that they cut off  $100(\alpha/2)$  percent tail areas of the  $\chi^2$  distribution, as shown in Figure 5.1.

Substituting  $\chi^2$  from (5.4.1) into (5.4.2) and rearranging the terms, we obtain

$$\Pr \left[ (n - 2) \frac{\hat{\sigma}^2}{\chi_{\alpha/2}^2} \leq \sigma^2 \leq (n - 2) \frac{\hat{\sigma}^2}{\chi_{1-\alpha/2}^2} \right] = 1 - \alpha \quad (5.4.3)$$

which gives the  $100(1 - \alpha)\%$  confidence interval for  $\sigma^2$ .

To illustrate, consider this example. From Chapter 3, Section 3.6, we obtain  $\hat{\sigma}^2 = 42.1591$  and  $df = 8$ . If  $\alpha$  is chosen at 5 percent, the chi-square table for 8 df gives the following critical values:  $\chi_{0.025}^2 = 17.5346$ , and  $\chi_{0.975}^2 = 2.1797$ . These values show that the probability of a chi-square value exceeding 17.5346 is 2.5 percent and that of 2.1797 is 97.5 percent. Therefore, the interval between these two values is the 95% confidence interval for  $\chi^2$ , as shown diagrammatically in Figure 5.1. (Note the skewed characteristic of the chi-square distribution.)

<sup>5</sup>For proof, see Robert V. Hogg and Allen T. Craig, *Introduction to Mathematical Statistics*, 2d ed., Macmillan, New York, 1965, p. 144.

Substituting the data of our example into (5.4.3), the reader should verify that the 95% confidence interval for  $\sigma^2$  is as follows:

$$19.2347 \leq \sigma^2 \leq 154.7336 \quad (5.4.4)$$

**The interpretation of this interval is:** If we establish 95% confidence limits on  $\sigma^2$  and if we maintain a priori that these limits will include true  $\sigma^2$ , we shall be right in the long run 95 percent of the time.

## 5.5 HYPOTHESIS TESTING: GENERAL COMMENTS

Having discussed the problem of point and interval estimation, we shall now consider the topic of hypothesis testing. In this section we discuss briefly some general aspects of this topic; **Appendix A** gives some additional details.

The problem of statistical hypothesis testing may be stated simply as follows: *Is a given observation or finding compatible with some stated hypothesis or not?* The word “compatible,” as used here, means “sufficiently” close to the hypothesized value so that we do not reject the stated hypothesis. Thus, if some theory or prior experience leads us to believe that the true slope coefficient  $\beta_2$  of the consumption–income example is unity, is the observed  $\hat{\beta}_2 = 0.5091$  obtained from the sample of Table 3.2 consistent with the stated hypothesis? If it is, we do not reject the hypothesis; otherwise, we may reject it.

In the language of statistics, the stated hypothesis is known as the **null hypothesis** and is denoted by the symbol  $H_0$ . The null hypothesis is usually tested against an **alternative hypothesis** (also known as **maintained hypothesis**) denoted by  $H_1$ , which may state, for example, that true  $\beta_2$  is different from unity. The alternative hypothesis may be **simple** or **composite**.<sup>6</sup> For example,  $H_1: \beta_2 = 1.5$  is a simple hypothesis, but  $H_1: \beta_2 \neq 1.5$  is a composite hypothesis.

The theory of hypothesis testing is concerned with developing rules or procedures for deciding whether to reject or not reject the null hypothesis. There are two *mutually complementary* approaches for devising such rules, namely, **confidence interval** and **test of significance**. Both these approaches predicate that the variable (statistic or estimator) under consideration has some probability distribution and that hypothesis testing involves making statements or assertions about the value(s) of the parameter(s) of such distribution. For example, we know that with the normality assumption  $\hat{\beta}_2$  is normally distributed with mean equal to  $\beta_2$  and variance given by (4.3.5). If we hypothesize that  $\beta_2 = 1$ , we are making an assertion about one

<sup>6</sup>A statistical hypothesis is called a **simple hypothesis** if it specifies the precise value(s) of the parameter(s) of a probability density function; otherwise, it is called a **composite hypothesis**. For example, in the normal pdf  $(1/\sigma\sqrt{2\pi}) \exp\{-\frac{1}{2}[(X-\mu)/\sigma]^2\}$ , if we assert that  $H_1: \mu = 15$  and  $\sigma = 2$ , it is a simple hypothesis; but if  $H_1: \mu = 15$  and  $\sigma > 15$ , it is a composite hypothesis, because the standard deviation does not have a specific value.

of the parameters of the normal distribution, namely, the mean. Most of the statistical hypotheses encountered in this text will be of this type—making assertions about one or more values of the parameters of some assumed probability distribution such as the normal,  $F$ ,  $t$ , or  $\chi^2$ . How this is accomplished is discussed in the following two sections.

**5.6 HYPOTHESIS TESTING:  
THE CONFIDENCE-INTERVAL APPROACH**

**Two-Sided or Two-Tail Test**

To illustrate the confidence-interval approach, once again we revert to the consumption–income example. As we know, the estimated marginal propensity to consume (MPC),  $\hat{\beta}_2$ , is 0.5091. Suppose we postulate that

$$H_0: \beta_2 = 0.3$$

$$H_1: \beta_2 \neq 0.3$$

that is, the true MPC is 0.3 under the null hypothesis but it is less than or greater than 0.3 under the alternative hypothesis. The null hypothesis is a simple hypothesis, whereas the alternative hypothesis is composite; actually it is what is known as a **two-sided hypothesis**. Very often such a two-sided alternative hypothesis reflects the fact that we do not have a strong a priori or theoretical expectation about the direction in which the alternative hypothesis should move from the null hypothesis.

Is the observed  $\hat{\beta}_2$  compatible with  $H_0$ ? To answer this question, let us refer to the confidence interval (5.3.9). We know that in the long run intervals like (0.4268, 0.5914) will contain the true  $\beta_2$  with 95 percent probability. Consequently, in the long run (i.e., repeated sampling) such intervals provide a range or limits within which the true  $\beta_2$  may lie with a confidence coefficient of, say, 95%. Thus, the confidence interval provides a set of plausible null hypotheses. Therefore, if  $\beta_2$  under  $H_0$  falls within the  $100(1 - \alpha)\%$  confidence interval, we do not reject the null hypothesis; if it lies outside the interval, we may reject it.<sup>7</sup> This range is illustrated schematically in Figure 5.2.

**Decision Rule:** Construct a  $100(1 - \alpha)\%$  confidence interval for  $\beta_2$ . If the  $\beta_2$  under  $H_0$  falls within this confidence interval, do not reject  $H_0$ , but if it falls outside this interval, reject  $H_0$ .

Following this rule, for our hypothetical example,  $H_0: \beta_2 = 0.3$  clearly lies outside the 95% confidence interval given in (5.3.9). Therefore, we can reject

<sup>7</sup>Always bear in mind that there is a  $100\alpha$  percent chance that the confidence interval does not contain  $\beta_2$  under  $H_0$  even though the hypothesis is correct. In short, there is a  $100\alpha$  percent chance of committing a **Type I error**. Thus, if  $\alpha = 0.05$ , there is a 5 percent chance that we could reject the null hypothesis even though it is true.



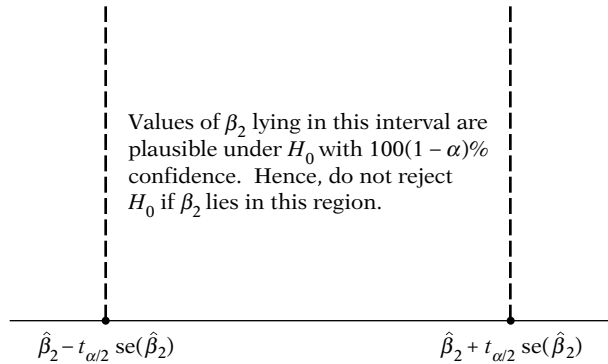


FIGURE 5.2 A  $100(1 - \alpha)\%$  confidence interval for  $\beta_2$ .

the hypothesis that the true MPC is 0.3, with 95% confidence. If the null hypothesis were true, the probability of our obtaining a value of MPC of as much as 0.5091 by sheer chance or fluke is at the most about 5 percent, a small probability.

In statistics, when we reject the null hypothesis, we say that our finding is **statistically significant**. On the other hand, when we do not reject the null hypothesis, we say that our finding is **not statistically significant**.

Some authors use a phrase such as “highly statistically significant.” By this they usually mean that when they reject the null hypothesis, the probability of committing a Type I error (i.e.,  $\alpha$ ) is a small number, usually 1 percent. But as our discussion of the **p value** in Section 5.8 will show, it is better to leave it to the researcher to decide whether a statistical finding is “significant,” “moderately significant,” or “highly significant.”

### One-Sided or One-Tail Test

Sometimes we have a strong a priori or theoretical expectation (or expectations based on some previous empirical work) that the alternative hypothesis is one-sided or unidirectional rather than two-sided, as just discussed. Thus, for our consumption–income example, one could postulate that

$$H_0: \beta_2 \leq 0.3 \quad \text{and} \quad H_1: \beta_2 > 0.3$$

Perhaps economic theory or prior empirical work suggests that the marginal propensity to consume is greater than 0.3. Although the procedure to test this hypothesis can be easily derived from (5.3.5), the actual mechanics are better explained in terms of the test-of-significance approach discussed next.<sup>8</sup>

<sup>8</sup>If you want to use the confidence interval approach, construct a  $(100 - \alpha)\%$  one-sided or one-tail confidence interval for  $\beta_2$ . Why?

## 5.7 HYPOTHESIS TESTING: THE TEST-OF-SIGNIFICANCE APPROACH

### Testing the Significance of Regression Coefficients: The $t$ Test

An *alternative but complementary approach* to the confidence-interval method of testing statistical hypotheses is the **test-of-significance approach** developed along independent lines by R. A. Fisher and jointly by Neyman and Pearson.<sup>9</sup> **Broadly speaking, a test of significance is a procedure by which sample results are used to verify the truth or falsity of a null hypothesis.** The key idea behind tests of significance is that of a **test statistic** (estimator) and the sampling distribution of such a statistic under the null hypothesis. The decision to accept or reject  $H_0$  is made on the basis of the value of the test statistic obtained from the data at hand.

As an illustration, recall that under the normality assumption the variable

$$\begin{aligned} t &= \frac{\hat{\beta}_2 - \beta_2}{\text{se}(\hat{\beta}_2)} \\ &= \frac{(\hat{\beta}_2 - \beta_2)\sqrt{\sum x_i^2}}{\hat{\sigma}} \end{aligned} \quad (5.3.2)$$

follows the  $t$  distribution with  $n - 2$  df. If the value of true  $\beta_2$  is specified under the null hypothesis, the  $t$  value of (5.3.2) can readily be computed from the available sample, and therefore it can serve as a test statistic. And since this test statistic follows the  $t$  distribution, confidence-interval statements such as the following can be made:

$$\Pr \left[ -t_{\alpha/2} \leq \frac{\hat{\beta}_2 - \beta_2^*}{\text{se}(\hat{\beta}_2)} \leq t_{\alpha/2} \right] = 1 - \alpha \quad (5.7.1)$$

where  $\beta_2^*$  is the value of  $\beta_2$  under  $H_0$  and where  $-t_{\alpha/2}$  and  $t_{\alpha/2}$  are the values of  $t$  (the **critical**  $t$  values) obtained from the  $t$  table for  $(\alpha/2)$  level of significance and  $n - 2$  df [cf. (5.3.4)]. The  $t$  table is given in **Appendix D**.

Rearranging (5.7.1), we obtain

$$\Pr [\beta_2^* - t_{\alpha/2} \text{se}(\hat{\beta}_2) \leq \hat{\beta}_2 \leq \beta_2^* + t_{\alpha/2} \text{se}(\hat{\beta}_2)] = 1 - \alpha \quad (5.7.2)$$

which gives the interval in which  $\hat{\beta}_2$  will fall with  $1 - \alpha$  probability, given  $\beta_2 = \beta_2^*$ . In the language of hypothesis testing, the  $100(1 - \alpha)\%$  confidence interval established in (5.7.2) is known as the **region of acceptance** (of

<sup>9</sup>Details may be found in E. L. Lehman, *Testing Statistical Hypotheses*, John Wiley & Sons, New York, 1959.

the null hypothesis) and the *region(s)* outside the confidence interval is (are) called the **region(s) of rejection** (of  $H_0$ ) or the **critical region(s)**. As noted previously, the confidence limits, the endpoints of the confidence interval, are also called **critical values**.

The intimate connection between the confidence-interval and test-of-significance approaches to hypothesis testing can now be seen by comparing (5.3.5) with (5.7.2). In the confidence-interval procedure we try to establish a range or an interval that has a certain probability of including the true but unknown  $\beta_2$ , whereas in the test-of-significance approach we hypothesize some value for  $\beta_2$  and try to see whether the computed  $\hat{\beta}_2$  lies within reasonable (confidence) limits around the hypothesized value.

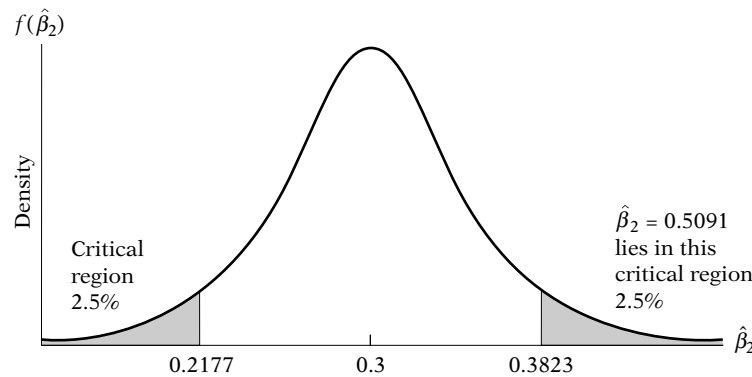
Once again let us revert to our consumption-income example. We know that  $\hat{\beta}_2 = 0.5091$ ,  $se(\hat{\beta}_2) = 0.0357$ , and  $df = 8$ . If we assume  $\alpha = 5$  percent,  $t_{\alpha/2} = 2.306$ . If we let  $H_0: \beta_2 = \beta_2^* = 0.3$  and  $H_1: \beta_2 \neq 0.3$ , (5.7.2) becomes

$$\Pr(0.2177 \leq \hat{\beta}_2 \leq 0.3823) = 0.95 \quad (5.7.3)^{10}$$

as shown diagrammatically in Figure 5.3. Since the observed  $\hat{\beta}_2$  lies in the critical region, we reject the null hypothesis that true  $\beta_2 = 0.3$ .

In practice, there is no need to estimate (5.7.2) explicitly. One can compute the  $t$  value in the middle of the double inequality given by (5.7.1) and see whether it lies between the critical  $t$  values or outside them. For our example,

$$t = \frac{0.5091 - 0.3}{0.0357} = 5.86 \quad (5.7.4)$$



**FIGURE 5.3** The 95% confidence interval for  $\hat{\beta}_2$  under the hypothesis that  $\beta_2 = 0.3$ .

<sup>10</sup>In Sec. 5.2, point 4, it was stated that we *cannot* say that the probability is 95 percent that the fixed interval (0.4268, 0.5914) includes the true  $\beta_2$ . But we can make the probabilistic statement given in (5.7.3) because  $\hat{\beta}_2$ , being an estimator, is a random variable.

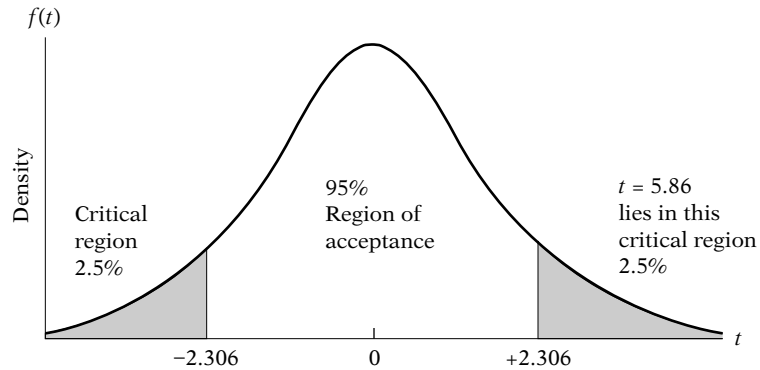


FIGURE 5.4 The 95% confidence interval for  $t(8 \text{ df})$ .

which clearly lies in the critical region of Figure 5.4. The conclusion remains the same; namely, we reject  $H_0$ .

Notice that if the estimated  $\beta_2 (= \hat{\beta}_2)$  is equal to the hypothesized  $\beta_2$ , the  $t$  value in (5.7.4) will be zero. However, as the estimated  $\beta_2$  value departs from the hypothesized  $\beta_2$  value,  $|t|$  (that is, the absolute  $t$  value; *note*:  $t$  can be positive as well as negative) will be increasingly large. *Therefore, a "large"  $|t|$  value will be evidence against the null hypothesis.* Of course, we can always use the  $t$  table to determine whether a particular  $t$  value is large or small; the answer, as we know, depends on the degrees of freedom as well as on the probability of Type I error that we are willing to accept. If you take a look at the  $t$  table given in **Appendix D**, you will observe that for any given value of  $df$  the probability of obtaining an increasingly large  $|t|$  value becomes progressively smaller. Thus, for 20  $df$  the probability of obtaining a  $|t|$  value of 1.725 or greater is 0.10 or 10 percent, but for the same  $df$  the probability of obtaining a  $|t|$  value of 3.552 or greater is only 0.002 or 0.2 percent.

Since we use the  $t$  distribution, the preceding testing procedure is called appropriately the  **$t$  test**. **In the language of significance tests, a statistic is said to be statistically significant if the value of the test statistic lies in the critical region. In this case the null hypothesis is rejected. By the same token, a test is said to be statistically insignificant if the value of the test statistic lies in the acceptance region.** In this situation, the null hypothesis is not rejected. In our example, the  $t$  test is significant and hence we reject the null hypothesis.

Before concluding our discussion of hypothesis testing, note that the testing procedure just outlined is known as a **two-sided, or two-tail**, test-of-significance procedure in that we consider the two extreme tails of the relevant probability distribution, the rejection regions, and reject the null hypothesis if it lies in either tail. But this happens because our  $H_1$  was a

two-sided composite hypothesis;  $\beta_2 \neq 0.3$  means  $\beta_2$  is either greater than or less than 0.3. But suppose prior experience suggests to us that the MPC is expected to be greater than 0.3. In this case we have:  $H_0: \beta_2 \leq 0.3$  and  $H_1: \beta_2 > 0.3$ . Although  $H_1$  is still a composite hypothesis, it is now one-sided. To test this hypothesis, we use the **one-tail test** (the right tail), as shown in Figure 5.5. (See also the discussion in Section 5.6.)

The test procedure is the same as before except that the upper confidence limit or critical value now corresponds to  $t_\alpha = t_{0.05}$ , that is, the 5 percent level. As Figure 5.5 shows, we need not consider the lower tail of the  $t$  distribution in this case. Whether one uses a two- or one-tail test of significance will depend upon how the alternative hypothesis is formulated, which, in turn, may depend upon some a priori considerations or prior empirical experience. (But more on this in Section 5.8.)

We can summarize the  $t$  test of significance approach to hypothesis testing as shown in Table 5.1.

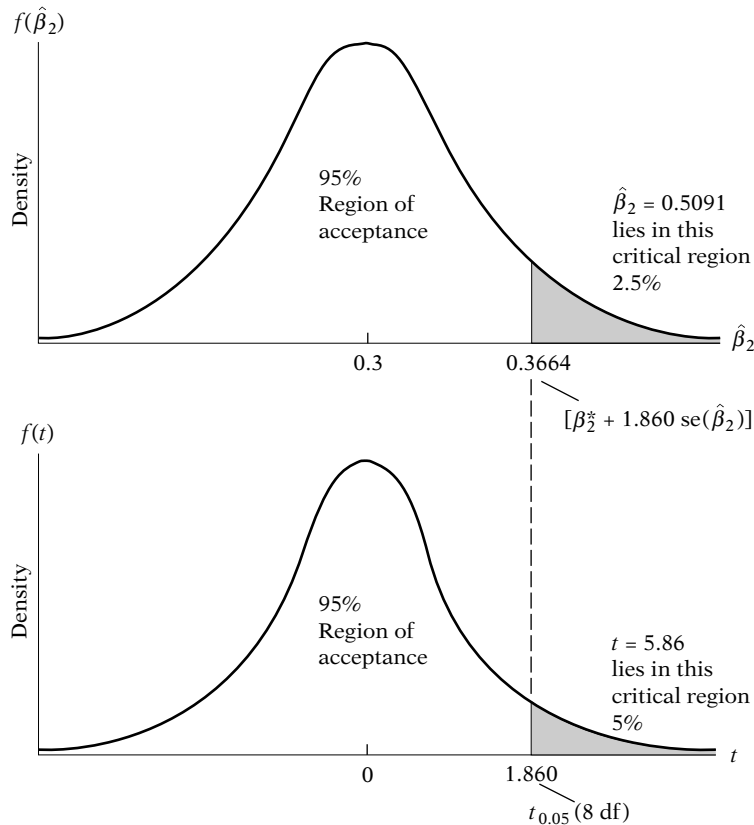


FIGURE 5.5 One-tail test of significance.

**TABLE 5.1** THE  $t$  TEST OF SIGNIFICANCE: DECISION RULES

Type of hypothesis	$H_0$ : the null hypothesis	$H_1$ : the alternative hypothesis	Decision rule: reject $H_0$ if
Two-tail	$\beta_2 = \beta_2^*$	$\beta_2 \neq \beta_2^*$	$ t  > t_{\alpha/2, df}$
Right-tail	$\beta_2 \leq \beta_2^*$	$\beta_2 > \beta_2^*$	$t > t_{\alpha, df}$
Left-tail	$\beta_2 \geq \beta_2^*$	$\beta_2 < \beta_2^*$	$t < -t_{\alpha, df}$

*Notes:*  $\beta_2^*$  is the hypothesized numerical value of  $\beta_2$ .  
 $|t|$  means the absolute value of  $t$ .  
 $t_{\alpha}$  or  $t_{\alpha/2}$  means the critical  $t$  value at the  $\alpha$  or  $\alpha/2$  level of significance.  
df: degrees of freedom,  $(n - 2)$  for the two-variable model,  $(n - 3)$  for the three-variable model, and so on.  
The same procedure holds to test hypotheses about  $\beta_1$ .

### Testing the Significance of $\sigma^2$ : The $\chi^2$ Test

As another illustration of the test-of-significance methodology, consider the following variable:

$$\chi^2 = (n - 2) \frac{\hat{\sigma}^2}{\sigma_0^2} \tag{5.4.1}$$

which, as noted previously, follows the  $\chi^2$  distribution with  $n - 2$  df. For the hypothetical example,  $\hat{\sigma}^2 = 42.1591$  and  $df = 8$ . If we postulate that  $H_0: \sigma^2 = 85$  vs.  $H_1: \sigma^2 \neq 85$ , Eq. (5.4.1) provides the test statistic for  $H_0$ . Substituting the appropriate values in (5.4.1), it can be found that under  $H_0$ ,  $\chi^2 = 3.97$ . If we assume  $\alpha = 5\%$ , the critical  $\chi^2$  values are 2.1797 and 17.5346. Since the computed  $\chi^2$  lies between these limits, the data support the null hypothesis and we do not reject it. (See Figure 5.1.) This test procedure is called the **chi-square test of significance**. The  $\chi^2$  test of significance approach to hypothesis testing is summarized in Table 5.2.

**TABLE 5.2** A SUMMARY OF THE  $\chi^2$  TEST

$H_0$ : the null hypothesis	$H_1$ : the alternative hypothesis	Critical region: reject $H_0$ if
$\sigma^2 = \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$\frac{df(\hat{\sigma}^2)}{\sigma_0^2} > \chi_{\alpha, df}^2$
$\sigma^2 = \sigma_0^2$	$\sigma^2 < \sigma_0^2$	$\frac{df(\hat{\sigma}^2)}{\sigma_0^2} < \chi_{(1-\alpha), df}^2$
$\sigma^2 = \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$	$\frac{df(\hat{\sigma}^2)}{\sigma_0^2} > \chi_{\alpha/2, df}^2$ or $< \chi_{(1-\alpha/2), df}^2$

*Note:*  $\sigma_0^2$  is the value of  $\sigma^2$  under the null hypothesis. The first subscript on  $\chi^2$  in the last column is the level of significance, and the second subscript is the degrees of freedom. These are critical chi-square values. Note that df is  $(n - 2)$  for the two-variable regression model,  $(n - 3)$  for the three-variable regression model, and so on.

## 5.8 HYPOTHESIS TESTING: SOME PRACTICAL ASPECTS

### The Meaning of “Accepting” or “Rejecting” a Hypothesis

If on the basis of a test of significance, say, the  $t$  test, we decide to “accept” the null hypothesis, all we are saying is that on the basis of the sample evidence we have no reason to reject it; we are not saying that the null hypothesis is true beyond any doubt. Why? To answer this, let us revert to our consumption–income example and assume that  $H_0: \beta_2$  (MPC) = 0.50. Now the estimated value of the MPC is  $\hat{\beta}_2 = 0.5091$  with a  $se(\hat{\beta}_2) = 0.0357$ . Then on the basis of the  $t$  test we find that  $t = (0.5091 - 0.50)/0.0357 = 0.25$ , which is insignificant, say, at  $\alpha = 5\%$ . Therefore, we say “accept”  $H_0$ . But now let us assume  $H_0: \beta_2 = 0.48$ . Applying the  $t$  test, we obtain  $t = (0.5091 - 0.48)/0.0357 = 0.82$ , which too is statistically insignificant. So now we say “accept” this  $H_0$ . Which of these two null hypotheses is the “truth”? We do not know. Therefore, in “accepting” a null hypothesis we should always be aware that another null hypothesis may be equally compatible with the data. It is therefore preferable to say that we *may* accept the null hypothesis rather than we (do) accept it. Better still,

... just as a court pronounces a verdict as “not guilty” rather than “innocent,” so the conclusion of a statistical test is “do not reject” rather than “accept.”<sup>11</sup>

### The “Zero” Null Hypothesis and the “2- $t$ ” Rule of Thumb

A null hypothesis that is commonly tested in empirical work is  $H_0: \beta_2 = 0$ , that is, the slope coefficient is zero. This “zero” null hypothesis is a kind of straw man, the objective being to find out whether  $Y$  is related at all to  $X$ , the explanatory variable. If there is no relationship between  $Y$  and  $X$  to begin with, then testing a hypothesis such as  $\beta_2 = 0.3$  or any other value is meaningless.

This null hypothesis can be easily tested by the confidence interval or the  $t$ -test approach discussed in the preceding sections. But very often such formal testing can be shortcut by adopting the “2- $t$ ” rule of significance, which may be stated as

**“2- $t$ ” Rule of Thumb.** If the number of degrees of freedom is 20 or more and if  $\alpha$ , the level of significance, is set at 0.05, then the null hypothesis  $\beta_2 = 0$  can be rejected if the  $t$  value [ $= \hat{\beta}_2/se(\hat{\beta}_2)$ ] computed from (5.3.2) exceeds 2 in absolute value.

The rationale for this rule is not too difficult to grasp. From (5.7.1) we know that we will reject  $H_0: \beta_2 = 0$  if

$$t = \hat{\beta}_2/se(\hat{\beta}_2) > t_{\alpha/2} \quad \text{when } \hat{\beta}_2 > 0$$

<sup>11</sup>Jan Kmenta, *Elements of Econometrics*, Macmillan, New York, 1971, p. 114.

or

$$t = \hat{\beta}_2 / \text{se}(\hat{\beta}_2) < -t_{\alpha/2} \quad \text{when } \hat{\beta}_2 < 0$$

or when

$$|t| = \left| \frac{\hat{\beta}_2}{\text{se}(\hat{\beta}_2)} \right| > t_{\alpha/2} \quad (5.8.1)$$

for the appropriate degrees of freedom.

Now if we examine the  $t$  table given in **Appendix D**, we see that for df of about 20 or more a computed  $t$  value in excess of 2 (in absolute terms), say, 2.1, is statistically significant at the 5 percent level, implying rejection of the null hypothesis. Therefore, if we find that for 20 or more df the computed  $t$  value is, say, 2.5 or 3, we do not even have to refer to the  $t$  table to assess the significance of the estimated slope coefficient. Of course, one can always refer to the  $t$  table to obtain the precise level of significance, and one should always do so when the df are fewer than, say, 20.

In passing, note that if we are testing the one-sided hypothesis  $\beta_2 = 0$  versus  $\beta_2 > 0$  or  $\beta_2 < 0$ , then we should reject the null hypothesis if

$$|t| = \left| \frac{\hat{\beta}_2}{\text{se}(\hat{\beta}_2)} \right| > t_{\alpha} \quad (5.8.2)$$

If we fix  $\alpha$  at 0.05, then from the  $t$  table we observe that for 20 or more df a  $t$  value in excess of 1.73 is statistically significant at the 5 percent level of significance (one-tail). Hence, whenever a  $t$  value exceeds, say, 1.8 (in absolute terms) and the df are 20 or more, one need not consult the  $t$  table for the statistical significance of the observed coefficient. Of course, if we choose  $\alpha$  at 0.01 or any other level, we will have to decide on the appropriate  $t$  value as the benchmark value. But by now the reader should be able to do that.

### Forming the Null and Alternative Hypotheses<sup>12</sup>

Given the null and the alternative hypotheses, testing them for statistical significance should no longer be a mystery. But how does one formulate these hypotheses? There are no hard-and-fast rules. Very often the phenomenon under study will suggest the nature of the null and alternative hypotheses. For example, consider the capital market line (CML) of portfolio theory, which postulates that  $E_i = \beta_1 + \beta_2 \sigma_i$ , where  $E$  = expected return on portfolio and  $\sigma$  = the standard deviation of return, a measure of risk. Since return and risk are expected to be positively related—the higher the risk, the

<sup>12</sup>For an interesting discussion about formulating hypotheses, see J. Bradford De Long and Kevin Lang, "Are All Economic Hypotheses False?" *Journal of Political Economy*, vol. 100, no. 6, 1992, pp. 1257–1272.



higher the return—the natural alternative hypothesis to the null hypothesis that  $\beta_2 = 0$  would be  $\beta_2 > 0$ . That is, one would not choose to consider values of  $\beta_2$  less than zero.

But consider the case of the demand for money. As we shall show later, one of the important determinants of the demand for money is income. Prior studies of the money demand functions have shown that the income elasticity of demand for money (the percent change in the demand for money for a 1 percent change in income) has typically ranged between 0.7 and 1.3. Therefore, in a new study of demand for money, if one postulates that the income-elasticity coefficient  $\beta_2$  is 1, the alternative hypothesis could be that  $\beta_2 \neq 1$ , a two-sided alternative hypothesis.

Thus, theoretical expectations or prior empirical work or both can be relied upon to formulate hypotheses. But no matter how the hypotheses are formed, *it is extremely important that the researcher establish these hypotheses before carrying out the empirical investigation*. Otherwise, he or she will be guilty of circular reasoning or self-fulfilling prophecies. That is, if one were to formulate hypotheses after examining the empirical results, there may be the temptation to form hypotheses that justify one's results. Such a practice should be avoided at all costs, at least for the sake of scientific objectivity. Keep in mind the Stigler quotation given at the beginning of this chapter!

### Choosing $\alpha$ , the Level of Significance

It should be clear from the discussion so far that whether we reject or do not reject the null hypothesis depends critically on  $\alpha$ , the level of significance or the *probability of committing a Type I error*—the probability of rejecting the true hypothesis. In **Appendix A** we discuss fully the nature of a Type I error, its relationship to a *Type II error* (the probability of accepting the false hypothesis) and why classical statistics generally concentrates on a Type I error. But even then, why is  $\alpha$  commonly fixed at the 1, 5, or at the most 10 percent levels? As a matter of fact, there is nothing sacrosanct about these values; any other values will do just as well.

In an introductory book like this it is not possible to discuss in depth why one chooses the 1, 5, or 10 percent levels of significance, for that will take us into the field of statistical decision making, a discipline unto itself. A brief summary, however, can be offered. As we discuss in **Appendix A**, for a given sample size, if we try to reduce a *Type I error*, a *Type II error* increases, and vice versa. That is, given the sample size, if we try to reduce the probability of rejecting the true hypothesis, we at the same time increase the probability of accepting the false hypothesis. So there is a tradeoff involved between these two types of errors, given the sample size. Now the only way we can decide about the tradeoff is to find out the relative costs of the two types of errors. Then,

If the error of rejecting the null hypothesis which is in fact true (Error Type I) is costly relative to the error of not rejecting the null hypothesis which is in fact

false (Error Type II), it will be rational to set the probability of the first kind of error low. If, on the other hand, the cost of making Error Type I is low relative to the cost of making Error Type II, it will pay to make the probability of the first kind of error high (thus making the probability of the second type of error low).<sup>13</sup>

Of course, the rub is that we rarely know the costs of making the two types of errors. Thus, applied econometricians generally follow the practice of setting the value of  $\alpha$  at a 1 or a 5 or at most a 10 percent level and choose a test statistic that would make the probability of committing a Type II error as small as possible. Since one minus the probability of committing a Type II error is known as the **power of the test**, this procedure amounts to maximizing the power of the test. (See **Appendix A** for a discussion of the power of a test.)

But all this problem with choosing the appropriate value of  $\alpha$  can be avoided if we use what is known as the ***p* value** of the test statistic, which is discussed next.

### The Exact Level of Significance: The *p* Value

As just noted, the Achilles heel of the classical approach to hypothesis testing is its arbitrariness in selecting  $\alpha$ . Once a test statistic (e.g., the *t* statistic) is obtained in a given example, why not simply go to the appropriate statistical table and find out the actual probability of obtaining a value of the test statistic as much as or greater than that obtained in the example? This probability is called the ***p* value** (i.e., **probability value**), also known as the **observed or exact level of significance** or the **exact probability of committing a Type I error**. More technically, the *p* value is defined as **the lowest significance level at which a null hypothesis can be rejected**.

To illustrate, let us return to our consumption–income example. Given the null hypothesis that the true MPC is 0.3, we obtained a *t* value of 5.86 in (5.7.4). What is the *p* value of obtaining a *t* value of as much as or greater than 5.86? Looking up the *t* table given in **Appendix D**, we observe that for 8 df the probability of obtaining such a *t* value must be much smaller than 0.001 (one-tail) or 0.002 (two-tail). By using the computer, it can be shown that the probability of obtaining a *t* value of 5.86 or greater (for 8 df) is about 0.000189.<sup>14</sup> This is the *p* value of the observed *t* statistic. This observed, or exact, level of significance of the *t* statistic is much smaller than the conventionally, and arbitrarily, fixed level of significance, such as 1, 5, or 10 percent. As a matter of fact, if we were to use the *p* value just computed,

<sup>13</sup>Jan Kmenta, *Elements of Econometrics*, Macmillan, New York, 1971, pp. 126–127.

<sup>14</sup>One can obtain the *p* value using electronic statistical tables to several decimal places. Unfortunately, the conventional statistical tables, for lack of space, cannot be that refined. Most statistical packages now routinely print out the *p* values.

and reject the null hypothesis that the true MPC is 0.3, the probability of our committing a Type I error is only about 0.02 percent, that is, only about 2 in 10,000!

As we noted earlier, if the data do not support the null hypothesis,  $|t|$  obtained under the null hypothesis will be “large” and therefore the  $p$  value of obtaining such a  $|t|$  value will be “small.” In other words, for a given sample size, as  $|t|$  increases, the  $p$  value decreases, and one can therefore reject the null hypothesis with increasing confidence.

What is the relationship of the  $p$  value to the level of significance  $\alpha$ ? If we make the habit of fixing  $\alpha$  equal to the  $p$  value of a test statistic (e.g., the  $t$  statistic), then there is no conflict between the two values. To put it differently, **it is better to give up fixing  $\alpha$  arbitrarily at some level and simply choose the  $p$  value of the test statistic.** It is preferable to leave it to the reader to decide whether to reject the null hypothesis at the given  $p$  value. If in an application the  $p$  value of a test statistic happens to be, say, 0.145, or 14.5 percent, and if the reader wants to reject the null hypothesis at this (exact) level of significance, so be it. Nothing is wrong with taking a chance of being wrong 14.5 percent of the time if you reject the true null hypothesis. Similarly, as in our consumption–income example, there is nothing wrong if the researcher wants to choose a  $p$  value of about 0.02 percent and not take a chance of being wrong more than 2 out of 10,000 times. After all, some investigators may be risk-lovers and some risk-aversers!

*In the rest of this text, we will generally quote the  $p$  value of a given test statistic.* Some readers may want to fix  $\alpha$  at some level and reject the null hypothesis if the  $p$  value is less than  $\alpha$ . That is their choice.

### Statistical Significance versus Practical Significance

Let us revert to our consumption–income example and now hypothesize that the true MPC is 0.61 ( $H_0: \beta_2 = 0.61$ ). On the basis of our sample result of  $\hat{\beta}_2 = 0.5091$ , we obtained the interval (0.4268, 0.5914) with 95 percent confidence. Since this interval does not include 0.61, we can, with 95 percent confidence, say that our estimate is statistically significant, that is, significantly different from 0.61.

But what is the practical or substantive significance of our finding? That is, what difference does it make if we take the MPC to be 0.61 rather than 0.5091? Is the 0.1009 difference between the two MPCs that important practically?

The answer to this question depends on what we really do with these estimates. For example, from macroeconomics we know that the income multiplier is  $1/(1 - \text{MPC})$ . Thus, if MPC is 0.5091, the multiplier is 2.04, but it is 2.56 if MPC is equal to 0.61. That is, if the government were to increase its expenditure by \$1 to lift the economy out of a recession, income will eventually increase by \$2.04 if the MPC is 0.5091 but by \$2.56 if the MPC is 0.61. And that difference could very well be crucial to resuscitating the economy.

*The point of all this discussion is that one should not confuse statistical significance with practical, or economic, significance.* As Goldberger notes:

When a null, say,  $\beta_j = 1$ , is specified, the likely intent is that  $\beta_j$  is close to 1, so close that for all practical purposes it may be treated *as if it were* 1. But whether 1.1 is “practically the same as” 1.0 is a matter of economics, not of statistics. One cannot resolve the matter by relying on a hypothesis test, because the test statistic  $[t = ](b_j - 1)/\hat{\sigma}_{b_j}$  measures the estimated coefficient in standard error units, which are not meaningful units in which to measure the economic parameter  $\beta_j - 1$ . It may be a good idea to reserve the term “significance” for the statistical concept, adopting “substantial” for the economic concept.<sup>15</sup>

The point made by Goldberger is important. As sample size becomes very large, issues of statistical significance become much less important but issues of economic significance become critical. Indeed, since with very large samples almost any null hypothesis will be rejected, there may be studies in which the magnitude of the point estimates may be the only issue.

### The Choice between Confidence-Interval and Test-of-Significance Approaches to Hypothesis Testing

In most applied economic analyses, the null hypothesis is set up as a straw man and the objective of the empirical work is to knock it down, that is, reject the null hypothesis. Thus, in our consumption-income example, the null hypothesis that the MPC  $\beta_2 = 0$  is patently absurd, but we often use it to dramatize the empirical results. Apparently editors of reputed journals do not find it exciting to publish an empirical piece that does not reject the null hypothesis. Somehow the finding that the MPC is statistically different from zero is more newsworthy than the finding that it is equal to, say, 0.7!

Thus, J. Bradford De Long and Kevin Lang argue that it is better for economists

... to concentrate on the magnitudes of coefficients and to report confidence levels and not significance tests. If all or almost all null hypotheses are false, there is little point in concentrating on whether or not an estimate is indistinguishable from its predicted value under the null. Instead, we wish to cast light on what models are good approximations, which requires that we know ranges of parameter values that are excluded by empirical estimates.<sup>16</sup>

In short, these authors prefer the confidence-interval approach to the test-of-significance approach. The reader may want to keep this advice in mind.<sup>17</sup>

<sup>15</sup>Arthur S. Goldberger, *A Course in Econometrics*, Harvard University Press, Cambridge, Massachusetts, 1991, p. 240. Note  $b_j$  is the OLS estimator of  $\beta_j$  and  $\hat{\sigma}_{b_j}$  is its standard error. For a corroborating view, see D. N. McCloskey, “The Loss Function Has Been Mislaid: The Rhetoric of Significance Tests,” *American Economic Review*, vol. 75, 1985, pp. 201–205. See also D. N. McCloskey and S. T. Ziliak, “The Standard Error of Regression,” *Journal of Economic Literature*, vol. 37, 1996, pp. 97–114.

<sup>16</sup>See their article cited in footnote 12, p. 1271.

<sup>17</sup>For a somewhat different perspective, see Carter Hill, William Griffiths, and George Judge, *Undergraduate Econometrics*, Wiley & Sons, New York, 2001, p. 108.

### 5.9 REGRESSION ANALYSIS AND ANALYSIS OF VARIANCE

In this section we study regression analysis from the point of view of the analysis of variance and introduce the reader to an illuminating and complementary way of looking at the statistical inference problem.

In Chapter 3, Section 3.5, we developed the following identity:

$$\sum y_i^2 = \sum \hat{y}_i^2 + \sum \hat{u}_i^2 = \hat{\beta}_2^2 \sum x_i^2 + \sum \hat{u}_i^2 \quad (3.5.2)$$

that is,  $TSS = ESS + RSS$ , which decomposed the total sum of squares (TSS) into two components: explained sum of squares (ESS) and residual sum of squares (RSS). A study of these components of TSS is known as the **analysis of variance** (ANOVA) from the regression viewpoint.

Associated with any sum of squares is its df, the number of independent observations on which it is based. TSS has  $n - 1$  df because we lose 1 df in computing the sample mean  $\bar{Y}$ . RSS has  $n - 2$  df. (Why?) (*Note:* This is true only for the two-variable regression model with the intercept  $\beta_1$  present.) ESS has 1 df (again true of the two-variable case only), which follows from the fact that  $ESS = \hat{\beta}_2^2 \sum x_i^2$  is a function of  $\hat{\beta}_2$  only, since  $\sum x_i^2$  is known.

Let us arrange the various sums of squares and their associated df in Table 5.3, which is the standard form of the AOV table, sometimes called the **ANOVA table**. Given the entries of Table 5.3, we now consider the following variable:

$$\begin{aligned} F &= \frac{\text{MSS of ESS}}{\text{MSS of RSS}} \\ &= \frac{\hat{\beta}_2^2 \sum x_i^2}{\sum \hat{u}_i^2 / (n - 2)} \\ &= \frac{\hat{\beta}_2^2 \sum x_i^2}{\hat{\sigma}^2} \end{aligned} \quad (5.9.1)$$

If we assume that the disturbances  $u_i$  are normally distributed, which we do under the CNLRM, and if the null hypothesis ( $H_0$ ) is that  $\beta_2 = 0$ , then it can be shown that the  $F$  variable of (5.9.1) follows the  $F$  distribution with

**TABLE 5.3** ANOVA TABLE FOR THE TWO-VARIABLE REGRESSION MODEL

Source of variation	SS*	df	MSS†
Due to regression (ESS)	$\sum \hat{y}_i^2 = \hat{\beta}_2^2 \sum x_i^2$	1	$\hat{\beta}_2^2 \sum x_i^2$
Due to residuals (RSS)	$\sum \hat{u}_i^2$	$n - 2$	$\frac{\sum \hat{u}_i^2}{n - 2} = \hat{\sigma}^2$
TSS	$\sum y_i^2$	$n - 1$	

\*SS means sum of squares.

†Mean sum of squares, which is obtained by dividing SS by their df.

1 df in the numerator and  $(n - 2)$  df in the denominator. (See Appendix 5A, Section 5A.3, for the proof. The general properties of the  $F$  distribution are discussed in **Appendix A**.)

What use can be made of the preceding  $F$  ratio? It can be shown<sup>18</sup> that

$$E\left(\hat{\beta}_2^2 \sum x_i^2\right) = \sigma^2 + \beta_2^2 \sum x_i^2 \tag{5.9.2}$$

and

$$E\frac{\sum \hat{u}_i^2}{n - 2} = E(\hat{\sigma}^2) = \sigma^2 \tag{5.9.3}$$

(Note that  $\beta_2$  and  $\sigma^2$  appearing on the right sides of these equations are the true parameters.) Therefore, if  $\beta_2$  is in fact zero, Eqs. (5.9.2) and (5.9.3) both provide us with identical estimates of true  $\sigma^2$ . In this situation, the explanatory variable  $X$  has no linear influence on  $Y$  whatsoever and the entire variation in  $Y$  is explained by the random disturbances  $u_i$ . If, on the other hand,  $\beta_2$  is not zero, (5.9.2) and (5.9.3) will be different and part of the variation in  $Y$  will be ascribable to  $X$ . Therefore, the  $F$  ratio of (5.9.1) provides a test of the null hypothesis  $H_0: \beta_2 = 0$ . Since all the quantities entering into this equation can be obtained from the available sample, this  $F$  ratio provides a test statistic to test the null hypothesis that true  $\beta_2$  is zero. All that needs to be done is to compute the  $F$  ratio and compare it with the critical  $F$  value obtained from the  $F$  tables at the chosen level of significance, or obtain the  **$p$  value** of the computed  $F$  statistic.

To illustrate, let us continue with our consumption-income example. The ANOVA table for this example is as shown in Table 5.4. The computed  $F$  value is seen to be 202.87. The  $p$  value of this  $F$  statistic corresponding to 1 and 8 df cannot be obtained from the  $F$  table given in **Appendix D**, but by using electronic statistical tables it can be shown that the  $p$  value is 0.0000001, an extremely small probability indeed. If you decide to choose the level-of-significance approach to hypothesis testing and fix  $\alpha$  at 0.01, or a 1 percent level, you can see that the computed  $F$  of 202.87 is obviously significant at this level. Therefore, if we reject the null hypothesis that  $\beta_2 = 0$ , the probability of committing a Type I error is very small. For all practical

**TABLE 5.4** ANOVA TABLE FOR THE CONSUMPTION-INCOME EXAMPLE

Source of variation	SS	df	MSS	
Due to regression (ESS)	8552.73	1	8552.73	$F = \frac{8552.73}{42.159}$
Due to residuals (RSS)	337.27	8	42.159	$= 202.87$
TSS	8890.00	9		

<sup>18</sup>For proof, see K. A. Brownlee, *Statistical Theory and Methodology in Science and Engineering*, John Wiley & Sons, New York, 1960, pp. 278–280.



purposes, our sample could not have come from a population with zero  $\beta_2$  value and we can conclude with great confidence that  $X$ , income, does affect  $Y$ , consumption expenditure.

Refer to Theorem 5.7 of Appendix 5A.1, which states that the square of the  $t$  value with  $k$  df is an  $F$  value with 1 df in the numerator and  $k$  df in the denominator. For our consumption–income example, if we assume  $H_0: \beta_2 = 0$ , then from (5.3.2) it can be easily verified that the estimated  $t$  value is 14.26. This  $t$  value has 8 df. Under the same null hypothesis, the  $F$  value was 202.87 with 1 and 8 df. Hence  $(14.24)^2 = F$  value, except for the rounding errors.

Thus, the  $t$  and the  $F$  tests provide us with two alternative but complementary ways of testing the null hypothesis that  $\beta_2 = 0$ . If this is the case, why not just rely on the  $t$  test and not worry about the  $F$  test and the accompanying analysis of variance? For the two-variable model there really is no need to resort to the  $F$  test. But when we consider the topic of multiple regression we will see that the  $F$  test has several interesting applications that make it a very useful and powerful method of testing statistical hypotheses.

### 5.10 APPLICATION OF REGRESSION ANALYSIS: THE PROBLEM OF PREDICTION

On the basis of the sample data of Table 3.2 we obtained the following sample regression:

$$\hat{Y}_i = 24.4545 + 0.5091X_i \quad (3.6.2)$$

where  $\hat{Y}_i$  is the estimator of true  $E(Y_i)$  corresponding to given  $X$ . What use can be made of this **historical regression**? One use is to “predict” or “forecast” the future consumption expenditure  $Y$  corresponding to some given level of income  $X$ . Now there are two kinds of predictions: (1) prediction of the conditional mean value of  $Y$  corresponding to a chosen  $X$ , say,  $X_0$ , that is the point on the population regression line itself (see Figure 2.2), and (2) prediction of an individual  $Y$  value corresponding to  $X_0$ . We shall call these two predictions the **mean prediction** and **individual prediction**.

#### Mean Prediction<sup>19</sup>

To fix the ideas, assume that  $X_0 = 100$  and we want to predict  $E(Y | X_0 = 100)$ . Now it can be shown that the historical regression (3.6.2) provides the point estimate of this mean prediction as follows:

$$\begin{aligned} \hat{Y}_0 &= \hat{\beta}_1 + \hat{\beta}_2 X_0 \\ &= 24.4545 + 0.5091(100) \\ &= 75.3645 \end{aligned} \quad (5.10.1)$$

<sup>19</sup>For the proofs of the various statements made, see App. 5A, Sec. 5A.4.

where  $\hat{Y}_0 =$  estimator of  $E(Y | X_0)$ . It can be proved that this point predictor is a best linear unbiased estimator (BLUE).

Since  $\hat{Y}_0$  is an estimator, it is likely to be different from its true value. The difference between the two values will give some idea about the prediction or forecast error. To assess this error, we need to find out the sampling distribution of  $\hat{Y}_0$ . It is shown in Appendix 5A, Section 5A.4, that  $\hat{Y}_0$  in Eq. (5.10.1) is normally distributed with mean  $(\beta_1 + \beta_2 X_0)$  and the variance is given by the following formula:

$$\text{var}(\hat{Y}_0) = \sigma^2 \left[ \frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\sum x_i^2} \right] \quad (5.10.2)$$

By replacing the unknown  $\sigma^2$  by its unbiased estimator  $\hat{\sigma}^2$ , we see that the variable

$$t = \frac{\hat{Y}_0 - (\beta_1 + \beta_2 X_0)}{\text{se}(\hat{Y}_0)} \quad (5.10.3)$$

follows the  $t$  distribution with  $n - 2$  df. The  $t$  distribution can therefore be used to derive confidence intervals for the true  $E(Y_0 | X_0)$  and test hypotheses about it in the usual manner, namely,

$$\Pr[\hat{\beta}_1 + \hat{\beta}_2 X_0 - t_{\alpha/2} \text{se}(\hat{Y}_0) \leq \beta_1 + \beta_2 X_0 \leq \hat{\beta}_1 + \hat{\beta}_2 X_0 + t_{\alpha/2} \text{se}(\hat{Y}_0)] = 1 - \alpha \quad (5.10.4)$$

where  $\text{se}(\hat{Y}_0)$  is obtained from (5.10.2).

For our data (see Table 3.3),

$$\begin{aligned} \text{var}(\hat{Y}_0) &= 42.159 \left[ \frac{1}{10} + \frac{(100 - 170)^2}{33,000} \right] \\ &= 10.4759 \end{aligned}$$

and

$$\text{se}(\hat{Y}_0) = 3.2366$$

Therefore, the 95% confidence interval for true  $E(Y | X_0) = \beta_1 + \beta_2 X_0$  is given by

$$75.3645 - 2.306(3.2366) \leq E(Y_0 | X = 100) \leq 75.3645 + 2.306(3.2366)$$

that is,

$$67.9010 \leq E(Y | X = 100) \leq 82.8381 \quad (5.10.5)$$



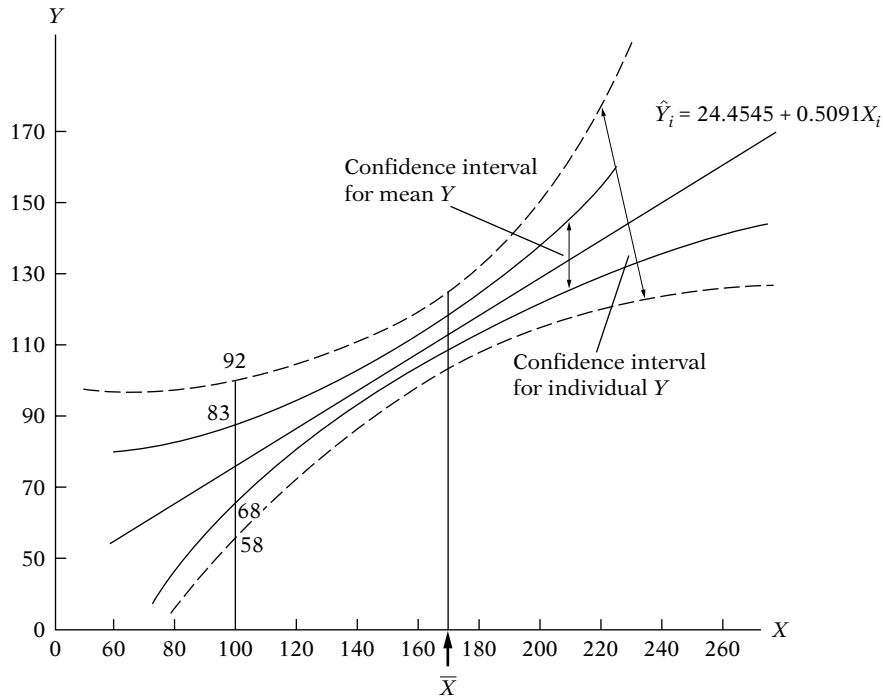


FIGURE 5.6 Confidence intervals (bands) for mean  $Y$  and individual  $Y$  values.

Thus, given  $X_0 = 100$ , in repeated sampling, 95 out of 100 intervals like (5.10.5) will include the true mean value; the single best estimate of the true mean value is of course the point estimate 75.3645.

If we obtain 95% confidence intervals like (5.10.5) for each of the  $X$  values given in Table 3.2, we obtain what is known as the **confidence interval**, or **confidence band**, for the population regression function, which is shown in Figure 5.6.

### Individual Prediction

If our interest lies in predicting an individual  $Y$  value,  $Y_0$ , corresponding to a given  $X$  value, say,  $X_0$ , then, as shown in Appendix 5, Section 5A.3, a best linear unbiased estimator of  $Y_0$  is also given by (5.10.1), but its variance is as follows:

$$\text{var}(Y_0 - \hat{Y}_0) = E[Y_0 - \hat{Y}_0]^2 = \sigma^2 \left[ 1 + \frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\sum x_i^2} \right] \quad (5.10.6)$$

It can be shown further that  $Y_0$  also follows the normal distribution with mean and variance given by (5.10.1) and (5.10.6), respectively. Substituting  $\hat{\sigma}^2$

for the unknown  $\sigma^2$ , it follows that

$$t = \frac{Y_0 - \hat{Y}_0}{\text{se}(Y_0 - \hat{Y}_0)}$$

also follows the  $t$  distribution. Therefore, the  $t$  distribution can be used to draw inferences about the true  $Y_0$ . Continuing with our consumption–income example, we see that the point prediction of  $Y_0$  is 75.3645, the same as that of  $\hat{Y}_0$ , and its variance is 52.6349 (the reader should verify this calculation). Therefore, the 95% confidence interval for  $Y_0$  corresponding to  $X_0 = 100$  is seen to be

$$(58.6345 \leq Y_0 | X_0 = 100 \leq 92.0945) \quad (5.10.7)$$

Comparing this interval with (5.10.5), we see that the confidence interval for individual  $Y_0$  is wider than that for the mean value of  $Y_0$ . (Why?) Computing confidence intervals like (5.10.7) conditional upon the  $X$  values given in Table 3.2, we obtain the 95% confidence band for the individual  $Y$  values corresponding to these  $X$  values. This confidence band along with the confidence band for  $\hat{Y}_0$  associated with the same  $X$ 's is shown in Figure 5.6.

Notice an important feature of the confidence bands shown in Figure 5.6. The width of these bands is smallest when  $X_0 = \bar{X}$ . (Why?) However, the width widens sharply as  $X_0$  moves away from  $\bar{X}$ . (Why?) This change would suggest that the predictive ability of the *historical* sample regression line falls markedly as  $X_0$  departs progressively from  $\bar{X}$ . **Therefore, one should exercise great caution in “extrapolating” the historical regression line to predict  $E(Y | X_0)$  or  $Y_0$  associated with a given  $X_0$  that is far removed from the sample mean  $\bar{X}$ .**

### 5.11 REPORTING THE RESULTS OF REGRESSION ANALYSIS

There are various ways of reporting the results of regression analysis, but in this text we shall use the following format, employing the consumption–income example of Chapter 3 as an illustration:

$$\begin{aligned}
 \hat{Y}_i &= 24.4545 & + & & 0.5091X_i \\
 \text{se} &= (6.4138) & & & (0.0357) & & r^2 = 0.9621 \\
 t &= (3.8128) & & & (14.2605) & & \text{df} = 8 \\
 p &= (0.002571) & & & (0.000000289) & & F_{1,8} = 202.87
 \end{aligned} \quad (5.11.1)$$

In Eq. (5.11.1) the figures in the first set of parentheses are the estimated standard errors of the regression coefficients, the figures in the second set are estimated  $t$  values computed from (5.3.2) under the null hypothesis that

the true population value of each regression coefficient individually is zero (e.g.,  $3.8128 = 24.4545 \div 6.4138$ ), and the figures in the third set are the estimated  $p$  values. Thus, for 8 df the probability of obtaining a  $t$  value of 3.8128 or greater is 0.0026 and the probability of obtaining a  $t$  value of 14.2605 or larger is about 0.0000003.

By presenting the  $p$  values of the estimated  $t$  coefficients, we can see at once the exact level of significance of each estimated  $t$  value. Thus, under the null hypothesis that the true population intercept value is zero, the exact probability (i.e., the  $p$  value) of obtaining a  $t$  value of 3.8128 or greater is only about 0.0026. Therefore, if we reject this null hypothesis, the probability of our committing a Type I error is about 26 in 10,000, a very small probability indeed. For all practical purposes we can say that the true population intercept is different from zero. Likewise, the  $p$  value of the estimated slope coefficient is zero for all practical purposes. If the true MPC were in fact zero, our chances of obtaining an MPC of 0.5091 would be practically zero. Hence we can reject the null hypothesis that the true MPC is zero.

Earlier we showed the intimate connection between the  $F$  and  $t$  statistics, namely,  $F_{1,k} = t_k^2$ . Under the null hypothesis that the true  $\beta_2 = 0$ , (5.11.1) shows that the  $F$  value is 202.87 (for 1 numerator and 8 denominator df) and the  $t$  value is about 14.24 (8 df); as expected, the former value is the square of the latter value, except for the roundoff errors. The ANOVA table for this problem has already been discussed.

## 5.12 EVALUATING THE RESULTS OF REGRESSION ANALYSIS

In Figure I.4 of the Introduction we sketched the anatomy of econometric modeling. Now that we have presented the results of regression analysis of our consumption–income example in (5.11.1), we would like to question the adequacy of the fitted model. How “good” is the fitted model? We need some criteria with which to answer this question.

First, are the signs of the estimated coefficients in accordance with theoretical or prior expectations? A priori,  $\beta_2$ , the marginal propensity to consume (MPC) in the consumption function, should be positive. In the present example it is. Second, if theory says that the relationship should be not only positive but also statistically significant, is this the case in the present application? As we discussed in Section 5.11, the MPC is not only positive but also statistically significantly different from zero; the  $p$  value of the estimated  $t$  value is extremely small. The same comments apply about the intercept coefficient. Third, how well does the regression model explain variation in the consumption expenditure? One can use  $r^2$  to answer this question. In the present example  $r^2$  is about 0.96, which is a very high value considering that  $r^2$  can be at most 1.

Thus, the model we have chosen for explaining consumption expenditure behavior seems quite good. But before we sign off, we would like to find out

whether our model satisfies the assumptions of CNLRM. We will not look at the various assumptions now because the model is patently so simple. But there is one assumption that we would like to check, namely, the normality of the disturbance term,  $u_i$ . Recall that the  $t$  and  $F$  tests used before require that the error term follow the normal distribution. Otherwise, the testing procedure will not be valid in small, or finite, samples.

### Normality Tests

Although several tests of normality are discussed in the literature, we will consider just three: (1) histogram of residuals; (2) normal probability plot (NPP), a graphical device; and (3) the **Jarque–Bera** test.

**Histogram of Residuals.** A histogram of residuals is a simple graphic device that is used to learn something about the shape of the PDF of a random variable. On the horizontal axis, we divide the values of the variable of interest (e.g., OLS residuals) into suitable intervals, and in each class interval we erect rectangles equal in height to the number of observations (i.e., frequency) in that class interval. If you mentally superimpose the bell-shaped normal distribution curve on the histogram, you will get some idea as to whether normal (PDF) approximation may be appropriate. A concrete example is given in Section 5.13 (see Figure 5.8). It is always a good practice to plot the histogram of the residuals as a rough and ready method of testing for the normality assumption.

**Normal Probability Plot.** A comparatively simple graphical device to study the shape of the probability density function (PDF) of a random variable is the **normal probability plot (NPP)** which makes use of *normal probability paper*, a specially designed graph paper. On the horizontal, or  $x$ , axis, we plot values of the variable of interest (say, OLS residuals,  $\hat{u}_i$ ), and on the vertical, or  $y$ , axis, we show the expected value of this variable if it were normally distributed. Therefore, if the variable is in fact from the normal population, the NPP will be approximately a straight line. The NPP of the residuals from our consumption–income regression is shown in Figure 5.7, which is obtained from the MINITAB software package, version 13. As noted earlier, if the fitted line in the NPP is approximately a straight line, one can conclude that the variable of interest is normally distributed. In Figure 5.7, we see that residuals from our illustrative example are approximately normally distributed, because a straight line seems to fit the data reasonably well.

MINITAB also produces the **Anderson–Darling normality test**, known as the  **$A^2$  statistic**. The underlying null hypothesis is that the variable under consideration is normally distributed. As Figure 5.7 shows, for our example, the computed  $A^2$  statistic is 0.394. The  $p$  value of obtaining such a value of  $A^2$  is 0.305, which is reasonably high. Therefore, we do not reject the

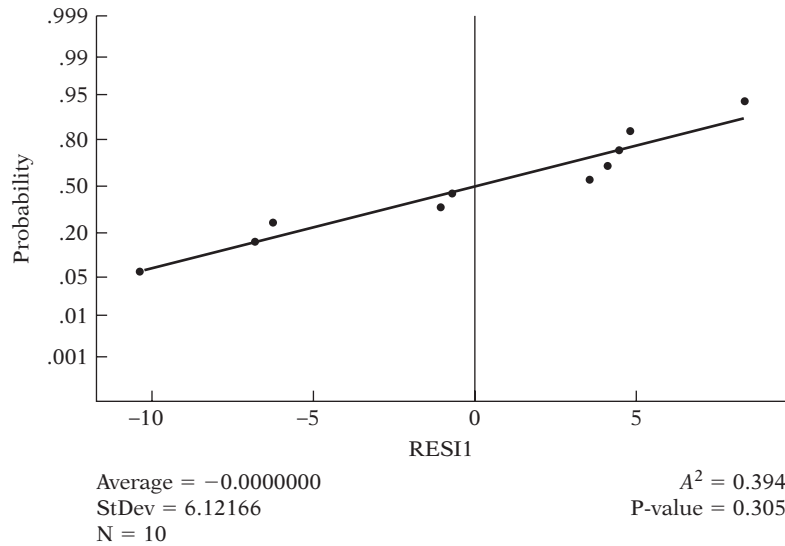


FIGURE 5.7 Residuals from consumption–income regression.

hypothesis that the residuals from our consumption–income example are normally distributed. Incidentally, Figure 5.7 shows the parameters of the (normal) distribution, the mean is approximately 0 and the standard deviation is about 6.12.

**Jarque–Bera (JB) Test of Normality.**<sup>20</sup> The JB test of normality is an *asymptotic*, or large-sample, test. It is also based on the OLS residuals. This test first computes the **skewness** and **kurtosis** (discussed in **Appendix A**) measures of the OLS residuals and uses the following test statistic:

$$JB = n \left[ \frac{S^2}{6} + \frac{(K - 3)^2}{24} \right] \quad (5.12.1)$$

where  $n$  = sample size,  $S$  = skewness coefficient, and  $K$  = kurtosis coefficient. For a normally distributed variable,  $S = 0$  and  $K = 3$ . Therefore, the JB test of normality is a test of the joint hypothesis that  $S$  and  $K$  are 0 and 3, respectively. In that case the value of the JB statistic is expected to be 0.

Under the null hypothesis that the residuals are normally distributed, Jarque and Bera showed that *asymptotically (i.e., in large samples) the JB statistic given in (5.12.1) follows the chi-square distribution with 2 df*. If the computed  $p$  value of the JB statistic in an application is sufficiently low, which will happen if the value of the statistic is very different from 0, one can reject the hypothesis that the residuals are normally distributed. But if

<sup>20</sup>See C. M. Jarque and A. K. Bera, “A Test for Normality of Observations and Regression Residuals,” *International Statistical Review*, vol. 55, 1987, pp. 163–172.

the  $p$  value is reasonably high, which will happen if the value of the statistic is close to zero, we do not reject the normality assumption.

The sample size in our consumption-income example is rather small. Hence, strictly speaking one should not use the JB statistic. If we mechanically apply the JB formula to our example, the JB statistic turns out to be 0.7769. The  $p$  value of obtaining such a value from the chi-square distribution with 2 df is about 0.68, which is quite high. In other words, we may not reject the normality assumption for our example. Of course, bear in mind the warning about the sample size.

### Other Tests of Model Adequacy

Remember that the CNLRM makes many more assumptions than the normality of the error term. As we examine econometric theory further, we will consider several tests of model adequacy (see Chapter 13). Until then, keep in mind that our regression modeling is based on several simplifying assumptions that may not hold in each and every case.

#### A CONCLUDING EXAMPLE

Let us return to Example 3.2 about food expenditure in India. Using the data given in (3.7.2) and adopting the format of (5.11.1), we obtain the following expenditure equation:

$$\begin{aligned} \widehat{\text{FoodExp}}_i &= 94.2087 + 0.4368 \text{ TotalExp}_i \\ \text{se} &= (50.8563) \quad (0.0783) \\ t &= (1.8524) \quad (5.5770) \\ p &= (0.0695) \quad (0.0000)^* \\ r^2 &= 0.3698; \quad \text{df} = 53 \\ F_{1,53} &= 31.1034 \quad (p \text{ value} = 0.0000)^* \end{aligned} \tag{5.12.2}$$

where \* denotes extremely small.

First, let us interpret this regression. As expected, there is a positive relationship between expenditure on food and total expenditure. If total expenditure went up by a rupee, on average, expenditure on food increased by about 44 paise. If total expenditure were zero, the average expenditure on food would be about 94 rupees. Of course, this mechanical interpretation of the intercept may not make much economic sense. The  $r^2$  value of about 0.37 means that 37 percent of the variation in food expenditure is explained by total expenditure, a proxy for income.

Suppose we want to test the null hypothesis that there is no relationship between food expenditure and total expenditure, that is, the true slope coefficient  $\beta_2 = 0$ . The estimated value of  $\beta_2$  is 0.4368. If the null hypothesis

were true, what is the probability of obtaining a value of 0.4368? Under the null hypothesis, we observe from (5.12.2) that the  $t$  value is 5.5770 and the  $p$  value of obtaining such a  $t$  value is practically zero. In other words, we can reject the null hypothesis resoundingly. But suppose the null hypothesis were that  $\beta_2 = 0.5$ . Now what? Using the  $t$  test we obtain:

$$t = \frac{0.4368 - 0.5}{0.0783} = -0.8071$$

The probability of obtaining a  $|t|$  of 0.8071 is greater than 20 percent. Hence we do not reject the hypothesis that the true  $\beta_2$  is 0.5.

Notice that, under the null hypothesis, the true slope coefficient is zero, the  $F$  value is 31.1034, as shown in (5.12.2). Under the same null hypothesis, we obtained a  $t$  value of 5.5770. If we square this value, we obtain 31.1029, which is about the same as the  $F$  value, again showing the close relationship between the  $t$  and the  $F$  statistic. (*Note:* The numerator df for the  $F$  statistic must be 1, which is the case here.)

Using the estimated residuals from the regression, what can we say about the probability distribution of the error term? The information is given in Figure 5.8. As the figure shows, the residuals from the food expenditure regression seem to be symmetrically distributed. Application of the Jarque-Bera test shows that the JB statistic is about 0.2576, and the probability of obtaining such a statistic under the normality assumption is about

(Continued)

A CONCLUDING EXAMPLE (Continued)

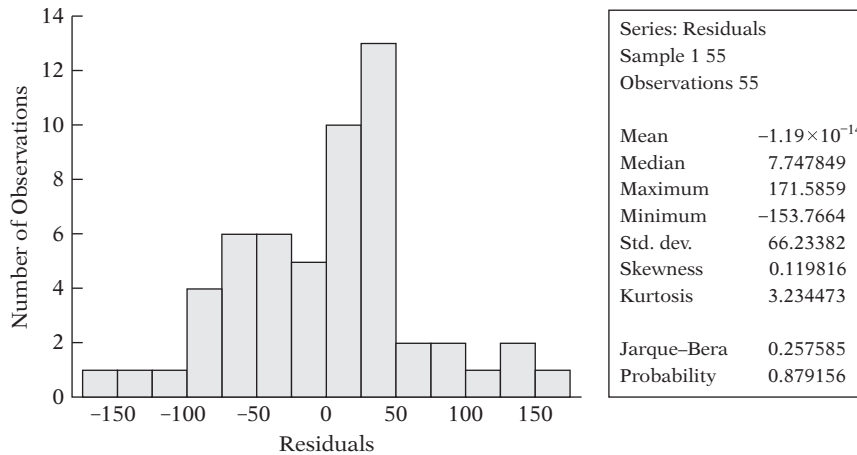


FIGURE 5.8 Residuals from the food expenditure regression.

88 percent. Therefore, we do not reject the hypothesis that the error terms are normally distributed. But keep in mind that the sample size of 55 observations may not be large enough.

We leave it to the reader to establish confidence intervals for the two regression coefficients as well as to obtain the normal probability plot and do mean and individual predictions.

### 5.13 SUMMARY AND CONCLUSIONS

1. Estimation and hypothesis testing constitute the two main branches of classical statistics. Having discussed the problem of estimation in Chapters 3 and 4, we have taken up the problem of hypothesis testing in this chapter.

2. Hypothesis testing answers this question: Is a given finding compatible with a stated hypothesis or not?

3. There are two mutually complementary approaches to answering the preceding question: **confidence interval** and **test of significance**.

4. Underlying the confidence-interval approach is the concept of **interval estimation**. An interval estimator is an interval or range constructed in such a manner that it has a specified probability of including within its limits the true value of the unknown parameter. The interval thus constructed is known as a **confidence interval**, which is often stated in percent form, such as 90 or 95%. The confidence interval provides a set of plausible hypotheses about the value of the unknown parameter. If the null-hypothesized value lies in the confidence interval, the hypothesis is not rejected, whereas if it lies outside this interval, the null hypothesis can be rejected.

5. In the **significance test** procedure, one develops a **test statistic** and examines its sampling distribution under the null hypothesis. The test statistic usually follows a well-defined probability distribution such as the normal,  $t$ ,  $F$ , or chi-square. Once a test statistic (e.g., the  $t$  statistic) is computed



from the data at hand, its  $p$  value can be easily obtained. The  $p$  value gives the exact probability of obtaining the estimated test statistic under the null hypothesis. If this  $p$  value is small, one can reject the null hypothesis, but if it is large one may not reject it. What constitutes a small or large  $p$  value is up to the investigator. In choosing the  $p$  value the investigator has to bear in mind the probabilities of committing **Type I** and **Type II errors**.

6. In practice, one should be careful in fixing  $\alpha$ , the probability of committing a **Type I error**, at arbitrary values such as 1, 5, or 10 percent. It is better to quote the  **$p$  value** of the test statistic. Also, the statistical significance of an estimate should not be confused with its practical significance.

7. Of course, hypothesis testing presumes that the model chosen for empirical analysis is adequate in the sense that it does not violate one or more assumptions underlying the classical normal linear regression model. Therefore, tests of model adequacy should precede tests of hypothesis. This chapter introduced one such test, the **normality test**, to find out whether the error term follows the normal distribution. Since in small, or finite, samples, the  $t$ ,  $F$ , and chi-square tests require the normality assumption, it is important that this assumption be checked formally.

8. If the model is deemed practically adequate, it may be used for forecasting purposes. But in forecasting the future values of the regressand, one should not go too far out of the sample range of the regressor values. Otherwise, forecasting errors can increase dramatically.

## EXERCISES

### Questions

- 5.1. State with reason whether the following statements are true, false, or uncertain. Be precise.
- The  $t$  test of significance discussed in this chapter requires that the sampling distributions of estimators  $\beta_1$  and  $\beta_2$  follow the normal distribution.
  - Even though the disturbance term in the CLRM is not normally distributed, the OLS estimators are still unbiased.
  - If there is no intercept in the regression model, the estimated  $u_i$  ( $= \hat{u}_i$ ) will not sum to zero.
  - The  $p$  value and the size of a test statistic mean the same thing.
  - In a regression model that contains the intercept, the sum of the residuals is always zero.
  - If a null hypothesis is not rejected, it is true.
  - The higher the value of  $\sigma^2$ , the larger is the variance of  $\hat{\beta}_2$  given in (3.3.1).
  - The conditional and unconditional means of a random variable are the same things.
  - In the two-variable PRF, if the slope coefficient  $\beta_2$  is zero, the intercept  $\beta_1$  is estimated by the sample mean  $\bar{Y}$ .
  - The conditional variance,  $\text{var}(Y_i | X_i) = \sigma^2$ , and the unconditional variance of  $Y$ ,  $\text{var}(Y) = \sigma_Y^2$ , will be the same if  $X$  had no influence on  $Y$ .



- 5.2. Set up the ANOVA table in the manner of Table 5.4 for the regression model given in (3.7.2) and test the hypothesis that there is no relationship between food expenditure and total expenditure in India.
- 5.3. From the data given in Table 2.6 on earnings and education, we obtained the following regression [see Eq. (3.7.3)]:

$$\widehat{\text{Meanwage}}_i = 0.7437 + 0.6416 \text{ Education}_i$$

$$\text{se} = (0.8355) \quad ( \quad )$$

$$t = ( \quad ) \quad (9.6536) \quad r^2 = 0.8944 \quad n = 13$$

- Fill in the missing numbers.
  - How do you interpret the coefficient 0.6416?
  - Would you reject the hypothesis that education has no effect whatsoever on wages? Which test do you use? And why? What is the  $p$  value of your test statistic?
  - Set up the ANOVA table for this example and test the hypothesis that the slope coefficient is zero. Which test do you use and why?
  - Suppose in the regression given above the  $r^2$  value was not given to you. Could you have obtained it from the other information given in the regression?
- 5.4. Let  $\rho^2$  represent the true population coefficient of correlation. Suppose you want to test the hypothesis that  $\rho^2 = 0$ . Verbally explain how you would test this hypothesis. *Hint:* Use Eq. (3.5.11). See also exercise 5.7.
- 5.5. What is known as the **characteristic line** of modern investment analysis is simply the regression line obtained from the following model:

$$r_{it} = \alpha_i + \beta_i r_{mt} + u_t$$

where  $r_{it}$  = the rate of return on the  $i$ th security in time  $t$   
 $r_{mt}$  = the rate of return on the market portfolio in time  $t$   
 $u_t$  = stochastic disturbance term

In this model  $\beta_i$  is known as the **beta coefficient** of the  $i$ th security, a measure of market (or systematic) risk of a security.\*

On the basis of 240 monthly rates of return for the period 1956–1976, Fogler and Ganapathy obtained the following characteristic line for IBM stock in relation to the market portfolio index developed at the University of Chicago†:

$$\hat{r}_{it} = 0.7264 + 1.0598 r_{mt} \quad r^2 = 0.4710$$

$$\text{se} = (0.3001) (0.0728) \quad \text{df} = 238$$

$$F_{1,238} = 211.896$$

- A security whose beta coefficient is greater than one is said to be a volatile or aggressive security. Was IBM a volatile security in the time period under study?

\*See Haim Levy and Marshall Sarnat, *Portfolio and Investment Selection: Theory and Practice*, Prentice-Hall International, Englewood Cliffs, N.J., 1984, Chap. 12.

†H. Russell Fogler and Sundaram Ganapathy, *Financial Econometrics*, Prentice Hall, Englewood Cliffs, N.J., 1982, p. 13.

b. Is the intercept coefficient significantly different from zero? If it is, what is its practical meaning?

5.6. Equation (5.3.5) can also be written as

$$\Pr[\hat{\beta}_2 - t_{\alpha/2} \text{se}(\hat{\beta}_2) < \beta_2 < \hat{\beta}_2 + t_{\alpha/2} \text{se}(\hat{\beta}_2)] = 1 - \alpha$$

That is, the weak inequality ( $\leq$ ) can be replaced by the strong inequality ( $<$ ). Why?

5.7. R. A. Fisher has derived the sampling distribution of the correlation coefficient defined in (3.5.13). If it is assumed that the variables  $X$  and  $Y$  are jointly normally distributed, that is, if they come from a bivariate normal distribution (see Appendix 4A, exercise 4.1), then under the assumption that the population correlation coefficient  $\rho$  is zero, it can be shown that  $t = r\sqrt{n-2}/\sqrt{1-r^2}$  follows Student's  $t$  distribution with  $n-2$  df.\* Show that this  $t$  value is identical with the  $t$  value given in (5.3.2) under the null hypothesis that  $\beta_2 = 0$ . Hence establish that under the same null hypothesis  $F = t^2$ . (See Section 5.9.)

## Problems

5.8. Consider the following regression output<sup>†</sup>:

$$\begin{aligned}\hat{Y}_i &= 0.2033 + 0.6560X_i \\ \text{se} &= (0.0976) \quad (0.1961) \\ r^2 &= 0.397 \quad \text{RSS} = 0.0544 \quad \text{ESS} = 0.0358\end{aligned}$$

where  $Y$  = labor force participation rate (LFPR) of women in 1972 and  $X$  = LFPR of women in 1968. The regression results were obtained from a sample of 19 cities in the United States.

a. How do you interpret this regression?

b. Test the hypothesis:  $H_0: \beta_2 = 1$  against  $H_1: \beta_2 > 1$ . Which test do you use? And why? What are the underlying assumptions of the test(s) you use?

c. Suppose that the LFPR in 1968 was 0.58 (or 58 percent). On the basis of the regression results given above, what is the mean LFPR in 1972? Establish a 95% confidence interval for the mean prediction.

d. How would you test the hypothesis that the error term in the population regression is normally distributed? Show the necessary calculations.

5.9. Table 5.5 gives data on average public teacher pay (annual salary in dollars) and spending on public schools per pupil (dollars) in 1985 for 50 states and the District of Columbia.

\*If  $\rho$  is in fact zero, Fisher has shown that  $t$  follows the same  $t$  distribution provided either  $X$  or  $Y$  is normally distributed. But if  $\rho$  is not equal to zero, both variables must be normally distributed. See R. L. Anderson and T. A. Bancroft, *Statistical Theory in Research*, McGraw-Hill, New York, 1952, pp. 87–88.

<sup>†</sup>Adapted from Samprit Chatterjee, Ali S. Hadi, and Bertram Price, *Regression Analysis by Example*, 3d ed., Wiley Interscience, New York, 2000, pp. 46–47.

**TABLE 5.5** AVERAGE SALARY AND PER PUPIL SPENDING (DOLLARS), 1985

Observation	Salary	Spending	Observation	Salary	Spending
1	19,583	3346	27	22,795	3366
2	20,263	3114	28	21,570	2920
3	20,325	3554	29	22,080	2980
4	26,800	4642	30	22,250	3731
5	29,470	4669	31	20,940	2853
6	26,610	4888	32	21,800	2533
7	30,678	5710	33	22,934	2729
8	27,170	5536	34	18,443	2305
9	25,853	4168	35	19,538	2642
10	24,500	3547	36	20,460	3124
11	24,274	3159	37	21,419	2752
12	27,170	3621	38	25,160	3429
13	30,168	3782	39	22,482	3947
14	26,525	4247	40	20,969	2509
15	27,360	3982	41	27,224	5440
16	21,690	3568	42	25,892	4042
17	21,974	3155	43	22,644	3402
18	20,816	3059	44	24,640	2829
19	18,095	2967	45	22,341	2297
20	20,939	3285	46	25,610	2932
21	22,644	3914	47	26,015	3705
22	24,624	4517	48	25,788	4123
23	27,186	4349	49	29,132	3608
24	33,990	5020	50	41,480	8349
25	23,382	3594	51	25,845	3766
26	20,627	2821			

Source: National Education Association, as reported by *Albuquerque Tribune*, Nov. 7, 1986.

To find out if there is any relationship between teacher's pay and per pupil expenditure in public schools, the following model was suggested:  $\text{Pay}_i = \beta_1 + \beta_2 \text{Spend}_i + u_i$ , where Pay stands for teacher's salary and Spend stands for per pupil expenditure.

- a. Plot the data and eyeball a regression line.
  - b. Suppose on the basis of a you decide to estimate the above regression model. Obtain the estimates of the parameters, their standard errors,  $r^2$ , RSS, and ESS.
  - c. Interpret the regression. Does it make economic sense?
  - d. Establish a 95% confidence interval for  $\beta_2$ . Would you reject the hypothesis that the true slope coefficient is 3.0?
  - e. Obtain the mean and individual forecast value of Pay if per pupil spending is \$5000. Also establish 95% confidence intervals for the true mean and individual values of Pay for the given spending figure.
  - f. How would you test the assumption of the normality of the error term? Show the test(s) you use.
- 5.10.** Refer to exercise 3.20 and set up the ANOVA tables and test the hypothesis that there is no relationship between productivity and real wage

compensation. Do this for both the business and nonfarm business sectors.

- 5.11. Refer to exercise 1.7.
- Plot the data with impressions on the vertical axis and advertising expenditure on the horizontal axis. What kind of relationship do you observe?
  - Would it be appropriate to fit a bivariate linear regression model to the data? Why or why not? If not, what type of regression model will you fit the data to? Do we have the necessary tools to fit such a model?
  - Suppose you do not plot the data and simply fit the bivariate regression model to the data. Obtain the usual regression output. Save the results for a later look at this problem.
- 5.12. Refer to exercise 1.1.
- Plot the U.S. Consumer Price Index (CPI) against the Canadian CPI. What does the plot show?
  - Suppose you want to predict the U.S. CPI on the basis of the Canadian CPI. Develop a suitable model.
  - Test the hypothesis that there is no relationship between the two CPIs. Use  $\alpha = 5\%$ . If you reject the null hypothesis, does that mean the Canadian CPI “causes” the U.S. CPI? Why or why not?
- 5.13. Refer to exercise 3.22.
- Estimate the two regressions given there, obtaining standard errors and the other usual output.
  - Test the hypothesis that the disturbances in the two regression models are normally distributed.
  - In the gold price regression, test the hypothesis that  $\beta_2 = 1$ , that is, there is a one-to-one relationship between gold prices and CPI (i.e., gold is a perfect hedge). What is the  $p$  value of the estimated test statistic?
  - Repeat step **c** for the NYSE Index regression. Is investment in the stock market a perfect hedge against inflation? What is the null hypothesis you are testing? What is its  $p$  value?
  - Between gold and stock, which investment would you choose? What is the basis of your decision?
- 5.14. Table 5.6 gives data on GNP and four definitions of the money stock for the United States for 1970–1983. Regressing GNP on the various definitions of money, we obtain the results shown in Table 5.7.
- The monetarists or quantity theorists maintain that nominal income (i.e., nominal GNP) is largely determined by changes in the quantity or the stock of money, although there is no consensus as to the “right” definition of money. Given the results in the preceding table, consider these questions:
- Which definition of money seems to be closely related to nominal GNP?
  - Since the  $r^2$  terms are uniformly high, does this fact mean that our choice for definition of money does not matter?
  - If the Fed wants to control the money supply, which one of these money measures is a better target for that purpose? Can you tell from the regression results?
- 5.15. Suppose the equation of an **indifference curve** between two goods is

$$X_i Y_i = \beta_1 + \beta_2 X_i$$

**TABLE 5.6** GNP AND FOUR MEASURES OF MONEY STOCK

Year	GNP, \$ billion	Money stock measure, \$ billion			
		M <sub>1</sub>	M <sub>2</sub>	M <sub>3</sub>	L
1970	992.70	216.6	628.2	677.5	816.3
1971	1,077.6	230.8	712.8	776.2	903.1
1972	1,185.9	252.0	805.2	886.0	1,023.0
1973	1,326.4	265.9	861.0	985.0	1,141.7
1974	1,434.2	277.6	908.5	1,070.5	1,249.3
1975	1,549.2	291.2	1,023.3	1,174.2	1,367.9
1976	1,718.0	310.4	1,163.6	1,311.9	1,516.6
1977	1,918.3	335.4	1,286.7	1,472.9	1,704.7
1978	2,163.9	363.1	1,389.1	1,647.1	1,910.6
1979	2,417.8	389.1	1,498.5	1,804.8	2,117.1
1980	2,631.7	414.9	1,632.6	1,990.0	2,326.2
1981	2,957.8	441.9	1,796.6	2,238.2	2,599.8
1982	3,069.3	480.5	1,965.4	2,462.5	2,870.8
1983	3,304.8	525.4	2,196.3	2,710.4	3,183.1

*Definitions:*  
M<sub>1</sub> = currency + demand deposits + travelers checks and other checkable deposits (OCDs)  
M<sub>2</sub> = M<sub>1</sub> + overnight RPs and Eurodollars + MMMF (money market mutual fund) balances + MMDAs (money market deposit accounts) + savings and small deposits  
M<sub>3</sub> = M<sub>2</sub> + large time deposits + term RPs + Institutional MMMF  
L = M<sub>3</sub> + other liquid assets

*Source:* *Economic Report of the President*, 1985, GNP data from Table B-1, p. 232; money stock data from Table B-61, p. 303.

**TABLE 5.7** GNP–MONEY STOCK REGRESSIONS, 1970–1983

1)	$\widehat{\text{GNP}}_t = -787.4723 + 8.0863 M_{1t}$ (77.9664) (0.2197)	$r^2 = 0.9912$
2)	$\widehat{\text{GNP}}_t = -44.0626 + 1.5875 M_{2t}$ (61.0134) (0.0448)	$r^2 = 0.9905$
3)	$\widehat{\text{GNP}}_t = 159.1366 + 1.2034 M_{3t}$ (42.9882) (0.0262)	$r^2 = 0.9943$
4)	$\widehat{\text{GNP}}_t = 164.2071 + 1.0290 L_t$ (44.7658) (0.0234)	$r^2 = 0.9938$

*Note:* The figures in parentheses are the estimated standard errors.

**TABLE 5.8**

Consumption of good X:	1	2	3	4	5
Consumption of good Y:	4	3.5	2.8	1.9	0.8

How would you estimate the parameters of this model? Apply the preceding model to the data in Table 5.8 and comment on your results.

**5.16.** Since 1986 the *Economist* has been publishing the Big Mac Index as a crude, and hilarious, measure of whether international currencies are at their “correct” exchange rate, as judged by the theory of **purchasing power parity (PPP)**. The PPP holds that a unit of currency should be able

to buy the same bundle of goods in all countries. The proponents of PPP argue that, in the long run, currencies tend to move toward their PPP. The *Economist* uses McDonald's Big Mac as a representative bundle and gives the information in Table 5.9.

Consider the following regression model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

where  $Y$  = actual exchange rate and  $X$  = implied PPP of the dollar.

**TABLE 5.9** THE HAMBURGER STANDARD

	Big Mac prices		Implied PPP* of the dollar	Actual \$ exchange rate April 17, 2001	Under (-)/ over (+) valuation against the dollar, %
	In local currency	In dollars			
United States <sup>†</sup>	\$2.54	2.54	—	—	—
Argentina	Peso2.50	2.50	0.98	1.00	-2
Australia	A\$3.00	1.52	1.18	1.98	-40
Brazil	Real3.60	1.64	1.42	2.19	-35
Britain	£1.99	2.85	1.28 <sup>‡</sup>	1.43 <sup>‡</sup>	12
Canada	C\$3.33	2.14	1.31	1.56	-16
Chile	Peso1260	2.10	496	601	-17
China	Yuan9.90	1.20	3.90	8.28	-53
Czech Rep	Koruna56.00	1.43	22.0	39.0	-44
Denmark	DKr24.75	2.93	9.74	8.46	15
Euro area	€2.57	2.27	0.99 <sup>§</sup>	0.88 <sup>§</sup>	-11
France	FFr18.5	2.49	7.28	7.44	-2
Germany	DM5.10	2.30	2.01	2.22	-9
Italy	Lire4300	1.96	1693	2195	-23
Spain	Pta395	2.09	156	189	-18
Hong Kong	HK\$10.70	1.37	4.21	7.80	-46
Hungary	Forint399	1.32	157	303	-48
Indonesia	Rupiah14700	1.35	5787	10855	-47
Japan	¥294	2.38	116	124	-6
Malaysia	M\$4.52	1.19	1.78	3.80	-53
Mexico	Peso21.9	2.36	8.62	9.29	-7
New Zealand	NZ\$3.60	1.46	1.42	2.47	-43
Philippines	Peso59.00	1.17	23.2	50.3	-54
Poland	Zloty5.90	1.46	2.32	4.03	-42
Russia	Rouble35.00	1.21	13.8	28.9	-52
Singapore	S\$3.30	1.82	1.30	1.81	-28
South Africa	Rand9.70	1.19	3.82	8.13	-53
South Korea	Won3000	2.27	1181	1325	-11
Sweden	SKr24.0	2.33	9.45	10.28	-8
Switzerland	SFr6.30	3.65	2.48	1.73	44
Taiwan	NT\$70.0	2.13	27.6	32.9	-16
Thailand	Baht55.0	1.21	21.7	45.5	-52

\*Purchasing power parity: local price divided by price in the United States.

<sup>†</sup>Average of New York, Chicago, San Francisco, and Atlanta.

<sup>‡</sup>Dollars per pound.

<sup>§</sup>Dollars per euro.

Source: McDonald's; *The Economist*, April 21, 2001.

- a. If the PPP holds, what values of  $\beta_1$  and  $\beta_2$  would you expect a priori?  
 b. Do the regression results support your expectation? What formal test do you use to test your hypothesis?  
 c. Should the *Economist* continue to publish the Big Mac Index? Why or why not?
- 5.17. Refer to the S.A.T. data given in exercise 2.16. Suppose you want to predict the male math ( $Y$ ) scores on the basis of the female math scores ( $X$ ) by running the following regression:

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

- a. Estimate the preceding model.  
 b. From the estimated residuals, find out if the normality assumption can be sustained.  
 c. Now test the hypothesis that  $\beta_2 = 1$ , that is, there is a one-to-one correspondence between male and female math scores.  
 d. Set up the ANOVA table for this problem.
- 5.18. Repeat the exercise in the preceding problem but let  $Y$  and  $X$  denote the male and female verbal scores, respectively.
- 5.19. Table 5.10 gives annual data on the Consumer Price Index (CPI) and the Wholesale Price Index (WPI), also called Producer Price Index (PPI), for the U.S. economy for the period 1960–1999.
- a. Plot the CPI on the vertical axis and the WPI on the horizontal axis. A priori, what kind of relationship do you expect between the two indexes? Why?

TABLE 5.10 CPI AND WPI, UNITED STATES, 1960–1999

Year	CPI	WPI	Year	CPI	WPI
1960	29.8	31.7	1980	86.3	93.8
1961	30.0	31.6	1981	94.0	98.8
1962	30.4	31.6	1982	97.6	100.5
1963	30.9	31.6	1983	101.3	102.3
1964	31.2	31.7	1984	105.3	103.5
1965	31.8	32.8	1985	109.3	103.6
1966	32.9	33.3	1986	110.5	99.70
1967	33.9	33.7	1987	115.4	104.2
1968	35.5	34.6	1988	120.5	109.0
1969	37.7	36.3	1989	126.1	113.0
1970	39.8	37.1	1990	133.8	118.7
1971	41.1	38.6	1991	137.9	115.9
1972	42.5	41.1	1992	141.9	117.6
1973	46.2	47.4	1993	145.8	118.6
1974	51.9	57.3	1994	149.7	121.9
1975	55.5	59.7	1995	153.5	125.7
1976	58.2	62.5	1996	158.6	128.8
1977	62.1	66.2	1997	161.3	126.7
1978	67.7	72.7	1998	163.9	122.7
1979	76.7	83.4	1999	168.3	128.0

Source: *Economic Report of the President*, 2000, pp. 373 and 379.

- b. Suppose you want to predict one of these indexes on the basis of the other index. Which will you use as the regressand and which as the regressor? Why?
- c. Run the regression you have decided in *b*. Show the standard output. Test the hypothesis that there is a one-to-one relationship between the two indexes.
- d. From the residuals obtained from the regression in *c*, can you entertain the hypothesis that the true error term is normally distributed? Show the tests you use.

## APPENDIX 5A

### 5A.1 PROBABILITY DISTRIBUTIONS RELATED TO THE NORMAL DISTRIBUTION

The *t*, **chi-square** ( $\chi^2$ ), and *F* probability distributions, whose salient features are discussed in **Appendix A**, are intimately related to the normal distribution. Since we will make heavy use of these probability distributions in the following chapters, we summarize their relationship with the normal distribution in the following theorem; the proofs, which are beyond the scope of this book, can be found in the references.<sup>1</sup>

**Theorem 5.1.** If  $Z_1, Z_2, \dots, Z_n$  are normally and independently distributed random variables such that  $Z_i \sim N(\mu_i, \sigma_i^2)$ , then the sum  $Z = \sum k_i Z_i$ , where  $k_i$  are constants not all zero, is also distributed normally with mean  $\sum k_i \mu_i$  and variance  $\sum k_i^2 \sigma_i^2$ ; that is,  $Z \sim N(\sum k_i \mu_i, \sum k_i^2 \sigma_i^2)$ . *Note:*  $\mu$  denotes the mean value.

In short, linear combinations of normal variables are themselves normally distributed. For example, if  $Z_1$  and  $Z_2$  are normally and independently distributed as  $Z_1 \sim N(10, 2)$  and  $Z_2 \sim N(8, 1.5)$ , then the linear combination  $Z = 0.8Z_1 + 0.2Z_2$  is also normally distributed with mean =  $0.8(10) + 0.2(8) = 9.6$  and variance =  $0.64(2) + 0.04(1.5) = 1.34$ , that is,  $Z \sim (9.6, 1.34)$ .

**Theorem 5.2.** If  $Z_1, Z_2, \dots, Z_n$  are normally distributed but are not independent, the sum  $Z = \sum k_i Z_i$ , where  $k_i$  are constants not all zero, is also normally distributed with mean  $\sum k_i \mu_i$  and variance  $[\sum k_i^2 \sigma_i^2 + 2 \sum k_i k_j \text{cov}(Z_i, Z_j), i \neq j]$ .

Thus, if  $Z_1 \sim N(6, 2)$  and  $Z_2 \sim N(7, 3)$  and  $\text{cov}(Z_1, Z_2) = 0.8$ , then the linear combination  $0.6Z_1 + 0.4Z_2$  is also normally distributed with mean =  $0.6(6) + 0.4(7) = 6.4$  and variance =  $[0.36(2) + 0.16(3) + 2(0.6)(0.4)(0.8)] = 1.584$ .

<sup>1</sup>For proofs of the various theorems, see Alexander M. Mood, Franklin A. Graybill, and Duane C. Bose, *Introduction to the Theory of Statistics*, 3d ed., McGraw-Hill, New York, 1974, pp. 239–249.



**Theorem 5.3.** If  $Z_1, Z_2, \dots, Z_n$  are normally and independently distributed random variables such that each  $Z_i \sim N(0, 1)$ , that is, a standardized normal variable, then  $\sum Z_i^2 = Z_1^2 + Z_2^2 + \dots + Z_n^2$  follows the chi-square distribution with  $n$  df. Symbolically,  $\sum Z_i^2 \sim \chi_n^2$ , where  $n$  denotes the degrees of freedom, df.

In short, “the sum of the squares of independent standard normal variables has a chi-square distribution with degrees of freedom equal to the number of terms in the sum.”<sup>2</sup>

**Theorem 5.4.** If  $Z_1, Z_2, \dots, Z_n$  are independently distributed random variables each following chi-square distribution with  $k_i$  df, then the sum  $\sum Z_i = Z_1 + Z_2 + \dots + Z_n$  also follows a chi-square distribution with  $k = \sum k_i$  df.

Thus, if  $Z_1$  and  $Z_2$  are independent  $\chi^2$  variables with df of  $k_1$  and  $k_2$ , respectively, then  $Z = Z_1 + Z_2$  is also a  $\chi^2$  variable with  $(k_1 + k_2)$  degrees of freedom. This is called the **reproductive property** of the  $\chi^2$  distribution.

**Theorem 5.5.** If  $Z_1$  is a standardized normal variable [ $Z_1 \sim N(0, 1)$ ] and another variable  $Z_2$  follows the chi-square distribution with  $k$  df and is independent of  $Z_1$ , then the variable defined as

$$t = \frac{Z_1}{\sqrt{Z_2}/\sqrt{k}} = \frac{Z_1\sqrt{k}}{\sqrt{Z_2}} = \frac{\text{standard normal variable}}{\sqrt{\text{independent chi-square variable/df}}} \sim t_k$$

follows Student’s  $t$  distribution with  $k$  df. *Note:* This distribution is discussed in **Appendix A** and is illustrated in Chapter 5.

Incidentally, note that as  $k$ , the df, increases indefinitely (i.e., as  $k \rightarrow \infty$ ), the Student’s  $t$  distribution approaches the standardized normal distribution.<sup>3</sup> As a matter of convention, the notation  $t_k$  means Student’s  $t$  distribution or variable with  $k$  df.

**Theorem 5.6.** If  $Z_1$  and  $Z_2$  are independently distributed chi-square variables with  $k_1$  and  $k_2$  df, respectively, then the variable

$$F = \frac{Z_1/k_1}{Z_2/k_2} \sim F_{k_1, k_2}$$

has the  $F$  distribution with  $k_1$  and  $k_2$  degrees of freedom, where  $k_1$  is known as the **numerator degrees of freedom** and  $k_2$  the **denominator degrees of freedom**.

<sup>2</sup>Ibid., p. 243.

<sup>3</sup>For proof, see Henri Theil, *Introduction to Econometrics*, Prentice-Hall, Englewood Cliffs, N.J., 1978, pp. 237–245.

Again as a matter of convention, the notation  $F_{k_1, k_2}$  means an  $F$  variable with  $k_1$  and  $k_2$  degrees of freedom, the df in the numerator being quoted first.

In other words, Theorem 5.6 states that the  $F$  variable is simply the ratio of two independently distributed chi-square variables divided by their respective degrees of freedom.

**Theorem 5.7.** The square of (Student's)  $t$  variable with  $k$  df has an  $F$  distribution with  $k_1 = 1$  df in the numerator and  $k_2 = k$  df in the denominator.<sup>4</sup> That is,

$$F_{1,k} = t_k^2$$

Note that for this equality to hold, the numerator df of the  $F$  variable must be 1. Thus,  $F_{1,4} = t_4^2$  or  $F_{1,23} = t_{23}^2$  and so on.

As noted, we will see the practical utility of the preceding theorems as we progress.

**Theorem 5.8.** For large denominator df, the numerator df times the  $F$  value is approximately equal to the chi-square value with the numerator df. Thus,

$$m F_{m,n} = \chi_m^2 \quad \text{as } n \rightarrow \infty$$

**Theorem 5.9.** For sufficiently large df, the chi-square distribution can be approximated by the standard normal distribution as follows:

$$Z = \sqrt{2\chi^2} - \sqrt{2k-1} \sim N(0, 1)$$

where  $k$  denotes df.

## 5A.2 DERIVATION OF EQUATION (5.3.2)

Let

$$Z_1 = \frac{\hat{\beta}_2 - \beta_2}{\text{se}(\hat{\beta}_2)} = \frac{(\hat{\beta}_2 - \beta_2)\sqrt{x_i^2}}{\sigma} \quad (1)$$

and

$$Z_2 = (n-2) \frac{\hat{\sigma}^2}{\sigma^2} \quad (2)$$

Provided  $\sigma$  is known,  $Z_1$  follows the standardized normal distribution; that is,  $Z_1 \sim N(0, 1)$ . (Why?)  $Z_2$ , follows the  $\chi^2$  distribution with  $(n-2)$  df.<sup>5</sup>

<sup>4</sup>For proof, see Eqs. (5.3.2) and (5.9.1).

<sup>5</sup>For proof, see Robert V. Hogg and Allen T. Craig, *Introduction to Mathematical Statistics*, 2d ed., Macmillan, New York, 1965, p. 144.

Furthermore, it can be shown that  $Z_2$  is distributed independently of  $Z_1$ .<sup>6</sup> Therefore, by virtue of Theorem 5.5, the variable

$$t = \frac{Z_1 \sqrt{n-2}}{\sqrt{Z_2}} \quad (3)$$

follows the  $t$  distribution with  $n-2$  df. Substitution of (1) and (2) into (3) gives Eq. (5.3.2).

### 5A.3 DERIVATION OF EQUATION (5.9.1)

Equation (1) shows that  $Z_1 \sim N(0, 1)$ . Therefore, by Theorem 5.3, the preceding quantity

$$Z_1^2 = \frac{(\hat{\beta}_2 - \beta_2)^2 \sum x_i^2}{\sigma^2}$$

follows the  $\chi^2$  distribution with 1 df. As noted in Section 5A.1,

$$Z_2 = (n-2) \frac{\hat{\sigma}^2}{\sigma^2} = \frac{\sum \hat{u}_i^2}{\sigma^2}$$

also follows the  $\chi^2$  distribution with  $n-2$  df. Moreover, as noted in Section 4.3,  $Z_2$  is distributed independently of  $Z_1$ . Then from Theorem 5.6, it follows that

$$F = \frac{Z_1^2/1}{Z_2/(n-2)} = \frac{(\hat{\beta}_2 - \beta_2)^2 (\sum x_i^2)}{\sum \hat{u}_i^2 / (n-2)}$$

follows the  $F$  distribution with 1 and  $n-2$  df, respectively. Under the null hypothesis  $H_0: \beta_2 = 0$ , the preceding  $F$  ratio reduces to Eq. (5.9.1).

### 5.A.4 DERIVATIONS OF EQUATIONS (5.10.2) AND (5.10.6)

#### Variance of Mean Prediction

Given  $X_i = X_0$ , the true mean prediction  $E(Y_0 | X_0)$  is given by

$$E(Y_0 | X_0) = \beta_1 + \beta_2 X_0 \quad (1)$$

We estimate (1) from

$$\hat{Y}_0 = \hat{\beta}_1 + \hat{\beta}_2 X_0 \quad (2)$$

Taking the expectation of (2), given  $X_0$ , we get

$$\begin{aligned} E(\hat{Y}_0) &= E(\hat{\beta}_1) + E(\hat{\beta}_2) X_0 \\ &= \beta_1 + \beta_2 X_0 \end{aligned}$$

<sup>6</sup>For proof, see J. Johnston, *Econometric Methods*, McGraw-Hill, 3d ed., New York, 1984, pp. 181–182. (Knowledge of matrix algebra is required to follow the proof.)

because  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are unbiased estimators. Therefore,

$$E(\hat{Y}_0) = E(Y_0 | X_0) = \beta_1 + \beta_2 X_0 \quad (3)$$

That is,  $\hat{Y}_0$  is an unbiased predictor of  $E(Y_0 | X_0)$ .

Now using the property that  $\text{var}(a + b) = \text{var}(a) + \text{var}(b) + 2 \text{cov}(a, b)$ , we obtain

$$\text{var}(\hat{Y}_0) = \text{var}(\hat{\beta}_1) + \text{var}(\hat{\beta}_2)X_0^2 + 2 \text{cov}(\hat{\beta}_1, \hat{\beta}_2)X_0 \quad (4)$$

Using the formulas for variances and covariance of  $\hat{\beta}_1$  and  $\hat{\beta}_2$  given in (3.3.1), (3.3.3), and (3.3.9) and manipulating terms, we obtain

$$\text{var}(\hat{Y}_0) = \sigma^2 \left[ \frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\sum x_i^2} \right] = (5.10.2)$$

#### Variance of Individual Prediction

We want to predict an individual  $Y$  corresponding to  $X = X_0$ ; that is, we want to obtain

$$Y_0 = \beta_1 + \beta_2 X_0 + u_0 \quad (5)$$

We predict this as

$$\hat{Y}_0 = \hat{\beta}_1 + \hat{\beta}_2 X_0 \quad (6)$$

The prediction error,  $Y_0 - \hat{Y}_0$ , is

$$\begin{aligned} Y_0 - \hat{Y}_0 &= \beta_1 + \beta_2 X_0 + u_0 - (\hat{\beta}_1 + \hat{\beta}_2 X_0) \\ &= (\beta_1 - \hat{\beta}_1) + (\beta_2 - \hat{\beta}_2)X_0 + u_0 \end{aligned} \quad (7)$$

Therefore,

$$\begin{aligned} E(Y_0 - \hat{Y}_0) &= E(\beta_1 - \hat{\beta}_1) + E(\beta_2 - \hat{\beta}_2)X_0 - E(u_0) \\ &= 0 \end{aligned}$$

because  $\hat{\beta}_1$ ,  $\hat{\beta}_2$  are unbiased,  $X_0$  is a fixed number, and  $E(u_0)$  is zero by assumption.

Squaring (7) on both sides and taking expectations, we get  $\text{var}(Y_0 - \hat{Y}_0) = \text{var}(\hat{\beta}_1) + X_0^2 \text{var}(\hat{\beta}_2) + 2X_0 \text{cov}(\hat{\beta}_1, \hat{\beta}_2) + \text{var}(u_0)$ . Using the variance and covariance formulas for  $\hat{\beta}_1$  and  $\hat{\beta}_2$  given earlier, and noting that  $\text{var}(u_0) = \sigma^2$ , we obtain

$$\text{var}(Y_0 - \hat{Y}_0) = \sigma^2 \left[ 1 + \frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\sum x_i^2} \right] = (5.10.6)$$

# 6

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## EXTENSIONS OF THE TWO-VARIABLE LINEAR REGRESSION MODEL

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Some aspects of linear regression analysis can be easily introduced within the framework of the two-variable linear regression model that we have been discussing so far. First we consider the case of **regression through the origin**, that is, a situation where the intercept term,  $\beta_1$ , is absent from the model. Then we consider the question of the **units of measurement**, that is, how the  $Y$  and  $X$  variables are measured and whether a change in the units of measurement affects the regression results. Finally, we consider the question of the **functional form** of the linear regression model. So far we have considered models that are linear in the parameters as well as in the variables. But recall that the regression theory developed in the previous chapters requires only that the parameters be linear; the variables may or may not enter linearly in the model. By considering models that are linear in the parameters but not necessarily in the variables, we show in this chapter how the two-variable models can deal with some interesting practical problems.

Once the ideas introduced in this chapter are grasped, their extension to multiple regression models is quite straightforward, as we shall show in Chapters 7 and 8.

### 6.1 REGRESSION THROUGH THE ORIGIN

There are occasions when the two-variable PRF assumes the following form:

$$Y_i = \beta_2 X_i + u_i \quad (6.1.1)$$

In this model the intercept term is absent or zero, hence the name **regression through the origin**.

As an illustration, consider the Capital Asset Pricing Model (CAPM) of modern portfolio theory, which, in its risk-premium form, may be expressed as<sup>1</sup>

$$(ER_i - r_f) = \beta_i(ER_m - r_f) \quad (6.1.2)$$

where  $ER_i$  = expected rate of return on security  $i$

$ER_m$  = expected rate of return on the market portfolio as represented by, say, the S&P 500 composite stock index

$r_f$  = risk-free rate of return, say, the return on 90-day Treasury bills

$\beta_i$  = the Beta coefficient, a measure of systematic risk, i.e., risk that cannot be eliminated through diversification. Also, a measure of the extent to which the  $i$ th security's rate of return moves with the market. A  $\beta_i > 1$  implies a volatile or aggressive security, whereas a  $\beta_i < 1$  a defensive security. (Note: Do not confuse this  $\beta_i$  with the slope coefficient of the two-variable regression,  $\beta_2$ .)

If capital markets work efficiently, then CAPM postulates that security  $i$ 's expected risk premium ( $= ER_i - r_f$ ) is equal to that security's  $\beta$  coefficient times the expected market risk premium ( $= ER_m - r_f$ ). If the CAPM holds, we have the situation depicted in Figure 6.1. The line shown in the figure is known as the **security market line (SML)**.

For empirical purposes, (6.1.2) is often expressed as

$$R_i - r_f = \beta_i(R_m - r_f) + u_i \quad (6.1.3)$$

or

$$R_i - r_f = \alpha_i + \beta_i(R_m - r_f) + u_i \quad (6.1.4)$$

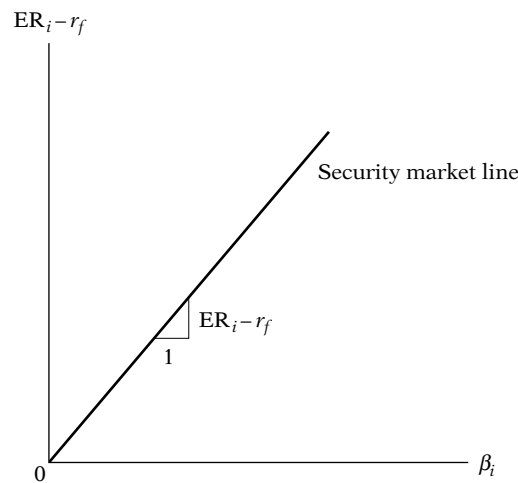
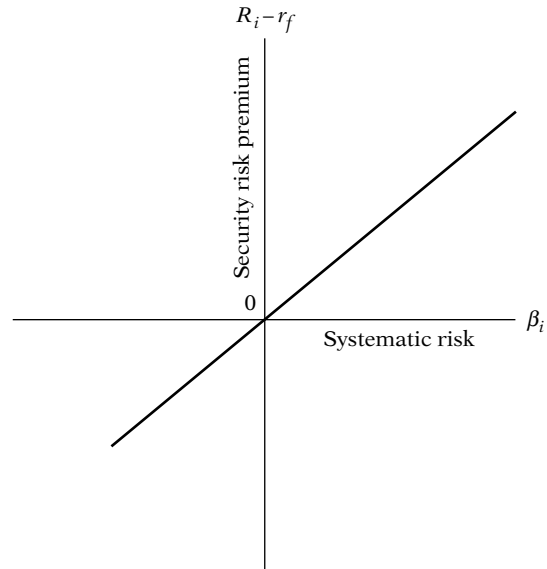


FIGURE 6.1 Systematic risk.

<sup>1</sup>See Haim Levy and Marshall Sarnat, *Portfolio and Investment Selection: Theory and Practice*, Prentice-Hall International, Englewood Cliffs, N.J., 1984, Chap. 14.



**FIGURE 6.2** The Market Model of Portfolio Theory (assuming  $\alpha_i = 0$ ).

The latter model is known as the **Market Model**.<sup>2</sup> If CAPM holds,  $\alpha_i$  is expected to be zero. (See Figure 6.2.)

In passing, note that in (6.1.4) the dependent variable,  $Y$ , is  $(R_i - r_f)$  and the explanatory variable,  $X$ , is  $\beta_i$ , the volatility coefficient, and *not*  $(R_m - r_f)$ . Therefore, to run regression (6.1.4), one must first estimate  $\beta_i$ , which is usually derived from the **characteristic line**, as described in exercise 5.5. (For further details, see exercise 8.28.)

As this example shows, sometimes the underlying theory dictates that the intercept term be absent from the model. Other instances where the zero-intercept model may be appropriate are Milton Friedman's permanent income hypothesis, which states that permanent consumption is proportional to permanent income; cost analysis theory, where it is postulated that the variable cost of production is proportional to output; and some versions of monetarist theory that state that the rate of change of prices (i.e., the rate of inflation) is proportional to the rate of change of the money supply.

How do we estimate models like (6.1.1), and what special problems do they pose? To answer these questions, let us first write the SRF of (6.1.1), namely,

$$Y_i = \hat{\beta}_2 X_i + \hat{u}_i \quad (6.1.5)$$

Now applying the OLS method to (6.1.5), we obtain the following formulas for  $\hat{\beta}_2$  and its variance (proofs are given in Appendix 6A, Section 6A.1):

$$\hat{\beta}_2 = \frac{\sum X_i Y_i}{\sum X_i^2} \quad (6.1.6)$$

<sup>2</sup>See, for instance, Diana R. Harrington, *Modern Portfolio Theory and the Capital Asset Pricing Model: A User's Guide*, Prentice Hall, Englewood Cliffs, N.J., 1983, p. 71.

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum X_i^2} \quad (6.1.7)$$

where  $\sigma^2$  is estimated by

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-1} \quad (6.1.8)$$

It is interesting to compare these formulas with those obtained when the intercept term is included in the model:

$$\hat{\beta}_2 = \frac{\sum x_i y_i}{\sum x_i^2} \quad (3.1.6)$$

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_i^2} \quad (3.3.1)$$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2} \quad (3.3.5)$$

The differences between the two sets of formulas should be obvious: In the model with the intercept term absent, we use **raw** sums of squares and cross products but in the intercept-present model, we use adjusted (from mean) sums of squares and cross products. Second, the df for computing  $\hat{\sigma}^2$  is  $(n-1)$  in the first case and  $(n-2)$  in the second case. (Why?)

Although the interceptless or zero intercept model may be appropriate on occasions, there are some features of this model that need to be noted. First,  $\sum \hat{u}_i$ , which is always zero for the model with the intercept term (the conventional model), need not be zero when that term is absent. In short,  $\sum \hat{u}_i$  need not be zero for the regression through the origin. Second,  $r^2$ , the coefficient of determination introduced in Chapter 3, which is always non-negative for the conventional model, can on occasions turn out to be *negative* for the interceptless model! This anomalous result arises because the  $r^2$  introduced in Chapter 3 explicitly assumes that the intercept is included in the model. Therefore, the conventionally computed  $r^2$  may not be appropriate for regression-through-the-origin models.<sup>3</sup>

### $r^2$ for Regression-through-Origin Model

As just noted, and as further discussed in Appendix 6A, Section 6A.1, the conventional  $r^2$  given in Chapter 3 is not appropriate for regressions that do not contain the intercept. But one can compute what is known as the **raw**  $r^2$  for such models, which is defined as

$$\text{raw } r^2 = \frac{(\sum X_i Y_i)^2}{\sum X_i^2 \sum Y_i^2} \quad (6.1.9)$$

<sup>3</sup>For additional discussion, see Dennis J. Aigner, *Basic Econometrics*, Prentice Hall, Englewood Cliffs, N.J., 1971, pp. 85–88.



*Note:* These are raw (i.e., not mean-corrected) sums of squares and cross products.

Although this raw  $r^2$  satisfies the relation  $0 < r^2 < 1$ , it is not directly comparable to the conventional  $r^2$  value. For this reason some authors do not report the  $r^2$  value for zero intercept regression models.

Because of these special features of this model, one needs to exercise great caution in using the zero intercept regression model. *Unless there is very strong a priori expectation*, one would be well advised to stick to the conventional, intercept-present model. This has a dual advantage. First, if the intercept term is included in the model but it turns out to be statistically insignificant (i.e., statistically equal to zero), for all practical purposes we have a regression through the origin.<sup>4</sup> Second, and more important, if in fact there is an intercept in the model but we insist on fitting a regression through the origin, we would be committing a **specification error**, thus violating Assumption 9 of the classical linear regression model.

AN ILLUSTRATIVE EXAMPLE:  
THE CHARACTERISTIC LINE OF PORTFOLIO  
THEORY

Table 6.1 gives data on the annual rates of return (%) on Afuture Fund, a mutual fund whose primary investment objective is maximum capital gain, and on the market portfolio, as measured by the Fisher Index, for the period 1971–1980.

In exercise 5.5 we introduced the *characteristic line* of investment analysis, which can be written as

$$Y_i = \alpha_i + \beta_i X_i + u_i \quad (6.1.10)$$

where  $Y_i$  = annual rate of return (%) on Afuture Fund  
 $X_i$  = annual rate of return (%) on the market portfolio

$\beta_i$  = slope coefficient, also known as the **Beta** coefficient in portfolio theory, and

$\alpha_i$  = the intercept

In the literature there is no consensus about the prior value of  $\alpha_i$ . Some empirical results have shown it to be positive and statistically significant and some have shown it to be not statistically significantly different from zero; in the latter case we could write the model as

$$Y_i = \beta_i X_i + u_i \quad (6.1.11)$$

that is, a regression through the origin.

**TABLE 6.1**  
ANNUAL RATES OF RETURN ON AFUTURE FUND  
AND ON THE FISHER INDEX (MARKET PORTFOLIO),  
1971–1980

Year	Return on Afuture Fund, % Y	Return on Fisher Index, % X
1971	67.5	19.5
1972	19.2	8.5
1973	-35.2	-29.3
1974	-42.0	-26.5
1975	63.7	61.9
1976	19.3	45.5
1977	3.6	9.5
1978	20.0	14.0
1979	40.3	35.3
1980	37.5	31.0

Source: Haim Levy and Marshall Sarnat, *Portfolio and Investment Selection: Theory and Practice*, Prentice-Hall International, Englewood Cliffs, N.J., 1984, pp. 730 and 738. These data were obtained by the authors from Weisenberg Investment Service, *Investment Companies*, 1981 edition.

(Continued)

<sup>4</sup>Henri Theil points out that if the intercept is in fact absent, the slope coefficient may be estimated with far greater precision than with the intercept term left in. See his *Introduction to Econometrics*, Prentice Hall, Englewood Cliffs, N.J., 1978, p. 76. See also the numerical example given next.

AN ILLUSTRATIVE EXAMPLE (Continued)

If we decide to use model (6.1.11), we obtain the following regression results

$$\hat{Y}_i = 1.0899 X_i$$

(0.1916)      raw  $r^2 = 0.7825$       **(6.1.12)**

$t = (5.6884)$

which shows that  $\beta_i$  is significantly greater than zero. The interpretation is that a 1 percent increase in the market rate of return leads on the average to about 1.09 percent increase in the rate of return on Afuture Fund.

How can we be sure that model (6.1.11), not (6.1.10), is appropriate, especially in view of the fact that there is no strong a priori belief in the hypothesis that  $\alpha_i$  is in fact zero? This can be checked by running the regression (6.1.10). Using the data given in Table 6.1, we obtained the following results:

$$\hat{Y}_i = 1.2797 + 1.0691X_i$$

(7.6886)    (0.2383)      **(6.1.13)**

$t = (0.1664)$     (4.4860)       $r^2 = 0.7155$

*Note:* The  $r^2$  values of (6.1.12) and (6.1.13) are *not* directly comparable. From these results one cannot reject the hypothesis that the true intercept is equal to zero, thereby justifying the use of (6.1.1), that is, regression through the origin.

In passing, note that there is not a great deal of difference in the results of (6.1.12) and (6.1.13), although the estimated standard error of  $\hat{\beta}$  is slightly lower for the regression-through-the-origin model, thus supporting Theil's argument given in footnote 4 that if  $\alpha_i$  is in fact zero, the slope coefficient may be measured with greater precision: using the data given in Table 6.1 and the regression results, the reader can easily verify that the 95% confidence interval for the slope coefficient of the regression-through-the-origin model is (0.6566, 1.5232) whereas for the model (6.1.13) it is (0.5195, 1.6186); that is, the former confidence interval is narrower than the latter.

## 6.2 SCALING AND UNITS OF MEASUREMENT

To grasp the ideas developed in this section, consider the data given in Table 6.2, which refers to U.S. gross private domestic investment (GPDI) and gross domestic product (GDP), in billions as well as millions of (chained) 1992 dollars.

**TABLE 6.2** GROSS PRIVATE DOMESTIC INVESTMENT AND GDP, UNITED STATES, 1988–1997

Observation	GPDI BL	GPDI M	GDP B	GDP M
1988	828.2000	828200.0	5865.200	5865200
1989	863.5000	863500.0	6062.000	6062000
1990	815.0000	815000.0	6136.300	6136300
1991	738.1000	738100.0	6079.400	6079400
1992	790.4000	790400.0	6244.400	6244400
1993	863.6000	863600.0	6389.600	6389600
1994	975.7000	975700.0	6610.700	6610700
1995	996.1000	996100.0	6761.600	6761600
1996	1084.1000	1084100.0	6994.800	6994800
1997	1206.4000	1206400.0	7269.800	7269800

*Note:* GPDI BL = gross private domestic investment, billions of 1992 dollars.  
 GPDI M = gross private domestic investments, millions of 1992 dollars.  
 GDP B = gross domestic product, billions of 1992 dollars.  
 GDP M = gross domestic product, millions of 1992 dollars.

*Source:* *Economic Report of the President*, 1999, Table B-2, p. 328.

Suppose in the regression of GPGDI on GDP one researcher uses data in billions of dollars but another expresses data in millions of dollars. Will the regression results be the same in both cases? If not, which results should one use? In short, do the units in which the regressand and regressor(s) are measured make any difference in the regression results? If so, what is the sensible course to follow in choosing units of measurement for regression analysis? To answer these questions, let us proceed systematically. Let

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \quad (6.2.1)$$

where  $Y = \text{GPGDI}$  and  $X = \text{GDP}$ . Define

$$Y_i^* = w_1 Y_i \quad (6.2.2)$$

$$X_i^* = w_2 X_i \quad (6.2.3)$$

where  $w_1$  and  $w_2$  are constants, called the **scale factors**;  $w_1$  may equal  $w_2$  or be different.

From (6.2.2) and (6.2.3) it is clear that  $Y_i^*$  and  $X_i^*$  are *rescaled*  $Y_i$  and  $X_i$ . Thus, if  $Y_i$  and  $X_i$  are measured in billions of dollars and one wants to express them in millions of dollars, we will have  $Y_i^* = 1000 Y_i$  and  $X_i^* = 1000 X_i$ ; here  $w_1 = w_2 = 1000$ .

Now consider the regression using  $Y_i^*$  and  $X_i^*$  variables:

$$Y_i^* = \hat{\beta}_1^* + \hat{\beta}_2^* X_i^* + \hat{u}_i^* \quad (6.2.4)$$

where  $Y_i^* = w_1 Y_i$ ,  $X_i^* = w_2 X_i$ , and  $\hat{u}_i^* = w_1 \hat{u}_i$ . (Why?)

We want to find out the relationships between the following pairs:

1.  $\hat{\beta}_1$  and  $\hat{\beta}_1^*$
2.  $\hat{\beta}_2$  and  $\hat{\beta}_2^*$
3.  $\text{var}(\hat{\beta}_1)$  and  $\text{var}(\hat{\beta}_1^*)$
4.  $\text{var}(\hat{\beta}_2)$  and  $\text{var}(\hat{\beta}_2^*)$
5.  $\hat{\sigma}^2$  and  $\hat{\sigma}^{*2}$
6.  $r_{xy}^2$  and  $r_{x^*y^*}^2$

From least-squares theory we know (see Chapter 3) that

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} \quad (6.2.5)$$

$$\hat{\beta}_2 = \frac{\sum x_i y_i}{\sum x_i^2} \quad (6.2.6)$$

$$\text{var}(\hat{\beta}_1) = \frac{\sum X_i^2}{n \sum x_i^2} \cdot \sigma^2 \quad (6.2.7)$$

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_i^2} \quad (6.2.8)$$

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-2} \quad (6.2.9)$$

Applying the OLS method to (6.2.4), we obtain similarly

$$\hat{\beta}_1^* = \bar{Y}^* - \hat{\beta}_2^* \bar{X}^* \quad (6.2.10)$$

$$\hat{\beta}_2^* = \frac{\sum x_i^* y_i^*}{\sum x_i^{*2}} \quad (6.2.11)$$

$$\text{var}(\hat{\beta}_1^*) = \frac{\sum X_i^{*2}}{n \sum x_i^{*2}} \cdot \sigma^{*2} \quad (6.2.12)$$

$$\text{var}(\hat{\beta}_2^*) = \frac{\sigma^{*2}}{\sum x_i^{*2}} \quad (6.2.13)$$

$$\hat{\sigma}^{*2} = \frac{\sum \hat{u}_i^{*2}}{(n-2)} \quad (6.2.14)$$

From these results it is easy to establish relationships between the two sets of parameter estimates. All that one has to do is recall these definitional relationships:  $Y_i^* = w_1 Y_i$  (or  $y_i^* = w_1 y_i$ );  $X_i^* = w_2 X_i$  (or  $x_i^* = w_2 x_i$ );  $\hat{u}_i^* = w_1 \hat{u}_i$ ;  $\bar{Y}^* = w_1 \bar{Y}$  and  $\bar{X}^* = w_2 \bar{X}$ . Making use of these definitions, the reader can easily verify that

$$\hat{\beta}_2^* = \left( \frac{w_1}{w_2} \right) \hat{\beta}_2 \quad (6.2.15)$$

$$\hat{\beta}_1^* = w_1 \hat{\beta}_1 \quad (6.2.16)$$

$$\hat{\sigma}^{*2} = w_1^2 \hat{\sigma}^2 \quad (6.2.17)$$

$$\text{var}(\hat{\beta}_1^*) = w_1^2 \text{var}(\hat{\beta}_1) \quad (6.2.18)$$

$$\text{var}(\hat{\beta}_2^*) = \left( \frac{w_1}{w_2} \right)^2 \text{var}(\hat{\beta}_2) \quad (6.2.19)$$

$$r_{xy}^2 = r_{x^*y^*}^2 \quad (6.2.20)$$

From the preceding results it should be clear that, given the regression results based on one scale of measurement, one can derive the results based on another scale of measurement once the scaling factors, the  $w$ 's, are known. In practice, though, one should choose the units of measurement sensibly; there is little point in carrying all those zeros in expressing numbers in millions or billions of dollars.

From the results given in (6.2.15) through (6.2.20) one can easily derive some special cases. For instance, if  $w_1 = w_2$ , that is, the scaling factors are identical, the slope coefficient and its standard error remain unaffected in going from the  $(Y_i, X_i)$  to the  $(Y_i^*, X_i^*)$  scale, which should be intuitively clear. However, the intercept and its standard error are both multiplied by  $w_1$ . But if the  $X$  scale is not changed (i.e.,  $w_2 = 1$ ) and the  $Y$  scale is changed by the factor  $w_1$ , the slope as well as the intercept coefficients and their respective standard errors are all multiplied by the same  $w_1$  factor. Finally, if the  $Y$  scale remains unchanged (i.e.,  $w_1 = 1$ ) but the  $X$  scale is changed by the factor  $w_2$ , the slope coefficient and its standard error are multiplied by the factor  $(1/w_2)$  but the intercept coefficient and its standard error remain unaffected.

It should, however, be noted that the transformation from the  $(Y, X)$  to the  $(Y^*, X^*)$  scale does not affect the properties of the OLS estimators discussed in the preceding chapters.

**A NUMERICAL EXAMPLE: THE RELATIONSHIP BETWEEN GPDI AND GDP, UNITED STATES, 1988–1997**

To substantiate the preceding theoretical results, let us return to the data given in Table 6.2 and examine the following results (numbers in parentheses are the estimated standard errors).

Both GPDI and GDP in billions of dollars:

$$\begin{aligned} \widehat{\text{GPDI}}_t &= -1026.498 + 0.3016 \text{ GDP}_t \\ \text{se} &= (257.5874) \quad (0.0399) \quad r^2 = 0.8772 \end{aligned} \quad (6.2.21)$$

Both GPDI and GDP in millions of dollars:

$$\begin{aligned} \widehat{\text{GPDI}}_t &= -1,026,498 + 0.3016 \text{ GDP}_t \\ \text{se} &= (257,587.4) \quad (0.0399) \quad r^2 = 0.8772 \end{aligned} \quad (6.2.22)$$

Notice that the intercept as well as its standard error is 1000 times the corresponding values in the regression (6.2.21) (note that  $w_1 = 1000$  in going from billions to millions of dollars), but the slope coefficient as well as its standard error is unchanged, in accordance with theory.

GPDI in billions of dollars and GDP in millions of dollars:

$$\begin{aligned} \widehat{\text{GPDI}}_t &= -1026.498 + 0.000301 \text{ GDP}_t \\ \text{se} &= (257.5874) \quad (0.0000399) \quad r^2 = 0.8772 \end{aligned} \quad (6.2.23)$$

As expected, the slope coefficient as well as its standard error is  $1/1000$  its value in (6.2.21), since only the  $X$ , or GDP, scale is changed.

GPDI in millions of dollars and GDP in billions of dollars:

$$\begin{aligned} \widehat{\text{GPDI}}_t &= -1,026,498 + 301.5826 \text{ GDP}_t \\ \text{se} &= (257,587.4) \quad (39.89989) \quad r^2 = 0.8772 \end{aligned} \quad (6.2.24)$$

Again notice that both the intercept and the slope coefficients as well as their respective standard errors are 1000 times their values in (6.2.21), in accordance with our theoretical results.

Notice that in all the regressions presented above the  $r^2$  value remains the same, which is not surprising because the  $r^2$  value is *invariant* to changes in the unit of measurement, as it is a pure, or dimensionless, number.

**A Word about Interpretation**

Since the slope coefficient  $\beta_2$  is simply the rate of change, it is measured in the units of the ratio

$$\frac{\text{Units of the dependent variable}}{\text{Units of the explanatory variable}}$$

Thus in regression (6.2.21) the interpretation of the slope coefficient 0.3016 is that if GDP changes by a unit, which is 1 billion dollars, GPDI on the average changes by 0.3016 billion dollars. In regression (6.2.23) a unit change in GDP, which is 1 million dollars, leads on average to a 0.000302 billion dollar change in GPDI. The two results are of course identical in the effects of GDP on GPDI; they are simply expressed in different units of measurement.

**6.3 REGRESSION ON STANDARDIZED VARIABLES**

We saw in the previous section that the units in which the regressand and regressor(s) are expressed affect the interpretation of the regression coefficients. This can be avoided if we are willing to express the regressand and regressor(s) as *standardized variables*. A variable is said to be standardized if we subtract the mean value of the variable from its individual values and divide the difference by the standard deviation of that variable.

Thus, in the regression of  $Y$  and  $X$ , if we redefine these variables as

$$Y_i^* = \frac{Y_i - \bar{Y}}{S_Y} \quad (6.3.1)$$

$$X_i^* = \frac{X_i - \bar{X}}{S_X} \quad (6.3.2)$$

where  $\bar{Y}$  = sample mean of  $Y$ ,  $S_Y$  = sample standard deviation of  $Y$ ,  $\bar{X}$  = sample mean of  $X$ , and  $S_X$  is the sample standard deviation of  $X$ ; the variables  $Y_i^*$  and  $X_i^*$  are called **standardized variables**.

*An interesting property of a standardized variable is that its mean value is always zero and its standard deviation is always 1.* (For proof, see Appendix 6A, Section 6A.2.)

As a result, it does not matter in what unit the regressand and regressor(s) are measured. Therefore, instead of running the standard (bivariate) regression:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (6.3.3)$$

we could run regression on the standardized variables as

$$Y_i^* = \beta_1^* + \beta_2^* X_i^* + u_i^* \quad (6.3.4)$$

$$= \beta_2^* X_i^* + u_i^* \quad (6.3.5)$$

since it is easy to show that, in the regression involving standardized regressand and regressor(s), the intercept term is always zero.<sup>5</sup> The regression coefficients of the standardized variables, denoted by  $\beta_1^*$  and  $\beta_2^*$ , are known in the literature as the **beta coefficients**.<sup>6</sup> Incidentally, notice that (6.3.5) is a regression through the origin.

How do we interpret the beta coefficients? The interpretation is that if the (standardized) regressor increases by one standard deviation, on average, the (standardized) regressand increases by  $\beta_2^*$  standard deviation units. Thus, unlike the traditional model (6.3.3), we measure the effect not in terms of the original units in which  $Y$  and  $X$  are expressed, but in standard deviation units.

To show the difference between (6.3.3) and (6.3.5), let us return to the GPD and GDP example discussed in the preceding section. The results of (6.2.21) discussed previously are reproduced here for convenience.

$$\begin{aligned} \widehat{\text{GPD}}_t &= -1026.498 + 0.3016 \text{ GDP}_t \\ \text{se} &= (257.5874) \quad (0.0399) \quad r^2 = 0.8872 \end{aligned} \quad (6.3.6)$$

where GPD and GDP are measured in billions of dollars.

The results corresponding to (6.3.5) are as follows, where the starred variables are standardized variables:

$$\begin{aligned} \widehat{\text{GPD}}_t^* &= 0.9387 \text{ GDP}_t^* \\ \text{se} &= (0.1149) \end{aligned} \quad (6.3.7)$$

We know how to interpret (6.3.6): If GDP goes up by a dollar, on average GPD goes up by about 30 cents. How about (6.3.7)? Here the interpretation is that if the (standardized) GDP increases by one standard deviation, on average, the (standardized) GPD increases by about 0.94 standard deviations.

What is the advantage of the standardized regression model over the traditional model? The advantage becomes more apparent if there is more than one regressor, a topic we will take up in Chapter 7. By standardizing all regressors, we put them on equal basis and therefore can compare them directly. If the coefficient of a standardized regressor is larger than that of another standardized regressor appearing in that model, then the latter contributes more relatively to the explanation of the regressand than the latter. In other words, we can use the beta coefficients as a measure of relative strength of the various regressors. But more on this in the next two chapters.

Before we leave this topic, two points may be noted. First, for the standardized regression (6.3.7) we have not given the  $r^2$  value because this is a regression through the origin for which the usual  $r^2$  is not applicable, as pointed out in Section 6.1. Second, there is an interesting relationship between the  $\beta$  coefficients of the conventional model and the beta coefficients.

<sup>5</sup>Recall from Eq. (3.1.7) that intercept = mean value of the dependent variable – slope times the mean value of the regressor. But for the standardized variables the mean values of the dependent variable and the regressor are zero. Hence the intercept value is zero.

<sup>6</sup>Do not confuse these beta coefficients with the beta coefficients of finance theory.

For the bivariate case, the relationship is as follows:

$$\hat{\beta}_2^* = \hat{\beta}_2 \left( \frac{S_x}{S_y} \right) \quad (6.3.8)$$

where  $S_x$  = the sample standard deviation of the  $X$  regressor and  $S_y$  = the sample standard deviation of the regressand. Therefore, one can crisscross between the  $\beta$  and beta coefficients if we know the (sample) standard deviation of the regressor and regressand. We will see in the next chapter that this relationship holds true in the multiple regression also. It is left as an exercise for the reader to verify (6.3.8) for our illustrative example.

#### 6.4 FUNCTIONAL FORMS OF REGRESSION MODELS

As noted in Chapter 2, this text is concerned primarily with models that are linear in the parameters; they may or may not be linear in the variables. In the sections that follow we consider some commonly used regression models that may be nonlinear in the variables but are linear in the parameters or that can be made so by suitable transformations of the variables. In particular, we discuss the following regression models:

1. The log-linear model
2. Semilog models
3. Reciprocal models
4. The logarithmic reciprocal model

We discuss the special features of each model, when they are appropriate, and how they are estimated. Each model is illustrated with suitable examples.

#### 6.5 HOW TO MEASURE ELASTICITY: THE LOG-LINEAR MODEL

Consider the following model, known as the **exponential regression model**:

$$Y_i = \beta_1 X_i^{\beta_2} e^{u_i} \quad (6.5.1)$$

which may be expressed alternatively as<sup>7</sup>

$$\ln Y_i = \ln \beta_1 + \beta_2 \ln X_i + u_i \quad (6.5.2)$$

where  $\ln$  = natural log (i.e., log to the base  $e$ , and where  $e = 2.718$ ).<sup>8</sup>

If we write (6.5.2) as

$$\ln Y_i = \alpha + \beta_2 \ln X_i + u_i \quad (6.5.3)$$

<sup>7</sup>Note these properties of the logarithms: (1)  $\ln(AB) = \ln A + \ln B$ , (2)  $\ln(A/B) = \ln A - \ln B$ , and (3)  $\ln(A^k) = k \ln A$ , assuming that  $A$  and  $B$  are positive, and where  $k$  is some constant.

<sup>8</sup>In practice one may use common logarithms, that is, log to the base 10. The relationship between the natural log and common log is:  $\ln_e X = 2.3026 \log_{10} X$ . By convention,  $\ln$  means natural logarithm, and  $\log$  means logarithm to the base 10; hence there is no need to write the subscripts  $e$  and 10 explicitly.



where  $\alpha = \ln \beta_1$ , this model is linear in the parameters  $\alpha$  and  $\beta_2$ , linear in the logarithms of the variables  $Y$  and  $X$ , and can be estimated by OLS regression. Because of this linearity, such models are called **log-log**, **double-log**, or **log-linear** models.

If the assumptions of the classical linear regression model are fulfilled, the parameters of (6.5.3) can be estimated by the OLS method by letting

$$Y_i^* = \alpha + \beta_2 X_i^* + u_i \quad (6.5.4)$$

where  $Y_i^* = \ln Y_i$  and  $X_i^* = \ln X_i$ . The OLS estimators  $\hat{\alpha}$  and  $\hat{\beta}_2$  obtained will be best linear unbiased estimators of  $\alpha$  and  $\beta_2$ , respectively.

One attractive feature of the log-log model, which has made it popular in applied work, is that the slope coefficient  $\beta_2$  measures the **elasticity** of  $Y$  with respect to  $X$ , that is, the percentage change in  $Y$  for a given (small) percentage change in  $X$ .<sup>9</sup> Thus, if  $Y$  represents the quantity of a commodity demanded and  $X$  its unit price,  $\beta_2$  measures the price elasticity of demand, a parameter of considerable economic interest. If the relationship between quantity demanded and price is as shown in Figure 6.3a, the double-log

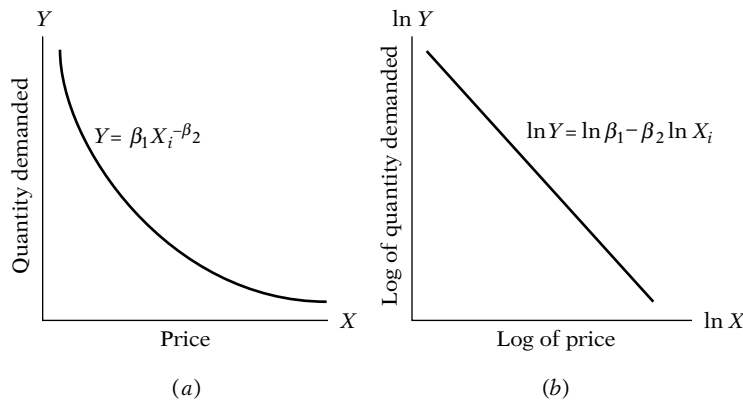


FIGURE 6.3 Constant-elasticity model.

<sup>9</sup>The elasticity coefficient, in calculus notation, is defined as  $(dY/Y)/(dX/X) = [(dY/dX)(X/Y)]$ . Readers familiar with differential calculus will readily see that  $\beta_2$  is in fact the elasticity coefficient.

A *technical note*: The calculus-minded reader will note that  $d(\ln X)/dX = 1/X$  or  $d(\ln X) = dX/X$ , that is, for infinitesimally small changes (note the differential operator  $d$ ) the change in  $\ln X$  is equal to the relative or proportional change in  $X$ . In practice, though, if the change in  $X$  is small, this relationship can be written as: change in  $\ln X \doteq$  relative change in  $X$ , where  $\doteq$  means approximately. Thus, for small changes,

$$(\ln X_t - \ln X_{t-1}) \doteq (X_t - X_{t-1})/X_{t-1} = \text{relative change in } X$$

Incidentally, the reader should note these terms, which will occur frequently: (1) **absolute change**, (2) **relative** or **proportional change**, and (3) **percentage change**, or **percent growth rate**. Thus,  $(X_t - X_{t-1})$  represents absolute change,  $(X_t - X_{t-1})/X_{t-1} = (X_t/X_{t-1} - 1)$  is relative or proportional change and  $[(X_t - X_{t-1})/X_{t-1}]100$  is the percentage change, or the growth rate.  $X_t$  and  $X_{t-1}$  are, respectively, the current and previous values of the variable  $X$ .

transformation as shown in Figure 6.3*b* will then give the estimate of the price elasticity ( $-\beta_2$ ).

Two special features of the log-linear model may be noted: The model assumes that the elasticity coefficient between  $Y$  and  $X$ ,  $\beta_2$ , remains constant throughout (why?), hence the alternative name **constant elasticity model**.<sup>10</sup> In other words, as Figure 6.3*b* shows, the change in  $\ln Y$  per unit change in  $\ln X$  (i.e., the elasticity,  $\beta_2$ ) remains the same no matter at which  $\ln X$  we measure the elasticity. Another feature of the model is that although  $\hat{\alpha}$  and  $\hat{\beta}_2$  are unbiased estimates of  $\alpha$  and  $\beta_2$ ,  $\hat{\beta}_1$  (the parameter entering the original model) when estimated as  $\hat{\beta}_1 = \text{antilog}(\hat{\alpha})$  is itself a biased estimator. In most practical problems, however, the intercept term is of secondary importance, and one need not worry about obtaining its unbiased estimate.<sup>11</sup>

In the two-variable model, the simplest way to decide whether the log-linear model fits the data is to plot the scattergram of  $\ln Y_i$  against  $\ln X_i$  and see if the scatter points lie approximately on a straight line, as in Figure 6.3*b*.

**AN ILLUSTRATIVE EXAMPLE:  
EXPENDITURE ON DURABLE GOODS  
IN RELATION TO TOTAL PERSONAL  
CONSUMPTION EXPENDITURE**

Table 6.3 presents data on total personal consumption expenditure (PCEXP), expenditure on durable goods (EXPDUR), expenditure on nondurable goods (EXPNONDUR), and expenditure on services (EXPSERVICES), all measured in 1992 billions of dollars.<sup>12</sup>

Suppose we wish to find the elasticity of expenditure on durable goods with respect to total personal consumption expenditure. Plotting the log of expenditure on durable goods against the log of total personal consumption expenditure, you will see that the relationship between the two variables is linear. Hence, the double-log model may be appropriate. The regression results

are as follows:

$$\begin{aligned} \ln \widehat{\text{EXPDUR}}_t &= -9.6971 + 1.9056 \ln \text{PCEXP}_t \\ \text{se} &= (0.4341) \quad (0.0514) \quad \mathbf{(6.5.5)} \\ t &= (-22.3370)^* \quad (37.0962)^* \quad r^2 = 0.9849 \end{aligned}$$

where \* indicates that the  $p$  value is extremely small.

As these results show, the elasticity of EXPDUR with respect to PCEXP is about 1.90, suggesting that if total personal expenditure goes up by 1 percent, on average, the expenditure on durable goods goes up by about 1.90 percent. Thus, expenditure on durable goods is very responsive to changes in personal consumption expenditure. This is one reason why producers of durable goods keep a keen eye on changes in personal income and personal consumption expenditure. In exercises 6.17 and 6.18, the reader is asked to carry out a similar exercise for nondurable goods expenditure and expenditure on services.

(Continued)

<sup>10</sup>A constant elasticity model will give a constant total revenue change for a given percentage change in price regardless of the absolute level of price. Readers should contrast this result with the elasticity conditions implied by a simple linear demand function,  $Y_i = \beta_1 + \beta_2 X_i + u_i$ . However, a simple linear function gives a constant quantity change per unit change in price. Contrast this with what the log-linear model implies for a given dollar change in price.

<sup>11</sup>Concerning the nature of the bias and what can be done about it, see Arthur S. Goldberger, *Topics in Regression Analysis*, Macmillan, New York, 1978, p. 120.

<sup>12</sup>Durable goods include motor vehicles and parts, furniture, and household equipment; nondurable goods include food, clothing, gasoline and oil, fuel oil and coal; and services include housing, electricity and gas, transportation, and medical care.

AN ILLUSTRATIVE EXAMPLE: . . . (Continued)

**TABLE 6.3**  
TOTAL PERSONAL EXPENDITURE AND CATEGORIES

Observation	EXPSERVICES	EXPDUR	EXPNONDUR	PCEXP
1993-I	2445.3	504.0	1337.5	4286.8
1993-II	2455.9	519.3	1347.8	4322.8
1993-III	2480.0	529.9	1356.8	4366.6
1993-IV	2494.4	542.1	1361.8	4398.0
1994-I	2510.9	550.7	1378.4	4439.4
1994-II	2531.4	558.8	1385.5	4472.2
1994-III	2543.8	561.7	1393.2	4498.2
1994-IV	2555.9	576.6	1402.5	4534.1
1995-I	2570.4	575.2	1410.4	4555.3
1995-II	2594.8	583.5	1415.9	4593.6
1995-III	2610.3	595.3	1418.5	4623.4
1995-IV	2622.9	602.4	1425.6	4650.0
1996-I	2648.5	611.0	1433.5	4692.1
1996-II	2668.4	629.5	1450.4	4746.6
1996-III	2688.1	626.5	1454.7	4768.3
1996-IV	2701.7	637.5	1465.1	4802.6
1997-I	2722.1	656.3	1477.9	4853.4
1997-II	2743.6	653.8	1477.1	4872.7
1997-III	2775.4	679.6	1495.7	4947.0
1997-IV	2804.8	648.8	1494.3	4981.0
1998-I	2829.3	710.3	1521.2	5055.1
1998-II	2866.8	729.4	1540.9	5130.2
1998-III	2904.8	733.7	1549.1	5181.8

Note: EXPSERVICES = expenditure on services, billions of 1992 dollars.

EXPDUR = expenditure on durable goods, billions of 1992 dollars.

EXPNONDUR = expenditure on nondurable goods, billions of 1992 dollars.

PCEXP = total personal consumption expenditure, billions of 1992 dollars.

Source: *Economic Report of the President*, 1999, Table B-17, p. 347.

## 6.6 SEMILOG MODELS: LOG-LIN AND LIN-LOG MODELS

### How to Measure the Growth Rate: The Log-Lin Model

Economists, businesspeople, and governments are often interested in finding out the rate of growth of certain economic variables, such as population, GNP, money supply, employment, productivity, and trade deficit.

Suppose we want to find out the growth rate of personal consumption expenditure on services for the data given in Table 6.3. Let  $Y_t$  denote real expenditure on services at time  $t$  and  $Y_0$  the initial value of the expenditure on services (i.e., the value at the end of 1992-IV). You may recall the following well-known compound interest formula from your introductory course in economics.

$$Y_t = Y_0(1 + r)^t \quad (6.6.1)$$

where  $r$  is the compound (i.e., over time) rate of growth of  $Y$ . Taking the natural logarithm of (6.6.1), we can write

$$\ln Y_t = \ln Y_0 + t \ln(1 + r) \quad (6.6.2)$$

Now letting

$$\beta_1 = \ln Y_0 \quad (6.6.3)$$

$$\beta_2 = \ln(1 + r) \quad (6.6.4)$$

we can write (6.6.2) as

$$\ln Y_t = \beta_1 + \beta_2 t \quad (6.6.5)$$

Adding the disturbance term to (6.6.5), we obtain<sup>13</sup>

$$\ln Y_t = \beta_1 + \beta_2 t + u_t \quad (6.6.6)$$

This model is like any other linear regression model in that the parameters  $\beta_1$  and  $\beta_2$  are linear. The only difference is that the regressand is the logarithm of  $Y$  and the regressor is “time,” which will take values of 1, 2, 3, etc.

Models like (6.6.6) are called **semilog models** because only one variable (in this case the regressand) appears in the logarithmic form. For descriptive purposes a model in which the regressand is logarithmic will be called a **log-lin model**. Later we will consider a model in which the regressand is linear but the regressor(s) are logarithmic and call it a **lin-log model**.

Before we present the regression results, let us examine the properties of model (6.6.5). In this model *the slope coefficient measures the constant proportional or relative change in  $Y$  for a given absolute change in the value of the regressor* (in this case the variable  $t$ ), that is,<sup>14</sup>

$$\beta_2 = \frac{\text{relative change in regressand}}{\text{absolute change in regressor}} \quad (6.6.7)$$

If we multiply the relative change in  $Y$  by 100, (6.6.7) will then give the percentage change, or the *growth rate*, in  $Y$  for an absolute change in  $X$ , the regressor. That is, 100 times  $\beta_2$  gives the growth rate in  $Y$ ; 100 times  $\beta_2$  is

<sup>13</sup>We add the error term because the compound interest formula will not hold exactly. Why we add the error after the logarithmic transformation is explained in Sec. 6.8.

<sup>14</sup>Using differential calculus one can show that  $\beta_2 = d(\ln Y)/dX = (1/Y)(dY/dX) = (dY/Y)/dX$ , which is nothing but (6.6.7). For small changes in  $Y$  and  $X$  this relation may be approximated by

$$\frac{(Y_t - Y_{t-1})/Y_{t-1}}{(X_t - X_{t-1})}$$

Note: Here  $X = t$ .

known in the literature as the **semielasticity** of  $Y$  with respect to  $X$ . (Question: To get the elasticity, what will we have to do?)

#### AN ILLUSTRATIVE EXAMPLE: THE RATE OF GROWTH EXPENDITURE ON SERVICES

To illustrate the growth model (6.6.6), consider the data on expenditure on services given in Table 6.3. The regression results are as follows:

$$\begin{aligned} \widehat{\ln EXS}_t &= 7.7890 + 0.00743t \\ \text{se} &= (0.0023) \quad (0.00017) \quad \text{(6.6.8)} \\ t &= (3387.619)^* \quad (44.2826)^* \quad r^2 = 0.9894 \end{aligned}$$

*Note:* EXS stands for expenditure on services and \* denotes that the  $p$  value is extremely small.

The interpretation of Eq. (6.6.8) is that over the quarterly period 1993:1 to 1998:3, expenditure on services increased at the (quarterly) rate of 0.743 percent. Roughly, this is equal to an annual growth rate of 2.97 percent. Since  $7.7890 = \log$  of EXS at the beginning of the study period, by taking its antilog we obtain 2413.90 (billion dollars) as the beginning value of EXS (i.e., the value at

the end of the fourth quarter of 1992). The regression line obtained in Eq. (6.6.8) is sketched in Figure 6.4.

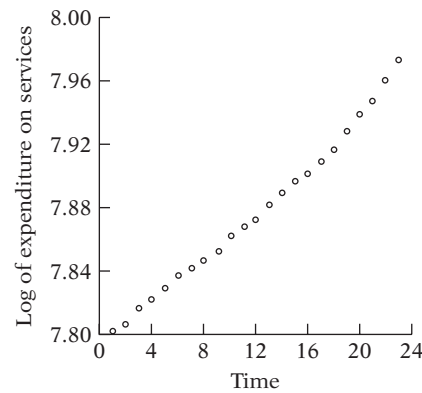


FIGURE 6.4

**Instantaneous versus Compound Rate of Growth.** The coefficient of the trend variable in the growth model (6.6.6),  $\beta_2$ , gives the **instantaneous** (at a point in time) rate of growth and not the **compound** (over a period of time) rate of growth. But the latter can be easily found from (6.6.4) by taking the antilog of the estimated  $\beta_2$  and subtracting 1 from it and multiplying the difference by 100. Thus, for our illustrative example, the estimated slope coefficient is 0.00743. Therefore,  $[\text{antilog}(0.00743) - 1] = 0.00746$  or 0.746 percent. Thus, in the illustrative example, the *compound rate of growth* on expenditure on services was about 0.746 percent per quarter, which is slightly higher than the instantaneous growth rate of 0.743 percent. This is of course due to the compounding effect.

**Linear Trend Model.** Instead of estimating model (6.6.6), researchers sometimes estimate the following model:

$$Y_t = \beta_1 + \beta_2 t + u_t \quad \text{(6.6.9)}$$

That is, instead of regressing the log of  $Y$  on time, they regress  $Y$  on time, where  $Y$  is the regressand under consideration. Such a model is called a **linear trend model** and the time variable  $t$  is known as the *trend variable*. If the slope coefficient in (6.6.9) is positive, there is an **upward trend** in  $Y$ , whereas if it is negative, there is a **downward trend** in  $Y$ .

For the expenditure on services data that we considered earlier, the results of fitting the linear trend model (6.6.9) are as follows:

$$\widehat{\text{EXS}}_t = 2405.848 + 19.6920t \quad (6.6.10)$$

$$t = (322.9855) \quad (36.2479) \quad r^2 = 0.9843$$

In contrast to Eq. (6.6.8), the interpretation of Eq. (6.6.10) is as follows: Over the quarterly period 1993-I to 1998-III, on average, expenditure on services increased at the absolute (*note: not relative*) rate of about 20 billion dollars per quarter. That is, there was an upward trend in the expenditure on services.

The choice between the growth rate model (6.6.8) and the linear trend model (6.6.10) will depend upon whether one is interested in the relative or absolute change in the expenditure on services, although for comparative purposes it is the relative change that is generally more relevant. In passing, *observe that we cannot compare the  $r^2$  values of models (6.6.8) and (6.6.10) because the regressands in the two models are different.* We will show in Chapter 7 how one compares the  $R^2$ 's of models like (6.6.8) and (6.6.10).

### The Lin-Log Model

Unlike the growth model just discussed, in which we were interested in finding the percent growth in  $Y$  for an absolute change in  $X$ , suppose we now want to find the absolute change in  $Y$  for a percent change in  $X$ . A model that can accomplish this purpose can be written as:

$$Y_i = \beta_1 + \beta_2 \ln X_i + u_i \quad (6.6.11)$$

For descriptive purposes we call such a model a **lin-log model**.

Let us interpret the slope coefficient  $\beta_2$ .<sup>15</sup> As usual,

$$\beta_2 = \frac{\text{change in } Y}{\text{change in } \ln X}$$

$$= \frac{\text{change in } Y}{\text{relative change in } X}$$

The second step follows from the fact that *a change in the log of a number is a relative change*.

<sup>15</sup>Again, using differential calculus, we have

$$\frac{dY}{dX} = \beta_2 \left( \frac{1}{X} \right)$$

Therefore,

$$\beta_2 = \frac{dY}{\frac{dX}{X}} = (6.6.12)$$

Symbolically, we have

$$\beta_2 = \frac{\Delta Y}{\Delta X/X} \tag{6.6.12}$$

where, as usual,  $\Delta$  denotes a small change. Equation (6.6.12) can be written, equivalently, as

$$\Delta Y = \beta_2(\Delta X/X) \tag{6.6.13}$$

This equation states that the absolute change in  $Y$  ( $= \Delta Y$ ) is equal to slope times the relative change in  $X$ . If the latter is multiplied by 100, then (6.6.13) gives the absolute change in  $Y$  for a percentage change in  $X$ . Thus, if  $(\Delta X/X)$  changes by 0.01 unit (or 1 percent), the absolute change in  $Y$  is  $0.01(\beta_2)$ ; if in an application one finds that  $\beta_2 = 500$ , the absolute change in  $Y$  is  $(0.01)(500) = 5.0$ . Therefore, when regression (6.6.11) is estimated by OLS, do not forget to multiply the value of the estimated slope coefficient by 0.01, or, what amounts to the same thing, divide it by 100. *If you do not keep this in mind, your interpretation in an application will be highly misleading.*

The practical question is: When is a lin-log model like (6.6.11) useful? An interesting application has been found in the so-called **Engel expenditure** models, named after the German statistician Ernst Engel, 1821–1896. (See exercise 6.10.) Engel postulated that “the total expenditure that is devoted to food tends to increase in arithmetic progression as total expenditure increases in geometric progression.”<sup>16</sup>

**AN ILLUSTRATIVE EXAMPLE**

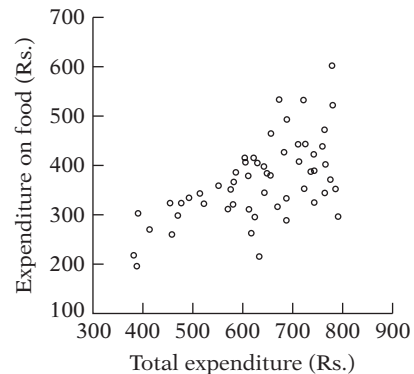
As an illustration of the lin-log model, let us revisit our example on food expenditure in India, Example 3.2. There we fitted a linear-in-variables model as a first approximation. But if we plot the data we obtain the plot in Figure 6.5. As this figure suggests, food expenditure increases more slowly as total expenditure increases, perhaps giving credence to Engel’s law. The results of fitting the lin-log model to the data are as follows:

$$\widehat{\text{FoodExp}}_i = -1283.912 + 257.2700 \ln \text{TotalExp}_i$$

$$t = (-4.3848)^* \quad (5.6625)^* \quad r^2 = 0.3769$$

**(6.6.14)**

Note: \* denotes an extremely small  $p$  value.



**FIGURE 6.5**

(Continued)

<sup>16</sup>See Chandan Mukherjee, Howard White, and Marc Wuyts, *Econometrics and Data Analysis for Developing Countries*, Routledge, London, 1998, p. 158. This quote is attributed to H. Working, “Statistical Laws of Family Expenditure,” *Journal of the American Statistical Association*, vol. 38, 1943, pp. 43–56.

## AN ILLUSTRATIVE EXAMPLE (Continued)

Interpreted in the manner described earlier, the slope coefficient of about 257 means that an increase in the total food expenditure of 1 percent, on average, leads to about 2.57 rupees increase in the expenditure on food of the 55 families included in the sample. (Note: We have divided the estimated slope coefficient by 100.)

Before proceeding further, note that if you want to compute the elasticity coefficient for the log–lin or lin–log models, you can do so from the definition of the elasticity

coefficient given before, namely,

$$\text{Elasticity} = \frac{dY}{dX} \frac{X}{Y}$$

As a matter of fact, once the functional form of a model is known, one can compute elasticities by applying the preceding definition. (Table 6.6, given later, summarizes the elasticity coefficients for the various models.)

## 6.7 RECIPROCAL MODELS

Models of the following type are known as **reciprocal** models.

$$Y_i = \beta_1 + \beta_2 \left( \frac{1}{X_i} \right) + u_i \quad (6.7.1)$$

Although this model is nonlinear in the variable  $X$  because it enters inversely or reciprocally, the model is linear in  $\beta_1$  and  $\beta_2$  and is therefore a linear regression model.<sup>17</sup>

This model has these features: As  $X$  increases indefinitely, the term  $\beta_2(1/X)$  approaches zero (note:  $\beta_2$  is a constant) and  $Y$  approaches the limiting or *asymptotic* value  $\beta_1$ . Therefore, models like (6.7.1) have built in them an **asymptote** or limit value that the dependent variable will take when the value of the  $X$  variable increases indefinitely.<sup>18</sup>

Some likely shapes of the curve corresponding to (6.7.1) are shown in Figure 6.6. As an illustration of Figure 6.6a, consider the data given in Table 6.4. These are cross-sectional data for 64 countries on child mortality and a few other variables. For now, concentrate on the variables, child mortality (CM) and per capita GNP, which are plotted in Figure 6.7.

As you can see, this figure resembles Figure 6.6a: As per capita GNP increases, one would expect child mortality to decrease because people can afford to spend more on health care, assuming all other factors remain constant. But the relationship is not a straight line one: As per capita GNP increases, initially there is dramatic drop in CM but the drop tapers off as per capita GNP continues to increase.

<sup>17</sup>If we let  $X_i^* = (1/X_i)$ , then (6.7.1) is linear in the parameters as well as the variables  $Y_i$  and  $X_i^*$ .

<sup>18</sup>The slope of (6.7.1) is:  $dY/dX = -\beta_2(1/X^2)$ , implying that if  $\beta_2$  is positive, the slope is negative throughout, and if  $\beta_2$  is negative, the slope is positive throughout. See Figures 6.6a and 6.6c, respectively.



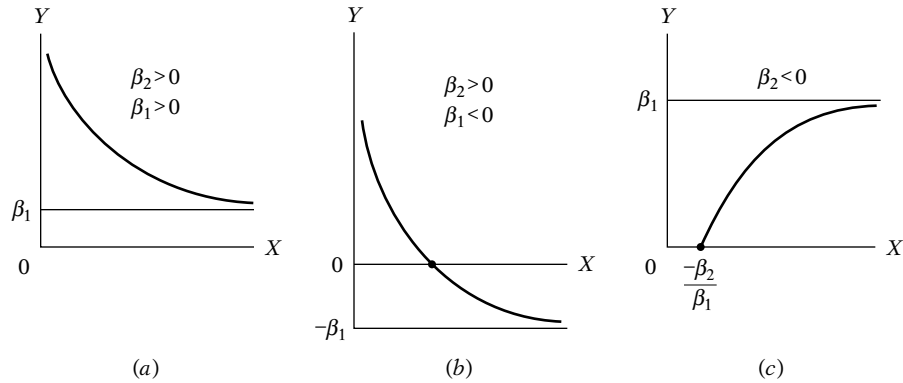


FIGURE 6.6 The reciprocal model:  $Y = \beta_1 + \beta_2 \left( \frac{1}{X} \right)$ .

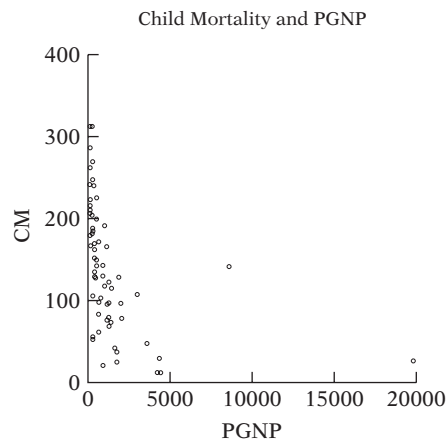


FIGURE 6.7 Relationship between child mortality and per capita GNP in 66 countries.

If we try to fit the reciprocal model (6.7.1), we obtain the following regression results:

$$\widehat{CM}_i = 81.79436 + 27,273.17 \left( \frac{1}{PGNP_i} \right) \tag{6.7.2}$$

$$se = (10.8321) \quad (3759.999)$$

$$t = (7.5511) \quad (7.2535) \quad r^2 = 0.4590$$

As per capita GNP increases indefinitely, child mortality approaches its asymptotic value of about 82 deaths per thousand. As explained in footnote 18, the positive value of the coefficient of  $(1/PGNP_i)$  implies that the rate of change of CM with respect to PGNP is negative.

One of the important applications of Figure 6.6b is the celebrated Phillips curve of macroeconomics. Using the data on percent rate of change of money wages ( $Y$ ) and the unemployment rate ( $X$ ) for the United Kingdom

**TABLE 6.4** FERTILITY AND OTHER DATA FOR 64 COUNTRIES

Observation	CM	FLFP	PGNP	TFR	Observation	CM	FLFP	PGNP	TFR
1	128	37	1870	6.66	33	142	50	8640	7.17
2	204	22	130	6.15	34	104	62	350	6.60
3	202	16	310	7.00	35	287	31	230	7.00
4	197	65	570	6.25	36	41	66	1620	3.91
5	96	76	2050	3.81	37	312	11	190	6.70
6	209	26	200	6.44	38	77	88	2090	4.20
7	170	45	670	6.19	39	142	22	900	5.43
8	240	29	300	5.89	40	262	22	230	6.50
9	241	11	120	5.89	41	215	12	140	6.25
10	55	55	290	2.36	42	246	9	330	7.10
11	75	87	1180	3.93	43	191	31	1010	7.10
12	129	55	900	5.99	44	182	19	300	7.00
13	24	93	1730	3.50	45	37	88	1730	3.46
14	165	31	1150	7.41	46	103	35	780	5.66
15	94	77	1160	4.21	47	67	85	1300	4.82
16	96	80	1270	5.00	48	143	78	930	5.00
17	148	30	580	5.27	49	83	85	690	4.74
18	98	69	660	5.21	50	223	33	200	8.49
19	161	43	420	6.50	51	240	19	450	6.50
20	118	47	1080	6.12	52	312	21	280	6.50
21	269	17	290	6.19	53	12	79	4430	1.69
22	189	35	270	5.05	54	52	83	270	3.25
23	126	58	560	6.16	55	79	43	1340	7.17
24	12	81	4240	1.80	56	61	88	670	3.52
25	167	29	240	4.75	57	168	28	410	6.09
26	135	65	430	4.10	58	28	95	4370	2.86
27	107	87	3020	6.66	59	121	41	1310	4.88
28	72	63	1420	7.28	60	115	62	1470	3.89
29	128	49	420	8.12	61	186	45	300	6.90
30	27	63	19830	5.23	62	47	85	3630	4.10
31	152	84	420	5.79	63	178	45	220	6.09
32	224	23	530	6.50	64	142	67	560	7.20

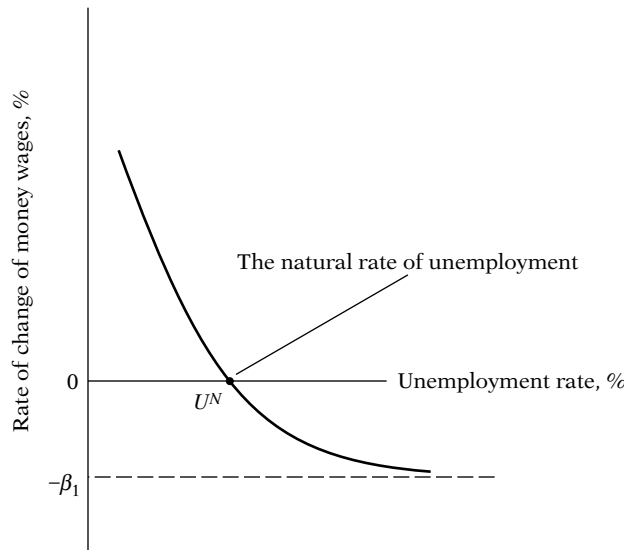
Note: CM = Child mortality, the number of deaths of children under age 5 in a year per 1000 live births.  
 FLFP = Female literacy rate, percent.  
 PGNP = per capita GNP in 1980.  
 TFR = total fertility rate, 1980–1985, the average number of children born to a woman, using age-specific fertility rates for a given year.

Source: Chandan Mukherjee, Howard White, and Marc Whyte, *Econometrics and Data Analysis for Developing Countries*, Routledge, London, 1998, p. 456.

for the period 1861–1957, Phillips obtained a curve whose general shape resembles Figure 6.6*b* (Figure 6.8).<sup>19</sup>

As Figure 6.8 shows, there is an asymmetry in the response of wage changes to the level of the unemployment rate: Wages rise faster for a unit change in unemployment if the unemployment rate is below  $U^n$ , which is

<sup>19</sup>A. W. Phillips, “The Relationship between Unemployment and the Rate of Change of Money Wages in the United Kingdom, 1861–1957,” *Economica*, November 1958, vol. 15, pp. 283–299. Note that the original curve did not cross the unemployment rate axis, but Fig. 6.8 represents a later version of the curve.



**FIGURE 6.8** The Phillips curve.

called the *natural rate of unemployment* by economists [defined as the rate of unemployment required to keep (wage) inflation constant], and then they fall for an equivalent change when the unemployment rate is above the natural rate,  $\beta_1$ , indicating the asymptotic floor for wage change. This particular feature of the Phillips curve may be due to institutional factors, such as union bargaining power, minimum wages, unemployment compensation, etc.

Since the publication of Phillips' article, there has been very extensive research on the Phillips curve at the theoretical as well as empirical levels. Space does not permit us to go into the details of the controversy surrounding the Phillips curve. The Phillips curve itself has gone through several incarnations. A comparatively recent formulation is provided by Olivier Blanchard.<sup>20</sup> If we let  $\pi_t$  denote the inflation rate at time  $t$ , which is defined as the percentage change in the price level as measured by a representative price index, such as the Consumer Price Index (CPI), and  $UN_t$  denote the unemployment rate at time  $t$ , then a modern version of the Phillips curve can be expressed in the following format:

$$\pi_t - \pi_t^e = \beta_2(UN_t - U^n) + u_t \quad (6.7.3)$$

where  $\pi_t$  = actual inflation rate at time  $t$   
 $\pi_t^e$  = expected inflation rate at time  $t$ , the expectation being formed in year  $(t - 1)$

<sup>20</sup>See Olivier Blanchard, *Macroeconomics*, Prentice Hall, Englewood Cliffs, N.J., 1997, Chap. 17.

$UN_t$  = actual unemployment rate prevailing at time  $t$   
 $U^n$  = natural rate of unemployment at time  $t$   
 $u_t$  = stochastic error term<sup>21</sup>

Since  $\pi_t^e$  is not directly observable, as a starting point one can make the simplifying assumption that  $\pi_t^e = \pi_{t-1}$ ; that is, the inflation expected this year is the inflation rate that prevailed in the last year; of course, more complicated assumptions about expectations formation can be made, and we will discuss this topic in Chapter 17, on distributed lag models.

Substituting this assumption into (6.7.3) and writing the regression model in the standard form, we obtain the following estimating equation:

$$\pi_t - \pi_{t-1} = \beta_1 + \beta_2 UN_t + u_t \quad (6.7.4)$$

where  $\beta_1 = -\beta_2 U^n$ . Equation (6.7.4) states that the change in the inflation rate between two time periods is linearly related to the current unemployment rate. A priori,  $\beta_2$  is expected to be negative (why?) and  $\beta_1$  is expected to be positive (this figures, since  $\beta_2$  is negative and  $U^n$  is positive).

Incidentally, the Phillips relationship given in (6.7.3) is known in the literature as the **modified Phillips curve**, or the **expectations-augmented Phillips curve** (to indicate that  $\pi_{t-1}$  stands for expected inflation), or the **accelerationist Phillips curve** (to suggest that a low unemployment rate leads to an increase in the inflation rate and hence an *acceleration* of the price level).

As an illustration of the modified Phillips curve, we present in Table 6.5 data on inflation as measured by year-to-year percentage in the Consumer Price Index (CPIflation) and the unemployment rate for the period 1960–1998. The unemployment rate represents the civilian unemployment rate. From these data we obtained the change in the inflation rate ( $\pi_t - \pi_{t-1}$ ) and plotted it against the civilian unemployment rate; we are using the CPI as a measure of inflation. The resulting graph appears in Figure 6.9.

As expected, the relation between the change in inflation rate and the unemployment rate is negative—a low unemployment rate leads to an increase in the inflation rate and therefore an acceleration of the price level, hence the name accelerationist Phillips curve.

Looking at Figure 6.9, it is not obvious whether a linear (straight line) regression model or a reciprocal model fits the data; there may be a curvilinear relationship between the two variables. We present below regressions based on both the models. However, keep in mind that for the reciprocal model the intercept term is expected to be negative and the slope positive, as noted in footnote 18.

$$\text{Linear model: } \widehat{(\pi_t - \pi_{t-1})} = 4.1781 - 0.6895 UN_t \quad (6.7.5)$$

$$t = (3.9521) \quad (-4.0692) \quad r^2 = 0.3150$$

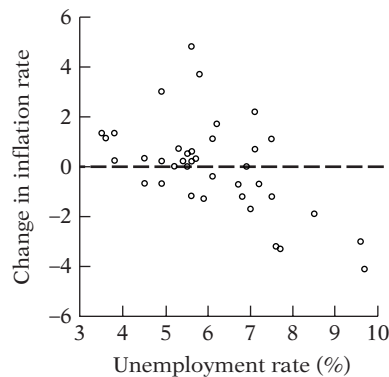
<sup>21</sup>Economists believe this error term represents some kind of supply shock, such as the OPEC oil embargoes of 1973 and 1979.

**TABLE 6.5** INFLATION RATE AND UNEMPLOYMENT RATE, UNITED STATES, 1960–1998

Observation	INFLRATE	UNRATE	Observation	INFLRATE	UNRATE
1960	1.7	5.5	1980	13.5	7.1
1961	1.0	6.7	1981	10.3	7.6
1962	1.0	5.5	1982	6.2	9.7
1963	1.3	5.7	1983	3.2	9.6
1964	1.3	5.2	1984	4.3	7.5
1965	1.6	4.5	1985	3.6	7.2
1966	2.9	3.8	1986	1.9	7.0
1967	3.1	3.8	1987	3.6	6.2
1968	4.2	3.6	1988	4.1	5.5
1969	5.5	3.5	1989	4.8	5.3
1970	5.7	4.9	1990	5.4	5.6
1971	4.4	5.9	1991	4.2	6.8
1972	3.2	5.6	1992	3.0	7.5
1973	6.2	4.9	1993	3.0	6.9
1974	11.0	5.6	1994	2.6	6.1
1975	9.1	8.5	1995	2.8	5.6
1976	5.8	7.7	1996	3.0	5.4
1977	6.5	7.1	1997	2.3	4.9
1978	7.6	6.1	1998	1.6	4.5
1979	11.3	5.8			

Note: The inflation rate is the percent year-to-year change in CPI. The unemployment rate is the civilian unemployment rate.

Source: *Economic Report of the President*, 1999, Table B-63, p. 399, for CPI changes and Table B-42, p. 376, for the unemployment rate.



**FIGURE 6.9** The modified Phillips curve.

Reciprocal model:

$$\widehat{(\pi_t - \pi_{t-1})} = -3.2514 + 18.5508 \left( \frac{1}{UN_t} \right) \quad (6.7.6)$$

$$t = (-2.9715) \quad (3.0625) \quad r^2 = 0.2067$$

All the estimated coefficients in both the models are *individually* statistically significant, all the *p* values being lower than the 0.005 level.

Model (6.7.5) shows that if the unemployment rate goes down by 1 percentage point, on average, the change in the inflation rate goes up by about 0.7 percentage points, and vice versa. Model (6.7.6) shows that even if the unemployment rate increases indefinitely, the most the change in the inflation rate will go down will be about 3.25 percentage points. Incidentally, from Eq. (6.7.5), we can compute the underlying natural rate of unemployment as:

$$U^n = \frac{\hat{\beta}_1}{-\hat{\beta}_2} = \frac{4.1781}{0.6895} = 6.0596 \quad (6.7.7)$$

That is, the natural rate of unemployment is about 6.06%. Economists put the natural rate between 5 to 6%, although in the recent past in the United States the actual rate has been much below this rate.

### Log Hyperbola or Logarithmic Reciprocal Model

We conclude our discussion of reciprocal models by considering the logarithmic reciprocal model, which takes the following form:

$$\ln Y_i = \beta_1 - \beta_2 \left( \frac{1}{X_i} \right) + u_i \quad (6.7.8)$$

Its shape is as depicted in Figure 6.10. As this figure shows, initially  $Y$  increases at an increasing rate (i.e., the curve is initially convex) and then it increases at a decreasing rate (i.e., the curve becomes concave).<sup>22</sup> Such a

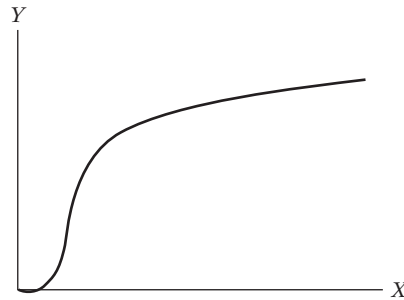


FIGURE 6.10 The log reciprocal model.

<sup>22</sup>From calculus, it can be shown that

$$\frac{d}{dX}(\ln Y) = -\beta_2 \left( -\frac{1}{X^2} \right) = \beta_2 \left( \frac{1}{X^2} \right)$$

But

$$\frac{d}{dX}(\ln Y) = \frac{1}{Y} \frac{dY}{dX}$$

Making this substitution, we obtain

$$\frac{dY}{dX} = \beta_2 \frac{Y}{X^2}$$

which is the slope of  $Y$  with respect to  $X$ .

model may therefore be appropriate to model a short-run production function. Recall from microeconomics that if labor and capital are the inputs in a production function and if we keep the capital input constant but increase the labor input, the short-run output–labor relationship will resemble Figure 6.10. (See Example 7.4, Chapter 7.)

## 6.8 CHOICE OF FUNCTIONAL FORM

In this chapter we discussed several functional forms an empirical model can assume, even within the confines of the linear-in-parameter regression models. The choice of a particular functional form may be comparatively easy in the two-variable case, because we can plot the variables and get some rough idea about the appropriate model. The choice becomes much harder when we consider the multiple regression model involving more than one regressor, as we will discover when we discuss this topic in the next two chapters. There is no denying that a great deal of skill and experience are required in choosing an appropriate model for empirical estimation. But some guidelines can be offered:

1. The underlying theory (e.g., the Phillips curve) may suggest a particular functional form.
2. It is good practice to find out the rate of change (i.e., the slope) of the regressand with respect to the regressor as well as to find out the elasticity of the regressand with respect to the regressor. For the various models considered in this chapter, we provide the necessary formulas for the slope and elasticity coefficients of the various models in Table 6.6. The knowledge of these formulas will help us to compare the various models.

TABLE 6.6

Model	Equation	Slope $\left(= \frac{dY}{dX}\right)$	Elasticity $\left(= \frac{dY}{dX} \frac{X}{Y}\right)$
Linear	$Y = \beta_1 + \beta_2 X$	$\beta_2$	$\beta_2 \left(\frac{X}{Y}\right)^*$
Log-linear	$\ln Y = \beta_1 + \beta_2 \ln X$	$\beta_2 \left(\frac{Y}{X}\right)$	$\beta_2$
Log-lin	$\ln Y = \beta_1 + \beta_2 X$	$\beta_2 (Y)$	$\beta_2 (X)^*$
Lin-log	$Y = \beta_1 + \beta_2 \ln X$	$\beta_2 \left(\frac{1}{X}\right)$	$\beta_2 \left(\frac{1}{Y}\right)^*$
Reciprocal	$Y = \beta_1 + \beta_2 \left(\frac{1}{X}\right)$	$-\beta_2 \left(\frac{1}{X^2}\right)$	$-\beta_2 \left(\frac{1}{XY}\right)^*$
Log reciprocal	$\ln Y = \beta_1 - \beta_2 \left(\frac{1}{X}\right)$	$\beta_2 \left(\frac{Y}{X^2}\right)$	$\beta_2 \left(\frac{1}{X}\right)^*$

Note: \* indicates that the elasticity is variable, depending on the value taken by  $X$  or  $Y$  or both. When no  $X$  and  $Y$  values are specified, in practice, very often these elasticities are measured at the mean values of these variables, namely,  $\bar{X}$  and  $\bar{Y}$ .

3. The coefficients of the model chosen should satisfy certain a priori expectations. For example, if we are considering the demand for automobiles as a function of price and some other variables, we should expect a negative coefficient for the price variable.

4. Sometime more than one model may fit a given set of data reasonably well. In the modified Phillips curve, we fitted both a linear and a reciprocal model to the same data. In both cases the coefficients were in line with prior expectations and they were all statistically significant. One major difference was that the  $r^2$  value of the linear model was larger than that of the reciprocal model. One may therefore give a slight edge to the linear model over the reciprocal model. *But make sure that in comparing two  $r^2$  values the dependent variable, or the regressand, of the two models is the same; the regressor(s) can take any form.* We will explain the reason for this in the next chapter.

5. In general *one should not overemphasize* the  $r^2$  measure in the sense that the higher the  $r^2$  the better the model. As we will discuss in the next chapter,  $r^2$  increases as we add more regressors to the model. What is of greater importance is the theoretical underpinning of the chosen model, the signs of the estimated coefficients and their statistical significance. If a model is good on these criteria, a model with a lower  $r^2$  may be quite acceptable. We will revisit this important topic in greater depth in Chapter 13.

#### \*6.9 A NOTE ON THE NATURE OF THE STOCHASTIC ERROR TERM: ADDITIVE VERSUS MULTIPLICATIVE STOCHASTIC ERROR TERM

Consider the following regression model, which is the same as (6.5.1) but without the error term:

$$Y_i = \beta_1 X_i^{\beta_2} \quad (6.9.1)$$

For estimation purposes, we can express this model in three different forms:

$$Y_i = \beta_1 X_i^{\beta_2} u_i \quad (6.9.2)$$

$$Y_i = \beta_1 X_i^{\beta_2} e^{u_i} \quad (6.9.3)$$

$$Y_i = \beta_1 X_i^{\beta_2} + u_i \quad (6.9.4)$$

Taking the logarithms on both sides of these equations, we obtain

$$\ln Y_i = \alpha + \beta_2 \ln X_i + \ln u_i \quad (6.9.2a)$$

$$\ln Y_i = \alpha + \beta_2 \ln X_i + u_i \quad (6.9.3a)$$

$$\ln Y_i = \ln (\beta_1 X_i^{\beta_2} + u_i) \quad (6.9.4a)$$

where  $\alpha = \ln \beta_1$ .

\*Optional



Models like (6.9.2) are *intrinsically linear (in-parameter)* regression models in the sense that by suitable (log) transformation the models can be made linear in the parameters  $\alpha$  and  $\beta_2$ . (Note: These models are nonlinear in  $\beta_1$ .) But model (6.9.4) is *intrinsically nonlinear-in-parameter*. There is no simple way to take the log of (6.9.4) because  $\ln(A + B) \neq \ln A + \ln B$ .

Although (6.9.2) and (6.9.3) are linear regression models and can be estimated by OLS or ML, we have to be careful about the properties of the stochastic error term that enters these models. Remember that the BLUE property of OLS requires that  $u_i$  has zero mean value, constant variance, and zero autocorrelation. For hypothesis testing, we further assume that  $u_i$  follows the normal distribution with mean and variance values just discussed. In short, we have assumed that  $u_i \sim N(0, \sigma^2)$ .

Now consider model (6.9.2). Its statistical counterpart is given in (6.9.2a). To use the classical normal linear regression model (CNLRM), we have to assume that

$$\ln u_i \sim N(0, \sigma^2) \quad (6.9.5)$$

Therefore, when we run the regression (6.9.2a), we will have to apply the normality tests discussed in Chapter 5 to the residuals obtained from this regression. Incidentally, note that if  $\ln u_i$  follows the normal distribution with zero mean and constant variance, then statistical theory shows that  $u_i$  in (6.9.2) must follow the **log-normal distribution** with mean  $e^{\sigma^2/2}$  and variance  $e^{\sigma^2}(e^{\sigma^2} - 1)$ .

As the preceding analysis shows, one has to pay very careful attention to the error term in transforming a model for regression analysis. As for (6.9.4), this model is a *nonlinear-in-parameter* regression model and will have to be solved by some iterative computer routine. Model (6.9.3) should not pose any problems for estimation.

To sum up, pay very careful attention to the disturbance term when you transform a model for regression analysis. Otherwise, a blind application of OLS to the transformed model will not produce a model with desirable statistical properties.

## 6.10 SUMMARY AND CONCLUSIONS

This chapter introduced several of the finer points of the classical linear regression model (CLRM).

1. Sometimes a regression model may not contain an explicit intercept term. Such models are known as **regression through the origin**. Although the algebra of estimating such models is simple, one should use such models with caution. In such models the sum of the residuals  $\sum \hat{u}_i$  is nonzero; additionally, the conventionally computed  $r^2$  may not be meaningful. Unless

there is a strong theoretical reason, it is better to introduce the intercept in the model explicitly.

2. The units and scale in which the regressand and the regressor(s) are expressed are very important because the interpretation of regression coefficients critically depends on them. In empirical research the researcher should not only quote the sources of data but also state explicitly how the variables are measured.

3. Just as important is the functional form of the relationship between the regressand and the regressor(s). Some of the important functional forms discussed in this chapter are (a) the log-linear or constant elasticity model, (b) semilog regression models, and (c) reciprocal models.

4. In the log-linear model both the regressand and the regressor(s) are expressed in the logarithmic form. The regression coefficient attached to the log of a regressor is interpreted as the elasticity of the regressand with respect to the regressor.

5. In the semilog model either the regressand or the regressor(s) are in the log form. In the semilog model where the regressand is logarithmic and the regressor  $X$  is time, the estimated slope coefficient (multiplied by 100) measures the (instantaneous) rate of growth of the regressand. Such models are often used to measure the growth rate of many economic phenomena. In the semilog model if the regressor is logarithmic, its coefficient measures the absolute rate of change in the regressand for a given percent change in the value of the regressor.

6. In the reciprocal models, either the regressand or the regressor is expressed in reciprocal, or inverse, form to capture nonlinear relationships between economic variables, as in the celebrated Phillips curve.

7. In choosing the various functional forms, great attention should be paid to the stochastic disturbance term  $u_i$ . As noted in Chapter 5, the CLRM explicitly assumes that the disturbance term has zero mean value and constant (homoscedastic) variance and that it is uncorrelated with the regressor(s). It is under these assumptions that the OLS estimators are BLUE. Further, under the CNLRM, the OLS estimators are also normally distributed. One should therefore find out if these assumptions hold in the functional form chosen for empirical analysis. After the regression is run, the researcher should apply diagnostic tests, such as the normality test, discussed in Chapter 5. This point cannot be overemphasized, for the classical tests of hypothesis, such as the  $t$ ,  $F$ , and  $\chi^2$ , rest on the assumption that the disturbances are normally distributed. This is especially critical if the sample size is small.

8. Although the discussion so far has been confined to two-variable regression models, the subsequent chapters will show that in many cases the extension to multiple regression models simply involves more algebra without necessarily introducing more fundamental concepts. That is why it is so very important that the reader have a firm grasp of the two-variable regression model.

## EXERCISES

## Questions

6.1. Consider the regression model

$$y_i = \beta_1 + \beta_2 x_i + u_i$$

where  $y_i = (Y_i - \bar{Y})$  and  $x_i = (X_i - \bar{X})$ . In this case, the regression line must pass through the origin. True or false? Show your calculations.

6.2. The following regression results were based on monthly data over the period January 1978 to December 1987:

$$\begin{aligned} \hat{Y}_i &= 0.00681 + 0.75815X_i \\ \text{se} &= (0.02596) \quad (0.27009) \\ t &= (0.26229) \quad (2.80700) \\ p \text{ value} &= (0.7984) \quad (0.0186) \quad r^2 = 0.4406 \\ \hat{Y}_i &= 0.76214X_i \\ \text{se} &= (0.265799) \\ t &= (2.95408) \\ p \text{ value} &= (0.0131) \quad r^2 = 0.43684 \end{aligned}$$

where  $Y$  = monthly rate of return on Texaco common stock, %, and  $X$  = monthly market rate of return, %.\*

- What is the difference between the two regression models?
  - Given the preceding results, would you retain the intercept term in the first model? Why or why not?
  - How would you interpret the slope coefficients in the two models?
  - What is the theory underlying the two models?
  - Can you compare the  $r^2$  terms of the two models? Why or why not?
  - The Jarque-Bera normality statistic for the first model in this problem is 1.1167 and for the second model it is 1.1170. What conclusions can you draw from these statistics?
  - The  $t$  value of the slope coefficient in the zero intercept model is about 2.95, whereas that with the intercept present is about 2.81. Can you rationalize this result?
- 6.3. Consider the following regression model:

$$\frac{1}{Y_i} = \beta_1 + \beta_2 \left( \frac{1}{X_i} \right) + u_i$$

*Note:* Neither  $Y$  nor  $X$  assumes zero value.

- Is this a linear regression model?
- How would you estimate this model?

\*The underlying data were obtained from the data diskette included in Ernst R. Berndt, *The Practice of Econometrics: Classic and Contemporary*, Addison-Wesley, Reading, Mass., 1991.

- c. What is the behavior of  $Y$  as  $X$  tends to infinity?  
 d. Can you give an example where such a model may be appropriate?
- 6.4. Consider the log-linear model:

$$\ln Y_i = \beta_1 + \beta_2 \ln X_i + u_i$$

Plot  $Y$  on the vertical axis and  $X$  on the horizontal axis. Draw the curves showing the relationship between  $Y$  and  $X$  when  $\beta_2 = 1$ , and when  $\beta_2 > 1$ , and when  $\beta_2 < 1$ .

- 6.5. Consider the following models:

$$\text{Model I: } Y_i = \beta_1 + \beta_2 X_i + u_i$$

$$\text{Model II: } Y_i^* = \alpha_1 + \alpha_2 X_i^* + u_i$$

where  $Y^*$  and  $X^*$  are standardized variables. Show that  $\hat{\alpha}_2 = \hat{\beta}_2(S_x/S_y)$  and hence *establish that although the regression slope coefficients are independent of the change of origin they are not independent of the change of scale.*

- 6.6. Consider the following models:

$$\ln Y_i^* = \alpha_1 + \alpha_2 \ln X_i^* + u_i^*$$

$$\ln Y_i = \beta_1 + \beta_2 \ln X_i + u_i$$

where  $Y_i^* = w_1 Y_i$  and  $X_i^* = w_2 X_i$ , the  $w$ 's being constants.

- a. Establish the relationships between the two sets of regression coefficients and their standard errors.  
 b. Is the  $r^2$  different between the two models?
- 6.7. Between regressions (6.6.8) and (6.6.10), which model do you prefer? Why?
- 6.8. For the regression (6.6.8), test the hypothesis that the slope coefficient is not significantly different from 0.005.
- 6.9. From the Phillips curve given in (6.7.3), is it possible to estimate the natural rate of unemployment? How?
- 6.10. The Engel expenditure curve relates a consumer's expenditure on a commodity to his or her total income. Letting  $Y$  = consumption expenditure on a commodity and  $X$  = consumer income, consider the following models:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

$$Y_i = \beta_1 + \beta_2(1/X_i) + u_i$$

$$\ln Y_i = \ln \beta_1 + \beta_2 \ln X_i + u_i$$

$$\ln Y_i = \ln \beta_1 + \beta_2(1/X_i) + u_i$$

$$Y_i = \beta_1 + \beta_2 \ln X_i + u_i$$

Which of these model(s) would you choose for the Engel expenditure curve and why? (*Hint*: Interpret the various slope coefficients, find out the expressions for elasticity of expenditure with respect to income, etc.)

6.11. Consider the following model:

$$Y_i = \frac{e^{\beta_1 + \beta_2 X_i}}{1 + e^{\beta_1 + \beta_2 X_i}}$$

As it stands, is this a linear regression model? If not, what “trick,” if any, can you use to make it a linear regression model? How would you interpret the resulting model? Under what circumstances might such a model be appropriate?

6.12. Graph the following models (for ease of exposition, we have omitted the observation subscript,  $i$ ):

a.  $Y = \beta_1 X^{\beta_2}$ , for  $\beta_2 > 1$ ,  $\beta_2 = 1$ ,  $0 < \beta_2 < 1$ , . . . .

b.  $Y = \beta_1 e^{\beta_2 X}$ , for  $\beta_2 > 0$  and  $\beta_2 < 0$ .

Discuss where such models might be appropriate.

**Problems**

6.13. You are given the data in Table 6.7.\* Fit the following model to these data and obtain the usual regression statistics and interpret the results:

$$\frac{100}{100 - Y_i} = \beta_1 + \beta_2 \left( \frac{1}{X_i} \right)$$

**TABLE 6.7**

$Y_i$	86	79	76	69	65	62	52	51	51	48
$X_i$	3	7	12	17	25	35	45	55	70	120

6.14. To measure the elasticity of substitution between capital and labor inputs Arrow, Chenery, Minhas, and Solow, the authors of the now famous CES (constant elasticity of substitution) production function, used the following model†:

$$\log \left( \frac{V}{L} \right) = \log \beta_1 + \beta_2 \log W + u$$

where  $(V/L)$  = value added per unit of labor

$L$  = labor input

$W$  = real wage rate

The coefficient  $\beta_2$  measures the elasticity of substitution between labor and capital (i.e., proportionate change in factor proportions/proportionate change in relative factor prices). From the data given in Table 6.8, verify that the estimated elasticity is 1.3338 and that it is not statistically significantly different from 1.

6.15. Table 6.9 gives data on the GDP (gross domestic product) deflator for domestic goods and the GDP deflator for imports for Singapore for the period 1968–1982. The GDP deflator is often used as an indicator of inflation in place of the CPI. Singapore is a small, open economy, heavily dependent on foreign trade for its survival.

\*Source: Adapted from J. Johnston, *Econometric Methods*, 3d ed., McGraw-Hill, New York, 1984, p. 87. Actually this is taken from an econometric examination of Oxford University in 1975.

†“Capital-Labor Substitution and Economic Efficiency,” *Review of Economics and Statistics*, August 1961, vol. 43, no. 5, pp. 225–254.

**TABLE 6.8**

Industry	$\log(V/L)$	$\log W$
Wheat flour	3.6973	2.9617
Sugar	3.4795	2.8532
Paints and varnishes	4.0004	3.1158
Cement	3.6609	3.0371
Glass and glassware	3.2321	2.8727
Ceramics	3.3418	2.9745
Plywood	3.4308	2.8287
Cotton textiles	3.3158	3.0888
Woolen textiles	3.5062	3.0086
Jute textiles	3.2352	2.9680
Chemicals	3.8823	3.0909
Aluminum	3.7309	3.0881
Iron and steel	3.7716	3.2256
Bicycles	3.6601	3.1025
Sewing machines	3.7554	3.1354

Source: Damodar Gujarati, "A Test of ACMS Production Function: Indian Industries, 1958," *Indian Journal of Industrial Relations*, vol. 2, no. 1, July 1966, pp. 95–97.

**TABLE 6.9**

Year	GDP deflator for domestic goods, $Y$	GDP deflator for imports, $X$
1968	1000	1000
1969	1023	1042
1970	1040	1092
1971	1087	1105
1972	1146	1110
1973	1285	1257
1974	1485	1749
1975	1521	1770
1976	1543	1889
1977	1567	1974
1978	1592	2015
1979	1714	2260
1980	1841	2621
1981	1959	2777
1982	2033	2735

Source: Colin Simkin, "Does Money Matter in Singapore?" *The Singapore Economic Review*, vol. XXIX, no. 1, April 1984, Table 6, p. 8.

To study the relationship between domestic and world prices, you are given the following models:

$$1. Y_t = \alpha_1 + \alpha_2 X_t + u_t$$

$$2. Y_t = \beta_2 X_t + u_t$$

where  $Y$  = GDP deflator for domestic goods and  $X$  = GDP deflator for imports.

- a. How would you choose between the two models a priori?  
 b. Fit both models to the data and decide which gives a better fit.  
 c. What other model(s) might be appropriate for the data?
- 6.16. Refer to the data given in exercise 6.15. The means of  $Y$  and  $X$  are 1456 and 1760, respectively, and the corresponding standard deviations are 346 and 641. Estimate the following regression:

$$Y_i^* = \alpha_1 + \alpha_2 X_i^* + u_i$$

where the starred variables are standardized variables, and interpret the results.

- 6.17. Refer to Table 6.3. Find out the rate of growth of expenditure on durable goods. What is the estimated *semielasticity*? Interpret your results. Would it make sense to run a double-log regression with expenditure on durable goods as the regressand and time as the regressor? How would you interpret the slope coefficient in this case.
- 6.18. From the data given in Table 6.3, find out the growth rate of expenditure on nondurable goods and compare your results with those obtained from problem 6.17.
- 6.19. Revisit exercise 1.7. Now that you know several functional forms, which one might be appropriate to study the relationship between advertising impressions retained and the amount of money spent on advertising? Show the necessary calculations.

## APPENDIX 6A

### 6A.1 DERIVATION OF LEAST-SQUARES ESTIMATORS FOR REGRESSION THROUGH THE ORIGIN

We want to minimize

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_2 X_i)^2 \quad (1)$$

with respect to  $\hat{\beta}_2$ .

Differentiating (1) with respect to  $\hat{\beta}_2$ , we obtain

$$\frac{d \sum \hat{u}_i^2}{d \hat{\beta}_2} = 2 \sum (Y_i - \hat{\beta}_2 X_i)(-X_i) \quad (2)$$

Setting (2) equal to zero and simplifying, we get

$$\hat{\beta}_2 = \frac{\sum X_i Y_i}{\sum X_i^2} \quad (6.1.6) = (3)$$

Now substituting the PRF:  $Y_i = \beta_2 X_i + u_i$  into this equation, we obtain

$$\begin{aligned} \hat{\beta}_2 &= \frac{\sum X_i (\beta_2 X_i + u_i)}{\sum X_i^2} \\ &= \beta_2 + \frac{\sum X_i u_i}{\sum X_i^2} \end{aligned} \quad (4)$$

[Note:  $E(\hat{\beta}_2) = \beta_2$ .] Therefore,

$$E(\hat{\beta}_2 - \beta_2)^2 = E \left[ \frac{\sum X_i u_i}{\sum X_i^2} \right]^2 \quad (5)$$

Expanding the right-hand side of (5) and noting that the  $X_i$  are nonstochastic and the  $u_i$  are homoscedastic and uncorrelated, we obtain

$$\text{var}(\hat{\beta}_2) = E(\hat{\beta}_2 - \beta_2)^2 = \frac{\sigma^2}{\sum X_i^2} \quad (6.1.7) = (6)$$

Incidentally, note that from (2) we get, after equating it to zero

$$\sum \hat{u}_i X_i = 0 \quad (7)$$

From Appendix 3A, Section 3A.1 we see that when the intercept term is present in the model, we get in addition to (7) the condition  $\sum \hat{u}_i = 0$ . From the mathematics just given it should be clear why the regression through the origin model may not have the error sum,  $\sum \hat{u}_i$ , equal to zero.

Suppose we want to impose the condition that  $\sum \hat{u}_i = 0$ . In that case we have

$$\begin{aligned} \sum Y_i &= \hat{\beta}_2 \sum X_i + \sum \hat{u}_i \\ &= \hat{\beta}_2 \sum X_i, \quad \text{since } \sum \hat{u}_i = 0 \text{ by construction} \end{aligned} \quad (8)$$

This expression then gives

$$\begin{aligned} \hat{\beta}_2 &= \frac{\sum Y_i}{\sum X_i} \\ &= \frac{\bar{Y}}{\bar{X}} = \frac{\text{mean value of } Y}{\text{mean value of } X} \end{aligned} \quad (9)$$

But this estimator is not the same as (3) above or (6.1.6). And since the  $\hat{\beta}_2$  of (3) is unbiased (why?), the  $\hat{\beta}_2$  of (9) cannot be unbiased.

The upshot is that, in regression through the origin, we cannot have both  $\sum \hat{u}_i X_i$  and  $\sum \hat{u}_i$  equal to zero, as in the conventional model. The only condition that is satisfied is that  $\sum \hat{u}_i X_i$  is zero.

Recall that

$$Y_i = \hat{Y}_i + \hat{u}_i \quad (2.6.3)$$

Summing this equation on both sides and dividing by  $N$ , the sample size, we obtain

$$\bar{Y} = \bar{\hat{Y}} + \bar{\hat{u}} \quad (10)$$

Since for the zero intercept model  $\sum \hat{u}_i$  and, therefore  $\bar{\hat{u}}$ , need not be zero, it then follows that

$$\bar{Y} \neq \bar{\hat{Y}} \quad (11)$$



that is, the mean of actual  $Y$  values need not be equal to the mean of the estimated  $Y$  values; the two mean values are identical for the intercept-present model, as can be seen from (3.1.10).

It was noted that, for the zero-intercept model,  $r^2$  can be negative, whereas for the conventional model it can never be negative. This condition can be shown as follows.

Using (3.5.5a), we can write

$$r^2 = 1 - \frac{\text{RSS}}{\text{TSS}} = 1 - \frac{\sum \hat{u}_i^2}{\sum y_i^2} \quad (12)$$

Now for the conventional, or intercept-present, model, Eq. (3.3.6) shows that

$$\text{RSS} = \sum \hat{u}_i^2 = \sum y_i^2 - \hat{\beta}_2^2 \sum x_i^2 \leq \sum y_i^2 \quad (13)$$

unless  $\hat{\beta}_2$  is zero (i.e.,  $X$  has no influence on  $Y$  whatsoever). That is, for the conventional model,  $\text{RSS} \leq \text{TSS}$ , or,  $r^2$  can never be negative.

For the zero-intercept model it can be shown analogously that

$$\text{RSS} = \sum \hat{u}_i^2 = \sum Y_i^2 - \hat{\beta}_2^2 \sum X_i^2 \quad (14)$$

(*Note:* The sums of squares of  $Y$  and  $X$  are not mean-adjusted.) Now there is no guarantee that this RSS will always be less than  $\sum y_i^2 = \sum Y_i^2 - N\bar{Y}^2$  (the TSS), which suggests that RSS can be greater than TSS, implying that  $r^2$ , as conventionally defined, can be negative. Incidentally, notice that in this case RSS will be greater than TSS if  $\hat{\beta}_2^2 \sum X_i^2 < N\bar{Y}^2$ .

## 6A.2 PROOF THAT A STANDARDIZED VARIABLE HAS ZERO MEAN AND UNIT VARIANCE

Consider the random variable (r.v.)  $Y$  with the (sample) mean value of  $\bar{Y}$  and (sample) standard deviation of  $S_y$ . Define

$$Y_i^* = \frac{Y_i - \bar{Y}}{S_y} \quad (15)$$

Hence  $Y_i^*$  is a standardized variable. Notice that standardization involves a dual operation: (1) change of the origin, which is the numerator of (15), and (2) change of scale, which is the denominator. Thus, standardization involves both a change of the origin and change of scale.

Now

$$\bar{Y}_i^* = \frac{1}{S_y} \frac{\sum (Y_i - \bar{Y})}{n} = 0 \quad (16)$$

since the sum of deviation of a variable from its mean value is always zero. Hence the mean value of the standardized value is zero. (*Note:* We could pull out the  $S_y$  term from the summation sign because its value is known.)

Now

$$\begin{aligned} S_{y^*}^2 &= \sum \frac{(Y_i - \bar{Y})^2 / (n-1)}{S_y^2} \\ &= \frac{1}{(n-1)S_y^2} \sum (Y_i - \bar{Y})^2 \\ &= \frac{(n-1)S_y^2}{(n-1)S_y^2} = 1 \end{aligned} \quad (17)$$

Note that

$$S_y^2 = \frac{\sum (Y_i - \bar{Y})^2}{n-1}$$

which is the sample variance of  $Y$ .



# 7

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## MULTIPLE REGRESSION ANALYSIS: THE PROBLEM OF ESTIMATION

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The two-variable model studied extensively in the previous chapters is often inadequate in practice. In our consumption–income example, for instance, it was assumed implicitly that only income  $X$  affects consumption  $Y$ . But economic theory is seldom so simple for, besides income, a number of other variables are also likely to affect consumption expenditure. An obvious example is wealth of the consumer. As another example, the demand for a commodity is likely to depend not only on its own price but also on the prices of other competing or complementary goods, income of the consumer, social status, etc. Therefore, we need to extend our simple two-variable regression model to cover models involving more than two variables. Adding more variables leads us to the discussion of multiple regression models, that is, models in which the dependent variable, or regressand,  $Y$  depends on two or more explanatory variables, or regressors.

The simplest possible multiple regression model is three-variable regression, with one dependent variable and two explanatory variables. In this and the next chapter we shall study this model. Throughout, we are concerned with multiple linear regression models, that is, models linear in the parameters; they may or may not be linear in the variables.

### 7.1 THE THREE-VARIABLE MODEL: NOTATION AND ASSUMPTIONS

Generalizing the two-variable population regression function (PRF) (2.4.2), we may write the three-variable PRF as

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i \quad (7.1.1)$$

where  $Y$  is the dependent variable,  $X_2$  and  $X_3$  the explanatory variables (or regressors),  $u$  the stochastic disturbance term, and  $i$  the  $i$ th observation; in case the data are time series, the subscript  $t$  will denote the  $t$ th observation.<sup>1</sup>

In Eq. (7.1.1)  $\beta_1$  is the intercept term. As usual, it gives the mean or average effect on  $Y$  of all the variables excluded from the model, although its mechanical interpretation is the average value of  $Y$  when  $X_2$  and  $X_3$  are set equal to zero. The coefficients  $\beta_2$  and  $\beta_3$  are called the **partial regression coefficients**, and their meaning will be explained shortly.

We continue to operate within the framework of the classical linear regression model (CLRM) first introduced in Chapter 3. Specifically, we assume the following:

Zero mean value of  $u_i$ , or

$$E(u_i | X_{2i}, X_{3i}) = 0 \quad \text{for each } i \quad (7.1.2)$$

No serial correlation, or

$$\text{cov}(u_i, u_j) = 0 \quad i \neq j \quad (7.1.3)$$

Homoscedasticity, or

$$\text{var}(u_i) = \sigma^2 \quad (7.1.4)$$

Zero covariance between  $u_i$  and each  $X$  variable, or

$$\text{cov}(u_i, X_{2i}) = \text{cov}(u_i, X_{3i}) = 0 \quad (7.1.5)^2$$

No specification bias, or

$$\text{The model is correctly specified} \quad (7.1.6)$$

No exact collinearity between the  $X$  variables, or

$$\text{No exact linear relationship between } X_2 \text{ and } X_3 \quad (7.1.7)$$

In addition, as in Chapter 3, we assume that the multiple regression model is *linear in the parameters*, that the values of the regressors are fixed in repeated sampling, and that there is sufficient variability in the values of the regressors.

The rationale for assumptions (7.1.2) through (7.1.6) is the same as that discussed in Section 3.2. Assumption (7.1.7), that there be no exact linear relationship between  $X_2$  and  $X_3$ , technically known as the assumption of

<sup>1</sup>For notational symmetry, Eq. (7.1.1) can also be written as

$$Y_i = \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

with the provision that  $X_{1i} = 1$  for all  $i$ .

<sup>2</sup>This assumption is automatically fulfilled if  $X_2$  and  $X_3$  are nonstochastic and (7.1.2) holds.

*no collinearity* or **no multicollinearity** if more than one exact linear relationship is involved, is new and needs some explanation.

Informally, no collinearity means none of the regressors can be written as *exact* linear combinations of the remaining regressors in the model.

Formally, no collinearity means that there exists no set of numbers,  $\lambda_2$  and  $\lambda_3$ , not both zero such that

$$\lambda_2 X_{2i} + \lambda_3 X_{3i} = 0 \quad (7.1.8)$$

If such an exact linear relationship exists, then  $X_2$  and  $X_3$  are said to be **collinear** or linearly dependent. On the other hand, if (7.1.8) holds true only when  $\lambda_2 = \lambda_3 = 0$ , then  $X_2$  and  $X_3$  are said to be *linearly independent*.

Thus, if

$$X_{2i} = -4X_{3i} \quad \text{or} \quad X_{2i} + 4X_{3i} = 0 \quad (7.1.9)$$

the two variables are linearly dependent, and if both are included in a regression model, we will have perfect collinearity or an exact linear relationship between the two regressors.

Although we shall consider the problem of multicollinearity in depth in Chapter 10, intuitively the logic behind the assumption of no multicollinearity is not too difficult to grasp. Suppose that in (7.1.1)  $Y$ ,  $X_2$ , and  $X_3$  represent consumption expenditure, income, and wealth of the consumer, respectively. In postulating that consumption expenditure is linearly related to income and wealth, economic theory presumes that wealth and income may have some independent influence on consumption. If not, there is no sense in including both income and wealth variables in the model. In the extreme, if there is an exact linear relationship between income and wealth, we have only one independent variable, not two, and there is no way to assess the *separate* influence of income and wealth on consumption. To see this clearly, let  $X_{3i} = 2X_{2i}$  in the consumption–income–wealth regression. Then the regression (7.1.1) becomes

$$\begin{aligned} Y_i &= \beta_1 + \beta_2 X_{2i} + \beta_3 (2X_{2i}) + u_i \\ &= \beta_1 + (\beta_2 + 2\beta_3) X_{2i} + u_i \\ &= \beta_1 + \alpha X_{2i} + u_i \end{aligned} \quad (7.1.10)$$

where  $\alpha = (\beta_2 + 2\beta_3)$ . That is, we in fact have a two-variable and not a three-variable regression. Moreover, if we run the regression (7.1.10) and obtain  $\alpha$ , there is no way to estimate the separate influence of  $X_2$  ( $= \beta_2$ ) and  $X_3$  ( $= \beta_3$ ) on  $Y$ , for  $\alpha$  gives the *combined influence* of  $X_2$  and  $X_3$  on  $Y$ .<sup>3</sup>

<sup>3</sup>Mathematically speaking,  $\alpha = (\beta_2 + 2\beta_3)$  is one equation in two unknowns and there is no *unique* way of estimating  $\beta_2$  and  $\beta_3$  from the estimated  $\alpha$ .

In short the assumption of no multicollinearity requires that in the PRF we include only those variables that are not exact linear functions of one or more variables in the model. Although we will discuss this topic more fully in Chapter 10, a couple of points may be noted here.

First, the assumption of no multicollinearity pertains to our theoretical (i.e., PRF) model. In practice, when we collect data for empirical analysis there is no guarantee that there will not be correlations among the regressors. As a matter of fact, in most applied work it is almost impossible to find two or more (economic) variables that may not be correlated to some extent, as we will show in our illustrative examples later in the chapter. What we require is that there be no exact relationships among the regressors, as in Eq. (7.1.9).

Second, keep in mind that we are talking only about perfect *linear* relationships between two or more variables. Multicollinearity does not rule out *nonlinear* relationships between variables. Suppose  $X_{3i} = X_{2i}^2$ . This does not violate the assumption of no perfect collinearity, as the relationship between the variables here is nonlinear.

## 7.2 INTERPRETATION OF MULTIPLE REGRESSION EQUATION

Given the assumptions of the classical regression model, it follows that, on taking the conditional expectation of  $Y$  on both sides of (7.1.1), we obtain

$$E(Y_i | X_{2i}, X_{3i}) = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} \quad (7.2.1)$$

In words, (7.2.1) gives the **conditional mean or expected value of  $Y$  conditional upon the given or fixed values of  $X_2$  and  $X_3$** . Therefore, as in the two-variable case, multiple regression analysis is regression analysis conditional upon the fixed values of the regressors, and what we obtain is the average or mean value of  $Y$  or the mean response of  $Y$  for the given values of the regressors.

## 7.3 THE MEANING OF PARTIAL REGRESSION COEFFICIENTS

As mentioned earlier, the regression coefficients  $\beta_2$  and  $\beta_3$  are known as **partial regression** or **partial slope coefficients**. The meaning of partial regression coefficient is as follows:  $\beta_2$  measures the *change* in the mean value of  $Y$ ,  $E(Y)$ , per unit change in  $X_2$ , holding the value of  $X_3$  constant. Put differently, it gives the “direct” or the “net” effect of a unit change in  $X_2$  on the mean value of  $Y$ , net of any effect that  $X_3$  may have on mean  $Y$ . Likewise,  $\beta_3$  measures the change in the mean value of  $Y$  per unit change in  $X_3$ , holding the value of  $X_2$  constant.<sup>4</sup> That is, it gives the “direct” or “net” effect of a unit

<sup>4</sup>The calculus-minded reader will notice at once that  $\beta_2$  and  $\beta_3$  are the partial derivatives of  $E(Y|X_2, X_3)$  with respect to  $X_2$  and  $X_3$ .

change in  $X_3$  on the mean value of  $Y$ , net of any effect that  $X_2$  may have on mean  $Y$ .<sup>5</sup>

How do we actually go about holding the influence of a regressor constant? To explain this, let us revert to our child mortality example. Recall that in that example,  $Y$  = child mortality (CM),  $X_2$  = per capita GNP (PGNP), and  $X_3$  = female literacy rate (FLR). Let us suppose we want to hold the influence of FLR constant. Since FLR may have some effect on CM as well as PGNP in any given concrete data, what we can do is to remove the (linear) influence of FLR from both CM and PGNP by running the regression of CM on FLR and that of PGNP on FLR separately and then looking at the residuals obtained from these regressions. Using the data given in Table 6.4, we obtain the following regressions:

$$\begin{aligned} \text{CM}_i &= 263.8635 - 2.3905 \text{FLR}_i + \hat{u}_{1i} \\ \text{se} &= (12.2249) \quad (0.2133) \quad r^2 = 0.6695 \end{aligned} \quad (7.3.1)$$

where  $\hat{u}_{1i}$  represents the residual term of this regression.

$$\begin{aligned} \text{PGNP}_i &= -39.3033 + 28.1427 \text{FLR}_i + \hat{u}_{2i} \\ \text{se} &= (734.9526) \quad (12.8211) \quad r^2 = 0.0721 \end{aligned} \quad (7.3.2)$$

where  $\hat{u}_{2i}$  represents the residual term of this regression.

Now

$$\hat{u}_{1i} = (\text{CM}_i - 263.8635 + 2.3905 \text{FLR}_i) \quad (7.3.3)$$

represents that part of CM left after removing from it the (linear) influence of FLR. Likewise,

$$\hat{u}_{2i} = (\text{PGNP}_i + 39.3033 - 28.1427 \text{FLR}_i) \quad (7.3.4)$$

represents that part of PGNP left after removing from it the (linear) influence of FLR.

Therefore, if we now regress  $\hat{u}_{1i}$  on  $\hat{u}_{2i}$ , which are “purified” of the (linear) influence of FLR, wouldn’t we obtain the net effect of PGNP on CM? That is indeed the case (see Appendix 7A, Section 7A.2). The regression results are as follows:

$$\begin{aligned} \hat{\hat{u}}_{1i} &= -0.0056\hat{u}_{2i} \\ \text{se} &= (0.0019) \quad r^2 = 0.1152 \end{aligned} \quad (7.3.5)$$

*Note:* This regression has no intercept term because the mean value of the OLS residuals  $\hat{u}_{1i}$  and  $\hat{u}_{2i}$  is zero (why?)

<sup>5</sup>Incidentally, the terms *holding constant*, *controlling for*, *allowing or accounting for the influence of*, *correcting the influence of*, and *sweeping out the influence of* are synonymous and will be used interchangeably in this text.



The slope coefficient of  $-0.0056$  now gives the “true” or net effect of a unit change in PGNP on CM or the true slope of CM with respect to PGNP. That is, it gives the partial regression coefficient of CM with respect to PGNP,  $\beta_2$ .

Readers who want to get the partial regression coefficient of CM with respect to FLR can replicate the above procedure by first regressing CM on PGNP and getting the residuals from this regression ( $\hat{u}_{1i}$ ), then regressing FLR on PGNP and obtaining the residuals from this regression ( $\hat{u}_{2i}$ ), and then regressing  $\hat{u}_{1i}$  on  $\hat{u}_{2i}$ . I am sure readers get the idea.

Do we have to go through this multistep procedure every time we want to find out the true partial regression coefficient? Fortunately, we do not have to do that, for the same job can be accomplished fairly quickly and routinely by the OLS procedure discussed in the next section. The multistep procedure just outlined is merely for pedagogic purposes to drive home the meaning of “partial” regression coefficient.

#### 7.4 OLS AND ML ESTIMATION OF THE PARTIAL REGRESSION COEFFICIENTS

To estimate the parameters of the three-variable regression model (7.1.1), we first consider the method of ordinary least squares (OLS) introduced in Chapter 3 and then consider briefly the method of maximum likelihood (ML) discussed in Chapter 4.

##### OLS Estimators

To find the OLS estimators, let us first write the sample regression function (SRF) corresponding to the PRF of (7.1.1) as follows:

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \hat{u}_i \quad (7.4.1)$$

where  $\hat{u}_i$  is the residual term, the sample counterpart of the stochastic disturbance term  $u_i$ .

As noted in Chapter 3, the OLS procedure consists in so choosing the values of the unknown parameters that the residual sum of squares (RSS)  $\sum \hat{u}_i^2$  is as small as possible. Symbolically,

$$\min \sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i})^2 \quad (7.4.2)$$

where the expression for the RSS is obtained by simple algebraic manipulations of (7.4.1).

The most straightforward procedure to obtain the estimators that will minimize (7.4.2) is to differentiate it with respect to the unknowns, set the resulting expressions to zero, and solve them simultaneously. As shown in Appendix 7A, Section 7A.1, this procedure gives the following *normal equations* [cf. Eqs. (3.1.4) and (3.1.5)]:

$$\bar{Y} = \hat{\beta}_1 + \hat{\beta}_2 \bar{X}_2 + \hat{\beta}_3 \bar{X}_3 \quad (7.4.3)$$

$$\sum Y_i X_{2i} = \hat{\beta}_1 \sum X_{2i} + \hat{\beta}_2 \sum X_{2i}^2 + \hat{\beta}_3 \sum X_{2i} X_{3i} \quad (7.4.4)$$

$$\sum Y_i X_{3i} = \hat{\beta}_1 \sum X_{3i} + \hat{\beta}_2 \sum X_{2i} X_{3i} + \hat{\beta}_3 \sum X_{3i}^2 \quad (7.4.5)$$

From Eq. (7.4.3) we see at once that

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}_2 - \hat{\beta}_3 \bar{X}_3 \quad (7.4.6)$$

which is the OLS estimator of the population intercept  $\beta_1$ .

Following the convention of letting the lowercase letters denote deviations from sample mean values, one can derive the following formulas from the normal equations (7.4.3) to (7.4.5):

$$\hat{\beta}_2 = \frac{(\sum y_i x_{2i})(\sum x_{3i}^2) - (\sum y_i x_{3i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} \quad (7.4.7)^6$$

$$\hat{\beta}_3 = \frac{(\sum y_i x_{3i})(\sum x_{2i}^2) - (\sum y_i x_{2i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} \quad (7.4.8)$$

which give the OLS estimators of the population partial regression coefficients  $\beta_2$  and  $\beta_3$ , respectively.

In passing, note the following: (1) Equations (7.4.7) and (7.4.8) are symmetrical in nature because one can be obtained from the other by interchanging the roles of  $X_2$  and  $X_3$ ; (2) the denominators of these two equations are identical; and (3) the three-variable case is a natural extension of the two-variable case.

### Variances and Standard Errors of OLS Estimators

Having obtained the OLS estimators of the partial regression coefficients, we can derive the variances and standard errors of these estimators in the manner indicated in Appendix 3A.3. As in the two-variable case, we need the standard errors for two main purposes: to establish confidence intervals and

<sup>6</sup>This estimator is equal to that of (7.3.5), as shown in App. 7A, Sec. 7A.2.

to test statistical hypotheses. The relevant formulas are as follows:<sup>7</sup>

$$\text{var}(\hat{\beta}_1) = \left[ \frac{1}{n} + \frac{\bar{X}_2^2 \sum x_{3i}^2 + \bar{X}_3^2 \sum x_{2i}^2 - 2\bar{X}_2\bar{X}_3 \sum x_{2i}x_{3i}}{\sum x_{2i}^2 \sum x_{3i}^2 - (\sum x_{2i}x_{3i})^2} \right] \cdot \sigma^2 \quad (7.4.9)$$

$$\text{se}(\hat{\beta}_1) = +\sqrt{\text{var}(\hat{\beta}_1)} \quad (7.4.10)$$

$$\text{var}(\hat{\beta}_2) = \frac{\sum x_{3i}^2}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i}x_{3i})^2} \sigma^2 \quad (7.4.11)$$

or, equivalently,

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2 (1 - r_{23}^2)} \quad (7.4.12)$$

where  $r_{23}$  is the sample coefficient of correlation between  $X_2$  and  $X_3$  as defined in Chapter 3.<sup>8</sup>

$$\text{se}(\hat{\beta}_2) = +\sqrt{\text{var}(\hat{\beta}_2)} \quad (7.4.13)$$

$$\text{var}(\hat{\beta}_3) = \frac{\sum x_{2i}^2}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i}x_{3i})^2} \sigma^2 \quad (7.4.14)$$

or, equivalently,

$$\text{var}(\hat{\beta}_3) = \frac{\sigma^2}{\sum x_{3i}^2 (1 - r_{23}^2)} \quad (7.4.15)$$

$$\text{se}(\hat{\beta}_3) = +\sqrt{\text{var}(\hat{\beta}_3)} \quad (7.4.16)$$

$$\text{cov}(\hat{\beta}_2, \hat{\beta}_3) = \frac{-r_{23}\sigma^2}{(1 - r_{23}^2)\sqrt{\sum x_{2i}^2}\sqrt{\sum x_{3i}^2}} \quad (7.4.17)$$

In all these formulas  $\sigma^2$  is the (homoscedastic) variance of the population disturbances  $u_i$ .

Following the argument of Appendix 3A, Section 3A.5, the reader can verify that an unbiased estimator of  $\sigma^2$  is given by

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n - 3} \quad (7.4.18)$$

<sup>7</sup>The derivations of these formulas are easier using matrix notation. Advanced readers may refer to **App. C**.

<sup>8</sup>Using the definition of  $r$  given in Chap. 3, we have

$$r_{23}^2 = \frac{(\sum x_{2i}x_{3i})^2}{\sum x_{2i}^2 \sum x_{3i}^2}$$

Note the similarity between this estimator of  $\sigma^2$  and its two-variable counterpart [ $\hat{\sigma}^2 = (\sum \hat{u}_i^2)/(n-2)$ ]. The degrees of freedom are now  $(n-3)$  because in estimating  $\sum \hat{u}_i^2$  we must first estimate  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$ , which consume 3 df. (The argument is quite general. Thus, in the four-variable case the df will be  $n-4$ .)

The estimator  $\hat{\sigma}^2$  can be computed from (7.4.18) once the residuals are available, but it can also be obtained more readily by using the following relation (for proof, see Appendix 7A, Section 7A.3):

$$\sum \hat{u}_i^2 = \sum y_i^2 - \hat{\beta}_2 \sum y_i x_{2i} - \hat{\beta}_3 \sum y_i x_{3i} \quad (7.4.19)$$

which is the three-variable counterpart of the relation given in (3.3.6).

### Properties of OLS Estimators

The properties of OLS estimators of the multiple regression model parallel those of the two-variable model. Specifically:

**1.** The three-variable regression line (surface) passes through the means  $\bar{Y}$ ,  $\bar{X}_2$ , and  $\bar{X}_3$ , which is evident from (7.4.3) [cf. Eq. (3.1.7) of the two-variable model]. This property holds generally. Thus in the  $k$ -variable linear regression model [a regressand and  $(k-1)$  regressors]

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i \quad (7.4.20)$$

we have

$$\hat{\beta}_1 = \bar{Y} - \beta_2 \bar{X}_2 - \beta_3 \bar{X}_3 - \cdots - \beta_k \bar{X}_k \quad (7.4.21)$$

**2.** The mean value of the estimated  $Y_i$  ( $= \hat{Y}_i$ ) is equal to the mean value of the actual  $Y_i$ , which is easy to prove:

$$\begin{aligned} \hat{Y}_i &= \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} \\ &= (\bar{Y} - \hat{\beta}_2 \bar{X}_2 - \hat{\beta}_3 \bar{X}_3) + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} \quad (\text{Why?}) \\ &= \bar{Y} + \hat{\beta}_2 (X_{2i} - \bar{X}_2) + \hat{\beta}_3 (X_{3i} - \bar{X}_3) \\ &= \bar{Y} + \hat{\beta}_2 x_{2i} + \hat{\beta}_3 x_{3i} \end{aligned} \quad (7.4.22)$$

where as usual small letters indicate values of the variables as deviations from their respective means.

Summing both sides of (7.4.22) over the sample values and dividing through by the sample size  $n$  gives  $\bar{\hat{Y}} = \bar{Y}$ . (Note:  $\sum x_{2i} = \sum x_{3i} = 0$ . Why?) Notice that by virtue of (7.4.22) we can write

$$\hat{y}_i = \hat{\beta}_2 x_{2i} + \hat{\beta}_3 x_{3i} \quad (7.4.23)$$

where  $\hat{y}_i = (\hat{Y}_i - \bar{Y})$ .

Therefore, the SRF (7.4.1) can be expressed in the *deviation form* as

$$y_i = \hat{y}_i + \hat{u}_i = \hat{\beta}_2 x_{2i} + \hat{\beta}_3 x_{3i} + \hat{u}_i \quad (7.4.24)$$

3.  $\sum \hat{u}_i = \bar{\hat{u}} = 0$ , which can be verified from (7.4.24). [*Hint*: Sum both sides of (7.4.24) over the sample values.]

4. The residuals  $\hat{u}_i$  are uncorrelated with  $X_{2i}$  and  $X_{3i}$ , that is,  $\sum \hat{u}_i X_{2i} = \sum \hat{u}_i X_{3i} = 0$  (see Appendix 7A.1 for proof).

5. The residuals  $\hat{u}_i$  are uncorrelated with  $\hat{Y}_i$ ; that is,  $\sum \hat{u}_i \hat{Y}_i = 0$ . Why? [*Hint*: Multiply (7.4.23) on both sides by  $\hat{u}_i$  and sum over the sample values.]

6. From (7.4.12) and (7.4.15) it is evident that as  $r_{23}$ , the correlation coefficient between  $X_2$  and  $X_3$ , increases toward 1, the variances of  $\hat{\beta}_2$  and  $\hat{\beta}_3$  increase for given values of  $\sigma^2$  and  $\sum x_{2i}^2$  or  $\sum x_{3i}^2$ . In the limit, when  $r_{23} = 1$  (i.e., perfect collinearity), these variances become infinite. The implications of this will be explored fully in Chapter 10, but intuitively the reader can see that as  $r_{23}$  increases it is going to be increasingly difficult to know what the true values of  $\beta_2$  and  $\beta_3$  are. [More on this in the next chapter, but refer to Eq. (7.1.10).]

7. It is also clear from (7.4.12) and (7.4.15) that for given values of  $r_{23}$  and  $\sum x_{2i}^2$  or  $\sum x_{3i}^2$ , the variances of the OLS estimators are directly proportional to  $\sigma^2$ ; that is, they increase as  $\sigma^2$  increases. Similarly, for given values of  $\sigma^2$  and  $r_{23}$ , the variance of  $\hat{\beta}_2$  is inversely proportional to  $\sum x_{2i}^2$ ; that is, the greater the variation in the sample values of  $X_2$ , the smaller the variance of  $\hat{\beta}_2$  and therefore  $\beta_2$  can be estimated more precisely. A similar statement can be made about the variance of  $\hat{\beta}_3$ .

8. Given the assumptions of the classical linear regression model, which are spelled out in Section 7.1, one can prove that the OLS estimators of the partial regression coefficients not only are linear and unbiased but also have minimum variance in the class of all linear unbiased estimators. In short, *they are BLUE*: Put differently, they satisfy the Gauss-Markov theorem. (The proof parallels the two-variable case proved in Appendix 3A, Section 3A.6 and will be presented more compactly using matrix notation in **Appendix C.**)

### Maximum Likelihood Estimators

We noted in Chapter 4 that under the assumption that  $u_i$ , the population disturbances, are normally distributed with zero mean and constant variance  $\sigma^2$ , the maximum likelihood (ML) estimators and the OLS estimators of the regression coefficients of the two-variable model are identical. This equality extends to models containing any number of variables. (For proof, see Appendix 7A, Section 7A.4.) However, this is not true of the estimator of  $\sigma^2$ . It can be shown that the ML estimator of  $\sigma^2$  is  $\sum \hat{u}_i^2/n$  regardless of the number of variables in the model, whereas the OLS estimator of  $\sigma^2$  is  $\sum \hat{u}_i^2/(n-2)$  in the two-variable case,  $\sum \hat{u}_i^2/(n-3)$  in the three-variable case, and  $\sum \hat{u}_i^2/(n-k)$  in the case of the  $k$ -variable model (7.4.20). In short, the OLS estimator of  $\sigma^2$  takes into account the number of degrees of freedom, whereas the ML estimator does not. Of course, if  $n$  is very large, the ML and OLS estimators of  $\sigma^2$  will tend to be close to each other. (Why?)

### 7.5 THE MULTIPLE COEFFICIENT OF DETERMINATION $R^2$ AND THE MULTIPLE COEFFICIENT OF CORRELATION $R$

In the two-variable case we saw that  $r^2$  as defined in (3.5.5) measures the goodness of fit of the regression equation; that is, it gives the proportion or percentage of the total variation in the dependent variable  $Y$  explained by the (single) explanatory variable  $X$ . This notation of  $r^2$  can be easily extended to regression models containing more than two variables. Thus, in the three-variable model we would like to know the proportion of the variation in  $Y$  explained by the variables  $X_2$  and  $X_3$  jointly. The quantity that gives this information is known as the **multiple coefficient of determination** and is denoted by  $R^2$ ; conceptually it is akin to  $r^2$ .

To derive  $R^2$ , we may follow the derivation of  $r^2$  given in Section 3.5. Recall that

$$\begin{aligned} Y_i &= \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \hat{u}_i \\ &= \hat{Y}_i + \hat{u}_i \end{aligned} \quad (7.5.1)$$

where  $\hat{Y}_i$  is the estimated value of  $Y_i$  from the fitted regression line and is an estimator of true  $E(Y_i | X_{2i}, X_{3i})$ . Upon shifting to lowercase letters to indicate deviations from the mean values, Eq. (7.5.1) may be written as

$$\begin{aligned} y_i &= \hat{\beta}_2 x_{2i} + \hat{\beta}_3 x_{3i} + \hat{u}_i \\ &= \hat{y}_i + \hat{u}_i \end{aligned} \quad (7.5.2)$$

Squaring (7.5.2) on both sides and summing over the sample values, we obtain

$$\begin{aligned} \sum y_i^2 &= \sum \hat{y}_i^2 + \sum \hat{u}_i^2 + 2 \sum \hat{y}_i \hat{u}_i \\ &= \sum \hat{y}_i^2 + \sum \hat{u}_i^2 \quad (\text{Why?}) \end{aligned} \quad (7.5.3)$$

Verbally, Eq. (7.5.3) states that the total sum of squares (TSS) equals the explained sum of squares (ESS) + the residual sum of squares (RSS). Now substituting for  $\sum \hat{u}_i^2$  from (7.4.19), we obtain

$$\sum y_i^2 = \sum \hat{y}_i^2 + \sum y_i^2 - \hat{\beta}_2 \sum y_i x_{2i} - \hat{\beta}_3 \sum y_i x_{3i}$$

which, on rearranging, gives

$$\text{ESS} = \sum \hat{y}_i^2 = \hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i} \quad (7.5.4)$$

Now, by definition

$$\begin{aligned} R^2 &= \frac{\text{ESS}}{\text{TSS}} \\ &= \frac{\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i}}{\sum y_i^2} \end{aligned} \quad (7.5.5)^9$$

[cf. (7.5.5) with (3.5.6)].

Since the quantities entering (7.5.5) are generally computed routinely,  $R^2$  can be computed easily. Note that  $R^2$ , like  $r^2$ , lies between 0 and 1. If it is 1, the fitted regression line explains 100 percent of the variation in  $Y$ . On the other hand, if it is 0, the model does not explain any of the variation in  $Y$ . Typically, however,  $R^2$  lies between these extreme values. The fit of the model is said to be “better” the closer is  $R^2$  to 1.

Recall that in the two-variable case we defined the quantity  $r$  as the coefficient of correlation and indicated that it measures the degree of (linear) association between two variables. The three-or-more-variable analogue of  $r$  is the coefficient of **multiple correlation**, denoted by  $R$ , and it is a measure of the degree of association between  $Y$  and all the explanatory variables jointly. Although  $r$  can be positive or negative,  $R$  is always taken to be positive. In practice, however,  $R$  is of little importance. The more meaningful quantity is  $R^2$ .

Before proceeding further, let us note the following relationship between  $R^2$  and the variance of a partial regression coefficient in the  $k$ -variable multiple regression model given in (7.4.20):

$$\text{var}(\hat{\beta}_j) = \frac{\sigma^2}{\sum x_j^2} \left( \frac{1}{1 - R_j^2} \right) \quad (7.5.6)$$

where  $\hat{\beta}_j$  is the partial regression coefficient of regressor  $X_j$  and  $R_j^2$  is the  $R^2$  in the regression of  $X_j$  on the remaining  $(k - 2)$  regressors. [Note: There are  $(k - 1)$  regressors in the  $k$ -variable regression model.] Although the utility of Eq. (7.5.6) will become apparent in Chapter 10 on multicollinearity, observe that this equation is simply an extension of the formula given in (7.4.12) or (7.4.15) for the three-variable regression model, one regressand and two regressors.

## 7.6 EXAMPLE 7.1: CHILD MORTALITY IN RELATION TO PER CAPITA GNP AND FEMALE LITERACY RATE

In Chapter 6 we considered the behavior of child mortality (CM) in relation to per capita GNP (PGNP). There we found that PGNP has a negative impact on CM, as one would expect. Now let us bring in female literacy as measured

<sup>9</sup>Note that  $R^2$  can also be computed as follows:

$$R^2 = 1 - \frac{\text{RSS}}{\text{TSS}} = 1 - \frac{\sum \hat{u}_i^2}{\sum y_i^2} = 1 - \frac{(n-3)\hat{\sigma}^2}{(n-1)S_y^2}$$

by the female literacy rate (FLR). A priori, we expect that FLR too will have a negative impact on CM. Now when we introduce both the variables in our model, we need to net out the influence of each of the regressors. That is, we need to estimate the (partial) regression coefficients of each regressor. Thus our model is:

$$CM_i = \beta_1 + \beta_2 PGNP_i + \beta_3 FLR_i + u_i \quad (7.6.1)$$

The necessary data are given in Table 6.4. Keep in mind that CM is the number of deaths of children under five per 1000 live births, PGNP is per capita GNP in 1980, and FLR is measured in percent. Our sample consists of 64 countries.

Using the Eviews3 statistical package, we obtained the following results:

$$\begin{aligned} \widehat{CM}_i &= 263.6416 - 0.0056 PGNP_i - 2.2316 FLR_i \\ \text{se} &= (11.5932) \quad (0.0019) \quad (0.2099) \quad R^2 = 0.7077 \quad (7.6.2) \\ &\quad \bar{R}^2 = 0.6981^* \end{aligned}$$

where figures in parentheses are the estimated standard errors. Before we interpret this regression, observe the partial slope coefficient of PGNP, namely,  $-0.0056$ . Is it not precisely the same as that obtained from the three-step procedure discussed in the previous section [see Eq. (7.3.5)]? But should that surprise you? Not only that, but the two standard errors are precisely the same, which is again unsurprising. But we did so without the three-step cumbersome procedure.

Let us now interpret these regression coefficients:  $-0.0056$  is the partial regression coefficient of PGNP and tells us that with the influence of FLR held constant, as PGNP increases, say, by a dollar, on average, child mortality goes down by 0.0056 units. To make it more economically interpretable, if the per capita GNP goes up by a thousand dollars, on average, the number of deaths of children under age 5 goes down by about 5.6 per thousand live births. The coefficient  $-2.2316$  tells us that holding the influence of PGNP constant, on average, the number of deaths of children under 5 goes down by about 2.23 per thousand live births as the female literacy rate increases by one percentage point. The intercept value of about 263, mechanically interpreted, means that if the values of PGNP and FLR rate were fixed at zero, the mean child mortality would be about 263 deaths per thousand live births. Of course, such an interpretation should be taken with a grain of salt. All one could infer is that if the two regressors were fixed at zero, child mortality will be quite high, which makes practical sense. The  $R^2$  value of about 0.71 means that about 71 percent of the variation in child mortality is explained by PGNP and FLR, a fairly high value considering that the maximum value of  $R^2$  can at most be 1. All told, the regression results make sense.

\*On this, see Sec. 7.8.



What about the statistical significance of the estimated coefficients? We will take this topic up in Chapter 8. As we will see there, in many ways this chapter will be an extension of Chapter 5, which dealt with the two-variable model. As we will also show, there are some important differences in statistical inference (i.e., hypothesis testing) between the two-variable and multi-variable regression models.

### Regression on Standardized Variables

In the preceding chapter we introduced the topic of regression on standardized variables and stated that the analysis can be extended to multivariable regressions. Recall that a variable is said to be standardized or in standard deviation units if it is expressed in terms of deviation from its mean and divided by its standard deviation.

For our child mortality example, the results are as follows:

$$\begin{aligned} \widehat{CM}^* &= -0.2026 PGNP_i^* - 0.7639 FLR_i^* && (7.6.3) \\ se &= (0.0713) \quad (0.0713) \quad r^2 = 0.7077 \end{aligned}$$

*Note:* The starred variables are standardized variables. Also note that there is no intercept in the model for reasons already discussed in the previous chapter.

As you can see from this regression, with FLR held constant, a standard deviation increase in PGNP leads, on average, to a 0.2026 standard deviation decrease in CM. Similarly, holding PGNP constant, a standard deviation increase in FLR, on average, leads to a 0.7639 standard deviation decrease in CM. Relatively speaking, female literacy has more impact on child mortality than per capita GNP. Here you will see the advantage of using standardized variables, for standardization puts all variables on equal footing because all standardized variables have zero means and unit variances.

### 7.7 SIMPLE REGRESSION IN THE CONTEXT OF MULTIPLE REGRESSION: INTRODUCTION TO SPECIFICATION BIAS

Recall that assumption (7.1.6) of the classical linear regression model states that the regression model used in the analysis is “correctly” specified; that is, there is no **specification bias or specification error** (see Chapter 3 for some introductory remarks). Although the topic of specification error will be discussed more fully in Chapter 13, the illustrative example given in the preceding section provides a splendid opportunity not only to drive home the importance of assumption (7.1.6) but also to shed additional light on the meaning of partial regression coefficient and provide a somewhat informal introduction to the topic of specification bias.

Assume that (7.6.1) is the “true” model explaining the behavior of child mortality in relation to per capita GNP and female literacy rate (FLR). But suppose we disregard FLR and estimate the following simple regression:

$$Y_i = \alpha_1 + \alpha_2 X_{2i} + u_{1i} \quad (7.7.1)$$

where  $Y = \text{CM}$  and  $X_2 = \text{PGNP}$ .

Since (7.6.1) is the true model, estimating (7.7.1) would constitute a specification error; the error here consists in *omitting* the variable  $X_3$ , the female literacy rate. Notice that we are using different parameter symbols (the alphas) in (7.7.1) to distinguish them from the true parameters (the betas) given in (7.6.1).

Now will  $\alpha_2$  provide an unbiased estimate of the true impact of PGNP, which is given by  $\beta_2$  in model (7.6.1)? In other words, will  $E(\hat{\alpha}_2) = \beta_2$ , where  $\hat{\alpha}_2$  is the estimated value of  $\alpha_2$ ? In other words, will the coefficient of PGNP in (7.7.1) provide an unbiased estimate of the true impact of PGNP on CM, knowing that we have omitted the variable  $X_3$  (FLR) from the model? As you would suspect, *in general*  $\hat{\alpha}_2$  will not be an unbiased estimator of the true  $\beta_2$ . To give a glimpse of the bias, let us run the regression (7.7.1), which gave the following results.

$$\begin{aligned} \widehat{\text{CM}}_i &= 157.4244 - 0.0114 \text{ PGNP}_i \\ \text{se} &= (9.8455) \quad (0.0032) \quad r^2 = 0.1662 \end{aligned} \quad (7.7.2)$$

Observe several things about this regression compared to the “true” multiple regression (7.6.1):

1. In absolute terms (i.e., disregarding the sign), the PGNP coefficient has increased from 0.0056 to 0.0114, almost a two-fold increase.
2. The standard errors are different.
3. The intercept values are different.
4. The  $r^2$  values are dramatically different, although it is generally the case that, as the number of regressors in the model increases, the  $r^2$  value increases.

Now suppose that you regress child mortality on female literacy rate, disregarding the influence of PGNP. You will obtain the following results:

$$\begin{aligned} \widehat{\text{CM}}_i &= 263.8635 - 2.3905 \text{ FLR}_i \\ \text{se} &= (21.2249) \quad (0.2133) \quad r^2 = 0.6696 \end{aligned} \quad (7.7.3)$$

Again if you compare the results of this (misspecified) regression with the “true” multiple regression, you will see that the results are different, although the difference here is not as noticeable as in the case of regression (7.7.2).

*The important point to note is that serious consequences can ensue if you misfit a model.* We will look into this topic more thoroughly in Chapter 13, on specification errors.

### 7.8 $R^2$ AND THE ADJUSTED $R^2$

An important property of  $R^2$  is that it is a nondecreasing function of the number of explanatory variables or regressors present in the model; as the number of regressors increases,  $R^2$  almost invariably increases and never decreases. Stated differently, an additional  $X$  variable will not decrease  $R^2$ . Compare, for instance, regression (7.7.2) or (7.7.3) with (7.6.2). To see this, recall the definition of the coefficient of determination:

$$\begin{aligned} R^2 &= \frac{\text{ESS}}{\text{TSS}} \\ &= 1 - \frac{\text{RSS}}{\text{TSS}} \\ &= 1 - \frac{\sum \hat{u}_i^2}{\sum y_i^2} \end{aligned} \quad (7.8.1)$$

Now  $\sum y_i^2$  is independent of the number of  $X$  variables in the model because it is simply  $\sum (Y_i - \bar{Y})^2$ . The RSS,  $\sum \hat{u}_i^2$ , however, depends on the number of regressors present in the model. Intuitively, it is clear that as the number of  $X$  variables increases,  $\sum \hat{u}_i^2$  is likely to decrease (at least it will not increase); hence  $R^2$  as defined in (7.8.1) will increase. In view of this, in comparing two regression models with the *same dependent variable* but differing number of  $X$  variables, one should be very wary of choosing the model with the highest  $R^2$ .

To compare two  $R^2$  terms, one must take into account the number of  $X$  variables present in the model. This can be done readily if we consider an alternative coefficient of determination, which is as follows:

$$\bar{R}^2 = 1 - \frac{\sum \hat{u}_i^2 / (n - k)}{\sum y_i^2 / (n - 1)} \quad (7.8.2)$$

where  $k$  = the number of parameters in the model *including the intercept term*. (In the three-variable regression,  $k = 3$ . Why?) The  $R^2$  thus defined is known as the **adjusted  $R^2$** , denoted by  $\bar{R}^2$ . The term *adjusted* means adjusted for the df associated with the sums of squares entering into (7.8.1):  $\sum \hat{u}_i^2$  has  $n - k$  df in a model involving  $k$  parameters, which include

the intercept term, and  $\sum y_i^2$  has  $n - 1$  df. (Why?) For the three-variable case, we know that  $\sum \hat{u}_i^2$  has  $n - 3$  df.

Equation (7.8.2) can also be written as

$$\bar{R}^2 = 1 - \frac{\hat{\sigma}^2}{S_Y^2} \quad (7.8.3)$$

where  $\hat{\sigma}^2$  is the residual variance, an unbiased estimator of true  $\sigma^2$ , and  $S_Y^2$  is the sample variance of  $Y$ .

It is easy to see that  $\bar{R}^2$  and  $R^2$  are related because, substituting (7.8.1) into (7.8.2), we obtain

$$\bar{R}^2 = 1 - (1 - R^2) \frac{n - 1}{n - k} \quad (7.8.4)$$

It is immediately apparent from Eq. (7.8.4) that (1) for  $k > 1$ ,  $\bar{R}^2 < R^2$  which implies that as the number of  $X$  variables increases, the adjusted  $R^2$  increases less than the unadjusted  $R^2$ ; and (2)  $\bar{R}^2$  can be negative, although  $R^2$  is necessarily nonnegative.<sup>10</sup> In case  $\bar{R}^2$  turns out to be negative in an application, its value is taken as zero.

Which  $R^2$  should one use in practice? As Theil notes:

... it is good practice to use  $\bar{R}^2$  rather than  $R^2$  because  $R^2$  tends to give an overly optimistic picture of the fit of the regression, particularly when the number of explanatory variables is not very small compared with the number of observations.<sup>11</sup>

But Theil's view is not uniformly shared, for he has offered no general theoretical justification for the "superiority" of  $\bar{R}^2$ . For example, Goldberger argues that the following  $R^2$ , call it **modified**  $R^2$ , will do just as well<sup>12</sup>:

$$\text{Modified } R^2 = (1 - k/n)R^2 \quad (7.8.5)$$

His advice is to report  $R^2$ ,  $n$ , and  $k$  and let the reader decide how to adjust  $R^2$  by allowing for  $n$  and  $k$ .

Despite this advice, it is the adjusted  $R^2$ , as given in (7.8.4), that is reported by most statistical packages along with the conventional  $R^2$ . The reader is well advised to treat  $\bar{R}^2$  as just another summary statistic.

<sup>10</sup>Note, however, that if  $R^2 = 1$ ,  $\bar{R}^2 = R^2 = 1$ . When  $R^2 = 0$ ,  $\bar{R}^2 = (1 - k)/(n - k)$ , in which case  $\bar{R}^2$  can be negative if  $k > 1$ .

<sup>11</sup>Henri Theil, *Introduction to Econometrics*, Prentice Hall, Englewood Cliffs, N.J., 1978, p. 135.

<sup>12</sup>Arthur S. Goldberger, *A Course in Econometrics*, Harvard University Press, Cambridge, Mass., 1991, p. 178. For a more critical view of  $R^2$ , see S. Cameron, "Why is the  $R$  Squared Adjusted Reported?", *Journal of Quantitative Economics*, vol. 9, no. 1, January 1993, pp. 183–186. He argues that "It [ $R^2$ ] is NOT a test statistic and there seems to be no clear intuitive justification for its use as a descriptive statistic. Finally, we should be clear that it is not an effective tool for the prevention of data mining" (p. 186).

Incidentally, for the child mortality regression (7.6.2), the reader should verify that  $\bar{R}^2$  is 0.6981, keeping in mind that in this example  $(n - 1) = 63$  and  $(n - k) = 60$ . As expected,  $\bar{R}^2$  of 0.6981 is less than  $R^2$  of 0.7077.

Besides  $R^2$  and adjusted  $R^2$  as goodness of fit measures, other criteria are often used to judge the adequacy of a regression model. Two of these are **Akaike's Information criterion** and **Amemiya's Prediction criteria**, which are used to select between competing models. We will discuss these criteria when we consider the problem of model selection in greater detail in a later chapter (see Chapter 13).

### Comparing Two $R^2$ Values

It is crucial to note that in comparing two models on the basis of the coefficient of determination, whether adjusted or not, *the sample size  $n$  and the dependent variable must be the same*; the explanatory variables may take any form. Thus for the models

$$\ln Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i \quad (7.8.6)$$

$$Y_i = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + u_i \quad (7.8.7)$$

the computed  $R^2$  terms cannot be compared. The reason is as follows: By definition,  $R^2$  measures the proportion of the variation in the dependent variable accounted for by the explanatory variable(s). Therefore, in (7.8.6)  $R^2$  measures the proportion of the *variation in  $\ln Y$*  explained by  $X_2$  and  $X_3$ , whereas in (7.8.7) it measures the proportion of the *variation in  $Y$* , and the two are not the same thing: As noted in Chapter 6, a change in  $\ln Y$  gives a relative or proportional change in  $Y$ , whereas a change in  $\widehat{Y}$  gives an absolute change. Therefore,  $\text{var } \widehat{Y}_i / \text{var } Y_i$  is not equal to  $\text{var } (\ln \widehat{Y}_i) / \text{var } (\ln Y_i)$ ; that is, the two coefficients of determination are not the same.<sup>13</sup>

How then does one compare the  $R^2$ 's of two models when the regressand is not in the same form? To answer this question, let us first consider a numerical example.

<sup>13</sup>From the definition of  $R^2$ , we know that

$$1 - R^2 = \frac{\text{RSS}}{\text{TSS}} = \frac{\sum \hat{u}_i^2}{\sum (Y_i - \bar{Y})^2}$$

for the linear model and

$$1 - R^2 = \frac{\sum \hat{u}_i^2}{\sum (\ln Y_i - \ln \bar{Y})^2}$$

for the log model. Since the denominators on the right-hand sides of these expressions are different, we cannot compare the two  $R^2$  terms directly.

As shown in Example 7.2, for the linear specification, the  $\text{RSS} = 0.1491$  (the residual sum of squares of coffee consumption), and for the log-linear specification, the  $\text{RSS} = 0.0226$  (the residual sum of squares of log of coffee consumption). These residuals are of different orders of magnitude and hence are not directly comparable.

**EXAMPLE 7.2**

COFFEE CONSUMPTION IN THE UNITED STATES, 1970–1980

Consider the data in Table 7.1. The data pertain to consumption of cups of coffee per day ( $Y$ ) and real retail price of coffee ( $X$ ) in the United States for years 1970–1980. Applying OLS to the data, we obtain the following regression results:

$$\hat{Y}_t = 2.6911 - 0.4795X_t \quad (7.8.8)$$

se = (0.1216) (0.1140)      RSS = 0.1491;  $r^2 = 0.6628$

The results make economic sense: As the price of coffee increases, on average, coffee consumption goes down by about half a cup per day. The  $r^2$  value of about 0.66 means that the price of coffee explains about 66 percent of the variation in coffee consumption. The reader can readily verify that the slope coefficient is statistically significant.

From the same data, the following double log, or constant elasticity, model can be estimated:

$$\widehat{\ln Y}_t = 0.7774 - 0.2530 \ln X_t \quad (7.8.9)$$

se = (0.0152) (0.0494)      RSS = 0.0226;  $r^2 = 0.7448$

Since this is a double log model, the slope coefficient gives a direct estimate of the price elasticity coefficient. In the present instance, it tells us that if the price of coffee per pound goes up by 1 percent, on average, per day coffee consumption goes down by about 0.25 percent. Remember that in the linear model (7.8.8) the slope coefficient only gives the rate of change of coffee consumption with respect to price. (How will you estimate the price elasticity for the

**TABLE 7.1**  
U.S. COFFEE CONSUMPTION ( $Y$ ) IN RELATION TO AVERAGE  
REAL RETAIL PRICE ( $X$ ),\* 1970–1980

Year	Y, Cups per person per day	X, \$ per lb
1970	2.57	0.77
1971	2.50	0.74
1972	2.35	0.72
1973	2.30	0.73
1974	2.25	0.76
1975	2.20	0.75
1976	2.11	1.08
1977	1.94	1.81
1978	1.97	1.39
1979	2.06	1.20
1980	2.02	1.17

\*Note: The nominal price was divided by the Consumer Price Index (CPI) for food and beverages, 1967 = 100.  
Source: The data for  $Y$  are from *Summary of National Coffee Drinking Study*, Data Group, Elkins Park, Penn., 1981; and the data on nominal  $X$  (i.e.,  $X$  in current prices) are from *Nielsen Food Index*, A. C. Nielsen, New York, 1981.

I am indebted to Scott E. Sandberg for collecting the data.

(Continued)

**EXAMPLE 7.2** (Continued)

linear model?) The  $r^2$  value of about 0.74 means that about 74 percent of the variation in the log of coffee demand is explained by the variation in the log of coffee price.

Since the  $r^2$  value of the linear model of 0.6628 is smaller than the  $r^2$  value of 0.7448 of the log-linear model, you might be tempted to choose the latter model because of its high  $r^2$  value. But for reasons already noted, we cannot do so. But if you do want to compare the two  $r^2$  values, you may proceed as follows:

1. Obtain  $\ln \hat{Y}_t$  from (7.8.9) for each observation; that is, obtain the estimated log value of each observation from this model. Take the antilog of these values and then compute  $r^2$  between these antilog values and actual  $Y_t$  in the manner indicated by Eq. (3.5.14). This  $r^2$  value is comparable to the  $r^2$  value of the linear model (7.8.8).
2. *Alternatively*, assuming all  $Y$  values are positive, take logarithms of the  $Y$  values,  $\ln Y$ . Obtain the estimated  $Y$  values,  $\hat{Y}_t$ , from the linear model (7.8.8), take the logarithms of these estimated  $Y$  values (i.e.,  $\ln \hat{Y}_t$ ) and compute the  $r^2$  between  $(\ln Y_t)$  and  $(\ln \hat{Y}_t)$  in the manner indicated in Eq. (3.5.14). This  $r^2$  value is comparable to the  $r^2$  value obtained from (7.8.9).

For our coffee example, we present the necessary raw data to compute the comparable  $r^2$ 's in Table 7.2. To compare the  $r^2$  value of the linear model (7.8.8) with that of (7.8.9), we first obtain log of  $(\hat{Y}_t)$  [given in column (6) of Table 7.2], then we obtain the log of actual  $Y$  values [given in column (5) of the table], and then compute  $r^2$  between these two sets of values using Eq. (3.5.14). The result is an  $r^2$  value of 0.7318, which is now comparable with the  $r^2$  value of the log-linear model of 0.7448. Now the difference between the two  $r^2$  values is very small.

On the other hand, if we want to compare the  $r^2$  value of the log-linear model with the linear model, we obtain  $\ln \hat{Y}_t$  for each observation from (7.8.9) [given in column (3) of the table], obtain their antilog values [given in column (4) of the table], and finally compute  $r^2$  between these antilog values and the actual  $Y$  values, using formula (3.5.14). This will give an  $r^2$  value of 0.7187, which is slightly higher than that obtained from the linear model (7.8.8), namely, 0.6628.

Using either method, it seems that the log-linear model gives a slightly better fit.

**TABLE 7.2** RAW DATA FOR COMPARING TWO  $R^2$  VALUES

Year	$Y_t$ (1)	$\hat{Y}_t$ (2)	$\ln \hat{Y}_t$ (3)	Antilog of $\ln \hat{Y}_t$ (4)	$\ln Y_t$ (5)	$\ln (\hat{Y}_t)$ (6)
1970	2.57	2.321887	0.843555	2.324616	0.943906	0.842380
1971	2.50	2.336272	0.853611	2.348111	0.916291	0.848557
1972	2.35	2.345863	0.860544	2.364447	0.854415	0.852653
1973	2.30	2.341068	0.857054	2.356209	0.832909	0.850607
1974	2.25	2.326682	0.846863	2.332318	0.810930	0.844443
1975	2.20	2.331477	0.850214	2.340149	0.788457	0.846502
1976	2.11	2.173233	0.757943	2.133882	0.746688	0.776216
1977	1.94	1.823176	0.627279	1.872508	0.662688	0.600580
1978	1.97	2.024579	0.694089	2.001884	0.678034	0.705362
1979	2.06	2.115689	0.731282	2.077742	0.722706	0.749381
1980	2.02	2.130075	0.737688	2.091096	0.703098	0.756157

*Notes:* Column (1): Actual  $Y$  values from Table 7.1  
 Column (2): Estimated  $Y$  values from the linear model (7.8.8)  
 Column (3): Estimated log  $Y$  values from the double-log model (7.8.9)  
 Column (4): Antilog of values in column (3)  
 Column (5): Log values of  $Y$  in column (1)  
 Column (6): Log values of  $\hat{Y}_t$  in column (2)



### Allocating $R^2$ among Regressors

Let us return to our child mortality example. We saw in (7.6.2) that the two regressors PGNP and FLR explain 0.7077 or 70.77 percent of the variation in child mortality. But now consider the regression (7.7.2) where we dropped the FLR variable and as a result the  $r^2$  value dropped to 0.1662. Does that mean the difference in the  $r^2$  value of 0.5415 ( $0.7077 - 0.1662$ ) is attributable to the dropped variable FLR? On the other hand, if you consider regression (7.7.3), where we dropped the PGNP variable, the  $r^2$  value drops to 0.6696. Does that mean the difference in the  $r^2$  value of 0.0381 ( $0.7077 - 0.6696$ ) is due to the omitted variable PGNP?

The question then is: Can we allocate the multiple  $R^2$  of 0.7077 between the two regressors, PGNP and FLR, in this manner? Unfortunately, we cannot do so, for the allocation depends on the order in which the regressors are introduced, as we just illustrated. Part of the problem here is that the two regressors are correlated, the correlation coefficient between the two being 0.2685 (verify it from the data given in Table 6.4). In most applied work with several regressors, correlation among them is a common problem. Of course, the problem will be very serious if there is perfect collinearity among the regressors.

The best practical advice is that there is little point in trying to allocate the  $R^2$  value to its constituent regressors.

### The “Game” of Maximizing $\bar{R}^2$

In concluding this section, a warning is in order: Sometimes researchers play the game of maximizing  $\bar{R}^2$ , that is, choosing the model that gives the highest  $\bar{R}^2$ . But this may be dangerous, for in regression analysis our objective is not to obtain a high  $\bar{R}^2$  per se but rather to obtain dependable estimates of the true population regression coefficients and draw statistical inferences about them. In empirical analysis it is not unusual to obtain a very high  $\bar{R}^2$  but find that some of the regression coefficients either are statistically insignificant or have signs that are contrary to a priori expectations. Therefore, the researcher should be more concerned about the logical or theoretical relevance of the explanatory variables to the dependent variable and their statistical significance. If in this process we obtain a high  $\bar{R}^2$ , well and good; on the other hand, if  $\bar{R}^2$  is low, it does not mean the model is necessarily bad.<sup>14</sup>

<sup>14</sup>Some authors would like to deemphasize the use of  $R^2$  as a measure of goodness of fit as well as its use for comparing two or more  $R^2$  values. See Christopher H. Achen, *Interpreting and Using Regression*, Sage Publications, Beverly Hills, Calif., 1982, pp. 58–67, and C. Granger and P. Newbold, “ $R^2$  and the Transformation of Regression Variables,” *Journal of Econometrics*, vol. 4, 1976, pp. 205–210. Incidentally, the practice of choosing a model on the basis of highest  $R^2$ , a kind of data mining, introduces what is known as **pretest bias**, which might destroy some of the properties of OLS estimators of the classical linear regression model. On this topic, the reader may want to consult George G. Judge, Carter R. Hill, William E. Griffiths, Helmut Lütkepohl, and Tsoung-Chao Lee, *Introduction to the Theory and Practice of Econometrics*, John Wiley, New York, 1982, Chap. 21.



As a matter of fact, Goldberger is very critical about the role of  $R^2$ . He has said:

From our perspective,  $R^2$  has a very modest role in regression analysis, being a measure of the goodness of fit of a sample LS [least-squares] linear regression in a body of data. Nothing in the CR [CLRM] model requires that  $R^2$  be high. Hence a high  $R^2$  is not evidence in favor of the model and a low  $R^2$  is not evidence against it.

In fact the most important thing about  $R^2$  is that it is not important in the CR model. The CR model is concerned with parameters in a population, not with goodness of fit in the sample. . . . If one insists on a measure of predictive success (or rather failure), then  $\sigma^2$  might suffice: after all, the parameter  $\sigma^2$  is the expected squared forecast error that would result if the population CEF [PRF] were used as the predictor. Alternatively, the squared standard error of forecast . . . at relevant values of  $x$  [regressors] may be informative.<sup>15</sup>

### 7.9 EXAMPLE 7.3: THE COBB–DOUGLAS PRODUCTION FUNCTION: MORE ON FUNCTIONAL FORM

In Section 6.4 we showed how with appropriate transformations we can convert nonlinear relationships into linear ones so that we can work within the framework of the classical linear regression model. The various transformations discussed there in the context of the two-variable case can be easily extended to multiple regression models. We demonstrate transformations in this section by taking up the multivariable extension of the two-variable log–linear model; others can be found in the exercises and in the illustrative examples discussed throughout the rest of this book. The specific example we discuss is the celebrated **Cobb–Douglas production function** of production theory.

The Cobb–Douglas production function, in its stochastic form, may be expressed as

$$Y_i = \beta_1 X_{2i}^{\beta_2} X_{3i}^{\beta_3} e^{u_i} \quad (7.9.1)$$

where  $Y$  = output

$X_2$  = labor input

$X_3$  = capital input

$u$  = stochastic disturbance term

$e$  = base of natural logarithm

From Eq. (7.9.1) it is clear that the relationship between output and the two inputs is nonlinear. However, if we log-transform this model, we obtain:

<sup>15</sup>Arther S. Goldberger, op. cit., pp. 177–178.

$$\begin{aligned}\ln Y_i &= \ln \beta_1 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i \\ &= \beta_0 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i\end{aligned}\quad (7.9.2)$$

where  $\beta_0 = \ln \beta_1$ .

Thus written, the model is linear in the parameters  $\beta_0$ ,  $\beta_2$ , and  $\beta_3$  and is therefore a linear regression model. Notice, though, it is nonlinear in the variables  $Y$  and  $X$  but linear in the logs of these variables. In short, (7.9.2) is a *log-log*, *double-log*, or *log-linear model*, the multiple regression counterpart of the two-variable log-linear model (6.5.3).

The properties of the Cobb–Douglas production function are quite well known:

1.  $\beta_2$  is the (partial) elasticity of output with respect to the labor input, that is, it measures the percentage change in output for, say, a 1 percent change in the labor input, holding the capital input constant (see exercise 7.9).

2. Likewise,  $\beta_3$  is the (partial) elasticity of output with respect to the capital input, holding the labor input constant.

3. The sum  $(\beta_2 + \beta_3)$  gives information about the *returns to scale*, that is, the response of output to a proportionate change in the inputs. If this sum is 1, then there are *constant returns to scale*, that is, doubling the inputs will double the output, tripling the inputs will triple the output, and so on. If the sum is less than 1, there are *decreasing returns to scale*—doubling the inputs will less than double the output. Finally, if the sum is greater than 1, there are *increasing returns to scale*—doubling the inputs will more than double the output.

Before proceeding further, note that whenever you have a log-linear regression model involving any number of variables the coefficient of each of the  $X$  variables measures the (partial) elasticity of the dependent variable  $Y$  with respect to that variable. Thus, if you have a  $k$ -variable log-linear model:

$$\ln Y_i = \beta_0 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + \cdots + \beta_k \ln X_{ki} + u_i \quad (7.9.3)$$

each of the (partial) regression coefficients,  $\beta_2$  through  $\beta_k$ , is the (partial) elasticity of  $Y$  with respect to variables  $X_2$  through  $X_k$ .<sup>16</sup>

To illustrate the Cobb–Douglas production function, we obtained the data shown in Table 7.3; these data are for the agricultural sector of Taiwan for 1958–1972.

Assuming that the model (7.9.2) satisfies the assumptions of the classical linear regression model,<sup>17</sup> we obtained the following regression by the OLS

<sup>16</sup>To see this, differentiate (7.9.3) partially with respect to the log of each  $X$  variable. Therefore,  $\partial \ln Y / \partial \ln X_2 = (\partial Y / \partial X_2)(X_2 / Y) = \beta_2$ , which, by definition, is the elasticity of  $Y$  with respect to  $X_2$  and  $\partial \ln Y / \partial \ln X_3 = (\partial Y / \partial X_3)(X_3 / Y) = \beta_3$ , which is the elasticity of  $Y$  with respect to  $X_3$ , and so on.

<sup>17</sup>Notice that in the Cobb–Douglas production function (7.9.1) we have introduced the stochastic error term in a special way so that in the resulting logarithmic transformation it enters in the usual linear form. On this, see Sec. 6.9.

**TABLE 7.3** REAL GROSS PRODUCT, LABOR DAYS, AND REAL CAPITAL INPUT IN THE AGRICULTURAL SECTOR OF TAIWAN, 1958–1972

Year	Real gross product (millions of NT \$)*, $Y$	Labor days (millions of days), $X_2$	Real capital input (millions of NT \$), $X_3$
1958	16,607.7	275.5	17,803.7
1959	17,511.3	274.4	18,096.8
1960	20,171.2	269.7	18,271.8
1961	20,932.9	267.0	19,167.3
1962	20,406.0	267.8	19,647.6
1963	20,831.6	275.0	20,803.5
1964	24,806.3	283.0	22,076.6
1965	26,465.8	300.7	23,445.2
1966	27,403.0	307.5	24,939.0
1967	28,628.7	303.7	26,713.7
1968	29,904.5	304.7	29,957.8
1969	27,508.2	298.6	31,585.9
1970	29,035.5	295.5	33,474.5
1971	29,281.5	299.0	34,821.8
1972	31,535.8	288.1	41,794.3

Source: Thomas Pei-Fan Chen, "Economic Growth and Structural Change in Taiwan—1952–1972, A Production Function Approach," unpublished Ph.D. thesis, Dept. of Economics, Graduate Center, City University of New York, June 1976, Table II.

\*New Taiwan dollars.

method (see Appendix 7A, Section 7A.5 for the computer printout):

$$\widehat{\ln Y}_i = -3.3384 + 1.4988 \ln X_{2i} + 0.4899 \ln X_{3i} \quad (7.9.4)$$

$$\begin{matrix} (2.4495) & (0.5398) & (0.1020) \\ t = (-1.3629) & (2.7765) & (4.8005) \\ & & R^2 = 0.8890 \quad df = 12 \\ & & \bar{R}^2 = 0.8705 \end{matrix}$$

From Eq. (7.9.4) we see that in the Taiwanese agricultural sector for the period 1958–1972 the output elasticities of labor and capital were 1.4988 and 0.4899, respectively. In other words, over the period of study, holding the capital input constant, a 1 percent increase in the labor input led on the average to about a 1.5 percent increase in the output. Similarly, holding the labor input constant, a 1 percent increase in the capital input led on the average to about a 0.5 percent increase in the output. Adding the two output elasticities, we obtain 1.9887, which gives the value of the returns to scale parameter. As is evident, over the period of the study, the Taiwanese agricultural sector was characterized by increasing returns to scale.<sup>18</sup>

<sup>18</sup>We abstain from the question of the appropriateness of the model from the theoretical viewpoint as well as the question of whether one can measure returns to scale from time series data.

From a purely statistical viewpoint, the estimated regression line fits the data quite well. The  $R^2$  value of 0.8890 means that about 89 percent of the variation in the (log of) output is explained by the (logs of) labor and capital. In Chapter 8, we shall see how the estimated standard errors can be used to test hypotheses about the “true” values of the parameters of the Cobb–Douglas production function for the Taiwanese economy.

### 7.10 POLYNOMIAL REGRESSION MODELS

We now consider a class of multiple regression models, the **polynomial regression models**, that have found extensive use in econometric research relating to cost and production functions. In introducing these models, we further extend the range of models to which the classical linear regression model can easily be applied.

To fix the ideas, consider Figure 7.1, which relates the short-run marginal cost (MC) of production ( $Y$ ) of a commodity to the level of its output ( $X$ ). The visually-drawn MC curve in the figure, the textbook U-shaped curve, shows that the relationship between MC and output is nonlinear. If we were to quantify this relationship from the given scatterpoints, how would we go about it? In other words, what type of econometric model would capture first the declining and then the increasing nature of marginal cost?

Geometrically, the MC curve depicted in Figure 7.1 represents a *parabola*. Mathematically, the parabola is represented by the following equation:

$$Y = \beta_0 + \beta_1 X + \beta_2 X^2 \quad (7.10.1)$$

which is called a *quadratic function*, or more generally, a *second-degree polynomial* in the variable  $X$ —the highest power of  $X$  represents the degree of the polynomial (if  $X^3$  were added to the preceding function, it would be a third-degree polynomial, and so on).

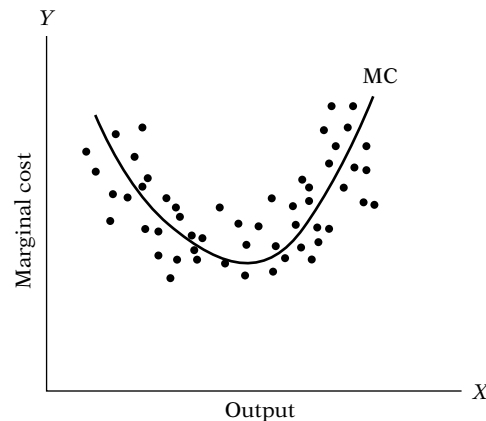


FIGURE 7.1 The U-shaped marginal cost curve.

The stochastic version of (7.10.1) may be written as

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i \quad (7.10.2)$$

which is called a *second-degree polynomial regression*.

The general *kth degree polynomial regression* may be written as

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \dots + \beta_k X_i^k + u_i \quad (7.10.3)$$

Notice that in these types of polynomial regressions there is only one explanatory variable on the right-hand side but it appears with various powers, thus making them multiple regression models. Incidentally, note that if  $X_i$  is assumed to be fixed or nonstochastic, the powered terms of  $X_i$  also become fixed or nonstochastic.

Do these models present any special estimation problems? Since the second-degree polynomial (7.10.2) or the *kth* degree polynomial (7.10.13) is linear in the parameters, the  $\beta$ 's, they can be estimated by the usual OLS or ML methodology. But what about the collinearity problem? Aren't the various  $X$ 's highly correlated since they are all powers of  $X$ ? Yes, but remember that terms like  $X^2$ ,  $X^3$ ,  $X^4$ , etc., are all nonlinear functions of  $X$  and hence, strictly speaking, do not violate the no multicollinearity assumption. In short, polynomial regression models can be estimated by the techniques presented in this chapter and present no new estimation problems.

**EXAMPLE 7.4**

**ESTIMATING THE TOTAL COST FUNCTION**

As an example of the polynomial regression, consider the data on output and total cost of production of a commodity in the short run given in Table 7.4. What type of regression model will fit these data? For this purpose, let us first draw the scattergram, which is shown in Figure 7.2.

From this figure it is clear that the relationship between total cost and output resembles the elongated S curve; notice how the total cost curve first increases gradually and then rapidly, as predicted by the celebrated law of *diminishing returns*. This S shape of the total cost curve can be captured by the following cubic or *third-degree polynomial*:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + u_i \quad (7.10.4)$$

where  $Y$  = total cost and  $X$  = output.

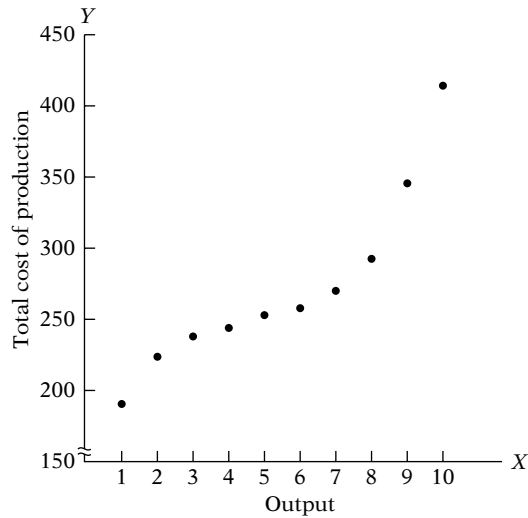
**TABLE 7.4** TOTAL COST ( $Y$ ) AND OUTPUT ( $X$ )

	Output	Total cost, \$
	1	193
	2	226
	3	240
	4	244
	5	257
	6	260
	7	274
	8	297
	9	350
	10	420

Given the data of Table 7.4, we can apply the OLS method to estimate the parameters of (7.10.4). But before we do that, let us find out what economic theory has to say about the short-run cubic cost function (7.10.4). Elementary price theory shows that in the short run the

*(Continued)*

**EXAMPLE 7.4** (Continued)

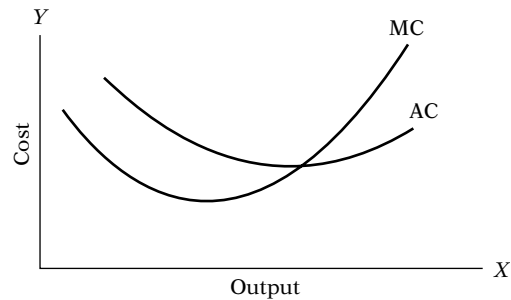
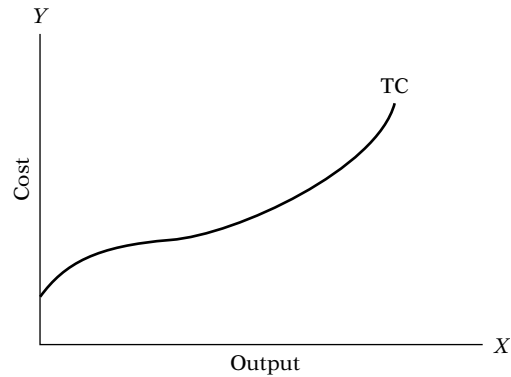


**FIGURE 7.2** The total cost curve.

marginal cost (MC) and average cost (AC) curves of production are typically U-shaped—initially, as output increases both MC and AC decline, but after a certain level of output they both turn upward, again the consequence of the law of diminishing return. This can be seen in Figure 7.3 (see also Figure 7.1). And since the MC and AC curves are derived from the total cost curve, the U-shaped nature of these curves puts some restrictions on the parameters of the total cost curve (7.10.4). As a matter of fact, it can be shown that the parameters of (7.10.4) must satisfy the following restrictions if one is to observe the typical U-shaped short-run marginal and average cost curves:<sup>19</sup>

1.  $\beta_0, \beta_1,$  and  $\beta_3 > 0$
  2.  $\beta_2 < 0$
  3.  $\beta_2^2 < 3\beta_1\beta_3$
- (7.10.5)**

All this theoretical discussion might seem a bit tedious. But this knowledge is extremely useful when we examine the empirical results, for if the empirical results do not agree with prior expectations, then, assuming we have not committed a specification error (i.e., chosen the



**FIGURE 7.3** Short-run cost functions.

wrong model), we will have to modify our theory or look for a new theory and start the empirical enquiry all over again. But as noted in the Introduction, this is the nature of any empirical investigation.

**Empirical Results**

When the third-degree polynomial regression was fitted to the data of Table 7.4, we obtained the following results:

$$\hat{Y}_i = 141.7667 + 63.4776X_i - 12.9615X_i^2 + 0.9396X_i^3$$

(6.3753) (4.7786) (0.9857) (0.0591)

$R^2 = 0.9983$

**(7.10.6)**  
(Continued)

<sup>19</sup>See Alpha C. Chiang, *Fundamental Methods of Mathematical Economics*, 3d ed., McGraw-Hill, New York, 1984, pp. 250–252.

**EXAMPLE 7.4** (Continued)

(Note: The figures in parentheses are the estimated standard errors.) Although we will examine the statistical significance of these results in the next chapter, the reader can verify that they are in conformity with the

theoretical expectations listed in (7.10.5). We leave it as an exercise for the reader to interpret the regression (7.10.6).

**EXAMPLE 7.5**GDP GROWTH RATE, 1960–1985 AND RELATIVE PER CAPITA GDP,  
IN 119 DEVELOPING COUNTRIES

As an additional economic example of the polynomial regression model, consider the following regression results<sup>20</sup>:

$$\widehat{\text{GDPG}}_i = 0.013 + 0.062 \text{RGDP} - 0.061 \text{RGDP}^2$$

$$\text{se} = (0.004) \quad (0.027) \quad (0.033) \quad (7.10.7)$$

$$R^2 = 0.053 \quad \text{adj } R^2 = 0.036$$

where GDPG = GDP growth rate, percent (average for 1960–1985), and RGDP = relative per capita GDP, 1960 (percentage of U.S. GDP per capita, 1960). The adjusted  $R^2$  (adj  $R^2$ ) tells us that, after taking into account the number of regressors, the model explains only about 3.6 percent of the variation in GDPG. Even the unadjusted  $R^2$  of 0.053 seems low. This might sound a disappointing value but, as we shall show in the next chapter, such low  $R^2$ 's are frequently encountered in cross-sectional data with a large number of observations. Besides, even an apparently low  $R^2$  value can be statistically significant (i.e., different from zero), as we will show in the next chapter.

As this regression shows, GDPG in developing countries increased as RGDP increased, but at a decreasing rate; that is, developing economies were not catching up with advanced economies.<sup>21</sup> This example shows how relatively simple econometric models can be used to shed light on important economic phenomena.

**\*7.11 PARTIAL CORRELATION COEFFICIENTS****Explanation of Simple and Partial Correlation Coefficients**

In Chapter 3 we introduced the coefficient of correlation  $r$  as a measure of the degree of linear association between two variables. For the three-variable

<sup>20</sup>Source: *The East Asian Economic Miracle: Economic Growth and Public Policy*, A World Bank Policy Research Report, Oxford University Press, U.K, 1993, p. 29.

<sup>21</sup>If you take the derivative of (7.10.7), you will obtain

$$\frac{d\text{GDPG}}{d\text{RGDP}} = 0.062 - 0.122 \text{RGDP}$$

showing that the rate of change of GDPG with respect to RGDP is declining. If you set this derivative to zero, you will get  $\text{RGDP} \approx 0.5082$ . Thus, if a country's GDP reaches about 51 percent of the U.S. GDP, the rate of growth of GDPG will crawl to zero.

\*Optional.

regression model we can compute three correlation coefficients:  $r_{12}$  (correlation between  $Y$  and  $X_2$ ),  $r_{13}$  (correlation coefficient between  $Y$  and  $X_3$ ), and  $r_{23}$  (correlation coefficient between  $X_2$  and  $X_3$ ); notice that we are letting the subscript 1 represent  $Y$  for notational convenience. These correlation coefficients are called **gross** or **simple correlation coefficients**, or **correlation coefficients of zero order**. These coefficients can be computed by the definition of correlation coefficient given in (3.5.13).

But now consider this question: Does, say,  $r_{12}$  in fact measure the “true” degree of (linear) association between  $Y$  and  $X_2$  when a third variable  $X_3$  may be associated with both of them? This question is analogous to the following question: Suppose the true regression model is (7.1.1) but we omit from the model the variable  $X_3$  and simply regress  $Y$  on  $X_2$ , obtaining the slope coefficient of, say,  $b_{12}$ . Will this coefficient be equal to the true coefficient  $\beta_2$  if the model (7.1.1) were estimated to begin with? The answer should be apparent from our discussion in Section 7.7. In general,  $r_{12}$  is not likely to reflect the true degree of association between  $Y$  and  $X_2$  in the presence of  $X_3$ . As a matter of fact, it is likely to give a false impression of the nature of association between  $Y$  and  $X_2$ , as will be shown shortly. Therefore, what we need is a correlation coefficient that is independent of the influence, if any, of  $X_3$  on  $X_2$  and  $Y$ . Such a correlation coefficient can be obtained and is known appropriately as the **partial correlation coefficient**. Conceptually, it is similar to the partial regression coefficient. We define

$r_{12.3}$  = partial correlation coefficient between  $Y$  and  $X_2$ , holding  $X_3$  constant

$r_{13.2}$  = partial correlation coefficient between  $Y$  and  $X_3$ , holding  $X_2$  constant

$r_{23.1}$  = partial correlation coefficient between  $X_2$  and  $X_3$ , holding  $Y$  constant

These partial correlations can be easily obtained from the simple or zero-order, correlation coefficients as follows (for proofs, see the exercises)<sup>22</sup>:

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}} \quad (7.11.1)$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{23}}{\sqrt{(1 - r_{12}^2)(1 - r_{23}^2)}} \quad (7.11.2)$$

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1 - r_{12}^2)(1 - r_{13}^2)}} \quad (7.11.3)$$

The partial correlations given in Eqs. (7.11.1) to (7.11.3) are called **first-order correlation coefficients**. By *order* we mean the number of secondary

<sup>22</sup>Most computer programs for multiple regression analysis routinely compute the simple correlation coefficients; hence the partial correlation coefficients can be readily computed.



subscripts. Thus  $r_{12.34}$  would be the correlation coefficient of order two,  $r_{12.345}$  would be the correlation coefficient of order three, and so on. As noted previously,  $r_{12}$ ,  $r_{13}$ , and so on are called *simple* or *zero-order correlations*. The interpretation of, say,  $r_{12.34}$  is that it gives the coefficient of correlation between  $Y$  and  $X_2$ , holding  $X_3$  and  $X_4$  constant.

### Interpretation of Simple and Partial Correlation Coefficients

In the two-variable case, the simple  $r$  had a straightforward meaning: It measured the degree of (linear) association (and not causation) between the dependent variable  $Y$  and the single explanatory variable  $X$ . But once we go beyond the two-variable case, we need to pay careful attention to the interpretation of the simple correlation coefficient. From (7.11.1), for example, we observe the following:

1. Even if  $r_{12} = 0$ ,  $r_{12.3}$  will not be zero unless  $r_{13}$  or  $r_{23}$  or both are zero.
2. If  $r_{12} = 0$  and  $r_{13}$  and  $r_{23}$  are nonzero and are of the same sign,  $r_{12.3}$  will be negative, whereas if they are of the opposite signs, it will be positive. An example will make this point clear. Let  $Y =$  crop yield,  $X_2 =$  rainfall, and  $X_3 =$  temperature. Assume  $r_{12} = 0$ , that is, no association between crop yield and rainfall. Assume further that  $r_{13}$  is positive and  $r_{23}$  is negative. Then, as (7.11.1) shows,  $r_{12.3}$  will be positive; that is, holding temperature constant, there is a positive association between yield and rainfall. This seemingly paradoxical result, however, is not surprising. Since temperature  $X_3$  affects both yield  $Y$  and rainfall  $X_2$ , in order to find out the net relationship between crop yield and rainfall, we need to remove the influence of the “nuisance” variable temperature. This example shows how one might be misled by the simple coefficient of correlation.
3. The terms  $r_{12.3}$  and  $r_{12}$  (and similar comparisons) need not have the same sign.
4. In the two-variable case we have seen that  $r^2$  lies between 0 and 1. The same property holds true of the squared partial correlation coefficients. Using this fact, the reader should verify that one can obtain the following expression from (7.11.1):

$$0 \leq r_{12}^2 + r_{13}^2 + r_{23}^2 - 2r_{12}r_{13}r_{23} \leq 1 \quad (7.11.4)$$

which gives the interrelationships among the three zero-order correlation coefficients. Similar expressions can be derived from Eqs. (7.9.3) and (7.9.4).

5. Suppose that  $r_{13} = r_{23} = 0$ . Does this mean that  $r_{12}$  is also zero? The answer is obvious from (7.11.4). The fact that  $Y$  and  $X_3$  and  $X_2$  and  $X_3$  are uncorrelated does not mean that  $Y$  and  $X_2$  are uncorrelated.

In passing, note that the expression  $r_{12.3}^2$  may be called the **coefficient of partial determination** and may be interpreted as the proportion of the variation in  $Y$  not explained by the variable  $X_3$  that has been explained

by the inclusion of  $X_2$  into the model (see exercise 7.5). Conceptually it is similar to  $R^2$ .

Before moving on, note the following relationships between  $R^2$ , simple correlation coefficients, and partial correlation coefficients:

$$R^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2} \quad (7.11.5)$$

$$R^2 = r_{12}^2 + (1 - r_{12}^2)r_{13.2}^2 \quad (7.11.6)$$

$$R^2 = r_{13}^2 + (1 - r_{13}^2)r_{12.3}^2 \quad (7.11.7)$$

In concluding this section, consider the following: It was stated previously that  $R^2$  will not decrease if an additional explanatory variable is introduced into the model, which can be seen clearly from (7.11.6). This equation states that the proportion of the variation in  $Y$  explained by  $X_2$  and  $X_3$  jointly is the sum of two parts: the part explained by  $X_2$  alone ( $= r_{12}^2$ ) and the part not explained by  $X_2$  ( $= 1 - r_{12}^2$ ) times the proportion that is explained by  $X_3$  after holding the influence of  $X_2$  constant. Now  $R^2 > r_{12}^2$  so long as  $r_{13.2}^2 > 0$ . At worst,  $r_{13.2}^2$  will be zero, in which case  $R^2 = r_{12}^2$ .

## 7.12 SUMMARY AND CONCLUSIONS

1. This chapter introduced the simplest possible multiple linear regression model, namely, the three-variable regression model. It is understood that the term *linear* refers to linearity in the parameters and not necessarily in the variables.

2. Although a three-variable regression model is in many ways an extension of the two-variable model, there are some new concepts involved, such as *partial regression coefficients*, *partial correlation coefficients*, *multiple correlation coefficient*, *adjusted and unadjusted (for degrees of freedom)  $R^2$* , *multicollinearity*, and *specification bias*.

3. This chapter also considered the functional form of the multiple regression model, such as the *Cobb–Douglas production function* and the *polynomial regression model*.

4. Although  $R^2$  and adjusted  $R^2$  are overall measures of how the chosen model fits a given set of data, their importance should not be overplayed. What is critical is the underlying theoretical expectations about the model in terms of a priori signs of the coefficients of the variables entering the model and, as it is shown in the following chapter, their statistical significance.

5. The results presented in this chapter can be easily generalized to a multiple linear regression model involving any number of regressors. But the algebra becomes very tedious. This tedium can be avoided by resorting to matrix algebra. For the interested reader, the extension to the  $k$ -variable

regression model using matrix algebra is presented in **Appendix C**, which is optional. But the general reader can read the remainder of the text without knowing much of matrix algebra.

## EXERCISES

### Questions

7.1. Consider the data in Table 7.5.

TABLE 7.5

$Y$	$X_2$	$X_3$
1	1	2
3	2	1
8	3	-3

Based on these data, estimate the following regressions:

$$Y_i = \alpha_1 + \alpha_2 X_{2i} + u_{1i} \quad (1)$$

$$Y_i = \lambda_1 + \lambda_3 X_{3i} + u_{2i} \quad (2)$$

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i \quad (3)$$

Note: Estimate only the coefficients and not the standard errors.

a. Is  $\alpha_2 = \beta_2$ ? Why or why not?

b. Is  $\lambda_3 = \beta_3$ ? Why or why not?

What important conclusion do you draw from this exercise?

7.2. From the following data estimate the partial regression coefficients, their standard errors, and the adjusted and unadjusted  $R^2$  values:

$$\bar{Y} = 367.693 \quad \bar{X}_2 = 402.760 \quad \bar{X}_3 = 8.0$$

$$\sum (Y_i - \bar{Y})^2 = 66042.269 \quad \sum (X_{2i} - \bar{X}_2)^2 = 84855.096$$

$$\sum (X_{3i} - \bar{X}_3)^2 = 280.000 \quad \sum (Y_i - \bar{Y})(X_{2i} - \bar{X}_2) = 74778.346$$

$$\sum (Y_i - \bar{Y})(X_{3i} - \bar{X}_3) = 4250.900 \quad \sum (X_{2i} - \bar{X}_2)(X_{3i} - \bar{X}_3) = 4796.000$$

$$n = 15$$

7.3. Show that Eq. (7.4.7) can also be expressed as

$$\begin{aligned} \hat{\beta}_2 &= \frac{\sum y_i (x_{2i} - b_{23} x_{3i})}{\sum (x_{2i} - b_{23} x_{3i})^2} \\ &= \frac{\text{net (of } x_3) \text{ covariation between } y \text{ and } x_2}{\text{net (of } x_3) \text{ variation in } x_2} \end{aligned}$$

where  $b_{23}$  is the slope coefficient in the regression of  $X_2$  on  $X_3$ . (Hint: Recall that  $b_{23} = \sum x_{2i} x_{3i} / \sum x_{3i}^2$ .)

- 7.4. In a multiple regression model you are told that the error term  $u_i$  has the following probability distribution, namely,  $u_i \sim N(0, 4)$ . How would you set up a *Monte Carlo* experiment to verify that the true variance is in fact 4?
- 7.5. Show that  $r_{12.3}^2 = (R^2 - r_{13}^2)/(1 - r_{13}^2)$  and interpret the equation.
- 7.6. If the relation  $\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = 0$  holds true for all values of  $X_1$ ,  $X_2$ , and  $X_3$ , find the values of the three partial correlation coefficients.
- 7.7. Is it possible to obtain the following from a set of data?
- $r_{23} = 0.9, r_{13} = -0.2, r_{12} = 0.8$
  - $r_{12} = 0.6, r_{23} = -0.9, r_{31} = -0.5$
  - $r_{21} = 0.01, r_{13} = 0.66, r_{23} = -0.7$
- 7.8. Consider the following model:

$$Y_i = \beta_1 + \beta_2 \text{ Education}_i + \beta_3 \text{ Years of experience} + u_i$$

Suppose you leave out the years of experience variable. What kinds of problems or biases would you expect? Explain verbally.

- 7.9. Show that  $\beta_2$  and  $\beta_3$  in (7.9.2) do, in fact, give output elasticities of labor and capital. (This question can be answered without using calculus; just recall the definition of the elasticity coefficient and remember that a change in the logarithm of a variable is a relative change, assuming the changes are rather small.)
- 7.10. Consider the three-variable linear regression model discussed in this chapter.
- Suppose you multiply all the  $X_2$  values by 2. What will be the effect of this rescaling, if any, on the estimates of the parameters and their standard errors?
  - Now instead of **a**, suppose you multiply all the  $Y$  values by 2. What will be the effect of this, if any, on the estimated parameters and their standard errors?
- 7.11. In general  $R^2 \neq r_{12}^2 + r_{13}^2$ , but it is so only if  $r_{23} = 0$ . Comment and point out the significance of this finding. [*Hint*: See Eq. (7.11.5).]
- 7.12. Consider the following models.\*

$$\text{Model A: } Y_t = \alpha_1 + \alpha_2 X_{2t} + \alpha_3 X_{3t} + u_{1t}$$

$$\text{Model B: } (Y_t - X_{2t}) = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_{2t}$$

- Will OLS estimates of  $\alpha_1$  and  $\beta_1$  be the same? Why?
  - Will OLS estimates of  $\alpha_3$  and  $\beta_3$  be the same? Why?
  - What is the relationship between  $\alpha_2$  and  $\beta_2$ ?
  - Can you compare the  $R^2$  terms of the two models? Why or why not?
- 7.13. Suppose you estimate the consumption function<sup>†</sup>

$$Y_i = \alpha_1 + \alpha_2 X_i + u_{1i}$$

and the savings function

$$Z_i = \beta_1 + \beta_2 X_i + u_{2i}$$

\*Adapted from Wojciech W. Charemza and Derek F. Deadman, *Econometric Practice: General to Specific Modelling, Cointegration and Vector Autoregression*, Edward Elgar, Brookfield, Vermont, 1992, p. 18.

<sup>†</sup>Adapted from Peter Kennedy, *A Guide to Econometrics*, 3d ed., The MIT Press, Cambridge, Massachusetts, 1992, p. 308, Question #9.

where  $Y$  = consumption,  $Z$  = savings,  $X$  = income, and  $X = Y + Z$ , that is, income is equal to consumption plus savings.

- What is the relationship, if any, between  $\alpha_2$  and  $\beta_2$ ? Show your calculations.
  - Will the residual sum of squares, RSS, be the same for the two models? Explain.
  - Can you compare the  $R^2$  terms of the two models? Why or why not?
- 7.14.** Suppose you express the Cobb–Douglas model given in (7.9.1) as follows:

$$Y_i = \beta_1 X_{2i}^{\beta_2} X_{3i}^{\beta_3} u_i$$

If you take the log-transform of this model, you will have  $\ln u_i$  as the disturbance term on the right-hand side.

- What probabilistic assumptions do you have to make about  $\ln u_i$  to be able to apply the classical normal linear regression model (CNLRM)? How would you test this with the data given in Table 7.3?
  - Do the same assumptions apply to  $u_i$ ? Why or why not?
- 7.15.** *Regression through the origin.* Consider the following regression through the origin:

$$Y_i = \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \hat{u}_i$$

- How would you go about estimating the unknowns?
- Will  $\sum \hat{u}_i$  be zero for this model? Why or why not?
- Will  $\sum \hat{u}_i X_{2i} = \sum \hat{u}_i X_{3i} = 0$  for this model?
- When would you use such a model?
- Can you generalize your results to the  $k$ -variable model?  
(*Hint:* Follow the discussion for the two-variable case given in Chapter 6.)

## Problems

- 7.16.** *The demand for roses.\** Table 7.6 gives quarterly data on these variables:

$Y$  = quantity of roses sold, dozens  
 $X_2$  = average wholesale price of roses, \$/dozen  
 $X_3$  = average wholesale price of carnations, \$/dozen  
 $X_4$  = average weekly family disposable income, \$/week  
 $X_5$  = the trend variable taking values of 1, 2, and so on, for the period 1971–III to 1975–II in the Detroit metropolitan area  
 You are asked to consider the following demand functions:

$$Y_t = \alpha_1 + \alpha_2 X_{2t} + \alpha_3 X_{3t} + \alpha_4 X_{4t} + \alpha_5 X_{5t} + u_t$$

$$\ln Y_t = \beta_1 + \beta_2 \ln X_{2t} + \beta_3 \ln X_{3t} + \beta_4 \ln X_{4t} + \beta_5 X_{5t} + u_t$$

- Estimate the parameters of the linear model and interpret the results.
- Estimate the parameters of the log-linear model and interpret the results.
- $\beta_2$ ,  $\beta_3$ , and  $\beta_4$  give, respectively, the *own-price*, *cross-price*, and *income elasticities* of demand. What are their a priori signs? Do the results concur with the a priori expectations?

\*I am indebted to Joe Walsh for collecting these data from a major wholesaler in the Detroit metropolitan area and subsequently processing them.

**TABLE 7.6**

Year and quarter	Y	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	X <sub>5</sub>
1971-III	11,484	2.26	3.49	158.11	1
-IV	9,348	2.54	2.85	173.36	2
1972-I	8,429	3.07	4.06	165.26	3
-II	10,079	2.91	3.64	172.92	4
-III	9,240	2.73	3.21	178.46	5
-IV	8,862	2.77	3.66	198.62	6
1973-I	6,216	3.59	3.76	186.28	7
-II	8,253	3.23	3.49	188.98	8
-III	8,038	2.60	3.13	180.49	9
-IV	7,476	2.89	3.20	183.33	10
1974-I	5,911	3.77	3.65	181.87	11
-II	7,950	3.64	3.60	185.00	12
-III	6,134	2.82	2.94	184.00	13
-IV	5,868	2.96	3.12	188.20	14
1975-I	3,160	4.24	3.58	175.67	15
-II	5,872	3.69	3.53	188.00	16

**d.** How would you compute the own-price, cross-price, and income elasticities for the linear model?

**e.** On the basis of your analysis, which model, if either, would you choose and why?

**7.17. Wildcat activity.** Wildcats are wells drilled to find and produce oil and/or gas in an improved area or to find a new reservoir in a field previously found to be productive of oil or gas or to extend the limit of a known oil or gas reservoir. Table 7.7 gives data on these variables\*:

Y = the number of wildcats drilled

X<sub>2</sub> = price at the wellhead in the previous period  
(in constant dollars, 1972 = 100)

X<sub>3</sub> = domestic output

X<sub>4</sub> = GNP constant dollars (1972 = 100)

X<sub>5</sub> = trend variable, 1948 = 1, 1949 = 2, . . . , 1978 = 31

See if the following model fits the data:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 \ln X_{3t} + \beta_4 X_{4t} + \beta_5 X_{5t} + u_t$$

**a.** Can you offer an a priori rationale to this model?

**b.** Assuming the model is acceptable, estimate the parameters of the model and their standard errors, and obtain  $R^2$  and  $\bar{R}^2$ .

**c.** Comment on your results in view of your prior expectations.

**d.** What other specification would you suggest to explain wildcat activity? Why?

\*I am indebted to Raymond Savino for collecting and processing the data.

TABLE 7.7

Thousands of wildcats, (Y)	Per barrel price, constant \$, (X <sub>2</sub> )	Domestic output (millions of barrels per day), (X <sub>3</sub> )	GNP, constant \$ billions, (X <sub>4</sub> )	Time, (X <sub>5</sub> )
8.01	4.89	5.52	487.67	1948 = 1
9.06	4.83	5.05	490.59	1949 = 2
10.31	4.68	5.41	533.55	1950 = 3
11.76	4.42	6.16	576.57	1951 = 4
12.43	4.36	6.26	598.62	1952 = 5
13.31	4.55	6.34	621.77	1953 = 6
13.10	4.66	6.81	613.67	1954 = 7
14.94	4.54	7.15	654.80	1955 = 8
16.17	4.44	7.17	668.84	1956 = 9
14.71	4.75	6.71	681.02	1957 = 10
13.20	4.56	7.05	679.53	1958 = 11
13.19	4.29	7.04	720.53	1959 = 12
11.70	4.19	7.18	736.86	1960 = 13
10.99	4.17	7.33	755.34	1961 = 14
10.80	4.11	7.54	799.15	1962 = 15
10.66	4.04	7.61	830.70	1963 = 16
10.75	3.96	7.80	874.29	1964 = 17
9.47	3.85	8.30	925.86	1965 = 18
10.31	3.75	8.81	980.98	1966 = 19
8.88	3.69	8.66	1,007.72	1967 = 20
8.88	3.56	8.78	1,051.83	1968 = 21
9.70	3.56	9.18	1,078.76	1969 = 22
7.69	3.48	9.03	1,075.31	1970 = 23
6.92	3.53	9.00	1,107.48	1971 = 24
7.54	3.39	8.78	1,171.10	1972 = 25
7.47	3.68	8.38	1,234.97	1973 = 26
8.63	5.92	8.01	1,217.81	1974 = 27
9.21	6.03	7.78	1,202.36	1975 = 28
9.23	6.12	7.88	1,271.01	1976 = 29
9.96	6.05	7.88	1,332.67	1977 = 30
10.78	5.89	8.67	1,385.10	1978 = 31

Source: Energy Information Administration, 1978 Report to Congress.

7.18. U.S. defense budget outlays, 1962–1981. In order to explain the U.S. defense budget, you are asked to consider the following model:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + \beta_4 X_{4t} + \beta_5 X_{5t} + u_t$$

where  $Y_t$  = defense budget-outlay for year  $t$ , \$ billions

$X_{2t}$  = GNP for year  $t$ , \$ billions

$X_{3t}$  = U.S. military sales/assistance in year  $t$ , \$ billions

$X_{4t}$  = aerospace industry sales, \$ billions

$X_{5t}$  = military conflicts involving more than 100,000 troops. This variable takes a value of 1 when 100,000 or more troops are

involved but is equal to zero when that number is under 100,000.

To test this model, you are given the data in Table 7.8.

- a. Estimate the parameters of this model and their standard errors and obtain  $R^2$ , modified  $R^2$ , and  $\bar{R}^2$ .
  - b. Comment on the results, taking into account any prior expectations you have about the relationship between  $Y$  and the various  $X$  variables.
  - c. What other variable(s) might you want to include in the model and why?
- 7.19.** *The demand for chicken in the United States, 1960–1982.* To study the per capita consumption of chicken in the United States, you are given the data in Table 7.9,

where  $Y$  = per capita consumption of chickens, lb

$X_2$  = real disposable income per capita, \$

$X_3$  = real retail price of chicken per lb, ¢

$X_4$  = real retail price of pork per lb, ¢

$X_5$  = real retail price of beef per lb, ¢

$X_6$  = composite real price of chicken substitutes per lb, ¢, which is a weighted average of the real retail prices per lb of pork and beef, the weights being the relative consumptions of beef and pork in total beef and pork consumption

**TABLE 7.8**

Year	Defense budget outlays, $Y$	GNP, $X_2$	U.S. military sales/assistance, $X_3$	Aerospace industry sales, $X_4$	Conflicts 100,000+, $X_5$
1962	51.1	560.3	0.6	16.0	0
1963	52.3	590.5	0.9	16.4	0
1964	53.6	632.4	1.1	16.7	0
1965	49.6	684.9	1.4	17.0	1
1966	56.8	749.9	1.6	20.2	1
1967	70.1	793.9	1.0	23.4	1
1968	80.5	865.0	0.8	25.6	1
1969	81.2	931.4	1.5	24.6	1
1970	80.3	992.7	1.0	24.8	1
1971	77.7	1,077.6	1.5	21.7	1
1972	78.3	1,185.9	2.95	21.5	1
1973	74.5	1,326.4	4.8	24.3	0
1974	77.8	1,434.2	10.3	26.8	0
1975	85.6	1,549.2	16.0	29.5	0
1976	89.4	1,718.0	14.7	30.4	0
1977	97.5	1,918.3	8.3	33.3	0
1978	105.2	2,163.9	11.0	38.0	0
1979	117.7	2,417.8	13.0	46.2	0
1980	135.9	2,633.1	15.3	57.6	0
1981	162.1	2,937.7	18.0	68.9	0

Source: The data were collected by Albert Lucchino from various government publications.



TABLE 7.9

Year	Y	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	X <sub>5</sub>	X <sub>6</sub>
1960	27.8	397.5	42.2	50.7	78.3	65.8
1961	29.9	413.3	38.1	52.0	79.2	66.9
1962	29.8	439.2	40.3	54.0	79.2	67.8
1963	30.8	459.7	39.5	55.3	79.2	69.6
1964	31.2	492.9	37.3	54.7	77.4	68.7
1965	33.3	528.6	38.1	63.7	80.2	73.6
1966	35.6	560.3	39.3	69.8	80.4	76.3
1967	36.4	624.6	37.8	65.9	83.9	77.2
1968	36.7	666.4	38.4	64.5	85.5	78.1
1969	38.4	717.8	40.1	70.0	93.7	84.7
1970	40.4	768.2	38.6	73.2	106.1	93.3
1971	40.3	843.3	39.8	67.8	104.8	89.7
1972	41.8	911.6	39.7	79.1	114.0	100.7
1973	40.4	931.1	52.1	95.4	124.1	113.5
1974	40.7	1,021.5	48.9	94.2	127.6	115.3
1975	40.1	1,165.9	58.3	123.5	142.9	136.7
1976	42.7	1,349.6	57.9	129.9	143.6	139.2
1977	44.1	1,449.4	56.5	117.6	139.2	132.0
1978	46.7	1,575.5	63.7	130.9	165.5	132.1
1979	50.6	1,759.1	61.6	129.8	203.3	154.4
1980	50.1	1,994.2	58.9	128.0	219.6	174.9
1981	51.7	2,258.1	66.4	141.0	221.6	180.8
1982	52.9	2,478.7	70.4	168.2	232.6	189.4

Source: Data on Y are from *Citibase* and on X<sub>2</sub> through X<sub>6</sub> are from the U.S. Department of Agriculture. I am indebted to Robert J. Fisher for collecting the data and for the statistical analysis.

Note: The real prices were obtained by dividing the nominal prices by the Consumer Price Index for food.

Now consider the following demand functions:

$$\ln Y_t = \alpha_1 + \alpha_2 \ln X_{2t} + \alpha_3 \ln X_{3t} + u_t \quad (1)$$

$$\ln Y_t = \gamma_1 + \gamma_2 \ln X_{2t} + \gamma_3 \ln X_{3t} + \gamma_4 \ln X_{4t} + u_t \quad (2)$$

$$\ln Y_t = \lambda_1 + \lambda_2 \ln X_{2t} + \lambda_3 \ln X_{3t} + \lambda_4 \ln X_{5t} + u_t \quad (3)$$

$$\ln Y_t = \theta_1 + \theta_2 \ln X_{2t} + \theta_3 \ln X_{3t} + \theta_4 \ln X_{4t} + \theta_5 \ln X_{5t} + u_t \quad (4)$$

$$\ln Y_t = \beta_1 + \beta_2 \ln X_{2t} + \beta_3 \ln X_{3t} + \beta_4 \ln X_{6t} + u_t \quad (5)$$

From microeconomic theory it is known that the demand for a commodity generally depends on the real income of the consumer, the real price of the commodity, and the real prices of competing or complementary commodities. In view of these considerations, answer the following questions.

- Which demand function among the ones given here would you choose, and why?
- How would you interpret the coefficients of  $\ln X_{2t}$  and  $\ln X_{3t}$  in these models?
- What is the difference between specifications (2) and (4)?
- What problems do you foresee if you adopt specification (4)? (*Hint:* Prices of both pork and beef are included along with the price of chicken.)

- e. Since specification (5) includes the composite price of beef and pork, would you prefer the demand function (5) to the function (4)? Why?
- f. Are pork and/or beef competing or substitute products to chicken? How do you know?
- g. Assume function (5) is the “correct” demand function. Estimate the parameters of this model, obtain their standard errors, and  $R^2$ ,  $\bar{R}^2$ , and modified  $R^2$ . Interpret your results.
- h. Now suppose you run the “incorrect” model (2). Assess the consequences of this mis-specification by considering the values of  $\gamma_2$  and  $\gamma_3$  in relation to  $\beta_2$  and  $\beta_3$ , respectively. (*Hint*: Pay attention to the discussion in Section 7.7.)
- 7.20.** In a study of turnover in the labor market, James F. Ragan, Jr., obtained the following results for the U.S. economy for the period of 1950–I to 1979–IV.\* (Figures in the parentheses are the estimated  $t$  statistics.)

$$\begin{aligned} \widehat{\ln Y}_t = & 4.47 - 0.34 \ln X_{2t} + 1.22 \ln X_{3t} + 1.22 \ln X_{4t} \\ & (4.28) \quad (-5.31) \quad (3.64) \quad (3.10) \\ & + 0.80 \ln X_{5t} - 0.0055 X_{6t} \quad \bar{R}^2 = 0.5370 \\ & (1.10) \quad (-3.09) \end{aligned}$$

*Note:* We will discuss the  $t$  statistics in the next chapter.

where  $Y$  = quit rate in manufacturing, defined as number of people leaving jobs voluntarily per 100 employees

$X_2$  = an instrumental or proxy variable for adult male unemployment rate

$X_3$  = percentage of employees younger than 25

$X_4 = N_{t-1}/N_{t-4}$  = ratio of manufacturing employment in quarter  $(t - 1)$  to that in quarter  $(t - 4)$

$X_5$  = percentage of women employees

$X_6$  = time trend (1950–I = 1)

- a. Interpret the foregoing results.
- b. Is the observed negative relationship between the logs of  $Y$  and  $X_2$  justifiable a priori?
- c. Why is the coefficient of  $\ln X_3$  positive?
- d. Since the trend coefficient is negative, there is a secular decline of what percent in the quit rate and why is there such a decline?
- e. Is the  $\bar{R}^2$  “too” low?
- f. Can you estimate the standard errors of the regression coefficients from the given data? Why or why not?
- 7.21.** Consider the following demand function for money in the United States for the period 1980–1998:

$$M_t = \beta_1 Y_t^{\beta_2} r_t^{\beta_3} e^{u_t}$$

\*Source: See Ragan’s article, “Turnover in the Labor Market: A Study of Quit and Layoff Rates,” *Economic Review*, Federal Reserve Bank of Kansas City, May 1981, pp. 13–22.

where  $M$  = real money demand, using the M2 definition of money  
 $Y$  = real GDP  
 $r$  = interest rate

To estimate the above demand for money function, you are given the data in Table 7.10.

*Note:* To convert nominal quantities into real quantities, divide  $M$  and GDP by CPI. There is no need to divide the interest rate variable by CPI. Also, note that we have given two interest rates, a short-term rate as measured by the 3-month treasury bill rate and the long-term rate as measured by the yield on 30-year treasury bond, as prior empirical studies have used both types of interest rates.

- Given the data, estimate the above demand function. What are the income and interest rate elasticities of demand for money?
- Instead of estimating the above demand function, suppose you were to fit the function  $(M/Y)_t = \alpha_1 r_t^{\alpha_2} e^{u_t}$ . How would you interpret the results? Show the necessary calculations.
- How will you decide which is a better specification? (*Note:* A formal statistical test will be given in Chapter 8.)

**7.22.** Table 7.11 gives data for the manufacturing sector of the Greek economy for the period 1961–1987.

**TABLE 7.10** DEMAND FOR MONEY IN THE UNITED STATES, 1980–1998

Observation	GDP	M2	CPI	LTRATE	TBRATE
1980	2795.6	1600.4	82.4	11.27	11.506
1981	3131.3	1756.1	90.9	13.45	14.029
1982	3259.2	1911.2	96.5	12.76	10.686
1983	3534.9	2127.8	99.6	11.18	8.630
1984	3932.7	2311.7	103.9	12.41	9.580
1985	4213.0	2497.4	107.6	10.79	7.480
1986	4452.9	2734.0	109.6	7.78	5.980
1987	4742.5	2832.8	113.6	8.59	5.820
1988	5108.3	2995.8	118.3	8.96	6.690
1989	5489.1	3159.9	124.0	8.45	8.120
1990	5803.2	3279.1	130.7	8.61	7.510
1991	5986.2	3379.8	136.2	8.14	5.420
1992	6318.9	3434.1	140.3	7.67	3.450
1993	6642.3	3487.5	144.5	6.59	3.020
1994	7054.3	3502.2	148.2	7.37	4.290
1995	7400.5	3649.3	152.4	6.88	5.510
1996	7813.2	3824.2	156.9	6.71	5.020
1997	8300.8	4046.7	160.5	6.61	5.070
1998	8759.9	4401.4	163.0	5.58	4.810

*Notes:* GDP: gross domestic product (\$ billions)

M2: M2 money supply.

CPI: Consumer Price Index (1982–1984 = 100).

LTRATE: long-term interest rate (30-year Treasury bond).

TBRATE: three-month Treasury bill rate (% per annum).

*Source:* *Economic Report of the President, 2000, Tables B-1, B-58, B-67, B-71.*

TABLE 7.11 GREEK INDUSTRIAL SECTOR

Observation	Output*	Capital	Labor†	Capital-to-labor ratio
1961	35.858	59.600	637.0	0.0936
1962	37.504	64.200	643.2	0.0998
1963	40.378	68.800	651.0	0.1057
1964	46.147	75.500	685.7	0.1101
1965	51.047	84.400	710.7	0.1188
1966	53.871	91.800	724.3	0.1267
1967	56.834	99.900	735.2	0.1359
1968	65.439	109.100	760.3	0.1435
1969	74.939	120.700	777.6	0.1552
1970	80.976	132.000	780.8	0.1691
1971	90.802	146.600	825.8	0.1775
1972	101.955	162.700	864.1	0.1883
1973	114.367	180.600	894.2	0.2020
1974	101.823	197.100	891.2	0.2212
1975	107.572	209.600	887.5	0.2362
1976	117.600	221.900	892.3	0.2487
1977	123.224	232.500	930.1	0.2500
1978	130.971	243.500	969.9	0.2511
1979	138.842	257.700	1006.9	0.2559
1980	135.486	274.400	1020.9	0.2688
1981	133.441	289.500	1017.1	0.2846
1982	130.388	301.900	1016.1	0.2971
1983	130.615	314.900	1008.1	0.3124
1984	132.244	327.700	985.1	0.3327
1985	137.318	339.400	977.1	0.3474
1986	137.468	349.492	1007.2	0.3470
1987	135.750	358.231	1000.0	0.3582

\*Billions of Drachmas at constant 1970 prices

†Thousands of workers per year.

Source: I am indebted to George K. Zestos of Christopher Newport University, Virginia, for the data.

- See if the Cobb–Douglas production function fits the data given in the table and interpret the results. What general conclusion do you draw?
- Now consider the following model:

$$\text{Output/labor} = A(K/L)^\beta e^u$$

where the regressand represents labor productivity and the regressor represents the capital labor ratio. What is the economic significance of such a relationship, if any? Estimate the parameters of this model and interpret your results.

- 7.23.** Refer to Example 3.3 and the data given in Table 2.6. Now consider the following models:

$$\ln(\text{hwage})_i = \beta_1 + \beta_2 \ln(\text{education})_i + \beta_3(\ln \text{education})^2 + u_i$$

where  $\ln$  = natural log. How would you interpret this model? Estimate this model, obtaining the usual statistics and comment on your results.

- Now consider the following model:

$$\ln(\text{hwage}) = \beta_1 + \beta_2 \ln(\text{education}) + \beta_3 \ln(\text{education}^2) + u_i$$

If you try to estimate this model, what problem(s) would you encounter? Try to estimate this model and see if your software package can estimate this model.

**7.24. Monte Carlo experiment:** Consider the following model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

You are told that  $\beta_1 = 262$ ,  $\beta_2 = -0.006$ ,  $\beta_3 = -2.4$ ,  $\sigma^2 = 42$ , and  $u_i \sim N(0, 42)$ . Generate 10 sets of 64 observations on  $u_i$  from the given normal distribution and use the 64 observations given in Table 6.4, where  $Y = \text{CM}$ ,  $X_2 = \text{PGNP}$ , and  $X_3 = \text{FLR}$  to generate 10 sets of the estimated  $\beta$  coefficients (each set will have the three estimated parameters). Take the averages of each of the estimated  $\beta$  coefficients and relate them to the true values of these coefficients given above. What overall conclusion do you draw?

## APPENDIX 7A

### 7A.1 DERIVATION OF OLS ESTIMATORS GIVEN IN EQUATIONS (7.4.3) TO (7.4.5)

Differentiating the equation

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i})^2 \quad (7.4.2)$$

partially with respect to the three unknowns and setting the resulting equations to zero, we obtain

$$\frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_1} = 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i})(-1) = 0$$

$$\frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_2} = 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i})(-X_{2i}) = 0$$

$$\frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_3} = 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i})(-X_{3i}) = 0$$

Simplifying these, we obtain Eqs. (7.4.3) to (7.4.5).

In passing, note that the three preceding equations can also be written as

$$\sum \hat{u}_i = 0$$

$$\sum \hat{u}_i X_{2i} = 0 \quad (\text{Why?})$$

$$\sum \hat{u}_i X_{3i} = 0$$

which show the properties of the least-squares fit, namely, that the residuals sum to zero and that they are uncorrelated with the explanatory variables  $X_2$  and  $X_3$ .

Incidentally, notice that to obtain the OLS estimators of the  $k$ -variable linear regression model (7.4.20) we proceed analogously. Thus, we first write

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \dots - \hat{\beta}_k X_{ki})^2$$

Differentiating this expression partially with respect to each of the  $k$  unknowns, setting the resulting equations equal to zero, and rearranging, we obtain the following  $k$  normal equations in the  $k$  unknowns:

$$\begin{aligned} \sum Y_i &= n\hat{\beta}_1 + \hat{\beta}_2 \sum X_{2i} + \hat{\beta}_3 \sum X_{3i} + \dots + \hat{\beta}_k \sum X_{ki} \\ \sum Y_i X_{2i} &= \hat{\beta}_1 \sum X_{2i} + \hat{\beta}_2 \sum X_{2i}^2 + \hat{\beta}_3 \sum X_{2i} X_{3i} + \dots + \hat{\beta}_k \sum X_{2i} X_{ki} \\ \sum Y_i X_{3i} &= \hat{\beta}_1 \sum X_{3i} + \hat{\beta}_2 \sum X_{2i} X_{3i} + \hat{\beta}_3 \sum X_{3i}^2 + \dots + \hat{\beta}_k \sum X_{3i} X_{ki} \\ &\dots \dots \dots \\ \sum Y_i X_{ki} &= \hat{\beta}_1 \sum X_{ki} + \hat{\beta}_2 \sum X_{2i} X_{ki} + \hat{\beta}_3 \sum X_{3i} X_{ki} + \dots + \hat{\beta}_k \sum X_{ki}^2 \end{aligned}$$

Or, switching to small letters, these equations can be expressed as

$$\begin{aligned} \sum y_i x_{2i} &= \hat{\beta}_2 \sum x_{2i}^2 + \hat{\beta}_3 \sum x_{2i} x_{3i} + \dots + \hat{\beta}_k \sum x_{2i} x_{ki} \\ \sum y_i x_{3i} &= \hat{\beta}_2 \sum x_{2i} x_{3i} + \hat{\beta}_3 \sum x_{3i}^2 + \dots + \hat{\beta}_k \sum x_{3i} x_{ki} \\ &\dots \dots \dots \\ \sum y_i x_{ki} &= \hat{\beta}_2 \sum x_{2i} x_{ki} + \hat{\beta}_3 \sum x_{3i} x_{ki} + \dots + \hat{\beta}_k \sum x_{ki}^2 \end{aligned}$$

It should further be noted that the  $k$ -variable model also satisfies these equations:

$$\begin{aligned} \sum \hat{u}_i &= 0 \\ \sum \hat{u}_i X_{2i} &= \sum \hat{u}_i X_{3i} = \dots = \sum \hat{u}_i X_{ki} = 0 \end{aligned}$$

**7A.2 EQUALITY BETWEEN THE COEFFICIENTS OF PGNP IN (7.3.5) AND (7.6.2)**

Letting  $Y = \text{CM}$ ,  $X_2 = \text{PGNP}$ , and  $X_3 = \text{FLR}$  and using the deviation form, write

$$y_i = b_{13} x_{3i} + \hat{u}_{1i} \tag{1}$$

$$x_{2i} = b_{23} x_{3i} + \hat{u}_{2i} \tag{2}$$

Now regress  $\hat{u}_{1i}$  on  $\hat{u}_{2i}$  to obtain:

$$a_1 = \frac{\sum \hat{u}_{1i} \hat{u}_{2i}}{\sum \hat{u}_{2i}^2} = -0.0056 \quad (\text{for our example}) \tag{3}$$

Note that because the  $\hat{u}$ 's are residuals, their mean values are zero. Using (1) and (2), we can write (3) as

$$a_1 = \frac{\sum (y_i - b_{13}x_{3i})(x_{2i} - b_{23}x_{3i})}{\sum (x_{2i} - b_{23}x_{3i})^2} \quad (4)$$

Expand the preceding expression, and note that

$$b_{23} = \frac{\sum x_{2i}x_{3i}}{\sum x_{3i}^2} \quad (5)$$

and

$$b_{13} = \frac{\sum y_i x_{3i}}{\sum x_{3i}^2} \quad (6)$$

Making these substitutions into (4), we get

$$\begin{aligned} \hat{\beta}_2 &= \frac{(\sum y_i x_{2i})(\sum x_{3i}^2) - (\sum y_i x_{3i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} \\ &= -0.0056 \quad (\text{for our example}) \end{aligned} \quad (7.4.7)$$

### 7A.3 DERIVATION OF EQUATION (7.4.19)

Recall that

$$\hat{u}_i = Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \hat{\beta}_3 X_{3i}$$

which can also be written as

$$\hat{u}_i = y_i - \hat{\beta}_2 x_{2i} - \hat{\beta}_3 x_{3i}$$

where small letters, as usual, indicate deviations from mean values.

Now

$$\begin{aligned} \sum \hat{u}_i^2 &= \sum (\hat{u}_i \hat{u}_i) \\ &= \sum \hat{u}_i (y_i - \hat{\beta}_2 x_{2i} - \hat{\beta}_3 x_{3i}) \\ &= \sum \hat{u}_i y_i \end{aligned}$$

where use is made of the fact that  $\sum \hat{u}_i x_{2i} = \sum \hat{u}_i x_{3i} = 0$ . (Why?) Also

$$\sum \hat{u}_i y_i = \sum y_i \hat{u}_i = \sum y_i (y_i - \hat{\beta}_2 x_{2i} - \hat{\beta}_3 x_{3i})$$

that is,

$$\sum \hat{u}_i^2 = \sum y_i^2 - \hat{\beta}_2 \sum y_i x_{2i} - \hat{\beta}_3 \sum y_i x_{3i}$$

which is the required result.





**7A.5 SAS OUTPUT OF THE COBB-DOUGLAS PRODUCTION FUNCTION (7.9.4)**

DEP VARIABLE: Y1

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	F VALUE	PROB > F
MODEL	2	0.538038	0.269019	48.069	0.0001
ERROR	12	0.067153	0.005596531		
C TOTAL	14	0.605196			

ROOT MSE	0.074810	R-SQUARE	0.8890
DEP MEAN	10.096535	ADJ R-SQ	0.8705
C.V.	0.7409469		

VARIABLE	DF	PARAMETER ESTIMATE	STANDARD ERROR	T FOR HO: PARAMETER = 0	PROB >  T
INTERCEP	1	-3.338455	2.449508	-1.363	0.1979
Y2	1	1.498767	0.539803	2.777	0.0168
Y3	1	0.489858	0.102043	4.800	0.0004

**COVARIANCE OF ESTIMATES**

COVB	INTERCEP	Y2	Y3
INTERCEP	6.000091	-1.26056	0.1121951
Y2	-1.26056	0.2913868	-0.0384272
Y3	0.01121951	-0.0384272	0.01041288

Y	X2	X3	Y1	Y2	Y3	Y1HAT	Y1RESID
16607.7	275.5	17803.7	9.7176	5.61859	9.7872	9.8768	-0.15920
17511.3	274.4	18096.8	9.7706	5.61459	9.8035	9.8788	-0.10822
20171.2	269.7	18271.8	9.9120	5.59731	9.8131	9.8576	0.05437
20932.9	267.0	19167.3	9.9491	5.58725	9.8610	9.8660	0.08307
20406.0	267.8	19647.6	9.9236	5.59024	9.8857	9.8826	0.04097
20831.6	275.0	20803.5	9.9442	5.61677	9.9429	9.9504	-0.00615
24806.3	283.0	22076.6	10.1189	5.64545	10.0023	10.0225	0.09640
26465.8	300.7	23445.2	10.1836	5.70611	10.0624	10.1428	0.04077
27403.0	307.5	24939.0	10.2184	5.72848	10.1242	10.2066	0.01180
28628.7	303.7	26713.7	10.2622	5.71604	10.1929	10.2217	0.04051
29904.5	304.7	29957.8	10.3058	5.71933	10.3075	10.2827	0.02304
27508.2	298.6	31585.9	10.2222	5.69910	10.3605	10.2783	-0.05610
29035.5	295.5	33474.5	10.2763	5.68867	10.4185	10.2911	-0.01487
29281.5	299.0	34821.8	10.2847	5.70044	10.4580	10.3281	-0.04341
31535.8	288.1	41794.3	10.3589	5.66331	10.6405	10.3619	-0.00299

**COLLINEARITY DIAGNOSTICS**

NUMBER	CONDITION EIGENVALUE	PORTION INDEX	PORTION INTERCEP	PORTION Y2	PORTION Y3
1	3.000	1.000	0.0000	0.0000	0.0000
2	.000375451	89.383	0.0491	0.0069	0.5959
3	.000024219	351.925	0.9509	0.0031	0.4040

**VARIANCE PROPORTIONS**

DURBIN-WATSON *d* 0.891  
1ST ORDER AUTOCORRELATION 0.366

Notes: Y1 = ln Y; Y2 = ln X2; Y3 = ln X3. The numbers under the heading PROB > |T| represent *p* values. See Chapter 10 for a discussion of collinearity diagnostics.

# 8

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## MULTIPLE REGRESSION ANALYSIS: THE PROBLEM OF INFERENCE

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This chapter, a continuation of Chapter 5, extends the ideas of interval estimation and hypothesis testing developed there to models involving three or more variables. Although in many ways the concepts developed in Chapter 5 can be applied straightforwardly to the multiple regression model, a few additional features are unique to such models, and it is these features that will receive more attention in this chapter.

### 8.1 THE NORMALITY ASSUMPTION ONCE AGAIN

We know by now that if our sole objective is point estimation of the parameters of the regression models, the method of ordinary least squares (OLS), which does not make any assumption about the probability distribution of the disturbances  $u_i$ , will suffice. But if our objective is estimation as well as inference, then, as argued in Chapters 4 and 5, we need to assume that the  $u_i$  follow some probability distribution.

For reasons already clearly spelled out, we assumed that the  $u_i$  follow the normal distribution with zero mean and constant variance  $\sigma^2$ . We continue to make the same assumption for multiple regression models. With the normality assumption and following the discussion of Chapters 4 and 7, we find that the OLS estimators of the partial regression coefficients, which are identical with the maximum likelihood (ML) estimators, are best linear unbiased estimators (BLUE).<sup>1</sup> Moreover, the estimators  $\hat{\beta}_2$ ,  $\hat{\beta}_3$ , and  $\hat{\beta}_1$  are

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<sup>1</sup>With the normality assumption, the OLS estimators  $\hat{\beta}_2$ ,  $\hat{\beta}_3$ , and  $\hat{\beta}_1$  are minimum-variance estimators in the entire class of unbiased estimators, whether linear or not. In short, they are BUE (best unbiased estimators). See C. R. Rao, *Linear Statistical Inference and Its Applications*, John Wiley & Sons, New York, 1965, p. 258.

themselves normally distributed with means equal to true  $\beta_2$ ,  $\beta_3$ , and  $\beta_1$  and the variances given in Chapter 7. Furthermore,  $(n - 3)\hat{\sigma}^2/\sigma^2$  follows the  $\chi^2$  distribution with  $n - 3$  df, and the three OLS estimators are distributed independently of  $\hat{\sigma}^2$ . The proofs follow the two-variable case discussed in Appendix 3. As a result and following Chapter 5, one can show that, upon replacing  $\sigma^2$  by its unbiased estimator  $\hat{\sigma}^2$  in the computation of the standard errors, each of the following variables

$$t = \frac{\hat{\beta}_1 - \beta_1}{\text{se}(\hat{\beta}_1)} \quad (8.1.1)$$

$$t = \frac{\hat{\beta}_2 - \beta_2}{\text{se}(\hat{\beta}_2)} \quad (8.1.2)$$

$$t = \frac{\hat{\beta}_3 - \beta_3}{\text{se}(\hat{\beta}_3)} \quad (8.1.3)$$

follows the  $t$  distribution with  $n - 3$  df.

Note that the df are now  $n - 3$  because in computing  $\sum \hat{u}_i^2$  and hence  $\hat{\sigma}^2$  we first need to estimate the three partial regression coefficients, which therefore put three restrictions on the residual sum of squares (RSS) (following this logic in the four-variable case there will be  $n - 4$  df, and so on). Therefore, the  $t$  distribution can be used to establish confidence intervals as well as test statistical hypotheses about the true population partial regression coefficients. Similarly, the  $\chi^2$  distribution can be used to test hypotheses about the true  $\sigma^2$ . To demonstrate the actual mechanics, we use the following illustrative example.

## 8.2 EXAMPLE 8.1: CHILD MORTALITY EXAMPLE REVISITED

In Chapter 7 we regressed child mortality (CM) on per capita GNP (PGNP) and the female literacy rate (FLR) for a sample of 64 countries. The regression results given in (7.6.2) are reproduced below with some additional information:

$$\begin{array}{rcccl} \widehat{CM}_i & = & 263.6416 & - & 0.0056 \text{ PGNP}_i & - & 2.2316 \text{ FLR}_i & & \\ \text{se} & = & (11.5932) & & (0.0019) & & (0.2099) & & \\ t & = & (22.7411) & & (-2.8187) & & (-10.6293) & & (8.2.1) \\ p \text{ value} & = & (0.0000)^* & & (0.0065) & & (0.0000)^* & & \\ & & & & & & R^2 = 0.7077 & & \bar{R}^2 = 0.6981 \end{array}$$

where \* denotes extremely low value.

In Eq. (8.2.1) we have followed the format first introduced in Eq. (5.11.1), where the figures in the first set of parentheses are the estimated standard errors, those in the second set are the  $t$  values under the null hypothesis that the relevant population coefficient has a value of zero, and those in the third are the estimated  $p$  values. Also given are  $R^2$  and adjusted  $R^2$  values. We have already interpreted this regression in Example 7.1.

What about the statistical significance of the observed results? Consider, for example, the coefficient of PGNP of  $-0.0056$ . Is this coefficient statistically significant, that is, statistically different from zero? Likewise, is the coefficient of FLR of  $-2.2316$  statistically significant? Are both coefficients statistically significant? To answer this and related questions, let us first consider the kinds of hypothesis testing that one may encounter in the context of a multiple regression model.

### 8.3 HYPOTHESIS TESTING IN MULTIPLE REGRESSION: GENERAL COMMENTS

Once we go beyond the simple world of the two-variable linear regression model, hypothesis testing assumes several interesting forms, such as the following:

1. Testing hypotheses about an individual partial regression coefficient (Section 8.4)
2. Testing the overall significance of the estimated multiple regression model, that is, finding out if all the partial slope coefficients are simultaneously equal to zero (Section 8.5)
3. Testing that two or more coefficients are equal to one another (Section 8.6)
4. Testing that the partial regression coefficients satisfy certain restrictions (Section 8.7)
5. Testing the stability of the estimated regression model over time or in different cross-sectional units (Section 8.8)
6. Testing the functional form of regression models (Section 8.9)

Since testing of one or more of these types occurs so commonly in empirical analysis, we devote a section to each type.

### 8.4 HYPOTHESIS TESTING ABOUT INDIVIDUAL REGRESSION COEFFICIENTS

If we invoke the assumption that  $u_i \sim N(0, \sigma^2)$ , then, as noted in Section 8.1, we can use the  $t$  test to test a hypothesis about any *individual* partial regression coefficient. To illustrate the mechanics, consider the child mortality regression, (8.2.1). Let us postulate that

$$H_0: \beta_2 = 0 \quad \text{and} \quad H_1: \beta_2 \neq 0$$

The null hypothesis states that, with  $X_3$  (female literacy rate) held constant,  $X_2$  (PGNP) has no (linear) influence on  $Y$  (child mortality).<sup>2</sup> To test the null hypothesis, we use the  $t$  test given in (8.1.2). Following Chapter 5 (see Table 5.1), if the computed  $t$  value exceeds the critical  $t$  value at the chosen level of significance, we may reject the null hypothesis; otherwise, we may not reject it. For our illustrative example, using (8.1.2) and noting that  $\beta_2 = 0$  under the null hypothesis, we obtain

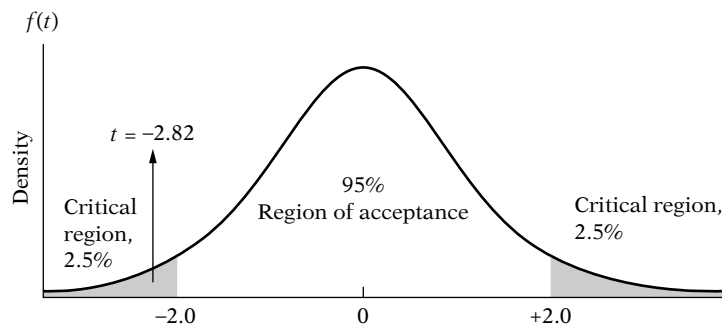
$$t = \frac{-0.0056}{0.0020} = -2.8187 \quad (8.4.1)$$

as shown in Eq. (8.2.1).

Notice that we have 64 observations. Therefore, the degrees of freedom in this example are 61 (why?). If you refer to the  $t$  table given in **Appendix D**, we do not have data corresponding to 61 df. The closest we have are for 60 df. If we use these df, and assume  $\alpha$ , the level of significance (i.e., the probability of committing a Type I error) of 5 percent, the critical  $t$  value is 2.0 for a two-tail test (look up  $t_{\alpha/2}$  for 60 df) or 1.671 for a one-tail test (look up  $t_{\alpha}$  for 60 df).

For our example, the alternative hypothesis is two-sided. Therefore, we use the two-tail  $t$  value. Since the computed  $t$  value of 2.8187 (in absolute terms) exceeds the critical  $t$  value of 2, we can reject the null hypothesis that PGNP has no effect on child mortality. To put it more positively, with the female literacy rate held constant, per capita GNP has a significant (negative) effect on child mortality, as one would expect a priori. Graphically, the situation is as shown in Figure 8.1.

In practice, one does not have to assume a particular value of  $\alpha$  to conduct hypothesis testing. One can simply use the  $p$  value given in (8.2.2),



**FIGURE 8.1** The 95% confidence interval for  $t$  (60 df).

<sup>2</sup>In most empirical investigations the null hypothesis is stated in this form, that is, taking the extreme position (a kind of straw man) that there is no relationship between the dependent variable and the explanatory variable under consideration. The idea here is to find out whether the relationship between the two is a trivial one to begin with.

which in the present case is 0.0065. The interpretation of this  $p$  value (i.e., the exact level of significance) is that if the null hypothesis were true, the probability of obtaining a  $t$  value of as much as 2.8187 or greater (in absolute terms) is only 0.0065 or 0.65 percent, which is indeed a small probability, much smaller than the artificially adopted value of  $\alpha = 5\%$ .

This example provides us an opportunity to decide whether we want to use a one-tail or a two-tail  $t$  test. Since a priori child mortality and per capita GNP are expected to be negatively related (why?), we should use the one-tail test. That is, our null and alternative hypothesis should be:

$$H_0: \beta_2 < 0 \quad \text{and} \quad H_1: \beta_2 \geq 0$$

As the reader knows by now, we can reject the null hypothesis on the basis of the one-tail  $t$  test in the present instance.

In Chapter 5 we saw the intimate connection between hypothesis testing and confidence interval estimation. For our example, the 95% confidence interval for  $\beta_2$  is:

$$\hat{\beta}_2 - t_{\alpha/2} \text{se}(\hat{\beta}_2) \leq \beta_2 \leq \hat{\beta}_2 + t_{\alpha/2} \text{se}(\hat{\beta}_2)$$

which in our example becomes

$$-0.0056 - 2(0.0020) \leq \beta_2 \leq -0.0056 + 2(0.0020)$$

that is,

$$-0.0096 \leq \beta_2 \leq -0.0016 \quad (8.4.2)$$

that is, the interval,  $-0.0096$  to  $-0.0016$  includes the true  $\beta_2$  coefficient with 95% confidence coefficient. Thus, if 100 samples of size 64 are selected and 100 confidence intervals like (8.4.2) are constructed, we expect 95 of them to contain the true population parameter  $\beta_2$ . Since the interval (8.4.2) does not include the null-hypothesized value of zero, we can reject the null hypothesis that the true  $\beta_2$  is zero with 95% confidence.

Thus, whether we use the  $t$  test of significance as in (8.4.1) or the confidence interval estimation as in (8.4.2), we reach the same conclusion. However, this should not be surprising in view of the close connection between confidence interval estimation and hypothesis testing.

Following the procedure just described, we can test hypotheses about the other parameters of our child mortality regression model. The necessary data are already provided in Eq. (8.2.1). For example, suppose we want to test the hypothesis that, with the influence of PGNP held constant, the female literacy rate has no effect whatsoever on child mortality. We can confidently reject this hypothesis, for under this null hypothesis the  $p$  value of obtaining an absolute  $t$  value of as much as 10.6 or greater is practically zero.

Before moving on, remember that the  $t$ -testing procedure is based on the assumption that the error term  $u_i$  follows the normal distribution. Although we cannot directly observe  $u_i$ , we can observe their proxy, the  $\hat{u}_i$ , that is, the

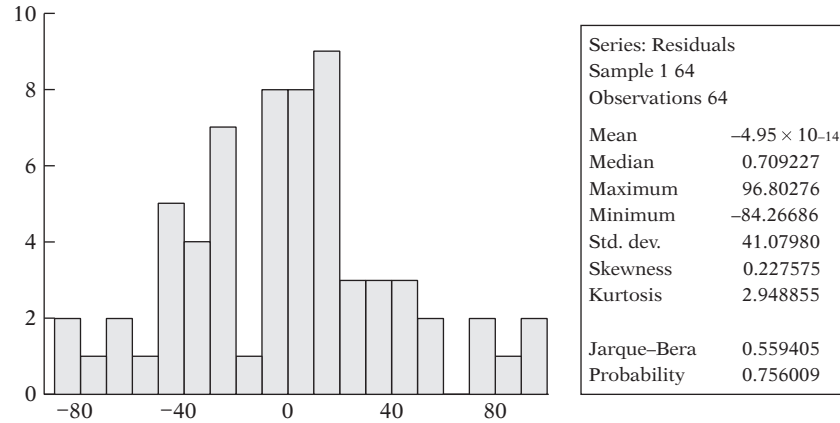


FIGURE 8.2 Histogram of residuals from regression (8.2.1).

residuals. For our mortality regression, the histogram of the residuals is as shown in Figure 8.2.

From the histogram it seems that the residuals are normally distributed. We can also compute the **Jarque-Bera (JB)** test of normality, as shown in Eq. (5.12.1). In our case the JB value is 0.5594 with a  $p$  value 0.76.<sup>3</sup> Therefore, it seems that the error term in our example follows the normal distribution. Of course, keep in mind that the JB test is a large-sample test and our sample of 64 observations may not be necessarily large.

### 8.5 TESTING THE OVERALL SIGNIFICANCE OF THE SAMPLE REGRESSION

Throughout the previous section we were concerned with testing the significance of the estimated partial regression coefficients individually, that is, under the separate hypothesis that each true population partial regression coefficient was zero. But now consider the following hypothesis:

$$H_0: \beta_2 = \beta_3 = 0 \quad (8.5.1)$$

This null hypothesis is a joint hypothesis that  $\beta_2$  and  $\beta_3$  are jointly or simultaneously equal to zero. A test of such a hypothesis is called a test of the **overall significance** of the observed or estimated regression line, that is, whether  $Y$  is linearly related to both  $X_2$  and  $X_3$ .

Can the joint hypothesis in (8.5.1) be tested by testing the significance of  $\hat{\beta}_2$  and  $\hat{\beta}_3$  individually as in Section 8.4? The answer is no, and the reasoning is as follows.

<sup>3</sup>For our example, the skewness value is 0.2276 and the kurtosis value is 2.9488. Recall that for a normally distributed variable the skewness and kurtosis values are, respectively, 0 and 3.

In testing the individual significance of an observed partial regression coefficient in Section 8.4, we assumed implicitly that each test of significance was based on a different (i.e., independent) sample. Thus, in testing the significance of  $\hat{\beta}_2$  under the hypothesis that  $\beta_2 = 0$ , it was assumed tacitly that the testing was based on a different sample from the one used in testing the significance of  $\hat{\beta}_3$  under the null hypothesis that  $\beta_3 = 0$ . But to test the joint hypothesis of (8.5.1), if we use the same sample data, we shall be violating the assumption underlying the test procedure.<sup>4</sup> The matter can be put differently: In (8.4.2) we established a 95% confidence interval for  $\beta_2$ . But if we use the same sample data to establish a confidence interval for  $\beta_3$ , say, with a confidence coefficient of 95%, we cannot assert that both  $\beta_2$  and  $\beta_3$  lie in their respective confidence intervals with a probability of  $(1 - \alpha)(1 - \alpha) = (0.95)(0.95)$ .

In other words, although the statements

$$\Pr [\hat{\beta}_2 - t_{\alpha/2} \text{se}(\hat{\beta}_2) \leq \beta_2 \leq \hat{\beta}_2 + t_{\alpha/2} \text{se}(\hat{\beta}_2)] = 1 - \alpha$$

$$\Pr [\hat{\beta}_3 - t_{\alpha/2} \text{se}(\hat{\beta}_3) \leq \beta_3 \leq \hat{\beta}_3 + t_{\alpha/2} \text{se}(\hat{\beta}_3)] = 1 - \alpha$$

are individually true, *it is not true* that the probability that the intervals

$$[\hat{\beta}_2 \pm t_{\alpha/2} \text{se}(\hat{\beta}_2), \hat{\beta}_3 \pm t_{\alpha/2} \text{se}(\hat{\beta}_3)]$$

simultaneously include  $\beta_2$  and  $\beta_3$  is  $(1 - \alpha)^2$ , because the intervals may not be independent when the same data are used to derive them. To state the matter differently,

... testing a series of single [individual] hypotheses is *not* equivalent to testing those same hypotheses jointly. The intuitive reason for this is that in a joint test of several hypotheses any single hypothesis is “affected” by the information in the other hypotheses.<sup>5</sup>

The upshot of the preceding argument is that for a given example (sample) only one confidence interval or only one test of significance can be obtained. How, then, does one test the simultaneous null hypothesis that  $\beta_2 = \beta_3 = 0$ ? The answer follows.

### The Analysis of Variance Approach to Testing the Overall Significance of an Observed Multiple Regression: The *F* Test

For reasons just explained, we cannot use the usual *t* test to test the joint hypothesis that the true partial slope coefficients are zero simultaneously. However, this joint hypothesis can be tested by the **analysis of variance** (ANOVA) technique first introduced in Section 5.9, which can be demonstrated as follows.

<sup>4</sup>In any given sample the cov ( $\hat{\beta}_2, \hat{\beta}_3$ ) may not be zero; that is,  $\hat{\beta}_2$  and  $\hat{\beta}_3$  may be correlated. See (7.4.17).

<sup>5</sup>Thomas B. Fomby, R. Carter Hill, and Stanley R. Johnson, *Advanced Econometric Methods*, Springer-Verlag, New York, 1984, p. 37.



**TABLE 8.1** ANOVA TABLE FOR THE THREE-VARIABLE REGRESSION

Source of variation	SS	df	MSS
Due to regression (ESS)	$\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i}$	2	$\frac{\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i}}{2}$
Due to residual (RSS)	$\sum \hat{u}_i^2$	$n - 3$	$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n - 3}$
Total	$\sum y_i^2$	$n - 1$	

Recall the identity

$$\sum y_i^2 = \hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i} + \sum \hat{u}_i^2 \quad (8.5.2)$$

TSS =        ESS        +    RSS

TSS has, as usual,  $n - 1$  df and RSS has  $n - 3$  df for reasons already discussed. ESS has 2 df since it is a function of  $\hat{\beta}_2$  and  $\hat{\beta}_3$ . Therefore, following the ANOVA procedure discussed in Section 5.9, we can set up Table 8.1.

Now it can be shown<sup>6</sup> that, under the assumption of normal distribution for  $u_i$  and the null hypothesis  $\beta_2 = \beta_3 = 0$ , the variable

$$F = \frac{(\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i})/2}{\sum \hat{u}_i^2 / (n - 3)} = \frac{\text{ESS/df}}{\text{RSS/df}} \quad (8.5.3)$$

is distributed as the  $F$  distribution with 2 and  $n - 3$  df.

What use can be made of the preceding  $F$  ratio? It can be proved<sup>7</sup> that under the assumption that the  $u_i \sim N(0, \sigma^2)$ ,

$$E \frac{\sum \hat{u}_i^2}{n - 3} = E(\hat{\sigma}^2) = \sigma^2 \quad (8.5.4)$$

With the additional assumption that  $\beta_2 = \beta_3 = 0$ , it can be shown that

$$\frac{E(\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i})}{2} = \sigma^2 \quad (8.5.5)$$

Therefore, if the null hypothesis is true, both (8.5.4) and (8.5.5) give identical estimates of true  $\sigma^2$ . This statement should not be surprising because if there is a trivial relationship between  $Y$  and  $X_2$  and  $X_3$ , the sole source of variation in  $Y$  is due to the random forces represented by  $u_i$ . If, however, the null hypothesis is false, that is,  $X_2$  and  $X_3$  definitely influence  $Y$ , the equality between (8.5.4) and (8.5.5) will not hold. In this case, the ESS will be

<sup>6</sup>See K. A. Brownlee, *Statistical Theory and Methodology in Science and Engineering*, John Wiley & Sons, New York, 1960, pp. 278–280.

<sup>7</sup>Ibid.

relatively larger than the RSS, taking due account of their respective df. Therefore, the  $F$  value of (8.5.3) provides a test of the null hypothesis that the true slope coefficients are simultaneously zero. If the  $F$  value computed from (8.5.3) exceeds the critical  $F$  value from the  $F$  table at the  $\alpha$  percent level of significance, we reject  $H_0$ ; otherwise we do not reject it. Alternatively, if the  $p$  value of the observed  $F$  is sufficiently low, we can reject  $H_0$ .

Table 8.2 summarizes the  $F$  test. Turning to our illustrative example, we obtain the ANOVA table, as shown in Table 8.3.

Using (8.5.3), we obtain

$$F = \frac{128,681.2}{1742.88} = 73.8325 \quad (8.5.6)$$

The  $p$  value of obtaining an  $F$  value of as much as 73.8325 or greater is almost zero, leading to the rejection of the hypothesis that together PGNP and FLR have no effect on child mortality. If you were to use the conventional 5 percent level-of-significance value, the critical  $F$  value for 2 df in the numerator and 60 df in the denominator (the actual df, however, are 61) is about 3.15 or about 4.98 if you were to use the 1 percent level of significance. Obviously, the observed  $F$  of about 74 far exceeds any of these critical  $F$  values.

TABLE 8.2 A SUMMARY OF THE  $F$  STATISTIC

Null hypothesis $H_0$	Alternative hypothesis $H_1$	Critical region Reject $H_0$ if
$\sigma_1^2 = \sigma_2^2$	$\sigma_1^2 > \sigma_2^2$	$\frac{S_1^2}{S_2^2} > F_{\alpha, ndf, ddf}$
$\sigma_1^2 = \sigma_2^2$	$\sigma_1^2 \neq \sigma_2^2$	$\frac{S_1^2}{S_2^2} > F_{\alpha/2, ndf, ddf}$ or $< F_{(1-\alpha/2), ndf, ddf}$

Notes:

- $\sigma_1^2$  and  $\sigma_2^2$  are the two population variances.
- $S_1^2$  and  $S_2^2$  are the two sample variances.
- $ndf$  and  $ddf$  denote, respectively, the numerator and denominator df.
- In computing the  $F$  ratio, put the larger  $S^2$  value in the numerator.
- The critical  $F$  values are given in the last column. The first subscript of  $F$  is the level of significance and the second subscript is the numerator and denominator df.
- Note that  $F_{(1-\alpha/2), ndf, ddf} = 1/F_{\alpha/2, ddf, ndf}$ .

TABLE 8.3 ANOVA TABLE FOR THE CHILD MORTALITY EXAMPLE

Source of variation	SS	df	MSS
Due to regression	257,362.4	2	128,681.2
Due to residuals	106,315.6	61	1742.88
Total	363,678	63	

We can generalize the preceding  $F$ -testing procedure as follows.

### Testing the Overall Significance of a Multiple Regression: The $F$ Test

**Decision Rule.** Given the  $k$ -variable regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i$$

To test the hypothesis

$$H_0: \beta_2 = \beta_3 = \cdots = \beta_k = 0$$

(i.e., all slope coefficients are simultaneously zero) versus

$$H_1: \text{Not all slope coefficients are simultaneously zero}$$

compute

$$F = \frac{\text{ESS}/\text{df}}{\text{RSS}/\text{df}} = \frac{\text{ESS}/(k-1)}{\text{RSS}/(n-k)} \quad (8.5.7)$$

If  $F > F_\alpha(k-1, n-k)$ , reject  $H_0$ ; otherwise you do not reject it, where  $F_\alpha(k-1, n-k)$  is the *critical  $F$*  value at the  $\alpha$  level of significance and  $(k-1)$  numerator df and  $(n-k)$  denominator df. Alternatively, if the  $p$  value of  $F$  obtained from (8.5.7) is sufficiently low, one can reject  $H_0$ .

Needless to say, in the three-variable case ( $Y$  and  $X_2, X_3$ )  $k$  is 3, in the four-variable case  $k$  is 4, and so on.

In passing, note that most regression packages routinely calculate the  $F$  value (given in the analysis of variance table) along with the usual regression output, such as the estimated coefficients, their standard errors,  $t$  values, etc. The null hypothesis for the  $t$  computation is usually assumed to be  $\beta_i = 0$ .

**Individual versus Joint Testing of Hypotheses.** In Section 8.4 we discussed the test of significance of a single regression coefficient and in Section 8.5 we have discussed the joint or overall test of significance of the estimated regression (i.e., all slope coefficients are simultaneously equal to zero). **We reiterate that these tests are different.** Thus, on the basis of the  $t$  test or confidence interval (of Section 8.4) it is possible to accept the hypothesis that a particular slope coefficient,  $\beta_k$ , is zero, and yet reject the joint hypothesis that all slope coefficients are zero.

The lesson to be learned is that the joint “message” of individual confidence intervals is no substitute for a joint confidence region [implied by the  $F$  test] in performing joint tests of hypotheses and making joint confidence statements.<sup>8</sup>

<sup>8</sup>Fomby et al., op. cit., p. 42.

**An Important Relationship between  $R^2$  and  $F$** 

There is an intimate relationship between the coefficient of determination  $R^2$  and the  $F$  test used in the analysis of variance. Assuming the normal distribution for the disturbances  $u_i$  and the null hypothesis that  $\beta_2 = \beta_3 = 0$ , we have seen that

$$F = \frac{\text{ESS}/2}{\text{RSS}/(n-3)} \quad (8.5.8)$$

is distributed as the  $F$  distribution with 2 and  $n - 3$  df.

More generally, in the  $k$ -variable case (including intercept), if we assume that the disturbances are normally distributed and that the null hypothesis is

$$H_0: \beta_2 = \beta_3 = \dots = \beta_k = 0 \quad (8.5.9)$$

then it follows that

$$F = \frac{\text{ESS}/(k-1)}{\text{RSS}/(n-k)} \quad (8.5.7) = (8.5.10)$$

follows the  $F$  distribution with  $k - 1$  and  $n - k$  df. (*Note:* The total number of parameters to be estimated is  $k$ , of which one is the intercept term.)

Let us manipulate (8.5.10) as follows:

$$\begin{aligned} F &= \frac{n-k}{k-1} \frac{\text{ESS}}{\text{RSS}} \\ &= \frac{n-k}{k-1} \frac{\text{ESS}}{\text{TSS} - \text{ESS}} \\ &= \frac{n-k}{k-1} \frac{\text{ESS}/\text{TSS}}{1 - (\text{ESS}/\text{TSS})} \\ &= \frac{n-k}{k-1} \frac{R^2}{1 - R^2} \\ &= \frac{R^2/(k-1)}{(1 - R^2)/(n-k)} \end{aligned} \quad (8.5.11)$$

where use is made of the definition  $R^2 = \text{ESS}/\text{TSS}$ . Equation (8.5.11) shows how  $F$  and  $R^2$  are related. These two vary directly. When  $R^2 = 0$ ,  $F$  is zero ipso facto. The larger the  $R^2$ , the greater the  $F$  value. In the limit, when  $R^2 = 1$ ,  $F$  is infinite. *Thus the  $F$  test, which is a measure of the overall significance of the estimated regression, is also a test of significance of  $R^2$ .* In other words, testing the null hypothesis (8.5.9) is equivalent to testing the null hypothesis that (the population)  $R^2$  is zero.

**TABLE 8.4** ANOVA TABLE IN TERMS OF  $R^2$ 

Source of variation	SS	df	MSS*
Due to regression	$R^2(\sum y_i^2)$	2	$R^2(\sum y_i^2)/2$
Due to residuals	$(1 - R^2)(\sum y_i^2)$	$n - 3$	$(1 - R^2)(\sum y_i^2)/(n - 3)$
Total	$\sum y_i^2$	$n - 1$	

\*Note that in computing the  $F$  value there is no need to multiply  $R^2$  and  $(1 - R^2)$  by  $\sum y_i^2$  because it drops out, as shown in (8.5.12).

For the three-variable case (8.5.11) becomes

$$F = \frac{R^2/2}{(1 - R^2)/(n - 3)} \quad (8.5.12)$$

By virtue of the close connection between  $F$  and  $R^2$ , the ANOVA Table 8.1 can be recast as Table 8.4.

For our illustrative example, using (8.5.12) we obtain:

$$F = \frac{0.7077/2}{(1 - 0.7077)/61} = 73.8726$$

which is about the same as obtained before, except for the rounding errors.

One advantage of the  $F$  test expressed in terms of  $R^2$  is its ease of computation: All that one needs to know is the  $R^2$  value. Therefore, the overall  $F$  test of significance given in (8.5.7) can be recast in terms of  $R^2$  as shown in Table 8.4.

### Testing the Overall Significance of a Multiple Regression in Terms of $R^2$

**Decision Rule.** Testing the overall significance of a regression in terms of  $R^2$ : Alternative but equivalent test to (8.5.7).

Given the  $k$ -variable regression model:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i$$

To test the hypothesis

$$H_0: \beta_2 = \beta_3 = \cdots = \beta_k = 0$$

versus

$$H_1: \text{Not all slope coefficients are simultaneously zero}$$

compute

$$F = \frac{R^2/(k - 1)}{(1 - R^2)/(n - k)} \quad (8.5.13)$$

If  $F > F_{\alpha(k-1, n-k)}$ , reject  $H_0$ ; otherwise you may accept  $H_0$  where  $F_{\alpha(k-1, n-k)}$  is the critical  $F$  value at the  $\alpha$  level of significance and  $(k - 1)$  numerator df and  $(n - k)$  denominator df. Alternatively, if the  $p$  value of  $F$  obtained from (8.5.13) is sufficiently low, reject  $H_0$ .

Before moving on, return to Example 7.5 in Chapter 7. From regression (7.10.7) we observe that RGDP (relative per capita GDP) and RGDP squared explain only about 5.3 percent of the variation in GDPG (GDP growth rate) in a sample of 119 countries. This  $R^2$  of 0.053 seems a “low” value. Is it really statistically different from zero? How do we find that out?

Recall our earlier discussion in “An Important Relationship between  $R^2$  and  $F$ ” about the relationship between  $R^2$  and the  $F$  value as given in (8.5.11) or (8.5.12) for the specific case of two regressors. As noted, if  $R^2$  is zero, then  $F$  is zero ipso facto, which will be the case if the regressors have no impact whatsoever on the regressand. Therefore, if we insert  $R^2 = 0.053$  into formula (8.5.12), we obtain

$$F = \frac{0.053/2}{(1 - 0.053)/116} = 3.2475 \quad (8.5.13)$$

Under the null hypothesis that  $R^2 = 0$ , the preceding  $F$  value follows the  $F$  distribution with 2 and 116 df in the numerator, respectively. (*Note:* There are 119 observations and two regressors.) From the  $F$  table we see that this  $F$  value is significant at about the 5 percent level; the  $p$  value is actually 0.0425. Therefore, we can reject the null hypothesis that the two regressors have no impact on the regressand, notwithstanding the fact that the  $R^2$  is only 0.053.

This example brings out an important empirical observation that in cross-sectional data involving several observations, one generally obtains low  $R^2$  because of the diversity of the cross-sectional units. Therefore, one should not be surprised or worried about finding low  $R^2$ 's in cross-sectional regressions. What is relevant is that the model is correctly specified, that the regressors have the correct (i.e., theoretically expected) signs, and that (hopefully) the regression coefficients are statistically significant. The reader should check that *individually* both the regressors in (7.10.7) are statistically significant at the 5 percent or better level (i.e., lower than 5 percent).

### The “Incremental” or “Marginal” Contribution of an Explanatory Variable

In Chapter 7 we stated that generally we cannot allocate the  $R^2$  value among the various regressors. In our child mortality example we found that the  $R^2$  was 0.7077 but we cannot say what part of this value is due to the regressor PGNP and what part is due to female literacy rate (FLR) because of possible correlation between the two regressors in the sample at hand. We can shed more light on this using the analysis of covariance technique.

For our illustrative example we found that *individually*  $X_2$  (PGNP) and  $X_3$  (FLR) were statistically significant on the basis of (*separate*)  $t$  tests. We have also found that on the basis of the  $F$  test *collectively* both the regressors have a significant effect on the regressand  $Y$  (child mortality).

Now suppose we introduce PGNP and FLR *sequentially*; that is, we first regress child mortality on PGNP and assess its significance and then add FLR to the model to find out whether it contributes anything (of course, the order in which PGNP and FLR enter can be reversed). By contribution we mean whether the addition of the variable to the model increases ESS (and hence  $R^2$ ) “significantly” in relation to the RSS. This contribution may appropriately be called the **incremental**, or **marginal**, contribution of an explanatory variable.

The topic of incremental contribution is an important one in practice. In most empirical investigations the researcher may not be completely sure whether it is worth adding an  $X$  variable to the model knowing that several other  $X$  variables are already present in the model. One does not wish to include variable(s) that contribute very little toward ESS. By the same token, one does not want to exclude variable(s) that substantially increase ESS. But how does one decide whether an  $X$  variable significantly reduces RSS? The analysis of variance technique can be easily extended to answer this question.

Suppose we first regress child mortality on PGNP and obtain the following regression:

$$\begin{aligned} \widehat{CM}_i &= 157.4244 - 0.0114 \text{ PGNP} && \text{(8.5.14)} \\ t &= (15.9894) \quad (-3.5156) && r^2 = 0.1662 \\ p \text{ value} &= (0.0000) \quad (0.0008) && \text{adj } r^2 = 0.1528 \end{aligned}$$

As these results show, PGNP has a significant effect on CM. The ANOVA table corresponding to the preceding regression is given in Table 8.5.

Assuming the disturbances  $u_i$  are normally distributed and the hypothesis that PGNP has no effect on CM, we obtain the  $F$  value of

$$F = \frac{60,449.5}{4890.7822} = 12.3598 \quad \text{(8.5.15)}$$

**TABLE 8.5** ANOVA TABLE FOR REGRESSION (8.5.14)

Source of variation	SS	df	MSS
ESS (due to PGNP)	60,449.5	1	60,449.5
RSS	303,228.5	62	4890.7822
Total	363,678	63	

which follows the  $F$  distribution with 1 and 62 df. This  $F$  value is highly significant, as the computed  $p$  value is 0.0008. Thus, as before, we reject the hypothesis that PGNP has no effect on CM. Incidentally, note that  $t^2 = (-3.5156)^2 = 12.3594$ , which is approximately the same as the  $F$  value of (8.5.15), where the  $t$  value is obtained from (8.5.14). But this should not be surprising in view of the fact that the square of the  $t$  statistic with  $n$  df is equal to the  $F$  value with 1 df in the numerator and  $n$  df in the denominator, a relationship first established in Chapter 5. Note that in the present example,  $n = 64$ .

Having run the regression (8.5.14), let us suppose we decide to add FLR to the model and obtain the multiple regression (8.2.1). The questions we want to answer are:

1. What is the marginal, or incremental, contribution of FLR, knowing that PGNP is already in the model and that it is significantly related to CM?
2. Is the incremental contribution of FLR statistically significant?
3. What is the criterion for adding variables to the model?

The preceding questions can be answered by the ANOVA technique. To see this, let us construct Table 8.6. In this table  $X_2$  refers to PGNP and  $X_3$  refers to FLR.

To assess the *incremental* contribution of  $X_3$  after allowing for the contribution of  $X_2$ , we form

$$\begin{aligned}
 F &= \frac{Q_2/\text{df}}{Q_4/\text{df}} \\
 &= \frac{\text{ESS}_{\text{new}} - \text{ESS}_{\text{old}}/\text{number of new regressors}}{\text{RSS}_{\text{new}}/\text{df} (= n - \text{number of parameters in the new model})} \\
 &= \frac{Q_2/1}{Q_4/12} \text{ for our example}
 \end{aligned}$$

(8.5.16)

**TABLE 8.6** ANOVA TABLE TO ASSESS INCREMENTAL CONTRIBUTION OF A VARIABLE(S)

Source of variation	SS	df	MSS
ESS due to $X_2$ alone	$Q_1 = \hat{\beta}_{12}^2 \sum x_2^2$	1	$\frac{Q_1}{1}$
ESS due to the addition of $X_3$	$Q_2 = Q_3 - Q_1$	1	$\frac{Q_2}{1}$
ESS due to both $X_2, X_3$	$Q_3 = \hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i}$	2	$\frac{Q_3}{2}$
RSS	$Q_4 = Q_5 - Q_3$	$n - 3$	$\frac{Q_4}{n - 3}$
Total	$Q_5 = \sum y_i^2$	$n - 1$	



**TABLE 8.7** ANOVA TABLE FOR THE ILLUSTRATIVE EXAMPLE: INCREMENTAL ANALYSIS

Source of variation	SS	df	MSS
ESS due to PGNP	60,449.5	1	60,449.5
ESS due to the addition of FLR	196,912.9	1	196,912.9
ESS due to PGNP and FLR	257,362.4	2	128,681.2
RSS	106,315.6	61	1742.8786
Total	363,678	63	

where  $ESS_{\text{new}} = ESS$  under the new model (i.e., after adding the new regressors =  $Q_3$ ),  $ESS_{\text{old}} = ESS$  under the old model (=  $Q_1$ ) and  $RSS_{\text{new}} = RSS$  under the new model (i.e., after taking into account all the regressors =  $Q_4$ ). For our illustrative example the results are as shown in Table 8.7.

Now applying (8.5.16), we obtain:

$$F = \frac{196,912.9}{1742.8786} = 112.9814 \quad (8.5.17)$$

Under the usual assumptions, this  $F$  value follows the  $F$  distribution with 1 and 62 df. The reader should check that this  $F$  value is highly significant, suggesting that addition of FLR to the model significantly increases ESS and hence the  $R^2$  value. Therefore, FLR should be added to the model. Again, note that if you square the value of the FLR coefficient in the multiple regression (8.2.1), which is  $(-10.6293)^2$ , you will obtain the  $F$  value of (8.5.17), save for the rounding errors.

Incidentally, the  $F$  ratio of (8.5.16) can be recast by using the  $R^2$  values only, as we did in (8.5.13). As exercise 8.2 shows, the  $F$  ratio of (8.5.16) is *equivalent* to the following  $F$  ratio:<sup>9</sup>

$$F = \frac{(R_{\text{new}}^2 - R_{\text{old}}^2)/df}{(1 - R_{\text{new}}^2)/df} = \frac{(R_{\text{new}}^2 - R_{\text{old}}^2)/\text{number of new regressors}}{(1 - R_{\text{new}}^2)/df (= n - \text{number of parameters in the new model})} \quad (8.5.18)$$

This  $F$  ratio follows the  $F$  distribution with the appropriate numerator and denominator df, 1 and 61 in our illustrative example.

For our example,  $R_{\text{new}}^2 = 0.7077$  [from Eq. (8.2.1)] and  $R_{\text{old}}^2 = 0.1662$  [from Eq. (8.5.14)]. Therefore,

$$F = \frac{(0.7077 - 0.1662)/1}{(1 - 0.7077)/61} = 113.05 \quad (8.5.19)$$

<sup>9</sup>The following  $F$  test is a special case of the more general  $F$  test given in (8.7.9) or (8.7.10) in Sec. 8.7.

which is about the same as that obtained from (8.5.17), except for the rounding errors. This  $F$  is highly significant, reinforcing our earlier finding that the variable FLR belongs in the model.

*A cautionary note:* If you use the  $R^2$  version of the  $F$  test given in (8.5.11), make sure that the dependent variable in the new and the old models is the same. If they are different, use the  $F$  test given in (8.5.16).

**When to Add a New Variable.** The  $F$ -test procedure just outlined provides a formal method of deciding whether a variable should be added to a regression model. Often researchers are faced with the task of choosing from several competing models **involving the same dependent variable** but with different explanatory variables. As a matter of ad hoc choice (because very often the theoretical foundation of the analysis is weak), these researchers frequently choose the model that gives the highest adjusted  $R^2$ . Therefore, if the inclusion of a variable increases  $\bar{R}^2$ , it is retained in the model although it does not reduce RSS significantly in the statistical sense. The question then becomes: When does the adjusted  $R^2$  increase? It can be shown that  $\bar{R}^2$  will increase if the  $t$  value of the coefficient of the newly added variable is larger than 1 in absolute value, where the  $t$  value is computed under the hypothesis that the population value of the said coefficient is zero [i.e., the  $t$  value computed from (5.3.2) under the hypothesis that the true  $\beta$  value is zero].<sup>10</sup> The preceding criterion can also be stated differently:  $\bar{R}^2$  will increase with the addition of an extra explanatory variable only if the  $F (= t^2)$  value of that variable exceeds 1.

Applying either criterion, the FLR variable in our child mortality example with a  $t$  value of  $-10.6293$  or an  $F$  value of  $112.9814$  should increase  $\bar{R}^2$ , which indeed it does—when FLR is added to the model,  $\bar{R}^2$  increases from  $0.1528$  to  $0.6981$ .

**When to Add a Group of Variables.** Can we develop a similar rule for deciding whether it is worth adding (or dropping) a group of variables from a model? The answer should be apparent from (8.5.18): *If adding (dropping) a group of variables to the model gives an  $F$  value greater (less) than 1,  $R^2$  will increase (decrease).* Of course, from (8.5.18) one can easily find out whether the addition (subtraction) of a group of variables significantly increases (decreases) the explanatory power of a regression model.

## 8.6 TESTING THE EQUALITY OF TWO REGRESSION COEFFICIENTS

Suppose in the multiple regression

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + u_i \quad (8.6.1)$$

<sup>10</sup>For proof, see Dennis J. Aigner, *Basic Econometrics*, Prentice Hall, Englewood Cliffs, N.J., 1971, pp. 91–92.

we want to test the hypotheses

$$\begin{aligned} H_0: \beta_3 &= \beta_4 & \text{or} & & (\beta_3 - \beta_4) &= 0 \\ H_1: \beta_3 &\neq \beta_4 & \text{or} & & (\beta_3 - \beta_4) &\neq 0 \end{aligned} \quad (8.6.2)$$

that is, the two slope coefficients  $\beta_3$  and  $\beta_4$  are equal.

Such a null hypothesis is of practical importance. For example, let (8.6.1) represent the demand function for a commodity where  $Y$  = amount of a commodity demanded,  $X_2$  = price of the commodity,  $X_3$  = income of the consumer, and  $X_4$  = wealth of the consumer. The null hypothesis in this case means that the income and wealth coefficients are the same. Or, if  $Y_i$  and the  $X$ 's are expressed in logarithmic form, the null hypothesis in (8.6.2) implies that the income and wealth elasticities of consumption are the same. (Why?)

How do we test such a null hypothesis? Under the classical assumptions, it can be shown that

$$t = \frac{(\hat{\beta}_3 - \hat{\beta}_4) - (\beta_3 - \beta_4)}{\text{se}(\hat{\beta}_3 - \hat{\beta}_4)} \quad (8.6.3)$$

follows the  $t$  distribution with  $(n - 4)$  df because (8.6.1) is a four-variable model or, more generally, with  $(n - k)$  df, where  $k$  is the total number of parameters estimated, including the constant term. The  $\text{se}(\hat{\beta}_3 - \hat{\beta}_4)$  is obtained from the following well-known formula (see **Appendix A** for details):

$$\text{se}(\hat{\beta}_3 - \hat{\beta}_4) = \sqrt{\text{var}(\hat{\beta}_3) + \text{var}(\hat{\beta}_4) - 2 \text{cov}(\hat{\beta}_3, \hat{\beta}_4)} \quad (8.6.4)$$

If we substitute the null hypothesis and the expression for the  $\text{se}(\hat{\beta}_3 - \hat{\beta}_4)$  into (8.6.3), our test statistic becomes

$$t = \frac{\hat{\beta}_3 - \hat{\beta}_4}{\sqrt{\text{var}(\hat{\beta}_3) + \text{var}(\hat{\beta}_4) - 2 \text{cov}(\hat{\beta}_3, \hat{\beta}_4)}} \quad (8.6.5)$$

Now the testing procedure involves the following steps:

1. Estimate  $\hat{\beta}_3$  and  $\hat{\beta}_4$ . Any standard computer package can do that.
2. Most standard computer packages routinely compute the variances and covariances of the estimated parameters.<sup>11</sup> From these estimates the standard error in the denominator of (8.6.5) can be easily obtained.
3. Obtain the  $t$  ratio from (8.6.5). Note the null hypothesis in the present case is  $(\beta_3 - \beta_4) = 0$ .
4. If the  $t$  variable computed from (8.6.5) exceeds the critical  $t$  value at the designated level of significance for given df, then you can reject the null hypothesis; otherwise, you do not reject it. Alternatively, if the  $p$  value of the

<sup>11</sup>The algebraic expression for the covariance formula is rather involved. **Appendix C** provides a compact expression for it, however, using matrix notation.

$t$  statistic from (8.6.5) is reasonably low, one can reject the null hypothesis. Note that the lower the  $p$  value, the greater the evidence against the null hypothesis. Therefore, when we say that a  $p$  value is low or reasonably low, we mean that it is less than the significance level, such as 10, 5, or 1 percent. Some personal judgment is involved in this decision.

**EXAMPLE 8.2**

## THE CUBIC COST FUNCTION REVISITED

Recall the cubic total cost function estimated in Section 7.10, which for convenience is reproduced below:

$$\begin{aligned} \hat{Y}_i &= 141.7667 + 63.4777X_i - 12.9615X_i^2 + 0.9396X_i^3 \\ \text{se} &= (6.3753) \quad (4.7786) \quad (0.9857) \quad (0.0591) \quad (7.10.6) \\ \text{cov}(\hat{\beta}_3, \hat{\beta}_4) &= -0.0576; \quad R^2 = 0.9983 \end{aligned}$$

where  $Y$  is total cost and  $X$  is output, and where the figures in parentheses are the estimated standard errors.

Suppose we want to test the hypothesis that the coefficients of the  $X^2$  and  $X^3$  terms in the cubic cost function are the same, that is,  $\beta_3 = \beta_4$  or  $(\beta_3 - \beta_4) = 0$ . In the regression (7.10.6) we have all the necessary output to conduct the  $t$  test of (8.6.5). The actual mechanics are as follows:

$$\begin{aligned} t &= \frac{\hat{\beta}_3 - \hat{\beta}_4}{\sqrt{\text{var}(\hat{\beta}_3) + \text{var}(\hat{\beta}_4) - 2 \text{cov}(\hat{\beta}_3, \hat{\beta}_4)}} \\ &= \frac{-12.9615 - 0.9396}{\sqrt{(0.9867)^2 + (0.0591)^2 - 2(-0.0576)}} \quad (8.6.6) \\ &= \frac{-13.9011}{1.0442} = -13.3130 \end{aligned}$$

The reader can verify that for 6 df (why?) the observed  $t$  value exceeds the critical  $t$  value even at the 0.002 (or 0.2 percent) level of significance (two-tail test); the  $p$  value is extremely small, 0.000006. Hence we can reject the hypothesis that the coefficients of  $X^2$  and  $X^3$  in the cubic cost function are identical.

**8.7 RESTRICTED LEAST SQUARES: TESTING LINEAR  
EQUALITY RESTRICTIONS**

There are occasions where economic theory may suggest that the coefficients in a regression model satisfy some linear equality restrictions. For instance, consider the Cobb–Douglas production function:

$$Y_i = \beta_1 X_{2i}^{\beta_2} X_{3i}^{\beta_3} e^{u_i} \quad (7.9.1) = (8.7.1)$$

where  $Y$  = output,  $X_2$  = labor input, and  $X_3$  = capital input. Written in log form, the equation becomes

$$\ln Y_i = \beta_0 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i \quad (8.7.2)$$

where  $\beta_0 = \ln \beta_1$ .

Now if there are constant returns to scale (equiproportional change in output for an equiproportional change in the inputs), economic theory would suggest that

$$\beta_2 + \beta_3 = 1 \quad (8.7.3)$$

which is an example of a linear equality restriction.<sup>12</sup>

How does one find out if there are constant returns to scale, that is, if the restriction (8.7.3) is valid? There are two approaches.

### The *t*-Test Approach

The simplest procedure is to estimate (8.7.2) in the usual manner without taking into account the restriction (8.7.3) explicitly. This is called the **unrestricted** or **unconstrained regression**. Having estimated  $\beta_2$  and  $\beta_3$  (say, by OLS method), a test of the hypothesis or restriction (8.7.3) can be conducted by the *t* test of (8.6.3), namely,

$$\begin{aligned} t &= \frac{(\hat{\beta}_2 + \hat{\beta}_3) - (\beta_2 + \beta_3)}{\text{se}(\hat{\beta}_2 + \hat{\beta}_3)} \\ &= \frac{(\hat{\beta}_2 + \hat{\beta}_3) - 1}{\sqrt{\text{var}(\hat{\beta}_2) + \text{var}(\hat{\beta}_3) + 2 \text{cov}(\hat{\beta}_2, \hat{\beta}_3)}} \end{aligned} \quad (8.7.4)$$

where  $(\beta_2 + \beta_3) = 1$  under the null hypothesis and where the denominator is the standard error of  $(\hat{\beta}_2 + \hat{\beta}_3)$ . Then following Section 8.6, if the *t* value computed from (8.7.4) exceeds the critical *t* value at the chosen level of significance, we reject the hypothesis of constant returns to scale; otherwise we do not reject it.

### The *F*-Test Approach: Restricted Least Squares

The preceding *t* test is a kind of postmortem examination because we try to find out whether the linear restriction is satisfied after estimating the “unrestricted” regression. A direct approach would be to incorporate the restriction (8.7.3) into the estimating procedure at the outset. In the present example, this procedure can be done easily. From (8.7.3) we see that

$$\beta_2 = 1 - \beta_3 \quad (8.7.5)$$

or

$$\beta_3 = 1 - \beta_2 \quad (8.7.6)$$

Therefore, using either of these equalities, we can eliminate one of the  $\beta$  coefficients in (8.7.2) and estimate the resulting equation. Thus, if we use (8.7.5), we can write the Cobb–Douglas production function as

$$\begin{aligned} \ln Y_i &= \beta_0 + (1 - \beta_3) \ln X_{2i} + \beta_3 \ln X_{3i} + u_i \\ &= \beta_0 + \ln X_{2i} + \beta_3 (\ln X_{3i} - \ln X_{2i}) + u_i \end{aligned}$$

<sup>12</sup>If we had  $\beta_2 + \beta_3 < 1$ , this relation would be an example of a linear inequality restriction. To handle such restrictions, one needs to use mathematical programming techniques.

or

$$(\ln Y_i - \ln X_{2i}) = \beta_0 + \beta_3(\ln X_{3i} - \ln X_{2i}) + u_i \quad (8.7.7)$$

or

$$\ln(Y_i/X_{2i}) = \beta_0 + \beta_3 \ln(X_{3i}/X_{2i}) + u_i \quad (8.7.8)$$

where  $(Y_i/X_{2i})$  = output/labor ratio and  $(X_{3i}/X_{2i})$  = capital labor ratio, quantities of great economic importance.

Notice how the original equation (8.7.2) is transformed. Once we estimate  $\beta_3$  from (8.7.7) or (8.7.8),  $\beta_2$  can be easily estimated from the relation (8.7.5). Needless to say, this procedure will guarantee that the sum of the estimated coefficients of the two inputs will equal 1. The procedure outlined in (8.7.7) or (8.7.8) is known as **restricted least squares (RLS)**. This procedure can be generalized to models containing any number of explanatory variables and more than one linear equality restriction. The generalization can be found in Theil.<sup>13</sup> (See also general  $F$  testing below.)

How do we compare the unrestricted and restricted least-squares regressions? In other words, how do we know that, say, the restriction (8.7.3) is valid? This question can be answered by applying the  $F$  test as follows. Let

$$\sum \hat{u}_{UR}^2 = \text{RSS of the unrestricted regression (8.7.2)}$$

$$\sum \hat{u}_R^2 = \text{RSS of the restricted regression (8.7.7)}$$

$m$  = number of linear restrictions (1 in the present example)

$k$  = number of parameters in the unrestricted regression

$n$  = number of observations

Then,

$$F = \frac{(\text{RSS}_R - \text{RSS}_{UR})/m}{\text{RSS}_{UR}/(n-k)} \quad (8.7.9)$$

$$= \frac{(\sum \hat{u}_R^2 - \sum \hat{u}_{UR}^2)/m}{\sum \hat{u}_{UR}^2/(n-k)}$$

follows the  $F$  distribution with  $m$ ,  $(n - k)$  df. (Note: UR and R stand for unrestricted and restricted, respectively.)

The  $F$  test above can also be expressed in terms of  $R^2$  as follows:

$$F = \frac{(R_{UR}^2 - R_R^2)/m}{(1 - R_{UR}^2)/(n-k)} \quad (8.7.10)$$

where  $R_{UR}^2$  and  $R_R^2$  are, respectively, the  $R^2$  values obtained from the unrestricted and restricted regressions, that is, from the regressions (8.7.2) and

<sup>13</sup>Henri Theil, *Principles of Econometrics*, John Wiley & Sons, New York, 1971, pp. 43–45.

(8.7.7). It should be noted that

$$R_{UR}^2 \geq R_R^2 \quad (8.7.11)$$

and

$$\sum \hat{u}_{UR}^2 \leq \sum \hat{u}_R^2 \quad (8.7.12)$$

In exercise 8.4 you are asked to justify these statements.

**A Cautionary Note:** In using (8.7.10) keep in mind that if the dependent variable in the restricted and unrestricted models is not the same,  $R_{UR}^2$  and  $R_R^2$  are not directly comparable. In that case, use the procedure described in Chapter 7 to render the two  $R^2$  values comparable (see Example 8.3 below) or use the  $F$  test given in (8.7.9).

### EXAMPLE 8.3

#### THE COBB–DOUGLAS PRODUCTION FUNCTION FOR THE MEXICAN ECONOMY, 1955–1974

By way of illustrating the preceding discussion consider the data given in Table 8.8. Attempting to fit the Cobb–Douglas production function to these data, yielded the following results:

$$\begin{aligned} \widehat{\ln \text{GDP}}_t &= -1.6524 + 0.3397 \ln \text{Labor}_t + 0.8460 \ln \text{Capital}_t & (8.7.13) \\ t &= (-2.7259) \quad (1.8295) \quad (9.0625) \\ p \text{ value} &= (0.0144) \quad (0.0849) \quad (0.0000) \\ & R^2 = 0.9951 \quad \text{RSS}_{UR} = 0.0136 \end{aligned}$$

where  $\text{RSS}_{UR}$  is the unrestricted RSS, as we have put no restrictions on estimating (8.7.13).

We have already seen in Chapter 7 how to interpret the coefficients of the Cobb–Douglas production function. As you can see, the output/labor elasticity is about 0.34 and the output/capital elasticity is about 0.85. If we add these coefficients, we obtain 1.19, suggesting that perhaps the Mexican economy during the stated time period was experiencing increasing returns to scale. Of course, we do not know if 1.19 is statistically different from 1.

To see if that is the case, let us impose the restriction of constant returns to scale, which gives the following regression:

$$\begin{aligned} \widehat{\ln (\text{GDP}/\text{Labor})}_t &= -0.4947 + 1.0153 \ln (\text{Capital}/\text{Labor})_t & (8.7.14) \\ t &= (-4.0612) \quad (28.1056) \\ p \text{ value} &= (0.0007) \quad (0.0000) \\ & R_R^2 = 0.9777 \quad \text{RSS}_R = 0.0166 \end{aligned}$$

where  $\text{RSS}_R$  is the restricted RSS, for we have imposed the restriction that there are constant returns to scale.

(Continued)

**EXAMPLE 9.3** (Continued)

Since the dependent variable in the preceding two regressions is different, we have to use the  $F$  test given in (8.7.9). We have the necessary data to obtain the  $F$  value.

$$\begin{aligned} F &= \frac{(RSS_R - RSS_{UR})/m}{RSS_{UR}/(n-k)} \\ &= \frac{(0.0166 - 0.0136)/1}{(0.0136)/(20 - 3)} \\ &= 3.75 \end{aligned}$$

Note in the present case  $m = 1$ , as we have imposed only one restriction and  $(n - k)$  is 17, since we have 20 observations and three parameters in the unrestricted regression.

This  $F$  value follows the  $F$  distribution with 1 df in the numerator and 17 df in the denominator. The reader can easily check that this  $F$  value is not significant at the 5% level. (See **Appendix D**, Table D.3.)

The conclusion then is that the Mexican economy was probably characterized by constant returns to scale over the sample period and therefore there may be no harm in using the restricted regression given in (8.7.14). As this regression shows, if capital/labor ratio increased by 1 percent, on average, labor productivity went up by about 1 percent.

**TABLE 8.8**  
REAL GDP, EMPLOYMENT, AND REAL FIXED CAPITAL—MEXICO

Year	GDP*	Employment <sup>†</sup>	Fixed capital <sup>‡</sup>
1955	114043	8310	182113
1956	120410	8529	193749
1957	129187	8738	205192
1958	134705	8952	215130
1959	139960	9171	225021
1960	150511	9569	237026
1961	157897	9527	248897
1962	165286	9662	260661
1963	178491	10334	275466
1964	199457	10981	295378
1965	212323	11746	315715
1966	226977	11521	337642
1967	241194	11540	363599
1968	260881	12066	391847
1969	277498	12297	422382
1970	296530	12955	455049
1971	306712	13338	484677
1972	329030	13738	520553
1973	354057	15924	561531
1974	374977	14154	609825

\*Millions of 1960 pesos;

<sup>†</sup>Thousands of people;

<sup>‡</sup>Millions of 1960 pesos.

Source: Victor J. Elias, *Sources of Growth: A Study of Seven Latin American Economies*, International Center for Economic Growth, ICS Press, San Francisco, 1992. Data from Tables E5, E12, and E14.



**General  $F$  Testing<sup>14</sup>**

The  $F$  test given in (8.7.10) or its equivalent (8.7.9) provides a general method of testing hypotheses about one or more parameters of the  $k$ -variable regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i \quad (8.7.15)$$

The  $F$  test of (8.5.16) or the  $t$  test of (8.6.3) is but a specific application of (8.7.10). Thus, hypotheses such as

$$H_0: \beta_2 = \beta_3 \quad (8.7.16)$$

$$H_0: \beta_3 + \beta_4 + \beta_5 = 3 \quad (8.7.17)$$

which involve some linear restrictions on the parameters of the  $k$ -variable model, or hypotheses such as

$$H_0: \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0 \quad (8.7.18)$$

which imply that some regressors are absent from the model, can all be tested by the  $F$  test of (8.7.10).

From the discussion in Sections 8.5 and 8.7, the reader will have noticed that the general strategy of  $F$  testing is this: There is a larger model, the *unconstrained model* (8.7.15), and then there is a smaller model, the *constrained* or *restricted model*, which is obtained from the larger model by deleting some variables from it, e.g., (8.7.18), or by putting some linear restrictions on one or more coefficients of the larger model, e.g., (8.7.16) or (8.7.17).

We then fit the unconstrained and constrained models to the data and obtain the respective coefficients of determination, namely,  $R_{UR}^2$  and  $R_R^2$ . We note the df in the unconstrained model ( $= n - k$ ) and also note the df in the constrained model ( $= m$ ),  $m$  being the number of linear restriction [e.g., 1 in (8.7.16) or (8.7.18)] or the number of regressors omitted from the model [e.g.,  $m = 4$  if (8.7.18) holds, since four regressors are assumed to be absent from the model]. We then compute the  $F$  ratio as indicated in (8.7.9) or (8.7.10) and use this *Decision Rule*: *If the computed  $F$  exceeds  $F_\alpha(m, n - k)$ , where  $F_\alpha(m, n - k)$  is the critical  $F$  at the  $\alpha$  level of significance, we reject the null hypothesis; otherwise we do not reject it.*

<sup>14</sup>If one is using the maximum likelihood approach to estimation, then a test similar to the one discussed shortly is the **likelihood ratio test**, which is slightly involved and is therefore discussed in the appendix to the chapter. For further discussion, see Theil, op. cit., pp. 179–184.

Let us illustrate:

**EXAMPLE 8.4**

THE DEMAND FOR CHICKEN IN THE UNITED STATES, 1960–1982

In exercise 7.19, among other things, you were asked to consider the following demand function for chicken:

$$\ln Y_t = \beta_1 + \beta_2 \ln X_{2t} + \beta_3 \ln X_{3t} + \beta_4 \ln X_{4t} + \beta_5 \ln X_{5t} + u_t \quad (8.7.19)$$

where  $Y$  = per capita consumption of chicken, lb,  $X_2$  = real disposable per capita income, \$,  $X_3$  = real retail price of chicken per lb, ¢,  $X_4$  = real retail price of pork per lb, ¢, and  $X_5$  = real retail price of beef per lb, ¢.

In this model  $\beta_2$ ,  $\beta_3$ ,  $\beta_4$ , and  $\beta_5$  are, respectively, the income, own-price, cross-price (pork), and cross-price (beef) elasticities. (Why?) According to economic theory,

$$\begin{aligned} \beta_2 &> 0 \\ \beta_3 &< 0 \\ \beta_4 &> 0, & \text{if chicken and pork are competing products} \\ &< 0, & \text{if chicken and pork are complementary products} \\ &= 0, & \text{if chicken and pork are unrelated products} \\ \beta_5 &> 0, & \text{if chicken and beef are competing products} \\ &< 0, & \text{if they are complementary products} \\ &= 0, & \text{if they are unrelated products} \end{aligned} \quad (8.7.20)$$

Suppose someone maintains that chicken and pork and beef are unrelated products in the sense that chicken consumption is not affected by the prices of pork and beef. In short,

$$H_0: \beta_4 = \beta_5 = 0 \quad (8.7.21)$$

Therefore, the constrained regression becomes

$$\ln Y_t = \beta_1 + \beta_2 \ln X_{2t} + \beta_3 \ln X_{3t} + u_t \quad (8.7.22)$$

Equation (8.7.19) is of course the unconstrained regression.

Using the data given in exercise 7.19, we obtain the following:

*Unconstrained regression*

$$\begin{aligned} \widehat{\ln Y}_t = & 2.1898 + 0.3425 \ln X_{2t} - 0.5046 \ln X_{3t} + 0.1485 \ln X_{4t} + 0.0911 \ln X_{5t} \\ & (0.1557) \quad (0.0833) \quad (0.1109) \quad (0.0997) \quad (0.1007) \\ & R_{UR}^2 = 0.9823 \quad (8.7.23) \end{aligned}$$

*Constrained regression*

$$\begin{aligned} \widehat{\ln Y}_t = & 2.0328 + 0.4515 \ln X_{2t} - 0.3772 \ln X_{3t} \\ & (0.1162) \quad (0.0247) \quad (0.0635) \\ & R_R^2 = 0.9801 \quad (8.7.24) \end{aligned}$$

(Continued)

**EXAMPLE 8.4** (Continued)

where the figures in parentheses are the estimated standard errors. *Note:* The  $R^2$  values of (8.7.23) and (8.7.24) are comparable since the dependent variable in the two models is the same.

Now the  $F$  ratio to test the hypothesis (8.7.21) is

$$F = \frac{(R_{UR}^2 - R_R^2) / m}{(1 - R_{UR}^2) / (n - k)} \quad (8.7.10)$$

The value of  $m$  in the present case is 2, since there are two restrictions involved:  $\beta_4 = 0$  and  $\beta_5 = 0$ . The denominator df,  $(n - k)$ , is 18, since  $n = 23$  and  $k = 5$  (5  $\beta$  coefficients).

Therefore, the  $F$  ratio is

$$\begin{aligned} F &= \frac{(0.9823 - 0.9801) / 2}{(1 - 0.9823) / 18} \\ &= 1.1224 \end{aligned} \quad (8.7.25)$$

which has the  $F$  distribution with 2 and 18 df.

At 5 percent, clearly this  $F$  value is not statistically significant [ $F_{0.5}(2, 18) = 3.55$ ]. The  $p$  value is 0.3472. Therefore, there is no reason to reject the null hypothesis—the demand for chicken does not depend on pork and beef prices. In short, we can accept the constrained regression (8.7.24) as representing the demand function for chicken.

Notice that the demand function satisfies a priori economic expectations in that the own-price elasticity is negative and that the income elasticity is positive. However, the estimated price elasticity, in absolute value, is statistically less than unity, implying that the demand for chicken is price inelastic. (Why?) Also, the income elasticity, although positive, is also statistically less than unity, suggesting that chicken is not a luxury item; by convention, an item is said to be a luxury item if its income elasticity is greater than one.

## 8.8 TESTING FOR STRUCTURAL OR PARAMETER STABILITY OF REGRESSION MODELS: THE CHOW TEST

When we use a regression model involving time series data, it may happen that there is a **structural change** in the relationship between the regressand  $Y$  and the regressors. By structural change, we mean that the values of the parameters of the model do not remain the same through the entire time period. Sometime the structural change may be due to external forces (e.g., the oil embargoes imposed by the OPEC oil cartel in 1973 and 1979 or the Gulf War of 1990–1991), or due to policy changes (such as the switch from a fixed exchange-rate system to a flexible exchange-rate system around 1973) or action taken by Congress (e.g., the tax changes initiated by President Reagan in his two terms in office or changes in the minimum wage rate) or to a variety of other causes.

How do we find out that a structural change has in fact occurred? To be specific, consider the data given in Table 8.9. This table gives data on disposable personal income and personal savings, in billions of dollars, for the United States for the period 1970–1995. Suppose we want to estimate a

**TABLE 8.9** SAVINGS AND PERSONAL DISPOSABLE INCOME (BILLIONS OF DOLLARS), UNITED STATES, 1970–1995

Observation	Savings	Income	Observation	Savings	Income
1970	61.0	727.1	1983	167.0	2522.4
1971	68.6	790.2	1984	235.7	2810.0
1972	63.6	855.3	1985	206.2	3002.0
1973	89.6	965.0	1986	196.5	3187.6
1974	97.6	1054.2	1987	168.4	3363.1
1975	104.4	1159.2	1988	189.1	3640.8
1976	96.4	1273.0	1989	187.8	3894.5
1977	92.5	1401.4	1990	208.7	4166.8
1978	112.6	1580.1	1991	246.4	4343.7
1979	130.1	1769.5	1992	272.6	4613.7
1980	161.8	1973.3	1993	214.4	4790.2
1981	199.1	2200.2	1994	189.4	5021.7
1982	205.5	2347.3	1995	249.3	5320.8

Source: *Economic Report of the President*, 1997, Table B-28, p. 332.

simple savings function that relates savings ( $Y$ ) to disposable personal income DPI ( $X$ ). Since we have the data, we can obtain an OLS regression of  $Y$  on  $X$ . But if we do that, we are maintaining that the relationship between savings and DPI has not changed much over the span of 26 years. That may be a tall assumption. For example, it is well known that in 1982 the United States suffered its worst peacetime recession. The civilian unemployment rate that year reached 9.7 percent, the highest since 1948. An event such as this might disturb the relationship between savings and DPI. To see if this happened, let us divide our sample data into two time periods: 1970–1981 and 1982–1995, the pre- and post-1982 recession periods.

Now we have three possible regressions:

$$\text{Time period 1970–1981: } Y_t = \lambda_1 + \lambda_2 X_t + u_{1t} \quad n_1 = 12 \quad (8.8.1)$$

$$\text{Time period 1982–1995: } Y_t = \gamma_1 + \gamma_2 X_t + u_{2t} \quad n_2 = 14 \quad (8.8.2)$$

$$\text{Time period 1970–1995: } Y_t = \alpha_1 + \alpha_2 X_t + u_t \quad n = (n_1 + n_2) = 26 \quad (8.8.3)$$

Regression (8.8.3) assumes that there is no difference between the two time periods and therefore estimates the relationship between savings and DPI for the entire time period consisting of 26 observations. In other words, this regression assumes that the intercept as well as the slope coefficient remains the same over the entire period; that is, there is no structural change. If this is in fact the situation, then  $\alpha_1 = \lambda_1 = \gamma_1$  and  $\alpha_2 = \lambda_2 = \gamma_2$ .

Regressions (8.8.1) and (8.8.2) assume that the regressions in the two time periods are different; that is, the intercept and the slope coefficients are different, as indicated by the subscripted parameters. In the preceding regressions, the  $u$ 's represent the error terms and the  $n$ 's represent the number of observations.

For the data given in Table 8.9, the empirical counterparts of the preceding three regressions are as follows:

$$\begin{aligned}\hat{Y}_t &= 1.0161 + 0.0803 X_t \\ t &= (0.0873) \quad (9.6015) && \text{(8.8.1a)} \\ R^2 &= 0.9021 \quad \text{RSS}_1 = 1785.032 \quad \text{df} = 10\end{aligned}$$

$$\begin{aligned}\hat{Y}_t &= 153.4947 + 0.0148X_t \\ t &= (4.6922) \quad (1.7707) && \text{(8.8.2a)} \\ R^2 &= 0.2971 \quad \text{RSS}_2 = 10,005.22 \quad \text{df} = 12\end{aligned}$$

$$\begin{aligned}\hat{Y}_t &= 62.4226 + 0.0376 X_t + \dots \\ t &= (4.8917) \quad (8.8937) + \dots && \text{(8.8.3a)} \\ R^2 &= 0.7672 \quad \text{RSS}_3 = 23,248.30 \quad \text{df} = 24\end{aligned}$$

In the preceding regressions, RSS denotes the residual sum of squares, and the figures in parentheses are the estimated  $t$  values.

A look at the estimated regressions suggests that the relationship between savings and DPI is not the same in the two subperiods. The slope in the preceding savings-income regressions represents the **marginal propensity to save (MPS)**, that is, the (mean) change in savings as a result of a dollar's increase in disposable personal income. In the period 1970–1981 the MPS was about 0.08, whereas in the period 1982–1995 it was about 0.02. Whether this change was due to the economic policies pursued by President Reagan is hard to say. This further suggests that the **pooled regression** (8.8.3a)—that is, the one that pools all the 26 observations and runs a common regression, disregarding possible differences in the two subperiods may not be appropriate. Of course, the preceding statements need to be supported by appropriate statistical test(s). Incidentally, the scattergrams and the estimated regression lines are as shown in Figure 8.3.

Now the possible differences, that is, structural changes, may be caused by differences in the intercept or the slope coefficient or both. How do we find that out? A visual feeling about this can be obtained as shown in Figure 8.2. But it would be useful to have a formal test.

This is where the **Chow test** comes in handy.<sup>15</sup> This test assumes that:

1.  $u_{1t} \sim N(0, \sigma^2)$  and  $u_{2t} \sim N(0, \sigma^2)$ . That is, the error terms in the subperiod regressions are normally distributed with the same (homoscedastic) variance  $\sigma^2$ .

<sup>15</sup>Gregory C. Chow, "Tests of Equality Between Sets of Coefficients in Two Linear Regressions," *Econometrica*, vol. 28, no. 3, 1960, pp. 591–605.

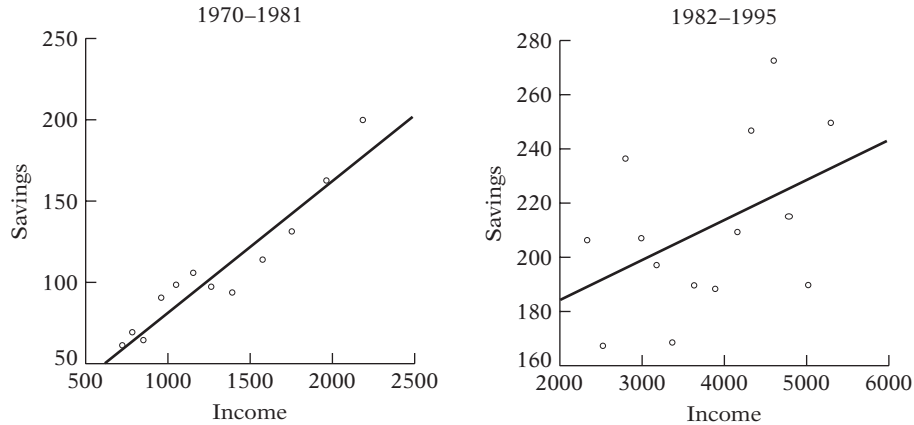


FIGURE 8.3

2. The two error terms  $u_{1t}$  and  $u_{2t}$  are independently distributed.

The mechanics of the Chow test are as follows:

1. Estimate regression (8.8.3), which is appropriate if there is no parameter instability, and obtain  $RSS_3$  with  $df = (n_1 + n_2 - k)$ , where  $k$  is the number of parameters estimated, 2 in the present case. For our example  $RSS_3 = 23,248.30$ . We call  $RSS_3$  the **restricted residual sum of squares (RSS<sub>R</sub>)** because it is obtained by imposing the restrictions that  $\lambda_1 = \gamma_1$  and  $\lambda_2 = \gamma_2$ , that is, the subperiod regressions are not different.

2. Estimate (8.8.1) and obtain its residual sum of squares,  $RSS_1$ , with  $df = (n_1 - k)$ . In our example,  $RSS_1 = 1785.032$  and  $df = 10$ .

3. Estimate (8.8.2) and obtain its residual sum of squares,  $RSS_2$ , with  $df = (n_2 - k)$ . In our example,  $RSS_2 = 10,005.22$  with  $df = 12$ .

4. Since the two sets of samples are deemed independent, we can add  $RSS_1$  and  $RSS_2$  to obtain what may be called the **unrestricted residual sum of squares (RSS<sub>UR</sub>)**, that is, obtain:

$$RSS_{UR} = RSS_1 + RSS_2 \quad \text{with } df = (n_1 + n_2 - 2k)$$

In the present case,

$$RSS_{UR} = (1785.032 + 10,005.22) = 11,790.252$$

5. Now the idea behind the Chow test is that if in fact there is no structural change [i.e., regressions (8.8.1) and (8.8.2) are essentially the same], then the  $RSS_R$  and  $RSS_{UR}$  should not be statistically different. Therefore, if we form the following ratio:

$$F = \frac{(RSS_R - RSS_{UR})/k}{(RSS_{UR})/(n_1 + n_2 - 2k)} \sim F_{[k, (n_1 + n_2 - 2k)]} \quad (8.8.4)$$

then Chow has shown that under the null hypothesis the regressions (8.8.1) and (8.8.2) are (statistically) the same (i.e., no structural change or break) and the  $F$  ratio given above follows the  $F$  distribution with  $k$  and  $(n_1 + n_2 - 2k)$  df in the numerator and denominator, respectively.

6. Therefore, we do not reject the null hypothesis of *parameter stability* (i.e., no structural change) if the computed  $F$  value in an application does not exceed the critical  $F$  value obtained from the  $F$  table at the chosen level of significance (or the  $p$  value). In this case we may be justified in using the pooled (restricted?) regression (8.8.3). Contrarily, if the computed  $F$  value exceeds the critical  $F$  value, we reject the hypothesis of parameter stability and conclude that the regressions (8.8.1) and (8.8.2) are different, in which case the pooled regression (8.8.3) is of dubious value, to say the least.

Returning to our example, we find that

$$\begin{aligned} F &= \frac{(23,248.30 - 11,790.252)/2}{(11,790.252)/22} \\ &= 10.69 \end{aligned} \tag{8.8.5}$$

From the  $F$  tables, we find that for 2 and 22 df the 1 percent critical  $F$  value is 5.72. Therefore, the probability of obtaining an  $F$  value of as much as or greater than 10.69 is much smaller than 1 percent; actually the  $p$  value is only 0.00057.

The Chow test therefore seems to support our earlier hunch that the savings-income relation has undergone a structural change in the United States over the period 1970–1995, assuming that the assumptions underlying the test are fulfilled. We will have more to say about this shortly.

Incidentally, note that the Chow test can be easily generalized to handle cases of more than one structural break. For example, if we believe that the savings-income relation changed after President Clinton took office in January 1992, we could divide our sample into three periods: 1970–1981, 1982–1991, 1992–1995, and carry out the Chow test. Of course, we will have four RSS terms, one for each subperiod and one for the pooled data. But the logic of the test remains the same. Data through 2001 are now available to extend the last period to 2001.

There are some caveats about the Chow test that must be kept in mind:

1. The assumptions underlying the test must be fulfilled. For example, one should find out if the error variances in the regressions (8.8.1) and (8.8.2) are the same. We will discuss this point shortly.

2. The Chow test will tell us only if the two regressions (8.8.1) and (8.8.2) are different, without telling us whether the difference is on account of the intercepts, or the slopes, or both. But in Chapter 9, on dummy variables, we will see how we can answer this question.

3. The Chow test assumes that we know the point(s) of structural break. In our example, we assumed it to be in 1982. However, if it is not possible to

determine when the structural change actually took place, we may have to use other methods.<sup>16</sup>

Before we leave the Chow test and our savings–income regression, let us examine one of the assumptions underlying the Chow test, namely, that the error variances in the two periods are the same. Since we cannot observe the true error variances, we can obtain their estimates from the RSS given in the regressions (8.8.1a) and (8.8.2a), namely,

$$\hat{\sigma}_1^2 = \frac{\text{RSS}_1}{n_1 - 2} = \frac{1785.032}{10} = 178.5032 \quad (8.8.6)$$

$$\hat{\sigma}_2^2 = \frac{\text{RSS}_2}{n_2 - 2} = \frac{10,005.22}{14 - 2} = 833.7683 \quad (8.8.7)$$

Notice that, since there are two parameters estimated in each equation, we deduct 2 from the number of observations to obtain the df. Given the assumptions underlying the Chow test,  $\hat{\sigma}_1^2$  and  $\hat{\sigma}_2^2$  are unbiased estimators of the true variances in the two subperiods. As a result, it can be shown that  $\sigma_1^2 = \sigma_2^2$ —that is, the variances in the two subpopulations are the same (as assumed by the Chow test)—then it can be shown that

$$\frac{(\hat{\sigma}_1^2/\sigma_1^2)}{(\hat{\sigma}_2^2/\sigma_2^2)} \sim F_{(n_1-k), (n_2-k)} \quad (8.8.8)$$

follows the  $F$  distribution with  $(n_1 - k)$  and  $(n_2 - k)$  df in the numerator and the denominator, respectively, in our example  $k = 2$ , since there are only two parameters in each subregression.

Of course,  $\sigma_1^2 = \sigma_2^2$ , the preceding  $F$  test reduces to computing

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} \quad (8.8.9)$$

*Note:* By convention we put the larger of the two estimated variances in the numerator. (See **Appendix A** for the details of the  $F$  and other probability distributions.)

Computing this  $F$  in an application and comparing it with the critical  $F$  value with the appropriate df, one can decide to reject or not reject the null hypothesis that the variances in the two subpopulations are the same. If the null hypothesis is not rejected, then one can use the Chow test.

Returning to our savings–income regression, we obtain the following result:

$$F = \frac{833.7683}{178.5032} = 4.6701 \quad (8.8.10)$$

<sup>16</sup>For a detailed discussion, see William H. Greene, *Econometric Analysis*, 4th ed., Prentice Hall, Englewood Cliffs, N.J., 2000, pp. 293–297.



Under the null hypothesis of equality of variances in the two subpopulations, this  $F$  value follows the  $F$  distribution with 12 and 10 df, in the numerator and denominator, respectively. (*Note:* We have put the larger of the two estimated variances in the numerator.) From the  $F$  tables in **Appendix D**, we see that the 5 and 1 percent critical  $F$  values for 12 and 10 df are 2.91 and 4.71, respectively. The computed  $F$  value is significant at the 5 percent level and is almost significant at the 1 percent level. Thus, our conclusion would be that the two subpopulation variances are not the same and, therefore, strictly speaking we should not use the Chow test.

Our purpose here has been to demonstrate the mechanics of the Chow test, which is used popularly in applied work. If the error variances in the two subpopulations are heteroscedastic, the Chow test can be modified. But the procedure is beyond the scope of this book.<sup>17</sup>

Another point we made earlier was that the Chow test is sensitive to the choice of the time at which the regression parameters might have changed. In our example, we assumed that the change probably took place in the recession year of 1982. If we had assumed it to be 1981, when Ronald Reagan began his presidency, we might find that the computed  $F$  value is different. As a matter of fact, in exercise 8.34 the reader is asked to check this out.

If we do not want to choose the point at which the break in the underlying relationship might have occurred, we could choose alternative methods, such as the **recursive residual test**. We will take this topic up in Chapter 13, the chapter on model specification analysis.

## 8.9 PREDICTION WITH MULTIPLE REGRESSION

In Section 5.10 we showed how the estimated two-variable regression model can be used for (1) *mean prediction*, that is, predicting the point on the population regression function (PRF), as well as for (2) *individual prediction*, that is, predicting an individual value of  $Y$  given the value of the regressor  $X = X_0$ , where  $X_0$  is the specified numerical value of  $X$ .

The estimated multiple regression too can be used for similar purposes, and the procedure for doing that is a straightforward extension of the two-variable case, except the formulas for estimating the variances and standard errors of the forecast value [comparable to (5.10.2) and (5.10.6) of the two-variable model] are rather involved and are better handled by the matrix methods discussed in **Appendix C**. Of course, most standard regression packages can do this routinely, so there is no need to look up the matrix formulation. It is given in **Appendix C** for the benefit of the mathematically inclined students. This appendix also gives a fully worked out example.

<sup>17</sup>For a discussion of the Chow test under heteroscedasticity, see William H. Greene, *Econometric Analysis*, 4th ed., Prentice Hall, Englewood Cliffs, N.J., 2000, pp. 292–293, and Adrian C. Darnell, *A Dictionary of Econometrics*, Edward Elgar, U.K., 1994, p. 51.

### \*8.10 THE TROIKA OF HYPOTHESIS TESTS: THE LIKELIHOOD RATIO (LR), WALD (W), AND LAGRANGE MULTIPLIER (LM) TESTS<sup>18</sup>

In this and the previous chapters we have, by and large, used the  $t$ ,  $F$ , and chi-square tests to test a variety of hypotheses in the context of linear (in-parameter) regression models. But once we go beyond the somewhat comfortable world of linear regression models, we need method(s) to test hypotheses that can handle regression models, linear or not.

The well-known trinity of **likelihood, Wald, and Lagrange multiplier tests** can accomplish this purpose. The interesting thing to note is that *asymptotically* (i.e., in large samples) all three tests are equivalent in that the test statistic associated with each of these tests follows the chi-square distribution.

Although we will discuss the **likelihood ratio test** in the appendix to this chapter, in general we will not use these tests in this textbook for the pragmatic reason that in small, or finite, samples, which is unfortunately what most researchers deal with, the  $F$  test that we have used so far will suffice. As Davidson and MacKinnon note:

For linear regression models, with or without normal errors, there is of course no need to look at LM, W and LR at all, since no information is gained from doing so over and above what is already contained in  $F$ .<sup>19</sup>

### \*8.11 TESTING THE FUNCTIONAL FORM OF REGRESSION: CHOOSING BETWEEN LINEAR AND LOG-LINEAR REGRESSION MODELS

The choice between a linear regression model (the regressand is a linear function of the regressors) or a log-linear regression model (the log of the regressand is a function of the logs of the regressors) is a perennial question in empirical analysis. We can use a test proposed by MacKinnon, White, and Davidson, which for brevity we call the **MWD test** to choose between the two models.<sup>20</sup>

To illustrate this test, assume the following

$H_0$ : *Linear Model*:  $Y$  is a linear function of regressors, the  $X$ 's.

$H_1$ : *Log-Linear Model*:  $\ln Y$  is a linear function of logs of regressors, the logs of  $X$ 's.

where, as usual,  $H_0$  and  $H_1$  denote the null and alternative hypotheses.

<sup>\*</sup>Optional.

<sup>18</sup>For an accessible discussion, see A. Buse, "The Likelihood Ratio, Wald and Lagrange Multiplier Tests: An Expository Note," *American Statistician*, vol. 36, 1982, pp. 153–157.

<sup>19</sup>Russell Davidson and James G. MacKinnon, *Estimation and Inference in Econometrics*, Oxford University Press, New York, 1993, p. 456.

<sup>20</sup>J. MacKinnon, H. White, and R. Davidson, "Tests for Model Specification in the Presence of Alternative Hypothesis; Some Further Results." *Journal of Econometrics*, vol. 21, 1983, pp. 53–70. A similar test is proposed in A. K. Bera and C. M. Jarque, "Model Specification Tests: A Simultaneous Approach," *Journal of Econometrics*, vol. 20, 1982, pp. 59–82.

The MWD test involves the following steps<sup>21</sup>:

**Step I:** Estimate the linear model and obtain the estimated  $Y$  values. Call them  $\hat{Y}_f$  (i.e.,  $\hat{Y}$ ).

**Step II:** Estimate the log-linear model and obtain the estimated  $\ln Y$  values; call them  $\ln \hat{Y}_f$  (i.e.,  $\widehat{\ln Y}$ ).

**Step III:** Obtain  $Z_1 = (\ln \hat{Y}_f - \ln Y_f)$ .

**Step IV:** Regress  $Z_1$  on  $X$ 's and  $Z_1$  obtained in Step III. Reject  $H_0$  if the coefficient of  $Z_1$  is statistically significant by the usual  $t$  test.

**Step V:** Obtain  $Z_2 = (\text{antilog of } \ln \hat{Y}_f - Y_f)$ .

**Step VI:** Regress  $Z_2$  on the logs of  $X$ 's and  $Z_2$ . Reject  $H_1$  if the coefficient of  $Z_2$  is statistically significant by the usual  $t$  test.

Although the MWD test seems involved, the logic of the test is quite simple. If the linear model is in fact the correct model, the constructed variable  $Z_1$  should not be statistically significant in Step IV, for in that case the estimated  $Y$  values from the linear model and those estimated from the log-linear model (after taking their antilog values for comparative purposes) should not be different. The same comment applies to the alternative hypothesis  $H_1$ .

#### EXAMPLE 8.5

##### THE DEMAND FOR ROSES

Refer to exercise 7.16 where we have presented data on the demand for roses in the Detroit metropolitan area for the period 1971–II to 1975–II. For illustrative purposes, we will consider the demand for roses as a function only of the prices of roses and carnations, leaving out the income variable for the time being. Now we consider the following models:

$$\text{Linear model: } Y_t = \alpha_1 + \alpha_2 X_{2t} + \alpha_3 X_{3t} + u_t \quad (8.11.1)$$

$$\text{Log-linear model: } \ln Y_t = \beta_1 + \beta_2 \ln X_{2t} + \beta_3 \ln X_{3t} + u_t \quad (8.11.2)$$

where  $Y$  is the quantity of roses in dozens,  $X_2$  is the average wholesale price of roses (\$/dozen), and  $X_3$  is the average wholesale price of carnations (\$/dozen). A priori,  $\alpha_2$  and  $\beta_2$  are expected to be negative (why?), and  $\alpha_3$  and  $\beta_3$  are expected to be positive (why?). As we know, the slope coefficients in the log-linear model are elasticity coefficients.

The regression results are as follows:

$$\begin{aligned} \hat{Y}_t &= 9734.2176 - 3782.1956X_{2t} + 2815.2515X_{3t} \\ t &= (3.3705) \quad (-6.6069) \quad (2.9712) \end{aligned} \quad (8.11.3)$$

$$F = 21.84 \quad R^2 = 0.77096$$

$$\begin{aligned} \widehat{\ln Y}_t &= 9.2278 - 1.7607 \ln X_{2t} + 1.3398 \ln X_{3t} \\ t &= (16.2349) \quad (-5.9044) \quad (2.5407) \end{aligned} \quad (8.11.4)$$

$$F = 17.50 \quad R^2 = 0.7292$$

(Continued)

<sup>21</sup>This discussion is based on William H. Greene, *ET. The Econometrics Toolkit Version 3*, Econometric Software, Bellport, New York, 1992, pp. 245–246.

**EXAMPLE 8.5** (Continued)

As these results show, both the linear and the log-linear models seem to fit the data reasonably well: The parameters have the expected signs and the  $t$  and  $R^2$  values are statistically significant.

To decide between these models on the basis of the **MWD test**, we first test the hypothesis that the true model is linear. Then, following Step IV of the test, we obtain the following regression:

$$\begin{aligned} \hat{Y}_t &= 9727.5685 - 3783.0623X_{2t} + 2817.7157X_{3t} + 85.2319Z_{1t} \\ t &= (3.2178) \quad (-6.3337) \quad (2.8366) \quad (0.0207) \quad \mathbf{(8.11.5)} \\ & \quad \quad \quad F = 13.44 \quad R^2 = 0.7707 \end{aligned}$$

Since the coefficient of  $Z_1$  is not statistically significant (the  $p$  value of the estimated  $t$  is 0.98), we do not reject the hypothesis that the true model is linear.

Suppose we switch gears and assume that the true model is log-linear. Following step VI of the MWD test, we obtain the following regression results:

$$\begin{aligned} \widehat{\ln Y}_t &= 9.1486 - 1.9699 \ln X_t + 1.5891 \ln X_{2t} - 0.0013Z_{2t} \\ t &= (17.0825) \quad (-6.4189) \quad (3.0728) \quad (-1.6612) \quad \mathbf{(8.11.6)} \\ & \quad \quad \quad F = 14.17 \quad R^2 = 0.7798 \end{aligned}$$

The coefficient of  $Z_2$  is statistically significant at about the 12 percent level ( $p$  value is 0.1225). Therefore, we can reject the hypothesis that the true model is log-linear at this level of significance. Of course, if one sticks to the conventional 1 or 5 percent significance levels, then one cannot reject the hypothesis that the true model is log-linear. As this example shows, it is quite possible that in a given situation we cannot reject either of the specifications.

**8.12 SUMMARY AND CONCLUSIONS**

1. This chapter extended and refined the ideas of interval estimation and hypothesis testing first introduced in Chapter 5 in the context of the two-variable linear regression model.

2. In a multiple regression, testing the *individual significance* of a partial regression coefficient (using the  $t$  test) and testing the *overall significance* of the regression (i.e.,  $H_0$ : all partial slope coefficients are zero or  $R^2 = 0$ ) are not the same thing.

3. In particular, the finding that one or more partial regression coefficients are statistically insignificant on the basis of the *individual  $t$  test* does not mean that all partial regression coefficients are also (collectively) statistically insignificant. The latter hypothesis can be tested only by the  $F$  test.

4. The  **$F$  test** is versatile in that it can test a variety of hypotheses, such as whether (1) an individual regression coefficient is statistically significant, (2) all partial slope coefficients are zero, (3) two or more coefficients are statistically equal, (4) the coefficients satisfy some linear restrictions, and (5) there is structural stability of the regression model.

5. As in the two-variable case, the multiple regression model can be used for the purpose of mean and or individual prediction.

## EXERCISES

## Questions

- 8.1. Suppose you want to study the behavior of sales of a product, say, automobiles over a number of years and suppose someone suggests you try the following models:

$$Y_t = \beta_0 + \beta_1 t$$

$$Y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2$$

where  $Y_t$  = sales at time  $t$  and  $t$  = time, measured in years. The first model postulates that sales is a linear function of time, whereas the second model states that it is a quadratic function of time.

- Discuss the properties of these models.
  - How would you decide between the two models?
  - In what situations will the quadratic model be useful?
  - Try to obtain data on automobile sales in the United States over the past 20 years and see which of the models fits the data better.
- 8.2. Show that the  $F$  ratio of (8.5.16) is equal to the  $F$  ratio of (8.5.18). (*Hint*:  $ESS/TSS = R^2$ .)
- 8.3. Show that  $F$  tests of (8.5.18) and (8.7.10) are equivalent.
- 8.4. Establish statements (8.7.11) and (8.7.12).
- 8.5. Consider the Cobb–Douglas production function

$$Y = \beta_1 L^{\beta_2} K^{\beta_3} \quad (1)$$

where  $Y$  = output,  $L$  = labor input, and  $K$  = capital input. Dividing (1) through by  $K$ , we get

$$(Y/K) = \beta_1 (L/K)^{\beta_2} K^{\beta_2 + \beta_3 - 1} \quad (2)$$

Taking the natural log of (2) and adding the error term, we obtain

$$\ln(Y/K) = \beta_0 + \beta_2 \ln(L/K) + (\beta_2 + \beta_3 - 1) \ln K + u_i \quad (3)$$

where  $\beta_0 = \ln \beta_1$ .

- Suppose you had data to run the regression (3). How would you test the hypothesis that there are constant returns to scale, i.e.,  $(\beta_2 + \beta_3) = 1$ ?
  - If there are constant returns to scale, how would you interpret regression (3)?
  - Does it make any difference whether we divide (1) by  $L$  rather than by  $K$ ?
- 8.6. **Critical values of  $R^2$  when true  $R^2 = 0$ .** Equation (8.5.11) gave the relationship between  $F$  and  $R^2$  under the hypothesis that all partial slope coefficients are simultaneously equal to zero (i.e.,  $R^2 = 0$ ). Just as we can find the critical  $F$  value at the  $\alpha$  level of significance from the  $F$  table, we can find the critical  $R^2$  value from the following relation:

$$R^2 = \frac{(k-1)F}{(k-1)F + (n-k)}$$

where  $k$  is the number of parameters in the regression model including the intercept and where  $F$  is the critical  $F$  value at the  $\alpha$  level of significance. If the observed  $R^2$  exceeds the critical  $R^2$  obtained from the preceding formula, we can reject the hypothesis that the true  $R^2$  is zero.

Establish the preceding formula and find out the critical  $R^2$  value (at  $\alpha = 5$  percent) for the regression (8.2.1).

- 8.7. From annual data for the years 1968–1987, the following regression results were obtained:

$$\hat{Y}_t = -859.92 + 0.6470X_{2t} - 23.195X_{3t} \quad R^2 = 0.9776 \quad (1)$$

$$\hat{Y}_t = -261.09 + 0.2452X_{2t} \quad R^2 = 0.9388 \quad (2)$$

where  $Y$  = U.S. expenditure on imported goods, billions of 1982 dollars,  $X_2$  = personal disposable income, billions of 1982 dollars, and  $X_3$  = trend variable. *True or false:* The standard error of  $X_3$  in (1) is 4.2750. Show your calculations. (*Hint:* Use the relationship between  $R^2$ ,  $F$ , and  $t$ .)

- 8.8. Suppose in the regression

$$\ln(Y_i/X_{2i}) = \alpha_1 + \alpha_2 \ln X_{2i} + \alpha_3 \ln X_{3i} + u_i$$

the values of the regression coefficients and their standard errors are known.\* From this knowledge, how would you estimate the parameters and standard errors of the following regression model?

$$\ln Y_i = \beta_1 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i$$

- 8.9. Assume the following:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{2i} X_{3i} + u_i$$

where  $Y$  is personal consumption expenditure,  $X_2$  is personal income, and  $X_3$  is personal wealth.† The term  $(X_{2i} X_{3i})$  is known as the **interaction term**. What is meant by this expression? How would you test the hypothesis that the marginal propensity to consume (MPC) (i.e.,  $\beta_2$ ) is independent of the wealth of the consumer?

- 8.10. You are given the following regression results:

$$\hat{Y}_t = 16,899 \quad - 2978.5X_{2t} \quad R^2 = 0.6149$$

$$t = \quad (8.5152) \quad (-4.7280)$$

$$\hat{Y}_t = 9734.2 \quad - 3782.2X_{2t} + 2815X_{3t} \quad R^2 = 0.7706$$

$$t = \quad (3.3705) \quad (-6.6070) \quad (2.9712)$$

Can you find out the sample size underlying these results? (*Hint:* Recall the relationship between  $R^2$ ,  $F$ , and  $t$  values.)

\*Adapted from Peter Kennedy, *A Guide to Econometrics*, the MIT Press, 3d ed., Cambridge, Mass., 1992, p. 310.

†Ibid., p. 327.

- 8.11.** Based on our discussion of *individual* and *joint* tests of hypothesis based, respectively, on the  $t$  and  $F$  tests, which of the following situations are likely?
1. Reject the joint null on the basis of the  $F$  statistic, but do not reject each separate null on the basis of the individual  $t$  tests.
  2. Reject the joint null on the basis of the  $F$  statistic, reject one individual hypothesis on the basis of the  $t$  test, and do not reject the other individual hypotheses on the basis of the  $t$  test.
  3. Reject the joint null hypothesis on the basis of the  $F$  statistic, and reject each separate null hypothesis on the basis of the individual  $t$  tests.
  4. Do not reject the joint null on the basis of the  $F$  statistic, and do not reject each separate null on the basis of individual  $t$  tests.
  5. Do not reject the joint null on the basis of the  $F$  statistic, reject one individual hypothesis on the basis of a  $t$  test, and do not reject the other individual hypotheses on the basis of the  $t$  test.
  6. Do not reject the joint null on the basis of the  $F$  statistic, but reject each separate null on the basis of individual  $t$  tests.\*

### Problems

- 8.12.** Refer to exercise 7.21.
- a. What are the real income and interest rate elasticities of real cash balances?
  - b. Are the preceding elasticities statistically significant individually?
  - c. Test the overall significance of the estimated regression.
  - d. Is the income elasticity of demand for real cash balances significantly different from unity?
  - e. Should the interest rate variable be retained in the model? Why?
- 8.13.** From the data for 46 states in the United States for 1992, Baltagi obtained the following regression results<sup>†</sup>:

$$\widehat{\log C} = 4.30 - 1.34 \log P + 0.17 \log Y$$

$$\text{se} = (0.91) \quad (0.32) \quad (0.20) \quad \bar{R}^2 = 0.27$$

where  $C$  = cigarette consumption, packs per year  
 $P$  = real price per pack  
 $Y$  = real disposable income per capita

- a. What is the elasticity of demand for cigarettes with respect to price? Is it statistically significant? If so, is it statistically different from one?
- b. What is the income elasticity of demand for cigarettes? Is it statistically significant? If not, what might be the reasons for it?
- c. How would you retrieve  $R^2$  from the *adjusted*  $R^2$  given above?

\*Quoted from Ernst R. Berndt, *The Practice of Econometrics: Classic and Contemporary*, Addison-Wesley, Reading, Mass., 1991, p. 79.

<sup>†</sup>See Badi H. Baltagi, *Econometrics*, Springer-Verlag, New York, 1998, p. 111.



- 8.14.** From a sample of 209 firms, Wooldridge obtained the following regression results\*:

$$\begin{aligned}\widehat{\log(\text{salary})} &= 4.32 + 0.280 \log(\text{sales}) + 0.0174 \text{roe} + 0.00024 \text{ros} \\ \text{se} &= (0.32) \quad (0.035) \quad (0.0041) \quad (0.00054) \\ R^2 &= 0.283\end{aligned}$$

where salary = salary of CEO  
sales = annual firm sales  
roe = return on equity in percent  
ros = return on firm's stock

and where figures in the parentheses are the estimated standard errors.

- Interpret the preceding regression taking into account any prior expectations that you may have about the signs of the various coefficients.
  - Which of the coefficients are *individually* statistically significant at the 5 percent level?
  - What is the overall significance of the regression? Which test do you use? And why?
  - Can you interpret the coefficients of roe and ros as elasticity coefficients? Why or why not?
- 8.15.** Assuming that  $Y$  and  $X_2, X_3, \dots, X_k$  are jointly normally distributed and assuming that the null hypothesis is that the population partial correlations are individually equal to zero, R. A. Fisher has shown that

$$t = \frac{r_{12.34\dots k} \sqrt{n-k-2}}{\sqrt{1-r_{12.34\dots k}^2}}$$

follows the  $t$  distribution with  $n - k - 2$  df, where  $k$  is the  $k$ th-order partial correlation coefficient and where  $n$  is the total number of observations. (Note:  $r_{12.3}$  is a first-order partial correlation coefficient,  $r_{12.34}$  is a second-order partial correlation coefficient, and so on.) Refer to exercise 7.2. Assuming  $Y$  and  $X_2$  and  $X_3$  to be jointly normally distributed, compute the three partial correlations  $r_{12.3}$ ,  $r_{13.2}$ , and  $r_{23.1}$  and test their significance under the hypothesis that the corresponding population correlations are individually equal to zero.

- 8.16.** In studying the demand for farm tractors in the United States for the periods 1921–1941 and 1948–1957, Griliches<sup>†</sup> obtained the following results:

$$\begin{aligned}\widehat{\log Y_t} &= \text{constant} - 0.519 \log X_{2t} - 4.933 \log X_{3t} \quad R^2 = 0.793 \\ &(0.231) \quad (0.477)\end{aligned}$$

where  $Y_t$  = value of stock of tractors on farms as of January 1, in 1935–1939 dollars,  $X_2$  = index of prices paid for tractors divided by an

\*See Jeffrey M. Wooldridge, *Introductory Econometrics*, South-Western Publishing Co., 2000, pp. 154–155.

<sup>†</sup>Z. Griliches, "The Demand for a Durable Input: Farm Tractors in the United States, 1921–1957," in *The Demand for Durable Goods*, Arnold C. Harberger (ed.), The University of Chicago Press, Chicago, 1960, Table 1, p. 192.



index of prices received for all crops at time  $t - 1$ ,  $X_3$  = interest rate prevailing in year  $t - 1$ , and the estimated standard errors are given in the parentheses.

- a. Interpret the preceding regression.
  - b. Are the estimated slope coefficients individually statistically significant? Are they significantly different from unity?
  - c. Use the analysis of variance technique to test the significance of the overall regression. *Hint:* Use the  $R^2$  variant of the ANOVA technique.
  - d. How would you compute the interest-rate elasticity of demand for farm tractors?
  - e. How would you test the significance of estimated  $R^2$ ?
- 8.17. Consider the following wage-determination equation for the British economy\* for the period 1950–1969:

$$\hat{W}_t = 8.582 + 0.364(\text{PF})_t + 0.004(\text{PF})_{t-1} - 2.560U_t$$

$$(1.129) \quad (0.080) \quad (0.072) \quad (0.658)$$

$$R^2 = 0.873 \quad \text{df} = 15$$

where  $W$  = wages and salaries per employee  
 $\text{PF}$  = prices of final output at factor cost  
 $U$  = unemployment in Great Britain as a percentage of the total number of employees of Great Britain  
 $t$  = time

(The figures in the parentheses are the estimated standard errors.)

- a. Interpret the preceding equation.
  - b. Are the estimated coefficients individually significant?
  - c. What is the rationale for the introduction of  $(\text{PF})_{t-1}$ ?
  - d. Should the variable  $(\text{PF})_{t-1}$  be dropped from the model? Why?
  - e. How would you compute the elasticity of wages and salaries per employee with respect to the unemployment rate  $U$ ?
- 8.18. A variation of the wage-determination equation given in exercise 8.17 is as follows†:

$$\hat{W}_t = 1.073 + 5.288V_t - 0.116X_t + 0.054M_t + 0.046M_{t-1}$$

$$(0.797) \quad (0.812) \quad (0.111) \quad (0.022) \quad (0.019)$$

$$R^2 = 0.934 \quad \text{df} = 14$$

where  $W$  = wages and salaries per employee  
 $V$  = unfilled job vacancies in Great Britain as a percentage of the total number of employees in Great Britain  
 $X$  = gross domestic product per person employed  
 $M$  = import prices  
 $M_{t-1}$  = import prices in the previous (or lagged) year

(The estimated standard errors are given in the parentheses.)

\*Taken from *Prices and Earnings in 1951–1969: An Econometric Assessment*, Dept. of Employment, HMSO, 1971, Eq. (19), p. 35.

†Ibid., Eq. (67), p. 37.

- a. Interpret the preceding equation.
  - b. Which of the estimated coefficients are individually statistically significant?
  - c. What is the rationale for the introduction of the  $X$  variable? A priori, is the sign of  $X$  expected to be negative?
  - d. What is the purpose of introducing both  $M_t$  and  $M_{t-1}$  in the model?
  - e. Which of the variables may be dropped from the model? Why?
  - f. Test the overall significance of the observed regression.
- 8.19. For the demand for chicken function estimated in (8.7.24), is the estimated income elasticity equal to 1? Is the price elasticity equal to  $-1$ ?
- 8.20. For the demand function (8.7.24) how would you test the hypothesis that the income elasticity is equal in value but opposite in sign to the price elasticity of demand? Show the necessary calculations. [Note:  $\text{cov}(\hat{\beta}_2, \hat{\beta}_3) = -0.00142$ .]
- 8.21. Refer to the demand for roses function of exercise 7.16. Confining your considerations to the logarithmic specification,
- a. What is the estimated own-price elasticity of demand (i.e., elasticity with respect to the price of roses)?
  - b. Is it statistically significant?
  - c. If so, is it significantly different from unity?
  - d. A priori, what are the expected signs of  $X_3$  (price of carnations) and  $X_4$  (income)? Are the empirical results in accord with these expectations?
  - e. If the coefficients of  $X_3$  and  $X_4$  are statistically insignificant, what may be the reasons?
- 8.22. Refer to exercise 7.17 relating to wildcat activity.
- a. Is each of the estimated slope coefficients individually statistically significant at the 5 percent level?
  - b. Would you reject the hypothesis that  $R^2 = 0$ ?
  - c. What is the instantaneous rate of growth of wildcat activity over the period 1948–1978? The corresponding compound rate of growth?
- 8.23. Refer to the U.S. defense budget outlay regression estimated in exercise 7.18.
- a. Comment generally on the estimated regression results.
  - b. Set up the ANOVA table and test the hypothesis that all the partial slope coefficients are zero.
- 8.24. The following is known as the **transcendental production function** (TPF), a generalization of the well-known Cobb–Douglas production function:

$$Y_i = \beta_1 L^{\beta_2} k^{\beta_3} e^{\beta_4 L + \beta_5 K}$$

where  $Y$  = output,  $L$  = labor input, and  $K$  = capital input.

After taking logarithms and adding the stochastic disturbance term, we obtain the stochastic TPF as

$$\ln Y_i = \beta_0 + \beta_2 \ln L_i + \beta_3 \ln K_i + \beta_4 L_i + \beta_5 K_i + u_i$$

where  $\beta_0 = \ln \beta_1$ .

- a. What are the properties of this function?
- b. For the TPF to reduce to the Cobb–Douglas production function, what must be the values of  $\beta_4$  and  $\beta_5$ ?

- c. If you had the data, how would you go about finding out whether the TPF reduces to the Cobb–Douglas production function? What testing procedure would you use?
- d. See if the TPF fits the data given in Table 8.8. Show your calculations.
- 8.25. *Energy prices and capital formation: United States, 1948–1978.* To test the hypothesis that a rise in the price of energy relative to output leads to a decline in the productivity of *existing* capital and labor resources, John A. Tatom estimated the following production function for the United States for the quarterly period 1948–I to 1978–II\*:

$$\widehat{\ln(y/k)} = 1.5492 + 0.7135 \ln(h/k) - 0.1081 \ln(P_e/P) + 0.0045t \quad R^2 = 0.98$$

(16.33)      (21.69)      (–6.42)      (15.86)

where  $y$  = real output in the private business sector  
 $k$  = a measure of the flow of capital services  
 $h$  = person hours in the private business sector  
 $P_e$  = producer price index for fuel and related products  
 $P$  = private business sector price deflator  
 $t$  = time.

The numbers in parentheses are  $t$  statistics.

- a. Do the results support the author's hypothesis?
- b. Between 1972 and 1977 the relative price of energy,  $(P_e/P)$ , increased by 60 percent. From the estimated regression, what is the loss in productivity?
- c. After allowing for the changes in  $(h/k)$  and  $(P_e/P)$ , what has been the trend rate of growth of productivity over the sample period?
- d. How would you interpret the coefficient value of 0.7135?
- e. Does the fact that each estimated partial slope coefficient is individually statistically significant (why?) mean we can reject the hypothesis that  $R^2 = 0$ ? Why or why not?
- 8.26. *The demand for cable.* Table 8.10 gives data used by a telephone cable manufacturer to predict sales to a major customer for the period 1968–1983.†

The variables in the table are defined as follows:

$Y$  = annual sales in MPF, million paired feet  
 $X_2$  = gross national product (GNP), \$, billions  
 $X_3$  = housing starts, thousands of units  
 $X_4$  = unemployment rate, %  
 $X_5$  = prime rate lagged 6 months  
 $X_6$  = Customer line gains, %

\*See his "Energy Prices and Capital Formation: 1972–1977," *Review*, Federal Reserve Bank of St. Louis, vol. 61, no. 5, May 1979, p. 4.

†I am indebted to Daniel J. Reardon for collecting and processing the data.

TABLE 8.10 REGRESSION VARIABLES

Year	$X_2$ , GNP	$X_3$ , housing starts	$X_4$ , unemployment, %	$X_5$ , prime rate lag, 6 mos.	$X_6$ , customer line gains, %	$Y$ , total plastic purchases (MPF)
1968	1051.8	1503.6	3.6	5.8	5.9	5873
1969	1078.8	1486.7	3.5	6.7	4.5	7852
1970	1075.3	1434.8	5.0	8.4	4.2	8189
1971	1107.5	2035.6	6.0	6.2	4.2	7497
1972	1171.1	2360.8	5.6	5.4	4.9	8534
1973	1235.0	2043.9	4.9	5.9	5.0	8688
1974	1217.8	1331.9	5.6	9.4	4.1	7270
1975	1202.3	1160.0	8.5	9.4	3.4	5020
1976	1271.0	1535.0	7.7	7.2	4.2	6035
1977	1332.7	1961.8	7.0	6.6	4.5	7425
1978	1399.2	2009.3	6.0	7.6	3.9	9400
1979	1431.6	1721.9	6.0	10.6	4.4	9350
1980	1480.7	1298.0	7.2	14.9	3.9	6540
1981	1510.3	1100.0	7.6	16.6	3.1	7675
1982	1492.2	1039.0	9.2	17.5	0.6	7419
1983	1535.4	1200.0	8.8	16.0	1.5	7923

You are to consider the following model:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + \beta_4 X_{4t} + \beta_5 X_{5t} + \beta_6 X_{6t} + u_t$$

- a. Estimate the preceding regression.
  - b. What are the expected signs of the coefficients of this model?
  - c. Are the empirical results in accordance with prior expectations?
  - d. Are the estimated partial regression coefficients individually statistically significant at the 5 percent level of significance?
  - e. Suppose you first regress  $Y$  on  $X_2$ ,  $X_3$ , and  $X_4$  only and then decide to add the variables  $X_5$  and  $X_6$ . How would you find out if it is worth adding the variables  $X_5$  and  $X_6$ ? Which test do you use? Show the necessary calculations.
- 8.27. Marc Nerlove has estimated the following cost function for electricity generation\*:

$$Y = AX^\beta P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} u \quad (1)$$

where  $Y$  = total cost of production  
 $X$  = output in kilowatt hours  
 $P_1$  = price of labor input  
 $P_2$  = price of capital input  
 $P_3$  = price of fuel  
 $u$  = disturbance term

\*Marc Nerlove, "Returns to Scale in Electric Supply," in Carl Christ, ed., *Measurement in Economics*, Stanford University Press, Palo Alto, Calif., 1963. The notation has been changed.



where  $r_f$  is the risk-free rate of return, we see that  $\hat{\gamma}_1$  is an estimate of  $r_f$  and  $\hat{\gamma}_2$  is an estimate of  $(ER_m - r_f)$ , the market risk premium.

Thus, in the empirical testing of CAPM,  $\bar{R}_i$  and  $\hat{\beta}_i$  are used as estimators of  $ER_i$  and  $\beta_i$ , respectively. Now if CAPM holds, statistically,

$$\hat{\gamma}_1 = r_f$$

$$\hat{\gamma}_2 = R_m - r_f, \text{ the estimator of } (ER_m - r_f)$$

Next consider an alternative model:

$$\bar{R}_i = \hat{\gamma}_1 + \hat{\gamma}_2 \hat{\beta}_i + \hat{\gamma}_3 s_{e_i}^2 + u_i \quad (4)$$

where  $s_{e_i}^2$  is the residual variance of the  $i$ th security from the first-stage regression. Then, if CAPM is valid,  $\hat{\gamma}_3$  should not be significantly different from zero.

To test the CAPM, Levy ran regressions (2) and (4) on a sample of 101 stocks for the period 1948–1968 and obtained the following results\*:

$$\begin{aligned} \hat{R}_i = & 0.109 + 0.037\beta_i \\ & (0.009) \quad (0.008) \\ t = & (12.0) \quad (5.1) \quad R^2 = 0.21 \end{aligned} \quad (2')$$

$$\begin{aligned} \hat{R}_i = & 0.106 + 0.0024\hat{\beta}_i + 0.201s_{e_i}^2 \\ & (0.008) \quad (0.007) \quad (0.038) \\ t = & (13.2) \quad (3.3) \quad (5.3) \quad R^2 = 0.39 \end{aligned} \quad (4')$$

- a. Are these results supportive of the CAPM?
  - b. Is it worth adding the variable  $s_{e_i}^2$  to the model? How do you know?
  - c. If the CAPM holds,  $\hat{\gamma}_1$  in (2)' should approximate the average value of the risk-free rate,  $r_f$ . The estimated value is 10.9 percent. Does this seem a reasonable estimate of the risk-free rate of return during the observation period, 1948–1968? (You may consider the rate of return on Treasury bills or a similar comparatively risk-free asset.)
  - d. If the CAPM holds, the market risk premium  $(\bar{R}_m - r_f)$  from (2)' is about 3.7 percent. If  $r_f$  is assumed to be 10.9 percent, this implies  $\bar{R}_m$  for the sample period was about 14.6 percent. Does this sound a reasonable estimate?
  - e. What can you say about the CAPM generally?
- 8.29.** Refer to exercise 7.21c. Now that you have the necessary tools, which test(s) would you use to choose between the two models. Show the necessary computations. Note that the dependent variables in the two models are different.
- 8.30.** Refer to Example 8.3. Use the  $t$  test as shown in (8.7.4) to find out if there were constant returns to scale in the Mexican economy for the period of the study.

\*H. Levy, "Equilibrium in an Imperfect Market: A constraint on the number of securities in the portfolio," *American Economic Review*, vol. 68, no. 4, September 1978, pp. 643–658.

**8.31.** Return to the child mortality example that we have discussed several times. In regression (7.6.2) we regressed child mortality (CM) on per capita GNP (PGNP) and female literacy rate (FLR). Now we extend this model by including total fertility rate (TFR). The data on all these variables are already given in Table 6.4. We reproduce regression (7.6.2) and give results of the extended regression model below:

$$1. \widehat{CM}_i = 263.6416 - 0.0056 \text{ PGNP}_i - 2.2316 \text{ FLR}_i \quad (7.6.2)$$

$$\text{se} = (11.5932) \quad (0.0019) \quad (0.2099) \quad R^2 = 0.7077$$

$$2. \widehat{CM}_i = 168.3067 - 0.0055 \text{ PGNP}_i - 1.7680 \text{ FLR}_i + 12.8686 \text{ TFR}_i$$

$$\text{se} = (32.8916) \quad (0.0018) \quad (0.2480) \quad (?)$$

$$R^2 = 0.7474$$

- a. How would you interpret the coefficient of TFR? A priori, would you expect a positive or negative relationship between CM and TFR? Justify your answer.
  - b. Have the coefficient values of PGNP and FR changed between the two equations? If so, what may be the reason(s) for such a change? Is the observed difference statistically significant? Which test do you use and why?
  - c. How would you choose between models 1 and 2? Which statistical test would you use to answer this question? Show the necessary calculations.
  - d. We have not given the standard error of the coefficient of TFR. Can you find it out? (*Hint*: Recall the relationship between the  $t$  and  $F$  distributions.)
- 8.32.** Return to exercise 1.7, which gave data on advertising impressions retained and advertising expenditure for a sample of 21 firms. In exercise 5.11 you were asked to plot these data and decide on an appropriate model about the relationship between impressions and advertising expenditure. Letting  $Y$  represent impressions retained and  $X$  the advertising expenditure, the following regressions were obtained:

$$\text{Model I: } \hat{Y}_i = 22.163 + 0.3631X_i$$

$$\text{se} = (7.089) \quad (0.0971) \quad r^2 = 0.424$$

$$\text{Model II: } \hat{Y}_i = 7.059 + 1.0847X_i - 0.0040X_i^2$$

$$\text{se} = (9.986) \quad (0.3699) \quad (0.0019) \quad R^2 = 0.53$$

- a. Interpret both models.
- b. Which is a better model? Why?
- c. Which statistical test(s) would you use to choose between the two models?
- d. Are there “diminishing returns” to advertising expenditure, that is, after a certain level of advertising expenditure (the saturation level) it does not pay to advertise? Can you find out what that level of expenditure might be? Show the necessary calculations.

- 8.33.** In regression (7.9.4), we presented the results of the Cobb–Douglas production function fitted to the Taiwanese agricultural sector for the years 1958–1972. On the basis of that regression, find out if there are constant returns to scale in that sector, using
- The  $t$  test given in (8.7.4). You are told that the covariance between the two slope estimators is  $-0.03843$ .
  - The  $F$  test given in (8.7.9).
  - Is there a difference in the two test results? And what is your conclusion regarding the returns to scale in the agriculture sector of Taiwan over the sample period?
- 8.34** Reconsider the savings–income regression in Section 8.8. Suppose we divide the sample into two periods as 1970–1982 and 1983–1995. Using the Chow test, decide if there is a structural change in the savings–income regression in the two periods. Comparing your results with those given in Section 8.8, what overall conclusion do you draw about the sensitivity of the Chow test to the choice of the break point that divides the sample into two (or more) periods?

## \*APPENDIX 8A

### Likelihood Ratio (LR) Test

The **LR test** is based on the maximum likelihood (ML) principle discussed in Appendix 4A, where we showed how one obtains the ML estimators of the two-variable regression model. The principle can be straightforwardly extended to the multiple regression model. Under the assumption that the disturbances  $u_i$  are normally distributed, we showed that, for the two-variable regression model, the OLS and ML estimators of the regression coefficients are identical, but the estimated error variances are different. The OLS estimator of  $\sigma^2$  is  $\sum \hat{u}_i^2 / (n - 2)$  but the ML estimator is  $\sum \hat{u}_i^2 / n$ , the former being unbiased and the latter biased, although in large samples the bias tends to disappear.

The same is true in the multiple regression case. To illustrate, consider the three-variable regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i \quad (1)$$

Corresponding to Eq. (5) of Appendix 4A, the log-likelihood function for the model (1) can be written as:

$$\ln \text{LF} = -\frac{n}{2} \sigma^2 - \frac{n}{2} \ln(2\pi) - \frac{1}{2} \sum (Y_i - \beta_1 - \beta_2 X_{2i} - \beta_3 X_{3i})^2 \quad (2)$$

---

\*Optional.



As shown in Appendix 4A, differentiating this function with respect to  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$ , and  $\sigma^2$ , setting the resulting expressions to zero, and solving, we obtain the ML estimators of these estimators. The ML estimators of  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$  will be identical to OLS estimators, which are already given in Eqs. (7.4.6) to (7.4.8), but the error variance will be different in that the residual sum of squares (RSS) will be divided by  $n$  rather than by  $(n - 3)$ , as in the case of OLS.

Now let us suppose that our null hypothesis  $H_0$  is that  $\beta_3$ , the coefficient of  $X_3$ , is zero. In this case, log LF given in (2) will become

$$\ln \text{LF} = -\frac{n}{2}\sigma^2 - \frac{n}{2} \ln(2\pi) - \frac{1}{2} \sum (Y_i - \beta_1 - \beta_2 X_{2i})^2 \quad (3)$$

Equation (3) is known as the **restricted log-likelihood function (RLLF)** because it is estimated with the restriction that a priori  $\beta_3$  is zero, whereas Eq. (1) is known as the **unrestricted log LF (ULLF)** because a priori there are no restrictions put on the parameters. To test the validity of the a priori restriction that  $\beta_3$  is zero, the LR test obtains the following test statistic:

$$\lambda = 2(\text{ULLF} - \text{RLLF}) \quad (4)^*$$

where ULLF and RLLF are, respectively, the unrestricted log-likelihood function [Eq. (1)] and the restricted log-likelihood function [Eq. (3)]. If the sample size is large, it can be shown that the test statistic  $\lambda$  given in (4) follows the chi-square ( $\chi^2$ ) distribution with df equal to the number of restrictions imposed by the null hypothesis, 1 in the present case.

The basic idea behind the LR test is simple: If the a priori restriction(s) are valid, the restricted and unrestricted (log) LF should not be different, in which case  $\lambda$  in (4) will be zero. But if that is not the case, the two LFs will diverge. And since in a large sample we know that  $\lambda$  follows the chi-square distribution, we can find out if the divergence is statistically significant, say, at a 1 or 5 percent level of significance. Or else, we can find out the  $p$  value of the estimated  $\lambda$ .

Let us illustrate the LR test with our child mortality example. If we regress child mortality (CM) on per capita GNP (PGNP) and female literacy rate (FLR) as we did in (8.2.1), we obtain ULLF of  $-328.1012$ , but if we regress CM on PGNP only, we obtain the RLLF of  $-361.6396$ . In absolute value (i.e., disregarding the sign), the former is smaller than the latter, which makes sense since we have an additional variable in the former model.

The question now is whether it is worth adding the FLR variable. If it is not, the restricted and unrestricted LLF should not differ much, but if it is, the LLFs will be different. To see if this difference is statistically significant,

\*This expression can also be expressed as  $-2(\text{RLLF} - \text{ULLF})$  or as  $-2 \ln(\text{RLLF}/\text{ULLF})$ .

we now use the LR test given in (4), which gives:

$$\lambda = 2[-328.1012 - (-361.6396)] = 67.0768$$

Asymptotically, this is distributed as the chi-square distribution with 1 df (because we have only one restriction imposed when we omitted the FLR variable from the full model). The  $p$  value of obtaining such a chi-square value for 1 df is almost zero, leading to the conclusion that the FLR variable should *not* be excluded from the model. In other words, the restricted regression in the present instance is not valid.

Because of the mathematical complexity of the Wald and LM tests, we will not discuss them here. But as noted in the text, asymptotically, the LR, Wald, and LM tests give identical answers, the choice of the test depending on computational convenience.

# PART TWO

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## RELAXING THE ASSUMPTIONS OF THE CLASSICAL MODEL

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In Part I we considered at length the classical normal linear regression model and showed how it can be used to handle the twin problems of statistical inference, namely, estimation and hypothesis testing, as well as the problem of prediction. But recall that this model is based on several simplifying assumptions, which are as follows.

Assumption 1. The regression model is linear in the parameters.

Assumption 2. The values of the regressors, the  $X$ 's, are fixed in repeated sampling.

Assumption 3. For given  $X$ 's, the mean value of the disturbance  $u_i$  is zero.

Assumption 4. For given  $X$ 's, the variance of  $u_i$  is constant or homoscedastic.

Assumption 5. For given  $X$ 's, there is no autocorrelation in the disturbances.

Assumption 6. If the  $X$ 's are stochastic, the disturbance term and the (stochastic)  $X$ 's are independent or at least uncorrelated.

Assumption 7. The number of observations must be greater than the number of regressors.

Assumption 8. There must be sufficient variability in the values taken by the regressors.

Assumption 9. The regression model is correctly specified.

Assumption 10. There is no exact linear relationship (i.e., multicollinearity) in the regressors.

Assumption 11. The stochastic (disturbance) term  $u_i$  is normally distributed.

Before proceeding further, let us note that most textbooks list fewer than 11 assumptions. For example, assumptions 7 and 8 are taken for granted rather than spelled out explicitly. We decided to state them explicitly because distinguishing between the assumptions required for OLS to have desirable statistical properties (such as BLUE) and the conditions required for OLS to be useful seems sensible. For example, OLS estimators are BLUE even if assumption 8 is not satisfied. But in that case the standard errors of the OLS estimators will be large relative to their coefficients (i.e., the  $t$  ratios will be small), thereby making it difficult to assess the contribution of one or more regressors to the explained sum of squares.

As Wetherill notes, in practice two major types of problems arise in applying the classical linear regression model: (1) those due to assumptions about the specification of the model and about the disturbances  $u_i$  and (2) those due to assumptions about the data.<sup>1</sup> In the first category are Assumptions 1, 2, 3, 4, 5, 9, and 11. Those in the second category include Assumptions 6, 7, 8, and 10. In addition, data problems, such as outliers (unusual or untypical observations) and errors of measurement in the data, also fall into the second category.

With respect to problems arising from the assumptions about disturbances and model specifications, three major questions arise: (1) How severe must the departure be from a particular assumption before it really matters? For example, if  $u_i$  are not exactly normally distributed, what level of departure from this assumption can one accept before the BLUE property of the OLS estimators is destroyed? (2) How do we find out whether a particular assumption is in fact violated in a concrete case? Thus, how does one find out if the disturbances are normally distributed in a given application? We have already discussed the **Anderson-Darling**, **chi-square**, and **Jarque-Bera** tests of normality. (3) What remedial measures can we take if one or more of the assumptions are false? For example, if the assumption of homoscedasticity is found to be false in an application, what do we do then?

With regard to problems attributable to assumptions about the data, we also face similar questions. (1) How serious is a particular problem? For example, is multicollinearity so severe that it makes estimation and inference very difficult? (2) How do we find out the severity of the data problem? For example, how do we decide whether the inclusion or exclusion of an observation or observations that may represent outliers will make a tremendous difference in the analysis? (3) Can some of the data problems be easily remedied? For example, can one have access to the original data to find out the sources of errors of measurement in the data?

Unfortunately, satisfactory answers cannot be given to all these questions. What we will do in the rest of Part II is to look at some of the assumptions more critically, but not all will receive full scrutiny. In particular, we will not discuss in depth the following: Assumptions 2, 3, 6, and 11. The

<sup>1</sup>G. Barrie Wetherill, *Regression Analysis with Applications*, Chapman and Hall, New York, 1986, pp. 14–15.

reasons are as follows:

**Assumptions 2 and 6: Fixed versus stochastic regressors.** Remember that our regression analysis is based on the assumption that the regressors are nonstochastic and assume fixed values in repeated sampling. There is a good reason for this strategy. Unlike scientists in the physical sciences, as noted in Chapter 1, economists generally have no control over the data they use. More often than not, economists depend on secondary data, that is, data collected by someone else, such as the government and private organizations. Therefore, the practical strategy to follow is to assume that for the problem at hand the values of the explanatory variables are given even though the variables themselves may be intrinsically stochastic or random. Hence, the results of the regression analysis are conditional upon these given values.

But suppose that we cannot regard the  $X$ 's as truly nonstochastic or fixed. This is the case of **random** or **stochastic regressors**. Now the situation is rather involved. The  $u_i$ , by assumption, are stochastic. If the  $X$ 's too are stochastic, then we must specify how the  $X$ 's and  $u_i$  are distributed. If we are willing to make Assumption 6 (i.e., the  $X$ 's, although random, are distributed independently of, or at least uncorrelated with,  $u_i$ ), then for all practical purposes we can continue to operate as if the  $X$ 's were nonstochastic. As Kmenta notes:

*Thus, relaxing the assumption that  $X$  is nonstochastic and replacing it by the assumption that  $X$  is stochastic but independent of  $[u]$  does not change the desirable properties and feasibility of least squares estimation.<sup>2</sup>*

Therefore, we will retain Assumption 2 or Assumption 6 until we come to deal with simultaneous equations models in Part IV.<sup>3</sup>

**Assumption 3: Zero mean value of  $u_i$ .** Recall the  $k$ -variable linear regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i \quad (1)$$

Let us now assume that

$$E(u_i | X_{2i}, X_{3i}, \dots, X_{ki}) = w \quad (2)$$

where  $w$  is a constant; note in the standard model  $w = 0$ , but now we let it be any constant.

<sup>2</sup>Jan Kmenta, *Elements of Econometrics*, 2d ed., Macmillan, New York, 1986, p. 338. (Emphasis in the original.)

<sup>3</sup>A technical point may be noted here. Instead of the strong assumption that the  $X$ 's and  $u$  are independent, we may use the weaker assumption that the values of  $X$  variables and  $u$  are uncorrelated contemporaneously (i.e., at the same point in time). In this case OLS estimators may be biased but they are **consistent**, that is, as the sample size increases indefinitely, the estimators converge on their true values. If, however, the  $X$ 's and  $u$  are contemporaneously correlated, the OLS estimators are biased as well as inconsistent. In Chap. 17 we will show how the method of **instrumental variables** can sometimes be used to obtain consistent estimators in this situation.

Taking the conditional expectation of (1), we obtain

$$\begin{aligned} E(Y_i|X_{2i}, X_{3i}, \dots, X_{ki}) &= \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \dots + \beta_k X_{ki} + w \\ &= (\beta_1 + w) + \beta_2 X_{2i} + \beta_3 X_{3i} + \dots + \beta_k X_{ki} \quad (3) \\ &= \alpha + \beta_2 X_{2i} + \beta_3 X_{3i} + \dots + \beta_k X_{ki} \end{aligned}$$

where  $\alpha = (\beta_1 + w)$  and where in taking the expectations one should note that the  $X$ 's are treated as constants. (Why?)

Therefore, if Assumption 3 is not fulfilled, we see that we cannot estimate the original intercept  $\beta_1$ ; what we obtain is  $\alpha$ , which contains  $\beta_1$  and  $E(u_i) = w$ . In short, we obtain a *biased* estimate of  $\beta_1$ .

But as we have noted on many occasions, in many practical situations the intercept term,  $\beta_1$ , is of little importance; the more meaningful quantities are the slope coefficients, which remain unaffected even if Assumption 3 is violated.<sup>4</sup> Besides, in many applications the intercept term has no physical interpretation.

**Assumption 11: Normality of  $u$ .** This assumption is not essential if our objective is estimation only. As noted in Chapter 3, the OLS estimators are BLUE regardless of whether the  $u_i$  are normally distributed or not. With the normality assumption, however, we were able to establish that the OLS estimators of the regression coefficients follow the normal distribution, that  $(n - k)\hat{\sigma}^2/\sigma^2$  has the  $\chi^2$  distribution, and that one could use the  $t$  and  $F$  tests to test various statistical hypotheses regardless of the sample size.

But what happens if the  $u_i$  are not normally distributed? We then rely on the following extension of the central limit theorem; recall that it was the central limit theorem we invoked to justify the normality assumption in the first place:

If the disturbances  $[u_i]$  are independently and identically distributed with zero mean and [constant] variance  $\sigma^2$  and if the explanatory variables are constant in repeated samples, the [O]LS coefficient estimators are asymptotically normally distributed with means equal to the corresponding  $\beta$ 's.<sup>5</sup>

Therefore, the usual test procedures—the  $t$  and  $F$  tests—are still valid *asymptotically*, that is, in the large sample, but not in the finite or small samples.

The fact that if the disturbances are not normally distributed the OLS estimators are still normally distributed asymptotically (under the assumption of homoscedastic variance and fixed  $X$ 's) is of little comfort to practicing

<sup>4</sup>It is very important to note that this statement is true only if  $E(u_i) = w$  for each  $i$ . However, if  $E(u_i) = w_i$ , that is, a different constant for each  $i$ , the partial slope coefficients may be biased as well as inconsistent. In this case violation of Assumption 3 will be critical. For proof and further details, see Peter Schmidt, *Econometrics*, Marcel Dekker, New York, 1976, pp. 36–39.

<sup>5</sup>Henri Theil, *Introduction to Econometrics*, Prentice-Hall, Englewood Cliffs, N.J., 1978, p. 240. It must be noted the assumptions of fixed  $X$ 's and constant  $\sigma^2$  are crucial for this result.

economists, who often do not have the luxury of large-sample data. Therefore, the normality assumption becomes extremely important for the purposes of hypothesis testing and prediction. Hence, with the twin problems of estimation and hypothesis testing in mind, and given the fact that small samples are the rule rather than the exception in most economic analyses, we shall continue to use the normality assumption.<sup>6</sup>

Of course, this means that when we deal with a finite sample, we must explicitly test for the normality assumption. We have already considered the **Anderson-Darling** and the **Jarque-Bera tests** of normality. The reader is strongly urged to apply these or other tests of normality to regression residuals. Keep in mind that in finite samples without the normality assumption the usual  $t$  and  $F$  statistics may not follow the  $t$  and  $F$  distributions.

We are left with Assumptions 1, 4, 5, 7, 8, 9, 10. Assumptions 7, 8, and 10 are closely related and are discussed in the chapter on multicollinearity (Chapter 10). Assumption 4 is discussed in the chapter on heteroscedasticity (Chapter 11). Assumption 5 is discussed in the chapter on autocorrelation (Chapter 12). Assumption 9 is discussed in the chapter on model specification and diagnostic testing (Chapter 13). Because of its specialized nature and mathematical demands, Assumption 1 is discussed as a special topic in Part III (Chapter 14).

For pedagogical reasons, in each of these chapters we follow a common format, namely, (1) identify the nature of the problem, (2) examine its consequences, (3) suggest methods of detecting it, and (4) consider remedial measures so that they may lead to estimators that possess the desirable statistical properties discussed in Part I.

A cautionary note is in order: As noted earlier, satisfactory answers to all the problems arising out of the violation of the assumptions of the CLRM do not exist. Moreover, there may be more than one solution to a particular problem, and often it is not clear which method is best. Besides, in a particular application more than one violation of the CLRM may be involved. Thus, specification bias, multicollinearity, and heteroscedasticity may coexist in an application, and there is no single omnipotent test that will solve all the problems simultaneously.<sup>7</sup> Furthermore, a particular test that was popular at one time may not be in vogue later because somebody found a flaw in the earlier test. But this is how science progresses. Econometrics is no exception.

<sup>6</sup>In passing, note that the effects of departure from normality and related topics are often discussed under the topic of **robust estimation** in the literature, a topic beyond the scope of this book.

<sup>7</sup>This is not for lack of trying. See A. K. Bera and C. M. Jarque, "Efficient Tests for Normality, Homoscedasticity and Serial Independence of Regression Residuals: Monte Carlo Evidence," *Economic Letters*, vol. 7, 1981, pp. 313–318.

# 9

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## DUMMY VARIABLE REGRESSION MODELS

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In Chapter 1 we discussed briefly the four types of variables that one generally encounters in empirical analysis: These are: **ratio scale**, **interval scale**, **ordinal scale**, and **nominal scale**. The types of variables that we have encountered in the preceding chapters were essentially *ratio scale*. But this should not give the impression that regression models can deal only with ratio scale variables. Regression models can also handle other types of variables mentioned previously. In this chapter, we consider models that may involve not only ratio scale variables but also **nominal scale** variables. Such variables are also known as **indicator variables**, **categorical variables**, **qualitative variables**, or **dummy variables**.<sup>1</sup>

### 9.1 THE NATURE OF DUMMY VARIABLES

In regression analysis the dependent variable, or regressand, is frequently influenced not only by ratio scale variables (e.g., income, output, prices, costs, height, temperature) but also by variables that are essentially qualitative, or nominal scale, in nature, such as sex, race, color, religion, nationality, geographical region, political upheavals, and party affiliation. For example, holding all other factors constant, female workers are found to earn less than their male counterparts or nonwhite workers are found to earn less than whites.<sup>2</sup> This pattern may result from sex or racial discrimination, but whatever the reason, qualitative variables such as sex and race seem to

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<sup>1</sup>We will discuss ordinal scale variables in Chap. 15.

<sup>2</sup>For a review of the evidence on this subject, see Bruce E. Kaufman and Julie L. Hotchkiss, *The Economics of Labor Market*, 5th ed., Dryden Press, New York, 2000.



influence the regressand and clearly should be included among the explanatory variables, or the regressors.

Since such variables usually indicate the presence or absence of a “quality” or an attribute, such as male or female, black or white, Catholic or non-Catholic, Democrat or Republican, they are essentially *nominal scale* variables. One way we could “quantify” such attributes is by constructing artificial variables that take on values of 1 or 0, 1 indicating the presence (or possession) of that attribute and 0 indicating the absence of that attribute. For example 1 may indicate that a person is a female and 0 may designate a male; or 1 may indicate that a person is a college graduate, and 0 that the person is not, and so on. Variables that assume such 0 and 1 values are called **dummy variables**.<sup>3</sup> *Such variables are thus essentially a device to classify data into mutually exclusive categories such as male or female.*

Dummy variables can be incorporated in regression models just as easily as quantitative variables. As a matter of fact, a regression model may contain regressors that are all exclusively dummy, or qualitative, in nature. Such models are called **Analysis of Variance (ANOVA) models**.<sup>4</sup>

## 9.2 ANOVA MODELS

To illustrate the ANOVA models, consider the following example.

### EXAMPLE 9.1

#### PUBLIC SCHOOL TEACHERS' SALARIES BY GEOGRAPHICAL REGION

Table 9.1 gives data on average salary (in dollars) of public school teachers in 50 states and the District of Columbia for the year 1985. These 51 areas are classified into three geographical regions: (1) Northeast and North Central (21 states in all), (2) South (17 states in all), and (3) West (13 states in all). For the time being, do not worry about the format of the table and the other data given in the table.

Suppose we want to find out if the average annual salary (AAS) of public school teachers differs among the three geographical regions of the country. If you take the simple arithmetic average of the average salaries of the teachers in the three regions, you will find that these averages for the three regions are as follows: \$24,424.14 (Northeast and North Central), \$22,894 (South), and \$26,158.62 (West). These numbers look different, but are they

(Continued)

<sup>3</sup>It is not absolutely essential that dummy variables take the values of 0 and 1. The pair (0,1) can be transformed into any other pair by a linear function such that  $Z = a + bD$  ( $b \neq 0$ ), where  $a$  and  $b$  are constants and where  $D = 1$  or 0. When  $D = 1$ , we have  $Z = a + b$ , and when  $D = 0$ , we have  $Z = a$ . Thus the pair (0, 1) becomes  $(a, a + b)$ . For example, if  $a = 1$  and  $b = 2$ , the dummy variables will be (1, 3). *This expression shows that qualitative, or dummy, variables do not have a natural scale of measurement.* That is why they are described as nominal scale variables.

<sup>4</sup>ANOVA models are used to assess the statistical significance of the relationship between a quantitative regressand and qualitative or dummy regressors. They are often used to compare the differences in the mean values of two or more groups or categories, and are therefore more general than the  $t$  test which can be used to compare the means of two groups or categories only.

**EXAMPLE 9.1** (Continued)

**TABLE 9.1** AVERAGE SALARY OF PUBLIC SCHOOL TEACHERS, BY STATE, 1986

Salary	Spending	$D_2$	$D_3$	Salary	Spending	$D_2$	$D_3$
19,583	3346	1	0	22,795	3366	0	1
20,263	3114	1	0	21,570	2920	0	1
20,325	3554	1	0	22,080	2980	0	1
26,800	4642	1	0	22,250	3731	0	1
29,470	4669	1	0	20,940	2853	0	1
26,610	4888	1	0	21,800	2533	0	1
30,678	5710	1	0	22,934	2729	0	1
27,170	5536	1	0	18,443	2305	0	1
25,853	4168	1	0	19,538	2642	0	1
24,500	3547	1	0	20,460	3124	0	1
24,274	3159	1	0	21,419	2752	0	1
27,170	3621	1	0	25,160	3429	0	1
30,168	3782	1	0	22,482	3947	0	0
26,525	4247	1	0	20,969	2509	0	0
27,360	3982	1	0	27,224	5440	0	0
21,690	3568	1	0	25,892	4042	0	0
21,974	3155	1	0	22,644	3402	0	0
20,816	3059	1	0	24,640	2829	0	0
18,095	2967	1	0	22,341	2297	0	0
20,939	3285	1	0	25,610	2932	0	0
22,644	3914	1	0	26,015	3705	0	0
24,624	4517	0	1	25,788	4123	0	0
27,186	4349	0	1	29,132	3608	0	0
33,990	5020	0	1	41,480	8349	0	0
23,382	3594	0	1	25,845	3766	0	0
20,627	2821	0	1				

Note:  $D_2 = 1$  for states in the Northeast and North Central; 0 otherwise.

$D_3 = 1$  for states in the South; 0 otherwise.

Source: National Educational Association, as reported by *Albuquerque Tribune*, Nov. 7, 1986.

statistically different from one another? There are various statistical techniques to compare two or more mean values, which generally go by the name of **analysis of variance**.<sup>5</sup> But the same objective can be accomplished within the framework of regression analysis.

To see this, consider the following model:

$$Y_i = \beta_1 + \beta_2 D_{2i} + \beta_3 D_{3i} + u_i \quad (9.2.1)$$

where  $Y_i$  = (average) salary of public school teacher in state  $i$

$D_{2i} = 1$  if the state is in the Northeast or North Central  
= 0 otherwise (i.e., in other regions of the country)

$D_{3i} = 1$  if the state is in the South  
= 0 otherwise (i.e., in other regions of the country)

(Continued)

<sup>5</sup>For an applied treatment, see John Fox, *Applied Regression Analysis, Linear Models, and Related Methods*, Sage Publications, 1997, Chap. 8.

**EXAMPLE 9.1** (Continued)

Note that (9.2.1) is like any multiple regression model considered previously, except that, instead of quantitative regressors, we have only qualitative, or dummy, regressors, taking the value of 1 if the observation belongs to a particular category and 0 if it does not belong to that category or group. *Hereafter, we shall designate all dummy variables by the letter  $D$ .* Table 9.1 shows the dummy variables thus constructed.

What does the model (9.2.1) tell us? Assuming that the error term satisfies the usual OLS assumptions, on taking expectation of (9.2.1) on both sides, we obtain:

Mean salary of public school teachers in the Northeast and North Central:

$$E(Y_i | D_{2i} = 1, D_{3i} = 0) = \beta_1 + \beta_2 \quad (9.2.2)$$

Mean salary of public school teachers in the South:

$$E(Y_i | D_{2i} = 0, D_{3i} = 1) = \beta_1 + \beta_3 \quad (9.2.3)$$

You might wonder how we find out the mean salary of teachers in the West. If you guessed that this is equal to  $\beta_1$ , you would be absolutely right, for

Mean salary of public school teachers in the West:

$$E(Y_i | D_{2i} = 0, D_{3i} = 0) = \beta_1 \quad (9.2.4)$$

In other words, the mean salary of public school teachers in the West is given by the intercept,  $\beta_1$ , in the multiple regression (9.2.1), and the “slope” coefficients  $\beta_2$  and  $\beta_3$  tell by how much the mean salaries of teachers in the Northeast and North Central and in the South differ from the mean salary of teachers in the West. But how do we know if these differences are statistically significant? Before we answer this question, let us present the results based on the regression (9.2.1). Using the data given in Table 9.1, we obtain the following results:

$$\begin{array}{rcccc} \hat{Y}_i = & 26,158.62 & - & 1734.473D_{2i} & - & 3264.615D_{3i} \\ \text{se} = & (1128.523) & & (1435.953) & & (1499.615) \\ t = & (23.1759) & & (-1.2078) & & (-2.1776) \\ & (0.0000)* & & (0.2330)* & & (0.0349)* & R^2 = 0.0901 \end{array} \quad (9.2.5)$$

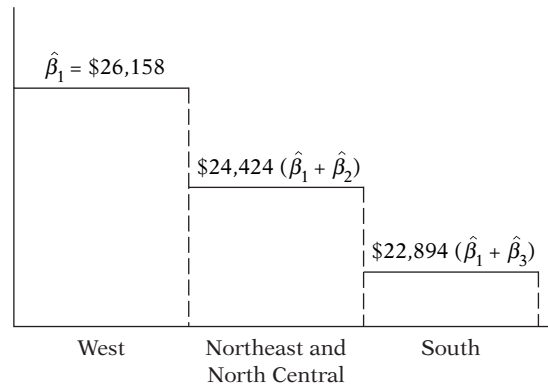
where \* indicates the  $p$  values.

As these regression results show, the mean salary of teachers in the West is about \$26,158, that of teachers in the Northeast and North Central is lower by about \$1734, and that of teachers in the South is lower by about \$3265. The actual mean salaries in the last two regions can be easily obtained by adding these differential salaries to the mean salary of teachers in the West, as shown in Eqs. (9.2.3) and (9.2.4). Doing this, we will find that the mean salaries in the latter two regions are about \$24,424 and \$22,894.

But how do we know that these mean salaries are statistically different from the mean salary of teachers in the West, the comparison category? That is easy enough. All we have to do is to find out if each of the “slope” coefficients in (9.2.5) is statistically significant. As can be seen from this regression, the estimated slope coefficient for Northeast and North Central is not statistically significant, as its  $p$  value is 23 percent, whereas that of the South is statistically significant, as the  $p$  value is only about 3.5 percent. Therefore, the overall conclusion is that statistically the mean salaries of public school teachers in the West and the Northeast and North Central are about the same but the mean salary of teachers in the South is statistically significantly lower by about \$3265. Diagrammatically, the situation is shown in Figure 9.1.

A caution is in order in interpreting these differences. The dummy variables will simply point out the differences, if they exist, but they do not suggest the reasons for the differences.

(Continued)

**EXAMPLE 9.1** (Continued)**FIGURE 9.1**

Average salary (in dollars) of public school teachers in three regions.

Differences in educational levels, in cost of living indexes, in gender and race may all have some effect on the observed differences. Therefore, unless we take into account all the other variables that may affect a teacher's salary, we will not be able to pin down the cause(s) of the differences.

From the preceding discussion, it is clear that all one has to do is see if the coefficients attached to the various dummy variables are individually statistically significant. This example also shows how easy it is to incorporate qualitative, or dummy, regressors in the regression models.

**Caution in the Use of Dummy Variables**

Although they are easy to incorporate in the regression models, one must use the dummy variables carefully. In particular, consider the following aspects:

1. In Example 9.1, to distinguish the three regions, we used only two dummy variables,  $D_2$  and  $D_3$ . Why did we not use three dummies to distinguish the three regions? Suppose we do that and write the model (9.2.1) as:

$$Y_i = \alpha + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} + u_i \quad (9.2.6)$$

where  $D_{1i}$  takes a value of 1 for states in the West and 0 otherwise. Thus, we now have a dummy variable for each of the three geographical regions. Using the data in Table 9.1, if you were to run the regression (9.2.6), the computer will "refuse" to run the regression (try it).<sup>6</sup> Why? The reason is that in

<sup>6</sup>Actually you will get a message saying that the data matrix is singular.

the setup of (9.2.6) where you have a dummy variable for each category or group and also an intercept, you have a case of **perfect collinearity**, that is, exact linear relationships among the variables. Why? Refer to Table 9.1. Imagine that now we add the  $D_1$  column, taking the value of 1 whenever a state is in the West and 0 otherwise. Now if you add the three  $D$  columns horizontally, you will obtain a column that has 51 ones in it. But since the value of the intercept  $\alpha$  is (implicitly) 1 for each observation, you will have a column that also contains 51 ones. In other words, the sum of the three  $D$  columns will simply reproduce the intercept column, thus leading to perfect collinearity. In this case, estimation of the model (9.2.6) is impossible.

The message here is: **If a qualitative variable has  $m$  categories, introduce only  $(m - 1)$  dummy variables.** In our example, since the qualitative variable “region” has three categories, we introduced only two dummies. If you do not follow this rule, you will fall into what is called the **dummy variable trap**, that is, the situation of perfect collinearity or perfect multicollinearity, if there is more than one exact relationship among the variables. This rule also applies if we have more than one qualitative variable in the model, an example of which is presented later. Thus we should restate the preceding rule as: **For each qualitative regressor the number of dummy variables introduced must be one less than the categories of that variable.** Thus, if in Example 9.1 we had information about the gender of the teacher, we would use an additional dummy variable (but not two) taking a value of 1 for female and 0 for male or vice versa.

2. The category for which no dummy variable is assigned is known as the **base, benchmark, control, comparison, reference, or omitted category**. And all comparisons are made in relation to the benchmark category.

3. The intercept value ( $\beta_1$ ) represents the *mean value* of the benchmark category. In Example 9.1, the benchmark category is the Western region. Hence, in the regression (9.2.5) the intercept value of about 26,159 represents the mean salary of teachers in the Western states.

4. The coefficients attached to the dummy variables in (9.2.1) are known as the **differential intercept coefficients** because they tell by how much the value of the intercept that receives the value of 1 differs from the intercept coefficient of the benchmark category. For example, in (9.2.5), the value of about  $-1734$  tells us that the mean salary of teachers in the Northeast or North Central is smaller by about \$1734 than the mean salary of about \$26,159 for the benchmark category, the West.

5. If a qualitative variable has more than one category, as in our illustrative example, the choice of the benchmark category is strictly up to the researcher. Sometimes the choice of the benchmark is dictated by the particular problem at hand. In our illustrative example, we could have chosen the South as the benchmark category. In that case the regression results given in (9.2.5) will change, because now all comparisons are made in relation to the South. Of course, this will not change the overall conclusion of our example (why?). In this case, the intercept value will be about \$22,894, which is the mean salary of teachers in the South.

6. We warned above about the dummy variable trap. There is a way to circumvent this trap by introducing as many dummy variables as the number of categories of that variable, *provided we do not introduce the intercept in such a model*. Thus, if we drop the intercept term from (9.2.6), and consider the following model,

$$Y_i = \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} + u_i \quad (9.2.7)$$

we do not fall into the dummy variable trap, as there is no longer perfect collinearity. *But make sure that when you run this regression, you use the no-intercept option in your regression package.*

How do we interpret regression (9.2.7)? If you take the expectation of (9.2.7), you will find that:

$\beta_1$  = mean salary of teachers in the West

$\beta_2$  = mean salary of teachers in the Northeast and North Central.

$\beta_3$  = mean salary of teachers in the South.

In other words, *with the intercept suppressed, and allowing a dummy variable for each category, we obtain directly the mean values of the various categories.* The results of (9.2.7) for our illustrative example are as follows:

$$\begin{aligned} \hat{Y}_i &= 26,158.62D_{1i} + 24,424.14D_{2i} + 22,894D_{3i} \\ \text{se} &= (1128.523) \quad (887.9170) \quad (986.8645) \\ t &= (23.1795)^* \quad (27.5072)^* \quad (23.1987)^* \\ R^2 &= 0.0901 \end{aligned} \quad (9.2.8)$$

where \* indicates that the  $p$  values of these  $t$  ratios are very small. As you can see, the dummy coefficients give directly the mean (salary) values in the three regions, West, Northeast and North Central, and South.

7. Which is a better method of introducing a dummy variable: (1) introduce a dummy for each category and omit the intercept term or (2) include the intercept term and introduce only  $(m - 1)$  dummies, where  $m$  is the number of categories of the dummy variable? As Kennedy notes:

Most researchers find the equation with an intercept more convenient because it allows them to address more easily the questions in which they usually have the most interest, namely, whether or not the categorization makes a difference, and if so, by how much. If the categorization does make a difference, by how much is measured directly by the dummy variable coefficient estimates. Testing whether or not the categorization is relevant can be done by running a  $t$  test of a dummy variable coefficient against zero (or, to be more general, an  $F$  test on the appropriate set of dummy variable coefficient estimates).<sup>7</sup>

<sup>7</sup>Peter Kennedy, *A Guide to Econometrics*, 4th ed., MIT Press, Cambridge, Mass., 1998, p. 223.

### 9.3 ANOVA MODELS WITH TWO QUALITATIVE VARIABLES

In the previous section we considered an ANOVA model with one qualitative variable with three categories. In this section we consider another ANOVA model, but with two qualitative variables, and bring out some additional points about dummy variables.

#### EXAMPLE 9.2

##### HOURLY WAGES IN RELATION TO MARITAL STATUS AND REGION OF RESIDENCE

From a sample of 528 persons in May 1985, the following regression results were obtained<sup>8</sup>:

$$\begin{array}{r} \hat{Y}_i = 8.8148 + 1.0997D_{2i} - 1.6729D_{3i} \\ \text{se} = (0.4015) \quad (0.4642) \quad (0.4854) \\ t = (21.9528) \quad (2.3688) \quad (-3.4462) \quad (9.3.1) \\ \quad (0.0000)^* \quad (0.0182)^* \quad (0.0006)^* \\ R^2 = 0.0322 \end{array}$$

where  $Y$  = hourly wage (\$)

$D_2$  = married status, 1 = married, 0 = otherwise

$D_3$  = region of residence; 1 = South, 0 = otherwise

and \* denotes the  $p$  values.

In this example we have two qualitative regressors, each with two categories. Hence we have assigned a single dummy variable for each category.

Which is the benchmark category here? Obviously, it is unmarried, non-South residence. In other words, unmarried persons who do not live in the South are the omitted category. Therefore, all comparisons are made in relation to this group. The mean hourly wage in this benchmark is about \$8.81. Compared with this, the average hourly wage of those who are married is higher by about \$1.10, for an actual average wage of \$9.91 (= 8.81 + 1.10). By contrast, for those who live in the South, the average hourly wage is lower by about \$1.67, for an actual average hourly wage of \$7.14.

Are the preceding average hourly wages statistically different compared to the base category? They are, for all the differential intercepts are statistically significant, as their  $p$  values are quite low.

The point to note about this example is this: *Once you go beyond one qualitative variable, you have to pay close attention to the category that is treated as the base category, since all comparisons are made in relation to that category. This is especially important when you have several qualitative regressors, each with several categories.* But the mechanics of introducing several qualitative variables should be clear by now.

### 9.4 REGRESSION WITH A MIXTURE OF QUANTITATIVE AND QUALITATIVE REGRESSORS: THE ANCOVA MODELS

ANOVA models of the type discussed in the preceding two sections, although common in fields such as sociology, psychology, education, and market research, are not that common in economics. Typically, in most economic research a regression model contains some explanatory variables that are quantitative and some that are qualitative. Regression models containing an admixture of quantitative and qualitative variables are called **analysis of covariance (ANCOVA) models**. ANCOVA models are an extension of the ANOVA models in that they provide a method of statistically controlling the effects of quantitative regressors, called **covariates** or **control**

<sup>8</sup>The data are obtained from the data disk in Arthur S. Goldberger, *Introductory Econometrics*, Harvard University Press, Cambridge, Mass., 1998. We have already considered these data in Chap. 2.



**variables**, in a model that includes both quantitative and qualitative, or dummy, regressors. We now illustrate the ANCOVA models.

To motivate the analysis, let us reconsider Example 9.1 by maintaining that the average salary of public school teachers may not be different in the three regions if we take into account any variables that cannot be standardized across the regions. Consider, for example, the variable *expenditure on public schools by local authorities*, as public education is primarily a local and state question. To see if this is the case, we develop the following model:

$$Y_i = \beta_1 + \beta_2 D_{2i} + \beta_3 D_{3i} + \beta_4 X_i + u_i \quad (9.4.1)$$

where  $Y_i$  = average annual salary of public school teachers in state (\$)

$X_i$  = spending on public school per pupil (\$)

$D_{2i}$  = 1, if the state is in the Northeast or North Central  
= 0, otherwise

$D_{3i}$  = 1, if the state is in the South  
= 0, otherwise

The data on  $X$  are given in Table 9.1. Keep in mind that we are treating the West as the benchmark category. Also, note that besides the two qualitative regressors, we have a quantitative variable,  $X$ , which in the context of the ANCOVA models is known as a **covariate**, as noted earlier.

**EXAMPLE 9.3**

TEACHER'S SALARY IN RELATION TO REGION AND SPENDING ON PUBLIC SCHOOL PER PUPIL

From the data in Table 9.1, the results of the model (9.4.1) are as follows:

$$\begin{array}{ccccccc} \hat{Y}_i & = & 13,269.11 & - & 1673.514D_{2i} & - & 1144.157D_{3i} & + & 3.2889X_i \\ \text{se} & = & (1395.056) & & (801.1703) & & (861.1182) & & (0.3176) \\ t & = & (9.5115)^* & & (-2.0889)^* & & (-1.3286)^{**} & & (10.3539)^* \end{array} \quad (9.4.2)$$

$R^2 = 0.7266$

where \* indicates  $p$  values less than 5 percent, and \*\* indicates  $p$  values greater than 5 percent.

As these results suggest, *ceteris paribus*: as public expenditure goes up by a dollar, on average, a public school teacher's salary goes up by about \$3.29. Controlling for spending on education, we now see that the differential intercept coefficient is significant for the Northeast and North-Central region, but not for the South. These results are different from those of (9.2.5). But this should not be surprising, for in (9.2.5) we did not account for the covariate, differences in per pupil public spending on education. Diagrammatically, we have the situation shown in Figure 9.2.

Note that although we have shown three regression lines for the three regions, statistically the regression lines are the same for the West and the South. Also note that the three regression lines are drawn parallel (why?).

(Continued)



EXAMPLE 9.3 (Continued)

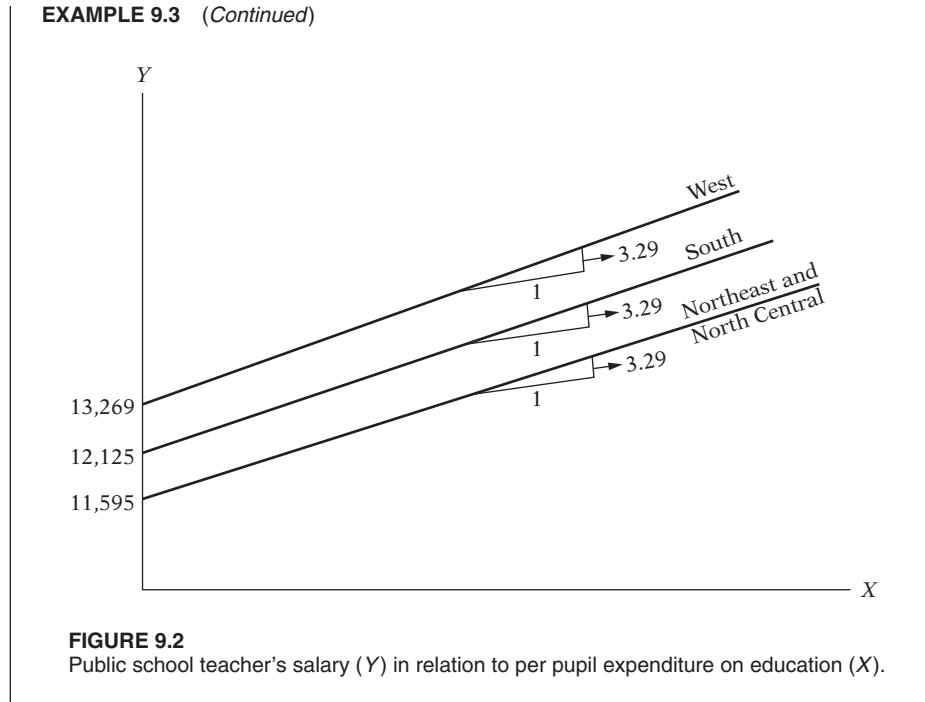


FIGURE 9.2  
Public school teacher's salary ( $Y$ ) in relation to per pupil expenditure on education ( $X$ ).

### 9.5 THE DUMMY VARIABLE ALTERNATIVE TO THE CHOW TEST<sup>9</sup>

In Section 8.8 we discussed the Chow test to examine the structural stability of a regression model. The example we discussed there related to the relationship between savings and income in the United States over the period 1970–1995. We divided the sample period into two, 1970–1981 and 1982–1995, and showed on the basis of the Chow test that there was a difference in the regression of savings on income between the two periods.

However, we could not tell whether the difference in the two regressions was because of differences in the intercept terms or the slope coefficients or both. Very often this knowledge itself is very useful.

Referring to Eqs. (8.8.1) and (8.8.2), we see that there are four possibilities, which we illustrate in Figure 9.3.

1. Both the intercept and the slope coefficients are the same in the two regressions. This, the case of **coincident regressions**, is shown in Figure 9.3a.

2. Only the intercepts in the two regressions are different but the slopes are the same. This is the case of **parallel regressions**, which is shown in Figure 9.3b.

<sup>9</sup>The material in this section draws on the author's articles, "Use of Dummy Variables in Testing for Equality between Sets of Coefficients in Two Linear Regressions: A Note," and "Use of Dummy Variables . . . A Generalization," both published in the *American Statistician*, vol. 24, nos. 1 and 5, 1970, pp. 50–52 and 18–21.

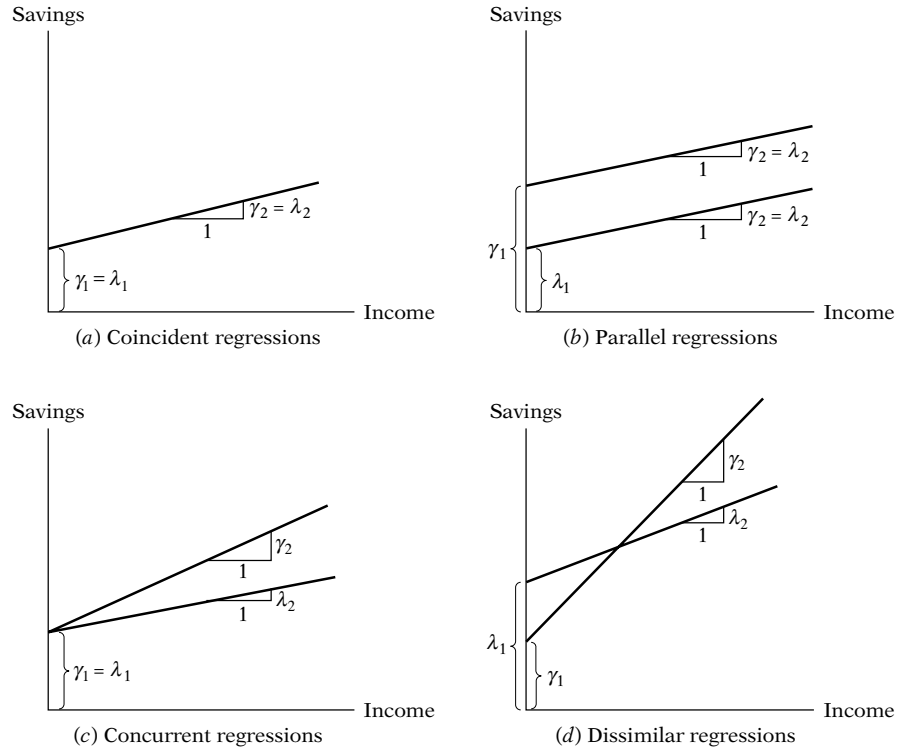


FIGURE 9.3 Plausible savings–income regressions.

3. The intercepts in the two regressions are the same, but the slopes are different. This is the situation of **concurrent regressions** (Figure 9.3c).

4. Both the intercepts and slopes in the two regressions are different. This is the case of **dissimilar regressions**, which is shown in Figure 9.3d.

The multistep Chow test procedure discussed in Section 8.8, as noted earlier, tells us only if two (or more) regressions are different without telling us what is the source of the difference. The source of difference, if any, can be pinned down by pooling all the observations (26 in all) and running just one multiple regression as shown below<sup>10</sup>:

$$Y_t = \alpha_1 + \alpha_2 D_t + \beta_1 X_t + \beta_2 (D_t X_t) + u_t \quad (9.5.1)$$

where  $Y$  = savings

$X$  = income

$t$  = time

$D = 1$  for observations in 1982–1995

$= 0$ , otherwise (i.e., for observations in 1970–1981)

<sup>10</sup>As in the Chow test, the pooling technique assumes homoscedasticity, that is,  $\sigma_1^2 = \sigma_2^2 = \sigma^2$ .

**TABLE 9.2** SAVINGS AND INCOME DATA, UNITED STATES, 1970–1995

Observation	Savings	Income	Dum
1970	61	727.1	0
1971	68.6	790.2	0
1972	63.6	855.3	0
1973	89.6	965	0
1974	97.6	1054.2	0
1975	104.4	1159.2	0
1976	96.4	1273	0
1977	92.5	1401.4	0
1978	112.6	1580.1	0
1979	130.1	1769.5	0
1980	161.8	1973.3	0
1981	199.1	2200.2	0
1982	205.5	2347.3	1
1983	167	2522.4	1
1984	235.7	2810	1
1985	206.2	3002	1
1986	196.5	3187.6	1
1987	168.4	3363.1	1
1988	189.1	3640.8	1
1989	187.8	3894.5	1
1990	208.7	4166.8	1
1991	246.4	4343.7	1
1992	272.6	4613.7	1
1993	214.4	4790.2	1
1994	189.4	5021.7	1
1995	249.3	5320.8	1

Note: Dum = 1 for observations beginning in 1982; 0 otherwise.  
Savings and income figures are in billions of dollars.  
Source: *Economic Report of the President*, 1997, Table B-28, p. 332.

Table 9.2 shows the structure of the data matrix.

To see the implications of (9.5.1), and, assuming, as usual, that  $E(u_t) = 0$ , we obtain:

*Mean savings function for 1970–1981:*

$$E(Y_t | D_t = 0, X_t) = \alpha_1 + \beta_1 X_t \quad (9.5.2)$$

*Mean savings function for 1982–1995:*

$$E(Y_t | D_t = 1, X_t) = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) X_t \quad (9.5.3)$$

The reader will notice that these are the same functions as (8.8.1) and (8.8.2), with  $\lambda_1 = \alpha_1$ ,  $\lambda_2 = \beta_1$ ,  $\gamma_1 = (\alpha_1 + \alpha_2)$ , and  $\gamma_2 = (\beta_1 + \beta_2)$ . Therefore, estimating (9.5.1) is equivalent to estimating the two individual savings functions (8.8.1) and (8.8.2).

In (9.5.1),  $\alpha_2$  is the **differential intercept**, as previously, and  $\beta_2$  is the **differential slope coefficient** (also called the **slope drifter**), indicating by how much the slope coefficient of the second period's savings function (the

category that receives the dummy value of 1) differs from that of the first period. Notice how the introduction of the dummy variable  $D$  in the **interactive**, or **multiplicative form** ( $D$  multiplied by  $X$ ) enables us to differentiate between slope coefficients of the two periods, just as the introduction of the dummy variable in the **additive form** enabled us to distinguish between the intercepts of the two periods.

**EXAMPLE 9.4**STRUCTURAL DIFFERENCES IN THE U.S. SAVINGS-INCOME REGRESSION,  
THE DUMMY VARIABLE APPROACH

Before we proceed further, let us first present the regression results of model (9.5.1) applied to the U.S. savings-income data.

$$\begin{aligned} \hat{Y}_t &= 1.0161 + 152.4786D_t + 0.0803X_t - 0.0655(D_tX_t) \\ \text{se} &= (20.1648) \quad (33.0824) \quad (0.0144) \quad (0.0159) & (9.5.4) \\ t &= (0.0504)** \quad (4.6090)* \quad (5.5413)* \quad (-4.0963)* \\ R^2 &= 0.8819 \end{aligned}$$

where \* indicates  $p$  values less than 5 percent and \*\* indicates  $p$  values greater than 5 percent.

As these regression results show, both the differential intercept and slope coefficients are statistically significant, strongly suggesting that the savings-income regressions for the two time periods are different, as in Figure 9.3d.

From (9.5.4), we can derive equations (9.5.2) and (9.5.3), which are:

*Savings-income regression, 1970–1981:*

$$\hat{Y}_t = 1.0161 + 0.0803X_t \quad (9.5.5)$$

*Savings-income regression, 1982–1995:*

$$\begin{aligned} \hat{Y}_t &= (1.0161 + 152.4786) + (0.0803 - 0.0655)X_t \\ &= 153.4947 + 0.0148X_t \end{aligned} \quad (9.5.6)$$

These are precisely the results we obtained in (8.8.1a) and (8.8.2a), which should not be surprising. These regressions are already shown in Figure 8.3.

The advantages of the dummy variable technique [i.e., estimating (9.5.1)] over the Chow test [i.e., estimating the three regressions (8.8.1), (8.8.2), and (8.8.3)] can now be seen readily:

1. We need to run only a single regression because the individual regressions can easily be derived from it in the manner indicated by equations (9.5.2) and (9.5.3).
2. The single regression (9.5.1) can be used to test a variety of hypotheses. Thus if the *differential intercept* coefficient  $\alpha_2$  is statistically insignificant, we may accept the hypothesis that the two regressions have the same intercept, that is, the two regressions are concurrent (see Figure 9.3c). Similarly, if the *differential slope* coefficient  $\beta_2$  is statistically insignificant but  $\alpha_2$  is significant, we may not reject the hypothesis that the two regressions have the same slope, that is, the two regression lines are parallel (cf. Figure 9.3b). The test of the stability of the entire regression (i.e.,  $\alpha_2 = \beta_2 = 0$ , simultaneously) can be made by the usual  $F$  test (recall the restricted least-squares  $F$  test). If this hypothesis is not rejected, the regression lines will be coincident, as shown in Figure 9.3a.

(Continued)

**EXAMPLE 9.4** (Continued)

3. The Chow test does not explicitly tell us *which* coefficient, intercept, or slope is different, or whether (as in this example) both are different in the two periods. That is, one can obtain a significant Chow test because the *slope* only is different or the *intercept* only is different, or both are different. In other words, we cannot tell, via the Chow test, which one of the four possibilities depicted in Figure 9.2 exists in a given instance. In this respect, the dummy variable approach has a distinct advantage, for it not only tells if the two are different but also pinpoints the source(s) of the difference—whether it is due to the intercept or the slope or both. In practice, the knowledge that two regressions differ in this or that coefficient is as important as, if not more than, the plain knowledge that they are different.
4. Finally, since pooling (i.e., including all the observations in one regression) increases the degrees of freedom, it may improve the relative precision of the estimated parameters. Of course, keep in mind that every addition of a dummy variable will consume one degree of freedom.

**9.6 INTERACTION EFFECTS USING DUMMY VARIABLES**

Dummy variables are a flexible tool that can handle a variety of interesting problems. To see this, consider the following model:

$$Y_i = \alpha_1 + \alpha_2 D_{2i} + \alpha_3 D_{3i} + \beta X_i + u_i \quad (9.6.1)$$

where  $Y$  = hourly wage in dollars

$X$  = education (years of schooling)

$D_2 = 1$  if female, 0 otherwise

$D_3 = 1$  if nonwhite and non-Hispanic, 0 otherwise

In this model gender and race are qualitative regressors and education is a quantitative regressor.<sup>11</sup> Implicit in this model is the assumption that the differential effect of the gender dummy  $D_2$  is constant across the two categories of race and the differential effect of the race dummy  $D_3$  is also constant across the two sexes. That is to say, if the mean salary is higher for males than for females, this is so whether they are nonwhite/non-Hispanic or not. Likewise, if, say, nonwhite/non-Hispanics have lower mean wages, this is so whether they are females or males.

In many applications such an assumption may be untenable. A female nonwhite/non-Hispanic may earn lower wages than a male nonwhite/non-Hispanic. In other words, there may be **interaction** between the two qualitative variables  $D_2$  and  $D_3$ . Therefore their effect on mean  $Y$  may not be simply **additive** as in (9.6.1) but **multiplicative** as well, as in the following model.

$$\hat{Y}_i = \alpha_1 + \alpha_2 D_{2i} + \alpha_3 D_{3i} + \alpha_4 (D_{2i} D_{3i}) + \beta X_i + u_i \quad (9.6.2)$$

where the variables are as defined for model (9.6.1).

From (9.6.2), we obtain:

$$E(Y_i | D_{2i} = 1, D_{3i} = 1, X_i) = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + \beta X_i \quad (9.6.3)$$

<sup>11</sup>If we were to define education as less than high school, high school, and more than high school, we could then use two dummies to represent the three classes.

which is the mean hourly wage function for female nonwhite/non-Hispanic workers. Observe that

$$\begin{aligned}\alpha_2 &= \text{differential effect of being a female} \\ \alpha_3 &= \text{differential effect of being a nonwhite/non-Hispanic} \\ \alpha_4 &= \text{differential effect of being a female nonwhite/non-Hispanic}\end{aligned}$$

which shows that the mean hourly wages of female nonwhite/non-Hispanics is different (by  $\alpha_4$ ) from the mean hourly wages of females or nonwhite/non-Hispanics. If, for instance, all the three differential dummy coefficients are negative, this would imply that female nonwhite/non-Hispanic workers earn much lower mean hourly wages than female or nonwhite/non-Hispanic workers as compared with the base category, which in the present example is male white or Hispanic.

Now the reader can see how the **interaction dummy** (i.e., the product of two qualitative or dummy variables) modifies the effect of the two attributes considered individually (i.e., additively).

#### EXAMPLE 9.5

##### AVERAGE HOURLY EARNINGS IN RELATION TO EDUCATION, GENDER, AND RACE

Let us first present the regression results based on model (9.6.1). Using the data that were used to estimate regression (9.3.1), we obtained the following results:

$$\begin{aligned}\hat{Y}_i &= -0.2610 - 2.3606D_{2i} - 1.7327D_{3i} + 0.8028X_i \\ t &= (-0.2357)** \quad (-5.4873)* \quad (-2.1803)* \quad (9.9094)* \quad (9.6.4) \\ &R^2 = 0.2032 \quad n = 528\end{aligned}$$

where \* indicates  $p$  values less than 5 percent and \*\* indicates  $p$  values greater than 5 percent.

The reader can check that the differential intercept coefficients are statistically significant, that they have the expected signs (why?), and that education has a strong positive effect on hourly wage, an unsurprising finding.

As (9.6.4) shows, *ceteris paribus*, the average hourly earnings of females are lower by about \$2.36, and the average hourly earnings of nonwhite non-Hispanic workers are also lower by about \$1.73.

We now consider the results of model (9.6.2), which includes the interaction dummy.

$$\begin{aligned}\hat{Y}_i &= -0.26100 - 2.3606D_{2i} - 1.7327D_{3i} + 2.1289D_{2i}D_{3i} + 0.8028X_i \\ t &= (-0.2357)** \quad (-5.4873)* \quad (-2.1803)* \quad (1.7420)** \quad (9.9095)** \quad (9.6.5) \\ &R^2 = 0.2032 \quad n = 528\end{aligned}$$

where \* indicates  $p$  values less than 5 percent and \*\* indicates  $p$  values greater than 5 percent.

As you can see, the two additive dummies are still statistically significant, but the interactive dummy is not at the conventional 5 percent level; the actual  $p$  value of the interaction dummy is about the 8 percent level. If you think this is a low enough probability, then the results of (9.6.5) can be interpreted as follows: Holding the level of education constant, if you add the three dummy coefficients you will obtain:  $-1.964$  ( $= -2.3605 - 1.7327 + 2.1289$ ), which means that mean hourly wages of nonwhite/non-Hispanic female workers is lower by about \$1.96, which is between the value of  $-2.3605$  (gender difference alone) and  $-1.7327$  (race difference alone).

The preceding example clearly reveals the role of interaction dummies when two or more qualitative regressors are included in the model. It is important to note that in the model (9.6.5) we are assuming that the rate of increase of hourly earnings with respect to education (of about 80 cents per additional year of schooling) remains constant across gender and race. But this may not be the case. If you want to test for this, you will have to introduce differential slope coefficients (see exercise 9.25)

### 9.7 THE USE OF DUMMY VARIABLES IN SEASONAL ANALYSIS

Many economic time series based on monthly or quarterly data exhibit seasonal patterns (regular oscillatory movements). Examples are sales of department stores at Christmas and other major holiday times, demand for money (or cash balances) by households at holiday times, demand for ice cream and soft drinks during summer, prices of crops right after harvesting season, demand for air travel, etc. Often it is desirable to remove the seasonal factor, or *component*, from a time series so that one can concentrate on the other components, such as the trend.<sup>12</sup> The process of removing the seasonal component from a time series is known as **deseasonalization** or **seasonal adjustment**, and the time series thus obtained is called the **deseasonalized**, or **seasonally adjusted**, time series. Important economic time series, such as the unemployment rate, the consumer price index (CPI), the producer's price index (PPI), and the index of industrial production, are usually published in seasonally adjusted form.

There are several methods of deseasonalizing a time series, but we will consider only one of these methods, namely, the *method of dummy variables*.<sup>13</sup> To illustrate how the dummy variables can be used to deseasonalize economic time series, consider the data given in Table 9.3. This table gives quarterly data for the years 1978–1995 on the sale of four major appliances, dishwashers, garbage disposers, refrigerators, and washing machines, all data in thousands of units. The table also gives data on durable goods expenditure in 1982 billions of dollars.

To illustrate the dummy technique, we will consider only the sales of refrigerators over the sample period. But first let us look at the data, which is shown in Figure 9.4. This figure suggests that perhaps there is a seasonal pattern in the data associated with the various quarters. To see if this is the case, consider the following model:

$$Y_t = \alpha_1 D_{1t} + \alpha_2 D_{2t} + \alpha_3 D_{3t} + \alpha_4 D_{4t} + u_t \quad (9.7.1)$$

where  $Y_t$  = sales of refrigerators (in thousands) and the  $D$ 's are the dummies, taking a value of 1 in the relevant quarter and 0 otherwise. *Note that*

<sup>12</sup>A time series may contain four components: a **seasonal**, a **cyclical**, a **trend**, and one that is strictly random.

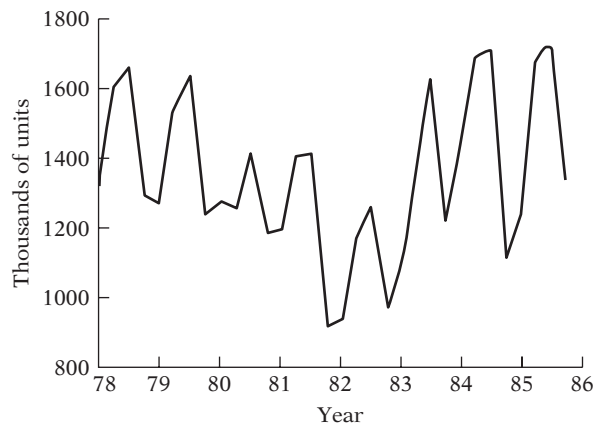
<sup>13</sup>For the various methods of seasonal adjustment, see, for instance, Francis X. Diebold, *Elements of Forecasting*, 2d ed., South-Western Publishers, 2001, Chap. 5.

**TABLE 9.3** QUARTERLY DATA ON APPLIANCE SALES (IN THOUSANDS) AND EXPENDITURE ON DURABLE GOODS (1978-I TO 1985-IV)

DISH	DISP	FRIG	WASH	DUR	DISH	DISP	FRIG	WASH	DUR
841	798	1317	1271	252.6	480	706	943	1036	247.7
957	837	1615	1295	272.4	530	582	1175	1019	249.1
999	821	1662	1313	270.9	557	659	1269	1047	251.8
960	858	1295	1150	273.9	602	837	973	918	262
894	837	1271	1289	268.9	658	867	1102	1137	263.3
851	838	1555	1245	262.9	749	860	1344	1167	280
863	832	1639	1270	270.9	827	918	1641	1230	288.5
878	818	1238	1103	263.4	858	1017	1225	1081	300.5
792	868	1277	1273	260.6	808	1063	1429	1326	312.6
589	623	1258	1031	231.9	840	955	1699	1228	322.5
657	662	1417	1143	242.7	893	973	1749	1297	324.3
699	822	1185	1101	248.6	950	1096	1117	1198	333.1
675	871	1196	1181	258.7	838	1086	1242	1292	344.8
652	791	1410	1116	248.4	884	990	1684	1342	350.3
628	759	1417	1190	255.5	905	1028	1764	1323	369.1
529	734	919	1125	240.4	909	1003	1328	1274	356.4

Note: DISH = dishwashers; DISP = garbage disposers; FRIG = refrigerators; WASH = dishwashers; DUR = durable goods expenditure, billions of 1992 dollars.

Source: *Business Statistics and Survey of Current Business*, Department of Commerce (various issues).



**FIGURE 9.4** Sales of refrigerators 1978–1985 (quarterly).

to avoid the dummy variable trap, we are assigning a dummy to each quarter of the year, but omitting the intercept term. If there is any seasonal effect in a given quarter, that will be indicated by a statistically significant  $t$  value of the dummy coefficient for that quarter.<sup>14</sup>

<sup>14</sup>Note a technical point. This method of assigning a dummy to each quarter assumes that the seasonal factor, if present, is deterministic and not stochastic. We will revisit this topic when we discuss time series econometrics in Part V of this book.



Notice that in (9.7.1) we are regressing  $Y$  effectively on an intercept, except that we allow for a different intercept in each season (i.e., quarter). As a result, the dummy coefficient of each quarter will give us the mean refrigerator sales in each quarter or season (why?).

**EXAMPLE 9.6**

SEASONALITY IN REFRIGERATOR SALES

From the data on refrigerator sales given in Table 9.3, we obtain the following regression results:

$$\hat{Y}_t = 1222.125D_{1t} + 1467.500D_{2t} + 1569.750D_{3t} + 1160.000D_{4t}$$

$$t = (20.3720) \quad (24.4622) \quad (26.1666) \quad (19.3364) \quad (9.7.2)$$

$$R^2 = 0.5317$$

*Note:* We have not given the standard errors of the estimated coefficients, as each standard error is equal to 59.9904, because all the dummies take only a value of 1 or zero.

The estimated  $\alpha$  coefficients in (9.7.2) represent the average, or *mean*, sales of refrigerators (in thousands of units) in each season (i.e., quarter). Thus, the average sale of refrigerators in the first quarter, in thousands of units, is about 1222, that in the second quarter about 1468, that in the third quarter about 1570, and that in the fourth quarter about 1160.

**TABLE 9.4** U.S. REFRIGERATOR SALES (THOUSANDS), 1978–1995 (QUARTERLY)

FRIG	DUR	$D_2$	$D_3$	$D_4$	FRIG	DUR	$D_2$	$D_3$	$D_4$
1317	252.6	0	0	0	943	247.7	0	0	0
1615	272.4	1	0	0	1175	249.1	1	0	0
1662	270.9	0	1	0	1269	251.8	0	1	0
1295	273.9	0	0	1	973	262.0	0	0	1
1271	268.9	0	0	0	1102	263.3	0	0	0
1555	262.9	1	0	0	1344	280.0	1	0	0
1639	270.9	0	1	0	1641	288.5	0	1	0
1238	263.4	0	0	1	1225	300.5	0	0	1
1277	260.6	0	0	0	1429	312.6	0	0	0
1258	231.9	1	0	0	1699	322.5	1	0	0
1417	242.7	0	1	0	1749	324.3	0	1	0
1185	248.6	0	0	1	1117	333.1	0	0	1
1196	258.7	0	0	0	1242	344.8	0	0	0
1410	248.4	1	0	0	1684	350.3	1	0	0
1417	255.5	0	1	0	1764	369.1	0	1	0
919	240.4	0	0	1	1328	356.4	0	0	1

*Note:* FRIG = refrigerator sales, thousands  
 DUR = durable goods expenditure, billions of 1992 dollars  
 $D_2$  = 1 in the second quarter, 0 otherwise  
 $D_3$  = 1 in the third quarter, 0 otherwise  
 $D_4$  = 1 in the fourth quarter, 0 otherwise

*Source:* *Business Statistics and Survey of Current Business*, Department of Commerce (various issues).

(Continued)

**EXAMPLE 9.6** (Continued)

Incidentally, instead of assigning a dummy for each quarter and suppressing the intercept term to avoid the dummy variable trap, we could assign only three dummies and include the intercept term. Suppose we treat the first quarter as the reference quarter and assign dummies to the second, third, and fourth quarters. This produces the following regression results (see Table 9.4 for the data setup):

$$\begin{aligned} \hat{Y}_t &= 1222.1250 + 245.3750D_{2t} + 347.6250D_{3t} - 62.1250D_{4t} \\ t &= (20.3720)^* \quad (2.8922)^* \quad (4.0974)^* \quad (-0.7322)^{**} \quad (9.7.3) \\ R^2 &= 0.5318 \end{aligned}$$

where \* indicates  $p$  values less than 5 percent and \*\* indicates  $p$  values greater than 5 percent.

Since we are treating the first quarter as the benchmark, the coefficients attached to the various dummies are now *differential intercepts*, showing by how much the *average value* of  $Y$  in the quarter that receives a dummy value of 1 differs from that of the benchmark quarter. Put differently, the coefficients on the seasonal dummies will give the seasonal increase or decrease in the average value of  $Y$  relative to the base season. If you add the various differential intercept values to the benchmark average value of 1222.125, you will get the average value for the various quarters. Doing so, you will reproduce exactly Eq. (9.7.2), except for the rounding errors.

But now you will see the value of treating one quarter as the benchmark quarter, for (9.7.3) shows that the average value of  $Y$  for the fourth quarter is not statistically different from the average value for the first quarter, as the dummy coefficient for the fourth quarter is not statistically significant. Of course, your answer will change, depending on which quarter you treat as the benchmark quarter, but the overall conclusion will not change.

How do we obtain the deseasonalized time series of refrigerator sales? This can be done easily. You estimate the values of  $Y$  from model (9.7.2) [or (9.7.3)] for each observation and subtract them from the actual values of  $Y$ , that is, you obtain  $(Y_t - \hat{Y}_t)$  which are simply the residuals from the regression (9.7.2). We show them in Table 9.5.<sup>15</sup>

What do these residuals represent? They represent the remaining components of the refrigerator time series, namely, the trend, cycle, and random components (but see the caution given in footnote 15).

Since models (9.7.2) and (9.7.3) do not contain any covariates, will the picture change if we bring in a quantitative regressor in the model? Since expenditure on durable goods has an important factor influence on the demand for refrigerators, let us expand our model (9.7.3) by bringing in this variable. The data for durable goods expenditure in billions of 1982 dollars are already given in Table 9.3. This is our (quantitative)  $X$  variable in the model. The regression results are as follows

$$\begin{aligned} \hat{Y}_t &= 456.2440 + 242.4976D_{2t} + 325.2643D_{3t} - 86.0804D_{4t} + 2.7734X_t \\ t &= (2.5593)^* \quad (3.6951)^* \quad (4.9421)^* \quad (-1.3073)^{**} \quad (4.4496)^* \quad (9.7.4) \\ R^2 &= 0.7298 \end{aligned}$$

where \* indicates  $p$  values less than 5 percent and \*\* indicates  $p$  values greater than 5 percent.

(Continued)

<sup>15</sup>Of course, this assumes that the dummy variables technique is an appropriate method of deseasonalizing a time series and that a time series (TS) can be represented as:  $TS = s + c + t + u$ , where  $s$  represents the seasonal,  $t$  the trend,  $c$  the cyclical, and  $u$  the random component. However, if the time series is of the form,  $TS = (s)(c)(t)(u)$ , where the four components enter multiplicatively, the preceding method of deseasonalization is inappropriate, for that method assumes that the four components of a time series are additive. But we will have more to say about this topic in the chapters on time series econometrics.

**EXAMPLE 9.6** (Continued)

**TABLE 9.5**  
REFRIGERATOR SALES REGRESSION: ACTUAL, FITTED, AND RESIDUAL  
VALUES (EQ. 9.7.3)

	Actual	Fitted	Residuals	Residual graph 0	
1978-I	1317	1222.12	94.875	.	*
1978-II	1615	1467.50	147.500	.	*
1978-III	1662	1569.75	92.250	.	*
1978-IV	1295	1160.00	135.000	.	*
1979-I	1271	1222.12	48.875	.	*
1979-II	1555	1467.50	87.500	.	*
1979-III	1639	1569.75	69.250	.	*
1979-IV	1238	1160.00	78.000	.	*
1980-I	1277	1222.12	54.875	.	*
1980-II	1258	1467.50	-209.500	*	.
1980-III	1417	1569.75	-152.750	*	.
1980-IV	1185	1160.00	25.000	.	*
1981-I	1196	1222.12	-26.125	.	*
1981-II	1410	1467.50	-57.500	.	*
1981-III	1417	1569.75	-152.750	.	*
1981-IV	919	1160.00	-241.000	*	.
1982-I	943	1222.12	-279.125	*	.
1982-II	1175	1467.50	-292.500	*	.
1982-III	1269	1569.75	-300.750	*	.
1982-IV	973	1160.00	-187.000	*	.
1983-I	1102	1222.12	-120.125	.	*
1983-II	1344	1467.50	-123.500	.	*
1983-III	1641	1569.75	71.250	.	*
1983-IV	1225	1160.00	65.000	.	*
1984-I	1429	1222.12	206.875	.	*
1984-II	1699	1467.50	231.500	.	*
1984-III	1749	1569.75	179.250	.	*
1984-IV	1117	1160.00	-43.000	.	*
1985-I	1242	1222.12	19.875	.	*
1985-II	1684	1467.50	216.500	.	*
1985-III	1764	1569.75	194.250	.	*
1985-IV	1328	1160.00	168.000	.	*
				-	0 +

Again, keep in mind that we are treating the first quarter as our base. As in (9.7.3), we see that the differential intercept coefficients for the second and third quarters are statistically different from that of the first quarter, but the intercepts of the fourth quarter and the first quarter are statistically about the same. The coefficient of  $X$  (durable goods expenditure) of about 2.77 tells us that, allowing for seasonal effects, if expenditure on durable goods goes up by a dollar, on average, sales of refrigerators go up by about 2.77 units, that is, approximately 3 units; bear in mind that refrigerators are in thousands of units and  $X$  is in (1982) billions of dollars.

(Continued)

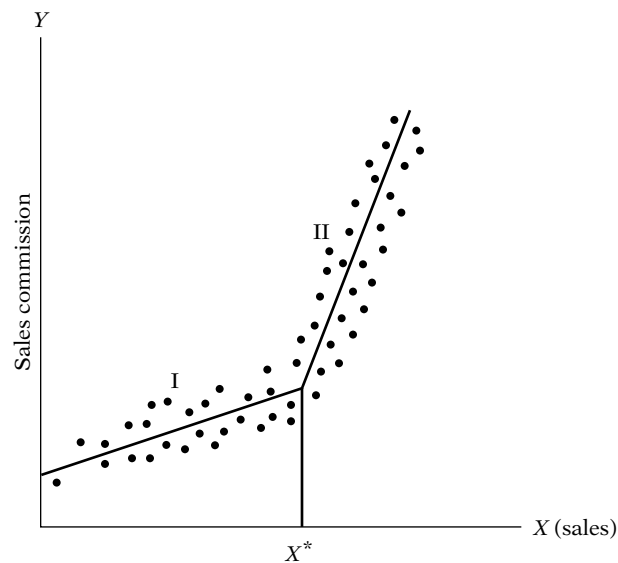
**EXAMPLE 9.6** (Continued)

An interesting question here is: Just as sales of refrigerators exhibit seasonal patterns, would not expenditure on durable goods also exhibit seasonal patterns? How then do we take into account seasonality in  $X$ ? The interesting thing about (9.7.4) is that the dummy variables in that model not only remove the seasonality in  $Y$  but also the seasonality, if any, in  $X$ . (This follows from a well-known theorem in statistics, known as the **Frisch-Waugh theorem**.<sup>16</sup>) So to speak, we kill (deseasonalize) two birds (two series) with one stone (the dummy technique).

If you want an informal proof of the preceding statement, just follow these steps: (1) Run the regression of  $Y$  on the dummies as in (9.7.2) or (9.7.3) and save the residuals, say,  $S_1$ ; these residuals represent deseasonalized  $Y$ . (2) Run a similar regression for  $X$  and obtain the residuals from this regression, say,  $S_2$ ; these residuals represent deseasonalized  $X$ . (3) Regress  $S_1$  on  $S_2$ . You will find that the slope coefficient in this regression is precisely the coefficient of  $X$  in the regression (9.7.4).

**9.8 PIECEWISE LINEAR REGRESSION**

To illustrate yet another use of dummy variables, consider Figure 9.5, which shows how a hypothetical company remunerates its sales representatives. It pays commissions based on sales in such a manner that up to a certain level, the *target*, or *threshold*, level  $X^*$ , there is one (stochastic) commission structure and beyond that level another. (*Note:* Besides sales, other factors affect sales commission. Assume that these other factors are represented



**FIGURE 9.5** Hypothetical relationship between sales commission and sales volume. (*Note:* The intercept on the  $Y$  axis denotes minimum guaranteed commission.)

<sup>16</sup>For proof, see Adrian C. Darnell, *A Dictionary of Econometrics*, Edward Elgar, Lyme, U.K., 1995, pp. 150–152.

by the stochastic disturbance term.) More specifically, it is assumed that sales commission increases linearly with sales until the threshold level  $X^*$ , after which also it increases linearly with sales but at a much steeper rate. Thus, we have a **piecewise linear regression** consisting of two linear pieces or segments, which are labeled I and II in Figure 9.5, and the commission function changes its slope at the threshold value. Given the data on commission, sales, and the value of the threshold level  $X^*$ , the technique of dummy variables can be used to estimate the (differing) slopes of the two segments of the piecewise linear regression shown in Figure 9.5. We proceed as follows:

$$Y_i = \alpha_1 + \beta_1 X_i + \beta_2 (X_i - X^*) D_i + u_i \quad (9.8.1)$$

where  $Y_i$  = sales commission

$X_i$  = volume of sales generated by the sales person

$X^*$  = threshold value of sales also known as a **knot** (known in advance)<sup>17</sup>

$$D = 1 \quad \text{if } X_i > X^* \\ = 0 \quad \text{if } X_i < X^*$$

Assuming  $E(u_i) = 0$ , we see at once that

$$E(Y_i | D_i = 0, X_i, X^*) = \alpha_1 + \beta_1 X_i \quad (9.8.2)$$

which gives the mean sales commission up to the target level  $X^*$  and

$$E(Y_i | D_i = 1, X_i, X^*) = \alpha_1 - \beta_2 X^* + (\beta_1 + \beta_2) X_i \quad (9.8.3)$$

which gives the mean sales commission beyond the target level  $X^*$ .

Thus,  $\beta_1$  gives the slope of the regression line in segment I, and  $\beta_1 + \beta_2$  gives the slope of the regression line in segment II of the piecewise linear regression shown in Figure 9.5. A test of the hypothesis that there is no break in the regression at the threshold value  $X^*$  can be conducted easily by noting the statistical significance of the estimated differential slope coefficient  $\beta_2$  (see Figure 9.6).

Incidentally, the piecewise linear regression we have just discussed is an example of a more general class of functions known as **spline functions**.<sup>18</sup>

<sup>17</sup>The threshold value may not always be apparent, however. An ad hoc approach is to plot the dependent variable against the explanatory variable(s) and observe if there seems to be a sharp change in the relation after a given value of  $X$  (i.e.,  $X^*$ ). An analytical approach to finding the break point can be found in the so-called **switching regression models**. But this is an advanced topic and a textbook discussion may be found in Thomas Fomby, R. Carter Hill, and Stanley Johnson, *Advanced Econometric Methods*, Springer-Verlag, New York, 1984, Chap. 14.

<sup>18</sup>For an accessible discussion on splines (i.e., piecewise polynomials of order  $k$ ), see Douglas C. Montgomery, Elizabeth A. Peck, and G. Geoffrey Vining, *Introduction to Linear Regression Analysis*, John Wiley & Sons, 3d ed., New York, 2001, pp. 228–230.

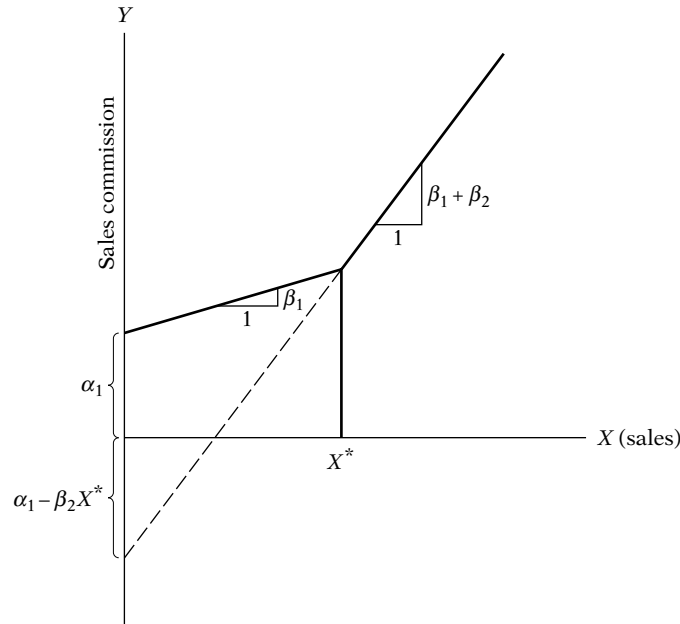


FIGURE 9.6 Parameters of the piecewise linear regression.

**EXAMPLE 9.7**

TOTAL COST IN RELATION TO OUTPUT

As an example of the application of the piecewise linear regression, consider the hypothetical total cost–total output data given in Table 9.6. We are told that the total cost may change its slope at the output level of 5500 units.

Letting  $Y$  in (9.8.4) represent total cost and  $X$  total output, we obtain the following results:

$$\hat{Y}_i = -145.72 + 0.2791X_i + 0.0945(X_i - X_i^*)D_i$$

$$t = (-0.8245) \quad (6.0669) \quad (1.1447) \quad \text{(9.8.4)}$$

$$R^2 = 0.9737 \quad X^* = 5500$$

As these results show, the marginal cost of production is about 28 cents per unit and although it is about 37 cents (28 + 9) for output over 5500 units, the difference between the two is not statistically significant because the dummy variable is not significant at, say, the

**TABLE 9.6**

HYPOTHETICAL DATA ON OUTPUT AND TOTAL COST

Total cost, dollars	Output, units
256	1,000
414	2,000
634	3,000
778	4,000
1,003	5,000
1,839	6,000
2,081	7,000
2,423	8,000
2,734	9,000
2,914	10,000

5 percent level. For all practical purposes, then, one can regress total cost on total output, dropping the dummy variable.

## 9.9 PANEL DATA REGRESSION MODELS

Recall that in Chapter 1 we discussed a variety of data that are available for empirical analysis, such as *cross-section*, *time series*, *pooled* (combination of time series and cross-section data), and *panel data*. The technique of dummy variable can be easily extended to pooled and panel data. Since the use of panel data is becoming increasingly common in applied work, we will consider this topic in some detail in Chapter 16.

## 9.10 SOME TECHNICAL ASPECTS OF THE DUMMY VARIABLE TECHNIQUE

### The Interpretation of Dummy Variables in Semilogarithmic Regressions

In Chapter 6 we discussed the log–lin models, where the regressand is logarithmic and the regressors are linear. In such a model, the slope coefficients of the regressors give the *semi*elasticity, that is, the percentage change in the regressand for a unit change in the regressor. *This is only so if the regressor is quantitative*. What happens if a regressor is a dummy variable? To be specific, consider the following model:

$$\ln Y_i = \beta_1 + \beta_2 D_i + u_i \quad (9.10.1)$$

where  $Y$  = hourly wage rate (\$) and  $D = 1$  for female and 0 for male.

How do we interpret such a model? Assuming  $E(u_i) = 0$ , we obtain:

*Wage function for male workers:*

$$E(\ln Y_i | D_i = 0) = \beta_1 \quad (9.10.2)$$

*Wage function for female workers:*

$$E(\ln Y_i | D_i = 1) = \beta_1 + \beta_2 \quad (9.10.3)$$

Therefore, the intercept  $\beta_1$  gives the *mean log hourly earnings* and the “slope” coefficient gives the difference in the mean log hourly earnings of male and females. This is a rather awkward way of stating things. But if we take the antilog of  $\beta_1$ , what we obtain is *not* the mean hourly wages of male workers, but their **median** wages. As you know, *mean*, *median*, and *mode* are the three measures of central tendency of a random variable. And if we take the antilog of  $(\beta_1 + \beta_2)$ , we obtain the median hourly wages of female workers.

**EXAMPLE 9.8**LOGARITHM OF HOURLY WAGES  
IN RELATION TO GENDER

To illustrate (9.10.1), we use the data that underlie Example 9.2. The regression results based on 528 observations are as follows:

$$\begin{aligned}\widehat{\ln Y}_i &= 2.1763 - 0.2437D_i \\ t &= (72.2943)^* \quad (-5.5048)^* \quad (9.10.4) \\ R^2 &= 0.0544\end{aligned}$$

where \* indicates  $p$  values are practically zero.

Taking the antilog of 2.1763, we find 8.8136 (\$), which is the median hourly earnings of male workers, and taking the antilog of  $[(2.1763 - 0.2437) = 1.92857]$ ,

we obtain 6.8796 (\$), which is the median hourly earnings of female workers. Thus, the female workers' median hourly earnings is lower by about 21.94 percent compared to their male counterparts  $[(8.8136 - 6.8796)/8.8136]$ .

Interestingly, we can obtain semielasticity for a dummy regressor directly by the device suggested by Halvorsen and Palmquist.<sup>19</sup> Take the antilog (to base  $e$ ) of the estimated dummy coefficient and subtract 1 from it and multiply the difference by 100. (For the underlying logic, see Appendix 9.A.1.) Therefore, if you take the antilog of  $-0.2437$ , you will obtain 0.78366. Subtracting 1 from this gives  $-0.2163$ , after multiplying this by 100, we get  $-21.63$  percent, suggesting that a female worker's ( $D = 1$ ) median salary is lower than that of her male counterpart by about 21.63 percent, the same as we obtained previously, save the rounding errors.

**Dummy Variables and Heteroscedasticity**

Let us revisit our savings–income regression for the United States for the periods 1970–1981 and 1982–1995 and for the entire period 1970–1995. In testing for structural stability using the dummy technique, we assumed that the error  $\text{var}(u_{1i}) = \text{var}(u_{2i}) = \sigma^2$ , that is, the error variances in the two periods were the same. This was also the assumption underlying the Chow test. If this assumption is not valid—that is, the error variances in the two subperiods are different—it is quite possible to draw misleading conclusions. Therefore, one must first check on the equality of variances in the subperiod, using suitable statistical techniques. Although we will discuss this topic more thoroughly in the chapter on heteroscedasticity, in Chapter 8 we showed how the  $F$  test can be used for this purpose.<sup>20</sup> (See our discussion of the Chow test in that chapter.) As we showed there, it seems the error variances in the two periods are not the same. Hence, the results of both the Chow test and the dummy variable technique presented before may not be entirely reliable. Of course, our purpose here is to illustrate the various techniques that one can use to handle a problem (e.g., the problem of structural stability). In any particular application, these techniques may not be valid. But that is par for most statistical techniques. Of course, one can take appropriate remedial actions to resolve the problem, as we will do in the chapter on heteroscedasticity later (however, see exercise 9.28).

<sup>19</sup>Robert Halvorsen and Raymond Palmquist, “The Interpretation of Dummy Variables in Semilogarithmic Equations,” *American Economic Review*, vol. 70, no. 3, pp. 474–475.

<sup>20</sup>The Chow test procedure can be performed even in the presence of heteroscedasticity, but then one will have to use the **Wald test**. The mathematics involved behind the test is somewhat involved. But in the chapter on heteroscedasticity, we will revisit this topic.



### Dummy Variables and Autocorrelation

Besides homoscedasticity, the classical linear regression model assumes that the error term in the regression models is uncorrelated. But what happens if that is not the case, especially in models involving dummy regressors? Since we will discuss the topic of autocorrelation in depth in the chapter on autocorrelation, we will defer the answer to this question until then.

### What Happens if the Dependent Variable Is a Dummy Variable?

So far we have considered models in which the regressand is quantitative and the regressors are quantitative or qualitative or both. But there are occasions where the regressand can also be qualitative or dummy. Consider, for example, the decision of a worker to participate in the labor force. The decision to participate is of the yes or no type, yes if the person decides to participate and no otherwise. Thus, the labor force participation variable is a dummy variable. Of course, the decision to participate in the labor force depends on several factors, such as the starting wage rate, education, and conditions in the labor market (as measured by the unemployment rate).

Can we still use OLS to estimate regression models where the regressand is dummy? Yes, mechanically, we can do so. But there are several statistical problems that one faces in such models. And since there are alternatives to OLS estimation that do not face these problems, we will discuss this topic in a later chapter (see Chapter 15 on logit and probit models). In that chapter we will also discuss models in which the regressand has more than two categories; for example, the decision to travel to work by car, bus, or train, or the decision to work part-time, full time, or not work at all. Such models are called **polytomous dependent variable** models in contrast to **dichotomous dependent variable models** in which the dependent variable has only two categories.

### 9.11 TOPICS FOR FURTHER STUDY

Several topics related to dummy variables are discussed in the literature that are rather advanced, including (1) **random, or varying, parameters models**, (2) **switching regression models**, and (3) **disequilibrium models**.

In the regression models considered in this text it is assumed that the parameters, the  $\beta$ 's, are unknown but fixed entities. The random coefficient models—and there are several versions of them—assume the  $\beta$ 's can be random too. A major reference work in this area is by Swamy.<sup>21</sup>

In the dummy variable model using both differential intercepts and slopes, it is implicitly assumed that we know the point of break. Thus, in our savings–income example for 1970–1995, we divided the period into

<sup>21</sup>P. A. V. B. Swamy, *Statistical Inference in Random Coefficient Regression Models*, Springer-Verlag, Berlin, 1971.

1970–1981 and 1982–1995, the pre- and postrecession periods, under the belief that the recession in 1982 changed the relation between savings and income. Sometimes it is not easy to pinpoint when the break took place. The technique of **switching regression models (SRM)** is developed for such situations. SRM treats the breakpoint as a random variable and through an iterative process determines when the break might have actually taken place. The seminal work in this area is by Goldfeld and Quandt.<sup>22</sup>

Special estimation techniques are required to deal with what are known as **disequilibrium situations**, that is, situations where markets do not clear (i.e., demand is not equal to supply). The classic example is that of demand for and supply of a commodity. The demand for a commodity is a function of its price and other variables, and the supply of the commodity is a function of its price and other variables, some of which are different from those entering the demand function. Now the quantity actually bought and sold of the commodity may not necessarily be equal to the one obtained by equating the demand to supply, thus leading to disequilibrium. For a thorough discussion of **disequilibrium models**, the reader may refer to Quandt.<sup>23</sup>

## 9.12 SUMMARY AND CONCLUSIONS

1. Dummy variables, taking values of 1 and zero (or their linear transforms), are a means of introducing qualitative regressors in regression models.

2. Dummy variables are a data-classifying device in that they divide a sample into various subgroups based on qualities or attributes (gender, marital status, race, religion, etc. ) and *implicitly* allow one to run individual regressions for each subgroup. If there are differences in the response of the regressand to the variation in the qualitative variables in the various subgroups, they will be reflected in the differences in the intercepts or slope coefficients, or both, of the various subgroup regressions.

3. Although a versatile tool, the dummy variable technique needs to be handled carefully. *First*, if the regression contains a constant term, the number of dummy variables must be one less than the number of classifications of each qualitative variable. *Second*, the coefficient attached to the dummy variables must *always* be interpreted in relation to the base, or reference, group—that is, the group that receives the value of zero. The base chosen will depend on the purpose of research at hand. *Finally*, if a model has several qualitative variables with several classes, introduction of dummy variables can consume a large number of degrees of freedom. Therefore, one should always weigh the number of dummy variables to be introduced against the total number of observations available for analysis.

<sup>22</sup>S. Goldfeld and R. Quandt, *Nonlinear Methods in Econometrics*, North Holland, Amsterdam, 1972.

<sup>23</sup>Richard E. Quandt, *The Econometrics of Disequilibrium*, Basil Blackwell, New York, 1988.

4. Among its various applications, this chapter considered but a few. These included (1) comparing two (or more) regressions, (2) deseasonalizing time series data, (3) interactive dummies, (4) interpretation of dummies in semilog models, and (4) piecewise linear regression models.

5. We also sounded cautionary notes in the use of dummy variables in situations of heteroscedasticity and autocorrelation. But since we will cover these topics fully in subsequent chapters, we will revisit these topics then.

## EXERCISES

### Questions

- 9.1. If you have monthly data over a number of years, how many dummy variables will you introduce to test the following hypotheses:
- All the 12 months of the year exhibit seasonal patterns.
  - Only February, April, June, August, October, and December exhibit seasonal patterns.
- 9.2. Consider the following regression results ( $t$  ratios are in parentheses)\*:

$$\begin{aligned} \hat{Y}_i = & 1286 & + & 104.97X_{2i} & - & 0.026X_{3i} & + & 1.20X_{4i} & + & 0.69X_{5i} \\ t = & (4.67) & & (3.70) & & (-3.80) & & (0.24) & & (0.08) \\ & & & -19.47X_{6i} & + & 266.06X_{7i} & & -118.64X_{8i} & - & 110.61X_{9i} \\ & & & (-0.40) & & (6.94) & & (-3.04) & & (-6.14) \\ & & & & & & & R^2 = 0.383 & & n = 1543 \end{aligned}$$

where  $Y$  = wife's annual desired hours of work, calculated as usual hours of work per year plus weeks looking for work

$X_2$  = after-tax real average hourly earnings of wife

$X_3$  = husband's previous year after-tax real annual earnings

$X_4$  = wife's age in years

$X_5$  = years of schooling completed by wife

$X_6$  = attitude variable, 1 = if respondent felt that it was all right for a woman to work if she desired and her husband agrees, 0 = otherwise

$X_7$  = attitude variable, 1 = if the respondent's husband favored his wife's working, 0 = otherwise

$X_8$  = number of children less than 6 years of age

$X_9$  = number of children in age groups 6 to 13

- Do the signs of the coefficients of the various nondummy regressors make economic sense? Justify your answer.
- How would you interpret the dummy variables,  $X_6$  and  $X_7$ ? Are these dummies statistically significant? Since the sample is quite large, you may use the "2- $t$ " rule of thumb to answer the question.

\*Jane Leuthold, "The Effect of Taxation on the Hours Worked by Married Women," *Industrial and Labor Relations Review*, no. 4, July 1978, pp. 520-526 (notation changed to suit our format).

- c. Why do you think that age and education variables are not significant factors in a woman's labor force participation decision in this study?
- 9.3. Consider the following regression results.\* (The actual data are in Table 9.7.)

$$\widehat{UN}_t = 2.7491 + 1.1507D_t - 1.5294V_t - 0.8511(D_tV_t)$$

$$t = (26.896) \quad (3.6288) \quad (-12.5552) \quad (-1.9819)$$

$$R^2 = 0.9128$$

TABLE 9.7 DATA MATRIX FOR REGRESSION, IN EXERCISE 9.3

Year and quarter	Unemployment rate UN, %	Job vacancy rate V, %	D	DV	Year and quarter	Unemployment rate UN, %	Job vacancy rate V, %	D	DV
1958-IV	1.915	0.510	0	0	1965-I	1.201	0.997	0	0
1959-I	1.876	0.541	0	0	-II	1.192	1.035	0	0
-II	1.842	0.541	0	0	-III	1.259	1.040	0	0
-III	1.750	0.690	0	0	-IV	1.192	1.086	0	0
-IV	1.648	0.771	0	0	1966-I	1.089	1.101	0	0
1960-I	1.450	0.836	0	0	-II	1.101	1.058	0	0
-II	1.393	0.908	0	0	-III	1.243	0.987	0	0
-III	1.322	0.968	0	0	-IV	1.623	0.819	1	0.819
-IV	1.260	0.998	0	0	1967-I	1.821	0.740	1	0.740
1961-I	1.171	0.968	0	0	-II	1.990	0.661	1	0.661
-II	1.182	0.964	0	0	-III	2.114	0.660	1	0.660
-III	1.221	0.952	0	0	-IV	2.115	0.698	1	0.698
-IV	1.340	0.849	0	0	1968-I	2.150	0.695	1	0.695
1962-I	1.411	0.748	0	0	-II	2.141	0.732	1	0.732
-II	1.600	0.658	0	0	-III	2.167	0.749	1	0.749
-III	1.780	0.562	0	0	-IV	2.107	0.800	1	0.800
-IV	1.941	0.510	0	0	1969-I	2.104	0.783	1	0.783
1963-I	2.178	0.510	0	0	-II	2.056	0.800	1	0.800
-II	2.067	0.544	0	0	-III	2.170	0.794	1	0.794
-III	1.942	0.568	0	0	-IV	2.161	0.790	1	0.790
-IV	1.764	0.677	0	0	1970-I	2.225	0.757	1	0.757
1964-I	1.532	0.794	0	0	-II	2.241	0.746	1	0.746
-II	1.455	0.838	0	0	-III	2.366	0.739	1	0.739
-III	1.409	0.885	0	0	-IV	2.324	0.707	1	0.707
-IV	1.296	0.978	0	0	1971-I	2.516*	0.583*	1	0.583*
					-II	2.909*	0.524*	1	0.524*

\*Preliminary estimates.

Source: Damodar Gujarati, "The Behaviour of Unemployment and Unfilled Vacancies: Great Britain, 1958-1971," *The Economic Journal*, vol. 82, March 1972, p. 202.

\*Damodar Gujarati, "The Behaviour of Unemployment and Unfilled Vacancies: Great Britain, 1958-1971," *The Economic Journal*, vol. 82, March 1972, pp. 195-202.

where UN = unemployment rate, %  
 $V$  = job vacancy rate, %  
 $D = 1$ , for period beginning in 1966–IV  
 $= 0$ , for period before 1966–IV  
 $t$  = time, measured in quarters

*Note:* In the fourth quarter of 1966, the then Labor government liberalized the National Insurance Act by replacing the flat-rate system of short-term unemployment benefits by a mixed system of flat-rate and (previous) earnings-related benefits, which increased the level of unemployment benefits.

- a. What are your prior expectations about the relationship between the unemployment and vacancy rates?
  - b. Holding the job vacancy rate constant, what is the average unemployment rate in the period beginning in the fourth quarter of 1966? Is it statistically different from the period before 1966 fourth quarter? How do you know?
  - c. Are the slopes in the pre- and post-1966 fourth quarter statistically different? How do you know?
  - d. Is it safe to conclude from this study that generous unemployment benefits lead to higher unemployment rates? Does this make economic sense?
- 9.4. From annual data for 1972–1979, William Nordhaus estimated the following model to explain the OPEC's oil price behavior (standard errors in parentheses).\*

$$\hat{y}_t = 0.3x_{1t} + 5.22x_{2t}$$

$$\text{se} = (0.03) \quad (0.50)$$

where  $y$  = difference between current and previous year's price (dollars per barrel)  
 $x_1$  = difference between current year's spot price and OPEC's price in the previous year  
 $x_2 = 1$  for 1974 and 0 otherwise

Interpret this result and show the results graphically. What do these results suggest about OPEC's monopoly power?

- 9.5. Consider the following model

$$Y_i = \alpha_1 + \alpha_2 D_i + \beta X_i + u_i$$

where  $Y$  = annual salary of a college professor  
 $X$  = years of teaching experience  
 $D$  = dummy for gender

Consider three ways of defining the dummy variable.

- a.  $D = 1$  for male, 0 for female.
- b.  $D = 1$  for female, 2 for male.
- c.  $D = 1$  for female,  $-1$  for male.

Interpret the preceding regression model for each dummy assignment. Is one method preferable to another? Justify your answer.

\*"Oil and Economic Performance in Industrial Countries," *Brookings Papers on Economic Activity*, 1980, pp. 341–388.

- 9.6. Refer to regression (9.7.3). How would you test the hypothesis that the coefficients of  $D_2$  and  $D_3$  are the same? And that the coefficients of  $D_2$  and  $D_4$  are the same? If the coefficient of  $D_3$  is statistically different from that of  $D_2$  and the coefficient of  $D_4$  is different from that of  $D_2$ , does that mean that the coefficients  $D_3$  and  $D_4$  are also different?

*Hint:*  $\text{var}(A \pm B) = \text{var}(A) + \text{var}(B) \pm 2 \text{cov}(A, B)$

- 9.7. Refer to the U.S. savings-income example discussed in the chapter.
- How would you obtain the standard errors of the regression coefficients given in (9.5.5) and (9.5.6), which were obtained from the pooled regression (9.5.4)?
  - To obtain numerical answers, what additional information, if any, is required?
- 9.8. In his study on the labor hours spent by the FDIC (Federal Deposit Insurance Corporation) on 91 bank examinations, R. J. Miller estimated the following function\*:

$$\widehat{\ln Y} = 2.41 + 0.3674 \ln X_1 + 0.2217 \ln X_2 + 0.0803 \ln X_3 \\ - 0.1755D_1 + 0.2799D_2 + 0.5634D_3 - 0.2572D_4 \\ \begin{matrix} (0.0477) & (0.0628) & (0.0287) \\ (0.2905) & (0.1044) & (0.1657) & (0.0787) \end{matrix} \\ R^2 = 0.766$$

where  $Y$  = FDIC examiner labor hours

$X_1$  = total assets of bank

$X_2$  = total number of offices in bank

$X_3$  = ratio of classified loans to total loans for bank

$D_1$  = 1 if management rating was "good"

$D_2$  = 1 if management rating was "fair"

$D_3$  = 1 if management rating was "satisfactory"

$D_4$  = 1 if examination was conducted jointly with the state

The figures in parentheses are the estimated standard errors.

- Interpret these results.
  - Is there any problem in interpreting the dummy variables in this model since  $Y$  is in the log form?
  - How would you interpret the dummy coefficients?
- 9.9. To assess the effect of the Fed's policy of deregulating interest rates beginning in July 1979, Sidney Langer, a student of mine, estimated the following model for the quarterly period of 1975-III to 1983-II.†

$$\hat{Y}_t = 8.5871 - 0.1328P_t - 0.7102Un_t - 0.2389M_t \\ \begin{matrix} \text{se} (1.9563) & (0.0992) & (0.1909) & (0.0727) \\ & & + 0.6592Y_{t-1} + 2.5831Dum_t & R^2 = 0.9156 \\ & & (0.1036) & (0.7549) \end{matrix}$$

\*"Examination of Man-Hour Cost for Independent, Joint, and Divided Examination Programs," *Journal of Bank Research*, vol. 11, 1980, pp. 28-35. Note: The notations have been altered to conform with our notations.

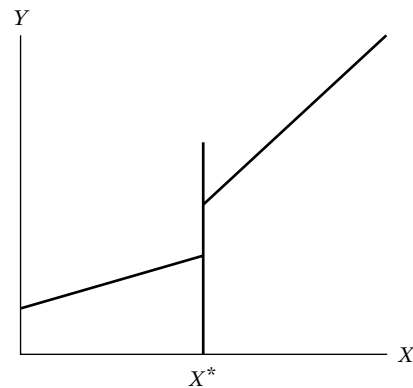
†Sidney Langer, "Interest Rate Deregulation and Short-Term Interest Rates," unpublished term paper.

where  $Y$  = 3-month Treasury bill rate  
 $P$  = expected rate of inflation  
 $Un$  = seasonally adjusted unemployment rate  
 $M$  = changes in the monetary base  
 $Dum$  = dummy, taking value of 1 for observations beginning July 1, 1979

- a. Interpret these results.
  - b. What has been the effect of rate deregulation? Do the results make economic sense?
  - c. The coefficients of  $P_t$ ,  $Un_t$ , and  $M_t$  are negative. Can you offer an economic rationale?
- 9.10.** Refer to the piecewise regression discussed in the text. Suppose there not only is a change in the slope coefficient at  $X^*$  but also the regression line jumps, as shown in Figure 9.7. How would you modify (9.8.1) to take into account the jump in the regression line at  $X^*$ ?
- 9.11.** *Determinants of price per ounce of cola.* Cathy Schaefer, a student of mine, estimated the following regression from cross-sectional data of 77 observations\*:

$$P_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} + \mu_i$$

where  $P_i$  = price per ounce of cola  
 $D_{1i}$  = 001 if discount store  
           = 010 if chain store  
           = 100 if convenience store  
 $D_{2i}$  = 10 if branded good  
           = 01 if unbranded good  
 $D_{3i}$  = 0001 if 67.6 ounce (2 liter) bottle  
           = 0010 if 28–33.8 ounce bottles (*Note:* 33.8 oz = 1 liter)  
           = 0100 if 16-ounce bottle  
           = 1000 if 12-ounce can



**FIGURE 9.7** Discontinuous piecewise linear regression.

\*Cathy Schaefer, "Price Per Ounce of Cola Beverage as a Function of Place of Purchase, Size of Container, and Branded or Unbranded Product," unpublished term project.

The results were as follows:

$$\begin{aligned} \hat{P}_i &= 0.0143 - 0.000004D_{1i} + 0.0090D_{2i} + 0.00001D_{3i} \\ Se &= \quad \quad (0.00001) \quad (0.00011) \quad (0.00000) \\ t &= \quad \quad (-0.3837) \quad (8.3927) \quad (5.8125) \\ R^2 &= 0.6033 \end{aligned}$$

*Note:* The standard errors are shown only to five decimal places.

- a. Comment on the way the dummies have been introduced in the model.
  - b. Assuming the dummy setup is acceptable, how would you interpret the results?
  - c. The coefficient of  $D_3$  is positive and statistically significant. How do you rationalize this result?
- 9.12.** From data for 101 countries on per capita income in dollars ( $X$ ) and life expectancy in years ( $Y$ ) in the early 1970s, Sen and Srivastava obtained the following regression results\*:

$$\begin{aligned} \hat{Y}_i &= -2.40 + 9.39 \ln X_i - 3.36 [D_i(\ln X_i - 7)] \\ se &= (4.73) \quad (0.859) \quad (2.42) \quad R^2 = 0.752 \end{aligned}$$

where  $D_i = 1$  if  $\ln X_i > 7$ , and  $D_i = 0$  otherwise. *Note:* When  $\ln X_i = 7$ ,  $X = \$1097$  (approximately).

- a. What might be the reason(s) for introducing the income variable in the log form?
  - b. How would you interpret the coefficient 9.39 of  $\ln X_i$ ?
  - c. What might be the reason for introducing the regressor  $D_i(\ln X_i - 7)$ ? How do you explain this regressor verbally? And how do you interpret the coefficient  $-3.36$  of this regressor (*Hint:* linear piecewise regression)?
  - d. Assuming per capita income of \$1097 as the dividing line between poorer and richer countries, how would you derive the regression for countries whose per capita is less than \$1097 and the regression for countries whose per capita income is greater than \$1097?
  - e. What general conclusions do you draw from the regression result presented in this problem?
- 9.13.** Consider the following model:

$$Y_i = \beta_1 + \beta_2 D_i + u_i$$

where  $D_i = 0$  for the first 20 observations and  $D_i = 1$  for the remaining 30 observations. You are also told that  $\text{var}(u_i^2) = 300$ .

- a. How would you interpret  $\beta_1$  and  $\beta_2$ ?
- b. What are the mean values of the two groups?
- c. How would you compute the variance of  $(\hat{\beta}_1 + \hat{\beta}_2)$ ? *Note:* You are given that the  $\text{cov}(\hat{\beta}_1, \hat{\beta}_2) = -15$ .

\*Ashish Sen and Muni Srivastava, *Regression Analysis: Theory, Methods, and Applications*, Springer-Verlag, New York, 1990, p. 92. Notation changed.



- 9.14.** To assess the effect of state right-to-work laws (which do not require membership in the union as a precondition of employment) on union membership, the following regression results were obtained, from the data for 50 states in the United States for 1982\*:

$$\widehat{PVT}_i = 19.8066 - 9.3917 RTW_i$$

$$t = (17.0352) \quad (-5.1086)$$

$$r^2 = 0.3522$$

where PVT = percentage of private sector employees in unions, 1982, and RTW = 1 if right-to-work law exists, 0 otherwise. *Note:* In 1982, twenty states had right-to-work laws.

- A priori, what is the expected relationship between PVT and RTW?
  - Do the regression results support the prior expectations?
  - Interpret the regression results.
  - What was the average percent of private sector employees in unions in the states that did not have the right-to-work laws?
- 9.15.** In the following regression model:

$$Y_i = \beta_1 + \beta_2 D_i + u_i$$

$Y$  represents hourly wage in dollars and  $D$  is the dummy variable, taking a value of 1 for a college graduate and a value of 0 for a high-school graduate. Using the OLS formulas given in Chapter 3, show that  $\hat{\beta}_1 = \bar{Y}_{hg}$  and  $\hat{\beta}_2 = \bar{Y}_{cg} - \bar{Y}_{hg}$ , where the subscripts have the following meanings: hg = high-school graduate, cg = college graduate. In all, there are  $n_1$  high-school graduates and  $n_2$  college graduates, for a total sample of  $n = n_1 + n_2$ .

- 9.16.** To study the rate of growth of population in Belize over the period 1970–1992, Mukherjee et al. estimated the following models†:

$$\text{Model I: } \widehat{\ln(\text{Pop})}_t = 4.73 + 0.024t$$

$$t = (781.25) \quad (54.71)$$

$$\text{Model II: } \widehat{\ln(\text{Pop})}_t = 4.77 + 0.015t - 0.075D_t + 0.011(D_t t)$$

$$t = (2477.92) \quad (34.01) \quad (-17.03) \quad (25.54)$$

where Pop = population in millions,  $t$  = trend variable,  $D_t = 1$  for observations beginning in 1978 and 0 before 1978, and ln stands for natural logarithm.

- In Model I, what is the rate of growth of Belize's population over the sample period?
- Are the population growth rates statistically different pre- and post-1978? How do you know? If they are different, what are the growth rates for 1972–1977 and 1978–1992?

\*The data used in the regression results were obtained from N. M. Meltz, "Interstate and Interprovincial Differences in Union Density," *Industrial Relations*, vol. 28, no. 2, 1989, pp. 142–158.

†Chandan Mukherjee, Howard White, and Marc Wuyts, *Econometrics and Data Analysis for Developing Countries*, Routledge, London, 1998, pp. 372–375. Notations adapted.

## Problems

- 9.17. Using the data given in Table 9.7, test the hypothesis that the error variances in the two subperiods 1958–IV to 1966–III and 1966–IV to 1971–II are the same.
- 9.18. Using the methodology discussed in Chapter 8, compare the unrestricted and restricted regressions (9.7.3) and (9.7.4); that is, test for the validity of the imposed restrictions.
- 9.19. In the U.S. savings–income regression (9.5.4) discussed in the chapter, suppose that instead of using 1 and 0 values for the dummy variable you use  $Z_i = a + bD_i$ , where  $D_i = 1$  and 0,  $a = 2$ , and  $b = 3$ . Compare your results.
- 9.20. Continuing with the savings–income regression (9.5.4), suppose you were to assign  $D_i = 0$  to observations in the second period and  $D_i = 1$  to observations in the first period. How would the results shown in (9.5.4) change?
- 9.21. Use the data given in Table 9.2 and consider the following model:

$$\ln \text{Savings}_i = \beta_1 + \beta_2 \ln \text{Income}_i + \beta_3 \ln D_i + u_i$$

where  $\ln$  stands for natural log and where  $D_i = 1$  for 1970–1981 and 10 for 1982–1995.

- What is the rationale behind assigning dummy values as suggested?
  - Estimate the preceding model and interpret your results.
  - What are the intercept values of the savings function in the two subperiods and how do you interpret them?
- 9.22. Refer to the quarterly appliance sales data given in Table 9.3. Consider the following model:

$$\text{Sales}_i = \alpha_1 + \alpha_2 D_{2i} + \alpha_3 D_{3i} + \alpha_4 D_{4i} + u_i$$

where the  $D$ 's are dummies taking 1 and 0 values for quarters II through IV.

- Estimate the preceding model for dishwashers, disposers, and washing machines individually.
  - How would you interpret the estimated slope coefficients?
  - How would you use the estimated  $\alpha$ 's to deseasonalize the sales data for individual appliances?
- 9.23. Reestimate the model in exercise 9.22 by adding the regressor, expenditure on durable goods.
- Is there a difference in the regression results you obtained in exercise 9.22 and in this exercise? If so, what explains the difference?
  - If there is seasonality in the durable goods expenditure data, how would you account for it?
- 9.24. Table 9.8 gives data on quadrennial presidential elections in the United States from 1916 to 1996.\*
- Using the data given in Table 9.6, develop a suitable model to predict the Democratic share of the two-party presidential vote.
  - How would you use this model to predict the outcome of a presidential election?

\*These data were originally compiled by Ray Fair of Yale University, who has been predicting the outcome of presidential elections for several years. The data are reproduced from Samprit Chatterjee, Ali S. Hadi, and Petram Price, *Regression Analysis by Example*, 3d ed., John Wiley & Sons, New York, 2000, pp. 150–151.

**TABLE 9.8** DATA ON U.S. PRESIDENTIAL ELECTIONS, 1916–1996

Year	<i>V</i>	<i>W</i>	<i>D</i>	<i>G</i>	<i>I</i>	<i>N</i>	<i>P</i>
1916	0.5168	0	1	2.229	1	3	4.252
1920	0.3612	1	0	-11.463	1	5	16.535
1924	0.4176	0	-1	-3.872	-1	10	5.161
1928	0.4118	0	0	4.623	-1	7	0.183
1932	0.5916	0	-1	-14.901	-1	4	7.069
1936	0.6246	0	1	11.921	1	9	2.362
1940	0.5500	0	1	3.708	1	8	0.028
1944	0.5377	1	1	4.119	1	14	5.678
1948	0.5237	1	1	1.849	1	5	8.722
1952	0.4460	0	0	0.627	1	6	2.288
1956	0.4224	0	-1	-1.527	-1	5	1.936
1960	0.5009	0	0	0.114	-1	5	1.932
1964	0.6134	0	1	5.054	1	10	1.247
1968	0.4960	0	0	4.836	1	7	3.215
1972	0.3821	0	-1	6.278	-1	4	4.766
1976	0.5105	0	0	3.663	-1	4	7.657
1980	0.4470	0	1	-3.789	1	5	8.093
1984	0.4083	0	-1	5.387	-1	7	5.403
1988	0.4610	0	0	2.068	-1	6	3.272
1992	0.5345	0	-1	2.293	-1	1	3.692
1996	0.5474	0	1	2.918	1	3	2.268

*Notes:*

- Year Election year
- V* Democratic share of the two-party presidential vote
- I* Indicator variable (1 if there is a Democratic incumbent at the time of the election and -1 if there is a Republican incumbent)
- D* Indicator variable (1 if a Democratic incumbent is running for election, -1 if a Republican incumbent is running for election, and 0 otherwise)
- W* Indicator variable (1 for the elections of 1920, 1944, and 1948, and 0 otherwise)
- G* Growth rate of real per capita GDP in the first three quarters of the election year
- P* Absolute value of the growth rate of the GDP deflator in the first 15 quarters of the administration
- N* Number of quarters in the first 15 quarters of the administration in which the growth rate of real per capita GDP is greater than 3.2%

c. Chatterjee et al. suggested considering the following model as a trial model to predict presidential elections:

$$V = \beta_0 + \beta_1 I + \beta_2 D + \beta_3 W + \beta_4 (GI) + \beta_5 P + \beta_6 N + u$$

Estimate this model and comment on the results in relation to the results of the model you have chosen.

- 9.25.** Refer to regression (9.6.4). Test the hypothesis that the rate of increase of average hourly earnings with respect to education differs by gender and race. (*Hint:* Use multiplicative dummies.)
- 9.26.** Refer to the regression (9.3.1). How would you modify the model to find out if there is any interaction between the gender and the region of residence dummies? Present the results based on this model and compare them with those given in (9.3.1).
- 9.27.** In the model  $Y_i = \beta_1 + \beta_2 D_i + u_i$ , let  $D_i = 0$  for the first 40 observations and  $D_i = 1$  for the remaining 60 observations. You are told that  $u_i$  has zero

mean and a variance of 100. What are the mean values and variances of the two sets of observations?\*

- 9.28. Refer to the U.S. savings–income regression discussed in the chapter. As an alternative to (9.5.1), consider the following model:

$$\ln Y_t = \beta_1 + \beta_2 D_t + \beta_3 X_t + \beta_4 (D_t X_t) + u_t$$

where  $Y$  is savings and  $X$  is income.

- Estimate the preceding model and compare the results with those given in (9.5.4). Which is a better model?
- How would you interpret the dummy coefficient in this model?
- As we will see in the chapter on heteroscedasticity, very often a log transformation of the dependent variable reduces heteroscedasticity in the data. See if this is the case in the present example by running the regression of log of  $Y$  on  $X$  for the two periods and see if the estimated error variances in the two periods are statistically the same. If they are, the Chow test can be used to pool the data in the manner indicated in the chapter.

## APPENDIX 9A

### Semilogarithmic Regression with Dummy Regressor

In Section 9.10 we noted that in models of the type

$$\ln Y_i = \beta_1 + \beta_2 D_i \quad (1)$$

the relative change in  $Y$  (i.e., semielasticity), with respect to the dummy regressor taking values of 1 or 0, can be obtained as (antilog of estimated  $\beta_2$ ) – 1 times 100, that is, as

$$(e^{\hat{\beta}_2} - 1) \times 100 \quad (2)$$

The proof is as follows: Since  $\ln$  and  $\exp (= e)$  are inverse functions, we can write (1) as:

$$\ln Y_i = \beta_1 + \ln(e^{\beta_2 D_i}) \quad (3)$$

Now when  $D = 0$ ,  $e^{\beta_2 D_i} = 1$  and when  $D = 1$ ,  $e^{\beta_2 D_i} = e^{\beta_2}$ . Therefore, in going from state 0 to state 1,  $\ln Y_i$  changes by  $(e^{\beta_2} - 1)$ . But a change in the log of a variable is a relative change, which after multiplication by 100 becomes a percentage change. Hence the percentage change is  $(e^{\beta_2} - 1) \times 100$ , as claimed. (Note:  $\ln_e e = 1$ , that is, the log of  $e$  to base  $e$  is 1, just as the log of 10 to base 10 is 1. Recall that log to base  $e$  is called the *natural* log and that log to base 10 is called the *common* log.)

\*This example is adapted from Peter Kennedy, *A Guide to Econometrics*, 4th ed., MIT Press, Cambridge, Mass., 1998, p. 347.

# 10

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## MULTICOLLINEARITY: WHAT HAPPENS IF THE REGRESSORS ARE CORRELATED?

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There is no pair of words that is more misused both in econometrics texts and in the applied literature than the pair “multi-collinearity problem.” That many of our explanatory variables are highly collinear is a fact of life. And it is completely clear that there are experimental designs  $\mathbf{X}'\mathbf{X}$  [i.e., data matrix] which would be much preferred to the designs the natural experiment has provided us [i.e., the sample at hand]. But a complaint about the apparent malevolence of nature is not at all constructive, and the *ad hoc* cures for a bad design, such as stepwise regression or ridge regression, can be disastrously inappropriate. Better that we should rightly accept the fact that our non-experiments [i.e., data not collected by designed experiments] are sometimes not very informative about parameters of interest.<sup>1</sup>

Assumption 10 of the *classical linear regression model* (CLRM) is that there is no **multicollinearity** among the regressors included in the regression model. In this chapter we take a critical look at this assumption by seeking answers to the following questions:

1. What is the nature of multicollinearity?
2. Is multicollinearity really a problem?
3. What are its practical consequences?
4. How does one detect it?
5. What remedial measures can be taken to alleviate the problem of multicollinearity?

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<sup>1</sup>Edward E. Leamer, “Model Choice and Specification Analysis,” in Zvi Griliches and Michael D. Intriligator, eds., *Handbook of Econometrics*, vol. I, North Holland Publishing Company, Amsterdam, 1983, pp. 300–301.

In this chapter we also discuss Assumption 7 of the CLRM, namely, that the number of observations in the sample must be greater than the number of regressors, and Assumption 8, which requires that there be sufficient variability in the values of the regressors, for they are intimately related to the assumption of no multicollinearity. Arthur Goldberger has christened Assumption 7 as the problem of **micronumerosity**,<sup>2</sup> which simply means small sample size.

### 10.1 THE NATURE OF MULTICOLLINEARITY

The term *multicollinearity* is due to Ragnar Frisch.<sup>3</sup> Originally it meant the existence of a “perfect,” or exact, linear relationship among some or all explanatory variables of a regression model.<sup>4</sup> For the  $k$ -variable regression involving explanatory variable  $X_1, X_2, \dots, X_k$  (where  $X_1 = 1$  for all observations to allow for the intercept term), an exact linear relationship is said to exist if the following condition is satisfied:

$$\lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_k X_k = 0 \quad (10.1.1)$$

where  $\lambda_1, \lambda_2, \dots, \lambda_k$  are constants such that not all of them are zero simultaneously.<sup>5</sup>

Today, however, the term multicollinearity is used in a broader sense to include the case of perfect multicollinearity, as shown by (10.1.1), as well as the case where the  $X$  variables are intercorrelated but not perfectly so, as follows<sup>6</sup>:

$$\lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_k X_k + v_i = 0 \quad (10.1.2)$$

where  $v_i$  is a stochastic error term.

To see the difference between *perfect* and *less than perfect* multicollinearity, assume, for example, that  $\lambda_2 \neq 0$ . Then, (10.1.1) can be written as

$$X_{2i} = -\frac{\lambda_1}{\lambda_2} X_{1i} - \frac{\lambda_3}{\lambda_2} X_{3i} - \dots - \frac{\lambda_k}{\lambda_2} X_{ki} \quad (10.1.3)$$

<sup>2</sup>See his *A Course in Econometrics*, Harvard University Press, Cambridge, Mass., 1991, p. 249.

<sup>3</sup>Ragnar Frisch, *Statistical Confluence Analysis by Means of Complete Regression Systems*, Institute of Economics, Oslo University, publ. no. 5, 1934.

<sup>4</sup>Strictly speaking, *multicollinearity* refers to the existence of more than one exact linear relationship, and *collinearity* refers to the existence of a single linear relationship. But this distinction is rarely maintained in practice, and multicollinearity refers to both cases.

<sup>5</sup>The chances of one's obtaining a sample of values where the regressors are related in this fashion are indeed very small in practice except by design when, for example, the number of observations is smaller than the number of regressors or if one falls into the “dummy variable trap” as discussed in Chap. 9. See exercise 10.2.

<sup>6</sup>If there are only two explanatory variables, *intercorrelation* can be measured by the zero-order or simple correlation coefficient. But if there are more than two  $X$  variables, intercorrelation can be measured by the partial correlation coefficients or by the multiple correlation coefficient  $R$  of one  $X$  variable with all other  $X$  variables taken together.

which shows how  $X_2$  is exactly linearly related to other variables or how it can be derived from a linear combination of other  $X$  variables. In this situation, the coefficient of correlation between the variable  $X_2$  and the linear combination on the right side of (10.1.3) is bound to be unity.

Similarly, if  $\lambda_2 \neq 0$ , Eq. (10.1.2) can be written as

$$X_{2i} = -\frac{\lambda_1}{\lambda_2}X_{1i} - \frac{\lambda_3}{\lambda_2}X_{3i} - \dots - \frac{\lambda_k}{\lambda_2}X_{ki} - \frac{1}{\lambda_2}v_i \quad (10.1.4)$$

which shows that  $X_2$  is not an exact linear combination of other  $X$ 's because it is also determined by the stochastic error term  $v_i$ .

As a numerical example, consider the following hypothetical data:

$X_2$	$X_3$	$X_3^*$
10	50	52
15	75	75
18	90	97
24	120	129
30	150	152

It is apparent that  $X_{3i} = 5X_{2i}$ . Therefore, there is perfect collinearity between  $X_2$  and  $X_3$  since the coefficient of correlation  $r_{23}$  is unity. The variable  $X_3^*$  was created from  $X_3$  by simply adding to it the following numbers, which were taken from a table of random numbers: 2, 0, 7, 9, 2. Now there is no longer perfect collinearity between  $X_2$  and  $X_3^*$ . However, the two variables are highly correlated because calculations will show that the coefficient of correlation between them is 0.9959.

The preceding algebraic approach to multicollinearity can be portrayed succinctly by the Ballentine (recall Figure 3.9, reproduced in Figure 10.1). In this figure the circles  $Y$ ,  $X_2$ , and  $X_3$  represent, respectively, the variations in  $Y$  (the dependent variable) and  $X_2$  and  $X_3$  (the explanatory variables). The degree of collinearity can be measured by the extent of the overlap (shaded area) of the  $X_2$  and  $X_3$  circles. In Figure 10.1a there is no overlap between  $X_2$  and  $X_3$ , and hence no collinearity. In Figure 10.1b through 10.1e there is a “low” to “high” degree of collinearity—the greater the overlap between  $X_2$  and  $X_3$  (i.e., the larger the shaded area), the higher the degree of collinearity. In the extreme, if  $X_2$  and  $X_3$  were to overlap completely (or if  $X_2$  were completely inside  $X_3$ , or vice versa), collinearity would be perfect.

In passing, note that multicollinearity, as we have defined it, refers only to linear relationships among the  $X$  variables. It does not rule out nonlinear relationships among them. For example, consider the following regression model:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + u_i \quad (10.1.5)$$

where, say,  $Y$  = total cost of production and  $X$  = output. The variables  $X_i^2$  (output squared) and  $X_i^3$  (output cubed) are obviously functionally related

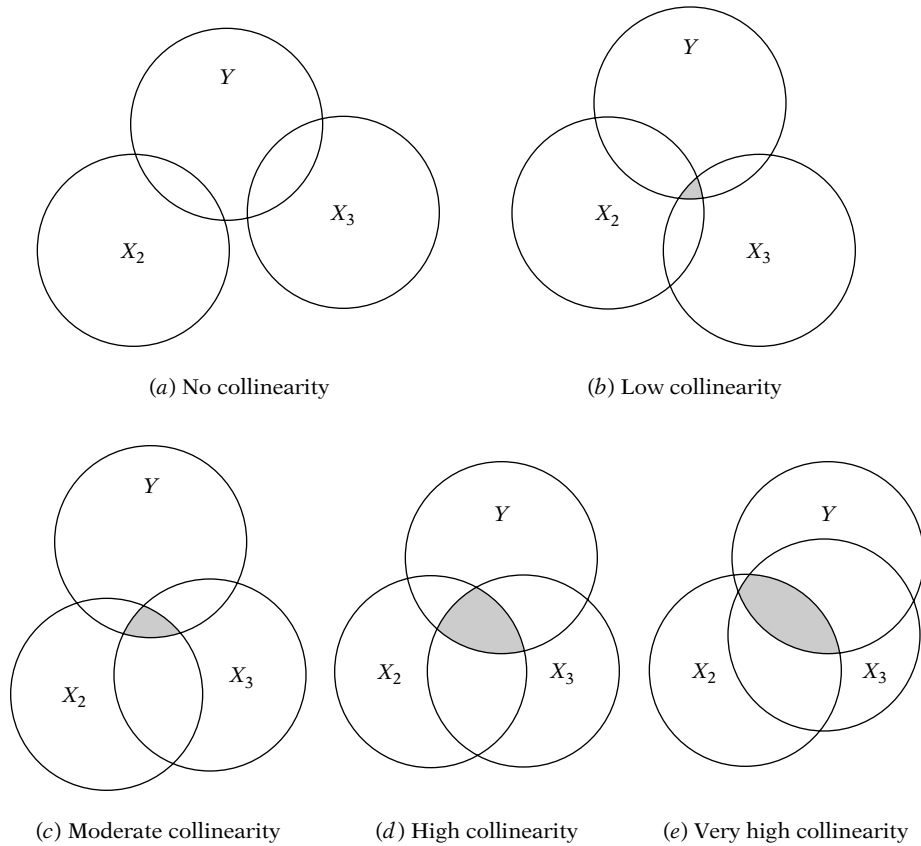


FIGURE 10.1 The Ballentine view of multicollinearity.

to  $X_i$ , but the relationship is nonlinear. Strictly, therefore, models such as (10.1.5) do not violate the assumption of no multicollinearity. However, in concrete applications, the conventionally measured correlation coefficient will show  $X_i$ ,  $X_i^2$ , and  $X_i^3$  to be highly correlated, which, as we shall show, will make it difficult to estimate the parameters of (10.1.5) with greater precision (i.e., with smaller standard errors).

Why does the classical linear regression model assume that there is no multicollinearity among the  $X$ 's? The reasoning is this: **If multicollinearity is perfect in the sense of (10.1.1), the regression coefficients of the  $X$  variables are indeterminate and their standard errors are infinite. If multicollinearity is less than perfect, as in (10.1.2), the regression coefficients, although determinate, possess large standard errors (in relation to the coefficients themselves), which means the coefficients cannot be estimated with great precision or accuracy.** The proofs of these statements are given in the following sections.



There are several sources of multicollinearity. As Montgomery and Peck note, multicollinearity may be due to the following factors<sup>7</sup>:

1. *The data collection method employed*, for example, sampling over a limited range of the values taken by the regressors in the population.
2. *Constraints on the model or in the population being sampled*. For example, in the regression of electricity consumption on income ( $X_2$ ) and house size ( $X_3$ ) there is a physical constraint in the population in that families with higher incomes generally have larger homes than families with lower incomes.
3. *Model specification*, for example, adding polynomial terms to a regression model, especially when the range of the  $X$  variable is small.
4. *An overdetermined model*. This happens when the model has more explanatory variables than the number of observations. This could happen in medical research where there may be a small number of patients about whom information is collected on a large number of variables.

An additional reason for multicollinearity, especially in time series data, may be that the regressors included in the model share a *common trend*, that is, they all increase or decrease over time. Thus, in the regression of consumption expenditure on income, wealth, and population, the regressors income, wealth, and population may all be growing over time at more or less the same rate, leading to collinearity among these variables.

## 10.2 ESTIMATION IN THE PRESENCE OF PERFECT MULTICOLLINEARITY

It was stated previously that in the case of perfect multicollinearity the regression coefficients remain indeterminate and their standard errors are infinite. This fact can be demonstrated readily in terms of the three-variable regression model. Using the deviation form, where all the variables are expressed as deviations from their sample means, we can write the three-variable regression model as

$$y_i = \hat{\beta}_2 x_{2i} + \hat{\beta}_3 x_{3i} + \hat{u}_i \quad (10.2.1)$$

Now from Chapter 7 we obtain

$$\hat{\beta}_2 = \frac{(\sum y_i x_{2i})(\sum x_{3i}^2) - (\sum y_i x_{3i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} \quad (7.4.7)$$

<sup>7</sup>Douglas Montgomery and Elizabeth Peck, *Introduction to Linear Regression Analysis*, John Wiley & Sons, New York, 1982, pp. 289–290. See also R. L. Mason, R. F. Gunst, and J. T. Webster, “Regression Analysis and Problems of Multicollinearity,” *Communications in Statistics A*, vol. 4, no. 3, 1975, pp. 277–292; R. F. Gunst, and R. L. Mason, “Advantages of Examining Multicollinearities in Regression Analysis,” *Biometrics*, vol. 33, 1977, pp. 249–260.

$$\hat{\beta}_3 = \frac{(\sum y_i x_{3i})(\sum x_{2i}^2) - (\sum y_i x_{2i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2) - (\sum x_{2i} x_{3i})^2} \quad (7.4.8)$$

Assume that  $X_{3i} = \lambda X_{2i}$ , where  $\lambda$  is a nonzero constant (e.g., 2, 4, 1.8, etc.). Substituting this into (7.4.7), we obtain

$$\begin{aligned} \hat{\beta}_2 &= \frac{(\sum y_i x_{2i})(\lambda^2 \sum x_{2i}^2) - (\lambda \sum y_i x_{2i})(\lambda \sum x_{2i}^2)}{(\sum x_{2i}^2)(\lambda^2 \sum x_{2i}^2) - \lambda^2 (\sum x_{2i}^2)^2} \\ &= \frac{0}{0} \end{aligned} \quad (10.2.2)$$

which is an indeterminate expression. The reader can verify that  $\hat{\beta}_3$  is also indeterminate.<sup>8</sup>

Why do we obtain the result shown in (10.2.2)? Recall the meaning of  $\hat{\beta}_2$ : It gives the rate of change in the average value of  $Y$  as  $X_2$  changes by a unit, holding  $X_3$  constant. But if  $X_3$  and  $X_2$  are perfectly collinear, there is no way  $X_3$  can be kept constant: As  $X_2$  changes, so does  $X_3$  by the factor  $\lambda$ . What it means, then, is that there is no way of disentangling the separate influences of  $X_2$  and  $X_3$  from the given sample: For practical purposes  $X_2$  and  $X_3$  are indistinguishable. In applied econometrics this problem is most damaging since the entire intent is to separate the partial effects of each  $X$  upon the dependent variable.

To see this differently, let us substitute  $X_{3i} = \lambda X_{2i}$  into (10.2.1) and obtain the following [see also (7.1.9)]:

$$\begin{aligned} y_i &= \hat{\beta}_2 x_{2i} + \hat{\beta}_3 (\lambda x_{2i}) + \hat{u}_i \\ &= (\hat{\beta}_2 + \lambda \hat{\beta}_3) x_{2i} + \hat{u}_i \\ &= \hat{\alpha} x_{2i} + \hat{u}_i \end{aligned} \quad (10.2.3)$$

where

$$\hat{\alpha} = (\hat{\beta}_2 + \lambda \hat{\beta}_3) \quad (10.2.4)$$

Applying the usual OLS formula to (10.2.3), we get

$$\hat{\alpha} = (\hat{\beta}_2 + \lambda \hat{\beta}_3) = \frac{\sum x_{2i} y_i}{\sum x_{2i}^2} \quad (10.2.5)$$

Therefore, although we can estimate  $\alpha$  uniquely, there is no way to estimate  $\beta_2$  and  $\beta_3$  uniquely; mathematically

$$\hat{\alpha} = \hat{\beta}_2 + \lambda \hat{\beta}_3 \quad (10.2.6)$$

<sup>8</sup>Another way of seeing this is as follows: By definition, the coefficient of correlation between  $X_2$  and  $X_3$ ,  $r_{23}$ , is  $\sum x_{2i} x_{3i} / \sqrt{\sum x_{2i}^2 \sum x_{3i}^2}$ . If  $r_{23}^2 = 1$ , i.e., perfect collinearity between  $X_2$  and  $X_3$ , the denominator of (7.4.7) will be zero, making estimation of  $\beta_2$  (or of  $\beta_3$ ) impossible.

gives us only one equation in two unknowns (note  $\lambda$  is given) and there is an infinity of solutions to (10.2.6) for given values of  $\hat{\alpha}$  and  $\lambda$ . To put this idea in concrete terms, let  $\hat{\alpha} = 0.8$  and  $\lambda = 2$ . Then we have

$$0.8 = \hat{\beta}_2 + 2\hat{\beta}_3 \quad (10.2.7)$$

or

$$\hat{\beta}_2 = 0.8 - 2\hat{\beta}_3 \quad (10.2.8)$$

Now choose a value of  $\hat{\beta}_3$  arbitrarily, and we will have a solution for  $\hat{\beta}_2$ . Choose another value for  $\hat{\beta}_3$ , and we will have another solution for  $\hat{\beta}_2$ . No matter how hard we try, there is no unique value for  $\hat{\beta}_2$ .

The upshot of the preceding discussion is that in the case of perfect multicollinearity one cannot get a unique solution for the individual regression coefficients. But notice that one can get a unique solution for linear combinations of these coefficients. The linear combination  $(\beta_2 + \lambda\beta_3)$  is uniquely estimated by  $\alpha$ , given the value of  $\lambda$ .<sup>9</sup>

In passing, note that in the case of perfect multicollinearity the variances and standard errors of  $\hat{\beta}_2$  and  $\hat{\beta}_3$  individually are infinite. (See exercise 10.21.)

### 10.3 ESTIMATION IN THE PRESENCE OF “HIGH” BUT “IMPERFECT” MULTICOLLINEARITY

The perfect multicollinearity situation is a pathological extreme. Generally, there is no exact linear relationship among the  $X$  variables, especially in data involving economic time series. Thus, turning to the three-variable model in the deviation form given in (10.2.1), instead of exact multicollinearity, we may have

$$x_{3i} = \lambda x_{2i} + v_i \quad (10.3.1)$$

where  $\lambda \neq 0$  and where  $v_i$  is a stochastic error term such that  $\sum x_{2i}v_i = 0$ . (Why?)

Incidentally, the Ballentines shown in Figure 10.1*b* to 10.1*e* represent cases of imperfect collinearity.

In this case, estimation of regression coefficients  $\beta_2$  and  $\beta_3$  may be possible. For example, substituting (10.3.1) into (7.4.7), we obtain

$$\hat{\beta}_2 = \frac{\sum(y_i x_{2i})(\lambda^2 \sum x_{2i}^2 + \sum v_i^2) - (\lambda \sum y_i x_{2i} + \sum y_i v_i)(\lambda \sum x_{2i}^2)}{\sum x_{2i}^2 (\lambda^2 \sum x_{2i}^2 + \sum v_i^2) - (\lambda \sum x_{2i}^2)^2} \quad (10.3.2)$$

where use is made of  $\sum x_{2i}v_i = 0$ . A similar expression can be derived for  $\hat{\beta}_3$ .

<sup>9</sup>In econometric literature, a function such as  $(\beta_2 + \lambda\beta_3)$  is known as an **estimable function**.

Now, unlike (10.2.2), there is no reason to believe a priori that (10.3.2) cannot be estimated. Of course, if  $v_i$  is sufficiently small, say, very close to zero, (10.3.1) will indicate almost perfect collinearity and we shall be back to the indeterminate case of (10.2.2).

#### 10.4 MULTICOLLINEARITY: MUCH ADO ABOUT NOTHING? THEORETICAL CONSEQUENCES OF MULTICOLLINEARITY

Recall that if the assumptions of the classical model are satisfied, the OLS estimators of the regression estimators are BLUE (or BUE, if the normality assumption is added). Now it can be shown that even if multicollinearity is very high, as in the case of *near multicollinearity*, the OLS estimators still retain the property of BLUE.<sup>10</sup> Then what is the multicollinearity fuss all about? As Christopher Achen remarks (note also the Leamer quote at the beginning of this chapter):

Beginning students of methodology occasionally worry that their independent variables are correlated—the so-called multicollinearity problem. But multicollinearity violates no regression assumptions. Unbiased, consistent estimates will occur, and their standard errors will be correctly estimated. The only effect of multicollinearity is to make it hard to get coefficient estimates with small standard error. But having a small number of observations also has that effect, as does having independent variables with small variances. (In fact, at a theoretical level, multicollinearity, few observations and small variances on the independent variables are essentially all the same problem.) Thus “What should I do about multicollinearity?” is a question like “What should I do if I don’t have many observations?” No statistical answer can be given.<sup>11</sup>

To drive home the importance of sample size, Goldberger coined the term **micronumerosity**, to counter the exotic polysyllabic name *multicollinearity*. According to Goldberger, **exact micronumerosity** (the counterpart of exact multicollinearity) arises when  $n$ , the sample size, is zero, in which case any kind of estimation is impossible. *Near micronumerosity*, like near multicollinearity, arises when the number of observations barely exceeds the number of parameters to be estimated.

Leamer, Achen, and Goldberger are right in bemoaning the lack of attention given to the sample size problem and the undue attention to the multicollinearity problem. Unfortunately, in applied work involving secondary data (i.e., data collected by some agency, such as the GNP data collected by the government), an individual researcher may not be able to do much about the size of the sample data and may have to face “estimating problems

<sup>10</sup>Since near multicollinearity per se does not violate the other assumptions listed in Chap. 7, the OLS estimators are BLUE as indicated there.

<sup>11</sup>Christopher H. Achen, *Interpreting and Using Regression*, Sage Publications, Beverly Hills, Calif., 1982, pp. 82–83.

important enough to warrant our treating it [i.e., multicollinearity] as a violation of the CLR [classical linear regression] model.”<sup>12</sup>

First, it is true that even in the case of near multicollinearity the OLS estimators are unbiased. But unbiasedness is a multisample or repeated sampling property. What it means is that, keeping the values of the  $X$  variables fixed, if one obtains repeated samples and computes the OLS estimators for each of these samples, the average of the sample values will converge to the true population values of the estimators as the number of samples increases. But this says nothing about the properties of estimators in any given sample.

Second, it is also true that collinearity does not destroy the property of minimum variance: In the class of all linear unbiased estimators, the OLS estimators have minimum variance; that is, they are efficient. But this does not mean that the variance of an OLS estimator will necessarily be small (in relation to the value of the estimator) in any given sample, as we shall demonstrate shortly.

Third, *multicollinearity is essentially a sample (regression) phenomenon* in the sense that even if the  $X$  variables are not linearly related in the population, they may be so related in the particular sample at hand: When we postulate the theoretical or population regression function (PRF), we believe that all the  $X$  variables included in the model have a separate or independent influence on the dependent variable  $Y$ . But it may happen that in any given sample that is used to test the PRF some or all of the  $X$  variables are so highly collinear that we cannot isolate their individual influence on  $Y$ . So to speak, our sample lets us down, although the theory says that all the  $X$ 's are important. In short, our sample may not be “rich” enough to accommodate all  $X$  variables in the analysis.

As an illustration, reconsider the consumption–income example of Chapter 3. Economists theorize that, besides income, the wealth of the consumer is also an important determinant of consumption expenditure. Thus, we may write

$$\text{Consumption}_i = \beta_1 + \beta_2 \text{Income}_i + \beta_3 \text{Wealth}_i + u_i$$

Now it may happen that when we obtain data on income and wealth, the two variables may be highly, if not perfectly, correlated: Wealthier people generally tend to have higher incomes. Thus, although in theory income and wealth are logical candidates to explain the behavior of consumption expenditure, in practice (i.e., in the sample) it may be difficult to disentangle the separate influences of income and wealth on consumption expenditure.

Ideally, to assess the individual effects of wealth and income on consumption expenditure we need a sufficient number of sample observations of wealthy individuals with low income, and high-income individuals with

<sup>12</sup>Peter Kennedy, *A Guide to Econometrics*, 3d ed., The MIT Press, Cambridge, Mass., 1992, p. 177.

low wealth (recall Assumption 8). Although this may be possible in cross-sectional studies (by increasing the sample size), it is very difficult to achieve in aggregate time series work.

For all these reasons, the fact that the OLS estimators are BLUE despite multicollinearity is of little consolation in practice. We must see what happens or is likely to happen in any given sample, a topic discussed in the following section.

### 10.5 PRACTICAL CONSEQUENCES OF MULTICOLLINEARITY

In cases of near or high multicollinearity, one is likely to encounter the following consequences:

1. Although BLUE, the OLS estimators have large variances and covariances, making precise estimation difficult.
2. Because of consequence 1, the confidence intervals tend to be much wider, leading to the acceptance of the “zero null hypothesis” (i.e., the true population coefficient is zero) more readily.
3. Also because of consequence 1, the  $t$  ratio of one or more coefficients tends to be statistically insignificant.
4. Although the  $t$  ratio of one or more coefficients is statistically insignificant,  $R^2$ , the overall measure of goodness of fit, can be very high.
5. The OLS estimators and their standard errors can be sensitive to small changes in the data.

The preceding consequences can be demonstrated as follows.

#### Large Variances and Covariances of OLS Estimators

To see large variances and covariances, recall that for the model (10.2.1) the variances and covariances of  $\hat{\beta}_2$  and  $\hat{\beta}_3$  are given by

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2 (1 - r_{23}^2)} \quad (7.4.12)$$

$$\text{var}(\hat{\beta}_3) = \frac{\sigma^2}{\sum x_{3i}^2 (1 - r_{23}^2)} \quad (7.4.15)$$

$$\text{cov}(\hat{\beta}_2, \hat{\beta}_3) = \frac{-r_{23}\sigma^2}{(1 - r_{23}^2)\sqrt{\sum x_{2i}^2 \sum x_{3i}^2}} \quad (7.4.17)$$

where  $r_{23}$  is the coefficient of correlation between  $X_2$  and  $X_3$ .

It is apparent from (7.4.12) and (7.4.15) that as  $r_{23}$  tends toward 1, that is, as collinearity increases, the variances of the two estimators increase and in the limit when  $r_{23} = 1$ , they are infinite. It is equally clear from (7.4.17) that as  $r_{23}$  increases toward 1, the covariance of the two estimators also increases in absolute value. [Note:  $\text{cov}(\hat{\beta}_2, \hat{\beta}_3) \equiv \text{cov}(\hat{\beta}_3, \hat{\beta}_2)$ .]

The speed with which variances and covariances increase can be seen with the **variance-inflating factor (VIF)**, which is defined as

$$\text{VIF} = \frac{1}{(1 - r_{23}^2)} \tag{10.5.1}$$

VIF shows how the variance of an estimator is *inflated* by the presence of multicollinearity. As  $r_{23}^2$  approaches 1, the VIF approaches infinity. That is, as the extent of collinearity increases, the variance of an estimator increases, and in the limit it can become infinite. As can be readily seen, if there is no collinearity between  $X_2$  and  $X_3$ , VIF will be 1.

Using this definition, we can express (7.4.12) and (7.4.15) as

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2} \text{VIF} \tag{10.5.2}$$

$$\text{var}(\hat{\beta}_3) = \frac{\sigma^2}{\sum x_{3i}^2} \text{VIF} \tag{10.5.3}$$

which show that the variances of  $\hat{\beta}_2$  and  $\hat{\beta}_3$  are directly proportional to the VIF.

To give some idea about how fast the variances and covariances increase as  $r_{23}$  increases, consider Table 10.1, which gives these variances and covariances for selected values of  $r_{23}$ . As this table shows, increases in  $r_{23}$

**TABLE 10.1** THE EFFECT OF INCREASING  $r_{23}$  ON VAR ( $\hat{\beta}_2$ ) AND COV ( $\hat{\beta}_2, \hat{\beta}_3$ )

Value of $r_{23}$ (1)	VIF (2)	var ( $\hat{\beta}_2$ ) (3)*	$\frac{\text{var}(\hat{\beta}_2)(r_{23} \neq 0)}{\text{var}(\hat{\beta}_2)(r_{23} = 0)}$ (4)	cov ( $\hat{\beta}_2, \hat{\beta}_3$ ) (5)
0.00	1.00	$\frac{\sigma^2}{\sum x_{2i}^2} = A$	—	0
0.50	1.33	$1.33 \times A$	1.33	$0.67 \times B$
0.70	1.96	$1.96 \times A$	1.96	$1.37 \times B$
0.80	2.78	$2.78 \times A$	2.78	$2.22 \times B$
0.90	5.76	$5.26 \times A$	5.26	$4.73 \times B$
0.95	10.26	$10.26 \times A$	10.26	$9.74 \times B$
0.97	16.92	$16.92 \times A$	16.92	$16.41 \times B$
0.99	50.25	$50.25 \times A$	50.25	$49.75 \times B$
0.995	100.00	$100.00 \times A$	100.00	$99.50 \times B$
0.999	500.00	$500.00 \times A$	500.00	$499.50 \times B$

Note:  $A = \frac{\sigma^2}{\sum x_{2i}^2}$   
 $B = \frac{-\sigma^2}{\sqrt{\sum x_{2i}^2 \sum x_{3i}^2}}$   
 × = times

\*To find out the effect of increasing  $r_{23}$  on var ( $\hat{\beta}_3$ ), note that  $A = \sigma^2 / \sum x_{3i}^2$  when  $r_{23} = 0$ , but the variance and covariance magnifying factors remain the same.

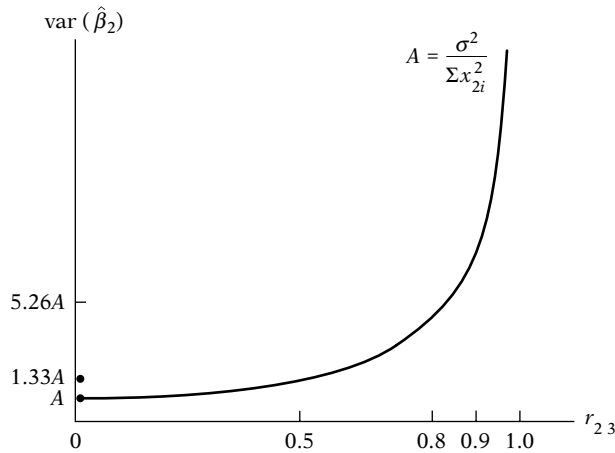


FIGURE 10.2 The behavior of  $\text{var}(\hat{\beta}_2)$  as a function of  $r_{23}$ .

have a dramatic effect on the estimated variances and covariances of the OLS estimators. When  $r_{23} = 0.50$ , the  $\text{var}(\hat{\beta}_2)$  is 1.33 times the variance when  $r_{23}$  is zero, but by the time  $r_{23}$  reaches 0.95 it is about 10 times as high as when there is no collinearity. And lo and behold, an increase of  $r_{23}$  from 0.95 to 0.995 makes the estimated variance 100 times that when collinearity is zero. The same dramatic effect is seen on the estimated covariance. All this can be seen in Figure 10.2.

The results just discussed can be easily extended to the  $k$ -variable model. In such a model, the variance of the  $k$ th coefficient, as noted in (7.5.6), can be expressed as:

$$\text{var}(\hat{\beta}_j) = \frac{\sigma^2}{\sum x_j^2} \left( \frac{1}{1 - R_j^2} \right) \quad (7.5.6)$$

where  $\hat{\beta}_j$  = (estimated) partial regression coefficient of regressor  $X_j$   
 $R_j^2 = R^2$  in the regression of  $X_j$  on the remaining  $(k - 2)$  regressions  
 [Note: There are  $(k - 1)$  regressors in the  $k$ -variable regression model.]  
 $\sum x_j^2 = \sum (X_j - \bar{X}_j)^2$

We can also write (7.5.6) as

$$\text{var}(\hat{\beta}_j) = \frac{\sigma^2}{\sum x_j^2} \text{VIF}_j \quad (10.5.4)$$

As you can see from this expression,  $\text{var}(\hat{\beta}_j)$  is proportional to  $\sigma^2$  and VIF but inversely proportional to  $\sum x_j^2$ . Thus, whether  $\text{var}(\hat{\beta}_j)$  is large or small



will depend on the three ingredients: (1)  $\sigma^2$ , (2) VIF, and (3)  $\sum x_j^2$ . The last one, which ties in with Assumption 8 of the classical model, states that the larger the variability in a regressor, the smaller the variance of the coefficient of that regressor, assuming the other two ingredients are constant, and therefore the greater the precision with which that coefficient can be estimated.

Before proceeding further, it may be noted that the inverse of the VIF is called **tolerance** (TOL). That is,

$$\text{TOL}_j = \frac{1}{\text{VIF}_j} = (1 - R_j^2) \tag{10.5.5}$$

When  $R_j^2 = 1$  (i.e., perfect collinearity),  $\text{TOL}_j = 0$  and when  $R_j^2 = 0$  (i.e., no collinearity whatsoever),  $\text{TOL}_j$  is 1. Because of the intimate connection between VIF and TOL, one can use them interchangeably.

### Wider Confidence Intervals

Because of the large standard errors, the confidence intervals for the relevant population parameters tend to be larger, as can be seen from Table 10.2. For example, when  $r_{23} = 0.95$ , the confidence interval for  $\beta_2$  is larger than when  $r_{23} = 0$  by a factor of  $\sqrt{10.26}$ , or about 3.

Therefore, in cases of high multicollinearity, the sample data may be compatible with a diverse set of hypotheses. Hence, the probability of accepting a false hypothesis (i.e., type II error) increases.

**TABLE 10.2** THE EFFECT OF INCREASING COLLINEARITY ON THE 95% CONFIDENCE INTERVAL FOR  $\beta_2$ :  $\hat{\beta}_2 \pm 1.96 \text{ se}(\hat{\beta}_2)$

Value of $r_{23}$	95% confidence interval for $\beta_2$
0.00	$\hat{\beta}_2 \pm 1.96 \sqrt{\frac{\sigma^2}{\sum x_{2i}^2}}$
0.50	$\hat{\beta}_2 \pm 1.96 \sqrt{(1.33)} \sqrt{\frac{\sigma^2}{\sum x_{2i}^2}}$
0.95	$\hat{\beta}_2 \pm 1.96 \sqrt{(10.26)} \sqrt{\frac{\sigma^2}{\sum x_{2i}^2}}$
0.995	$\hat{\beta}_2 \pm 1.96 \sqrt{(100)} \sqrt{\frac{\sigma^2}{\sum x_{2i}^2}}$
0.999	$\hat{\beta}_2 \pm 1.96 \sqrt{(500)} \sqrt{\frac{\sigma^2}{\sum x_{2i}^2}}$

*Note:* We are using the normal distribution because  $\sigma^2$  is assumed for convenience to be known. Hence the use of 1.96, the 95% confidence factor for the normal distribution.

The standard errors corresponding to the various  $r_{23}$  values are obtained from Table 10.1.

**“Insignificant”  $t$  Ratios**

Recall that to test the null hypothesis that, say,  $\beta_2 = 0$ , we use the  $t$  ratio, that is,  $\hat{\beta}_2/\text{se}(\hat{\beta}_2)$ , and compare the estimated  $t$  value with the critical  $t$  value from the  $t$  table. But as we have seen, in cases of high collinearity the estimated standard errors increase dramatically, thereby making the  $t$  values smaller. Therefore, in such cases, one will increasingly accept the null hypothesis that the relevant true population value is zero.<sup>13</sup>

**A High  $R^2$  but Few Significant  $t$  Ratios**

Consider the  $k$ -variable linear regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i$$

In cases of high collinearity, it is possible to find, as we have just noted, that one or more of the partial slope coefficients are individually statistically insignificant on the basis of the  $t$  test. Yet the  $R^2$  in such situations may be so high, say, in excess of 0.9, that on the basis of the  $F$  test one can convincingly reject the hypothesis that  $\beta_2 = \beta_3 = \cdots = \beta_k = 0$ . Indeed, this is one of the signals of multicollinearity—insignificant  $t$  values but a high overall  $R^2$  (and a significant  $F$  value)!

We shall demonstrate this signal in the next section, but this outcome should not be surprising in view of our discussion on individual vs. joint testing in Chapter 8. As you may recall, the real problem here is the covariances between the estimators, which, as formula (7.4.17) indicates, are related to the correlations between the regressors.

**Sensitivity of OLS Estimators and Their Standard Errors to Small Changes in Data**

As long as multicollinearity is not perfect, estimation of the regression coefficients is possible but the estimates and their standard errors become very sensitive to even the slightest change in the data.

To see this, consider Table 10.3. Based on these data, we obtain the following multiple regression:

$$\begin{aligned} \hat{Y}_i &= 1.1939 + 0.4463X_{2i} + 0.0030X_{3i} \\ &\quad (0.7737) \quad (0.1848) \quad (0.0851) \\ t &= (1.5431) \quad (2.4151) \quad (0.0358) \quad \quad \quad (10.5.6) \\ &\quad \quad \quad R^2 = 0.8101 \quad r_{23} = 0.5523 \\ &\quad \quad \quad \text{cov}(\hat{\beta}_2, \hat{\beta}_3) = -0.00868 \quad \text{df} = 2 \end{aligned}$$

<sup>13</sup>In terms of the confidence intervals,  $\beta_2 = 0$  value will lie increasingly in the acceptance region as the degree of collinearity increases.

**TABLE 10.3**  
HYPOTHETICAL DATA ON  $Y$ ,  $X_2$ , AND  $X_3$ 

$Y$	$X_2$	$X_3$
1	2	4
2	0	2
3	4	12
4	6	0
5	8	16

**TABLE 10.4**  
HYPOTHETICAL DATA ON  $Y$ ,  $X_2$ , AND  $X_3$ 

$Y$	$X_2$	$X_3$
1	2	4
2	0	2
3	4	0
4	6	12
5	8	16

Regression (10.5.6) shows that none of the regression coefficients is individually significant at the conventional 1 or 5 percent levels of significance, although  $\hat{\beta}_2$  is significant at the 10 percent level on the basis of a one-tail  $t$  test.

Now consider Table 10.4. The only difference between Tables 10.3 and 10.4 is that the third and fourth values of  $X_3$  are interchanged. Using the data of Table 10.4, we now obtain

$$\begin{aligned} \hat{Y}_i &= 1.2108 + 0.4014X_{2i} + 0.0270X_{3i} \\ &\quad (0.7480) \quad (0.2721) \quad (0.1252) \\ t &= (1.6187) \quad (1.4752) \quad (0.2158) \qquad \qquad \qquad (10.5.7) \\ R^2 &= 0.8143 \quad r_{23} = 0.8285 \\ \text{cov}(\hat{\beta}_2, \hat{\beta}_3) &= -0.0282 \quad \text{df} = 2 \end{aligned}$$

As a result of a slight change in the data, we see that  $\hat{\beta}_2$ , which was statistically significant before at the 10 percent level of significance, is no longer significant even at that level. Also note that in (10.5.6)  $\text{cov}(\hat{\beta}_2, \hat{\beta}_3) = -0.00868$  whereas in (10.5.7) it is  $-0.0282$ , a more than threefold increase. All these changes may be attributable to increased multicollinearity: In (10.5.6)  $r_{23} = 0.5523$ , whereas in (10.5.7) it is  $0.8285$ . Similarly, the standard errors of  $\hat{\beta}_2$  and  $\hat{\beta}_3$  increase between the two regressions, a usual symptom of collinearity.

We noted earlier that in the presence of high collinearity one cannot estimate the individual regression coefficients precisely but that linear combinations of these coefficients may be estimated more precisely. This fact can be substantiated from the regressions (10.5.6) and (10.5.7). In the first regression the sum of the two partial slope coefficients is  $0.4493$  and in the second it is  $0.4284$ , practically the same. Not only that, their standard errors are practically the same,  $0.1550$  vs.  $0.1823$ .<sup>14</sup> Note, however, the coefficient of  $X_3$  has changed dramatically, from  $0.003$  to  $0.027$ .

<sup>14</sup>These standard errors are obtained from the formula

$$\text{se}(\hat{\beta}_2 + \hat{\beta}_3) = \sqrt{\text{var}(\hat{\beta}_2) + \text{var}(\hat{\beta}_3) + 2 \text{cov}(\hat{\beta}_2, \hat{\beta}_3)}$$

Note that increasing collinearity increases the variances of  $\hat{\beta}_2$  and  $\hat{\beta}_3$ , but these variances may be offset if there is high negative covariance between the two, as our results clearly point out.

**Consequences of Micronumerosity**

In a parody of the consequences of multicollinearity, and in a tongue-in-cheek manner, Goldberger cites exactly similar consequences of micronumerosity, that is, analysis based on small sample size.<sup>15</sup> The reader is advised to read Goldberger's analysis to see why he regards micronumerosity as being as important as multicollinearity.

**10.6 AN ILLUSTRATIVE EXAMPLE: CONSUMPTION EXPENDITURE IN RELATION TO INCOME AND WEALTH**

To illustrate the various points made thus far, let us reconsider the consumption-income example of Chapter 3. In Table 10.5 we reproduce the data of Table 3.2 and add to it data on wealth of the consumer. If we assume that consumption expenditure is linearly related to income and wealth, then, from Table 10.5 we obtain the following regression:

$$\begin{aligned} \hat{Y}_i &= 24.7747 + 0.9415X_{2i} - 0.0424X_{3i} \\ &\quad (6.7525) \quad (0.8229) \quad (0.0807) \\ t &= (3.6690) \quad (1.1442) \quad (-0.5261) \qquad (10.6.1) \\ R^2 &= 0.9635 \quad \bar{R}^2 = 0.9531 \quad df = 7 \end{aligned}$$

Regression (10.6.1) shows that income and wealth together explain about 96 percent of the variation in consumption expenditure, and yet neither of the slope coefficients is individually statistically significant. Moreover, not only is the wealth variable statistically insignificant but also it has the wrong

**TABLE 10.5** HYPOTHETICAL DATA ON CONSUMPTION EXPENDITURE  $Y$ , INCOME  $X_2$ , AND WEALTH  $X_3$

$Y$ , \$	$X_2$ , \$	$X_3$ , \$
70	80	810
65	100	1009
90	120	1273
95	140	1425
110	160	1633
115	180	1876
120	200	2052
140	220	2201
155	240	2435
150	260	2686

<sup>15</sup>Goldberger, op. cit., pp. 248–250.

TABLE 10.6 ANOVA TABLE FOR THE CONSUMPTION-INCOME-WEALTH EXAMPLE

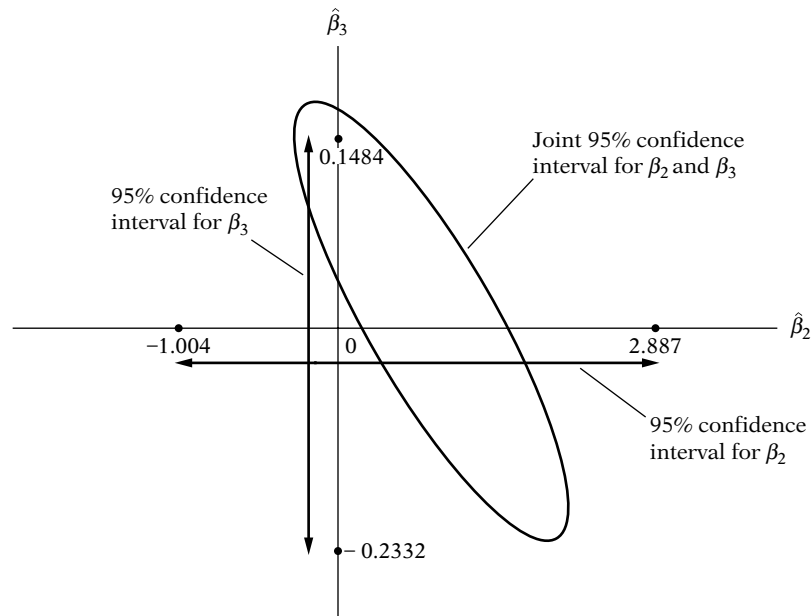
Source of variation	SS	df	MSS
Due to regression	8,565.5541	2	4,282.7770
Due to residual	324.4459	7	46.3494

sign. A priori, one would expect a positive relationship between consumption and wealth. Although  $\hat{\beta}_2$  and  $\hat{\beta}_3$  are individually statistically insignificant, if we test the hypothesis that  $\beta_2 = \beta_3 = 0$  simultaneously, this hypothesis can be rejected, as Table 10.6 shows. Under the usual assumption we obtain

$$F = \frac{4282.7770}{46.3494} = 92.4019 \quad (10.6.2)$$

This  $F$  value is obviously highly significant.

It is interesting to look at this result geometrically. (See Figure 10.3.) Based on the regression (10.6.1), we have established the individual 95% confidence intervals for  $\beta_2$  and  $\beta_3$  following the usual procedure discussed in Chapter 8. As these intervals show, individually each of them includes the value of zero. Therefore, *individually* we can accept the hypothesis that the

FIGURE 10.3 Individual confidence intervals for  $\beta_2$  and  $\beta_3$  and joint confidence interval (ellipse) for  $\beta_2$  and  $\beta_3$ .

two partial slopes are zero. But, when we establish the joint confidence interval to test the hypothesis that  $\beta_2 = \beta_3 = 0$ , that hypothesis cannot be accepted since the joint confidence interval, actually an ellipse, does not include the origin.<sup>16</sup> As already pointed out, when collinearity is high, tests on individual regressors are not reliable; in such cases it is the overall  $F$  test that will show if  $Y$  is related to the various regressors.

Our example shows dramatically what multicollinearity does. The fact that the  $F$  test is significant but the  $t$  values of  $X_2$  and  $X_3$  are individually insignificant means that the two variables are so highly correlated that it is impossible to isolate the individual impact of either income or wealth on consumption. As a matter of fact, if we regress  $X_3$  on  $X_2$ , we obtain

$$\begin{aligned}\hat{X}_{3i} &= 7.5454 + 10.1909X_{2i} \\ &\quad (29.4758) \quad (0.1643) \\ t &= (0.2560) \quad (62.0405) \quad R^2 = 0.9979\end{aligned}\tag{10.6.3}$$

which shows that there is almost perfect collinearity between  $X_3$  and  $X_2$ .

Now let us see what happens if we regress  $Y$  on  $X_2$  only:

$$\begin{aligned}\hat{Y}_i &= 24.4545 + 0.5091X_{2i} \\ &\quad (6.4138) \quad (0.0357) \\ t &= (3.8128) \quad (14.2432) \quad R^2 = 0.9621\end{aligned}\tag{10.6.4}$$

In (10.6.1) the income variable was statistically insignificant, whereas now it is highly significant. If instead of regressing  $Y$  on  $X_2$ , we regress it on  $X_3$ , we obtain

$$\begin{aligned}\hat{Y}_i &= 24.411 + 0.0498X_{3i} \\ &\quad (6.874) \quad (0.0037) \\ t &= (3.551) \quad (13.29) \quad R^2 = 0.9567\end{aligned}\tag{10.6.5}$$

We see that wealth has now a significant impact on consumption expenditure, whereas in (10.6.1) it had no effect on consumption expenditure.

Regressions (10.6.4) and (10.6.5) show very clearly that in situations of extreme multicollinearity dropping the highly collinear variable will often make the other  $X$  variable statistically significant. This result would suggest that a way out of extreme collinearity is to drop the collinear variable, but we shall have more to say about it in Section 10.8.

<sup>16</sup>As noted in Sec. 5.3, the topic of joint confidence interval is rather involved. The interested reader may consult the reference cited there.

## 10.7 DETECTION OF MULTICOLLINEARITY

Having studied the nature and consequences of multicollinearity, the natural question is: How does one know that collinearity is present in any given situation, especially in models involving more than two explanatory variables? Here it is useful to bear in mind Kmenta's warning:

1. Multicollinearity is a question of degree and not of kind. The meaningful distinction is not between the presence and the absence of multicollinearity, but between its various degrees.

2. Since multicollinearity refers to the condition of the explanatory variables that are assumed to be nonstochastic, it is a feature of the sample and not of the population.

Therefore, we do not "test for multicollinearity" but can, if we wish, measure its degree in any particular sample.<sup>17</sup>

Since multicollinearity is essentially a sample phenomenon, arising out of the largely nonexperimental data collected in most social sciences, we do not have one unique method of detecting it or measuring its strength. What we have are some rules of thumb, some informal and some formal, but rules of thumb all the same. We now consider some of these rules.

**1. High  $R^2$  but few significant  $t$  ratios.** As noted, this is the "classic" symptom of multicollinearity. If  $R^2$  is high, say, in excess of 0.8, the  $F$  test in most cases will reject the hypothesis that the partial slope coefficients are simultaneously equal to zero, but the individual  $t$  tests will show that none or very few of the partial slope coefficients are statistically different from zero. This fact was clearly demonstrated by our consumption-income-wealth example.

Although this diagnostic is sensible, its disadvantage is that "it is too strong in the sense that multicollinearity is considered as harmful only when all of the influences of the explanatory variables on  $Y$  cannot be disentangled."<sup>18</sup>

**2. High pair-wise correlations among regressors.** Another suggested rule of thumb is that if the pair-wise or zero-order correlation coefficient between two regressors is high, say, in excess of 0.8, then multicollinearity is a serious problem. The problem with this criterion is that, although high zero-order correlations may suggest collinearity, it is not necessary that they be high to have collinearity in any specific case. To put the matter somewhat technically, *high zero-order correlations are a sufficient but not a necessary condition for the existence of multicollinearity because it can exist even though the zero-order or simple correlations are comparatively low* (say, less than 0.50). To see this relationship, suppose we have a four-variable model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + u_i$$

<sup>17</sup>Jan Kmenta, *Elements of Econometrics*, 2d ed., Macmillan, New York, 1986, p. 431.

<sup>18</sup>*Ibid.*, p. 439.

and suppose that

$$X_{4i} = \lambda_2 X_{2i} + \lambda_3 X_{3i}$$

where  $\lambda_2$  and  $\lambda_3$  are constants, not both zero. Obviously,  $X_4$  is an exact linear combination of  $X_2$  and  $X_3$ , giving  $R_{4,23}^2 = 1$ , the coefficient of determination in the regression of  $X_4$  on  $X_2$  and  $X_3$ .

Now recalling the formula (7.11.5) from Chapter 7, we can write

$$R_{4,23}^2 = \frac{r_{42}^2 + r_{43}^2 - 2r_{42}r_{43}r_{23}}{1 - r_{23}^2} \quad (10.7.1)$$

But since  $R_{4,23}^2 = 1$  because of perfect collinearity, we obtain

$$1 = \frac{r_{42}^2 + r_{43}^2 - 2r_{42}r_{43}r_{23}}{1 - r_{23}^2} \quad (10.7.2)$$

It is not difficult to see that (10.7.2) is satisfied by  $r_{42} = 0.5$ ,  $r_{43} = 0.5$ , and  $r_{23} = -0.5$ , which are not very high values.

Therefore, in models involving more than two explanatory variables, the simple or zero-order correlation will not provide an infallible guide to the presence of multicollinearity. Of course, if there are only two explanatory variables, the zero-order correlations will suffice.

**3. Examination of partial correlations.** Because of the problem just mentioned in relying on zero-order correlations, Farrar and Glauber have suggested that one should look at the partial correlation coefficients.<sup>19</sup> Thus, in the regression of  $Y$  on  $X_2$ ,  $X_3$ , and  $X_4$ , a finding that  $R_{1,234}^2$  is very high but  $r_{12,34}^2$ ,  $r_{13,24}^2$ , and  $r_{14,23}^2$  are comparatively low may suggest that the variables  $X_2$ ,  $X_3$ , and  $X_4$  are highly intercorrelated and that at least one of these variables is superfluous.

Although a study of the partial correlations may be useful, there is no guarantee that they will provide an infallible guide to multicollinearity, for it may happen that both  $R^2$  and all the partial correlations are sufficiently high. But more importantly, C. Robert Wichers has shown<sup>20</sup> that the Farrar-Glauber partial correlation test is ineffective in that a given partial correlation may be compatible with different multicollinearity patterns. The Farrar-Glauber test has also been severely criticized by T. Krishna Kumar<sup>21</sup> and John O'Hagan and Brendan McCabe.<sup>22</sup>

<sup>19</sup>D. E. Farrar and R. R. Glauber, "Multicollinearity in Regression Analysis: The Problem Revisited," *Review of Economics and Statistics*, vol. 49, 1967, pp. 92-107.

<sup>20</sup>"The Detection of Multicollinearity: A Comment," *Review of Economics and Statistics*, vol. 57, 1975, pp. 365-366.

<sup>21</sup>"Multicollinearity in Regression Analysis," *Review of Economics and Statistics*, vol. 57, 1975, pp. 366-368.

<sup>22</sup>"Tests for the Severity of Multicollinearity in Regression Analysis: A Comment," *Review of Economics and Statistics*, vol. 57, 1975, pp. 368-370.



**4. Auxiliary regressions.** Since multicollinearity arises because one or more of the regressors are exact or approximately linear combinations of the other regressors, one way of finding out which  $X$  variable is related to other  $X$  variables is to regress each  $X_i$  on the remaining  $X$  variables and compute the corresponding  $R^2$ , which we designate as  $R_i^2$ ; each one of these regressions is called an **auxiliary regression**, auxiliary to the main regression of  $Y$  on the  $X$ 's. Then, following the relationship between  $F$  and  $R^2$  established in (8.5.11), the variable

$$F_i = \frac{R_{x_i \cdot x_2 x_3 \dots x_k}^2 / (k - 2)}{(1 - R_{x_i \cdot x_2 x_3 \dots x_k}^2) / (n - k + 1)} \quad (10.7.3)$$

follows the  $F$  distribution with  $k - 2$  and  $n - k + 1$  df. In Eq. (10.7.3)  $n$  stands for the sample size,  $k$  stands for the number of explanatory variables including the intercept term, and  $R_{x_i \cdot x_2 x_3 \dots x_k}^2$  is the coefficient of determination in the regression of variable  $X_i$  on the remaining  $X$  variables.<sup>23</sup>

If the computed  $F$  exceeds the critical  $F_i$  at the chosen level of significance, it is taken to mean that the particular  $X_i$  is collinear with other  $X$ 's; if it does not exceed the critical  $F_i$ , we say that it is not collinear with other  $X$ 's, in which case we may retain that variable in the model. If  $F_i$  is statistically significant, we will still have to decide whether the particular  $X_i$  should be dropped from the model. This question will be taken up in Section 10.8.

But this method is not without its drawbacks, for

... if the multicollinearity involves only a few variables so that the auxiliary regressions do not suffer from extensive multicollinearity, the estimated coefficients may reveal the nature of the linear dependence among the regressors. Unfortunately, if there are several complex linear associations, this curve fitting exercise may not prove to be of much value as it will be difficult to identify the separate interrelationships.<sup>24</sup>

Instead of formally testing all auxiliary  $R^2$  values, one may adopt **Klien's rule of thumb**, which suggests that multicollinearity may be a troublesome problem only if the  $R^2$  obtained from an auxiliary regression is greater than the overall  $R^2$ , that is, that obtained from the regression of  $Y$  on all the regressors.<sup>25</sup> Of course, like all other rules of thumb, this one should be used judiciously.

**5. Eigenvalues and condition index.** If you examine the SAS output of the Cobb–Douglas production function given in Appendix 7A.5 you will see

<sup>23</sup>For example,  $R_{x_2}^2$  can be obtained by regressing  $X_{2i}$  as follows:  $X_{2i} = a_1 + a_3 X_{3i} + a_4 X_{4i} + \dots + a_k X_{ki} + \hat{u}_i$ .

<sup>24</sup>George G. Judge, R. Carter Hill, William E. Griffiths, Helmut Lütkepohl, and Tsoung-Chao Lee, *Introduction to the Theory and Practice of Econometrics*, John Wiley & Sons, New York, 1982, p. 621.

<sup>25</sup>Lawrence R. Klien, *An Introduction to Econometrics*, Prentice-Hall, Englewood Cliffs, N.J., 1962, p. 101.

that SAS uses *eigenvalues* and the *condition index* to diagnose multicollinearity. We will not discuss eigenvalues here, for that would take us into topics in matrix algebra that are beyond the scope of this book. From these eigenvalues, however, we can derive what is known as the **condition number  $k$**  defined as

$$k = \frac{\text{Maximum eigenvalue}}{\text{Minimum eigenvalue}}$$

and the **condition index (CI)** defined as

$$\text{CI} = \sqrt{\frac{\text{Maximum eigenvalue}}{\text{Minimum eigenvalue}}} = \sqrt{k}$$

**Then we have this rule of thumb.** If  $k$  is between 100 and 1000 there is moderate to strong multicollinearity and if it exceeds 1000 there is severe multicollinearity. Alternatively, if the CI ( $= \sqrt{k}$ ) is between 10 and 30, there is moderate to strong multicollinearity and if it exceeds 30 there is severe multicollinearity.

For the illustrative example,  $k = 3.0/0.00002422$  or about 123,864, and  $\text{CI} = \sqrt{123,864} =$  about 352; both  $k$  and the CI therefore suggest severe multicollinearity. Of course,  $k$  and CI can be calculated between the maximum eigenvalue and any other eigenvalue, as is done in the printout. (*Note:* The printout does not explicitly compute  $k$ , but that is simply the square of CI.) Incidentally, note that a low eigenvalue (in relation to the maximum eigenvalue) is generally an indication of near-linear dependencies in the data.

Some authors believe that the condition index is the best available multicollinearity diagnostic. But this opinion is not shared widely. For us, then, the CI is just a rule of thumb, a bit more sophisticated perhaps. But for further details, the reader may consult the references.<sup>26</sup>

**6. Tolerance and variance inflation factor.** We have already introduced TOL and VIF. As  $R_j^2$ , the coefficient of determination in the regression of regressor  $X_j$  on the remaining regressors in the model, increases toward unity, that is, as the collinearity of  $X_j$  with the other regressors increases, VIF also increases and in the limit it can be infinite.

Some authors therefore use the VIF as an indicator of multicollinearity. The larger the value of  $\text{VIF}_j$ , the more “troublesome” or collinear the variable  $X_j$ . **As a rule of thumb**, if the VIF of a variable exceeds 10, which will happen if  $R_j^2$  exceeds 0.90, that variable is said to be highly collinear.<sup>27</sup>

Of course, one could use  $\text{TOL}_j$  as a measure of multicollinearity in view of its intimate connection with  $\text{VIF}_j$ . The closer is  $\text{TOL}_j$  to zero, the greater the degree of collinearity of that variable with the other regressors. On the

<sup>26</sup>See especially D. A. Belsley, E. Kuh, and R. E. Welsch, *Regression Diagnostics: Identifying Influential Data and Sources of Collinearity*, John Wiley & Sons, New York, 1980, Chap. 3. However, this book is not for the beginner.

<sup>27</sup>See David G. Kleinbaum, Lawrence L. Kupper, and Keith E. Muller, *Applied Regression Analysis and other Multivariate Methods*, 2d ed., PWS-Kent, Boston, Mass., 1988, p. 210.

other hand, the closer  $TOL_j$  is to 1, the greater the evidence that  $X_j$  is not collinear with the other regressors.

VIF (or tolerance) as a measure of collinearity is not free of criticism. As (10.5.4) shows,  $\text{var}(\hat{\beta}_j)$  depends on three factors:  $\sigma^2$ ,  $\sum x_j^2$ , and  $VIF_j$ . A high VIF can be counterbalanced by a low  $\sigma^2$  or a high  $\sum x_j^2$ . To put it differently, a high VIF is neither necessary nor sufficient to get high variances and high standard errors. Therefore, high multicollinearity, as measured by a high VIF, may not necessarily cause high standard errors. In all this discussion, the terms *high* and *low* are used in a relative sense.

To conclude our discussion of detecting multicollinearity, we stress that the various methods we have discussed are essentially in the nature of “fishing expeditions,” for we cannot tell which of these methods will work in any particular application. Alas, not much can be done about it, for multicollinearity is specific to a given sample over which the researcher may not have much control, especially if the data are nonexperimental in nature—the usual fate of researchers in the social sciences.

Again as a parody of multicollinearity, Goldberger cites numerous ways of detecting micronumerosity, such as developing critical values of the sample size,  $n^*$ , such that micronumerosity is a problem only if the actual sample size,  $n$ , is smaller than  $n^*$ . The point of Goldberger’s parody is to emphasize that small sample size and lack of variability in the explanatory variables may cause problems that are at least as serious as those due to multicollinearity.

## 10.8 REMEDIAL MEASURES

What can be done if multicollinearity is serious? We have two choices: (1) do nothing or (2) follow some rules of thumb.

### Do Nothing

The “do nothing” school of thought is expressed by Blanchard as follows<sup>28</sup>:

When students run their first ordinary least squares (OLS) regression, the first problem that they usually encounter is that of multicollinearity. Many of them conclude that there is something wrong with OLS; some resort to new and often creative techniques to get around the problem. But, we tell them, this is wrong. Multicollinearity is God’s will, not a problem with OLS or statistical technique in general.

What Blanchard is saying is that multicollinearity is essentially a data deficiency problem (micronumerosity, again) and some times we have no choice over the data we have available for empirical analysis.

Also, it is not that all the coefficients in a regression model are statistically insignificant. Moreover, even if we cannot estimate one or more regression coefficients with greater precision, a linear combination of them (i.e., estimable function) can be estimated relatively efficiently. As we saw in

<sup>28</sup>Blanchard, O. J., Comment, *Journal of Business and Economic Statistics*, vol. 5, 1967, pp. 449–451. The quote is reproduced from Peter Kennedy, *A Guide to Econometrics*, 4th ed., MIT Press, Cambridge, Mass., 1998, p. 190.

(10.2.3), we can estimate  $\alpha$  uniquely, even if we cannot estimate its two components given there individually. Sometimes this is the best we can do with a given set of data.<sup>29</sup>

### Rule-of-Thumb Procedures

One can try the following rules of thumb to address the problem of multicollinearity, the success depending on the severity of the collinearity problem.

**1. A priori information.** Suppose we consider the model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

where  $Y$  = consumption,  $X_2$  = income, and  $X_3$  = wealth. As noted before, income and wealth variables tend to be highly collinear. But suppose a priori we believe that  $\beta_3 = 0.10\beta_2$ ; that is, the rate of change of consumption with respect to wealth is one-tenth the corresponding rate with respect to income. We can then run the following regression:

$$\begin{aligned} Y_i &= \beta_1 + \beta_2 X_{2i} + 0.10\beta_2 X_{3i} + u_i \\ &= \beta_1 + \beta_2 X_i + u_i \end{aligned}$$

where  $X_i = X_{2i} + 0.1X_{3i}$ . Once we obtain  $\hat{\beta}_2$ , we can estimate  $\hat{\beta}_3$  from the postulated relationship between  $\beta_2$  and  $\beta_3$ .

How does one obtain a priori information? It could come from previous empirical work in which the collinearity problem happens to be less serious or from the relevant theory underlying the field of study. For example, in the Cobb–Douglas–type production function (7.9.1), if one expects constant returns to scale to prevail, then  $(\beta_2 + \beta_3) = 1$ , in which case we could run the regression (8.7.14), regressing the output-labor ratio on the capital-labor ratio. If there is collinearity between labor and capital, as generally is the case in most sample data, such a transformation may reduce or eliminate the collinearity problem. But a warning is in order here regarding imposing such a priori restrictions, “. . . since in general we will want to test economic theory’s a priori predictions rather than simply impose them on data for which they may not be true.”<sup>30</sup> However, we know from Section 8.7 how to test for the validity of such restrictions explicitly.

**2. Combining cross-sectional and time series data.** A variant of the extraneous or a priori information technique is the combination of cross-sectional and time-series data, known as *pooling the data*. Suppose we want

<sup>29</sup>For an interesting discussion on this, see Conlisk, J., “When Collinearity is Desirable,” *Western Economic Journal*, vol. 9, 1971, pp. 393–407.

<sup>30</sup>Mark B. Stewart and Kenneth F. Wallis, *Introductory Econometrics*, 2d ed., John Wiley & Sons, A Halstead Press Book, New York, 1981, p. 154.

to study the demand for automobiles in the United States and assume we have time series data on the number of cars sold, average price of the car, and consumer income. Suppose also that

$$\ln Y_t = \beta_1 + \beta_2 \ln P_t + \beta_3 \ln I_t + u_t$$

where  $Y$  = number of cars sold,  $P$  = average price,  $I$  = income, and  $t$  = time. Our objective is to estimate the price elasticity  $\beta_2$  and income elasticity  $\beta_3$ .

In time series data the price and income variables generally tend to be highly collinear. Therefore, if we run the preceding regression, we shall be faced with the usual multicollinearity problem. A way out of this has been suggested by Tobin.<sup>31</sup> He says that if we have cross-sectional data (for example, data generated by consumer panels, or budget studies conducted by various private and governmental agencies), we can obtain a fairly reliable estimate of the income elasticity  $\beta_3$  because in such data, which are at a point in time, the prices do not vary much. Let the cross-sectionally estimated income elasticity be  $\hat{\beta}_3$ . Using this estimate, we may write the preceding time series regression as

$$Y_t^* = \beta_1 + \beta_2 \ln P_t + u_t$$

where  $Y^* = \ln Y - \hat{\beta}_3 \ln I$ , that is,  $Y^*$  represents that value of  $Y$  after removing from it the effect of income. We can now obtain an estimate of the price elasticity  $\beta_2$  from the preceding regression.

Although it is an appealing technique, pooling the time series and cross-sectional data in the manner just suggested may create problems of interpretation, because we are assuming implicitly that the cross-sectionally estimated income elasticity is the same thing as that which would be obtained from a pure time series analysis.<sup>32</sup> Nonetheless, the technique has been used in many applications and is worthy of consideration in situations where the cross-sectional estimates do not vary substantially from one cross section to another. An example of this technique is provided in exercise 10.26.

**3. Dropping a variable(s) and specification bias.** When faced with severe multicollinearity, one of the “simplest” things to do is to drop one of the collinear variables. Thus, in our consumption–income–wealth illustration, when we drop the wealth variable, we obtain regression (10.6.4), which shows that, whereas in the original model the income variable was statistically insignificant, it is now “highly” significant.

But in dropping a variable from the model we may be committing a **specification bias** or **specification error**. Specification bias arises from

<sup>31</sup>J. Tobin, “A Statistical Demand Function for Food in the U.S.A.,” *Journal of the Royal Statistical Society*, Ser. A, 1950, pp. 113–141.

<sup>32</sup>For a thorough discussion and application of the pooling technique, see Edwin Kuh, *Capital Stock Growth: A Micro-Econometric Approach*, North-Holland Publishing Company, Amsterdam, 1963, Chaps. 5 and 6.

incorrect specification of the model used in the analysis. Thus, if economic theory says that income and wealth should both be included in the model explaining the consumption expenditure, dropping the wealth variable would constitute specification bias.

Although we will discuss the topic of specification bias in Chapter 13, we caught a glimpse of it in Section 7.7. If, for example, the true model is

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

but we mistakenly fit the model

$$Y_i = b_1 + b_{12} X_{2i} + \hat{u}_i \quad (10.8.1)$$

then it can be shown that (see Appendix 13A.1)

$$E(b_{12}) = \beta_2 + \beta_3 b_{32} \quad (10.8.2)$$

where  $b_{32}$  = slope coefficient in the regression of  $X_3$  on  $X_2$ . Therefore, it is obvious from (10.8.2) that  $b_{12}$  will be a biased estimate of  $\beta_2$  as long as  $b_{32}$  is different from zero (it is assumed that  $\beta_3$  is different from zero; otherwise there is no sense in including  $X_3$  in the original model).<sup>33</sup> Of course, if  $b_{32}$  is zero, we have no multicollinearity problem to begin with. It is also clear from (10.8.2) that if both  $b_{32}$  and  $\beta_3$  are positive (or both are negative),  $E(b_{12})$  will be greater than  $\beta_2$ ; hence, on the average  $b_{12}$  will overestimate  $\beta_2$ , leading to a positive bias. Similarly, if the product  $b_{32}\beta_3$  is negative, on the average  $b_{12}$  will underestimate  $\beta_2$ , leading to a negative bias.

From the preceding discussion it is clear that dropping a variable from the model to alleviate the problem of multicollinearity may lead to the specification bias. Hence the remedy may be worse than the disease in some situations because, whereas multicollinearity may prevent precise estimation of the parameters of the model, omitting a variable may seriously mislead us as to the true values of the parameters. Recall that OLS estimators are BLUE despite near collinearity.

**4. Transformation of variables.** Suppose we have time series data on consumption expenditure, income, and wealth. One reason for high multicollinearity between income and wealth in such data is that over time both the variables tend to move in the same direction. One way of minimizing this dependence is to proceed as follows.

If the relation

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_t \quad (10.8.3)$$

<sup>33</sup>Note further that if  $b_{32}$  does not approach zero as the sample size is increased indefinitely, then  $b_{12}$  will be not only biased but also inconsistent.

holds at time  $t$ , it must also hold at time  $t - 1$  because the origin of time is arbitrary anyway. Therefore, we have

$$Y_{t-1} = \beta_1 + \beta_2 X_{2,t-1} + \beta_3 X_{3,t-1} + u_{t-1} \quad (10.8.4)$$

If we subtract (10.8.4) from (10.8.3), we obtain

$$Y_t - Y_{t-1} = \beta_2(X_{2t} - X_{2,t-1}) + \beta_3(X_{3t} - X_{3,t-1}) + v_t \quad (10.8.5)$$

where  $v_t = u_t - u_{t-1}$ . Equation (10.8.5) is known as the **first difference form** because we run the regression, not on the original variables, but on the differences of successive values of the variables.

The first difference regression model often reduces the severity of multicollinearity because, although the levels of  $X_2$  and  $X_3$  may be highly correlated, there is no a priori reason to believe that their differences will also be highly correlated.

As we shall see in the chapters on **time series econometrics**, an incidental advantage of the first-difference transformation is that it may make a nonstationary time series stationary. In those chapters we will see the importance of stationary time series. As noted in Chapter 1, loosely speaking, a time series, say,  $Y_t$ , is stationary if its mean and variance do not change systematically over time.

Another commonly used transformation in practice is the **ratio transformation**. Consider the model:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_t \quad (10.8.6)$$

where  $Y$  is consumption expenditure in real dollars,  $X_2$  is GDP, and  $X_3$  is total population. Since GDP and population grow over time, they are likely to be correlated. One “solution” to this problem is to express the model on a per capita basis, that is, by dividing (10.8.4) by  $X_3$ , to obtain:

$$\frac{Y_t}{X_{3t}} = \beta_1 \left( \frac{1}{X_{3t}} \right) + \beta_2 \left( \frac{X_{2t}}{X_{3t}} \right) + \beta_3 + \left( \frac{u_t}{X_{3t}} \right) \quad (10.8.7)$$

Such a transformation may reduce collinearity in the original variables.

But the first-difference or ratio transformations are not without problems. For instance, the error term  $v_t$  in (10.8.5) may not satisfy one of the assumptions of the classical linear regression model, namely, that the disturbances are serially uncorrelated. As we will see in Chapter 12, if the original disturbance term  $u_t$  is serially uncorrelated, the error term  $v_t$  obtained previously will in most cases be serially correlated. Therefore, the remedy may be worse than the disease. Moreover, there is a loss of one observation due to the differencing procedure, and therefore the degrees of freedom are



reduced by one. In a small sample, this could be a factor one would wish at least to take into consideration. Furthermore, the first-differencing procedure may not be appropriate in cross-sectional data where there is no logical ordering of the observations.

Similarly, in the ratio model (10.8.7), the error term

$$\left( \frac{u_t}{X_{3t}} \right)$$

will be heteroscedastic, if the original error term  $u_t$  is homoscedastic, as we shall see in Chapter 11. Again, the remedy may be worse than the disease of collinearity.

In short, one should be careful in using the first difference or ratio method of transforming the data to resolve the problem of multicollinearity.

**5. Additional or new data.** Since multicollinearity is a sample feature, it is possible that in another sample involving the same variables collinearity may not be so serious as in the first sample. Sometimes simply increasing the size of the sample (if possible) may attenuate the collinearity problem. For example, in the three-variable model we saw that

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2 (1 - r_{23}^2)}$$

Now as the sample size increases,  $\sum x_{2i}^2$  will generally increase. (Why?) Therefore, for any given  $r_{23}$ , the variance of  $\hat{\beta}_2$  will decrease, thus decreasing the standard error, which will enable us to estimate  $\beta_2$  more precisely.

As an illustration, consider the following regression of consumption expenditure  $Y$  on income  $X_2$  and wealth  $X_3$  based on 10 observations<sup>34</sup>:

$$\begin{aligned} \hat{Y}_i &= 24.377 + 0.8716X_{2i} - 0.0349X_{3i} \\ t &= (3.875) \quad (2.7726) \quad (-1.1595) \quad R^2 = 0.9682 \end{aligned} \quad (10.8.8)$$

The wealth coefficient in this regression not only has the wrong sign but is also statistically insignificant at the 5 percent level. But when the sample size was increased to 40 observations (micronumerosity?), the following results were obtained:

$$\begin{aligned} \hat{Y}_i &= 2.0907 + 0.7299X_{2i} + 0.0605X_{3i} \\ t &= (0.8713) \quad (6.0014) \quad (2.0014) \quad R^2 = 0.9672 \end{aligned} \quad (10.8.9)$$

Now the wealth coefficient not only has the correct sign but also is statistically significant at the 5 percent level.

<sup>34</sup>I am indebted to Albert Zucker for providing the results given in the following regressions.



Obtaining additional or “better” data is not always that easy, for as Judge et al. note:

Unfortunately, economists seldom can obtain additional data without bearing large costs, much less choose the values of the explanatory variables they desire. In addition, when adding new variables in situations that are not controlled, we must be aware of adding observations that were generated by a process other than that associated with the original data set; that is, we must be sure that the economic structure associated with the new observations is the same as the original structure.<sup>35</sup>

**6. Reducing collinearity in polynomial regressions.** In Section 7.10 we discussed polynomial regression models. A special feature of these models is that the explanatory variable(s) appear with various powers. Thus, in the total cubic cost function involving the regression of total cost on output,  $(\text{output})^2$ , and  $(\text{output})^3$ , as in (7.10.4), the various output terms are going to be correlated, making it difficult to estimate the various slope coefficients precisely.<sup>36</sup> In practice though, it has been found that if the explanatory variable(s) are expressed in the deviation form (i.e., deviation from the mean value), multicollinearity is substantially reduced. But even then the problem may persist,<sup>37</sup> in which case one may want to consider techniques such as **orthogonal polynomials**.<sup>38</sup>

**7. Other methods of remedying multicollinearity.** Multivariate statistical techniques such as **factor analysis** and **principal components** or techniques such as **ridge regression** are often employed to “solve” the problem of multicollinearity. Unfortunately, these techniques are beyond the scope of this book, for they cannot be discussed competently without resorting to matrix algebra.<sup>39</sup>

## 10.9 IS MULTICOLLINEARITY NECESSARILY BAD? MAYBE NOT IF THE OBJECTIVE IS PREDICTION ONLY

It has been said that if the sole purpose of regression analysis is prediction or forecasting, then multicollinearity is not a serious problem because the higher the  $R^2$ , the better the prediction.<sup>40</sup> But this may be so “. . . as long as

<sup>35</sup>Judge et al., op. cit., p. 625. See also Sec. 10.9.

<sup>36</sup>As noted, since the relationship between  $X$ ,  $X^2$ , and  $X^3$  is nonlinear, polynomial regressions do not violate the assumption of no multicollinearity of the classical model, strictly speaking.

<sup>37</sup>See R. A. Bradley and S. S. Srivastava, “Correlation and Polynomial Regression,” *American Statistician*, vol. 33, 1979, pp. 11–14.

<sup>38</sup>See Norman Draper and Harry Smith, *Applied Regression Analysis*, 2d ed., John Wiley & Sons, New York, 1981, pp. 266–274.

<sup>39</sup>A readable account of these techniques from an applied viewpoint can be found in Samprit Chatterjee and Bertram Price, *Regression Analysis by Example*, John Wiley & Sons, New York, 1977, Chaps. 7 and 8. See also H. D. Vinod, “A Survey of Ridge Regression and Related Techniques for Improvements over Ordinary Least Squares,” *Review of Economics and Statistics*, vol. 60, February 1978, pp. 121–131.

<sup>40</sup>See R. C. Geary, “Some Results about Relations between Stochastic Variables: A Discussion Document,” *Review of International Statistical Institute*, vol. 31, 1963, pp. 163–181.

the values of the explanatory variables for which predictions are desired obey the same near-exact linear dependencies as the original design [data] matrix  $X$ .”<sup>41</sup> Thus, if in an estimated regression it was found that  $X_2 = 2X_3$  approximately, then in a future sample used to forecast  $Y$ ,  $X_2$  should also be approximately equal to  $2X_3$ , a condition difficult to meet in practice (see footnote 35), in which case prediction will become increasingly uncertain.<sup>42</sup> Moreover, if the objective of the analysis is not only prediction but also reliable estimation of the parameters, serious multicollinearity will be a problem because we have seen that it leads to large standard errors of the estimators.

In one situation, however, multicollinearity may not pose a serious problem. This is the case when  $R^2$  is high and the regression coefficients are individually significant as revealed by the higher  $t$  values. Yet, multicollinearity diagnostics, say, the condition index, indicate that there is serious collinearity in the data. When can such a situation arise? As Johnston notes:

This can arise if individual coefficients happen to be numerically well in excess of the true value, so that the effect still shows up in spite of the inflated standard error and/or because the true value itself is so large that even an estimate on the downside still shows up as significant.<sup>43</sup>

### 10.10 AN EXTENDED EXAMPLE: THE LONGLEY DATA

We conclude this chapter by analyzing the data collected by Longley.<sup>44</sup> Although originally collected to assess the computational accuracy of least-squares estimates in several computer programs, the Longley data has become the workhorse to illustrate several econometric problems, including multicollinearity. The data are reproduced in Table 10.7. The data are time series for the years 1947–1962 and pertain to  $Y$  = number of people employed, in thousands;  $X_1$  = GNP implicit price deflator;  $X_2$  = GNP, millions of dollars;  $X_3$  = number of people unemployed in thousands,  $X_4$  = number of people in the armed forces,  $X_5$  = noninstitutionalized population over 14 years of age; and  $X_6$  = year, equal to 1 in 1947, 2 in 1948, and 16 in 1962.

<sup>41</sup>Judge et al., op. cit., p. 619. You will also find on this page proof of why, despite collinearity, one can obtain better mean predictions if the existing collinearity structure also continues in the future samples.

<sup>42</sup>For an excellent discussion, see E. Malinvaud, *Statistical Methods of Econometrics*, 2d ed., North-Holland Publishing Company, Amsterdam, 1970, pp. 220–221.

<sup>43</sup>J. Johnston, *Econometric Methods*, 3d ed., McGraw-Hill, New York, 1984, p. 249.

<sup>44</sup>Longley, J. “An Appraisal of Least-Squares Programs from the Point of the User,” *Journal of the American Statistical Association*, vol. 62, 1967, pp. 819–841.

TABLE 10.7 LONGLEY DATA

Observation	y	X <sub>1</sub>	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	X <sub>5</sub>	Time
1947	60,323	830	234,289	2356	1590	107,608	1
1948	61,122	885	259,426	2325	1456	108,632	2
1949	60,171	882	258,054	3682	1616	109,773	3
1950	61,187	895	284,599	3351	1650	110,929	4
1951	63,221	962	328,975	2099	3099	112,075	5
1952	63,639	981	346,999	1932	3594	113,270	6
1953	64,989	990	365,385	1870	3547	115,094	7
1954	63,761	1000	363,112	3578	3350	116,219	8
1955	66,019	1012	397,469	2904	3048	117,388	9
1956	67,857	1046	419,180	2822	2857	118,734	10
1957	68,169	1084	442,769	2936	2798	120,445	11
1958	66,513	1108	444,546	4681	2637	121,950	12
1959	68,655	1126	482,704	3813	2552	123,366	13
1960	69,564	1142	502,601	3931	2514	125,368	14
1961	69,331	1157	518,173	4806	2572	127,852	15
1962	70,551	1169	554,894	4007	2827	130,081	16

Source: See footnote 44.

Assume that our objective is to predict Y on the basis of the six X variables. Using Eviews3, we obtain the following regression results:

Dependent Variable: Y  
Sample: 1947-1962

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-3482259.	890420.4	-3.910803	0.0036
X <sub>1</sub>	15.06187	84.91493	0.177376	0.8631
X <sub>2</sub>	-0.035819	0.033491	-1.069516	0.3127
X <sub>3</sub>	-2.020230	0.488400	-4.136427	0.0025
X <sub>4</sub>	-1.033227	0.214274	-4.821985	0.0009
X <sub>5</sub>	-0.051104	0.226073	-0.226051	0.8262
X <sub>6</sub>	1829.151	455.4785	4.015890	0.0030
R-squared	0.995479	Mean dependent var		65317.00
Adjusted R-squared	0.992465	S.D. dependent var		3511.968
S.E. of regression	304.8541	Akaike info criterion		14.57718
Sum squared resid	836424.1	Schwarz criterion		14.91519
Log likelihood	-109.6174	F-statistic		330.2853
Durbin-Watson stat	2.559488	Prob(F-statistic)		0.000000

A glance at these results would suggest that we have the collinearity problem, for the  $R^2$  value is very high, but quite a few variables are statistically insignificant ( $X_1$ ,  $X_2$ , and  $X_5$ ), a classic symptom of multicollinearity. To shed more light on this, we show in Table 10.8 the intercorrelations among the six regressors.

TABLE 10.8 INTERCORRELATIONS

	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$
$X_1$	1.000000	0.991589	0.620633	0.464744	0.979163	0.991149
$X_2$	0.991589	1.000000	0.604261	0.446437	0.991090	0.995273
$X_3$	0.620633	0.604261	1.000000	-0.177421	0.686552	0.668257
$X_4$	0.464744	0.446437	-0.177421	1.000000	0.364416	0.417245
$X_5$	0.979163	0.991090	0.686552	0.364416	1.000000	0.993953
$X_6$	0.991149	0.995273	0.668257	0.417245	0.993953	1.000000

This table gives what is called the **correlation matrix**. In this table the entries on the main diagonal (those running from the upper left-hand corner to the lower right-hand corner) give the correlation of one variable with itself, which is always 1 by definition, and the entries off the main diagonal are the pair-wise correlations among the  $X$  variables. If you take the first row of this table, this gives the correlation of  $X_1$  with the other  $X$  variables. For example, 0.991589 is the correlation between  $X_1$  and  $X_2$ , 0.620633 is the correlation between  $X_1$  and  $X_3$ , and so on.

As you can see, several of these pair-wise correlations are quite high, suggesting that there may be a severe collinearity problem. Of course, remember the warning given earlier that such pair-wise correlations may be a sufficient but not a necessary condition for the existence of multicollinearity.

To shed further light on the nature of the multicollinearity problem, let us run the auxiliary regressions, that is the regression of each  $X$  variable on the remaining  $X$  variables. To save space, we will present only the  $R^2$  values obtained from these regressions, which are given in Table 10.9. Since the  $R^2$  values in the auxiliary regressions are very high (with the possible exception of the regression of  $X_4$ ) on the remaining  $X$  variables, it seems that we do have a serious collinearity problem. The same information is obtained from the tolerance factors. As noted previously, the closer the tolerance factor is to zero, the greater is the evidence of collinearity.

Applying Klein's rule of thumb, we see that the  $R^2$  values obtained from the auxiliary regressions exceed the overall  $R^2$  value (that is the one obtained from the regression of  $Y$  on all the  $X$  variables) of 0.9954 in 3 out of

TABLE 10.9  $R^2$  VALUES FROM THE AUXILIARY REGRESSIONS

Dependent variable	$R^2$ value	Tolerance (TOL) = $1 - R^2$
$X_1$	0.9926	0.0074
$X_2$	0.9994	0.0006
$X_3$	0.9702	0.0298
$X_4$	0.7213	0.2787
$X_5$	0.9970	0.0030
$X_6$	0.9986	0.0014

6 auxiliary regressions, again suggesting that indeed the Longley data are plagued by the multicollinearity problem. Incidentally, applying the  $F$  test given in (10.7.3) the reader should verify that the  $R^2$  values given in the preceding tables are all statistically significantly different from zero.

We noted earlier that the OLS estimators and their standard errors are sensitive to small changes in the data. In exercise 10.32 the reader is asked to rerun the regression of  $Y$  on all the six  $X$  variables but drop the last data observations, that is, run the regression for the period 1947–1961. You will see how the regression results change by dropping just a single year’s observations.

Now that we have established that we have the multicollinearity problem, what “remedial” actions can we take? Let us reconsider our original model. First of all, we could express GNP not in nominal terms, but in real terms, which we can do by dividing nominal GNP by the implicit price deflator. Second, since noninstitutional population over 14 years of age grows over time because of natural population growth, it will be highly correlated with time, the variable  $X_6$  in our model. Therefore, instead of keeping both these variables, we will keep the variable  $X_5$  and drop  $X_6$ . Third, there is no compelling reason to include  $X_3$ , the number of people unemployed; perhaps the unemployment rate would have been a better measure of labor market conditions. But we have no data on the latter. So, we will drop the variable  $X_3$ . Making these changes, we obtain the following regression results (RGNP = real GNP)<sup>45</sup>:

Dependent Variable: Y  
Sample: 1947–1962

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	65720.37	10624.81	6.185558	0.0000
RGNP	9.736496	1.791552	5.434671	0.0002
$X_4$	-0.687966	0.322238	-2.134965	0.0541
$X_5$	-0.299537	0.141761	-2.112965	0.0562
R-squared	0.981404	Mean dependent var		65317.00
Adjusted R-squared	0.976755	S.D. dependent var		3511.968
S.E. of regression	535.4492	Akaike info criterion		15.61641
Sum squared resid	3440470.	Schwarz criterion		15.80955
Log likelihood	-120.9313	F-statistic		211.0972
Durbin-Watson stat	1.654069	Prob(F-statistic)		0.000000

Although the  $R^2$  value has declined slightly compared with the original  $R^2$ , it is still very high. Now all the estimated coefficients are significant and the signs of the coefficients make economic sense.

<sup>45</sup>The coefficient of correlation between  $X_5$  and  $X_6$  is about 0.9939, a very high correlation indeed.

We leave it for the reader to devise alternative models and see how the results change. Also keep in mind the warning sounded earlier about using the ratio method of transforming the data to alleviate the problem of collinearity. We will revisit this question in Chapter 11.

### 10.11 SUMMARY AND CONCLUSIONS

1. One of the assumptions of the classical linear regression model is that there is no multicollinearity among the explanatory variables, the  $X$ 's. Broadly interpreted, multicollinearity refers to the situation where there is either an exact or approximately exact linear relationship among the  $X$  variables.

2. The consequences of multicollinearity are as follows: If there is perfect collinearity among the  $X$ 's, their regression coefficients are indeterminate and their standard errors are not defined. If collinearity is high but not perfect, estimation of regression coefficients is possible but their standard errors tend to be large. As a result, the population values of the coefficients cannot be estimated precisely. However, if the objective is to estimate linear combinations of these coefficients, *the estimable functions*, this can be done even in the presence of perfect multicollinearity.

3. Although there are no sure methods of detecting collinearity, there are several indicators of it, which are as follows:

- (a) The clearest sign of multicollinearity is when  $R^2$  is very high but none of the regression coefficients is statistically significant on the basis of the conventional  $t$  test. This case is, of course, extreme.
- (b) In models involving just two explanatory variables, a fairly good idea of collinearity can be obtained by examining the zero-order, or simple, correlation coefficient between the two variables. If this correlation is high, multicollinearity is generally the culprit.
- (c) However, the zero-order correlation coefficients can be misleading in models involving more than two  $X$  variables since it is possible to have low zero-order correlations and yet find high multicollinearity. In situations like these, one may need to examine the partial correlation coefficients.
- (d) If  $R^2$  is high but the partial correlations are low, multicollinearity is a possibility. Here one or more variables may be superfluous. But if  $R^2$  is high and the partial correlations are also high, multicollinearity may not be readily detectable. Also, as pointed out by C. Robert, Krishna Kumar, John O'Hagan, and Brendan McCabe, there are some statistical problems with the partial correlation test suggested by Farrar and Glauber.
- (e) Therefore, one may regress each of the  $X_i$  variables on the remaining  $X$  variables in the model and find out the corresponding coefficients of determination  $R_i^2$ . A high  $R_i^2$  would suggest that  $X_i$

is highly correlated with the rest of the  $X$ 's. Thus, one may drop that  $X_i$  from the model, provided it does not lead to serious specification bias.

4. Detection of multicollinearity is half the battle. The other half is concerned with how to get rid of the problem. Again there are no sure methods, only a few rules of thumb. Some of these rules are as follows: (1) using extraneous or prior information, (2) combining cross-sectional and time series data, (3) omitting a highly collinear variable, (4) transforming data, and (5) obtaining additional or new data. Of course, which of these rules will work in practice will depend on the nature of the data and severity of the collinearity problem.

5. We noted the role of multicollinearity in prediction and pointed out that unless the collinearity structure continues in the future sample it is hazardous to use the estimated regression that has been plagued by multicollinearity for the purpose of forecasting.

6. Although multicollinearity has received extensive (some would say excessive) attention in the literature, an equally important problem encountered in empirical research is that of micronumerosity, smallness of sample size. According to Goldberger, "When a research article complains about multicollinearity, readers ought to see whether the complaints would be convincing if "micronumerosity" were substituted for "multicollinearity."<sup>46</sup> He suggests that the reader ought to decide how small  $n$ , the number of observations, is before deciding that one has a small-sample problem, just as one decides how high an  $R^2$  value is in an auxiliary regression before declaring that the collinearity problem is very severe.

## EXERCISES

### Questions

- 10.1. In the  $k$ -variable linear regression model there are  $k$  normal equations to estimate the  $k$  unknowns. These normal equations are given in **Appendix C**. Assume that  $X_k$  is a perfect linear combination of the remaining  $X$  variables. How would you show that in this case it is impossible to estimate the  $k$  regression coefficients?
- 10.2. Consider the set of hypothetical data in Table 10.10. Suppose you want to fit the model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

to the data.

- Can you estimate the three unknowns? Why or why not?
- If not, what linear functions of these parameters, the estimable functions, can you estimate? Show the necessary calculations.

<sup>46</sup>Goldberger, op. cit., p. 250.



TABLE 10.10

Y	X <sub>2</sub>	X <sub>3</sub>
-10	1	1
-8	2	3
-6	3	5
-4	4	7
-2	5	9
0	6	11
2	7	13
4	8	15
6	9	17
8	10	19
10	11	21

**10.3.** Refer to the child mortality example discussed in Chapter 8. The example there involved the regression of child mortality (CM) rate on per capita GNP (PGNP) and female literacy rate (FLR). Now suppose we add the variable, total fertility rate (TFR). This gives the following regression results.

Dependent Variable: CM

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	168.3067	32.89165	5.117003	0.0000
PGNP	-0.005511	0.001878	-2.934275	0.0047
FLR	-1.768029	0.248017	-7.128663	0.0000
TFR	12.86864	4.190533	3.070883	0.0032
R-squared	0.747372	Mean dependent var	141.5000	
Adjusted R-squared	0.734740	S.D. dependent var	75.97807	
S.E. of regression	39.13127	Akaike info criterion	10.23218	
Sum squared resid	91875.38	Schwarz criterion	10.36711	
Log likelihood	-323.4298	F-statistic	59.16767	
Durbin-Watson stat	2.170318	Prob(F-statistic)	0.000000	

- a. Compare these regression results with those given in Eq. (8.2.1). What changes do you see? And how do you account for them?
  - b. Is it worth adding the variable TFR to the model? Why?
  - c. Since all the individual *t* coefficients are statistically significant, can we say that we do not have a collinearity problem in the present case?
- 10.4.** If the relation  $\lambda_1 X_{1i} + \lambda_2 X_{2i} + \lambda_3 X_{3i} = 0$  holds true for all values of  $\lambda_1, \lambda_2,$  and  $\lambda_3,$  estimate  $r_{1,2,3}, r_{1,3,2},$  and  $r_{2,3,1}.$  Also find  $R^2_{1,2,3}, R^2_{2,1,3},$  and  $R^2_{3,1,2}.$  What is the degree of multicollinearity in this situation? *Note:*  $R^2_{1,2,3}$  is the coefficient of determination in the regression of *Y* on  $X_2$  and  $X_3.$  Other  $R^2$  values are to be interpreted similarly.
- 10.5.** Consider the following model:

$$Y_t = \beta_1 + \beta_2 X_t + \beta_3 X_{t-1} + \beta_4 X_{t-2} + \beta_5 X_{t-3} + \beta_6 X_{t-4} + u_t$$

where *Y* = consumption, *X* = income, and *t* = time. The preceding model postulates that consumption expenditure at time *t* is a function not only



of income at time  $t$  but also of income through previous periods. Thus, consumption expenditure in the first quarter of 2000 is a function of income in that quarter and the four quarters of 1999. Such models are called **distributed lag models**, and we shall discuss them in a later chapter.

- a. Would you expect multicollinearity in such models and why?
  - b. If collinearity is expected, how would you resolve the problem?
- 10.6.** Consider the illustrative example of Section 10.6. How would you reconcile the difference in the marginal propensity to consume obtained from (10.6.1) and (10.6.4)?
- 10.7.** In data involving economic time series such as GNP, money supply, prices, income, unemployment, etc., multicollinearity is usually suspected. Why?
- 10.8.** Suppose in the model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

that  $r_{23}$ , the coefficient of correlation between  $X_2$  and  $X_3$ , is zero. Therefore, someone suggests that you run the following regressions:

$$Y_i = \alpha_1 + \alpha_2 X_{2i} + u_{1i}$$

$$Y_i = \gamma_1 + \gamma_3 X_{3i} + u_{2i}$$

- a. Will  $\hat{\alpha}_2 = \hat{\beta}_2$  and  $\hat{\gamma}_3 = \hat{\beta}_3$ ? Why?
  - b. Will  $\hat{\beta}_1$  equal  $\hat{\alpha}_1$  or  $\hat{\gamma}_1$  or some combination thereof?
  - c. Will  $\text{var}(\hat{\beta}_2) = \text{var}(\hat{\alpha}_2)$  and  $\text{var}(\hat{\beta}_3) = \text{var}(\hat{\gamma}_3)$ ?
- 10.9.** Refer to the illustrative example of Chapter 7 where we fitted the Cobb–Douglas production function to the Taiwanese agricultural sector. The results of the regression given in (7.9.4) show that both the labor and capital coefficients are individually statistically significant.
- a. Find out whether the variables labor and capital are highly correlated.
  - b. If your answer to (a) is affirmative, would you drop, say, the labor variable from the model and regress the output variable on capital input only?
  - c. If you do so, what kind of specification bias is committed? Find out the nature of this bias.
- 10.10.** Refer to Example 7.4. For this problem the correlation matrix is as follows:

	$X_i$	$X_i^2$	$X_i^3$
$X_i$	1	0.9742	0.9284
$X_i^2$		1.0	0.9872
$X_i^3$			1.0

- a. “Since the zero-order correlations are very high, there must be serious multicollinearity.” Comment.
- b. Would you drop variables  $X_i^2$  and  $X_i^3$  from the model?
- c. If you drop them, what will happen to the value of the coefficient of  $X_i$ ?

- 10.11. Stepwise regression.** In deciding on the “best” set of explanatory variables for a regression model, researchers often follow the method of stepwise regression. In this method one proceeds either by introducing the  $X$  variables one at a time (**stepwise forward regression**) or by including all the possible  $X$  variables in one multiple regression and rejecting them one at a time (**stepwise backward regression**). The decision to add or drop a variable is usually made on the basis of the contribution of that variable to the ESS, as judged by the  $F$  test. Knowing what you do now about multicollinearity, would you recommend either procedure? Why or why not?\*
- 10.12.** State *with reason* whether the following statements are true, false, or uncertain:
- Despite perfect multicollinearity, OLS estimators are BLUE.
  - In cases of high multicollinearity, it is not possible to assess the individual significance of one or more partial regression coefficients.
  - If an auxiliary regression shows that a particular  $R_i^2$  is high, there is definite evidence of high collinearity.
  - High pair-wise correlations do not suggest that there is high multicollinearity.
  - Multicollinearity is harmless if the objective of the analysis is prediction only.
  - Ceteris paribus, the higher the VIF is, the larger the variances of OLS estimators.
  - The tolerance (TOL) is a better measure of multicollinearity than the VIF.
  - You will not obtain a high  $R^2$  value in a multiple regression if all the partial slope coefficients are *individually* statistically insignificant on the basis of the usual  $t$  test.
  - In the regression of  $Y$  on  $X_2$  and  $X_3$ , suppose there is little variability in the values of  $X_3$ . This would increase  $\text{var}(\hat{\beta}_3)$ . In the extreme, if all  $X_3$  are identical,  $\text{var}(\hat{\beta}_3)$  is infinite.
- 10.13. a.** Show that if  $r_{1i} = 0$  for  $i = 2, 3, \dots, k$  then

$$R_{1,2,3,\dots,k} = 0$$

- b.** What is the importance of this finding for the regression of variable  $X_1(=Y)$  on  $X_2, X_3, \dots, X_k$ ?
- 10.14.** Suppose all the zero-order correlation coefficients of  $X_1(=Y), X_2, \dots, X_k$  are equal to  $r$ .
- What is the value of  $R_{1,2,3,\dots,k}^2$ ?
  - What are the values of the first-order correlation coefficients?
- \*\***10.15.** In matrix notation it can be shown (see **Appendix C**) that

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

- What happens to  $\hat{\beta}$  when there is perfect collinearity among the  $X$ 's?
- How would you know if perfect collinearity exists?

\*See if your reasoning agrees with that of Arthur S. Goldberg and D. B. Jochems, “Note on Stepwise Least-Squares,” *Journal of the American Statistical Association*, vol. 56, March 1961, pp. 105–110.

\*\*Optional.

\*10.16. Using matrix notation, it can be shown

$$\text{var-cov}(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

What happens to this var-cov matrix:

- When there is perfect multicollinearity?
- When collinearity is high but not perfect?

\*10.17. Consider the following **correlation matrix**:

$$\mathbf{R} = \begin{matrix} & X_2 & X_3 & \cdots & X_k \\ X_2 & \left[ \begin{array}{cccc} 1 & r_{23} & \cdots & r_{2k} \\ r_{32} & 1 & \cdots & r_{3k} \\ \cdots & \cdots & \cdots & \cdots \\ r_{k2} & r_{k3} & \cdots & 1 \end{array} \right] \\ X_3 & & & & \\ \vdots & & & & \\ X_k & & & & \end{matrix}$$

How would you find out from the correlation matrix whether (a) there is perfect collinearity, (b) there is less than perfect collinearity, and (c) the  $X$ 's are uncorrelated.

*Hint:* You may use  $|\mathbf{R}|$  to answer these questions, where  $|\mathbf{R}|$  denotes the determinant of  $\mathbf{R}$ .

\*10.18. **Orthogonal explanatory variables.** Suppose in the model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i$$

$X_2$  to  $X_k$  are all uncorrelated. Such variables are called **orthogonal variables**. If this is the case:

- What will be the structure of the  $(\mathbf{X}'\mathbf{X})$  matrix?
- How would you obtain  $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ ?
- What will be the nature of the var-cov matrix of  $\hat{\boldsymbol{\beta}}$ ?
- Suppose you have run the regression and afterward you want to introduce another orthogonal variable, say,  $X_{k+1}$  into the model. Do you have to recompute all the previous coefficients  $\hat{\beta}_1$  to  $\hat{\beta}_k$ ? Why or why not?

10.19. Consider the following model:

$$\text{GNP}_t = \beta_1 + \beta_2 M_t + \beta_3 M_{t-1} + \beta_4 (M_t - M_{t-1}) + u_t$$

where  $\text{GNP}_t = \text{GNP}$  at time  $t$ ,  $M_t = \text{money supply}$  at time  $t$ ,  $M_{t-1} = \text{money supply}$  at time  $(t-1)$ , and  $(M_t - M_{t-1}) = \text{change in the money supply between time } t \text{ and time } (t-1)$ . This model thus postulates that the level of GNP at time  $t$  is a function of the money supply at time  $t$  and time  $(t-1)$  as well as the change in the money supply between these time periods.

- Assuming you have the data to estimate the preceding model, would you succeed in estimating all the coefficients of this model? Why or why not?
- If not, what coefficients can be estimated?

\*Optional.

- c. Suppose that the  $\beta_3 M_{t-1}$  term were absent from the model. Would your answer to (a) be the same?
- d. Repeat (c), assuming that the term  $\beta_2 M_t$  were absent from the model.
- 10.20.** Show that (7.4.7) and (7.4.8) can also be expressed as

$$\hat{\beta}_2 = \frac{(\sum y_i x_{2i})(\sum x_{3i}^2) - (\sum y_i x_{3i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2)(1 - r_{23}^2)}$$

$$\hat{\beta}_3 = \frac{(\sum y_i x_{3i})(\sum x_{2i}^2) - (\sum y_i x_{2i})(\sum x_{2i} x_{3i})}{(\sum x_{2i}^2)(\sum x_{3i}^2)(1 - r_{23}^2)}$$

where  $r_{23}$  is the coefficient of correlation between  $X_2$  and  $X_3$ .

- 10.21.** Using (7.4.12) and (7.4.15), show that when there is perfect collinearity, the variances of  $\hat{\beta}_2$  and  $\hat{\beta}_3$  are infinite.
- 10.22.** Verify that the standard errors of the sums of the slope coefficients estimated from (10.5.6) and (10.5.7) are, respectively, 0.1549 and 0.1825. (See Section 10.5.)
- 10.23.** For the  $k$ -variable regression model, it can be shown that the variance of the  $k$ th ( $k = 2, 3, \dots, K$ ) partial regression coefficient given in (7.5.6) can also be expressed as\*

$$\text{var}(\hat{\beta}_k) = \frac{1}{n - k} \frac{\sigma_y^2}{\sigma_k^2} \left( \frac{1 - R^2}{1 - R_k^2} \right)$$

where  $\sigma_y^2$  = variance of  $Y$ ,  $\sigma_k^2$  = variance of the  $k$ th explanatory variable,  $R_k^2 = R^2$  from the regression of  $X_k$  on the remaining  $X$  variables, and  $R^2$  = coefficient of determination from the multiple regression, that is, regression of  $Y$  on all the  $X$  variables.

- a. Other things the same, if  $\sigma_k^2$  increases, what happens to  $\text{var}(\hat{\beta}_k)$ ? What are the implications for the multicollinearity problem?
- b. What happens to the preceding formula when collinearity is perfect?
- c. True or false: "The variance of  $\hat{\beta}_k$  decreases as  $R^2$  rises, so that the effect of a high  $R_k^2$  can be offset by a high  $R^2$ ."
- 10.24.** From the annual data for the U.S. manufacturing sector for 1899–1922, Dougherty obtained the following regression results<sup>†</sup>:

$$\begin{aligned} \widehat{\log Y} &= 2.81 - 0.53 \log K + 0.91 \log L + 0.047t \\ \text{se} &= (1.38) \quad (0.34) \quad (0.14) \quad (0.021) \quad (1) \\ & \quad \quad \quad \quad \quad \quad R^2 = 0.97 \quad F = 189.8 \end{aligned}$$

where  $Y$  = index of real output,  $K$  = index of real capital input,  $L$  = index of real labor input,  $t$  = time or trend.

\*This formula is given by R. Stone, "The Analysis of Market Demand," *Journal of the Royal Statistical Society*, vol. B7, 1945, p. 297. Also recall (7.5.6). For further discussion, see Peter Kennedy, *A Guide to Econometrics*, 2d ed., The MIT Press, Cambridge, Mass., 1985, p. 156.

<sup>†</sup>Christopher Dougherty, *Introduction to Econometrics*, Oxford University Press, New York, 1992, pp. 159–160.

Using the same data, he also obtained the following regression:

$$\widehat{\log(Y/L)} = -0.11 + 0.11 \log(K/L) + 0.006t$$

$$\text{se} = (0.03) \quad (0.15) \quad (0.006) \quad (2)$$

$$R^2 = 0.65 \quad F = 19.5$$

- a. Is there multicollinearity in regression (1)? How do you know?
  - b. In regression (1), what is the a priori sign of  $\log K$ ? Do the results conform to this expectation? Why or why not?
  - c. How would you justify the functional form of regression (1)? (*Hint*: Cobb–Douglas production function.)
  - d. Interpret regression (1). What is the role of the trend variable in this regression?
  - e. What is the logic behind estimating regression (2)?
  - f. If there was multicollinearity in regression (1), has that been reduced by regression (2)? How do you know?
  - g. If regression (2) is a restricted version of regression (1), what restriction is imposed by the author? (*Hint*: returns to scale.) How do you know if this restriction is valid? Which test do you use? Show all your calculations.
  - h. Are the  $R^2$  values of the two regressions comparable? Why or why not? How would you make them comparable, if they are not comparable in the present form?
- 10.25.** Critically evaluate the following statements:
- a. “In fact, multicollinearity is not a modeling error. It is a condition of deficient data.”\*
  - b. “If it is not feasible to obtain more data, then one must accept the fact that the data one has contain a limited amount of information and must simplify the model accordingly. Trying to estimate models that are too complicated is one of the most common mistakes among inexperienced applied econometricians.”†
  - c. “It is common for researchers to claim that multicollinearity is at work whenever their hypothesized signs are not found in the regression results, when variables that they know *a priori* to be important have insignificant  $t$  values, or when various regression results are changed substantively whenever an explanatory variable is deleted. Unfortunately, none of these conditions is either necessary or sufficient for the existence of collinearity, and furthermore none provides any useful suggestions as to what kind of extra information might be required to solve the estimation problem they present.”‡

\*Samprit Chatterjee, Ali S. Hadi, and Betram Price, *Regression Analysis by Example*, 3d ed., John Wiley & Sons, New York, 2000, p. 226.

†Russel Davidson and James G. MacKinnon, *Estimation and Inference in Econometrics*, Oxford University Press, New York, 1993, p. 186.

‡Peter Kennedy, *A Guide to Econometrics*, 4th ed., MIT Press, Cambridge, Mass., 1998, p. 187.

- d. “. . . any time series regression containing more than four independent variables results in garbage.”\*

**Problems**

- 10.26.** Klein and Goldberger attempted to fit the following regression model to the U.S. economy:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + u_i$$

where  $Y$  = consumption,  $X_2$  = wage income,  $X_3$  = nonwage, nonfarm income, and  $X_4$  = farm income. But since  $X_2$ ,  $X_3$ , and  $X_4$  are expected to be highly collinear, they obtained estimates of  $\beta_3$  and  $\beta_4$  from cross-sectional analysis as follows:  $\beta_3 = 0.75\beta_2$  and  $\beta_4 = 0.625\beta_2$ . Using these estimates, they reformulated their consumption function as follows:

$$Y_i = \beta_1 + \beta_2(X_{2i} + 0.75X_{3i} + 0.625X_{4i}) + u_i = \beta_1 + \beta_2 Z_i + u_i$$

where  $Z_i = X_{2i} + 0.75X_{3i} + 0.625X_{4i}$ .

- a. Fit the modified model to the data in Table 10.11 and obtain estimates of  $\beta_1$  to  $\beta_4$ .  
b. How would you interpret the variable  $Z$ ?
- 10.27.** Table 10.12 gives data on imports, GDP, and the Consumer Price Index (CPI) for the United States over the period 1970–1998. You are asked to consider the following model:

$$\ln \text{Imports}_t = \beta_1 + \beta_2 \ln \text{GDP}_t + \beta_3 \ln \text{CPI}_t + u_t$$

- a. Estimate the parameters of this model using the data given in the table.  
b. Do you suspect that there is multicollinearity in the data?

**TABLE 10.11**

Year	Y	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	Year	Y	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>
1936	62.8	43.41	17.10	3.96	1946	95.7	76.73	28.26	9.76
1937	65.0	46.44	18.65	5.48	1947	98.3	75.91	27.91	9.31
1938	63.9	44.35	17.09	4.37	1948	100.3	77.62	32.30	9.85
1939	67.5	47.82	19.28	4.51	1949	103.2	78.01	31.39	7.21
1940	71.3	51.02	23.24	4.88	1950	108.9	83.57	35.61	7.39
1941	76.6	58.71	28.11	6.37	1951	108.5	90.59	37.58	7.98
1945*	86.3	87.69	30.29	8.96	1952	111.4	95.47	35.17	7.42

\*The data for the war years 1942–1944 are missing. The data for other years are billions of 1939 dollars.  
Source: L. R. Klein and A. S. Goldberger, *An Economic Model of the United States, 1929–1952*, North Holland Publishing Company, Amsterdam, 1964, p. 131.

\*This quote attributed to the late econometrician Zvi Griliches, is obtained from Ernst R. Berndt, *The Practice of Econometrics: Classic and Contemporary*, Addison Wesley, Reading, Mass., 1991, p. 224.

**TABLE 10.12** U.S. IMPORTS, GDP, AND CPI, 1970–1998

Observation	CPI	GDP	Imports	Observation	CPI	GDP	Imports
1970	38.8	1039.7	39,866	1985	107.6	4213.0	338,088
1971	40.5	1128.6	45,579	1986	109.6	4452.9	368,425
1972	41.8	1240.4	55,797	1987	113.6	4742.5	409,765
1973	44.4	1385.5	70,499	1988	118.3	5108.3	447,189
1974	49.3	1501.0	103,811	1989	124.0	5489.1	477,365
1975	53.8	1635.2	98,185	1990	130.7	5803.2	498,337
1976	56.9	1823.9	124,228	1991	136.2	5986.2	490,981
1977	60.6	2031.4	151,907	1992	140.3	6318.9	536,458
1978	65.2	2295.9	176,002	1993	144.5	6642.3	589,441
1979	72.6	2566.4	212,007	1994	148.2	7054.3	668,590
1980	82.4	2795.0	249,750	1995	152.4	7400.5	749,574
1981	90.9	3131.3	265,067	1996	156.9	7813.2	803,327
1982	96.5	3259.2	247,642	1997	160.5	8300.8	876,366
1983	99.6	3534.9	268,901	1998	163.0	8759.9	917,178
1984	103.9	3932.7	332,418				

- c. Regress: (1)  $\ln \text{Imports}_t = A_1 + A_2 \ln \text{GDP}_t$   
(2)  $\ln \text{Imports}_t = B_1 + B_2 \ln \text{CPI}_t$   
(3)  $\ln \text{GDP}_t = C_1 + C_2 \ln \text{CPI}_t$

On the basis of these regressions, what can you say about the nature of multicollinearity in the data?

- d. Suppose there is multicollinearity in the data but  $\hat{\beta}_2$  and  $\hat{\beta}_3$  are individually significant at the 5 percent level and the overall  $F$  test is also significant. In this case should we worry about the collinearity problem?

**10.28.** Refer to Exercise 7.19 about the demand function for chicken in the United States.

- a. Using the log-linear, or double-log, model, estimate the various auxiliary regressions. How many are there?  
b. From these auxiliary regressions, how do you decide which of the regressor(s) are highly collinear? Which test do you use? Show the details of your calculations.  
c. If there is significant collinearity in the data, which variable(s) would you drop to reduce the severity of the collinearity problem? If you do that, what econometric problems do you face?  
d. Do you have any suggestions, other than dropping variables, to ameliorate the collinearity problem? Explain.

**10.29.** Table 10.13 gives data on new passenger cars sold in the United States as a function of several variables.

- a. Develop a suitable linear or log-linear model to estimate a demand function for automobiles in the United States.  
b. If you decide to include all the regressors given in the table as explanatory variables, do you expect to face the multicollinearity problem? Why?  
c. If you do, how would you go about resolving the problem? State your assumptions clearly and show all the calculations explicitly.

TABLE 10.13

Year	Y	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	X <sub>5</sub>	X <sub>6</sub>
1971	10,227	112.0	121.3	776.8	4.89	79,367
1972	10,872	111.0	125.3	839.6	4.55	82,153
1973	11,350	111.1	133.1	949.8	7.38	85,064
1974	8,775	117.5	147.7	1,038.4	8.61	86,794
1975	8,539	127.6	161.2	1,142.8	6.16	85,846
1976	9,994	135.7	170.5	1,252.6	5.22	88,752
1977	11,046	142.9	181.5	1,379.3	5.50	92,017
1978	11,164	153.8	195.3	1,551.2	7.78	96,048
1979	10,559	166.0	217.7	1,729.3	10.25	98,824
1980	8,979	179.3	247.0	1,918.0	11.28	99,303
1981	8,535	190.2	272.3	2,127.6	13.73	100,397
1982	7,980	197.6	286.6	2,261.4	11.20	99,526
1983	9,179	202.6	297.4	2,428.1	8.69	100,834
1984	10,394	208.5	307.6	2,670.6	9.65	105,005
1985	11,039	215.2	318.5	2,841.1	7.75	107,150
1986	11,450	224.4	323.4	3,022.1	6.31	109,597

Y = new passenger cars sold (thousands), seasonally unadjusted  
X<sub>2</sub> = new cars, Consumer Price Index, 1967 = 100, seasonally unadjusted  
X<sub>3</sub> = Consumer Price Index, all items, all urban consumers, 1967 = 100, seasonally unadjusted  
X<sub>4</sub> = the personal disposable income (PDI), billions of dollars, unadjusted for seasonal variation  
X<sub>5</sub> = the interest rate, percent, finance company paper placed directly  
X<sub>6</sub> = the employed civilian labor force (thousands), unadjusted for seasonal variation

Source: *Business Statistics, 1986*, A Supplement to the *Current Survey of Business*, U.S. Department of Commerce.

**10.30.** To assess the feasibility of a guaranteed annual wage (negative income tax), the Rand Corporation conducted a study to assess the response of labor supply (average hours of work) to increasing hourly wages.\* The data for this study were drawn from a national sample of 6000 households with a male head earnings less than \$15,000 annually. The data were divided into 39 demographic groups for analysis. These data are given in Table 10.14. Because data for four demographic groups were missing for some variables, the data given in the table refer to only 35 demographic groups. The definitions of the various variables used in the analysis are given at the end of the table.

- Regress average hours worked during the year on the variables given in the table and interpret your regression.
- Is there evidence of multicollinearity in the data? How do you know?
- Compute the variance inflation factors (VIF) and TOL measures for the various regressors.
- If there is the multicollinearity problem, what remedial action, if any, would you take?
- What does this study tell about the feasibility of a negative income tax?

**10.31.** Table 10.15 gives data on the crime rate in 47 states in the United States for 1960. Try to develop a suitable model to explain the crime rate in relation to the 14 socioeconomic variables given in the table. Pay particular attention to the collinearity problem in developing your model.

\*D. H. Greenberg and M. Kosters, *Income Guarantees and the Working Poor*, Rand Corporation, R-579-OEO, December 1970.



**TABLE 10.14** HOURS OF WORK AND OTHER DATA FOR 35 GROUPS

Observation	Hours	Rate	ERSP	ERNO	NEIN	Assets	Age	DEP	School
1	2157	2.905	1121	291	380	7250	38.5	2.340	10.5
2	2174	2.970	1128	301	398	7744	39.3	2.335	10.5
3	2062	2.350	1214	326	185	3068	40.1	2.851	8.9
4	2111	2.511	1203	49	117	1632	22.4	1.159	11.5
5	2134	2.791	1013	594	730	12710	57.7	1.229	8.8
6	2185	3.040	1135	287	382	7706	38.6	2.602	10.7
7	2210	3.222	1100	295	474	9338	39.0	2.187	11.2
8	2105	2.493	1180	310	255	4730	39.9	2.616	9.3
9	2267	2.838	1298	252	431	8317	38.9	2.024	11.1
10	2205	2.356	885	264	373	6789	38.8	2.662	9.5
11	2121	2.922	1251	328	312	5907	39.8	2.287	10.3
12	2109	2.499	1207	347	271	5069	39.7	3.193	8.9
13	2108	2.796	1036	300	259	4614	38.2	2.040	9.2
14	2047	2.453	1213	297	139	1987	40.3	2.545	9.1
15	2174	3.582	1141	414	498	10239	40.0	2.064	11.7
16	2067	2.909	1805	290	239	4439	39.1	2.301	10.5
17	2159	2.511	1075	289	308	5621	39.3	2.486	9.5
18	2257	2.516	1093	176	392	7293	37.9	2.042	10.1
19	1985	1.423	553	381	146	1866	40.6	3.833	6.6
20	2184	3.636	1091	291	560	11240	39.1	2.328	11.6
21	2084	2.983	1327	331	296	5653	39.8	2.208	10.2
22	2051	2.573	1194	279	172	2806	40.0	2.362	9.1
23	2127	3.262	1226	314	408	8042	39.5	2.259	10.8
24	2102	3.234	1188	414	352	7557	39.8	2.019	10.7
25	2098	2.280	973	364	272	4400	40.6	2.661	8.4
26	2042	2.304	1085	328	140	1739	41.8	2.444	8.2
27	2181	2.912	1072	304	383	7340	39.0	2.337	10.2
28	2186	3.015	1122	30	352	7292	37.2	2.046	10.9
29	2188	3.010	990	366	374	7325	38.4	2.847	10.6
30	2077	1.901	350	209	95	1370	37.4	4.158	8.2
31	2196	3.009	947	294	342	6888	37.5	3.047	10.6
32	2093	1.899	342	311	120	1425	37.5	4.512	8.1
33	2173	2.959	1116	296	387	7625	39.2	2.342	10.5
34	2179	2.971	1128	312	397	7779	39.4	2.341	10.5
35	2200	2.980	1126	204	393	7885	39.2	2.341	10.6

Notes: Hours = average hours worked during the year

Rate = average hourly wage (dollars)

ERSP = average yearly earnings of spouse (dollars)

ERNO = average yearly earnings of other family members (dollars)

NEIN = average yearly nonearned income

Assets = average family asset holdings (bank account, etc.) (dollars)

Age = average age of respondent

Dep = average number of dependents

School = average highest grade of school completed

Source: D. H. Greenberg and M. Kusters, *Income Guarantees and the Working Poor*, The Rand Corporation, R-579-OEO, December 1970.

**10.32.** Refer to the Longley data given in Section 10.10. Repeat the regression given in the table there by omitting the data for 1962; that is, run the regression for the period 1947–1961. Compare the two regressions. What general conclusion can you draw from this exercise?

**TABLE 10.15** U.S. CRIME DATA FOR 47 STATES IN 1960

Observation	<i>R</i>	Age	<i>S</i>	ED	EX <sub>0</sub>	EX <sub>1</sub>	LF	<i>M</i>	<i>N</i>	NW	<i>U</i> <sub>1</sub>	<i>U</i> <sub>2</sub>	<i>W</i>	<i>X</i>
1	79.1	151	1	91	58	56	510	950	33	301	108	41	394	261
2	163.5	143	0	113	103	95	583	1012	13	102	96	36	557	194
3	57.8	142	1	89	45	44	533	969	18	219	94	33	318	250
4	196.9	136	0	121	149	141	577	994	157	80	102	39	673	167
5	123.4	141	0	121	109	101	591	985	18	30	91	20	578	174
6	68.2	121	0	110	118	115	547	964	25	44	84	29	689	126
7	96.3	127	1	111	82	79	519	982	4	139	97	38	620	168
8	155.5	131	1	109	115	109	542	969	50	179	79	35	472	206
9	85.6	157	1	90	65	62	553	955	39	286	81	28	421	239
10	70.5	140	0	118	71	68	632	1029	7	15	100	24	526	174
11	167.4	124	0	105	121	116	580	966	101	106	77	35	657	170
12	84.9	134	0	108	75	71	595	972	47	59	83	31	580	172
13	51.1	128	0	113	67	60	624	972	28	10	77	25	507	206
14	66.4	135	0	117	62	61	595	986	22	46	77	27	529	190
15	79.8	152	1	87	57	53	530	986	30	72	92	43	405	264
16	94.6	142	1	88	81	77	497	956	33	321	116	47	427	247
17	53.9	143	0	110	66	63	537	977	10	6	114	35	487	166
18	92.9	135	1	104	123	115	537	978	31	170	89	34	631	165
19	75.0	130	0	116	128	128	536	934	51	24	78	34	627	135
20	122.5	125	0	108	113	105	567	985	78	94	130	58	626	166
21	74.2	126	0	108	74	67	602	984	34	12	102	33	557	195
22	43.9	157	1	89	47	44	512	962	22	423	97	34	288	276
23	121.6	132	0	96	87	83	564	953	43	92	83	32	513	227
24	96.8	131	0	116	78	73	574	1038	7	36	142	42	540	176
25	52.3	130	0	116	63	57	641	984	14	26	70	21	486	196
26	199.3	131	0	121	160	143	631	1071	3	77	102	41	674	152
27	34.2	135	0	109	69	71	540	965	6	4	80	22	564	139
28	121.6	152	0	112	82	76	571	1018	10	79	103	28	537	215
29	104.3	119	0	107	166	157	521	938	168	89	92	36	637	154
30	69.6	166	1	89	58	54	521	973	46	254	72	26	396	237
31	37.3	140	0	93	55	54	535	1045	6	20	135	40	453	200
32	75.4	125	0	109	90	81	586	964	97	82	105	43	617	163
33	107.2	147	1	104	63	64	560	972	23	95	76	24	462	233
34	92.3	126	0	118	97	97	542	990	18	21	102	35	589	166
35	65.3	123	0	102	97	87	526	948	113	76	124	50	572	158
36	127.2	150	0	100	109	98	531	964	9	24	87	38	559	153
37	83.1	177	1	87	58	56	638	974	24	349	76	28	382	254
38	56.6	133	0	104	51	47	599	1024	7	40	99	27	425	225
39	82.6	149	1	88	61	54	515	953	36	165	86	35	395	251
40	115.1	145	1	104	82	74	560	981	96	126	88	31	488	228
41	88.0	148	0	122	72	66	601	998	9	19	84	20	590	144
42	54.2	141	0	109	56	54	523	968	4	2	107	37	489	170
43	82.3	162	1	99	75	70	522	996	40	208	73	27	496	224
44	103.0	136	0	121	95	96	574	1012	29	36	111	37	622	162
45	45.5	139	1	88	46	41	480	968	19	49	135	53	457	249
46	50.8	126	0	104	106	97	599	989	40	24	78	25	593	171
47	84.9	130	0	121	90	91	623	1049	3	22	113	40	588	160

Definitions of variables:

- R* = crime rate, number of offenses reported to police per million population
- Age = number of males of age 14–24 per 1000 population
- S* = indicator variable for southern states (0 = no, 1 = yes)
- ED = mean number of years of schooling times 10 for persons age 25 or older.
- EX<sub>0</sub> = 1960 per capita expenditure on police by state and local government
- EX<sub>1</sub> = 1959 per capita expenditure on police by state and local government
- LF = labor force participation rate per 1000 civilian urban males age 14–24
- M* = number of males per 1000 females
- N* = state population size in hundred thousands
- NW = number of nonwhites per 1000 population
- U*<sub>1</sub> = unemployment rate of urban males per 1000 of age 14–24
- U*<sub>2</sub> = unemployment rate of urban males per 1000 of age 35–39
- W* = median value of transferable goods and assets or family income in tens of dollars
- X* = the number of families per 1000 earnings 1/2 the median income

Observation = state (47 states for the year 1960)

Source: W. Vandaele, "Participation in Illegitimate Activities: Erlich Revisited," in A. Blumstein, J. Cohen, and Nagin, D., eds., *Deterrence and Incapacitation*, National Academy of Sciences, 1978, pp. 270–335.

# 11

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## HETEROSCEDASTICITY: WHAT HAPPENS IF THE ERROR VARIANCE IS NONCONSTANT?

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An important assumption of the classical linear regression model (Assumption 4) is that the disturbances  $u_i$  appearing in the population regression function are homoscedastic; that is, they all have the same variance. In this chapter we examine the validity of this assumption and find out what happens if this assumption is not fulfilled. As in Chapter 10, we seek answers to the following questions:

1. What is the nature of heteroscedasticity?
2. What are its consequences?
3. How does one detect it?
4. What are the remedial measures?

### 11.1 THE NATURE OF HETEROSCEDASTICITY

As noted in Chapter 3, one of the important assumptions of the classical linear regression model is that the variance of each disturbance term  $u_i$ , conditional on the chosen values of the explanatory variables, is some constant number equal to  $\sigma^2$ . This is the assumption of **homoscedasticity**, or *equal (homo) spread (scedasticity)*, that is, *equal variance*. Symbolically,

$$E(u_i^2) = \sigma^2 \quad i = 1, 2, \dots, n \quad (11.1.1)$$

Diagrammatically, in the two-variable regression model homoscedasticity can be shown as in Figure 3.4, which, for convenience, is reproduced as

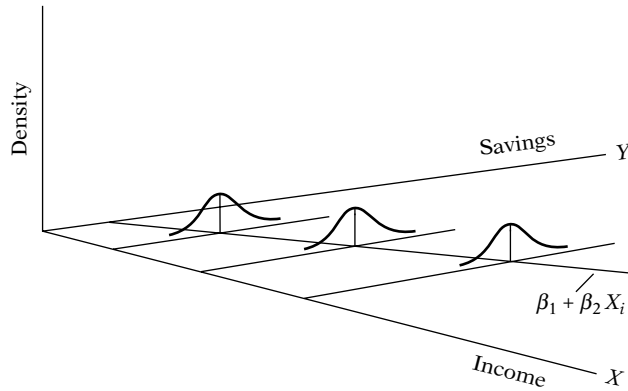


FIGURE 11.1 Homoscedastic disturbances.

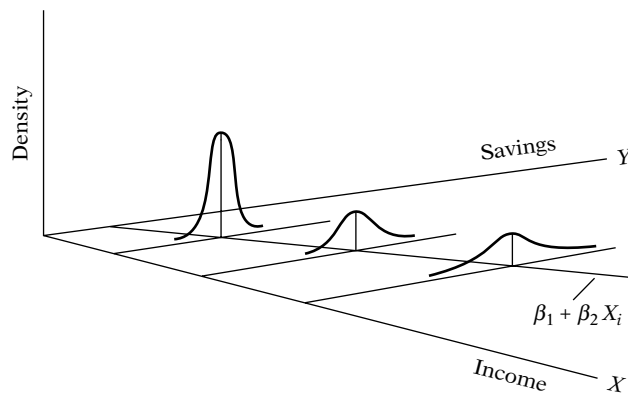


FIGURE 11.2 Heteroscedastic disturbances.

Figure 11.1. As Figure 11.1 shows, the conditional variance of  $Y_i$  (which is equal to that of  $u_i$ ), conditional upon the given  $X_i$ , remains the same regardless of the values taken by the variable  $X$ .

In contrast, consider Figure 11.2, which shows that the conditional variance of  $Y_i$  increases as  $X$  increases. Here, the variances of  $Y_i$  are not the same. Hence, there is heteroscedasticity. Symbolically,

$$E(u_i^2) = \sigma_i^2 \quad (11.1.2)$$

Notice the subscript of  $\sigma^2$ , which reminds us that the conditional variances of  $u_i$  (= conditional variances of  $Y_i$ ) are no longer constant.

To make the difference between homoscedasticity and heteroscedasticity clear, assume that in the two-variable model  $Y_i = \beta_1 + \beta_2 X_i + u_i$ ,  $Y$  represents savings and  $X$  represents income. Figures 11.1 and 11.2 show that as income increases, savings on the average also increase. But in Figure 11.1

the variance of savings remains the same at all levels of income, whereas in Figure 11.2 it increases with income. It seems that in Figure 11.2 the higher-income families on the average save more than the lower-income families, but there is also more variability in their savings.

There are several reasons why the variances of  $u_i$  may be variable, some of which are as follows.<sup>1</sup>

1. Following the *error-learning models*, as people learn, their errors of behavior become smaller over time. In this case,  $\sigma_i^2$  is expected to decrease. As an example, consider Figure 11.3, which relates the number of typing errors made in a given time period on a test to the hours put in typing practice. As Figure 11.3 shows, as the number of hours of typing practice increases, the average number of typing errors as well as their variances decreases.

2. As incomes grow, people have more *discretionary income*<sup>2</sup> and hence more scope for choice about the disposition of their income. Hence,  $\sigma_i^2$  is likely to increase with income. Thus in the regression of savings on income one is likely to find  $\sigma_i^2$  increasing with income (as in Figure 11.2) because people have more choices about their savings behavior. Similarly, companies with larger profits are generally expected to show greater variability in their dividend policies than companies with lower profits. Also, *growth-oriented* companies are likely to show more variability in their dividend payout ratio than established companies.

3. As data collecting techniques improve,  $\sigma_i^2$  is likely to decrease. Thus, banks that have sophisticated data processing equipment are likely to

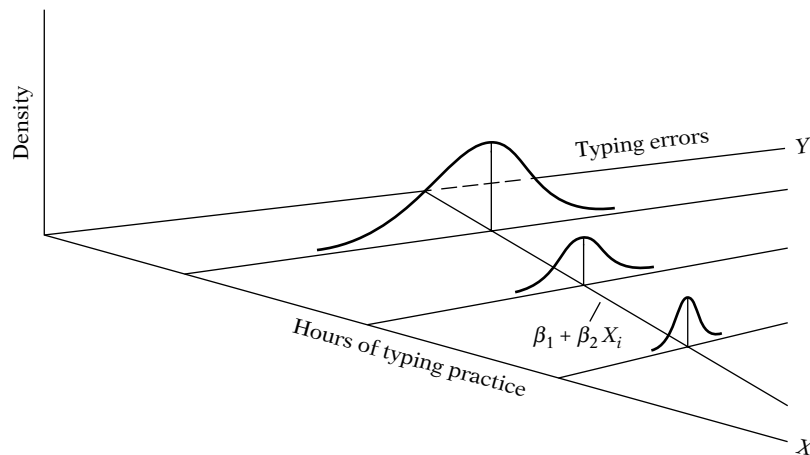


FIGURE 11.3 Illustration of heteroscedasticity.

<sup>1</sup>See Stefan Valavanis, *Econometrics*, McGraw-Hill, New York, 1959, p. 48.

<sup>2</sup>As Valavanis puts it, "Income grows, and people now barely discern dollars whereas previously they discerned dimes," *ibid.*, p. 48.

commit fewer errors in the monthly or quarterly statements of their customers than banks without such facilities.

4. Heteroscedasticity can also arise as a result of the presence of **outliers**. An outlying observation, or outlier, is an observation that is much different (either very small or very large) in relation to the observations in the sample. More precisely, an outlier is an observation from a different population to that generating the remaining sample observations.<sup>3</sup> The inclusion or exclusion of such an observation, especially if the sample size is small, can substantially alter the results of regression analysis.

As an example, consider the scattergram given in Figure 11.4. Based on the data given in exercise 11.22, this figure plots percent rate of change of stock prices ( $Y$ ) and consumer prices ( $X$ ) for the post-World War II period through 1969 for 20 countries. In this figure the observation on  $Y$  and  $X$  for Chile can be regarded as an outlier because the given  $Y$  and  $X$  values are much larger than for the rest of the countries. In situations such as this, it would be hard to maintain the assumption of homoscedasticity. In exercise 11.22, you are asked to find out what happens to the regression results if the observations for Chile are dropped from the analysis.

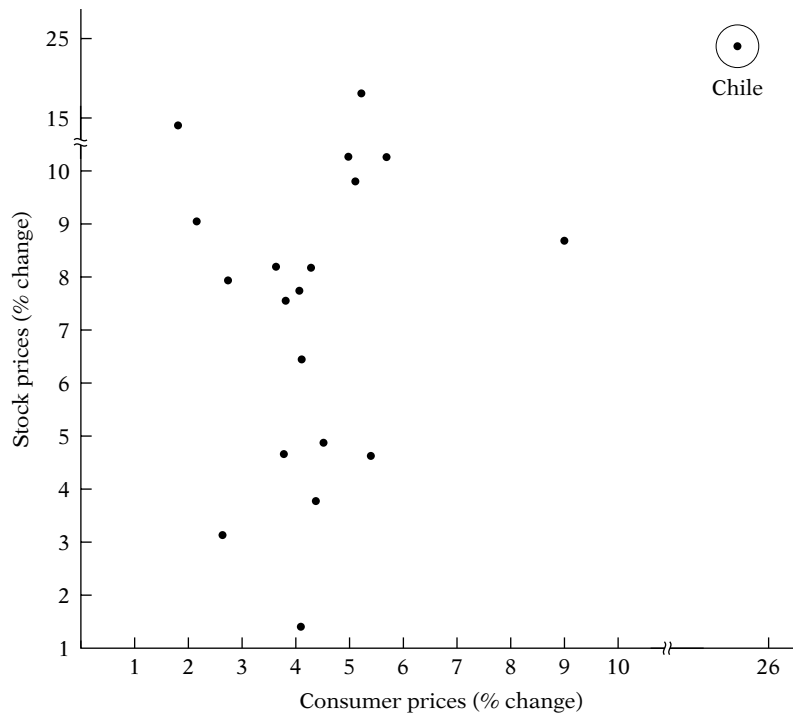


FIGURE 11.4 The relationship between stock prices and consumer prices.

<sup>3</sup>I am indebted to Michael McAleer for pointing this out to me.

5. Another source of heteroscedasticity arises from violating Assumption 9 of CLRM, namely, that the regression model is correctly specified. Although we will discuss the topic of specification errors more fully in Chapter 13, very often what looks like heteroscedasticity may be due to the fact that some important variables are omitted from the model. Thus, in the demand function for a commodity, if we do not include the prices of commodities complementary to or competing with the commodity in question (the omitted variable bias), the residuals obtained from the regression may give the distinct impression that the error variance may not be constant. But if the omitted variables are included in the model, that impression may disappear.

As a concrete example, recall our study of advertising impressions retained ( $Y$ ) in relation to advertising expenditure ( $X$ ). (See exercise 8.32.) If you regress  $Y$  on  $X$  only and observe the residuals from this regression, you will see one pattern, but if you regress  $Y$  on  $X$  and  $X^2$ , you will see another pattern, which can be seen clearly from Figure 11.5. We have already seen that  $X^2$  belongs in the model. (See exercise 8.32.)

6. Another source of heteroscedasticity is **skewness** in the distribution of one or more regressors included in the model. Examples are economic variables such as income, wealth, and education. It is well known that the distribution of income and wealth in most societies is uneven, with the bulk of the income and wealth being owned by a few at the top.

7. Other sources of heteroscedasticity: As David Hendry notes, heteroscedasticity can also arise because of (1) incorrect data transformation (e.g., ratio or first difference transformations) and (2) incorrect functional form (e.g., linear versus log-linear models).<sup>4</sup>

Note that the problem of heteroscedasticity is likely to be more common in cross-sectional than in time series data. In cross-sectional data, one

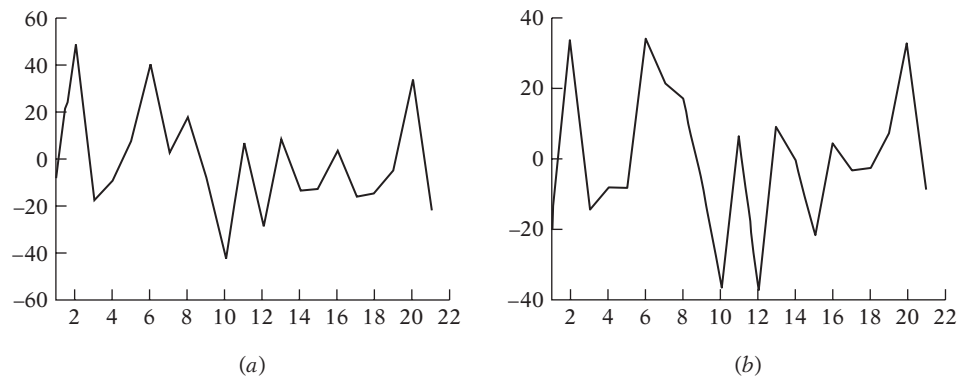


FIGURE 11.5 Residuals from the regression of (a) impressions of advertising expenditure and (b) impression on Adexp and Adexp<sup>2</sup>.

<sup>4</sup>David F. Hendry, *Dynamic Econometrics*, Oxford University Press, 1995, p. 45.

usually deals with members of a population at a given point in time, such as individual consumers or their families, firms, industries, or geographical subdivisions such as state, country, city, etc. Moreover, these members may be of different sizes, such as small, medium, or large firms or low, medium, or high income. In time series data, on the other hand, the variables tend to be of similar orders of magnitude because one generally collects the data for the same entity over a period of time. Examples are GNP, consumption expenditure, savings, or employment in the United States, say, for the period 1950 to 2000.

As an illustration of heteroscedasticity likely to be encountered in cross-sectional analysis, consider Table 11.1. This table gives data on compensation per employee in 10 nondurable goods manufacturing industries, classified by the employment size of the firm or the establishment for the year 1958. Also given in the table are average productivity figures for nine employment classes.

Although the industries differ in their output composition, Table 11.1 shows clearly that on the average large firms pay more than the small firms.

**TABLE 11.1**  
COMPENSATION PER EMPLOYEE (\$) IN NONDURABLE MANUFACTURING INDUSTRIES ACCORDING TO  
EMPLOYMENT SIZE OF ESTABLISHMENT, 1958

Industry	Employment size (average number of employees)								
	1-4	5-9	10-19	20-49	50-99	100-249	250-499	500-999	1000-2499
Food and kindred products	2994	3295	3565	3907	4189	4486	4676	4968	5342
Tobacco products	1721	2057	3336	3320	2980	2848	3072	2969	3822
Textile mill products	3600	3657	3674	3437	3340	3334	3225	3163	3168
Apparel and related products	3494	3787	3533	3215	3030	2834	2750	2967	3453
Paper and allied products	3498	3847	3913	4135	4445	4885	5132	5342	5326
Printing and publishing	3611	4206	4695	5083	5301	5269	5182	5395	5552
Chemicals and allied products	3875	4660	4930	5005	5114	5248	5630	5870	5876
Petroleum and coal products	4616	5181	5317	5337	5421	5710	6316	6455	6347
Rubber and plastic products	3538	3984	4014	4287	4221	4539	4721	4905	5481
Leather and leather products	3016	3196	3149	3317	3414	3254	3177	3346	4067
Average compensation	3396	3787	4013	4104	4146	4241	4388	4538	4843
Standard deviation	742.2	851.4	727.8	805.06	929.9	1080.6	1241.2	1307.7	1110.5
Average productivity	9355	8584	7962	8275	8389	9418	9795	10,281	11,750

Source: *The Census of Manufacturers*, U.S. Department of Commerce, 1958 (computed by author).



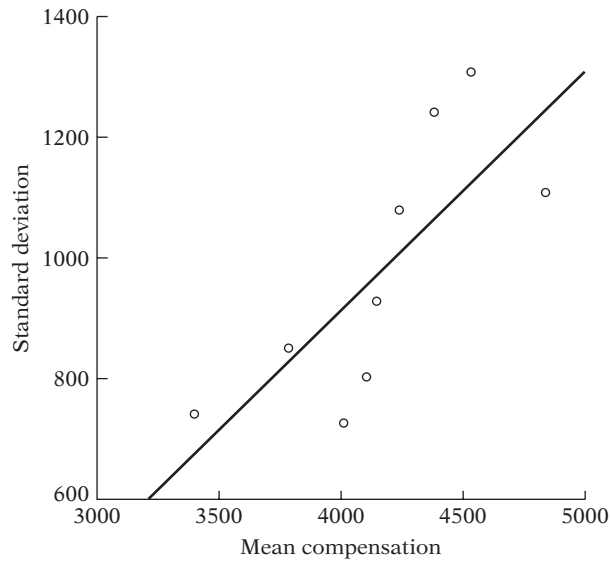


FIGURE 11.6 Standard deviation of compensation and mean compensation.

As an example, firms employing one to four employees paid on the average about \$3396, whereas those employing 1000 to 2499 employees on the average paid about \$4843. But notice that there is considerable variability in earning among various employment classes as indicated by the estimated standard deviations of earnings. This can be seen also from Figure 11.6, which plots the standard deviation of compensation and average compensation in each employment class. As can be seen clearly, on average, the standard deviation of compensation increases with the average value of compensation.

## 11.2 OLS ESTIMATION IN THE PRESENCE OF HETEROSCEDASTICITY

What happens to OLS estimators and their variances if we introduce heteroscedasticity by letting  $E(u_i^2) = \sigma_i^2$  but retain all other assumptions of the classical model? To answer this question, let us revert to the two-variable model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

Applying the usual formula, the OLS estimator of  $\beta_2$  is

$$\begin{aligned} \hat{\beta}_2 &= \frac{\sum x_i y_i}{\sum x_i^2} \\ &= \frac{n \sum X_i Y_i - \sum X_i \sum Y_i}{n \sum X_i^2 - (\sum X_i)^2} \end{aligned} \quad (11.2.1)$$

but its variance is now given by the following expression (see Appendix 11A, Section 11A.1):

$$\text{var}(\hat{\beta}_2) = \frac{\sum x_i^2 \sigma_i^2}{(\sum x_i^2)^2} \quad (11.2.2)$$

which is obviously different from the usual variance formula obtained under the assumption of homoscedasticity, namely,

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_i^2} \quad (11.2.3)$$

Of course, if  $\sigma_i^2 = \sigma^2$  for each  $i$ , the two formulas will be identical. (Why?)

Recall that  $\hat{\beta}_2$  is best linear unbiased estimator (BLUE) if the assumptions of the classical model, including homoscedasticity, hold. Is it still BLUE when we drop only the homoscedasticity assumption and replace it with the assumption of heteroscedasticity? It is easy to prove that  $\hat{\beta}_2$  is still linear and unbiased. As a matter of fact, as shown in Appendix 3A, Section 3A.2, to establish the unbiasedness of  $\hat{\beta}_2$  it is not necessary that the disturbances ( $u_i$ ) be homoscedastic. In fact, the variance of  $u_i$ , homoscedastic or heteroscedastic, plays no part in the determination of the unbiasedness property. Recall that in Appendix 3A, Section 3A.7, we showed that  $\hat{\beta}_2$  is a consistent estimator under the assumptions of the classical linear regression model. Although we will not prove it, it can be shown that  $\hat{\beta}_2$  is a consistent estimator despite heteroscedasticity; that is, as the sample size increases indefinitely, the estimated  $\beta_2$  converges to its true value. Furthermore, it can also be shown that under certain conditions (called regularity conditions),  $\hat{\beta}_2$  is *asymptotically normally distributed*. Of course, what we have said about  $\hat{\beta}_2$  also holds true of other parameters of a multiple regression model.

Granted that  $\hat{\beta}_2$  is still linear unbiased and consistent, is it “efficient” or “best”; that is, does it have minimum variance in the class of unbiased estimators? And is that minimum variance given by Eq. (11.2.2)? The answer is *no* to both the questions:  $\hat{\beta}_2$  is no longer best and the minimum variance is not given by (11.2.2). Then what is BLUE in the presence of heteroscedasticity? The answer is given in the following section.

### 11.3 THE METHOD OF GENERALIZED LEAST SQUARES (GLS)

Why is the usual OLS estimator of  $\beta_2$  given in (11.2.1) not best, although it is still unbiased? Intuitively, we can see the reason from Table 11.1. As the table shows, there is considerable variability in the earnings between employment classes. If we were to regress per-employee compensation on the size of employment, we would like to make use of the knowledge that there is considerable interclass variability in earnings. Ideally, we would like to devise

the estimating scheme in such a manner that observations coming from populations with greater variability are given less weight than those coming from populations with smaller variability. Examining Table 11.1, we would like to weight observations coming from employment classes 10–19 and 20–49 more heavily than those coming from employment classes like 5–9 and 250–499, for the former are more closely clustered around their mean values than the latter; thereby enabling us to estimate the PRF more accurately.

Unfortunately, the usual OLS method does not follow this strategy and therefore does not make use of the “information” contained in the unequal variability of the dependent variable  $Y$ , say, employee compensation of Table 11.1: It assigns equal weight or importance to each observation. But a method of estimation, known as **generalized least squares (GLS)**, takes such information into account explicitly and is therefore capable of producing estimators that are BLUE. To see how this is accomplished, let us continue with the now-familiar two-variable model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (11.3.1)$$

which for ease of algebraic manipulation we write as

$$Y_i = \beta_1 X_{0i} + \beta_2 X_i + u_i \quad (11.3.2)$$

where  $X_{0i} = 1$  for each  $i$ . The reader can see that these two formulations are identical.

Now assume that the heteroscedastic variances  $\sigma_i^2$  are *known*. Divide (11.3.2) through by  $\sigma_i$  to obtain

$$\frac{Y_i}{\sigma_i} = \beta_1 \left( \frac{X_{0i}}{\sigma_i} \right) + \beta_2 \left( \frac{X_i}{\sigma_i} \right) + \left( \frac{u_i}{\sigma_i} \right) \quad (11.3.3)$$

which for ease of exposition we write as

$$Y_i^* = \beta_1^* X_{0i}^* + \beta_2^* X_i^* + u_i^* \quad (11.3.4)$$

where the starred, or transformed, variables are the original variables divided by (the known)  $\sigma_i$ . We use the notation  $\beta_1^*$  and  $\beta_2^*$ , the parameters of the transformed model, to distinguish them from the usual OLS parameters  $\beta_1$  and  $\beta_2$ .

What is the purpose of transforming the original model? To see this, notice the following feature of the transformed error term  $u_i^*$ :

$$\begin{aligned} \text{var}(u_i^*) &= E(u_i^*)^2 = E\left(\frac{u_i}{\sigma_i}\right)^2 \\ &= \frac{1}{\sigma_i^2} E(u_i^2) \quad \text{since } \sigma_i^2 \text{ is known} \\ &= \frac{1}{\sigma_i^2} (\sigma_i^2) \quad \text{since } E(u_i^2) = \sigma_i^2 \\ &= 1 \end{aligned} \quad (11.3.5)$$

which is a constant. That is, the variance of the transformed disturbance term  $u_i^*$  is now homoscedastic. Since we are still retaining the other assumptions of the classical model, the finding that it is  $u^*$  that is homoscedastic suggests that if we apply OLS to the transformed model (11.3.3) it will produce estimators that are BLUE. In short, the estimated  $\beta_1^*$  and  $\beta_2^*$  are now BLUE and not the OLS estimators  $\hat{\beta}_1$  and  $\hat{\beta}_2$ .

This procedure of transforming the original variables in such a way that the transformed variables satisfy the assumptions of the classical model and then applying OLS to them is known as the method of generalized least squares (GLS). *In short, GLS is OLS on the transformed variables that satisfy the standard least-squares assumptions.* The estimators thus obtained are known as **GLS estimators**, and it is these estimators that are BLUE.

The actual mechanics of estimating  $\beta_1^*$  and  $\beta_2^*$  are as follows. First, we write down the SRF of (11.3.3)

$$\frac{Y_i}{\sigma_i} = \hat{\beta}_1^* \left( \frac{X_{0i}}{\sigma_i} \right) + \hat{\beta}_2^* \left( \frac{X_i}{\sigma_i} \right) + \left( \frac{\hat{u}_i}{\sigma_i} \right)$$

or

$$Y_i^* = \hat{\beta}_1^* X_{0i}^* + \hat{\beta}_2^* X_i^* + \hat{u}_i^* \quad (11.3.6)$$

Now, to obtain the GLS estimators, we minimize

$$\sum \hat{u}_i^{2*} = \sum (Y_i^* - \hat{\beta}_1^* X_{0i}^* - \hat{\beta}_2^* X_i^*)^2$$

that is,

$$\sum \left( \frac{\hat{u}_i}{\sigma_i} \right)^2 = \sum \left[ \left( \frac{Y_i}{\sigma_i} \right) - \hat{\beta}_1^* \left( \frac{X_{0i}}{\sigma_i} \right) - \hat{\beta}_2^* \left( \frac{X_i}{\sigma_i} \right) \right]^2 \quad (11.3.7)$$

The actual mechanics of minimizing (11.3.7) follow the standard calculus techniques and are given in Appendix 11A, Section 11A.2. As shown there, the GLS estimator of  $\beta_2^*$  is

$$\hat{\beta}_2^* = \frac{(\sum w_i)(\sum w_i X_i Y_i) - (\sum w_i X_i)(\sum w_i Y_i)}{(\sum w_i)(\sum w_i X_i^2) - (\sum w_i X_i)^2} \quad (11.3.8)$$

and its variance is given by

$$\text{var}(\hat{\beta}_2^*) = \frac{\sum w_i}{(\sum w_i)(\sum w_i X_i^2) - (\sum w_i X_i)^2} \quad (11.3.9)$$

where  $w_i = 1/\sigma_i^2$ .

### Difference between OLS and GLS

Recall from Chapter 3 that in OLS we minimize

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2 \quad (11.3.10)$$

but in GLS we minimize the expression (11.3.7), which can also be written as

$$\sum w_i \hat{u}_i^2 = \sum w_i (Y_i - \hat{\beta}_1^* X_{0i} - \hat{\beta}_2^* X_i)^2 \quad (11.3.11)$$

where  $w_i = 1/\sigma_i^2$  [verify that (11.3.11) and (11.3.7) are identical].

Thus, in GLS we minimize a *weighted sum of residual squares* with  $w_i = 1/\sigma_i^2$  acting as the weights, but in OLS we minimize an unweighted or (what amounts to the same thing) equally weighted RSS. As (11.3.7) shows, in GLS the weight assigned to each observation is inversely proportional to its  $\sigma_i$ , that is, observations coming from a population with larger  $\sigma_i$  will get relatively smaller weight and those from a population with smaller  $\sigma_i$  will get proportionately larger weight in minimizing the RSS (11.3.11). To see the difference between OLS and GLS clearly, consider the hypothetical scattergram given in Figure 11.7.

In the (unweighted) OLS, each  $\hat{u}_i^2$  associated with points A, B, and C will receive the same weight in minimizing the RSS. Obviously, in this case the  $\hat{u}_i^2$  associated with point C will dominate the RSS. But in GLS the extreme observation C will get relatively smaller weight than the other two observations. As noted earlier, this is the right strategy, for in estimating the

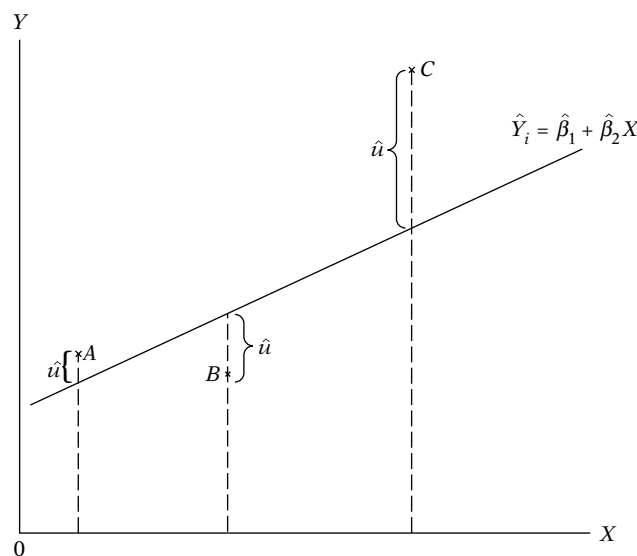


FIGURE 11.7 Hypothetical scattergram.

population regression function (PRF) more reliably we would like to give more weight to observations that are closely clustered around their (population) mean than to those that are widely scattered about.

Since (11.3.11) minimizes a weighted RSS, it is appropriately known as **weighted least squares (WLS)**, and the estimators thus obtained and given in (11.3.8) and (11.3.9) are known as **WLS estimators**. But WLS is just a special case of the more general estimating technique, GLS. In the context of heteroscedasticity, one can treat the two terms WLS and GLS interchangeably. In later chapters we will come across other special cases of GLS.

In passing, note that if  $w_i = w$ , a constant for all  $i$ ,  $\hat{\beta}_2^*$  is identical with  $\hat{\beta}_2$  and  $\text{var}(\hat{\beta}_2^*)$  is identical with the usual (i.e., homoscedastic)  $\text{var}(\hat{\beta}_2)$  given in (11.2.3), which should not be surprising. (Why?) (See exercise 11.8.)

#### 11.4 CONSEQUENCES OF USING OLS IN THE PRESENCE OF HETEROSCEDASTICITY

As we have seen, both  $\hat{\beta}_2^*$  and  $\hat{\beta}_2$  are (linear) unbiased estimators: In repeated sampling, on the average,  $\hat{\beta}_2^*$  and  $\hat{\beta}_2$  will equal the true  $\beta_2$ ; that is, they are both unbiased estimators. But we know that it is  $\hat{\beta}_2^*$  that is efficient, that is, has the smallest variance. What happens to our confidence interval, hypotheses testing, and other procedures if we continue to use the OLS estimator  $\hat{\beta}_2$ ? We distinguish two cases.

##### OLS Estimation Allowing for Heteroscedasticity

Suppose we use  $\hat{\beta}_2$  and use the variance formula given in (11.2.2), which takes into account heteroscedasticity explicitly. Using this variance, and assuming  $\sigma_i^2$  are known, can we establish confidence intervals and test hypotheses with the usual  $t$  and  $F$  tests? The answer generally is no because it can be shown that  $\text{var}(\hat{\beta}_2^*) \leq \text{var}(\hat{\beta}_2)$ ,<sup>5</sup> which means that confidence intervals based on the latter will be unnecessarily larger. As a result, the  $t$  and  $F$  tests are likely to give us inaccurate results in that  $\text{var}(\hat{\beta}_2)$  is overly large and what appears to be a statistically insignificant coefficient (because the  $t$  value is smaller than what is appropriate) may in fact be significant if the correct confidence intervals were established on the basis of the GLS procedure.

##### OLS Estimation Disregarding Heteroscedasticity

The situation can become serious if we not only use  $\hat{\beta}_2$  but also continue to use the usual (homoscedastic) variance formula given in (11.2.3) even if heteroscedasticity is present or suspected: Note that this is the more likely

<sup>5</sup>A formal proof can be found in Phoebus J. Dhrymes, *Introductory Econometrics*, Springer-Verlag, New York, 1978, pp. 110–111. In passing, note that the loss of efficiency of  $\hat{\beta}_2$  [i.e., by how much  $\text{var}(\hat{\beta}_2)$  exceeds  $\text{var}(\hat{\beta}_2^*)$ ] depends on the sample values of the  $X$  variables and the value of  $\sigma_i^2$ .

case of the two we discuss here, because running a standard OLS regression package and ignoring (or being ignorant of) heteroscedasticity will yield variance of  $\hat{\beta}_2$  as given in (11.2.3). First of all,  $\text{var}(\hat{\beta}_2)$  given in (11.2.3) is a *biased* estimator of  $\text{var}(\hat{\beta}_2)$  given in (11.2.2), that is, on the average it overestimates or underestimates the latter, and *in general* we cannot tell whether the bias is positive (overestimation) or negative (underestimation) because it depends on the nature of the relationship between  $\sigma_i^2$  and the values taken by the explanatory variable  $X$ , as can be seen clearly from (11.2.2) (see exercise 11.9). The bias arises from the fact that  $\hat{\sigma}^2$ , the conventional estimator of  $\sigma^2$ , namely,  $\sum \hat{u}_i^2 / (n - 2)$  is no longer an unbiased estimator of the latter when heteroscedasticity is present (see Appendix 11A.3). As a result, we can no longer rely on the conventionally computed confidence intervals and the conventionally employed  $t$  and  $F$  tests.<sup>6</sup> **In short, if we persist in using the usual testing procedures despite heteroscedasticity, whatever conclusions we draw or inferences we make may be very misleading.**

To throw more light on this topic, we refer to a **Monte Carlo** study conducted by Davidson and MacKinnon.<sup>7</sup> They consider the following simple model, which in our notation is

$$Y_i = \beta_1 + \beta_2 X_i + u_i \tag{11.4.1}$$

They assume that  $\beta_1 = 1$ ,  $\beta_2 = 1$ , and  $u_i \sim N(0, X_i^\alpha)$ . As the last expression shows, they assume that the error variance is heteroscedastic and is related to the value of the regressor  $X$  with power  $\alpha$ . If, for example,  $\alpha = 1$ , the error variance is proportional to the value of  $X$ ; if  $\alpha = 2$ , the error variance is proportional to the square of the value of  $X$ , and so on. In Section 11.6 we will consider the logic behind such a procedure. Based on 20,000 replications and allowing for various values for  $\alpha$ , they obtain the standard errors of the two regression coefficients using OLS [see Eq. (11.2.3)], OLS allowing for heteroscedasticity [see Eq. (11.2.2)], and GLS [see Eq. (11.3.9)]. We quote their results for selected values of  $\alpha$ :

Value of $\alpha$	Standard error of $\hat{\beta}_1$			Standard error of $\hat{\beta}_2$		
	OLS	OLS <sub>het</sub>	GLS	OLS	OLS <sub>het</sub>	GLS
0.5	0.164	0.134	0.110	0.285	0.277	0.243
1.0	0.142	0.101	0.048	0.246	0.247	0.173
2.0	0.116	0.074	0.0073	0.200	0.220	0.109
3.0	0.100	0.064	0.0013	0.173	0.206	0.056
4.0	0.089	0.059	0.0003	0.154	0.195	0.017

Note: OLS<sub>het</sub> means OLS allowing for heteroscedasticity.

<sup>6</sup>From (5.3.6) we know that the  $100(1 - \alpha)\%$  confidence interval for  $\beta_2$  is  $[\hat{\beta}_2 \pm t_{\alpha/2} \text{se}(\hat{\beta}_2)]$ . But if  $\text{se}(\hat{\beta}_2)$  cannot be estimated unbiasedly, what trust can we put in the conventionally computed confidence interval?

<sup>7</sup>Russell Davidson and James G. MacKinnon, *Estimation and Inference in Econometrics*, Oxford University Press, New York, 1993, pp. 549–550.

*The most striking feature of these results is that OLS, with or without correction for heteroscedasticity, consistently overestimates the true standard error obtained by the (correct) GLS procedure, especially for large values of  $\alpha$ , thus establishing the superiority of GLS.* These results also show that if we do not use GLS and rely on OLS—allowing for or not allowing for heteroscedasticity—the picture is mixed. The usual OLS standard errors are either too large (for the intercept) or generally too small (for the slope coefficient) in relation to those obtained by OLS allowing for heteroscedasticity. The message is clear: In the presence of heteroscedasticity, use GLS. However, for reasons explained later in the chapter, in practice it is not always easy to apply GLS. Also, as we discuss later, unless heteroscedasticity is very severe, one may not abandon OLS in favor of GLS or WLS.

From the preceding discussion it is clear that heteroscedasticity is potentially a serious problem and the researcher needs to know whether it is present in a given situation. If its presence is detected, then one can take corrective action, such as using the weighted least-squares regression or some other technique. Before we turn to examining the various corrective procedures, however, we must first find out whether heteroscedasticity is present or likely to be present in a given case. This topic is discussed in the following section.

### A Technical Note

Although we have stated that, in cases of heteroscedasticity, it is the GLS, not the OLS, that is BLUE, there are examples where OLS can be BLUE, despite heteroscedasticity.<sup>8</sup> But such examples are infrequent in practice.

## 11.5 DETECTION OF HETEROSCEDASTICITY

As with multicollinearity, the important practical question is: How does one know that heteroscedasticity is present in a specific situation? Again, as in the case of multicollinearity, there are no hard-and-fast rules for detecting heteroscedasticity, only a few rules of thumb. But this situation is inevitable because  $\sigma_i^2$  can be known only if we have the entire  $Y$  population corresponding to the chosen  $X$ 's, such as the population shown in Table 2.1 or Table 11.1. But such data are an exception rather than the rule in most

<sup>8</sup>The reason for this is that the Gauss–Markov theorem provides the sufficient (but not necessary) condition for OLS to be efficient. The necessary and sufficient condition for OLS to be BLUE is given by **Kruskal's Theorem**. But this topic is beyond the scope of this book. I am indebted to Michael McAleer for bringing this to my attention. For further details, see Denzil G. Fiebig, Michael McAleer, and Robert Bartels, "Properties of Ordinary Least Squares Estimators in Regression Models with Nonspherical Disturbances," *Journal of Econometrics*, vol. 54, No. 1–3, Oct.–Dec., 1992, pp. 321–334. For the mathematically inclined student, I discuss this topic further in **App. C**, using matrix algebra.



economic investigations. In this respect the econometrician differs from scientists in fields such as agriculture and biology, where researchers have a good deal of control over their subjects. More often than not, in economic studies there is only one sample  $Y$  value corresponding to a particular value of  $X$ . And there is no way one can know  $\sigma_i^2$  from just one  $Y$  observation. Therefore, in most cases involving econometric investigations, heteroscedasticity may be a matter of intuition, educated guesswork, prior empirical experience, or sheer speculation.

With the preceding caveat in mind, let us examine some of the informal and formal methods of detecting heteroscedasticity. As the following discussion will reveal, most of these methods are based on the examination of the OLS residuals  $\hat{u}_i$  since they are the ones we observe, and not the disturbances  $u_i$ . One hopes that they are good estimates of  $u_i$ , a hope that may be fulfilled if the sample size is fairly large.

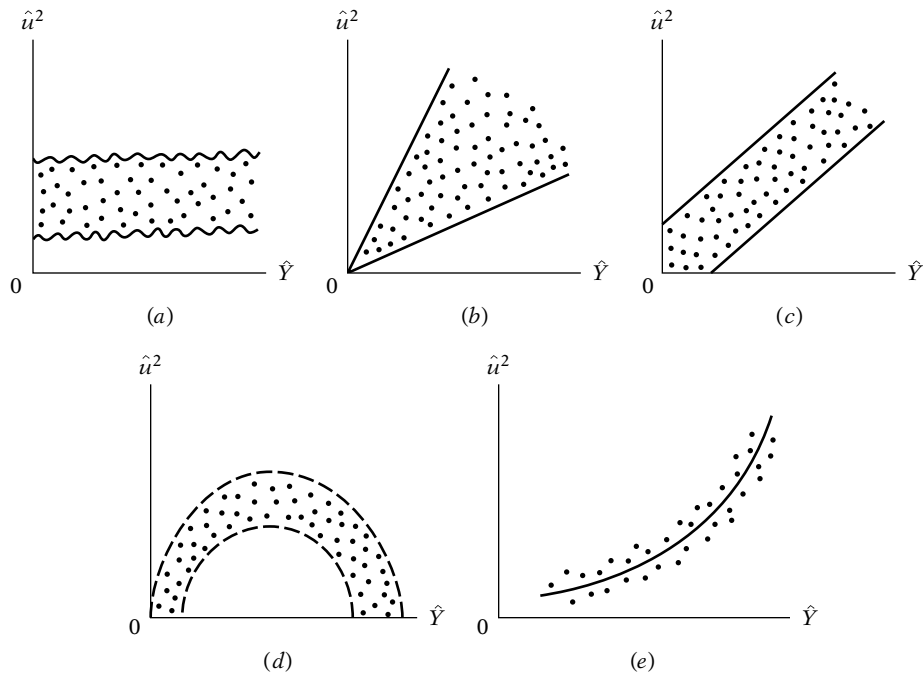
### Informal Methods

**Nature of the Problem** Very often the nature of the problem under consideration suggests whether heteroscedasticity is likely to be encountered. For example, following the pioneering work of Prais and Houthakker on family budget studies, where they found that the residual variance around the regression of consumption on income increased with income, one now generally assumes that in similar surveys one can expect unequal variances among the disturbances.<sup>9</sup> As a matter of fact, in cross-sectional data involving heterogeneous units, heteroscedasticity may be the rule rather than the exception. Thus, in a cross-sectional analysis involving the investment expenditure in relation to sales, rate of interest, etc., heteroscedasticity is generally expected if small-, medium-, and large-size firms are sampled together.

As a matter of fact, we have already come across examples of this. In Chapter 2 we discussed the relationship between mean, or average, hourly wages in relation to years of schooling in the United States. In that chapter we also discussed the relationship between expenditure on food and total expenditure for 55 families in India (see exercise 11.16).

**Graphical Method** If there is no a priori or empirical information about the nature of heteroscedasticity, in practice one can do the regression analysis on the assumption that there is no heteroscedasticity and then do a postmortem examination of the residual squared  $\hat{u}_i^2$  to see if they exhibit any systematic pattern. Although  $\hat{u}_i^2$  are not the same thing as  $u_i^2$ , they can be

<sup>9</sup>S. J. Prais and H. S. Houthakker, *The Analysis of Family Budgets*, Cambridge University Press, New York, 1955.



**FIGURE 11.8** Hypothetical patterns of estimated squared residuals.

used as proxies especially if the sample size is sufficiently large.<sup>10</sup> An examination of the  $\hat{u}_i^2$  may reveal patterns such as those shown in Figure 11.8.

In Figure 11.8,  $\hat{u}_i^2$  are plotted against  $\hat{Y}_i$ , the estimated  $Y_i$  from the regression line, the idea being to find out whether the estimated mean value of  $Y$  is systematically related to the squared residual. In Figure 11.8a we see that there is no systematic pattern between the two variables, suggesting that perhaps no heteroscedasticity is present in the data. Figure 11.8b to e, however, exhibits definite patterns. For instance, Figure 11.8c suggests a linear relationship, whereas Figure 11.8d and e indicates a quadratic relationship between  $\hat{u}_i^2$  and  $\hat{Y}_i$ . Using such knowledge, albeit informal, one may transform the data in such a manner that the transformed data do not exhibit heteroscedasticity. In Section 11.6 we shall examine several such transformations.

Instead of plotting  $\hat{u}_i^2$  against  $\hat{Y}_i$ , one may plot them against one of the explanatory variables, especially if plotting  $\hat{u}_i^2$  against  $\hat{Y}_i$  results in the pattern shown in Figure 11.8a. Such a plot, which is shown in Figure 11.9, may reveal patterns similar to those given in Figure 11.8. (In the case of the two-variable model, plotting  $\hat{u}_i^2$  against  $\hat{Y}_i$  is equivalent to plotting it against

<sup>10</sup>For the relationship between  $\hat{u}_i$  and  $u_i$ , see E. Malinvaud, *Statistical Methods of Econometrics*, North Holland Publishing Company, Amsterdam, 1970, pp. 88–89.

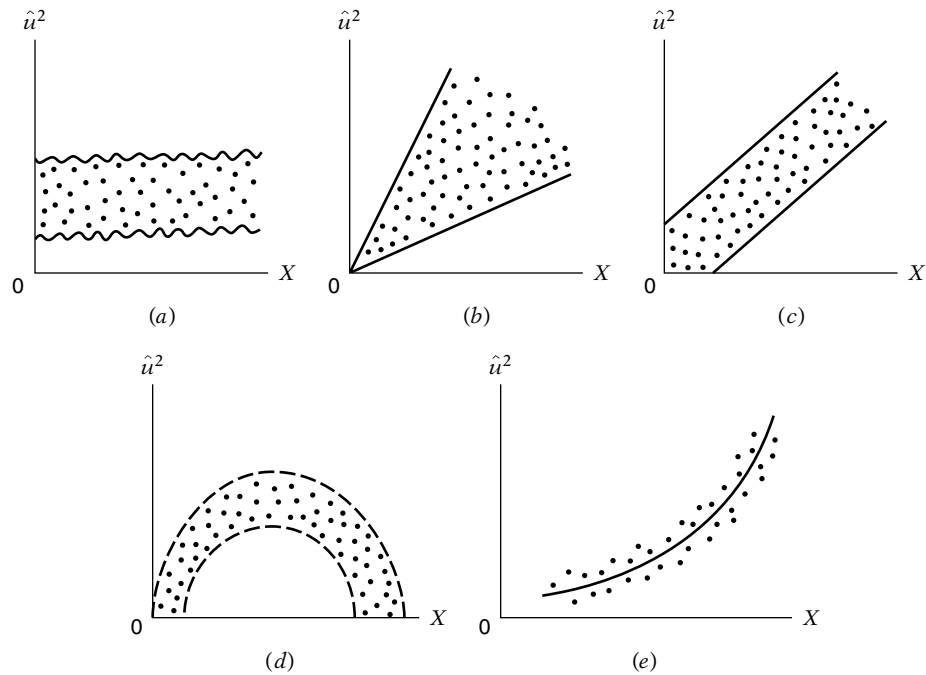


FIGURE 11.9 Scattergram of estimated squared residuals against  $X$ .

$X_i$ , and therefore Figure 11.9 is similar to Figure 11.8. But this is not the situation when we consider a model involving two or more  $X$  variables; in this instance,  $\hat{u}_i^2$  may be plotted against any  $X$  variable included in the model.)

A pattern such as that shown in Figure 11.9c, for instance, suggests that the variance of the disturbance term is linearly related to the  $X$  variable. Thus, if in the regression of savings on income one finds a pattern such as that shown in Figure 11.9c, it suggests that the heteroscedastic variance may be *proportional* to the value of the income variable. This knowledge may help us in transforming our data in such a manner that in the regression on the transformed data the variance of the disturbance is homoscedastic. We shall return to this topic in the next section.

### Formal Methods

**Park Test**<sup>11</sup> Park formalizes the graphical method by suggesting that  $\sigma_i^2$  is some function of the explanatory variable  $X_i$ . The functional form he

<sup>11</sup>R. E. Park, "Estimation with Heteroscedastic Error Terms," *Econometrica*, vol. 34, no. 4, October 1966, p. 888. The Park test is a special case of the general test proposed by A. C. Harvey in "Estimating Regression Models with Multiplicative Heteroscedasticity," *Econometrica*, vol. 44, no. 3, 1976, pp. 461–465.

suggested was

$$\sigma_i^2 = \sigma^2 X_i^\beta e^{v_i}$$

or

$$\ln \sigma_i^2 = \ln \sigma^2 + \beta \ln X_i + v_i \quad (11.5.1)$$

where  $v_i$  is the stochastic disturbance term.

Since  $\sigma_i^2$  is generally not known, Park suggests using  $\hat{u}_i^2$  as a proxy and running the following regression:

$$\begin{aligned} \ln \hat{u}_i^2 &= \ln \sigma^2 + \beta \ln X_i + v_i \\ &= \alpha + \beta \ln X_i + v_i \end{aligned} \quad (11.5.2)$$

If  $\beta$  turns out to be statistically significant, it would suggest that heteroscedasticity is present in the data. If it turns out to be insignificant, we may accept the assumption of homoscedasticity. The Park test is thus a two-stage procedure. In the first stage we run the OLS regression disregarding the heteroscedasticity question. We obtain  $\hat{u}_i$  from this regression, and then in the second stage we run the regression (11.5.2).

Although empirically appealing, the Park test has some problems. Goldfeld and Quandt have argued that the error term  $v_i$  entering into (11.5.2) may not satisfy the OLS assumptions and may itself be heteroscedastic.<sup>12</sup> Nonetheless, as a strictly exploratory method, one may use the Park test.

**EXAMPLE 11.1**

**RELATIONSHIP BETWEEN COMPENSATION AND PRODUCTIVITY**

To illustrate the Park approach, we use the data given in Table 11.1 to run the following regression:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

where  $Y$  = average compensation in thousands of dollars,  $X$  = average productivity in thousands of dollars, and  $i$  =  $i$ th employment size of the establishment. The results of the regression were as follows:

$$\begin{aligned} \hat{Y}_i &= 1992.3452 + 0.2329X_i \\ \text{se} &= (936.4791) \quad (0.0998) \\ t &= (2.1275) \quad (2.333) \quad R^2 = 0.4375 \end{aligned} \quad (11.5.3)$$

The results reveal that the estimated slope coefficient is significant at the 5 percent level on the basis of a one-tail  $t$  test. The equation shows that as labor productivity increases by, say, a dollar, labor compensation on the average increases by about 23 cents.

The residuals obtained from regression (11.5.3) were regressed on  $X_i$  as suggested in Eq. (11.5.2), giving the following results:

$$\begin{aligned} \widehat{\ln \hat{u}_i^2} &= 35.817 - 2.8099 \ln X_i \\ \text{se} &= (38.319) \quad (4.216) \\ t &= (0.934) \quad (-0.667) \quad R^2 = 0.0595 \end{aligned} \quad (11.5.4)$$

Obviously, there is no statistically significant relationship between the two variables. Following the Park test, one may conclude that there is no heteroscedasticity in the error variance.<sup>13</sup>

<sup>12</sup>Stephen M. Goldfeld and Richard E. Quandt, *Nonlinear Methods in Econometrics*, North Holland Publishing Company, Amsterdam, 1972, pp. 93–94.

<sup>13</sup>The particular functional form chosen by Park is only suggestive. A different functional form may reveal significant relationships. For example, one may use  $\hat{u}_i^2$  instead of  $\ln \hat{u}_i^2$  as the dependent variable.

**Glejser Test**<sup>14</sup> The Glejser test is similar in spirit to the Park test. After obtaining the residuals  $\hat{u}_i$  from the OLS regression, Glejser suggests regressing the absolute values of  $\hat{u}_i$  on the  $X$  variable that is thought to be closely associated with  $\sigma_i^2$ . In his experiments, Glejser used the following functional forms:

$$|\hat{u}_i| = \beta_1 + \beta_2 X_i + v_i$$

$$|\hat{u}_i| = \beta_1 + \beta_2 \sqrt{X_i} + v_i$$

$$|\hat{u}_i| = \beta_1 + \beta_2 \frac{1}{X_i} + v_i$$

$$|\hat{u}_i| = \beta_1 + \beta_2 \frac{1}{\sqrt{X_i}} + v_i$$

$$|\hat{u}_i| = \sqrt{\beta_1 + \beta_2 X_i} + v_i$$

$$|\hat{u}_i| = \sqrt{\beta_1 + \beta_2 X_i^2} + v_i$$

where  $v_i$  is the error term.

Again as an empirical or practical matter, one may use the Glejser approach. But Goldfeld and Quandt point out that the error term  $v_i$  has some problems in that its expected value is nonzero, it is serially correlated (see Chapter 12), and ironically it is heteroscedastic.<sup>15</sup> An additional difficulty with the Glejser method is that models such as

$$|\hat{u}_i| = \sqrt{\beta_1 + \beta_2 X_i} + v_i$$

and

$$|\hat{u}_i| = \sqrt{\beta_1 + \beta_2 X_i^2} + v_i$$

are nonlinear in the parameters and therefore cannot be estimated with the usual OLS procedure.

Glejser has found that for large samples the first four of the preceding models give generally satisfactory results in detecting heteroscedasticity. As a practical matter, therefore, the Glejser technique may be used for large samples and may be used in the small samples strictly as a qualitative device to learn something about heteroscedasticity.

<sup>14</sup>H. Glejser, "A New Test for Heteroscedasticity," *Journal of the American Statistical Association*, vol. 64, 1969, pp. 316–323.

<sup>15</sup>For details, see Goldfeld and Quandt, *op. cit.*, Chap. 3.

**EXAMPLE 11.2**RELATIONSHIP BETWEEN COMPENSATION AND PRODUCTIVITY:  
THE GLEJSER TEST

Continuing with Example 11.1, the absolute value of the residuals obtained from regression (11.5.3) were regressed on average productivity ( $X$ ), giving the following results:

$$\begin{aligned} |\hat{u}_i| &= 407.2783 - 0.0203X_i \\ \text{se} &= (633.1621) \quad (0.0675) \quad r^2 = 0.0127 \\ t &= (0.6432) \quad (-0.3012) \end{aligned} \quad (11.5.5)$$

As you can see from this regression, there is no relationship between the absolute value of the residuals and the regressor, average productivity. This reinforces the conclusion based on the Park test.

**Spearman's Rank Correlation Test.** In exercise 3.8 we defined the Spearman's rank correlation coefficient as

$$r_s = 1 - 6 \left[ \frac{\sum d_i^2}{n(n^2 - 1)} \right] \quad (11.5.6)$$

where  $d_i$  = difference in the ranks assigned to two different characteristics of the  $i$ th individual or phenomenon and  $n$  = number of individuals or phenomena ranked. The preceding rank correlation coefficient can be used to detect heteroscedasticity as follows: Assume  $Y_i = \beta_0 + \beta_1 X_i + u_i$ .

**Step 1.** Fit the regression to the data on  $Y$  and  $X$  and obtain the residuals  $\hat{u}_i$ .

**Step 2.** Ignoring the sign of  $\hat{u}_i$ , that is, taking their absolute value  $|\hat{u}_i|$ , rank both  $|\hat{u}_i|$  and  $X_i$  (or  $\hat{Y}_i$ ) according to an ascending or descending order and compute the Spearman's rank correlation coefficient given previously.

**Step 3.** Assuming that the population rank correlation coefficient  $\rho_s$  is zero and  $n > 8$ , the significance of the sample  $r_s$  can be tested by the  $t$  test as follows<sup>16</sup>:

$$t = \frac{r_s \sqrt{n-2}}{\sqrt{1-r_s^2}} \quad (11.5.7)$$

with  $df = n - 2$ .

<sup>16</sup>See G. Udny Yule and M. G. Kendall, *An Introduction to the Theory of Statistics*, Charles Griffin & Company, London, 1953, p. 455.

If the computed  $t$  value exceeds the critical  $t$  value, we may accept the hypothesis of heteroscedasticity; otherwise we may reject it. If the regression model involves more than one  $X$  variable,  $r_s$  can be computed between  $|\hat{u}_i|$  and each of the  $X$  variables separately and can be tested for statistical significance by the  $t$  test given in Eq. (11.5.7).

**EXAMPLE 11.3**

**ILLUSTRATION OF THE RANK CORRELATION TEST**

To illustrate the rank correlation test, consider the data given in Table 11.2. The data pertain to the average annual return ( $E_i$ , %) and the standard deviation of annual return ( $\sigma_i$ , %) of 10 mutual funds.

The capital market line (CML) of portfolio theory postulates a linear relationship between expected return ( $E_i$ ) and risk (as measured by the standard deviation,  $\sigma_i$ ) of a portfolio as follows:

$$E_i = \beta_1 + \beta_2 \sigma_i$$

Using the data in Table 11.2, the preceding model was estimated and the residuals from this model were computed. Since the data relate to 10 mutual funds of differing sizes and investment goals, a priori one might expect heteroscedasticity. To test this hypothesis, we apply the

rank correlation test. The necessary calculations are given in Table 11.2.

Applying formula (11.5.6), we obtain

$$r_s = 1 - 6 \frac{110}{10(100 - 1)} = 0.3333 \tag{11.5.8}$$

Applying the  $t$  test given in (11.5.7), we obtain

$$t = \frac{(0.3333)(\sqrt{8})}{\sqrt{1 - 0.1110}} = 0.9998 \tag{11.5.9}$$

For 8 df this  $t$  value is not significant even at the 10% level of significance; the  $p$  value is 0.17. Thus, there is no evidence of a systematic relationship between the explanatory variable and the absolute values of the residuals, which might suggest that there is no heteroscedasticity.

**TABLE 11.2**  
RANK CORRELATION TEST OF HETEROSCEDASTICITY

Name of mutual fund	$E_i$ , average annual return, %	$\sigma_i$ , standard deviation of annual return, %	$\hat{E}_i^*$	$ \hat{u}_i ^\dagger$ residuals, $ (E_i - \hat{E}_i) $	Rank of $ \hat{u}_i $	Rank of $\sigma_i$	$d$ , difference between two rankings	$d^2$
Boston Fund	12.4	12.1	11.37	1.03	9	4	5	25
Delaware Fund	14.4	21.4	15.64	1.24	10	9	1	1
Equity Fund	14.6	18.7	14.40	0.20	4	7	-3	9
Fundamental Investors	16.0	21.7	15.78	0.22	5	10	-5	25
Investors Mutual	11.3	12.5	11.56	0.26	6	5	1	1
Loomis-Sales Mutual Fund	10.0	10.4	10.59	0.59	7	2	5	25
Massachusetts Investors Trust	16.2	20.8	15.37	0.83	8	8	0	0
New England Fund	10.4	10.2	10.50	0.10	3	1	2	4
Putnam Fund of Boston	13.1	16.0	13.16	0.06	2	6	-4	16
Wellington Fund	11.3	12.0	11.33	0.03	1	3	-2	4
Total							0	110

\*Obtained from the regression:  $\hat{E}_i = 5.8194 + 0.4590 \sigma_i$ .

†Absolute value of the residuals.

Note: The ranking is in ascending order of values.

**Goldfeld-Quandt Test.**<sup>17</sup> This popular method is applicable if one assumes that the heteroscedastic variance,  $\sigma_i^2$ , is positively related to *one* of the explanatory variables in the regression model. For simplicity, consider the usual two-variable model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

Suppose  $\sigma_i^2$  is positively related to  $X_i$  as

$$\sigma_i^2 = \sigma^2 X_i^2 \quad (11.5.10)$$

where  $\sigma^2$  is a constant.<sup>18</sup>

Assumption (11.5.10) postulates that  $\sigma_i^2$  is proportional to the square of the  $X$  variable. Such an assumption has been found quite useful by Prais and Houthakker in their study of family budgets. (See Section 11.6.)

If (11.5.10) is appropriate, it would mean  $\sigma_i^2$  would be larger, the larger the values of  $X_i$ . If that turns out to be the case, heteroscedasticity is most likely to be present in the model. To test this explicitly, Goldfeld and Quandt suggest the following steps:

**Step 1.** Order or rank the observations according to the values of  $X_i$ , beginning with the lowest  $X$  value.

**Step 2.** Omit  $c$  central observations, where  $c$  is specified a priori, and divide the remaining  $(n - c)$  observations into two groups each of  $(n - c)/2$  observations.

**Step 3.** Fit separate OLS regressions to the first  $(n - c)/2$  observations and the last  $(n - c)/2$  observations, and obtain the respective residual sums of squares  $RSS_1$  and  $RSS_2$ ,  $RSS_1$  representing the RSS from the regression corresponding to the smaller  $X_i$  values (the small variance group) and  $RSS_2$  that from the larger  $X_i$  values (the large variance group). These RSS each have

$$\frac{(n - c)}{2} - k \quad \text{or} \quad \left( \frac{n - c - 2k}{2} \right) \text{ df}$$

where  $k$  is the number of parameters to be estimated, including the intercept. (Why?) For the two-variable case  $k$  is of course 2.

**Step 4.** Compute the ratio

$$\lambda = \frac{RSS_2/\text{df}}{RSS_1/\text{df}} \quad (11.5.11)$$

If  $u_i$  are assumed to be normally distributed (which we usually do), and if the assumption of homoscedasticity is valid, then it can be shown that  $\lambda$  of (11.5.10) follows the  $F$  distribution with numerator and denominator df each of  $(n - c - 2k)/2$ .

<sup>17</sup>Goldfeld and Quandt, op. cit., Chap. 3.

<sup>18</sup>This is only one plausible assumption. Actually, what is required is that  $\sigma_i^2$  be monotonically related to  $X_i$ .



If in an application the computed  $\lambda (= F)$  is greater than the critical  $F$  at the chosen level of significance, we can reject the hypothesis of homoscedasticity, that is, we can say that heteroscedasticity is very likely.

Before illustrating the test, a word about omitting the  $c$  central observations is in order. These observations are omitted to sharpen or accentuate the difference between the small variance group (i.e.,  $RSS_1$ ) and the large variance group (i.e.,  $RSS_2$ ). But the ability of the Goldfeld–Quandt test to do this successfully depends on how  $c$  is chosen.<sup>19</sup> For the two-variable model the Monte Carlo experiments done by Goldfeld and Quandt suggest that  $c$  is about 8 if the sample size is about 30, and it is about 16 if the sample size is about 60. But Judge et al. note that  $c = 4$  if  $n = 30$  and  $c = 10$  if  $n$  is about 60 have been found satisfactory in practice.<sup>20</sup>

Before moving on, it may be noted that in case there is more than one  $X$  variable in the model, the ranking of observations, the first step in the test, can be done according to any one of them. Thus in the model:  $Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + u_i$ , we can rank-order the data according to any one of these  $X$ 's. If a priori we are not sure which  $X$  variable is appropriate, we can conduct the test on each of the  $X$  variables, or via a Park test, in turn, on each  $X$ .

**EXAMPLE 11.4**

## THE GOLDFELD–QUANDT TEST

To illustrate the Goldfeld–Quandt test, we present in Table 11.3 data on consumption expenditure in relation to income for a cross section of 30 families. Suppose we postulate that consumption expenditure is linearly related to income but that heteroscedasticity is present in the data. We further postulate that the nature of heteroscedasticity is as given in (11.5.10). The necessary reordering of the data for the application of the test is also presented in Table 11.3.

Dropping the middle 4 observations, the OLS regressions based on the first 13 and the last 13 observations and their associated residual sums of squares are as shown next (standard errors in the parentheses).

Regression based on the first 13 observations:

$$\hat{Y}_i = 3.4094 + 0.6968X_i$$

(8.7049) (0.0744)     $r^2 = 0.8887$      $RSS_1 = 377.17$      $df = 11$

Regression based on the last 13 observations:

$$\hat{Y}_i = -28.0272 + 0.7941X_i$$

(30.6421) (0.1319)     $r^2 = 0.7681$      $RSS_2 = 1536.8$      $df = 11$

(Continued)

<sup>19</sup>Technically, the **power** of the test depends on how  $c$  is chosen. In statistics, the *power of a test* is measured by the probability of rejecting the null hypothesis when it is false [i.e., by  $1 - \text{Prob}(\text{type II error})$ ]. Here the null hypothesis is that the variances of the two groups are the same, i.e., homoscedasticity. For further discussion, see M. M. Ali and C. Giaccotto, "A Study of Several New and Existing Tests for Heteroscedasticity in the General Linear Model," *Journal of Econometrics*, vol. 26, 1984, pp. 355–373.

<sup>20</sup>George G. Judge, R. Carter Hill, William E. Griffiths, Helmut Lütkepohl, and Tsoung-Chao Lee, *Introduction to the Theory and Practice of Econometrics*, John Wiley & Sons, New York, 1982, p. 422.

**EXAMPLE 11.4** (Continued)

From these results we obtain

$$\lambda = \frac{RSS_2/df}{RSS_1/df} = \frac{1536.8/11}{377.17/11}$$

$$\lambda = 4.07$$

The critical  $F$  value for 11 numerator and 11 denominator  $df$  at the 5 percent level is 2.82. Since the estimated  $F (= \lambda)$  value exceeds the critical value, we may conclude that there is heteroscedasticity in the error variance. However, if the level of significance is fixed at 1 percent, we may not reject the assumption of homoscedasticity. (Why?) Note that the  $p$  value of the observed  $\lambda$  is 0.014.

**TABLE 11.3**  
HYPOTHETICAL DATA ON CONSUMPTION EXPENDITURE  $Y(\$)$  AND  
INCOME  $X(\$)$  TO ILLUSTRATE THE GOLDFELD–QUANDT TEST

Y	X	Data ranked by X values	
		Y	X
55	80	55	80
65	100	70	85
70	85	75	90
80	110	65	100
79	120	74	105
84	115	80	110
98	130	84	115
95	140	79	120
90	125	90	125
75	90	98	130
74	105	95	140
110	160	108	145
113	150	113	150
125	165	110	160
108	145	125	165
115	180	115	180
140	225	130	185
120	200	135	190
145	240	120	200
130	185	140	205
152	220	144	210
144	210	152	220
175	245	140	225
180	260	137	230
135	190	145	240
140	205	175	245
178	265	189	250
191	270	180	260
137	230	178	265
189	250	191	270

} Middle 4  
observations

**Breusch–Pagan–Godfrey Test.**<sup>21</sup> The success of the Goldfeld–Quandt test depends not only on the value of  $c$  (the number of central observations to be omitted) but also on identifying the correct  $X$  variable with which to order the observations. This limitation of this test can be avoided if we consider the **Breusch–Pagan–Godfrey (BPG) test**.

To illustrate this test, consider the  $k$ -variable linear regression model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i \quad (11.5.12)$$

Assume that the error variance  $\sigma_i^2$  is described as

$$\sigma_i^2 = f(\alpha_1 + \alpha_2 Z_{2i} + \cdots + \alpha_m Z_{mi}) \quad (11.5.13)$$

that is,  $\sigma_i^2$  is some function of the nonstochastic variables  $Z$ 's; some or all of the  $X$ 's can serve as  $Z$ 's. Specifically, assume that

$$\sigma_i^2 = \alpha_1 + \alpha_2 Z_{2i} + \cdots + \alpha_m Z_{mi} \quad (11.5.14)$$

that is,  $\sigma_i^2$  is a linear function of the  $Z$ 's. If  $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$ ,  $\sigma_i^2 = \alpha_1$ , which is a constant. Therefore, to test whether  $\sigma_i^2$  is homoscedastic, one can test the hypothesis that  $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$ . This is the basic idea behind the Breusch–Pagan test. The actual test procedure is as follows.

**Step 1.** Estimate (11.5.12) by OLS and obtain the residuals  $\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n$ .

**Step 2.** Obtain  $\tilde{\sigma}^2 = \sum \hat{u}_i^2 / n$ . Recall from Chapter 4 that this is the maximum likelihood (ML) estimator of  $\sigma^2$ . [Note: The OLS estimator is  $\sum \hat{u}_i^2 / (n - k)$ .]

**Step 3.** Construct variables  $p_i$  defined as

$$p_i = \hat{u}_i^2 / \tilde{\sigma}^2$$

which is simply each residual squared divided by  $\tilde{\sigma}^2$ .

**Step 4.** Regress  $p_i$  thus constructed on the  $Z$ 's as

$$p_i = \alpha_1 + \alpha_2 Z_{2i} + \cdots + \alpha_m Z_{mi} + v_i \quad (11.5.15)$$

where  $v_i$  is the residual term of this regression.

**Step 5.** Obtain the ESS (explained sum of squares) from (11.5.15) and define

$$\Theta = \frac{1}{2}(\text{ESS}) \quad (11.5.16)$$

Assuming  $u_i$  are normally distributed, one can show that if there is homoscedasticity and if the sample size  $n$  increases indefinitely, then

$$\Theta \underset{\text{asy}}{\sim} \chi_{m-1}^2 \quad (11.5.17)$$

<sup>21</sup>T. Breusch and A. Pagan, "A Simple Test for Heteroscedasticity and Random Coefficient Variation," *Econometrica*, vol. 47, 1979, pp. 1287–1294. See also L. Godfrey, "Testing for Multiplicative Heteroscedasticity," *Journal of Econometrics*, vol. 8, 1978, pp. 227–236. Because of similarity, these tests are known as Breusch–Pagan–Godfrey tests of heteroscedasticity.

that is,  $\Theta$  follows the chi-square distribution with  $(m - 1)$  degrees of freedom. (*Note: asy* means asymptotically.)

Therefore, if in an application the computed  $\Theta (= \chi^2)$  exceeds the critical  $\chi^2$  value at the chosen level of significance, one can reject the hypothesis of homoscedasticity; otherwise one does not reject it.

The reader may wonder why BPG chose  $\frac{1}{2}$ ESS as the test statistic. The reasoning is slightly involved and is left for the references.<sup>22</sup>

**EXAMPLE 11.5**

## THE BREUSCH-PAGAN-GODFREY (BPG) TEST

As an example, let us revisit the data (Table 11.3) that were used to illustrate the Goldfeld-Quandt heteroscedasticity test. Regressing  $Y$  on  $X$ , we obtain the following:

**Step 1.**

$$\hat{Y}_i = 9.2903 + 0.6378X_i$$

$$se = (5.2314) \quad (0.0286) \quad RSS = 2361.153 \quad R^2 = 0.9466 \quad (11.5.18)$$

**Step 2.**

$$\bar{\sigma}^2 = \sum \hat{u}_i^2 / 30 = 2361.153 / 30 = 78.7051$$

**Step 3.** Divide the squared residuals  $\hat{u}_i$  obtained from regression (11.5.18) by 78.7051 to construct the variable  $p_i$ .

**Step 4.** Assuming that  $p_i$  are linearly related to  $X_i (= Z_i)$  as per (11.5.14), we obtain the regression

$$\hat{p}_i = -0.7426 + 0.0101X_i$$

$$se = (0.7529) \quad (0.0041) \quad ESS = 10.4280 \quad R^2 = 0.18 \quad (11.5.19)$$

**Step 5.**

$$\Theta = \frac{1}{2}(ESS) = 5.2140 \quad (11.5.20)$$

Under the assumptions of the BPG test  $\Theta$  in (11.5.20) asymptotically follows the chi-square distribution with 1 df. [*Note:* There is only one regressor in (11.5.19).] Now from the chi-square table we find that for 1 df the 5 percent critical chi-square value is 3.8414 and the 1 percent critical  $\chi^2$  value is 6.6349. Thus, the observed chi-square value of 5.2140 is significant at the 5 percent but not the 1 percent level of significance. Therefore, we reach the same conclusion as the Goldfeld-Quandt test. But keep in mind that, strictly speaking, the BPG test is an asymptotic, or large-sample, test and in the present example 30 observations may not constitute a large sample. It should also be pointed out that in small samples the test is sensitive to the assumption that the disturbances  $u_i$  are normally distributed. Of course, we can test the normality assumption by the tests discussed in Chapter 5.<sup>23</sup>

<sup>22</sup>See Adrian C. Darnell, *A Dictionary of Econometrics*, Edward Elgar, Cheltenham, U.K., 1994, pp. 178–179.

<sup>23</sup>On this, see R. Koenker, "A Note on Studentizing a Test for Heteroscedasticity," *Journal of Econometrics*, vol. 17, 1981, pp. 1180–1200.

**White's General Heteroscedasticity Test.** Unlike the Goldfeld-Quandt test, which requires reordering the observations with respect to the  $X$  variable that supposedly caused heteroscedasticity, or the BPG test, which is sensitive to the normality assumption, the general test of heteroscedasticity proposed by White does not rely on the normality assumption and is easy to implement.<sup>24</sup> As an illustration of the basic idea, consider the following three-variable regression model (the generalization to the  $k$ -variable model is straightforward):

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i \quad (11.5.21)$$

The White test proceeds as follows:

**Step 1.** Given the data, we estimate (11.5.21) and obtain the residuals,  $\hat{u}_i$ .

**Step 2.** We then run the following (*auxiliary*) regression:

$$\hat{u}_i^2 = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + \alpha_4 X_{2i}^2 + \alpha_5 X_{3i}^2 + \alpha_6 X_{2i} X_{3i} + v_i \quad (11.5.22)^{25}$$

That is, the squared residuals from the original regression are regressed on the original  $X$  variables or regressors, their squared values, and the cross product(s) of the regressors. Higher powers of regressors can also be introduced. Note that there is a constant term in this equation even though the original regression may or may not contain it. Obtain the  $R^2$  from this (*auxiliary*) regression.

**Step 3.** Under the null hypothesis that there is no heteroscedasticity, it can be shown that sample size ( $n$ ) times the  $R^2$  obtained from the auxiliary regression *asymptotically* follows the chi-square distribution with df equal to the number of regressors (excluding the constant term) in the auxiliary regression. That is,

$$n \cdot R^2 \underset{\text{asy}}{\sim} \chi_{\text{df}}^2 \quad (11.5.23)$$

where df is as defined previously. In our example, there are 5 df since there are 5 regressors in the auxiliary regression.

**Step 4.** If the chi-square value obtained in (11.5.23) exceeds the critical chi-square value at the chosen level of significance, the conclusion is that there is heteroscedasticity. If it does not exceed the critical chi-square value, there is no heteroscedasticity, which is to say that in the auxiliary regression (11.5.21),  $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$  (see footnote 25).

<sup>24</sup>H. White, "A Heteroscedasticity Consistent Covariance Matrix Estimator and a Direct Test of Heteroscedasticity," *Econometrica*, vol. 48, 1980, pp. 817–818.

<sup>25</sup>Implied in this procedure is the assumption that the error variance of  $u_i$ ,  $\sigma_i^2$ , is functionally related to the regressors, their squares, and their cross products. If all the partial slope coefficients in this regression are simultaneously equal to zero, then the error variance is the homoscedastic constant equal to  $\alpha_1$ .

**EXAMPLE 11.6**

## WHITE'S HETEROSCEDASTICITY TEST

From cross-sectional data on 41 countries, Stephen Lewis estimated the following regression model<sup>26</sup>:

$$\ln Y_i = \beta_1 + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i \quad (11.5.24)$$

where  $Y$  = ratio of trade taxes (import and export taxes) to total government revenue,  $X_2$  = ratio of the sum of exports plus imports to GNP, and  $X_3$  = GNP per capita; and  $\ln$  stands for natural log. His hypotheses were that  $Y$  and  $X_2$  would be positively related (the higher the trade volume, the higher the trade tax revenue) and that  $Y$  and  $X_3$  would be negatively related (as income increases, government finds it is easier to collect direct taxes—e.g., income tax—than rely on trade taxes).

The empirical results supported the hypotheses. For our purpose, the important point is whether there is heteroscedasticity in the data. Since the data are cross-sectional involving a heterogeneity of countries, a priori one would expect heteroscedasticity in the error variance. By applying **White's heteroscedasticity test** to the residuals obtained from regression (11.5.24), the following results were obtained<sup>27</sup>:

$$\begin{aligned} \widehat{u}_i^2 = & -5.8417 + 2.5629 \ln \text{Trade}_i + 0.6918 \ln \text{GNP}_i \\ & -0.4081(\ln \text{Trade}_i)^2 - 0.0491(\ln \text{GNP}_i)^2 \\ & + 0.0015(\ln \text{Trade}_i)(\ln \text{GNP}_i) \quad R^2 = 0.1148 \end{aligned} \quad (11.5.25)$$

*Note:* The standard errors are not given, as they are not pertinent for our purpose here.

Now  $n \cdot R^2 = 41(0.1148) = 4.7068$ , which has, asymptotically, a chi-square distribution with 5 df (why?). The 5 percent critical chi-square value for 5 df is 11.0705, the 10 percent critical value is 9.2363, and the 25 percent critical value is 6.62568. For all practical purposes, one can conclude, on the basis of the White test, that there is no heteroscedasticity.

A comment is in order regarding the White test. If a model has several regressors, then introducing all the regressors, their squared (or higher-powered) terms, and their cross products can quickly consume degrees of freedom. Therefore, one must use caution in using the test.<sup>28</sup>

In cases where the White test statistic given in (11.5.25) is statistically significant, heteroscedasticity may not necessarily be the cause, but specification errors, about which more will be said in Chapter 13 (recall point 5 of Section 11.1). In other words, **the White test can be a test of (pure) heteroscedasticity or specification error or both**. It has been argued that if no cross-product terms are present in the White test procedure, then it is a test of pure heteroscedasticity. If cross-product terms are present, then it is a test of both heteroscedasticity and specification bias.<sup>29</sup>

<sup>26</sup>Stephen R. Lewis, "Government Revenue from Foreign Trade," *Manchester School of Economics and Social Studies*, vol. 31, 1963, pp. 39–47.

<sup>27</sup>These results, with change in notation, are reproduced from William F. Lott and Subhash C. Ray, *Applied Econometrics: Problems with Data Sets*, Instructor's Manual, Chap. 22, pp. 137–140.

<sup>28</sup>Sometimes the test can be modified to conserve degrees of freedom. See exercise 11.18.

<sup>29</sup>See Richard Harris, *Using Cointegration Analysis in Econometrics Modelling*, Prentice Hall & Harvester Wheatsheaf, U.K., 1995, p. 68.

**Other Tests of Heteroscedasticity.** There are several other tests of heteroscedasticity, each based on certain assumptions. The interested reader may want to consult the references.<sup>30</sup> We mention but one of these tests because of its simplicity. This is the **Koenker–Bassett (KB) test**. Like the Park, Breusch–Pagan–Godfrey, and White’s tests of heteroscedasticity, the KB test is based on the squared residuals,  $\hat{u}_i^2$ , but instead of being regressed on one or more regressors, the squared residuals are regressed on the squared estimated values of the regressand. Specifically, if the original model is:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i \quad (11.5.26)$$

you estimate this model, obtain  $\hat{u}_i$  from this model, and then estimate

$$\hat{u}_i^2 = \alpha_1 + \alpha_2 (\hat{Y}_i)^2 + v_i \quad (11.5.27)$$

where  $\hat{Y}_i$  are the estimated values from the model (11.5.26). The null hypothesis is that  $\alpha_2 = 0$ . If this is not rejected, then one could conclude that there is no heteroscedasticity. The null hypothesis can be tested by the usual  $t$  test or the  $F$  test. (Note that  $F_{1,k} = t_k^2$ .) If the model (11.5.26) is double log, then the squared residuals are regressed on  $(\log \hat{Y}_i)^2$ . One other advantage of the KB test is that it is applicable even if the error term in the original model (11.5.26) is not normally distributed. If you apply the KB test to Example 11.1, you will find that the slope coefficient in the regression of the squared residuals obtained from (11.5.3) on the estimated  $\hat{Y}_i^2$  from (11.5.3) is statistically not different from zero, thus reinforcing the Park test. This result should not be surprising since in the present instance we only have a single regressor. But the KB test is applicable if there is one regressor or many.

## 11.6 REMEDIAL MEASURES

As we have seen, heteroscedasticity does not destroy the unbiasedness and consistency properties of the OLS estimators, but they are no longer efficient, not even asymptotically (i.e., large sample size). This lack of efficiency makes the usual hypothesis-testing procedure of dubious value. Therefore, remedial measures may be called for. There are two approaches to remediation: when  $\sigma_i^2$  is known and when  $\sigma_i^2$  is not known.

### When $\sigma_i^2$ Is Known: The Method of Weighted Least Squares

As we have seen in Section 11.3, if  $\sigma_i^2$  is known, the most straightforward method of correcting heteroscedasticity is by means of weighted least squares, for the estimators thus obtained are BLUE.

<sup>30</sup>See M. J. Harrison and B. P. McCabe, “A Test for Heteroscedasticity Based on Ordinary Least Squares Residuals,” *Journal of the American Statistical Association*, vol. 74, 1979, pp. 494–499; J. Szroeter, “A Class of Parametric Tests for Heteroscedasticity in Linear Econometric Models,” *Econometrica*, vol. 46, 1978, pp. 1311–1327; M. A. Evans and M. L. King, “A Further Class of Tests for Heteroscedasticity,” *Journal of Econometrics*, vol. 37, 1988, pp. 265–276; R. Koenker and G. Bassett, “Robust Tests for Heteroscedasticity Based on Regression Quantiles,” *Econometrica*, vol. 50, 1982, pp. 43–61.



**EXAMPLE 11.7**

**ILLUSTRATION OF THE METHOD OF WEIGHTED LEAST SQUARES**

To illustrate the method, suppose we want to study the relationship between compensation and employment size for the data presented in Table 11.1. For simplicity, we measure employment size by 1 (1–4 employees), 2 (5–9 employees), . . . , 9 (1000–2499 employees), although we could also measure it by the midpoint of the various employment classes given in the table.

Now letting  $Y$  represent average compensation per employee (\$) and  $X$  the employment size, we run the following regression [see Eq. (11.3.6)]:

$$Y_i/\sigma_i = \hat{\beta}_1^*(1/\sigma_i) + \hat{\beta}_2^*(X_i/\sigma_i) + (\hat{u}_i/\sigma_i) \quad (11.6.1)$$

where  $\sigma_i$  are the standard deviations of wages as reported in Table 11.1. The necessary raw data to run this regression are given in Table 11.4.

Before going on to the regression results, note that (11.6.1) has no intercept term. (Why?) Therefore, one will have to use the regression-through-the-origin model to estimate  $\beta_1^*$  and  $\beta_2^*$ , a topic discussed in Chapter 6. But most computer packages these days have an option

to suppress the intercept term (see Minitab or Eviews, for example). Also note another interesting feature of (11.6.1): It has two explanatory variables,  $(1/\sigma_i)$  and  $(X_i/\sigma_i)$ , whereas if we were to use OLS, regressing compensation on employment size, that regression would have a single explanatory variable,  $X_i$ . (Why?)

The regression results of WLS are as follows:

$$\begin{aligned} \widehat{(Y_i/\sigma_i)} &= 3406.639(1/\sigma_i) + 154.153(X_i/\sigma_i) \\ &\quad (80.983) \quad (16.959) \quad (11.6.2) \\ t &= (42.066) \quad (9.090) \\ R^2 &= 0.9993^{31} \end{aligned}$$

For comparison, we give the usual or unweighted OLS regression results:

$$\begin{aligned} \hat{Y}_i &= 3417.833 + 148.767 X_i \\ &\quad (81.136) \quad (14.418) \quad (11.6.3) \\ t &= (42.125) \quad (10.318) \quad R^2 = 0.9383 \end{aligned}$$

In exercise 11.7 you are asked to compare these two regressions.

**TABLE 11.4**  
ILLUSTRATION OF WEIGHTED LEAST-SQUARES REGRESSION

Compensation, Y	Employment size, X	$\sigma_i$	$Y_i/\sigma_i$	$X_i/\sigma_i$
3396	1	743.7	4.5664	0.0013
3787	2	851.4	4.4480	0.0023
4013	3	727.8	5.5139	0.0041
4104	4	805.06	5.0978	0.0050
4146	5	929.9	4.4585	0.0054
4241	6	1080.6	3.9247	0.0055
4387	7	1243.2	3.5288	0.0056
4538	8	1307.7	3.4702	0.0061
4843	9	1112.5	4.3532	0.0081

Note: In regression (11.6.2), the dependent variable is  $(Y_i/\sigma_i)$  and the independent variables are  $(1/\sigma_i)$  and  $(X_i/\sigma_i)$ .

Source: Data on  $Y$  and  $\sigma_i$  (standard deviation of compensation) are from Table 11.1. Employment size: 1 = 1–4 employees, 2 = 5–9 employees, etc. The latter data are also from Table 11.1.

<sup>31</sup>As noted in footnote 3 of Chap. 6, the  $R^2$  of the regression through the origin is not directly comparable with the  $R^2$  of the intercept-present model. The reported  $R^2$  of 0.9993 takes this difference into account. (See the SAS package for further details about how the  $R^2$  is corrected to take into account the absence of the intercept term. See also App. 6A, Sec. 6A1.)



**When  $\sigma_i^2$  Is Not Known**

As noted earlier, if true  $\sigma_i^2$  are known, we can use the WLS method to obtain BLUE estimators. Since the true  $\sigma_i^2$  are rarely known, is there a way of obtaining *consistent* (in the statistical sense) estimates of the variances and covariances of OLS estimators even if there is heteroscedasticity? The answer is yes.

**White's Heteroscedasticity-Consistent Variances and Standard Errors.** White has shown that this estimate can be performed so that *asymptotically* valid (i.e., large-sample) statistical inferences can be made about the true parameter values.<sup>32</sup> We will not present the mathematical details, for they are beyond the scope of this book. However, Appendix 11A.4 outlines White's procedure. Nowadays, several computer packages present White's heteroscedasticity-corrected variances and standard errors along with the usual OLS variances and standard errors.<sup>33</sup> Incidentally, White's heteroscedasticity-corrected standard errors are also known as **robust standard errors**.

**EXAMPLE 11.8**

## ILLUSTRATION OF WHITE'S PROCEDURE

As an example, we quote the following results due to Greene<sup>34</sup>:

$$\hat{Y}_i = 832.91 - 1834.2 (\text{Income}) + 1587.04 (\text{Income})^2$$

OLS se =	(327.3)	(829.0)	(519.1)	
$t =$	(2.54)	(2.21)	(3.06)	<b>(11.6.4)</b>
White se =	(460.9)	(1243.0)	(830.0)	
$t =$	(1.81)	(-1.48)	(1.91)	

where  $Y$  = per capita expenditure on public schools by state in 1979 and  $\text{Income}$  = per capita income by state in 1979. The sample consisted of 50 states plus Washington, D.C.

As the preceding results show, (White's) heteroscedasticity-corrected standard errors are considerably larger than the OLS standard errors and therefore the estimated  $t$  values are much smaller than those obtained by OLS. On the basis of the latter, both the regressors are statistically significant at the 5 percent level, whereas on the basis of White estimators they are not. However, it should be pointed out that White's heteroscedasticity-corrected

<sup>32</sup>See H. White, op. cit.

<sup>33</sup>More technically, they are known as **heteroscedasticity-consistent covariance matrix estimators**.

<sup>34</sup>William H. Greene, *Econometric Analysis*, 2d ed., Macmillan, New York, 1993, p. 385.

standard errors can be larger or smaller than the uncorrected standard errors.

Since White's heteroscedasticity-consistent estimators of the variances are now available in established regression packages, it is recommended that the reader report them. As Wallace and Silver note:

Generally speaking, it is probably a good idea to use the WHITE option [available in regression programs] routinely, perhaps comparing the output with regular OLS output as a check to see whether heteroscedasticity is a serious problem in a particular set of data.<sup>35</sup>

**Plausible Assumptions about Heteroscedasticity Pattern.** Apart from being a large-sample procedure, one drawback of the White procedure is that the estimators thus obtained may not be so efficient as those obtained by methods that transform data to reflect specific types of heteroscedasticity. To illustrate this, let us revert to the two-variable regression model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

We now consider several assumptions about the pattern of heteroscedasticity.

**Assumption 1:** The error variance is proportional to  $X_i^2$ :

$$E(u_i^2) = \sigma^2 X_i^2 \quad (11.6.5)^{36}$$

If, as a matter of "speculation," graphical methods, or Park and Glejser approaches, it is believed that the variance of  $u_i$  is proportional to the square of the explanatory variable  $X$  (see Figure 11.10), one may transform the original model as follows. Divide the original model through by  $X_i$ :

$$\begin{aligned} \frac{Y_i}{X_i} &= \frac{\beta_1}{X_i} + \beta_2 + \frac{u_i}{X_i} \\ &= \beta_1 \frac{1}{X_i} + \beta_2 + v_i \end{aligned} \quad (11.6.6)$$

where  $v_i$  is the transformed disturbance term, equal to  $u_i/X_i$ . Now it is easy to verify that

$$\begin{aligned} E(v_i^2) &= E\left(\frac{u_i}{X_i}\right)^2 = \frac{1}{X_i^2} E(u_i^2) \\ &= \sigma^2 \quad \text{using (11.6.5)} \end{aligned}$$

<sup>35</sup>T. Dudley Wallace and J. Lew Silver, *Econometrics: An Introduction*, Addison-Wesley, Reading, Mass., 1988, p. 265.

<sup>36</sup>Recall that we have already encountered this assumption in our discussion of the Goldfeld-Quandt test.

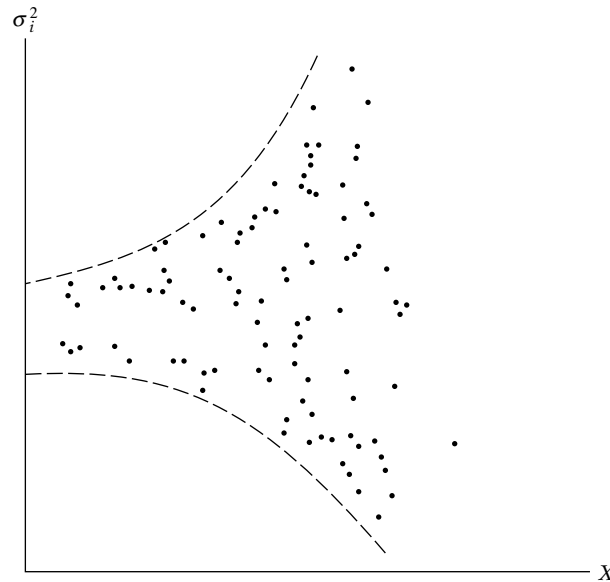


FIGURE 11.10 Error variance proportional to  $X^2$ .

Hence the variance of  $v_i$  is now homoscedastic, and one may proceed to apply OLS to the transformed equation (11.6.6), regressing  $Y_i/X_i$  on  $1/X_i$ .

Notice that in the transformed regression the intercept term  $\beta_2$  is the slope coefficient in the original equation and the slope coefficient  $\beta_1$  is the intercept term in the original model. Therefore, to get back to the original model we shall have to multiply the estimated (11.6.6) by  $X_i$ . An application of this transformation is given in exercise 11.20.

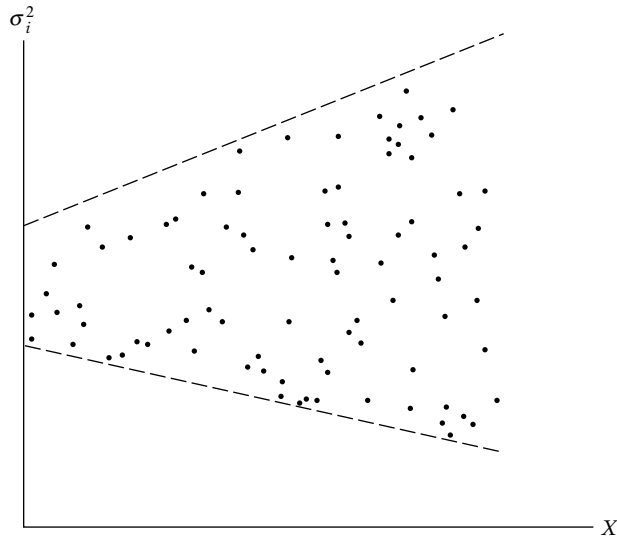
**Assumption 2:** The error variance is proportional to  $X_i$ . The **square root transformation:**

$$E(u_i^2) = \sigma^2 X_i \quad (11.6.7)$$

If it is believed that the variance of  $u_i$ , instead of being proportional to the squared  $X_i$ , is proportional to  $X_i$  itself, then the original model can be transformed as follows (see Figure 11.11):

$$\begin{aligned} \frac{Y_i}{\sqrt{X_i}} &= \frac{\beta_1}{\sqrt{X_i}} + \beta_2 \sqrt{X_i} + \frac{u_i}{\sqrt{X_i}} \\ &= \beta_1 \frac{1}{\sqrt{X_i}} + \beta_2 \sqrt{X_i} + v_i \end{aligned} \quad (11.6.8)$$

where  $v_i = u_i/\sqrt{X_i}$  and where  $X_i > 0$ .



**FIGURE 11.11** Error variance proportional to  $X$ .

Given assumption 2, one can readily verify that  $E(v_i^2) = \sigma^2$ , a homoscedastic situation. Therefore, one may proceed to apply OLS to (11.6.8), regressing  $Y_i/\sqrt{X_i}$  on  $1/\sqrt{X_i}$  and  $\sqrt{X_i}$ .

Note an important feature of the transformed model: It has no intercept term. Therefore, one will have to use the regression-through-the-origin model to estimate  $\beta_1$  and  $\beta_2$ . Having run (11.6.8), one can get back to the original model simply by multiplying (11.6.8) by  $\sqrt{X_i}$ .

**Assumption 3:** The error variance is proportional to the square of the mean value of  $Y$ .

$$E(u_i^2) = \sigma^2[E(Y_i)]^2 \quad (11.6.9)$$

Equation (11.6.9) postulates that the variance of  $u_i$  is proportional to the square of the expected value of  $Y$  (see Figure 11.8e). Now

$$E(Y_i) = \beta_1 + \beta_2 X_i$$

Therefore, if we transform the original equation as follows,

$$\begin{aligned} \frac{Y_i}{E(Y_i)} &= \frac{\beta_1}{E(Y_i)} + \beta_2 \frac{X_i}{E(Y_i)} + \frac{u_i}{E(Y_i)} \\ &= \beta_1 \left( \frac{1}{E(Y_i)} \right) + \beta_2 \frac{X_i}{E(Y_i)} + v_i \end{aligned} \quad (11.6.10)$$

where  $v_i = u_i/E(Y_i)$ , it can be seen that  $E(v_i^2) = \sigma^2$ ; that is, the disturbances  $v_i$  are homoscedastic. Hence, it is regression (11.6.10) that will satisfy the homoscedasticity assumption of the classical linear regression model.

The transformation (11.6.10) is, however, inoperational because  $E(Y_i)$  depends on  $\beta_1$  and  $\beta_2$ , which are unknown. Of course, we know  $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$ , which is an estimator of  $E(Y_i)$ . Therefore, we may proceed in two steps: First, we run the usual OLS regression, disregarding the heteroscedasticity problem, and obtain  $\hat{Y}_i$ . Then, using the estimated  $\hat{Y}_i$ , we transform our model as follows:

$$\frac{Y_i}{\hat{Y}_i} = \beta_1 \left( \frac{1}{\hat{Y}_i} \right) + \beta_2 \left( \frac{X_i}{\hat{Y}_i} \right) + v_i \quad (11.6.11)$$

where  $v_i = (u_i/\hat{Y}_i)$ . In Step 2, we run the regression (11.6.11). Although  $\hat{Y}_i$  are not exactly  $E(Y_i)$ , they are consistent estimators; that is, as the sample size increases indefinitely, they converge to true  $E(Y_i)$ . Hence, the transformation (11.6.11) will perform satisfactorily in practice if the sample size is reasonably large.

**Assumption 4:** A log transformation such as

$$\ln Y_i = \beta_1 + \beta_2 \ln X_i + u_i \quad (11.6.12)$$

very often reduces heteroscedasticity when compared with the regression  $Y_i = \beta_1 + \beta_2 X_i + u_i$ .

This result arises because log transformation compresses the scales in which the variables are measured, thereby reducing a tenfold difference between two values to a twofold difference. Thus, the number 80 is 10 times the number 8, but  $\ln 80 (= 4.3280)$  is about twice as large as  $\ln 8 (= 2.0794)$ .

An additional advantage of the log transformation is that the slope coefficient  $\beta_2$  measures the elasticity of  $Y$  with respect to  $X$ , that is, the percentage change in  $Y$  for a percentage change in  $X$ . For example, if  $Y$  is consumption and  $X$  is income,  $\beta_2$  in (11.6.12) will measure income elasticity, whereas in the original model  $\beta_2$  measures only the rate of change of mean consumption for a unit change in income. It is one reason why the log models are quite popular in empirical econometrics. (For some of the problems associated with log transformation, see exercise 11.4.)

To conclude our discussion of the remedial measures, we reemphasize that all the transformations discussed previously are ad hoc; we are essentially speculating about the nature of  $\sigma_i^2$ . Which of the transformations discussed previously will work will depend on the nature of the problem and the severity of heteroscedasticity. There are some additional problems with the transformations we have considered that should be borne

in mind:

1. When we go beyond the two-variable model, we may not know a priori which of the  $X$  variables should be chosen for transforming the data.<sup>37</sup>
2. Log transformation as discussed in Assumption 4 is not applicable if some of the  $Y$  and  $X$  values are zero or negative.<sup>38</sup>
3. Then there is the problem of **spurious correlation**. This term, due to Karl Pearson, refers to the situation where correlation is found to be present between the ratios of variables even though the original variables are uncorrelated or random.<sup>39</sup> Thus, in the model  $Y_i = \beta_1 + \beta_2 X_i + u_i$ ,  $Y$  and  $X$  may not be correlated but in the transformed model  $Y_i/X_i = \beta_1(1/X_i) + \beta_2$ ,  $Y_i/X_i$  and  $1/X_i$  are often found to be correlated.
4. When  $\sigma_i^2$  are not directly known and are estimated from one or more of the transformations that we have discussed earlier, all our testing procedures using the  $t$  tests,  $F$  tests, etc., are *strictly speaking valid only in large samples*. Therefore, one has to be careful in interpreting the results based on the various transformations in small or finite samples.<sup>40</sup>

## 11.7 CONCLUDING EXAMPLES

In concluding our discussion of heteroscedasticity we present two examples illustrating the main points made in this chapter.

### EXAMPLE 11.9

#### CHILD MORTALITY REVISITED

Let us return to the child mortality example we have considered on several occasions. From data for 64 countries, we obtained the regression results shown in Eq. (8.2.1). Since the data are cross sectional, involving diverse countries with different child mortality experiences, it is likely that we might encounter heteroscedasticity. To find this out, let us first consider the residuals obtained from Eq. (8.2.1). These residuals are plotted in Figure 11.12. From this figure it seems that the residuals do not show any distinct pattern that might suggest heteroscedasticity. Nonetheless, appearances can be deceptive. So, let us apply the Park, Glejser, and White tests to see if there is any evidence of heteroscedasticity.

#### Park Test

Since there are two regressors, GNP and FLR, we can regress the squared residuals from regression (8.2.1) on either of these variables. Or, we can regress them on the estimated CM values ( $= \widehat{CM}$ ) from regression (8.2.1). Using the latter, we obtained the following results.

$$\begin{aligned} \widehat{u}_i^2 &= 854.4006 + 5.7016 \widehat{CM}_i & (11.7.1) \\ t &= (1.2010) \quad (1.2428) & r^2 = 0.024 \end{aligned}$$

*Note:*  $\widehat{u}_i$  are the residuals obtained from regression (8.2.1) and  $\widehat{CM}$  are the estimated values of CM from regression (8.2.1).

As this regression shows, there is no systematic relation between the squared residuals and the estimated CM values (why?), suggesting that the assumption of

(Continued)

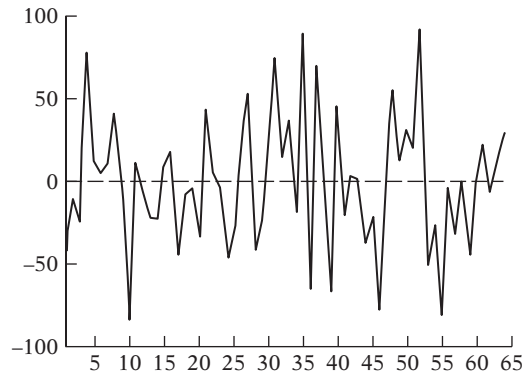
<sup>37</sup>However, as a practical matter, one may plot  $\widehat{u}_i^2$  against each variable and decide which  $X$  variable may be used for transforming the data. (See Fig. 11.9.)

<sup>38</sup>Sometimes we can use  $\ln(Y_i + k)$  or  $\ln(X_i + k)$ , where  $k$  is a positive number chosen in such a way that all the values of  $Y$  and  $X$  become positive.

<sup>39</sup>For example, if  $X_1$ ,  $X_2$ , and  $X_3$  are mutually uncorrelated  $r_{12} = r_{13} = r_{23} = 0$  and we find that the (values of the) ratios  $X_1/X_3$  and  $X_2/X_3$  are correlated, then there is spurious correlation. "More generally, correlation may be described as spurious if it is induced by the method of handling the data and is not present in the original material." M. G. Kendall and W. R. Buckland, *A Dictionary of Statistical Terms*, Hafner Publishing, New York, 1972, p. 143.

<sup>40</sup>For further details, see George G. Judge et al., op. cit., Sec. 14.4, pp. 415–420.

**EXAMPLE 11.9** (Continued)



**FIGURE 11.12** Residuals from regression (8.2.1).

homoscedasticity may be valid. Incidentally, regressing the log of the squared residual values on the log of CM did not change the conclusion.

**Glejser Test**

The absolute values of the residual obtained from (8.2.1), when regressed on the estimated CM value from the same regression, gave the following results:

$$|\hat{u}_i| = 22.3127 + 0.0646 \hat{CM}_i \quad (11.7.2)$$

$$t = (2.8086) \quad (1.2622) \quad r^2 = 0.0250$$

Again, there is not much systematic relationship between the absolute values of the residuals and the

estimated CM values, as the  $t$  value of the slope coefficient is not statistically significant.

**White Test**

Applying White's heteroscedasticity test with and without cross-product terms, we did not find any evidence of heteroscedasticity. We also reestimated (8.2.1) to obtain White's heteroscedasticity-consistent standard errors and  $t$  values, but the results were quite similar to those given in Eq. (8.2.1), which should not be surprising in view of the various heteroscedasticity tests we conducted earlier.

In sum, it seems that our child mortality regression (8.2.1) does not suffer from heteroscedasticity.

**EXAMPLE 11.10**

**R&D EXPENDITURE, SALES, AND PROFITS IN 18 INDUSTRY GROUPINGS IN THE UNITED STATES, 1988**

Table 11.5 gives data on research and development (R&D) expenditure, sales, and profits for 18 industry groupings in the United States, all figures in millions of dollars. Since the cross-sectional data presented in this table are quite heterogeneous, in a regression of R&D on sales (or profits), heteroscedasticity is likely. The regression results were as follows:

$$\widehat{R\&D}_i = 192.9931 + 0.0319 \text{Sales}_i \quad (11.7.3)$$

$$se = (533.9317) \quad (0.0083)$$

$$t = (0.3614) \quad (3.8433) \quad r^2 = 0.4783$$

Unsurprisingly, there is a significant positive relationship between R&D and sale.

To see if the regression (11.7.3) suffers from heteroscedasticity, we obtained the residuals,  $\hat{u}_i$ , and the squared residuals,  $\hat{u}_i^2$ , from the preceding regression and plotted them against sales, as shown in Figure 11.13. It seems from this figure that there is a systematic

(Continued)

**EXAMPLE 11.10** (Continued)**TABLE 11.5**INNOVATION IN AMERICA: RESEARCH AND DEVELOPMENT (R&D) EXPENDITURE  
IN THE UNITED STATES, 1988 (All Figures in Millions of Dollars)

Industry grouping	Sales	R&D expenses	Profits
1. Containers and packaging	6,375.3	62.5	185.1
2. Nonbank financial	11,626.4	92.9	1,569.5
3. Service industries	14,655.1	178.3	276.8
4. Metals and mining	21,869.2	258.4	2,828.1
5. Housing and construction	26,408.3	494.7	225.9
6. General manufacturing	32,405.6	1,083.0	3,751.9
7. Leisure time industries	35,107.7	1,620.6	2,884.1
8. Paper and forest products	40,295.4	421.7	4,645.7
9. Food	70,761.6	509.2	5,036.4
10. Health care	80,552.8	6,620.1	13,869.9
11. Aerospace	95,294.0	3,918.6	4,487.8
12. Consumer products	101,314.1	1,595.3	10,278.9
13. Electrical and electronics	116,141.3	6,107.5	8,787.3
14. Chemicals	122,315.7	4,454.1	16,438.8
15. Conglomerates	141,649.9	3,163.8	9,761.4
16. Office equipment and computers	175,025.8	13,210.7	19,774.5
17. Fuel	230,614.5	1,703.8	22,626.6
18. Automotive	293,543.0	9,528.2	18,415.4

Source: *Business Week*, Special 1989 Bonus Issue, R&D Scorecard, pp. 180–224.

Note: The industries are listed in increasing order of sales volume.

pattern between the residuals and squared residuals and sales, perhaps suggesting that there is heteroscedasticity. To test this formally, we used the Park, Glejser, and White tests, which gave the following results:

**Park Test**

$$\begin{aligned} |\hat{u}_i^2| &= -974,469.1 + 86.2321 \text{ Sales}_i \\ \text{se} &= (4,802,343) \quad (40.3625) \quad r^2 = 0.2219 \quad (11.7.4) \\ t &= \quad (-0.2029) \quad (2.1364) \end{aligned}$$

The Park test suggests that there is a statistically significant positive relationship between squared residuals and sales.

**Glejser Test**

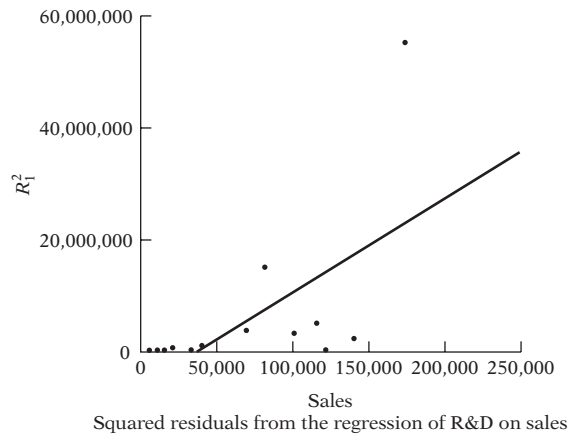
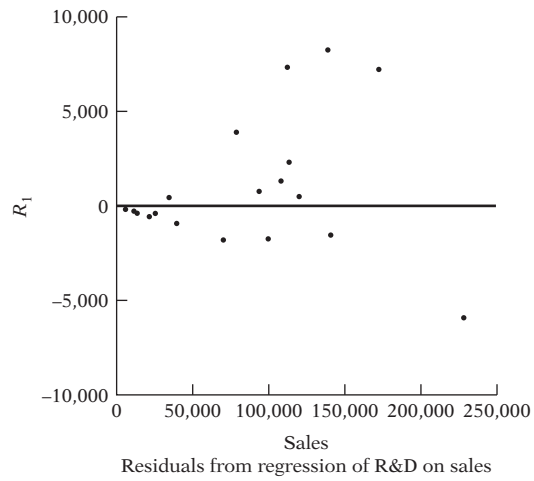
$$\begin{aligned} |\hat{u}_i| &= 578.5710 + 0.0119 \text{ Sales}_i \\ \text{se} &= (678.6950) \quad (0.0057) \quad r^2 = 0.214 \quad (11.7.5) \\ t &= \quad (0.8524) \quad (2.0877) \end{aligned}$$

The Glejser test also suggests that there is a systematic relationship between the absolute values of the residuals and sales, raising the possibility that the regression (11.7.3) suffers from heteroscedasticity.

(Continued)



**EXAMPLE 11.10** (Continued)



**FIGURE 11.13**  
Residuals  $R_1$  and squared residuals ( $R_1^2$ ) on sales.

**White Test**

$$\begin{aligned} \widehat{u}_i^2 &= -6,219,665 & + & 229.3508 \text{ Sales}_i & - & 0.000537 \text{ Sales}_i^2 \\ \text{se} &= (6,459,809) & & (126.2197) & & (0.0004) \\ t &= & (0.9628) & (1.8170) & & (-1.3425) \end{aligned} \tag{11.7.6}$$

$$R^2 = 0.2895$$

Using the  $R^2$  value and  $n = 18$ , we obtain  $nR^2 = 5.2124$ , which, under the null hypothesis of no heteroscedasticity, has a chi-square distribution with 2 df [because there are two regressors in (11.7.6)]. The  $p$  value of obtaining a chi-square value of as much as 5.2124 or greater

(Continued)

**EXAMPLE 11.10** (Continued)

is about 0.074. If this  $p$  value is deemed sufficiently low, the White test also suggests that there is heteroscedasticity.

In sum, then, on the basis of the residual graphs and the Park, Glejser, and White tests, it seems that our R&D regression (11.7.3) suffers from heteroscedasticity. Since the true error variance is unknown, we cannot use the method of weighted least squares to obtain heteroscedasticity-corrected standard errors and  $t$  values. Therefore, we will have to make some educated guesses about the nature of the error variance.

Looking at the residual graphs given in Figure 11.13, it seems that the error variance is proportional to sales as in Eq. (11.6.7), that is, the *square root transformation*. Effecting this transformation, we obtain the following results.

$$\frac{\widehat{\text{R\&D}}}{\sqrt{\text{Sales}}} = -246.6769 \frac{1}{\sqrt{\text{Sales}_i}} + 0.0367 \sqrt{\text{Sales}_i}$$

$$\begin{array}{lll} \text{se} = & (381.1285) & (0.0071) \quad R^2 = 0.3648 \quad (11.7.7) \\ t = & (-0.6472) & (5.1690) \end{array}$$

If you want, you can multiply the preceding equation by  $\sqrt{\text{Sales}_i}$  to get back to the original model. Comparing (11.7.7) with (11.7.3), you can see that the slope coefficients in the two equations are about the same, but their standard errors are different. In (11.7.3) it was 0.0083, whereas in (11.7.7) it is only 0.0071, a decrease of about 14 percent.

To conclude our example, we present below White's heteroscedasticity-consistent standard errors, as discussed in Section 11.6.

$$\widehat{\text{R\&D}}_i = 192.9931 + 0.0319 \text{ Sales}_i$$

$$\begin{array}{lll} \text{se} = & (533.9931) & (0.0101) \quad r^2 = 0.4783 \quad (11.7.8) \\ t = & (0.3614) & (3.1584) \end{array}$$

Comparing with the original (i.e., without correction for heteroscedasticity) regression (11.7.3), we see that although the parameter estimates have not changed (as we would expect), the standard error of the intercept coefficient has decreased and that of the slope coefficient has slightly increased. But remember that the White procedure is strictly a large-sample procedure, whereas we only have 18 observations.

## 11.8 A CAUTION ABOUT OVERREACTING TO HETEROSCEDASTICITY

Reverting to the R&D example discussed in the previous section, we saw that when we used the square root transformation to correct for heteroscedasticity in the original model (11.7.3), the standard error of the slope coefficient decreased and its  $t$  value increased. Is this change so significant that one should worry about it in practice? To put the matter differently, when should we really worry about the heteroscedasticity problem? As one author contends, "heteroscedasticity has never been a reason to throw out an otherwise good model."<sup>41</sup>

<sup>41</sup>N. Gregory Mankiw, "A Quick Refresher Course in Macroeconomics," *Journal of Economic Literature*, vol. XXVIII, December 1990, p. 1648.

Here it may be useful to bear in mind the caution sounded by John Fox:

. . . unequal error variance is worth correcting only when the problem is severe.

The impact of nonconstant error variance on the efficiency of ordinary least-squares estimator and on the validity of least-squares inference depends on several factors, including the sample size, the degree of variation in the  $\sigma_i^2$ , the configuration of the  $X$  [i.e., regressor] values, and the relationship between the error variance and the  $X$ 's. It is therefore not possible to develop wholly general conclusions concerning the harm produced by heteroscedasticity.<sup>42</sup>

Returning to the model (11.3.1), we saw earlier that variance of the slope estimator,  $\text{var}(\hat{\beta}_2)$ , is given by the usual formula shown in (11.2.3). Under GLS the variance of the slope estimator,  $\text{var}(\hat{\beta}_2^*)$ , is given by (11.3.9). We know that the latter is more efficient than the former. But how large does the former (i.e., OLS) variance have to be in relation to the GLS variance before one should really worry about it? As a rule of thumb, Fox suggests that we worry about this problem “. . . when the largest error variance is more than about 10 times the smallest.”<sup>43</sup> Thus, returning to the Monte Carlo simulations results of Davidson and MacKinnon presented earlier, consider the value of  $\alpha = 2$ . The variance of the estimated  $\beta_2$  is 0.04 under OLS and 0.012 under GLS, the ratio of the former to the latter thus being about 3.33.<sup>44</sup> According to the Fox rule, the severity of heteroscedasticity in this case may not be large enough to worry about it.

Also remember that, despite heteroscedasticity, OLS estimators are linear unbiased and are (under general conditions) asymptotically (i.e., in large samples) normally distributed.

As we will see when we discuss other violations of the assumptions of the classical linear regression model, the caution sounded in this section is appropriate as a general rule. Otherwise, one can go overboard.

## 11.9 SUMMARY AND CONCLUSIONS

**1.** A critical assumption of the classical linear regression model is that the disturbances  $u_i$  have all the same variance,  $\sigma^2$ . If this assumption is not satisfied, there is heteroscedasticity.

**2.** Heteroscedasticity does not destroy the unbiasedness and consistency properties of OLS estimators.

**3.** But these estimators are no longer minimum variance or efficient. That is, they are not BLUE.

**4.** The BLUE estimators are provided by the method of weighted least squares, provided the heteroscedastic error variances,  $\sigma_i^2$ , are known.

**5.** In the presence of heteroscedasticity, the variances of OLS estimators are not provided by the usual OLS formulas. But if we persist in using the

<sup>42</sup>John Fox, *Applied Regression Analysis, Linear Models, and Related Methods*, Sage Publications, California, 1997, p. 306.

<sup>43</sup>*Ibid.*, p. 307.

<sup>44</sup>Note that we have squared the standard errors to obtain the variances.

usual OLS formulas, the  $t$  and  $F$  tests based on them can be highly misleading, resulting in erroneous conclusions.

6. Documenting the consequences of heteroscedasticity is easier than detecting it. There are several diagnostic tests available, but one cannot tell for sure which will work in a given situation.

7. Even if heteroscedasticity is suspected and detected, it is not easy to correct the problem. If the sample is large, one can obtain White's heteroscedasticity corrected standard errors of OLS estimators and conduct statistical inference based on these standard errors.

8. Otherwise, on the basis of OLS residuals, one can make educated guesses of the likely pattern of heteroscedasticity and transform the original data in such a way that in the transformed data there is no heteroscedasticity.

## EXERCISES

### Questions

- 11.1. State *with brief reason* whether the following statements are true, false, or uncertain:
- In the presence of heteroscedasticity OLS estimators are biased as well as inefficient.
  - If heteroscedasticity is present, the conventional  $t$  and  $F$  tests are invalid.
  - In the presence of heteroscedasticity the usual OLS method always overestimates the standard errors of estimators.
  - If residuals estimated from an OLS regression exhibit a systematic pattern, it means heteroscedasticity is present in the data.
  - There is no general test of heteroscedasticity that is free of any assumption about which variable the error term is correlated with.
  - If a regression model is mis-specified (e.g., an important variable is omitted), the OLS residuals will show a distinct pattern.
  - If a regressor that has nonconstant variance is (incorrectly) omitted from a model, the (OLS) residuals will be heteroscedastic.
- 11.2. In a regression of average wages ( $W$ , \$) on the number of employees ( $N$ ) for a random sample of 30 firms, the following regression results were obtained\*:

$$\widehat{W} = 7.5 + 0.009N \quad (1)$$

$$t = \text{n.a.} \quad (16.10) \quad R^2 = 0.90$$

$$\widehat{W}/N = 0.008 + 7.8(1/N) \quad (2)$$

$$t = (14.43) \quad (76.58) \quad R^2 = 0.99$$

- How do you interpret the two regressions?
- What is the author assuming in going from Eq. (1) to (2)? Was he worried about heteroscedasticity? How do you know?

\*See Dominick Salvatore, *Managerial Economics*, McGraw-Hill, New York, 1989, p. 157.

- c. Can you relate the slopes and intercepts of the two models?  
 d. Can you compare the  $R^2$  values of the two models? Why or why not?  
 11.3. a. Can you estimate the parameters of the models

$$|\hat{u}_i| = \sqrt{\beta_1 + \beta_2 X_i} + v_i$$

$$|\hat{u}_i| = \sqrt{\beta_1 + \beta_2 X_i^2} + v_i$$

by the method of ordinary least squares? Why or why not?

- b. If not, can you suggest a method, informal or formal, of estimating the parameters of such models? (See Chapter 14.)  
 11.4. Although log models as shown in Eq. (11.6.12) often reduce heteroscedasticity, one has to pay careful attention to the properties of the disturbance term of such models. For example, the model

$$Y_i = \beta_1 X_i^{\beta_2} u_i \quad (1)$$

can be written as

$$\ln Y_i = \ln \beta_1 + \beta_2 \ln X_i + \ln u_i \quad (2)$$

- a. If  $\ln u_i$  is to have zero expectation, what must be the distribution of  $u_i$ ?  
 b. If  $E(u_i) = 1$ , will  $E(\ln u_i) = 0$ ? Why or why not?  
 c. If  $E(\ln u_i)$  is not zero, what can be done to make it zero?  
 11.5. Show that  $\beta_2^*$  of (11.3.8) can also be expressed as

$$\beta_2^* = \frac{\sum w_i y_i^* x_i^*}{\sum w_i x_i^{2*}}$$

and  $\text{var}(\beta_2^*)$  given in (11.3.9) can also be expressed as

$$\text{var}(\beta_2^*) = \frac{1}{\sum w_i x_i^{2*}}$$

where  $y_i^* = Y_i - \bar{Y}^*$  and  $x_i^* = X_i - \bar{X}^*$  represent deviations from the weighted means  $\bar{Y}^*$  and  $\bar{X}^*$  defined as

$$\bar{Y}^* = \frac{\sum w_i Y_i}{\sum w_i}$$

$$\bar{X}^* = \frac{\sum w_i X_i}{\sum w_i}$$

- 11.6. For pedagogic purposes Hanushek and Jackson estimate the following model:

$$C_t = \beta_1 + \beta_2 \text{GNP}_t + \beta_3 D_t + u_t \quad (1)$$

where  $C_t$  = aggregate private consumption expenditure in year  $t$ ,  $\text{GNP}_t$  = gross national product in year  $t$ , and  $D$  = national defense expenditures in year  $t$ , the objective of the analysis being to study the effect of defense expenditures on other expenditures in the economy.

Postulating that  $\sigma_t^2 = \sigma^2(\text{GNP}_t)^2$ , they transform (1) and estimate

$$C_t/\text{GNP}_t = \beta_1 (1/\text{GNP}_t) + \beta_2 + \beta_3 (D_t/\text{GNP}_t) + u_t/\text{GNP}_t \quad (2)$$

The empirical results based on the data for 1946–1975 were as follows (standard errors in the parentheses)\*:

$$\begin{aligned} \hat{C}_t &= 26.19 && + 0.6248 \text{ GNP}_t - 0.4398 D_t \\ (2.73) &&& (0.0060) \quad (0.0736) \quad R^2 = 0.999 \\ \widehat{C_t/\text{GNP}_t} &= 25.92 (1/\text{GNP}_t) + 0.6246 && - 0.4315 (D_t/\text{GNP}_t) \\ (2.22) &&& (0.0068) \quad (0.0597) \quad R^2 = 0.875 \end{aligned}$$

- a. What assumption is made by the authors about the nature of heteroscedasticity? Can you justify it?
  - b. Compare the results of the two regressions. Has the transformation of the original model improved the results, that is, reduced the estimated standard errors? Why or why not?
  - c. Can you compare the two  $R^2$  values? Why or why not? (*Hint*: Examine the dependent variables.)
- 11.7. Refer to the estimated regressions (11.6.2) and (11.6.3). The regression results are quite similar. What could account for this outcome?
- 11.8. Prove that if  $w_i = w$  a constant, for each  $i$ ,  $\beta_2^*$  and  $\hat{\beta}_2$  as well as their variance are identical.
- 11.9. Refer to formulas (11.2.2) and (11.2.3). Assume

$$\sigma_i^2 = \sigma^2 k_i$$

where  $\sigma^2$  is a constant and where  $k_i$  are *known* weights, not necessarily all equal.

Using this assumption, show that the variance given in (11.2.2) can be expressed as

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_i^2} \cdot \frac{\sum x_i^2 k_i}{\sum x_i^2}$$

The first term on the right side is the variance formula given in (11.2.3), that is,  $\text{var}(\hat{\beta}_2)$  under homoscedasticity. What can you say about the nature of the relationship between  $\text{var}(\hat{\beta}_2)$  under heteroscedasticity and under homoscedasticity? (*Hint*: Examine the second term on the right side of the preceding formula.) Can you draw any general conclusions about the relationships between (11.2.2) and (11.2.3)?

- 11.10. In the model

$$Y_i = \beta_2 X_i + u_i \quad (\text{Note: there is no intercept})$$

\*Eric A. Hanushek and John E. Jackson, *Statistical Methods for Social Scientists*, Academic, New York, 1977, p. 160.

you are told that  $\text{var}(u_i) = \sigma^2 X_i^2$ . Show that

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2 \sum X_i^4}{(\sum X_i^2)^2}$$

**Problems**

- 11.11.** For the data given in Table 11.1, regress average compensation  $Y$  on average productivity  $X$ , treating employment size as the unit of observation. Interpret your results, and see if your results agree with those given in (11.5.3).
- From the preceding regression obtain the residuals  $\hat{u}_i$ .
  - Following the Park test, regress  $\ln \hat{u}_i^2$  on  $\ln X_i$  and verify the regression (11.5.4).
  - Following the Glejser approach, regress  $|\hat{u}_i|$  on  $X_i$  and then regress  $|\hat{u}_i|$  on  $\sqrt{X_i}$  and comment on your results.
  - Find the rank correlation between  $|\hat{u}_i|$  and  $X_i$  and comment on the nature of heteroscedasticity, if any, present in the data.
- 11.12.** Table 11.6 gives data on the sales/cash ratio in U.S. manufacturing industries classified by the asset size of the establishment for the period 1971–I to 1973–IV. (The data are on a quarterly basis.) The sales/cash ratio may be regarded as a measure of income velocity in the corporate sector; that is, the number of times a dollar turns over.
- For each asset size compute the mean and standard deviation of the sales/cash ratio.
  - Plot the mean value against the standard deviation as computed in **a**, using asset size as the unit of observation.
  - By means of a suitable regression model decide whether standard deviation of the ratio increases with the mean value. If not, how would you rationalize the result?

**TABLE 11.6** ASSET SIZE (Millions of Dollars)

Year and quarter	1–10	10–25	25–50	50–100	100–250	250–1000	1000 +
1971–I	6.696	6.929	6.858	6.966	7.819	7.557	7.860
–II	6.826	7.311	7.299	7.081	7.907	7.685	7.351
–III	6.338	7.035	7.082	7.145	7.691	7.309	7.088
–IV	6.272	6.265	6.874	6.485	6.778	7.120	6.765
1972–I	6.692	6.236	7.101	7.060	7.104	7.584	6.717
–II	6.818	7.010	7.719	7.009	8.064	7.457	7.280
–III	6.783	6.934	7.182	6.923	7.784	7.142	6.619
–IV	6.779	6.988	6.531	7.146	7.279	6.928	6.919
1973–I	7.291	7.428	7.272	7.571	7.583	7.053	6.630
–II	7.766	9.071	7.818	8.692	8.608	7.571	6.805
–III	7.733	8.357	8.090	8.357	7.680	7.654	6.772
–IV	8.316	7.621	7.766	7.867	7.666	7.380	7.072

Source: Quarterly Financial Report for Manufacturing Corporations, Federal Trade Commission and the Securities and Exchange Commission, U.S. government, various issues (computed).

d. If there is a statistically significant relationship between the two, how would you transform the data so that there is no heteroscedasticity?

**11.13. Bartlett's homogeneity-of-variance test.\*** Suppose there are  $k$  independent sample variances  $s_1^2, s_2^2, \dots, s_k^2$  with  $f_1, f_2, \dots, f_k$  df, each from populations which are normally distributed with mean  $\mu$  and variance  $\sigma_i^2$ . Suppose further that we want to test the null hypothesis  $H_0: \sigma_1^2 = \sigma_2^2 = \dots = \sigma_k^2 = \sigma^2$ ; that is, each sample variance is an estimate of the same population variance  $\sigma^2$ .

If the null hypothesis is true, then

$$s^2 = \frac{\sum_{i=1}^k f_i s_i^2}{\sum f_i} = \frac{\sum f_i s_i^2}{f}$$

provides an estimate of the common (pooled) estimate of the population variance  $\sigma^2$ , where  $f_i = (n_i - 1)$ ,  $n_i$  being the number of observations in the  $i$ th group and where  $f = \sum_{i=1}^k f_i$ .

Bartlett has shown that the null hypothesis can be tested by the ratio  $A/B$ , which is approximately distributed as the  $\chi^2$  distribution with  $k - 1$  df, where

$$A = f \ln s^2 - \sum (f_i \ln s_i^2)$$

and

$$B = 1 + \frac{1}{3(k-1)} \left[ \sum \left( \frac{1}{f_i} \right) - \frac{1}{f} \right]$$

Apply Bartlett's test to the data of Table 11.1 and verify that the hypothesis that population variances of employee compensation are the same in each employment size of the establishment cannot be rejected at the 5 percent level of significance.

*Note:*  $f_i$ , the df for each sample variance, is 9, since  $n_i$  for each sample (i.e., employment class) is 10.

**11.14.** Consider the following regression-through-the origin model:

$$Y_i = \beta X_i + u_i, \quad \text{for } i = 1, 2$$

You are told that  $u_1 \sim N(0, \sigma^2)$  and  $u_2 \sim N(0, 2\sigma^2)$  and that they are statistically independent. If  $X_1 = +1$  and  $X_2 = -1$ , obtain the *weighted* least-squares (WLS) estimate of  $\beta$  and its variance. If in this situation you had assumed incorrectly that the two error variances are the same (say, equal to  $\sigma^2$ ), what would be the OLS estimator of  $\beta$ ? And its variance? Compare these estimates with the estimates obtained by the method of WLS? What general conclusion do you draw?<sup>†</sup>

**11.15.** Table 11.7 gives data on 81 cars about MPG (average miles per gallons), HP (engine horsepower), VOL (cubic feet of cab space), SP (top speed, miles per hour), and WT (vehicle weight in 100 lb).

\*See "Properties of Sufficiency and Statistical Tests," *Proceedings of the Royal Society of London A*, vol. 160, 1937, p. 268.

<sup>†</sup>Adapted from F. A. F. Seber, *Linear Regression Analysis*, John Wiley & Sons, New York, 1977, p. 64.



TABLE 11.7 PASSENGER CAR MILAGE DATA

Observation	MPG	SP	HP	VOL	WT	Observation	MPG	SP	HP	VOL	WT
1	65.4	96	49	89	17.5	42	32.2	106	95	106	30.0
2	56.0	97	55	92	20.0	43	32.2	109	102	92	30.0
3	55.9	97	55	92	20.0	44	32.2	106	95	88	30.0
4	49.0	105	70	92	20.0	45	31.5	105	93	102	30.0
5	46.5	96	53	92	20.0	46	31.5	108	100	99	30.0
6	46.2	105	70	89	20.0	47	31.4	108	100	111	30.0
7	45.4	97	55	92	20.0	48	31.4	107	98	103	30.0
8	59.2	98	62	50	22.5	49	31.2	120	130	86	30.0
9	53.3	98	62	50	22.5	50	33.7	109	115	101	35.0
10	43.4	107	80	94	22.5	51	32.6	109	115	101	35.0
11	41.1	103	73	89	22.5	52	31.3	109	115	101	35.0
12	40.9	113	92	50	22.5	53	31.3	109	115	124	35.0
13	40.9	113	92	99	22.5	54	30.4	133	180	113	35.0
14	40.4	103	73	89	22.5	55	28.9	125	160	113	35.0
15	39.6	100	66	89	22.5	56	28.0	115	130	124	35.0
16	39.3	103	73	89	22.5	57	28.0	102	96	92	35.0
17	38.9	106	78	91	22.5	58	28.0	109	115	101	35.0
18	38.8	113	92	50	22.5	59	28.0	104	100	94	35.0
19	38.2	106	78	91	22.5	60	28.0	105	100	115	35.0
20	42.2	109	90	103	25.0	61	27.7	120	145	111	35.0
21	40.9	110	92	99	25.0	62	25.6	107	120	116	40.0
22	40.7	101	74	107	25.0	63	25.3	114	140	131	40.0
23	40.0	111	95	101	25.0	64	23.9	114	140	123	40.0
24	39.3	105	81	96	25.0	65	23.6	117	150	121	40.0
25	38.8	111	95	89	25.0	66	23.6	122	165	50	40.0
26	38.4	110	92	50	25.0	67	23.6	122	165	114	40.0
27	38.4	110	92	117	25.0	68	23.6	122	165	127	40.0
28	38.4	110	92	99	25.0	69	23.6	122	165	123	40.0
29	46.9	90	52	104	27.5	70	23.5	148	245	112	40.0
30	36.3	112	103	107	27.5	71	23.4	160	280	50	40.0
31	36.1	103	84	114	27.5	72	23.4	121	162	135	40.0
32	36.1	103	84	101	27.5	73	23.1	121	162	132	40.0
33	35.4	111	102	97	27.5	74	22.9	110	140	160	45.0
34	35.3	111	102	113	27.5	75	22.9	110	140	129	45.0
35	35.1	102	81	101	27.5	76	19.5	121	175	129	45.0
36	35.1	106	90	98	27.5	77	18.1	165	322	50	45.0
37	35.0	106	90	88	27.5	78	17.2	140	238	115	45.0
38	33.2	109	102	86	30.0	79	17.0	147	263	50	45.0
39	32.9	109	102	86	30.0	80	16.7	157	295	119	45.0
40	32.3	120	130	92	30.0	81	13.2	130	236	107	55.0
41	32.2	106	95	113	30.0						

Note:

VOL = cubic feet of cab space  
 HP = engine horsepower  
 MPG = average miles per gallon  
 SP = top speed, miles per hour  
 WT = vehicle weight, hundreds of pounds  
 Observation = car observation number (Names of cars not disclosed)  
 Source: U.S. Environmental Protection Agency, 1991, Report EPA/AA/CTAB/91-02.

- a. Consider the following model:

$$\text{MPG}_i = \beta_1 + \beta_2\text{SP} + \beta_3\text{HP} + \beta_4\text{WT} + u_i$$

Estimate the parameters of this model and interpret the results. Do they make economic sense?

- b. Would you expect the error variance in the preceding model to be heteroscedastic? Why?
- c. Use the White test to find out if the error variance is heteroscedastic.
- d. Obtain White's heteroscedasticity-consistent standard errors and  $t$  values and compare your results with those obtained from OLS.
- e. If heteroscedasticity is established, how would you transform the data so that in the transformed data the error variance is homoscedastic? Show the necessary calculations.
- 11.16. Food expenditure in India.** In Table 2.8 we have given data on expenditure on food and total expenditure for 55 families in India.
- a. Regress expenditure on food on total expenditure, and examine the residuals obtained from this regression.
- b. Plot the residuals obtained in **a** against total expenditure and see if you observe any systematic pattern.
- c. If the plot in **b** suggests that there is heteroscedasticity, apply the Park, Glejser, and White tests to find out if the impression of heteroscedasticity observed in **b** is supported by these tests.
- d. Obtain White's heteroscedasticity-consistent standard errors and compare those with the OLS standard errors. Decide if it is worth correcting for heteroscedasticity in this example.
- 11.17.** Repeat exercise 11.16, but this time regress the logarithm of expenditure on food on the logarithm of total expenditure. If you observe heteroscedasticity in the linear model of exercise 11.16 but not in the log-linear model, what conclusion do you draw? Show all the necessary calculations.
- 11.18. A shortcut to White's test.** As noted in the text, the White test can consume degrees of freedom if there are several regressors and if we introduce all the regressors, their squared terms, and their cross products. Therefore, instead of estimating regressions like (11.5.22), why not simply run the following regression:

$$\hat{u}_i^2 = \alpha_1 + \alpha_2\hat{Y}_i + \alpha_3\hat{Y}_i^2 + v_i$$

where  $\hat{Y}_i$  are the estimated  $Y$  (i.e., regressand) values from whatever model you are estimating? After all,  $\hat{Y}_i$  is simply the weighted average of the regressors, with the estimated regression coefficients serving as the weights.

Obtain the  $R^2$  value from the preceding regression and use (11.5.22) to test the hypothesis that there is no heteroscedasticity.

Apply the preceding test to the food expenditure example of exercise 11.16.

- 11.19.** Return to the R&D example discussed in Section 11.7. Repeat the example using profits as the regressor. A priori, would you expect your

**TABLE 11.8** MEDIAN SALARIES OF FULL PROFESSORS IN  
STATISTICS, 2000–2001

Years in rank	Count	Median
0 to 1	11	\$69,000
2 to 3	20	\$70,500
4 to 5	26	\$74,050
6 to 7	33	\$82,600
8 to 9	18	\$91,439
10 to 11	26	\$83,127
12 to 13	31	\$84,700
14 to 15	15	\$82,601
16 to 17	22	\$93,286
18 to 19	23	\$90,400
20 to 21	13	\$98,200
22 to 24	29	\$100,000
25 to 27	22	\$99,662
28 to 32	22	\$116,012
33 or more	11	\$85,200

Source: American Statistical Association, "2000–2001 Salary Report of Academic Statisticians," *Amstat News*, Issue 282, December 2000, p. 4.

results to be different from those using sales as the regressor? Why or why not?

**11.20.** Table 11.8 gives data on median salaries of full professors in statistics in research universities in the United States for the academic year 2000–2001.

**a.** Plot median salaries against years in rank (as a measure of years of experience). For the plotting purposes, assume that the median salaries refer to the midpoint of years in rank. Thus, the salary \$74,050 in the range 4–5 refers to 4.5 years in the rank, and so on. For the last group, assume that the range is 33–35.

**b.** Consider the following regression models:

$$Y_i = \alpha_1 + \alpha_2 X_i + u_i \quad (1)$$

$$Y_i = \beta_1 + \beta_2 X_i + \beta_3 X_i^2 + v_i \quad (2)$$

where  $Y$  = median salary,  $X$  = year in rank (measured at midpoint of the range), and  $u$  and  $v$  are the error terms. Can you argue why model (2) might be preferable to model (1)? From the data given, estimate both the models.

**c.** If you observe heteroscedasticity in model (1) but not in model (2), what conclusion would you draw? Show the necessary computations.

**d.** If heteroscedasticity is observed in model (2), how would you transform the data so that in the transformed model there is no heteroscedasticity?

**11.21.** You are given the following data:

$$RSS_1 \text{ based on the first 30 observations} = 55, df = 25$$

$$RSS_2 \text{ based on the last 30 observations} = 140, df = 25$$

Carry out the Goldfeld–Quandt test of heteroscedasticity at the 5 percent level of significance.

**11.22.** Table 11.9 gives data on percent change per year for stock prices ( $Y$ ) and consumer prices ( $X$ ) for a cross section of 20 countries.

- a. Plot the data in a scattergram.
- b. Regress  $Y$  on  $X$  and examine the residuals from this regression. What do you observe?
- c. Since the data for Chile seem atypical (outlier?), repeat the regression in **b**, dropping the data on Chile. Now examine the residuals from this regression. What do you observe?
- d. If on the basis of the results in **b** you conclude that there was heteroscedasticity in error variance but on the basis of the results in **c** you reverse your conclusion, what general conclusions do you draw?

**TABLE 11.9** STOCK AND CONSUMER PRICES, POST-WORLD WAR II PERIOD (Through 1969)

Country	Rate of change, % per year	
	Stock prices, $Y$	Consumer prices, $X$
1. Australia	5.0	4.3
2. Austria	11.1	4.6
3. Belgium	3.2	2.4
4. Canada	7.9	2.4
5. Chile	25.5	26.4
6. Denmark	3.8	4.2
7. Finland	11.1	5.5
8. France	9.9	4.7
9. Germany	13.3	2.2
10. India	1.5	4.0
11. Ireland	6.4	4.0
12. Israel	8.9	8.4
13. Italy	8.1	3.3
14. Japan	13.5	4.7
15. Mexico	4.7	5.2
16. Netherlands	7.5	3.6
17. New Zealand	4.7	3.6
18. Sweden	8.0	4.0
19. United Kingdom	7.5	3.9
20. United States	9.0	2.1

Source: Phillip Cagan, *Common Stock Values and Inflation: The Historical Record of Many Countries*, National Bureau of Economic Research, Suppl., March 1974, Table 1, p. 4.

**APPENDIX 11A****11A.1 PROOF OF EQUATION (11.2.2)**

From Appendix 3A, Section 3A.3, we have

$$\begin{aligned}\text{var}(\hat{\beta}_2) &= E(k_1^2 u_1^2 + k_2^2 u_2^2 + \cdots + k_n^2 u_n^2 + 2 \text{ cross-product terms}) \\ &= E(k_1^2 u_1^2 + k_2^2 u_2^2 + \cdots + k_n^2 u_n^2)\end{aligned}$$

since the expectations of the cross-product terms are zero because of the assumption of no serial correlation,

$$\text{var}(\hat{\beta}_2) = k_1^2 E(u_1^2) + k_2^2 E(u_2^2) + \cdots + k_n^2 E(u_n^2)$$

since the  $k_i$  are known. (Why?)

$$\text{var}(\hat{\beta}_2) = k_1^2 \sigma_1^2 + k_2^2 \sigma_2^2 + \cdots + k_n^2 \sigma_n^2$$

since  $E(u_i^2) = \sigma_i^2$ .

$$\begin{aligned}\text{var}(\hat{\beta}_2) &= \sum k_i^2 \sigma_i^2 \\ &= \sum \left[ \left( \frac{x_i}{\sum x_i^2} \right)^2 \sigma_i^2 \right] \quad \text{since } k_i = \frac{x_i}{\sum x_i^2} \quad \text{(11.2.2)} \\ &= \frac{\sum x_i^2 \sigma_i^2}{(\sum x_i^2)^2}\end{aligned}$$

**11A.2 THE METHOD OF WEIGHTED LEAST SQUARES**

To illustrate the method, we use the two-variable model  $Y_i = \beta_1 + \beta_2 X_i + u_i$ . The unweighted least-squares method minimizes

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2 \quad (1)$$

to obtain the estimates, whereas the weighted least-squares method minimizes the weighted residual sum of squares:

$$\sum w_i \hat{u}_i^2 = \sum w_i (Y_i - \hat{\beta}_1^* - \hat{\beta}_2^* X_i)^2 \quad (2)$$

where  $\hat{\beta}_1^*$  and  $\hat{\beta}_2^*$  are the weighted least-squares estimators and where the weights  $w_i$  are such that

$$w_i = \frac{1}{\sigma_i^2} \quad (3)$$

that is, the weights are inversely proportional to the variance of  $u_i$  or  $Y_i$  conditional upon the given  $X_i$ , it being understood that  $\text{var}(u_i | X_i) = \text{var}(Y_i | X_i) = \sigma_i^2$ .

Differentiating (2) with respect to  $\hat{\beta}_1^*$  and  $\hat{\beta}_2^*$ , we obtain

$$\frac{\partial \sum w_i \hat{u}_i^2}{\partial \hat{\beta}_1^*} = 2 \sum w_i (Y_i - \hat{\beta}_1^* - \hat{\beta}_2^* X_i)(-1)$$

$$\frac{\partial \sum w_i \hat{u}_i^2}{\partial \hat{\beta}_2^*} = 2 \sum w_i (Y_i - \hat{\beta}_1^* - \hat{\beta}_2^* X_i)(-X_i)$$

Setting the preceding expressions equal to zero, we obtain the following two normal equations:

$$\sum w_i Y_i = \hat{\beta}_1^* \sum w_i + \hat{\beta}_2^* \sum w_i X_i \quad (4)$$

$$\sum w_i X_i Y_i = \hat{\beta}_1^* \sum w_i X_i + \hat{\beta}_2^* \sum w_i X_i^2 \quad (5)$$

Notice the similarity between these normal equations and the normal equations of the unweighted least squares.

Solving these equations simultaneously, we obtain

$$\hat{\beta}_1^* = \bar{Y}^* - \hat{\beta}_2^* \bar{X}^* \quad (6)$$

and

$$\hat{\beta}_2^* = \frac{(\sum w_j)(\sum w_i X_i Y_i) - (\sum w_i X_i)(\sum w_i Y_i)}{(\sum w_i)(\sum w_i X_i^2) - (\sum w_i X_i)^2} \quad (11.3.8) = (7)$$

The variance of  $\hat{\beta}_2^*$  shown in (11.3.9) can be obtained in the manner of the variance of  $\hat{\beta}_2$  shown in Appendix 3A, Section 3A.3.

*Note:*  $\bar{Y}^* = \sum w_i Y_i / \sum w_i$  and  $\bar{X}^* = \sum w_i X_i / \sum w_i$ . As can be readily verified, these weighted means coincide with the usual or unweighted means  $\bar{Y}$  and  $\bar{X}$  when  $w_i = w$ , a constant, for all  $i$ .

### 11A.3 PROOF THAT $E(\hat{\sigma}^2) \neq \sigma^2$ IN THE PRESENCE OF HETEROSCEDASTICITY

Consider the two-variable model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (1)$$

where  $\text{var}(u_i) = \sigma_i^2$

Now

$$\begin{aligned} \hat{\sigma}^2 &= \frac{\sum \hat{u}_i^2}{n-2} = \frac{\sum (Y_i - \hat{Y}_i)^2}{n-2} = \frac{\sum [\beta_1 + \beta_2 X_i + u_i - \hat{\beta}_1 - \hat{\beta}_2 X_i]^2}{n-2} \\ &= \frac{\sum [-(\hat{\beta}_1 - \beta_1) - (\hat{\beta}_2 - \beta_2) X_i + u_i]^2}{n-2} \end{aligned} \quad (2)$$

Noting that  $(\hat{\beta}_1 - \beta_1) = -(\hat{\beta}_2 - \beta_2)\bar{X} + \bar{u}$ , and substituting this into (2) and taking expectations on both sides, we get:

$$\begin{aligned} E(\hat{\sigma}^2) &= \frac{1}{n-2} \left\{ -\sum x_i^2 \text{var}(\hat{\beta}_2) + E\left[\sum (u_i - \bar{u})^2\right] \right\} \\ &= \frac{1}{n-2} \left[ -\frac{\sum x_i^2 \sigma_i^2}{\sum x_i^2} + \frac{(n-1)\sum \sigma_i^2}{n} \right] \end{aligned} \quad (3)$$

where use is made of (11.2.2).

As you can see from (3), if there is homoscedasticity, that is,  $\sigma_i^2 = \sigma^2$  for each  $i$ ,  $E(\hat{\sigma}^2) = \sigma^2$ . Therefore, the expected value of the conventionally computed  $\hat{\sigma}^2 = \sum \hat{u}_i^2 / (n-2)$  will not be equal to the true  $\sigma^2$  in the presence of heteroscedasticity.<sup>1</sup>

#### 11A.4 WHITE'S ROBUST STANDARD ERRORS

To give you some idea about White's heteroscedasticity-corrected standard errors, consider the two-variable regression model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad \text{var}(u_i) = \sigma_i^2. \quad (1)$$

As shown in (11.2.2),

$$\text{var}(\hat{\beta}_2) = \frac{\sum x_i^2 \sigma_i^2}{(\sum x_i^2)^2} \quad (2)$$

Since  $\sigma_i^2$  are not directly observable, White suggests using  $\hat{u}_i^2$ , the squared residual for each  $i$ , in place of  $\sigma_i^2$  and estimate the  $\text{var}(\hat{\beta}_2)$  as follows:

$$\text{var}(\hat{\beta}_2) = \frac{\sum x_i^2 \hat{u}_i^2}{(\sum x_i^2)^2} \quad (3)$$

White has shown that (3) is a consistent estimator of (2), that is, as the sample size increases indefinitely, (3) converges to (2).<sup>2</sup>

Incidentally, note that if your software package does not contain White's robust standard error procedure, you can do it as shown in (3) by first running the usual OLS regression, obtaining the residuals from this regression and then using formula (3).

White's procedure can be generalized to the  $k$ -variable regression model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i \quad (4)$$

<sup>1</sup>Further details can be obtained from Jan Kmenta, *Elements of Econometrics*, 2d. ed., Macmillan, New York, 1986, pp. 276–278.

<sup>2</sup>To be more precise,  $n$  times (3) converges in probability to  $E[(X_i - \mu_X)^2 u_i^2] / (\sigma_X^2)^2$ , which is the probability limit of  $n$  times (2), where  $n$  is the sample size,  $\mu_X$  is the expected value of  $X$ , and  $\sigma_X^2$  is the (population) variance of  $X$ . For more details, see Jeffrey M. Wooldridge, *Introductory Econometrics; A Modern Approach*, South-Western Publishing, 2000, p. 250.

The variance of any partial regression coefficient, say  $\hat{\beta}_j$ , is obtained as follows:

$$\text{var}(\hat{\beta}_j) = \frac{\sum \hat{w}_{ji}^2 \hat{u}_i^2}{(\sum \hat{w}_{ji}^2)^2} \quad (5)$$

where  $\hat{u}_i$  are the residuals obtained from the (original) regression (4) and  $\hat{w}_{ji}$  are the residuals obtained from the (auxiliary) regression of the regressor  $X_j$  on the remaining regressors in (4).

Obviously, this is a time-consuming procedure, for you will have to estimate (5) for each  $X$  variable. Of course, all this labor can be avoided if you have a statistical package that does this routinely. Packages such as PcGive, Eviews, Microfit, Shazam, Stata, and Limdep now obtain White's heteroscedasticity-robust standard errors very easily.



# 12

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## AUTOCORRELATION: WHAT HAPPENS IF THE ERROR TERMS ARE CORRELATED?

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The reader may recall that there are generally three types of data that are available for empirical analysis: (1) cross section, (2) time series, and (3) combination of cross section and time series, also known as pooled data. In developing the classical linear regression model (CLRM) in **Part I** we made several assumptions, which were discussed in Section 7.1. However, we noted that *not* all these assumptions would hold in every type of data. As a matter of fact, we saw in the previous chapter that the assumption of homoscedasticity, or equal error variance, may not be always tenable in cross-sectional data. In other words, cross-sectional data are often plagued by the problem of heteroscedasticity.

However, in cross-section studies, data are often collected on the basis of a random sample of cross-sectional units, such as households (in a consumption function analysis) or firms (in an investment study analysis) so that there is no prior reason to believe that the error term pertaining to one household or a firm is correlated with the error term of another household or firm. If by chance such a correlation is observed in cross-sectional units, it is called **spatial autocorrelation**, that is, correlation in space rather than over time. However, it is important to remember that, in cross-sectional analysis, the ordering of the data must have some logic, or economic interest, to make sense of any determination of whether (spatial) autocorrelation is present or not.

The situation, however, is likely to be very different if we are dealing with time series data, for the observations in such data follow a natural ordering over time so that successive observations are likely to exhibit intercorrelations, especially if the time interval between successive observations is

short, such as a day, a week, or a month rather than a year. If you observe stock price indexes, such as the Dow Jones or S&P 500 over successive days, it is not unusual to find that these indexes move up or down for several days in succession. Obviously, in situations like this, the assumption of **no auto, or serial, correlation** in the error terms that underlies the CLRM will be violated.

In this chapter we take a critical look at this assumption with a view to answering the following questions:

1. What is the nature of autocorrelation?
2. What are the theoretical and practical consequences of autocorrelation?
3. Since the assumption of no autocorrelation relates to the unobservable disturbances  $u_t$ , how does one know that there is autocorrelation in any given situation? Notice that we now use the subscript  $t$  to emphasize that we are dealing with time series data.
4. How does one remedy the problem of autocorrelation?

The reader will find this chapter in many ways similar to the preceding chapter on heteroscedasticity in that **under both heteroscedasticity and autocorrelation the usual OLS estimators, although linear, unbiased, and asymptotically (i.e., in large samples) normally distributed,<sup>1</sup> are no longer minimum variance among all linear unbiased estimators. In short, they are not efficient relative to other linear and unbiased estimators. Put differently, they may not be BLUE. As a result, the usual,  $t$ ,  $F$ , and  $\chi^2$  may not be valid.**

## 12.1 THE NATURE OF THE PROBLEM

The term **autocorrelation** may be defined as “correlation between members of series of observations ordered in time [as in time series data] or space [as in cross-sectional data].”<sup>2</sup> In the regression context, the classical linear regression model assumes that such autocorrelation does not exist in the disturbances  $u_i$ . Symbolically,

$$E(u_i u_j) = 0 \quad i \neq j \quad (3.2.5)$$

Put simply, the classical model assumes that the disturbance term relating to any observation is not influenced by the disturbance term relating to any other observation. For example, if we are dealing with quarterly time series data involving the regression of output on labor and capital inputs and if,

<sup>1</sup>On this, see William H. Greene, *Econometric Analysis*, 4th ed., Prentice Hall, N.J., 2000, Chap. 11, and Paul A. Rudd, *An Introduction to Classical Econometric Theory*, Oxford University Press, 2000, Chap. 19.

<sup>2</sup>Maurice G. Kendall and William R. Buckland, *A Dictionary of Statistical Terms*, Hafner Publishing Company, New York, 1971, p. 8.

say, there is a labor strike affecting output in one quarter, there is no reason to believe that this disruption will be carried over to the next quarter. That is, if output is lower this quarter, there is no reason to expect it to be lower next quarter. Similarly, if we are dealing with cross-sectional data involving the regression of family consumption expenditure on family income, the effect of an increase of one family's income on its consumption expenditure is not expected to affect the consumption expenditure of another family.

However, if there is such a dependence, we have autocorrelation. Symbolically,

$$E(u_i u_j) \neq 0 \quad i \neq j \quad (12.1.1)$$

In this situation, the disruption caused by a strike this quarter may very well affect output next quarter, or the increases in the consumption expenditure of one family may very well prompt another family to increase its consumption expenditure if it wants to keep up with the Joneses.

Before we find out why autocorrelation exists, it is essential to clear up some terminological questions. Although it is now a common practice to treat the terms **autocorrelation** and **serial correlation** synonymously, some authors prefer to distinguish the two terms. For example, Tintner defines autocorrelation as "lag correlation of a given series with itself, lagged by a number of time units," whereas he reserves the term serial correlation to "lag correlation between two different series."<sup>3</sup> Thus, correlation between two time series such as  $u_1, u_2, \dots, u_{10}$  and  $u_2, u_3, \dots, u_{11}$ , where the former is the latter series lagged by one time period, is *autocorrelation*, whereas correlation between time series such as  $u_1, u_2, \dots, u_{10}$  and  $v_2, v_3, \dots, v_{11}$ , where  $u$  and  $v$  are two different time series, is called *serial correlation*. Although the distinction between the two terms may be useful, in this book we shall treat them synonymously.

Let us visualize some of the plausible patterns of auto- and nonautocorrelation, which are given in Figure 12.1. Figure 12.1*a* to *d* shows that there is a discernible pattern among the  $u$ 's. Figure 12.1*a* shows a cyclical pattern; Figure 12.1*b* and *c* suggests an upward or downward linear trend in the disturbances; whereas Figure 12.1*d* indicates that both linear and quadratic trend terms are present in the disturbances. Only Figure 12.1*e* indicates no systematic pattern, supporting the nonautocorrelation assumption of the classical linear regression model.

The natural question is: Why does serial correlation occur? There are several reasons, some of which are as follows:

**Inertia.** A salient feature of most economic time series is inertia, or sluggishness. As is well known, time series such as GNP, price indexes, production, employment, and unemployment exhibit (business) cycles.

<sup>3</sup>Gerhard Tintner, *Econometrics*, John Wiley & Sons, New York, 1965.

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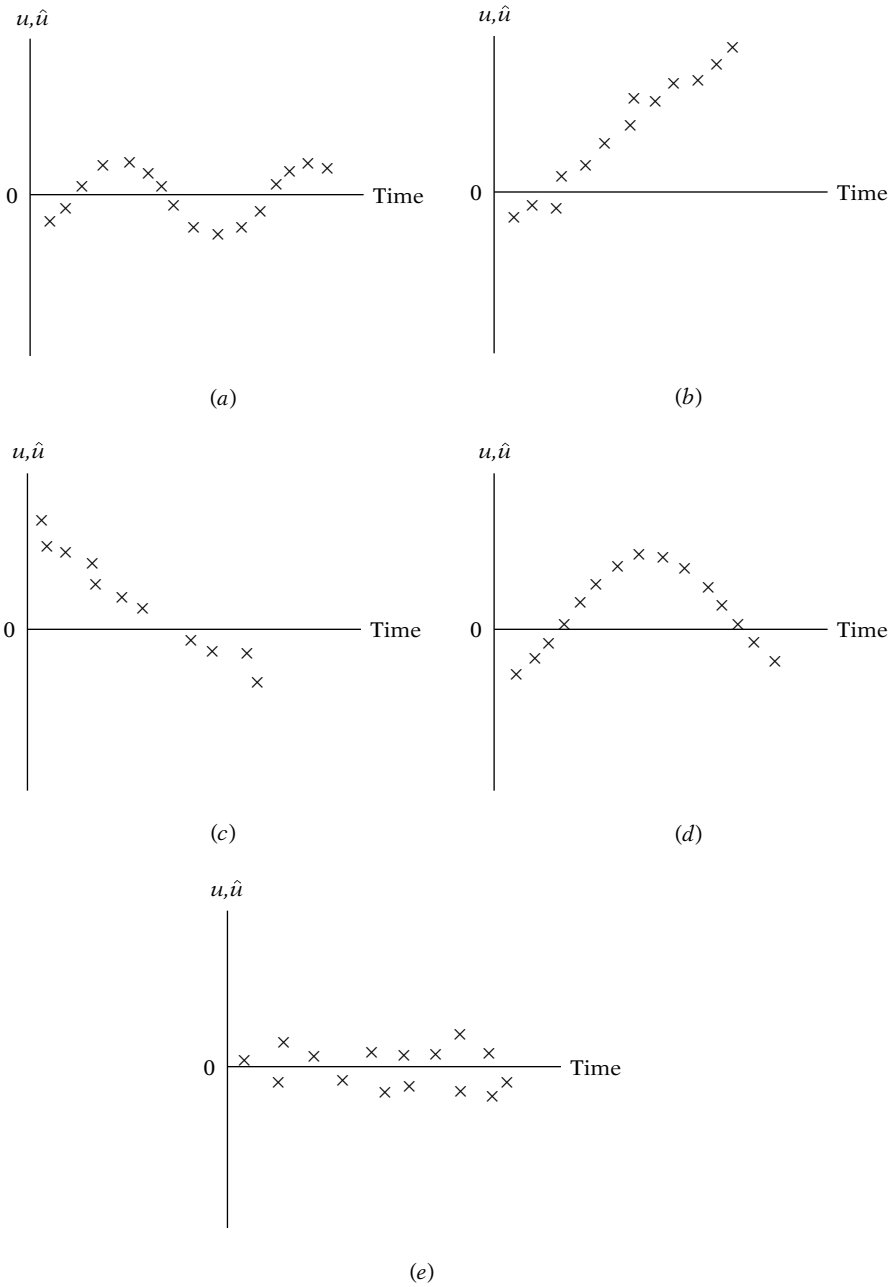


FIGURE 12.1 Patterns of autocorrelation and nonautocorrelation.

Starting at the bottom of the recession, when economic recovery starts, most of these series start moving upward. In this upswing, the value of a series at one point in time is greater than its previous value. Thus there is a “momentum” built into them, and it continues until something happens (e.g., increase in interest rate or taxes or both) to slow them down. Therefore, in regressions involving time series data, successive observations are likely to be interdependent.

**Specification Bias: Excluded Variables Case.** In empirical analysis the researcher often starts with a plausible regression model that may not be the most “perfect” one. After the regression analysis, the researcher does the postmortem to find out whether the results accord with a priori expectations. If not, surgery is begun. For example, the researcher may plot the residuals  $\hat{u}_i$  obtained from the fitted regression and may observe patterns such as those shown in Figure 12.1*a* to *d*. These residuals (which are proxies for  $u_i$ ) may suggest that some variables that were originally candidates but were not included in the model for a variety of reasons should be included. This is the case of **excluded variable** specification bias. Often the inclusion of such variables removes the correlation pattern observed among the residuals. For example, suppose we have the following demand model:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + \beta_4 X_{4t} + u_t \quad (12.1.2)$$

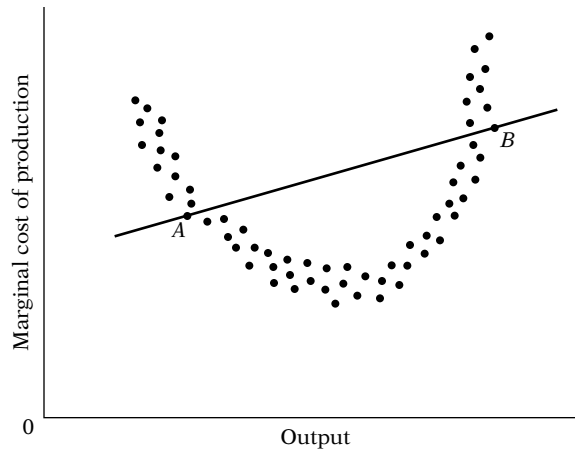
where  $Y$  = quantity of beef demanded,  $X_2$  = price of beef,  $X_3$  = consumer income,  $X_4$  = price of pork, and  $t$  = time.<sup>4</sup> However, for some reason we run the following regression:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + v_t \quad (12.1.3)$$

Now if (12.1.2) is the “correct” model or the “truth” or true relation, running (12.1.3) is tantamount to letting  $v_t = \beta_4 X_{4t} + u_t$ . And to the extent the price of pork affects the consumption of beef, the error or disturbance term  $v$  will reflect a systematic pattern, thus creating (false) autocorrelation. A simple test of this would be to run both (12.1.2) and (12.1.3) and see whether autocorrelation, if any, observed in model (12.1.3) disappears when (12.1.2) is run.<sup>5</sup> The actual mechanics of detecting autocorrelation will be discussed in Section 12.6 where we will show that a plot of the residuals from regressions (12.1.2) and (12.1.3) will often shed considerable light on serial correlation.

<sup>4</sup>As a matter of convention, we shall use the subscript  $t$  to denote time series data and the usual subscript  $i$  for cross-sectional data.

<sup>5</sup>If it is found that the real problem is one of specification bias, not autocorrelation, then as will be shown in Chap. 13, the OLS estimators of the parameters (12.1.3) may be biased as well as inconsistent.



**FIGURE 12.2** Specification bias: incorrect functional form.

**Specification Bias: Incorrect Functional Form.** Suppose the “true” or correct model in a cost-output study is as follows:

$$\text{Marginal cost}_i = \beta_1 + \beta_2 \text{output}_i + \beta_3 \text{output}_i^2 + u_i \quad (12.1.4)$$

but we fit the following model:

$$\text{Marginal cost}_i = \alpha_1 + \alpha_2 \text{output}_i + v_i \quad (12.1.5)$$

The marginal cost curve corresponding to the “true” model is shown in Figure 12.2 along with the “incorrect” linear cost curve.

As Figure 12.2 shows, between points *A* and *B* the linear marginal cost curve will consistently overestimate the true marginal cost, whereas beyond these points it will consistently underestimate the true marginal cost. This result is to be expected, because the disturbance term  $v_i$  is, in fact, equal to  $\text{output}_i^2 + u_i$ , and hence will catch the systematic effect of the  $\text{output}_i^2$  term on marginal cost. In this case,  $v_i$  will reflect autocorrelation because of the use of an incorrect functional form. In Chapter 13 we will consider several methods of detecting specification bias.

**Cobweb Phenomenon.** The supply of many agricultural commodities reflects the so-called cobweb phenomenon, where supply reacts to price with a lag of one time period because supply decisions take time to implement (the gestation period). Thus, at the beginning of this year’s planting of crops, farmers are influenced by the price prevailing last year, so that their supply function is

$$\text{Supply}_t = \beta_1 + \beta_2 P_{t-1} + u_t \quad (12.1.6)$$

Suppose at the end of period  $t$ , price  $P_t$  turns out to be lower than  $P_{t-1}$ . Therefore, in period  $t + 1$  farmers may very well decide to produce less than

they did in period  $t$ . Obviously, in this situation the disturbances  $u_t$  are not expected to be random because if the farmers overproduce in year  $t$ , they are likely to reduce their production in  $t + 1$ , and so on, leading to a Cobweb pattern.

**Lags.** In a time series regression of consumption expenditure on income, it is not uncommon to find that the consumption expenditure in the current period depends, among other things, on the consumption expenditure of the previous period. That is,

$$\text{Consumption}_t = \beta_1 + \beta_2 \text{income}_t + \beta_3 \text{consumption}_{t-1} + u_t \quad (12.1.7)$$

A regression such as (12.1.7) is known as **autoregression** because one of the explanatory variables is the lagged value of the dependent variable. (We shall study such models in Chapter 17.) The rationale for a model such as (12.1.7) is simple. Consumers do not change their consumption habits readily for psychological, technological, or institutional reasons. Now if we neglect the lagged term in (12.1.7), the resulting error term will reflect a systematic pattern due to the influence of lagged consumption on current consumption.

**“Manipulation” of Data.** In empirical analysis, the raw data are often “manipulated.” For example, in time series regressions involving quarterly data, such data are usually derived from the monthly data by simply adding three monthly observations and dividing the sum by 3. This averaging introduces smoothness into the data by dampening the fluctuations in the monthly data. Therefore, the graph plotting the quarterly data looks much smoother than the monthly data, and this smoothness may itself lead to a systematic pattern in the disturbances, thereby introducing autocorrelation. Another source of manipulation is **interpolation** or **extrapolation** of data. For example, the Census of Population is conducted every 10 years in this country, the last being in 2000 and the one before that in 1990. Now if there is a need to obtain data for some year within the intercensus period 1990–2000, the common practice is to interpolate on the basis of some ad hoc assumptions. All such data “massaging” techniques might impose upon the data a systematic pattern that might not exist in the original data.<sup>6</sup>

**Data Transformation.** As an example of this, consider the following model:

$$Y_t = \beta_1 + \beta_2 X_t + u_t \quad (12.1.8)$$

where, say,  $Y$  = consumption expenditure and  $X$  = income. Since (12.1.8) holds true at every time period, it holds true also in the previous time

<sup>6</sup>On this, see William H. Greene, op. cit., p. 526.

period,  $(t - 1)$ . So, we can write (12.1.8) as

$$Y_{t-1} = \beta_1 + \beta_2 X_{t-1} + u_{t-1} \quad (12.1.9)$$

$Y_{t-1}$ ,  $X_{t-1}$ , and  $u_{t-1}$  are known as the **lagged values** of  $Y$ ,  $X$ , and  $u$ , respectively, here lagged by one period. We will see the importance of the lagged values later in the chapter as well in several places in the text.

Now if we subtract (12.1.9) from (12.1.8), we obtain

$$\Delta Y_t = \beta_2 \Delta X_t + \Delta u_t \quad (12.1.10)$$

where  $\Delta$ , known as the **first difference operator**, tells us to take successive differences of the variables in question. Thus,  $\Delta Y_t = (Y_t - Y_{t-1})$ ,  $\Delta X_t = (X_t - X_{t-1})$ , and  $\Delta u_t = (u_t - u_{t-1})$ . For empirical purposes, we write (12.1.10) as

$$\Delta Y_t = \beta_2 \Delta X_t + v_t \quad (12.1.11)$$

where  $v_t = \Delta u_t = (u_t - u_{t-1})$ .

Equation (12.1.9) is known as the **level form** and Eq. (12.1.10) is known as the **(first) difference form**. Both forms are often used in empirical analysis. For example, if in (12.1.9)  $Y$  and  $X$  represent the logarithms of consumption expenditure and income, then in (12.1.10)  $\Delta Y$  and  $\Delta X$  will represent changes in the logs of consumption expenditure and income. But as we know, a change in the log of a variable is a relative change, or a percentage change, if the former is multiplied by 100. So, instead of studying relationships between variables in the level form, we may be interested in their relationships in the growth form.

Now if the error term in (12.1.8) satisfies the standard OLS assumptions, particularly the assumption of no autocorrelation, it can be shown that the error term  $v_t$  in (12.1.11) is autocorrelated. (The proof is given in Appendix 12A, Section 12A.1.) It may be noted here that models like (12.1.11) are known as **dynamic regression models**, that is, models involving lagged regressands. We will study such models in depth in Chapter 17.

The point of the preceding example is that sometimes autocorrelation may be induced as a result of transforming the original model.

**Nonstationarity.** We mentioned in Chapter 1 that, while dealing with time series data, we may have to find out if a given time series is stationary. Although we will discuss the topic of nonstationary time series more thoroughly in the chapters on time series econometrics in **Part V** of the text, loosely speaking, a time series is stationary if its characteristics (e.g., mean, variance, and covariance) are *time invariant*; that is, they do not change over time. If that is not the case, we have a nonstationary time series.

As we will discuss in **Part V**, in a regression model such as (12.1.8), it is quite possible that both  $Y$  and  $X$  are nonstationary and therefore the error  $u$  is also nonstationary.<sup>7</sup> In that case, the error term will exhibit autocorrelation.

<sup>7</sup>As we will also see in **Part V**, even though  $Y$  and  $X$  are nonstationary, it is possible to find  $u$  to be stationary. We will explore the implication of that later on.



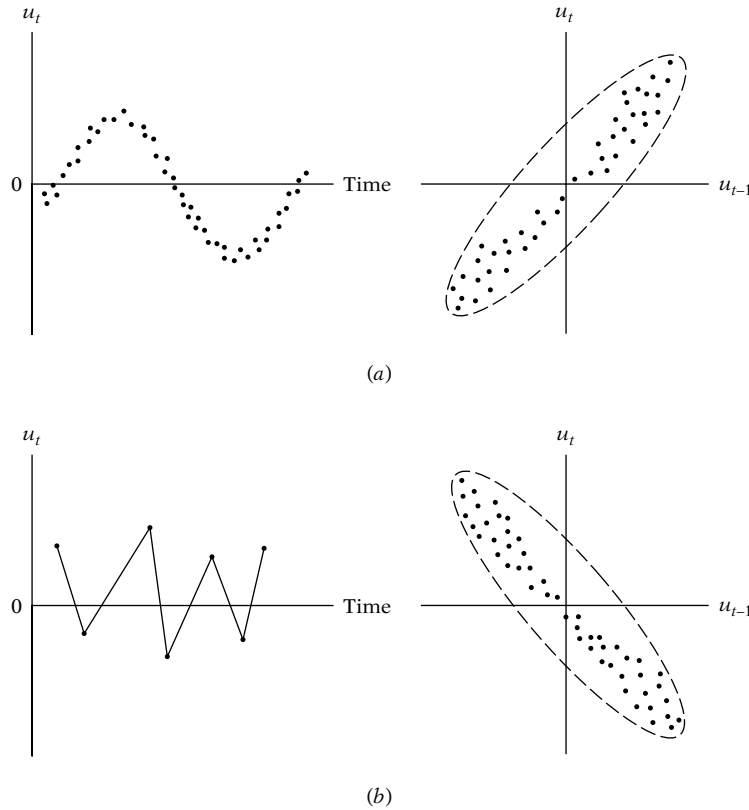


FIGURE 12.3 (a) Positive and (b) negative autocorrelation.

In summary, then, there are a variety of reasons why the error term in a regression model may be autocorrelated. In the rest of the chapter we investigate in some detail the problems posed by autocorrelation and what can be done about it.

It should be noted also that autocorrelation can be positive (Figure 12.3a) as well as negative, although most economic time series generally exhibit positive autocorrelation because most of them either move upward or downward over extended time periods and do not exhibit a constant up-and-down movement such as that shown in Figure 12.3b.

## 12.2 OLS ESTIMATION IN THE PRESENCE OF AUTOCORRELATION

What happens to the OLS estimators and their variances if we introduce autocorrelation in the disturbances by assuming that  $E(u_t u_{t+s}) \neq 0$  ( $s \neq 0$ ) but retain all the other assumptions of the classical model?<sup>8</sup> Note again that

<sup>8</sup>If  $s = 0$ , we obtain  $E(u_t^2)$ . Since  $E(u_t) = 0$  by assumption,  $E(u_t^2)$  will represent the variance of the error term, which obviously is nonzero (why?).

we are now using the subscript  $t$  on the disturbances to emphasize that we are dealing with time series data.

We revert once again to the two-variable regression model to explain the basic ideas involved, namely,  $Y_t = \beta_1 + \beta_2 X_t + u_t$ . To make any headway, we must assume the mechanism that generates  $u_t$ , for  $E(u_t u_{t+s}) \neq 0$  ( $s \neq 0$ ) is too general an assumption to be of any practical use. As a starting point, or first approximation, one can assume that the disturbance, or error, terms are generated by the following mechanism.

$$u_t = \rho u_{t-1} + \varepsilon_t \quad -1 < \rho < 1 \quad (12.2.1)$$

where  $\rho$  (= rho) is known as the **coefficient of autocovariance** and where  $\varepsilon_t$  is the stochastic disturbance term such that it satisfied the standard OLS assumptions, namely,

$$\begin{aligned} E(\varepsilon_t) &= 0 \\ \text{var}(\varepsilon_t) &= \sigma_\varepsilon^2 \\ \text{cov}(\varepsilon_t, \varepsilon_{t+s}) &= 0 \quad s \neq 0 \end{aligned} \quad (12.2.2)$$

In the engineering literature, an error term with the preceding properties is often called a **white noise error term**. What (12.2.1) postulates is that the value of the disturbance term in period  $t$  is equal to rho times its value in the previous period plus a purely random error term.

The scheme (12.2.1) is known as **Markov first-order autoregressive scheme**, or simply a **first-order autoregressive scheme**, usually denoted as **AR(1)**. The name *autoregressive* is appropriate because (12.2.1) can be interpreted as the regression of  $u_t$  on itself lagged one period. It is first order because  $u_t$  and its immediate past value are involved; that is, the maximum lag is 1. If the model were  $u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \varepsilon_t$ , it would be an AR(2), or second-order, autoregressive scheme, and so on. We will examine such higher-order schemes in the chapters on time series econometrics in **Part V**.

In passing, note that rho, the coefficient of autocovariance in (12.2.1), can also be interpreted as the **first-order coefficient of autocorrelation**, or more accurately, **the coefficient of autocorrelation at lag 1**.<sup>9</sup>

<sup>9</sup>This name can be easily justified. By definition, the (population) coefficient of correlation between  $u_t$  and  $u_{t-1}$  is

$$\begin{aligned} \rho &= \frac{E\{[u_t - E(u_t)][u_{t-1} - E(u_{t-1})]\}}{\sqrt{\text{var}(u_t)}\sqrt{\text{var}(u_{t-1})}} \\ &= \frac{E(u_t u_{t-1})}{\text{var}(u_{t-1})} \end{aligned}$$

since  $E(u_t) = 0$  for each  $t$  and  $\text{var}(u_t) = \text{var}(u_{t-1})$  because we are retaining the assumption of homoscedasticity. The reader can see that  $\rho$  is also the slope coefficient in the regression of  $u_t$  on  $u_{t-1}$ .

Given the AR(1) scheme, it can be shown that (see Appendix 12A, Section 12A.2)

$$\text{var}(u_t) = E(u_t^2) = \frac{\sigma_\varepsilon^2}{1 - \rho^2} \quad (12.2.3)$$

$$\text{cov}(u_t, u_{t+s}) = E(u_t u_{t+s}) = \rho^s \frac{\sigma_\varepsilon^2}{1 - \rho^2} \quad (12.2.4)$$

$$\text{cor}(u_t, u_{t+s}) = \rho^s \quad (12.2.5)$$

where  $\text{cov}(u_t, u_{t+s})$  means covariance between error terms  $s$  periods apart and where  $\text{cor}(u_t, u_{t+s})$  means correlation between error terms  $s$  periods apart. Note that because of the symmetry property of covariances and correlations,  $\text{cov}(u_t, u_{t+s}) = \text{cov}(u_t, u_{t-s})$  and  $\text{cor}(u_t, u_{t+s}) = \text{cor}(u_t, u_{t-s})$ .

Since  $\rho$  is a constant between  $-1$  and  $+1$ , (12.2.3) shows that under the AR(1) scheme, the variance of  $u_t$  is *still homoscedastic*, but  $u_t$  is correlated not only with its immediate past value but its values several periods in the past. It is *critical* to note that  $|\rho| < 1$ , that is, the absolute value of rho is less than one. If, for example, rho is one, the variances and covariances listed above are not defined. If  $|\rho| < 1$ , we say that the AR(1) process given in (12.2.1) is *stationary*; that is, the mean, variance, and covariance of  $u_t$  do not change over time. If  $|\rho|$  is less than one, then it is clear from (12.2.4) that the value of the covariance will decline as we go into the distant past. We will see the utility of the preceding results shortly.

One reason we use the AR(1) process is not only because of its simplicity compared to higher-order AR schemes, but also because in many applications it has proved to be quite useful. Additionally, a considerable amount of theoretical and empirical work has been done on the AR(1) scheme.

Now return to our two-variable regression model:  $Y_t = \beta_1 + \beta_2 X_t + u_t$ . We know from Chapter 3 that the OLS estimator of the slope coefficient is

$$\hat{\beta}_2 = \frac{\sum x_t y_t}{\sum x_t^2} \quad (12.2.6)$$

and its variance is given by

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_t^2} \quad (12.2.7)$$

where the small letters as usual denote deviation from the mean values.

Now under the AR(1) scheme, it can be shown that the variance of this estimator is:

$$\text{var}(\hat{\beta}_2)_{\text{AR1}} = \frac{\sigma^2}{\sum x_t^2} \left[ 1 + 2\rho \frac{\sum x_t x_{t-1}}{\sum x_t^2} + 2\rho^2 \frac{\sum x_t x_{t-2}}{\sum x_t^2} + \dots + 2\rho^{n-1} \frac{x_1 x_n}{\sum x_t^2} \right] \quad (12.2.8)$$

where  $\text{var}(\hat{\beta}_2)_{\text{AR1}}$  means the variance of  $\hat{\beta}_2$  under first-order autoregressive scheme.

A comparison of (12.2.8) with (12.2.7) shows the former is equal to the latter times a term that depends on  $\rho$  as well as the sample autocorrelations between the values taken by the regressor  $X$  at various lags.<sup>10</sup> And in general we cannot foretell whether  $\text{var}(\hat{\beta}_2)$  is less than or greater than  $\text{var}(\hat{\beta}_2)_{\text{AR1}}$  [but see Eq. (12.4.1) below]. Of course, if  $\rho$  is zero, the two formulas will coincide, as they should (why?). Also, if the correlations among the successive values of the regressor are very small, the usual OLS variance of the slope estimator will not be seriously biased. But, as a general principle, the two variances will not be the same.

To give some idea about the difference between the variances given in (12.2.7) and (12.2.8), assume that the regressor  $X$  also follows the first-order autoregressive scheme with a coefficient of autocorrelation of  $r$ . Then it can be shown that (12.2.8) reduces to:

$$\text{var}(\hat{\beta}_2)_{\text{AR}(1)} = \frac{\sigma^2}{\sum x_t^2} \left( \frac{1+r\rho}{1-r\rho} \right) = \text{var}(\hat{\beta}_2)_{\text{OLS}} \left( \frac{1+r\rho}{1-r\rho} \right) \quad (12.2.9)$$

If, for example,  $r = 0.6$  and  $\rho = 0.8$ , using (12.2.9) we can check that  $\text{var}(\hat{\beta}_2)_{\text{AR1}} = 2.8461 \text{var}(\hat{\beta}_2)_{\text{OLS}}$ . To put it another way,  $\text{var}(\hat{\beta}_2)_{\text{OLS}} = \frac{1}{2.8461} \text{var}(\hat{\beta}_2)_{\text{AR1}} = 0.3513 \text{var}(\hat{\beta}_2)_{\text{AR1}}$ . That is, the usual OLS formula [i.e., (12.2.7)] will underestimate the variance of  $(\hat{\beta}_2)_{\text{AR1}}$  by about 65 percent. As you will realize, this answer is specific for the given values of  $r$  and  $\rho$ . But the point of this exercise is to warn you that a blind application of the usual OLS formulas to compute the variances and standard errors of the OLS estimators could give seriously misleading results.

Suppose we continue to use the OLS estimator  $\hat{\beta}_2$  and adjust the usual variance formula by taking into account the AR(1) scheme. That is, we use  $\hat{\beta}_2$  given by (12.2.6) but use the variance formula given by (12.2.8). What now are the properties of  $\hat{\beta}_2$ ? It is easy to prove that  $\hat{\beta}_2$  is still linear and unbiased. As a matter of fact, as shown in Appendix 3A, Section 3A.2, the assumption of no serial correlation, like the assumption of no heteroscedasticity, is not required to prove that  $\hat{\beta}_2$  is unbiased. Is  $\hat{\beta}_2$  still BLUE? Unfortunately, it is not; in the class of linear unbiased estimators, it does not have minimum variance. In short,  $\hat{\beta}_2$ , although linear-unbiased, is not efficient (relatively speaking, of course). The reader will notice that this finding is quite similar to the finding that  $\hat{\beta}_2$  is less efficient in the presence of heteroscedasticity. There we saw that it was the weighted least-square estimator  $\hat{\beta}_2^*$  given in (11.3.8), a special case of the generalized least-squares (GLS) estimator, that was efficient. In the case of autocorrelation can we find an estimator that is BLUE? The answer is yes, as can be seen from the discussion in the following section.

<sup>10</sup>Note that the term  $r = \sum x_t x_{t+1} / \sum x_t^2$  is the correlation between  $X_t$  and  $X_{t+1}$  (or  $X_{t-1}$ , since the correlation coefficient is symmetric);  $r^2 = \sum x_t x_{t+2} / \sum x_t^2$  is the correlation between the  $X$ 's lagged two periods, and so on.

### 12.3 THE BLUE ESTIMATOR IN THE PRESENCE OF AUTOCORRELATION

Continuing with the two-variable model and assuming the AR(1) process, we can show that the BLUE estimator of  $\beta_2$  is given by the following expression<sup>11</sup>:

$$\hat{\beta}_2^{\text{GLS}} = \frac{\sum_{t=2}^n (x_t - \rho x_{t-1})(y_t - \rho y_{t-1})}{\sum_{t=2}^n (x_t - \rho x_{t-1})^2} + C \quad (12.3.1)$$

where  $C$  is a correction factor that may be disregarded in practice. Note that the subscript  $t$  now runs from  $t = 2$  to  $t = n$ . And its variance is given by

$$\text{var } \hat{\beta}_2^{\text{GLS}} = \frac{\sigma^2}{\sum_{t=2}^n (x_t - \rho x_{t-1})^2} + D \quad (12.3.2)$$

where  $D$  too is a correction factor that may also be disregarded in practice. (See exercise 12.18.)

The estimator  $\hat{\beta}_2^{\text{GLS}}$ , as the superscript suggests, is obtained by the method of GLS. As noted in Chapter 11, in GLS we incorporate any additional information we have (e.g., the nature of the heteroscedasticity or of the autocorrelation) directly into the estimating procedure by transforming the variables, whereas in OLS such side information is not directly taken into consideration. As the reader can see, the GLS estimator of  $\beta_2$  given in (12.3.1) incorporates the autocorrelation parameter  $\rho$  in the estimating formula, whereas the OLS formula given in (12.2.6) simply neglects it. Intuitively, this is the reason why the GLS estimator is BLUE and not the OLS estimator—the GLS estimator makes the most use of the available information.<sup>12</sup> It hardly needs to be added that if  $\rho = 0$ , there is no additional information to be considered and hence both the GLS and OLS estimators are identical.

*In short*, under autocorrelation, it is the GLS estimator given in (12.3.1) that is BLUE, and the minimum variance is now given by (12.3.2) and not by (12.2.8) and obviously not by (12.2.7).

**A Technical Note.** As we noted in the previous chapter, the Gauss–Markov theorem provides only the sufficient condition for OLS to be BLUE. The necessary and sufficient conditions for OLS to be BLUE are given by

<sup>11</sup>For proofs, see Jan Kmenta, *Elements of Econometrics*, Macmillan, New York, 1971, pp. 274–275. The correction factor  $C$  pertains to the first observation,  $(Y_1, X_1)$ . On this point see exercise 12.18.

<sup>12</sup>The formal proof that  $\hat{\beta}_2^{\text{GLS}}$  is BLUE can be found in Kmenta, *ibid.* But the tedious algebraic proof can be simplified considerably using matrix notation. See J. Johnston, *Econometric Methods*, 3d ed., McGraw-Hill, New York, 1984, pp. 291–293.

**Krushkal's theorem**, mentioned in the previous chapter. Therefore, in some cases it can happen that OLS is BLUE despite autocorrelation. But such cases are infrequent in practice.

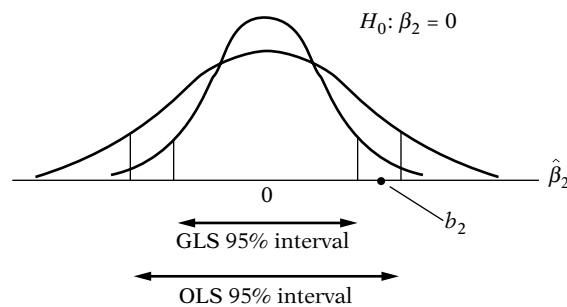
What happens if we blithely continue to work with the usual OLS procedure despite autocorrelation? The answer is provided in the following section.

### 12.4 CONSEQUENCES OF USING OLS IN THE PRESENCE OF AUTOCORRELATION

As in the case of heteroscedasticity, in the presence of autocorrelation the OLS estimators are still linear unbiased as well as consistent and asymptotically normally distributed, but they are no longer efficient (i.e., minimum variance). What then happens to our usual hypothesis testing procedures if we continue to use the OLS estimators? Again, as in the case of heteroscedasticity, we distinguish two cases. For pedagogical purposes we still continue to work with the two-variable model, although the following discussion can be extended to multiple regressions without much trouble.<sup>13</sup>

#### OLS Estimation Allowing for Autocorrelation

As noted,  $\hat{\beta}_2$  is not BLUE, and even if we use  $\text{var}(\hat{\beta}_2)_{AR1}$ , the confidence intervals derived from there are likely to be wider than those based on the GLS procedure. As Kmenta shows, this result is likely to be the case even if the sample size increases indefinitely.<sup>14</sup> That is,  $\hat{\beta}_2$  is not asymptotically efficient. The implication of this finding for hypothesis testing is clear: We are likely to declare a coefficient statistically insignificant (i.e., not different from zero) even though in fact (i.e., based on the correct GLS procedure) it may be. This difference can be seen clearly from Figure 12.4. In this figure we show the 95% OLS [AR(1)] and GLS confidence intervals assuming that true  $\beta_2 = 0$ . Consider a particular estimate of  $\beta_2$ , say,  $b_2$ . Since  $b_2$  lies in the



**FIGURE 12.4** GLS and OLS 95% confidence intervals.

<sup>13</sup>But matrix algebra becomes almost a necessity to avoid tedious algebraic manipulations.

<sup>14</sup>See Kmenta, op. cit., pp. 277–278.

OLS confidence interval, we could accept the hypothesis that true  $\beta_2$  is zero with 95% confidence. But if we were to use the (correct) GLS confidence interval, we could reject the null hypothesis that true  $\beta_2$  is zero, for  $b_2$  lies in the region of rejection.

**The message is: To establish confidence intervals and to test hypotheses, one should use GLS and not OLS even though the estimators derived from the latter are unbiased and consistent.** (However, see Section 12.11 later.)

### OLS Estimation Disregarding Autocorrelation

The situation is potentially very serious if we not only use  $\hat{\beta}_2$  but also continue to use  $\text{var}(\hat{\beta}_2) = \sigma^2 / \sum x_i^2$ , which completely disregards the problem of autocorrelation, that is, we mistakenly believe that the usual assumptions of the classical model hold true. Errors will arise for the following reasons:

1. The residual variance  $\hat{\sigma}^2 = \sum \hat{u}_i^2 / (n - 2)$  is likely to underestimate the true  $\sigma^2$ .
2. As a result, we are likely to overestimate  $R^2$ .
3. Even if  $\sigma^2$  is not underestimated,  $\text{var}(\hat{\beta}_2)$  may underestimate  $\text{var}(\hat{\beta}_2)_{\text{AR1}}$  [Eq. (12.2.8)], its variance under (first-order) autocorrelation, even though the latter is inefficient compared to  $\text{var}(\hat{\beta}_2)^{\text{GLS}}$ .
4. Therefore, the usual  $t$  and  $F$  tests of significance are no longer valid, and if applied, are likely to give seriously misleading conclusions about the statistical significance of the estimated regression coefficients.

To establish some of these propositions, let us revert to the two-variable model. We know from Chapter 3 that under the classical assumption

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{(n - 2)}$$

provides an unbiased estimator of  $\sigma^2$ , that is,  $E(\hat{\sigma}^2) = \sigma^2$ . But if there is autocorrelation, given by AR(1), it can be shown that

$$E(\hat{\sigma}^2) = \frac{\sigma^2 \{n - [2/(1 - \rho)] - 2\rho r\}}{n - 2} \quad (12.4.1)$$

where  $r = \sum_{i=1}^{n-1} x_i x_{i-1} / \sum_{i=1}^n x_i^2$ , which can be interpreted as the (sample) correlation coefficient between successive values of the  $X$ 's.<sup>15</sup> If  $\rho$  and  $r$  are both positive (not an unlikely assumption for most economic time series), it is apparent from (12.4.1) that  $E(\hat{\sigma}^2) < \sigma^2$ ; that is, the usual residual variance

<sup>15</sup>See S. M. Goldfeld and R. E. Quandt, *Nonlinear Methods in Econometrics*, North Holland Publishing Company, Amsterdam, 1972, p. 183. In passing, note that if the errors are positively autocorrelated, the  $R^2$  value tends to have an upward bias, that is, it tends to be larger than the  $R^2$  in the absence of such correlation.

formula, on average, will underestimate the true  $\sigma^2$ . In other words,  $\hat{\sigma}^2$  will be biased downward. Needless to say, this bias in  $\hat{\sigma}^2$  will be transmitted to  $\text{var}(\hat{\beta}_2)$  because in practice we estimate the latter by the formula  $\hat{\sigma}^2 / \sum x_i^2$ .

But even if  $\sigma^2$  is not underestimated,  $\text{var}(\hat{\beta}_2)$  is a *biased* estimator of  $\text{var}(\hat{\beta}_2)_{\text{AR1}}$ , which can be readily seen by comparing (12.2.7) with (12.2.8),<sup>16</sup> for the two formulas are not the same. As a matter of fact, if  $\rho$  is positive (which is true of most economic time series) and the  $X$ 's are positively correlated (also true of most economic time series), then it is clear that

$$\text{var}(\hat{\beta}_2) < \text{var}(\hat{\beta}_2)_{\text{AR1}} \quad (12.4.2)$$

that is, the usual OLS variance of  $\hat{\beta}_2$  underestimates its variance under AR(1) [see Eq. (12.2.9)]. Therefore, if we use  $\text{var}(\hat{\beta}_2)$ , we shall inflate the precision or accuracy (i.e., underestimate the standard error) of the estimator  $\hat{\beta}_2$ . As a result, in computing the  $t$  ratio as  $t = \hat{\beta}_2 / \text{se}(\hat{\beta}_2)$  (under the hypothesis that  $\beta_2 = 0$ ), we shall be overestimating the  $t$  value and hence the statistical significance of the estimated  $\beta_2$ . The situation is likely to get worse if additionally  $\sigma^2$  is underestimated, as noted previously.

To see how OLS is likely to underestimate  $\sigma^2$  and the variance of  $\hat{\beta}_2$ , let us conduct the following **Monte Carlo experiment**. Suppose in the two-variable model we “know” that the true  $\beta_1 = 1$  and  $\beta_2 = 0.8$ . Therefore, the stochastic PRF is

$$Y_t = 1.0 + 0.8X_t + u_t \quad (12.4.3)$$

Hence,

$$E(Y_t | X_t) = 1.0 + 0.8X_t \quad (12.4.4)$$

which gives the true population regression line. Let us assume that  $u_t$  are generated by the first-order autoregressive scheme as follows:

$$u_t = 0.7u_{t-1} + \varepsilon_t \quad (12.4.5)$$

where  $\varepsilon_t$  satisfy all the OLS assumptions. We assume further for convenience that the  $\varepsilon_t$  are normally distributed with zero mean and unit (= 1) variance. Equation (12.4.5) postulates that the successive disturbances are positively correlated, with a coefficient of autocorrelation of +0.7, a rather high degree of dependence.

Now, using a table of random normal numbers with zero mean and unit variance, we generated 10 random numbers shown in Table 12.1 and then by the scheme (12.4.5) we generated  $u_t$ . To start off the scheme, we need to specify the initial value of  $u$ , say,  $u_0 = 5$ .

Plotting the  $u_t$  generated in Table 12.1, we obtain Figure 12.5, which shows that initially each successive  $u_t$  is higher than its previous value and

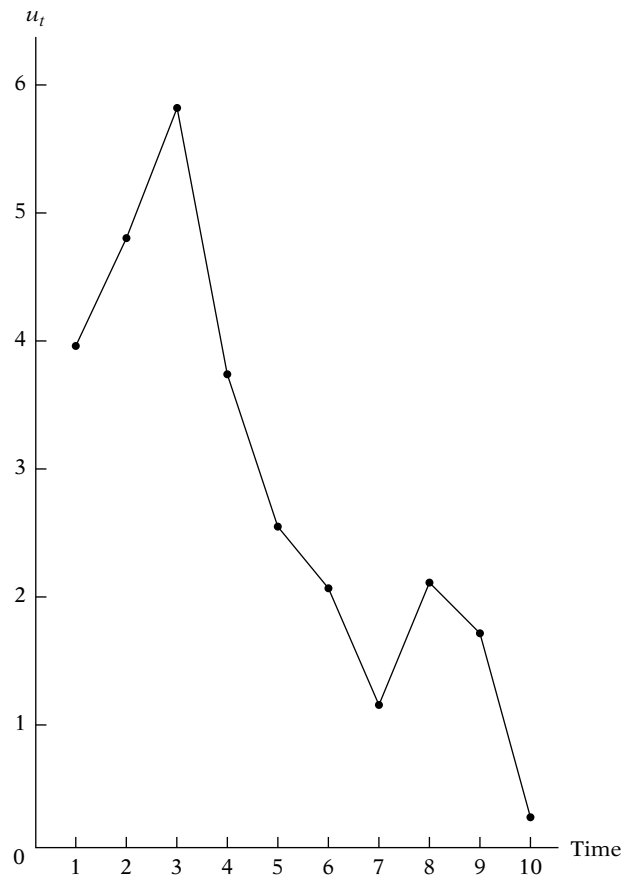
<sup>16</sup>For a formal proof, see Kmenta, op. cit., p. 281.



**TABLE 12.1** A HYPOTHETICAL EXAMPLE OF POSITIVELY AUTOCORRELATED ERROR TERMS

	$\varepsilon_t^*$	$u_t = 0.7u_{t-1} + \varepsilon_t$
0	0	$u_0 = 5$ (assumed)
1	0.464	$u_1 = 0.7(5) + 0.464 = 3.964$
2	2.026	$u_2 = 0.7(3.964) + 2.0262 = 4.8008$
3	2.455	$u_3 = 0.7(4.8010) + 2.455 = 5.8157$
4	-0.323	$u_4 = 0.7(5.8157) - 0.323 = 3.7480$
5	-0.068	$u_5 = 0.7(3.7480) - 0.068 = 2.5556$
6	0.296	$u_6 = 0.7(2.5556) + 0.296 = 2.0849$
7	-0.288	$u_7 = 0.7(2.0849) - 0.288 = 1.1714$
8	1.298	$u_8 = 0.7(1.1714) + 1.298 = 2.1180$
9	0.241	$u_9 = 0.7(2.1180) + 0.241 = 1.7236$
10	-0.957	$u_{10} = 0.7(1.7236) - 0.957 = 0.2495$

\*Obtained from *A Million Random Digits and One Hundred Thousand Deviates*, Rand Corporation, Santa Monica, Calif., 1950.



**FIGURE 12.5** Correlation generated by the scheme  $u_t = 0.7u_{t-1} + \varepsilon_t$  (Table 12.1).

**TABLE 12.2** GENERATION OF  $Y$  SAMPLE VALUES

$X_t$	$u_t^*$	$Y_t = 1.0 + 0.8X_t + u_t$
1	3.9640	$Y_1 = 1.0 + 0.8(1) + 3.9640 = 5.7640$
2	4.8010	$Y_2 = 1.0 + 0.8(2) + 4.8008 = 7.4008$
3	5.8157	$Y_3 = 1.0 + 0.8(3) + 5.8157 = 9.2157$
4	3.7480	$Y_4 = 1.0 + 0.8(4) + 3.7480 = 7.9480$
5	2.5556	$Y_5 = 1.0 + 0.8(5) + 2.5556 = 7.5556$
6	2.0849	$Y_6 = 1.0 + 0.8(6) + 2.0849 = 7.8849$
7	1.1714	$Y_7 = 1.0 + 0.8(7) + 1.1714 = 7.7714$
8	2.1180	$Y_8 = 1.0 + 0.8(8) + 2.1180 = 9.5180$
9	1.7236	$Y_9 = 1.0 + 0.8(9) + 1.7236 = 9.9236$
10	0.2495	$Y_{10} = 1.0 + 0.8(10) + 0.2495 = 9.2495$

\*Obtained from Table 12.1.

subsequently it is generally smaller than its previous value showing, in general, a positive autocorrelation.

Now suppose the values of  $X$  are fixed at 1, 2, 3, . . . , 10. Then, given these  $X$ 's, we can generate a sample of 10  $Y$  values from (12.4.3) and the values of  $u_t$  given in Table 12.1. The details are given in Table 12.2. Using the data of Table 12.2, if we regress  $Y$  on  $X$ , we obtain the following (sample) regression:

$$\begin{aligned} \hat{Y}_t &= 6.5452 + 0.3051X_t \\ &\quad (0.6153) \quad (0.0992) \\ t &= (10.6366) \quad (3.0763) \\ r^2 &= 0.5419 \quad \hat{\sigma}^2 = 0.8114 \end{aligned} \tag{12.4.6}$$

whereas the true regression line is as given by (12.4.4). Both the regression lines are given in Figure 12.6, which shows clearly how much the fitted regression line distorts the true regression line; it seriously underestimates the true slope coefficient but overestimates the true intercept. (But note that the OLS estimators are still unbiased.)

Figure 12.6 also shows why the true variance of  $u_t$  is likely to be underestimated by the estimator  $\hat{\sigma}^2$ , which is computed from the  $\hat{u}_t$ . The  $\hat{u}_t$  are generally close to the fitted line (which is due to the OLS procedure) but deviate substantially from the true PRF. Hence, they do not give a correct picture of  $u_t$ . To gain some insight into the extent of underestimation of true  $\sigma^2$ , suppose we conduct another sampling experiment. Keeping the  $X_t$  and  $\varepsilon_t$  given in Tables 12.1 and 12.2, let us assume  $\rho = 0$ , that is, no autocorrelation. The new sample of  $Y$  values thus generated is given in Table 12.3.

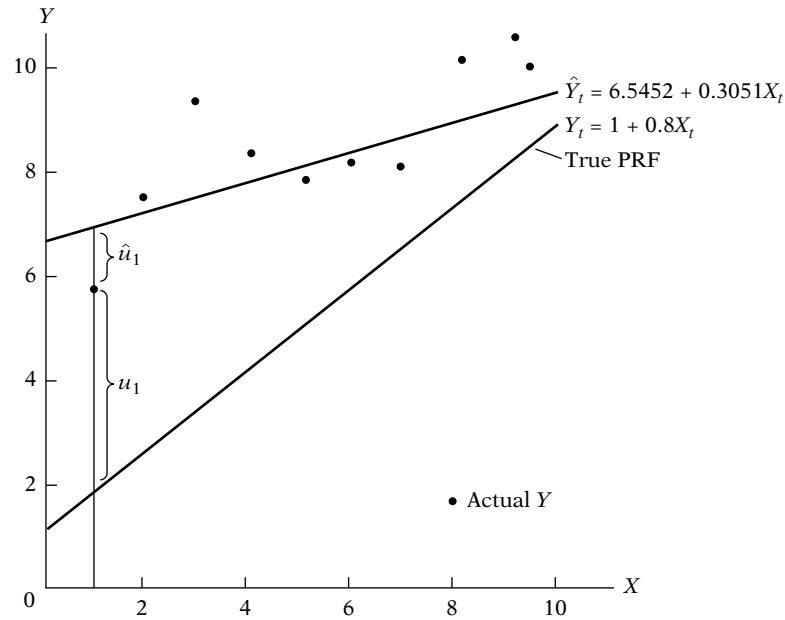


FIGURE 12.6 True PRF and the estimated regression line for the data of Table 12.2.

TABLE 12.3 SAMPLE OF Y VALUES WITH ZERO SERIAL CORRELATION

$X_t$	$\varepsilon_t = u_t^*$	$Y_t = 1.0 + 0.8X_t + \varepsilon_t$
1	0.464	2.264
2	2.026	4.626
3	2.455	5.855
4	-0.323	3.877
5	-0.068	4.932
6	0.296	6.096
7	-0.288	6.312
8	1.298	8.698
9	0.241	8.441
10	-0.957	8.043

\*Since there is no autocorrelation, the  $u_t$  and  $\varepsilon_t$  are identical. The  $\varepsilon_t$  are from Table 12.1.

The regression based on Table 12.3 is as follows:

$$\begin{aligned}
 \hat{Y}_t &= 2.5345 + 0.6145X_t \\
 &\quad (0.6796) \quad (0.1087) \\
 t &= (3.7910) \quad (5.6541) \\
 r^2 &= 0.7997 \quad \hat{\sigma}^2 = 0.9752
 \end{aligned}
 \tag{12.4.7}$$

This regression is much closer to the “truth” because the  $Y$ 's are now essentially random. Notice that  $\hat{\sigma}^2$  has increased from 0.8114 ( $\rho = 0.7$ ) to 0.9752 ( $\rho = 0$ ). Also notice that the standard errors of  $\hat{\beta}_1$  and  $\hat{\beta}_2$  have increased. This result is in accord with the theoretical results considered previously.

### 12.5 RELATIONSHIP BETWEEN WAGES AND PRODUCTIVITY IN THE BUSINESS SECTOR OF THE UNITED STATES, 1959–1998

Now that we have discussed the consequences of autocorrelation, the obvious question is, How do we detect it and how do we correct for it? Before we turn to these topics, it is useful to consider a concrete example. Table 12.4 gives data on indexes of real compensation per hour ( $Y$ ) and output per hour ( $X$ ) in the business sector of the U.S. economy for the period 1959–1998, the base of the indexes being 1992 = 100.

First plotting the data on  $Y$  and  $X$ , we obtain Figure 12.7. Since the relationship between real compensation and labor productivity is expected to be positive, it is not surprising that the two variables are positively related. What is surprising is that the relationship between the two is almost linear,

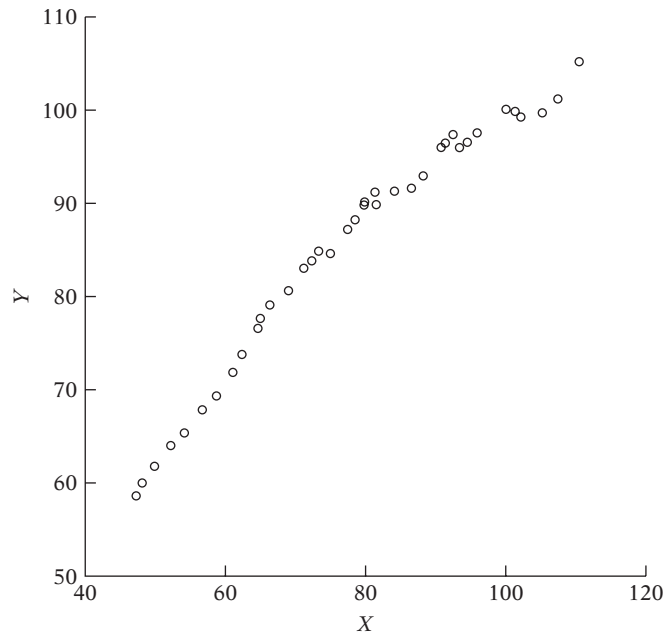
**TABLE 12.4** INDEXES OF REAL COMPENSATION AND PRODUCTIVITY, UNITED STATES, 1959–1998

Observation	Y	X	Observation	Y	X
1959	58.5	47.2	1979	90.0	79.7
1960	59.9	48.0	1980	89.7	79.8
1961	61.7	49.8	1981	89.8	81.4
1962	63.9	52.1	1982	91.1	81.2
1963	65.3	54.1	1983	91.2	84.0
1964	67.8	54.6	1984	91.5	86.4
1965	69.3	58.6	1985	92.8	88.1
1966	71.8	61.0	1986	95.9	90.7
1967	73.7	62.3	1987	96.3	91.3
1968	76.5	64.5	1988	97.3	92.4
1969	77.6	64.8	1989	95.8	93.3
1970	79.0	66.2	1990	96.4	94.5
1971	80.5	68.8	1991	97.4	95.9
1972	82.9	71.0	1992	100.0	100.0
1973	84.7	73.1	1993	99.9	100.1
1974	83.7	72.2	1994	99.7	101.4
1975	84.5	74.8	1995	99.1	102.2
1976	87.0	77.2	1996	99.6	105.2
1977	88.1	78.4	1997	101.1	107.5
1978	89.7	79.5	1998	105.1	110.5

Notes:  $X$  = index of output per hour, business sector (1992 = 100)

$Y$  = index of real compensation per hour, business sector (1992 = 100)

Source: *Economic Report of the President*, 2000, Table B-47, p. 362.



**FIGURE 12.7** Index of compensation ( $Y$ ) and index of productivity ( $X$ ), United States, 1959–1998.

although there is some hint that at higher values of productivity the relationship between the two may be slightly nonlinear. Therefore, we decided to estimate a linear as well as a log–linear model, with the following results:

$$\begin{aligned} \hat{Y}_t &= 29.5192 + 0.7136X_t \\ \text{se} &= (1.9423) \quad (0.0241) \\ t &= (15.1977) \quad (29.6066) \\ r^2 &= 0.9584 \quad d = 0.1229 \quad \hat{\sigma} = 2.6755 \end{aligned} \tag{12.5.1}$$

where  $d$  is the Durbin–Watson statistic, which will be discussed shortly.

$$\begin{aligned} \widehat{\ln Y}_t &= 1.5239 + 0.6716 \ln X_t \\ \text{se} &= (0.0762) \quad (0.0175) \\ t &= (19.9945) \quad (38.2892) \\ r^2 &= 0.9747 \quad d = 0.1542 \quad \hat{\sigma} = 0.0260 \end{aligned} \tag{12.5.2}$$

For discussion purposes, we will call (12.5.1) and (12.5.2) wages–productivity regressions.

Qualitatively, both the models give similar results. In both cases the estimated coefficients are “highly” significant, as indicated by the high  $t$  values. In the linear model, if the index of productivity goes up by a unit, on average, the index of compensation goes up by about 0.71 units. In the log-linear model, the slope coefficient being elasticity (why?), we find that if the index of productivity goes up by 1 percent, on average, the index of real compensation goes up by about 0.67 percent.

How reliable are the results given in (12.5.1) and (12.5.2) if there is autocorrelation? As stated previously, if there is autocorrelation, the estimated standard errors are biased, as a result of which the estimated  $t$  ratios are unreliable. We obviously need to find out if our data suffer from autocorrelation. In the following section we discuss several methods of detecting autocorrelation. We will illustrate these methods with the linear model (12.5.1) only, leaving the log-linear model (12.5.2) as an exercise.

## 12.6 DETECTING AUTOCORRELATION

### I. Graphical Method

Recall that the assumption of nonautocorrelation of the classical model relates to the population disturbances  $u_t$ , which are not directly observable. What we have instead are their proxies, the residuals  $\hat{u}_t$ , which can be obtained by the usual OLS procedure. Although the  $\hat{u}_t$  are not the same thing as  $u_t$ ,<sup>17</sup> very often a visual examination of the  $\hat{u}$ 's gives us some clues about the likely presence of autocorrelation in the  $u$ 's. Actually, a visual examination of  $\hat{u}_t$  or  $(\hat{u}_t^2)$  can provide useful information not only about autocorrelation but also about heteroscedasticity (as we saw in the preceding chapter), model inadequacy, or specification bias, as we shall see in the next chapter. As one author notes:

The importance of producing and analyzing plots of [residuals] as a standard part of statistical analysis cannot be overemphasized. Besides occasionally providing an easy to understand summary of a complex problem, they allow the simultaneous examination of the data as an aggregate while clearly displaying the behavior of individual cases.<sup>18</sup>

There are various ways of examining the residuals. We can simply plot them against time, the **time sequence plot**, as we have done in Figure 12.8, which shows the residuals obtained from the wages–productivity regression (12.5.1). The values of these residuals are given in Table 12.5 along with some other data.

<sup>17</sup>Even if the disturbances  $u_t$  are homoscedastic and uncorrelated, their estimators, the residuals,  $\hat{u}_t$ , are heteroscedastic and autocorrelated. On this, see G. S. Maddala, *Introduction to Econometrics*, 2d ed., Macmillan, New York, 1992, pp. 480–481. However, it can be shown that as the sample size increases indefinitely, the residuals tend to converge to their true values, the  $u_t$ 's. On this see, E. Malinvaud, *Statistical Methods of Econometrics*, 2d ed., North-Holland Publishers, Amsterdam, 1970, p. 88.

<sup>18</sup>Stanford Weisberg, *Applied Linear Regression*, John Wiley & Sons, New York, 1980, p. 120.

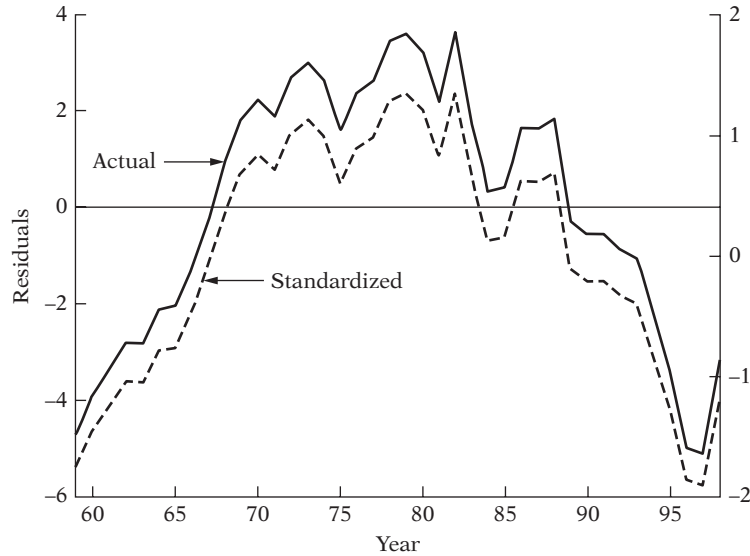


FIGURE 12.8 Residuals and standardized residuals from the wages–productivity regression (12.5.1).

TABLE 12.5 RESIDUALS: ACTUAL, STANDARDIZED, AND LAGGED

Observation	RES1	SRES1	RES1(-1)	Observation	RES1	SRES1	RES1(-1)
1959	-4.703979	-1.758168		1979	3.602089	1.346324	3.444821
1960	-3.874907	-1.448293	-4.703979	1980	3.230723	1.207521	3.602089
1961	-3.359494	-1.255651	-3.874907	1981	2.188868	0.818116	3.230723
1962	-2.800911	-1.046874	-3.359494	1982	3.631600	1.357354	2.188868
1963	-2.828229	-1.057084	-2.800911	1983	1.733354	0.647862	3.631600
1964	-2.112378	-0.789526	-2.828229	1984	0.320571	0.119817	1.733354
1965	-2.039697	-0.762361	-2.112378	1985	0.407350	0.152252	0.320571
1966	-1.252480	-0.468129	-2.039697	1986	1.651836	0.617393	0.407350
1967	-0.280237	-0.104742	-1.252480	1987	1.623640	0.606855	1.651836
1968	0.949713	0.354966	-0.280237	1988	1.838615	0.687204	1.623640
1969	1.835615	0.686083	0.949713	1989	-0.303679	-0.113504	1.838615
1970	2.236492	0.835915	1.835615	1990	-0.560070	-0.209333	-0.303679
1971	1.880977	0.703038	2.236492	1991	-0.559193	-0.209005	-0.560070
1972	2.710926	1.013241	1.880977	1992	-0.885197	-0.330853	-0.559193
1973	3.012241	1.125861	2.710926	1993	-1.056563	-0.394903	-0.885197
1974	2.654535	0.992164	3.012241	1994	-2.184320	-0.816416	-1.056563
1975	1.599020	0.597653	2.654535	1995	-3.355248	-1.254064	-2.184320
1976	2.386238	0.891885	1.599020	1996	-4.996226	-1.867399	-3.355248
1977	2.629847	0.982936	2.386238	1997	-5.137643	-1.920255	-4.996226
1978	3.444821	1.287543	2.629847	1998	-3.278621	-1.225424	-5.137643

Notes: RES 1 = residuals from regression (12.5.1).  
 SRES 1 = standardized residuals = RES1/2.6755.  
 RES(-1) = residuals lagged one period.

Alternatively, we can plot the **standardized residuals** against time, which are also shown in Figure 12.8 and Table 12.5. The standardized residuals are simply the residuals ( $\hat{u}_t$ ) divided by the standard error of the regression ( $\sqrt{\hat{\sigma}^2}$ ), that is, they are  $(\hat{u}_t/\hat{\sigma})$ . Notice that  $\hat{u}_t$  and  $\hat{\sigma}$  are measured in the units in which the regressand  $Y$  is measured. The values of the standardized residuals will therefore be pure numbers (devoid of units of measurement) and can be compared with the standardized residuals of other regressions. Moreover, the standardized residuals, like  $\hat{u}_t$ , have zero mean (why?) and *approximately* unit variance.<sup>19</sup> In large samples  $(\hat{u}_t/\hat{\sigma})$  is approximately normally distributed with zero mean and unit variance. For our example,  $\hat{\sigma} = 2.6755$ .

Examining the time sequence plot given in Figure 12.8, we observe that both  $\hat{u}_t$  and the standardized  $\hat{u}_t$  exhibit a pattern observed in Figure 12.1d, suggesting that perhaps  $u_t$  are not random.

To see this differently, we can plot  $\hat{u}_t$  against  $\hat{u}_{t-1}$ , that is, plot the residuals at time  $t$  against their value at time  $(t - 1)$ , a kind of empirical test of the AR(1) scheme. If the residuals are nonrandom, we should obtain pictures similar to those shown in Figure 12.3. This plot for our wages–productivity regression is as shown in Figure 12.9; the underlying data are given in

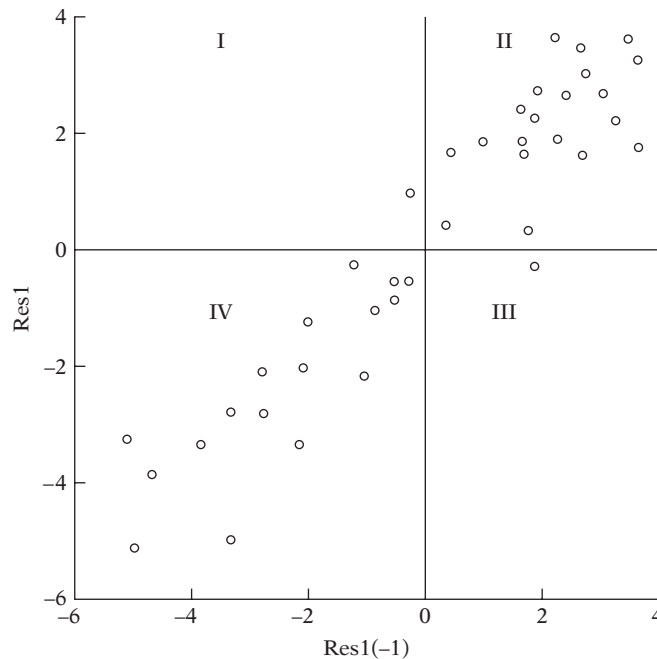


FIGURE 12.9 Current residuals versus lagged residuals.

<sup>19</sup>Actually, it is the so-called **Studentized** residuals that have a unit variance. But in practice the standardized residuals will give the same picture, and hence we may rely on them. On this, see Norman Draper and Harry Smith, *Applied Regression Analysis*, 3d ed., John Wiley & Sons, New York, 1998, pp. 207–208.



Table 12.5. As this figure reveals, most of the residuals are bunched in the second (northeast) and the fourth (southwest) quadrants, suggesting a strong positive correlation in the residuals.

The graphical method we have just discussed, although powerful and suggestive, is subjective or qualitative in nature. But there are several quantitative tests that one can use to supplement the purely qualitative approach. We now consider some of these tests.

## II. The Runs Test

If we carefully examine Figure 12.8, we notice a peculiar feature: Initially, we have several residuals that are negative, then there is a series of positive residuals, and then there are several residuals that are negative. If these residuals were purely random, could we observe such a pattern? Intuitively, it seems unlikely. This intuition can be checked by the so-called **runs test**, sometimes also known as the **Geary test**, a nonparametric test.<sup>20</sup>

To explain the runs test, let us simply note down the signs (+ or -) of the residuals obtained from the wages-productivity regression, which are given in the first column of Table 12.5.

$$(-\text{-----})(+\text{+++++++})(-\text{-----}) \tag{12.6.1}$$

Thus there are 9 negative residuals, followed by 21 positive residuals, followed by 10 negative residuals, for a total of 40 observations.

We now define a **run** as an uninterrupted sequence of one symbol or attribute, such as + or -. We further define the **length of a run** as the number of elements in it. In the sequence shown in (12.6.1), there are 3 runs: a run of 9 minuses (i.e., of length 9), a run of 21 pluses (i.e., of length 21) and a run of 10 minuses (i.e., of length 10). For a better visual effect, we have presented the various runs in parentheses.

By examining how runs behave in a strictly random sequence of observations, one can derive a test of randomness of runs. We ask this question: Are the 3 runs observed in our illustrative example consisting of 40 observations too many or too few compared with the number of runs expected in a strictly random sequence of 40 observations? If there are too many runs, it would mean that in our example the residuals change sign frequently, thus indicating negative serial correlation (cf. Figure 12.3*b*). Similarly, if there are too few runs, they may suggest positive autocorrelation, as in Figure 12.3*a*. A priori, then, Figure 12.8 would indicate positive correlation in the residuals.

<sup>20</sup>In **nonparametric** tests we make no assumptions about the (probability) distribution from which the observations are drawn. On the Geary test, see R. C. Geary, "Relative Efficiency of Count Sign Changes for Assessing Residual Autoregression in Least Squares Regression," *Biometrika*, vol. 57, 1970, pp. 123-127.

Now let

$N$  = total number of observations =  $N_1 + N_2$

$N_1$  = number of + symbols (i.e., + residuals)

$N_2$  = number of – symbols (i.e., – residuals)

$R$  = number of runs

Then under the null hypothesis that the successive outcomes (here, residuals) are independent, and assuming that  $N_1 > 10$  and  $N_2 > 10$ , the number of runs is (*asymptotically*) normally distributed with

$$\begin{aligned} \text{Mean: } E(R) &= \frac{2N_1N_2}{N} + 1 \\ \text{Variance: } \sigma_R^2 &= \frac{2N_1N_2(2N_1N_2 - N)}{(N)^2(N - 1)} \end{aligned} \quad (12.6.2)$$

Note:  $N = N_1 + N_2$ .

If the null hypothesis of randomness is sustainable, following the properties of the normal distribution, we should expect that

$$\text{Prob} [E(R) - 1.96\sigma_R \leq R \leq E(R) + 1.96\sigma_R] = 0.95 \quad (12.6.3)$$

That is, the probability is 95 percent that the preceding interval will include  $R$ . Therefore we have this rule:

**Decision Rule.** Do not reject the null hypothesis of randomness with 95% confidence if  $R$ , the number of runs, lies in the preceding confidence interval; reject the null hypothesis if the estimated  $R$  lies outside these limits. (Note: You can choose any level of confidence you want.)

Returning to our example, we know that  $N_1$ , the number of minuses, is 19 and  $N_2$ , the number of pluses, is 21 and  $R = 3$ . Using the formulas given in (12.6.2), we obtain:

$$\begin{aligned} E(R) &= 10.975 \\ \sigma_R^2 &= 9.6936 \\ \sigma_R &= 3.1134 \end{aligned} \quad (12.6.4)$$

The 95% confidence interval for  $R$  in our example is thus:

$$[10.975 \pm 1.96(3.1134)] = (4.8728, 17.0722)$$

Obviously, this interval does not include 3. Hence, we can *reject* the hypothesis that the residuals in our wages–productivity regression are random

with 95% confidence. In other words, the residuals exhibit autocorrelation. As a general rule, if there is positive autocorrelation, the number of runs will be few, whereas if there is negative autocorrelation, the number of runs will be many. Of course, from (12.6.2) we can find out whether we have too many runs or too few runs.

Swed and Eisenhart have developed special tables that give critical values of the runs expected in a random sequence of  $N$  observations if  $N_1$  or  $N_2$  is smaller than 20. These tables are given in **Appendix D**, Table D.6. Using these tables, the reader can verify that the residuals in our wages–productivity regression are indeed nonrandom; actually they are positively correlated.

### III. Durbin–Watson $d$ Test<sup>21</sup>

The most celebrated test for detecting serial correlation is that developed by statisticians Durbin and Watson. It is popularly known as the **Durbin–Watson  $d$  statistic**, which is defined as

$$d = \frac{\sum_{t=2}^{t=n} (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^{t=n} \hat{u}_t^2} \quad (12.6.5)$$

which is simply the ratio of the sum of squared differences in successive residuals to the RSS. Note that in the numerator of the  $d$  statistic the number of observations is  $n - 1$  because one observation is lost in taking successive differences.

A great advantage of the  $d$  statistic is that it is based on the estimated residuals, which are routinely computed in regression analysis. Because of this advantage, it is now a common practice to report the Durbin–Watson  $d$  along with summary measures, such as  $R^2$ , adjusted  $R^2$ ,  $t$ , and  $F$ . Although it is now routinely used, it is **important to note the assumptions underlying the  $d$  statistic**.

1. The regression model includes the intercept term. If it is not present, as in the case of the regression through the origin, it is essential to rerun the regression including the intercept term to obtain the RSS.<sup>22</sup>

2. The explanatory variables, the  $X$ 's, are nonstochastic, or fixed in repeated sampling.

3. The disturbances  $u_t$  are generated by the first-order autoregressive scheme:  $u_t = \rho u_{t-1} + \varepsilon_t$ . Therefore, it cannot be used to detect higher-order autoregressive schemes.

4. The error term  $u_t$  is assumed to be normally distributed.

<sup>21</sup>J. Durbin and G. S. Watson, "Testing for Serial Correlation in Least-Squares Regression," *Biometrika*, vol. 38, 1951, pp. 159–171.

<sup>22</sup>However, R. W. Farebrother has calculated  $d$  values when the intercept term is absent from the model. See his "The Durbin–Watson Test for Serial Correlation When There Is No Intercept in the Regression," *Econometrica*, vol. 48, 1980, pp. 1553–1563.

5. The regression model does not include the lagged value(s) of the dependent variable as one of the explanatory variables. Thus, the test is inapplicable in models of the following type:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + \cdots + \beta_k X_{kt} + \gamma Y_{t-1} + u_t \quad (12.6.6)$$

where  $Y_{t-1}$  is the one period lagged value of  $Y$ . Such models are known as **autoregressive models**, which we will study in Chapter 17.

6. There are no missing observations in the data. Thus, in our wages–productivity regression for the period 1959–1998, if observations for, say, 1978 and 1982 were missing for some reason, the  $d$  statistic makes no allowance for such missing observations.<sup>23</sup>

The exact sampling or probability distribution of the  $d$  statistic given in (12.6.5) is difficult to derive because, as Durbin and Watson have shown, it depends in a complicated way on the  $X$  values present in a given sample.<sup>24</sup> This difficulty should be understandable because  $d$  is computed from  $\hat{u}_t$ , which are, of course, dependent on the given  $X$ 's. Therefore, unlike the  $t$ ,  $F$ , or  $\chi^2$  tests, there is no unique critical value that will lead to the rejection or the acceptance of the null hypothesis that there is no first-order serial correlation in the disturbances  $u_i$ . However, Durbin and Watson were successful in deriving a lower bound  $d_L$  and an upper bound  $d_U$  such that if the computed  $d$  from (12.6.5) lies outside these critical values, a decision can be made regarding the presence of positive or negative serial correlation. Moreover, these limits depend only on the number of observations  $n$  and the number of explanatory variables and do not depend on the values taken by these explanatory variables. These limits, for  $n$  going from 6 to 200 and up to 20 explanatory variables, have been tabulated by Durbin and Watson and are reproduced in **Appendix D**, Table D.5 (up to 20 explanatory variables).

The actual test procedure can be explained better with the aid of Figure 12.10, which shows that the limits of  $d$  are 0 and 4. These can be established as follows. Expand (12.6.5) to obtain

$$d = \frac{\sum \hat{u}_t^2 + \sum \hat{u}_{t-1}^2 - 2 \sum \hat{u}_t \hat{u}_{t-1}}{\sum \hat{u}_t^2} \quad (12.6.7)$$

Since  $\sum \hat{u}_t^2$  and  $\sum \hat{u}_{t-1}^2$  differ in only one observation, they are approximately equal. Therefore, setting  $\sum \hat{u}_{t-1}^2 \approx \sum \hat{u}_t^2$ , (12.6.7) may be written as

$$d \approx 2 \left( 1 - \frac{\sum \hat{u}_t \hat{u}_{t-1}}{\sum \hat{u}_t^2} \right) \quad (12.6.8)$$

where  $\approx$  means approximately.

<sup>23</sup>For further details, see Gabor Korosi, Laszlo Matyas, and Istvan P. Szekey, *Practical Econometrics*, Avebury Press, England, 1992, pp. 88–89.

<sup>24</sup>But see the discussion on the “exact” Durbin–Watson test given later in the section.

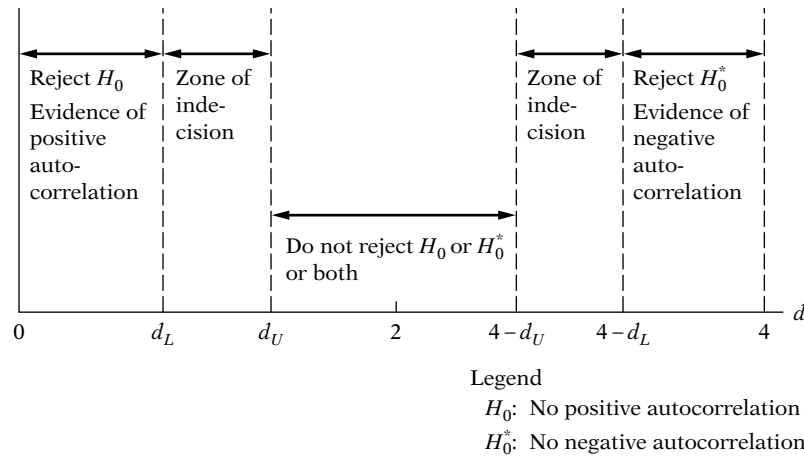


FIGURE 12.10 Durbin–Watson  $d$  statistic.

Now let us define

$$\hat{\rho} = \frac{\sum \hat{u}_t \hat{u}_{t-1}}{\sum \hat{u}_t^2} \quad (12.6.9)$$

as the sample first-order coefficient of autocorrelation, an estimator of  $\rho$ . (See footnote 9.) Using (12.6.9), we can express (12.6.8) as

$$d \approx 2(1 - \hat{\rho}) \quad (12.6.10)$$

But since  $-1 \leq \rho \leq 1$ , (12.6.10) implies that

$$0 \leq d \leq 4 \quad (12.6.11)$$

These are the bounds of  $d$ ; any estimated  $d$  value must lie within these limits.

It is apparent from Eq. (12.6.10) that if  $\hat{\rho} = 0$ ,  $d = 2$ ; that is, if there is no serial correlation (of the first-order),  $d$  is expected to be about 2. *Therefore, as a rule of thumb, if  $d$  is found to be 2 in an application, one may assume that there is no first-order autocorrelation, either positive or negative.* If  $\hat{\rho} = +1$ , indicating perfect positive correlation in the residuals,  $d \approx 0$ . Therefore, the closer  $d$  is to 0, the greater the evidence of positive serial correlation. This relationship should be evident from (12.6.5) because if there is positive autocorrelation, the  $\hat{u}_t$ 's will be bunched together and their differences will therefore tend to be small. As a result, the numerator sum of squares will be smaller in comparison with the denominator sum of squares, which remains a unique value for any given regression.

If  $\hat{\rho} = -1$ , that is, there is perfect negative correlation among successive residuals,  $d \approx 4$ . Hence, the closer  $d$  is to 4, the greater the evidence of negative serial correlation. Again, looking at (12.6.5), this is understandable. For if there is negative autocorrelation, a positive  $\hat{u}_t$  will tend to be followed by a negative  $\hat{u}_t$  and vice versa so that  $|\hat{u}_t - \hat{u}_{t-1}|$  will usually be greater than  $|\hat{u}_t|$ . Therefore, the numerator of  $d$  will be comparatively larger than the denominator.

The mechanics of the Durbin–Watson test are as follows, assuming that the assumptions underlying the test are fulfilled:

1. Run the OLS regression and obtain the residuals.
2. Compute  $d$  from (12.6.5). (Most computer programs now do this routinely.)
3. For the given sample size and given number of explanatory variables, find out the critical  $d_L$  and  $d_U$  values.
4. Now follow the decision rules given in Table 12.6. For ease of reference, these decision rules are also depicted in Figure 12.10.

To illustrate the mechanics, let us return to our wages–productivity regression. From the data given in Table 12.5 the estimated  $d$  value can be shown to be 0.1229, suggesting that there is positive serial correlation in the residuals. From the Durbin–Watson tables, we find that for 40 observations and one explanatory variable,  $d_L = 1.44$  and  $d_U = 1.54$  at the 5 percent level. Since the computed  $d$  of 0.1229 lies below  $d_L$ , we cannot reject the hypothesis that there is positive serial correlations in the residuals.

Although extremely popular, the  $d$  test has one great drawback in that, if it falls in the **indecisive zone**, one cannot conclude that (first-order) autocorrelation does or does not exist. To solve this problem, several authors have proposed modifications of the  $d$  test but they are rather involved and beyond the scope of this book.<sup>25</sup> In many situations, however, it has been found that the upper limit  $d_U$  is approximately the true significance limit and therefore in case  $d$  lies in the indecisive zone, one can use the following **modified  $d$  test**: Given the level of significance  $\alpha$ ,

1.  $H_0: \rho = 0$  versus  $H_1: \rho > 0$ . Reject  $H_0$  at  $\alpha$  level if  $d < d_U$ . That is, there is statistically significant positive autocorrelation.

**TABLE 12.6** DURBIN–WATSON  $d$  TEST: DECISION RULES

Null hypothesis	Decision	If
No positive autocorrelation	Reject	$0 < d < d_L$
No positive autocorrelation	No decision	$d_L \leq d \leq d_U$
No negative correlation	Reject	$4 - d_L < d < 4$
No negative correlation	No decision	$4 - d_U \leq d \leq 4 - d_L$
No autocorrelation, positive or negative	Do not reject	$d_U < d < 4 - d_U$

<sup>25</sup>For details, see Thomas B. Fomby, R. Carter Hill, and Stanley R. Johnson, *Advanced Econometric Methods*, Springer Verlag, New York, 1984, pp. 225–228.

2.  $H_0: \rho = 0$  versus  $H_1: \rho < 0$ . Reject  $H_0$  at  $\alpha$  level if the estimated  $(4 - d) < d_U$ , that is, there is statistically significant evidence of negative autocorrelation.

3.  $H_0: \rho = 0$  versus  $H_1: \rho \neq 0$ . Reject  $H_0$  at  $2\alpha$  level if  $d < d_U$  or  $(4 - d) < d_U$ , that is, there is statistically significant evidence of autocorrelation, positive or negative.

It may be pointed out that the indecisive zone narrows as the sample size increases, which can be seen clearly from the Durbin–Watson tables. For example, with 4 regressors and 20 observations, the 5 percent lower and upper  $d$  values are 0.894 and 1.828, respectively, but these values are 1.515 and 1.739 if the sample size is 75.

The computer program Shazam performs an *exact d test*, that is, it gives the  $p$  value, the exact probability of the computed  $d$  value. With modern computing facilities, it is no longer difficult to find the  $p$  value of the computed  $d$  statistic. Using SHAZAM (version 9) for our wages–productivity regression, we find the  $p$  value of the computed  $d$  of 0.1229 is practically zero, thereby reconfirming our earlier conclusion based on the Durbin–Watson tables.

The Durbin–Watson  $d$  test has become so venerable that practitioners often forget the assumptions underlying the test. In particular, the assumptions that (1) the explanatory variables, or regressors, are nonstochastic; (2) the error term follows the normal distribution; and (3) that the regression models do not include the lagged value(s) of the regressand are very important for the application of the  $d$  test.

If a regression model contains lagged value(s) of the regressand, the  $d$  value in such cases is often around 2, which would suggest that there is no (first-order) autocorrelation in such models. Thus, there is a built-in bias against discovering (first-order) autocorrelation in such models. This does not mean that autoregressive models do not suffer from the autocorrelation problem. As a matter of fact, Durbin has developed the so-called ***h test*** to test serial correlation in such models. But this test is not as powerful, in a statistical sense, as the **Breusch–Godfrey test** to be discussed shortly, so there is no need to use the ***h test***. However, because of its historical importance, it is discussed in exercise 12.36.

Also, if the error term  $u_t$  are not NIID, the routinely used  $d$  test may not be reliable.<sup>26</sup> In this respect the **runs test** discussed earlier has an advantage in that it does not make any (probability) distributional assumption about the error term. However, if the sample is large (technically infinite), we can use the Durbin–Watson  $d$ , for it can be shown that<sup>27</sup>

$$\sqrt{n} \left( 1 - \frac{1}{2}d \right) \approx N(0, 1) \quad (12.6.12)$$

<sup>26</sup>For an advanced discussion, see Ron C. Mittelhammer, George G. Judge, and Douglas J. Miller, *Econometric Foundations*, Cambridge University Press, New York, 2000, p. 550.

<sup>27</sup>See James Davidson, *Econometric Theory*, Blackwell Publishers, New York, 2000, p. 161.



That is, in large samples the  $d$  statistic as transformed in (12.6.12) follows the standard normal distribution. Incidentally, in view of the relationship between  $d$  and  $\hat{\rho}$ , the estimated first-order autocorrelation coefficient, shown in (12.6.10), it follows that

$$\sqrt{n}\hat{\rho} \approx N(0, 1) \quad (12.6.13)$$

that is, in large samples, the square root of the sample size times the estimated first-order autocorrelation coefficient also follows the standard normal distribution.

As an illustration of the test, for our wages–productivity example, we found that  $d = 0.1229$  with  $n = 40$ . Therefore, from (12.6.12) we find that

$$\sqrt{40} \left( 1 - \frac{0.1229}{2} \right) \approx 5.94$$

Asymptotically, if the null hypothesis of zero (first-order) autocorrelation were true, the probability of obtaining a  $Z$  value (i.e., a standardized normal variable) of as much as 5.94 or greater is extremely small. Recall that for a standard normal distribution, the (two-tail) critical 5 percent  $Z$  value is only 1.96 and the 1 percent critical  $Z$  value is about 2.58. Although our sample size is only 40, for practical purposes it may be large enough to use the normal approximation. The conclusion remains the same, namely, that the residuals from the wages–productivity regression suffer from autocorrelation.

But the most serious problem with the  $d$  test is the assumption that the regressors are nonstochastic, that is, their values are fixed in repeated sampling. If this is not the case, then the  $d$  test is not valid either in finite, or small, samples or in large samples.<sup>28</sup> And since this assumption is usually difficult to maintain in economic models involving time series data, one author contends that the Durbin–Watson statistic may not be useful in econometrics involving time series data.<sup>29</sup> In his view, more useful tests of autocorrelation are available, but they are all based on large samples. We discuss one such test below, the **Breusch–Godfrey test**.

#### IV. A General Test of Autocorrelation: The Breusch–Godfrey (BG) Test<sup>30</sup>

To avoid some of the pitfalls of the Durbin–Watson  $d$  test of autocorrelation, statisticians Breusch and Godfrey have developed a test of autocorrelation that is general in the sense that it allows for (1) nonstochastic regressors, such as the lagged values of the regressand; (2) higher-order autoregressive

<sup>28</sup>Ibid., p. 161.

<sup>29</sup>Fumio Hayashi, *Econometrics*, Princeton University Press, Princeton, N.J., 2000, p. 45.

<sup>30</sup>See, L. G. Godfrey, “Testing Against General Autoregressive and Moving Average Error Models When the Regressor include Lagged Dependent Variables,” *Econometrica*, vol. 46, 1978, pp. 1293–1302, and T. S. Breusch, “Testing for Autocorrelation in Dynamic Linear Models,” *Australian Economic Papers*, vol. 17, 1978, pp. 334–355.



schemes, such as AR(1), AR(2), etc.; and (3) simple or higher-order **moving averages** of white noise error terms, such as  $\varepsilon_t$  in (12.2.1).<sup>31</sup>

Without going into the mathematical details, which can be obtained from the references, the **BG test**, which is also known as the **LM test**,<sup>32</sup> proceeds as follows: We use the two-variable regression model to illustrate the test, although many regressors can be added to the model. Also, lagged values of the regressand can be added to the model. Let

$$Y_t = \beta_1 + \beta_2 X_t + u_t \quad (12.6.14)$$

Assume that the error term  $u_t$  follows the  $p$ th-order autoregressive, AR( $p$ ), scheme as follows:

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \cdots + \rho_p u_{t-p} + \varepsilon_t \quad (12.6.15)$$

where  $\varepsilon_t$  is a white noise error term as discussed previously. As you will recognize, this is simply the extension of the AR(1) scheme.

The null hypothesis  $H_0$  to be tested is that

$$H_0: \rho_1 = \rho_2 = \cdots = \rho_p = 0 \quad (12.6.16)$$

That is, there is no serial correlation of any order. The BG test involves the following steps:

1. Estimate (12.6.14) by OLS and obtain the residuals,  $\hat{u}_t$ .
2. Regress  $\hat{u}_t$  on the original  $X_t$  (if there is more than one  $X$  variable in the original model, include them also) and  $\hat{u}_{t-1}, \hat{u}_{t-2}, \dots, \hat{u}_{t-p}$ , where the latter are the lagged values of the estimated residuals in step 1. Thus, if  $p = 4$ , we will introduce four lagged values of the residuals as additional regressor in the model. Note that to run this regression we will have only  $(n - p)$  observations (why?). In short, run the following regression:

$$\hat{u}_t = \alpha_1 + \alpha_2 X_t + \hat{\rho}_1 \hat{u}_{t-1} + \hat{\rho}_2 \hat{u}_{t-2} + \cdots + \hat{\rho}_p \hat{u}_{t-p} + \varepsilon_t \quad (12.6.17)$$

and obtain  $R^2$  from this (auxiliary) regression.<sup>33</sup>

3. If the sample size is large (technically, infinite), Breusch and Godfrey have shown that

$$(n - p)R^2 \sim \chi_p^2 \quad (12.6.18)$$

<sup>31</sup>For example, in the regression  $Y_t = \beta_1 + \beta_2 X_t + u_t$  the error term can be represented as  $u_t = \varepsilon_t + \lambda_1 \varepsilon_{t-1} + \lambda_2 \varepsilon_{t-2}$ , which represents a three-period moving average of the white noise error term  $\varepsilon_t$ .

<sup>32</sup>The test is based on the **Lagrange Multiplier principle** briefly mentioned in Chap. 8.

<sup>33</sup>The reason that the original regressor  $X$  is included in the model is to allow for the fact that  $X$  may not be strictly nonstochastic. But if it is strictly nonstochastic, it may be omitted from the model. On this, see Jeffrey M. Wooldridge, *Introductory Econometrics: A Modern Approach*, South-Western Publishing Co., 200, p. 386.

That is, asymptotically,  $n - p$  times the  $R^2$  value obtained from the auxiliary regression (12.6.17) follows the chi-square distribution with  $p$  df. If in an application,  $(n - p)R^2$  exceeds the critical chi-square value at the chosen level of significance, we reject the null hypothesis, in which case at least one rho in (12.6.15) is statistically significantly different from zero.

The following *practical points* about the BG test may be noted:

1. The regressors included in the regression model may contain lagged values of the regressand  $Y$ , that is,  $Y_{t-1}$ ,  $Y_{t-2}$ , etc., may appear as explanatory variables. Contrast this model with the Durbin–Watson test restriction that there be no lagged values of the regressand among the regressors.

2. As noted earlier, the BG test is applicable even if the disturbances follow a  $p$ th-order **moving average (MA)** process, that is, the  $u_t$  are generated as follows:

$$u_t = \varepsilon_t + \lambda_1 \varepsilon_{t-1} + \lambda_2 \varepsilon_{t-2} + \cdots + \lambda_p \varepsilon_{t-p} \quad (12.6.19)$$

where  $\varepsilon_t$  is a white noise error term, that is, the error term that satisfies all the classical assumptions.

In the chapters on time series econometrics, we will study in some detail the  $p$ th-order autoregressive and moving average processes.

3. If in (12.6.15)  $p = 1$ , meaning first-order autoregression, then the BG test is known as **Durbin's  $M$  test**.

4. A drawback of the BG test is that the value of  $p$ , the length of the lag, cannot be specified a priori. Some experimentation with the  $p$  value is inevitable. Sometimes one can use the so-called **Akaike** and **Schwarz** information criteria to select the lag length. We will discuss these criteria in Chapter 13 and later in the chapters on time series econometrics.

#### ILLUSTRATION OF THE BG TEST: THE WAGES–PRODUCTIVITY RELATION

To illustrate the test, we will apply it to our illustrative example. Using an AR(6) scheme, we obtained the results shown in exercise 12.25. From the regression results given there, it can be seen that  $(n - p) = 34$  and  $R^2 = 0.8920$ . Therefore, multiplying these two, we obtain a chi-square value of 30.328. For 6 df (why?), the probability of obtaining a chi-square value of as much as 30.328 or greater is extremely small; the chi-square table in Appendix D.4 shows that the probability of ob-

taining a chi-square value of as much as 18.5476 or greater is only 0.005. Therefore, for the same df, the probability of obtaining a chi-square value of about 30 must be extremely small. As a matter of fact, the actual  $p$  value is almost zero.

Therefore, the conclusion is that, for our example, at least one of the six autocorrelations must be nonzero.

Trying varying lag lengths from 1 to 6, we find that only the AR(1) coefficient is significant, suggesting that there is no need to consider more than one lag. In essence the BG test in this case turns out to be **Durbin's  $m$  test**.

### Why So Many Tests of Autocorrelation?

The answer to this question is that “. . . no particular test has yet been judged to be unequivocally best [i.e., more powerful in the statistical sense], and thus the analyst is still in the unenviable position of considering a

varied collection of test procedures for detecting the presence or structure, or both, of autocorrelation.”<sup>34</sup> Of course, a similar argument can be made about the various tests of heteroscedasticity discussed in the previous chapter.

### 12.7 WHAT TO DO WHEN YOU FIND AUTOCORRELATION: REMEDIAL MEASURES

If after applying one or more of the diagnostic tests of autocorrelation discussed in the previous section, we find that there is autocorrelation, what then? We have four options:

1. Try to find out if the autocorrelation is **pure autocorrelation** and not the result of mis-specification of the model. As we discussed in Section 12.1, sometimes we observe patterns in residuals because the model is mis-specified—that is, it has excluded some important variables—or because its functional form is incorrect.

2. If it is pure autocorrelation, one can use appropriate transformation of the original model so that in the transformed model we do not have the problem of (pure) autocorrelation. As in the case of heteroscedasticity, we will have to use some type of **generalized least-square (GLS) method**.

3. In large samples, we can use the **Newey–West** method to obtain standard errors of OLS estimators that are corrected for autocorrelation. This method is actually an extension of White’s heteroscedasticity-consistent standard errors method that we discussed in the previous chapter.

4. In some situations we can continue to use the OLS method.

Because of the importance of each of these topics, we devote a section to each one.

### 12.8 MODEL MIS-SPECIFICATION VERSUS PURE AUTOCORRELATION

Let us return to our wages–productivity regression given in (12.5.1). There we saw that the  $d$  value was 0.1229 and based on the Durbin–Watson  $d$  test we concluded that there was positive correlation in the error term. Could this correlation have arisen because our model was not correctly specified? Since the data underlying regression (12.5.1) is time series data, it is quite possible that both wages and productivity exhibit trends. If that is the case,

<sup>34</sup>Ron C. Mittelhammer et al., op. cit., p. 547. Recall that the **power of a statistical test** is one minus the probability of committing a Type II error, that is, one minus the probability of accepting a false hypothesis. The maximum power of a test is 1 and the minimum is 0. The closer of the power of a test is to zero, the worse is that test, and the closer it is to 1, the more powerful is that test. What these authors are essentially saying is that there is no single most powerful test of autocorrelation.

then we need to include the time or trend,  $t$ , variable in the model to see the relationship between wages and productivity net of the trends in the two variables.

To test this, we included the trend variable in (12.5.1) and obtained the following results

$$\begin{aligned}\hat{Y}_t &= 1.4752 + 1.3057X_t - 0.9032t \\ \text{se} &= (13.18) \quad (0.2765) \quad (0.4203) \\ t &= (0.1119) \quad (4.7230) \quad (-2.1490) \\ R^2 &= 0.9631; \quad d = 0.2046\end{aligned}\tag{12.8.1}$$

The interpretation of this model is straightforward: Over time, the index of real wages has been decreasing by about 0.90 units per year. After allowing for this, if the productivity index went up by a unit, on average, the real wage index went up by about 1.30 units, although this number is not statistically different from one (why?). What is interesting to note is that even allowing for the trend variable, the  $d$  value is still very low, suggesting that (12.8.1) suffers from pure autocorrelation and not necessarily specification error.

How do we know that (12.8.1) is the correct specification? To test this, we regress  $Y$  on  $X$  and  $X^2$  to test for the possibility that the real wage index may be nonlinearly related to the productivity index. The results of this regression are as follows:

$$\begin{aligned}\hat{Y}_t &= -16.2181 + 1.9488X_t - 0.0079X_t^2 \\ t &= (-5.4891) \quad (24.9868) \quad (-15.9363) \\ R^2 &= 0.9947 \quad d = 1.02\end{aligned}\tag{12.8.2}$$

These results are interesting. All the coefficients are statistically highly significant, the  $p$  values being extremely small. From the negative quadratic term, it seems that although the real wage index increases as the productivity index increases, it increases at a decreasing rate. But look at the  $d$  value. It still suggests positive autocorrelation in the residuals, for  $d_L = 1.391$  and  $d_U = 1.60$  and the estimated  $d$  value lies below  $d_L$ .

It may be safe to conclude from the preceding analysis that our wages–productivity regression probably suffers from pure autocorrelation and not necessarily from specification bias. Knowing the consequences of autocorrelation, we may therefore want to take some corrective action. We will do so shortly.

Incidentally, for all the wages–productivity regressions that we have presented above, we applied the **Jarque–Bera test of normality** and found that the residuals were normally distributed, which is comforting because the  $d$  test assumes normality of the error term.

### 12.9 CORRECTING FOR (PURE) AUTOCORRELATION: THE METHOD OF GENERALIZED LEAST SQUARES (GLS)

Knowing the consequences of autocorrelation, especially the lack of efficiency of OLS estimators, we may need to remedy the problem. The remedy depends on the knowledge one has about the nature of interdependence among the disturbances, that is, knowledge about the structure of autocorrelation.

As a starter, consider the two-variable regression model:

$$Y_t = \beta_1 + \beta_2 X_t + u_t \quad (12.9.1)$$

and assume that the error term follows the AR(1) scheme, namely,

$$u_t = \rho u_{t-1} + \varepsilon_t \quad -1 < \rho < 1 \quad (12.9.2)$$

Now we consider two cases: (1)  $\rho$  is known and (2)  $\rho$  is not known but has to be estimated.

#### When $\rho$ Is Known

If the coefficient of first-order autocorrelation is known, the problem of autocorrelation can be easily solved. If (12.9.1) holds true at time  $t$ , it also holds true at time  $(t - 1)$ . Hence,

$$Y_{t-1} = \beta_1 + \beta_2 X_{t-1} + u_{t-1} \quad (12.9.3)$$

Multiplying (12.9.3) by  $\rho$  on both sides, we obtain

$$\rho Y_{t-1} = \rho \beta_1 + \rho \beta_2 X_{t-1} + \rho u_{t-1} \quad (12.9.4)$$

Subtracting (12.9.4) from (12.9.1) gives

$$(Y_t - \rho Y_{t-1}) = \beta_1(1 - \rho) + \beta_2(X_t - \rho X_{t-1}) + \varepsilon_t \quad (12.9.5)$$

where  $\varepsilon_t = (u_t - \rho u_{t-1})$

We can express (12.9.5) as

$$Y_t^* = \beta_1^* + \beta_2^* X_t^* + \varepsilon_t \quad (12.9.6)$$

where  $\beta_1^* = \beta_1(1 - \rho)$ ,  $Y_t^* = (Y_t - \rho Y_{t-1})$ ,  $X_t^* = (X_t - \rho X_{t-1})$ , and  $\beta_2^* = \beta_2$ .

Since the error term in (12.9.6) satisfies the usual OLS assumptions, we can apply OLS to the transformed variables  $Y^*$  and  $X^*$  and obtain estimators with all the optimum properties, namely, BLUE. In effect, running (12.9.6) is tantamount to using generalized least squares (GLS) discussed in the previous chapter—recall that GLS is nothing but OLS applied to the transformed model that satisfies the classical assumptions.

Regression (12.9.5) is known as the **generalized, or quasi, difference equation**. It involves regressing  $Y$  on  $X$ , not in the original form, but in the **difference form**, which is obtained by subtracting a proportion ( $= \rho$ ) of the value of a variable in the previous time period from its value in the current time period. In this differencing procedure we lose one observation because the first observation has no antecedent. To avoid this loss of one observation, the first observation on  $Y$  and  $X$  is transformed as follows<sup>35</sup>:  $Y_1\sqrt{1-\rho^2}$  and  $X_1\sqrt{1-\rho^2}$ . This transformation is known as the **Prais-Winsten transformation**.

### When $\rho$ Is Not Known

Although conceptually straightforward to apply, the method of generalized difference given in (12.9.5) is difficult to implement because  $\rho$  is rarely known in practice. Therefore, we need to find ways of estimating  $\rho$ . We have several possibilities.

**The First-Difference Method.** Since  $\rho$  lies between 0 and  $\pm 1$ , one could start from two extreme positions. At one extreme, one could assume that  $\rho = 0$ , that is, no (first-order) serial correlation, and at the other extreme we could let  $\rho = \pm 1$ , that is, perfect positive or negative correlation. As a matter of fact, when a regression is run, one generally assumes that there is no autocorrelation and then lets the Durbin-Watson or other test show whether this assumption is justified. If, however,  $\rho = +1$ , the generalized difference equation (12.9.5) reduces to the **first-difference equation**:

$$Y_t - Y_{t-1} = \beta_2(X_t - X_{t-1}) + (u_t - u_{t-1})$$

or

$$\Delta Y_t = \beta_2 \Delta X_t + \varepsilon_t \quad (12.9.7)$$

where  $\Delta$  is the first-difference operator introduced in (12.1.10)

Since the error term in (12.9.7) is free from (first-order) serial correlation (why?), to run the regression (12.9.7) all one has to do is form the first differences of both the regressand and regressor(s) and run the regression on these first differences.

The first difference transformation may be appropriate if the coefficient of autocorrelation is very high, say in excess of 0.8, or the Durbin-Watson  $d$  is quite low. Maddala has proposed this rough rule of thumb: *Use the first difference form whenever  $d < R^2$* .<sup>36</sup> This is the case in our wages-productivity

<sup>35</sup>The loss of one observation may not be very serious in large samples but can make a substantial difference in the results in small samples. Without transforming the first observation as indicated, the error variance will not be homoscedastic. On this see, Jeffrey Wooldridge, op. cit., p. 388. For some Monte Carlo results on the importance of the first observation, see Russell Davidson and James G. MacKinnon, *Estimation and Inference in Econometrics*, Oxford University Press, New York, 1993, Table 10.1, p. 349.

<sup>36</sup>Maddala, op. cit., p. 232.

regression (12.5.1), where we found that  $d = 0.1229$  and  $r^2 = 0.9584$ . The first-difference regression for our illustrative example will be presented shortly.

An interesting feature of the first-difference model (12.9.7) is that **there is no intercept in it**. Hence, to estimate (12.9.7), you have to use the **regression through the origin** routine (that is, suppress the intercept term), which is now available in most software packages. If, however, you forget to drop the intercept term in the model and estimate the following model that includes the intercept term

$$\Delta Y_t = \beta_1 + \beta_2 \Delta X_t + \varepsilon_t \quad (12.9.8)$$

then the original model must have a *trend* in it and  $\beta_1$  represents the coefficient of the trend variable.<sup>37</sup> Therefore, one “accidental” benefit of introducing the intercept term in the first-difference model is to test for the presence of a trend variable in the original model.

Returning to our wages–productivity regression (12.5.1), and given the AR(1) scheme and a low  $d$  value in relation to  $r^2$ , we rerun (12.5.1) in the first-difference form without the intercept term; remember that (12.5.1) is in the *level form*. The results are as follows<sup>38</sup>:

$$\begin{aligned} \widehat{\Delta Y}_t &= 0.7199 \Delta X_t \\ t &= (9.2073) \quad r^2 = 0.3610 \quad d = 1.5096 \end{aligned} \quad (12.9.9)$$

Compared with the level form regression (12.5.1), we see that the slope coefficient has not changed much, but the  $r^2$  value has dropped considerably. This is generally the case because by taking the first differences we are essentially studying the behavior of variables around their (linear) trend values. Of course, we cannot compare the  $r^2$  of (12.9.9) directly with that of the  $r^2$  of (12.5.1) because the dependent variables in the two models are different.<sup>39</sup> Also, notice that compared with the original regression, the  $d$  value has increased dramatically, perhaps indicating that there is little autocorrelation in the first-difference regression.<sup>40</sup>

Another interesting aspect of the first-difference transformation relates to the *stationarity* properties of the underlying time series. Return to Eq. (12.2.1), which describes the AR(1) scheme. Now if in fact  $\rho = 1$ , then it is clear from Eqs. (12.2.3) and (12.2.4) that the series  $u_t$  is *nonstationary*, for the variances and covariances become infinite. That is why, when we

<sup>37</sup>This is easy to show. Let  $Y_t = \alpha_1 + \beta_1 t + \beta_2 X_t + u_t$ . Therefore,  $Y_{t-1} = \alpha + \beta_1(t-1) + \beta_2 X_{t-1} + u_{t-1}$ . Subtracting the latter from the former, you will obtain:  $\Delta Y_t = \beta_1 + \beta_2 \Delta X_t + \varepsilon_t$ , which shows that the intercept term in this equation is indeed the coefficient of the trend variable in the original model. Remember that we are assuming that  $\rho = 1$ .

<sup>38</sup>In exercise 12.38 you are asked to run this model, including the constant term.

<sup>39</sup>The comparison of  $r^2$  in the level and first-difference form is slightly involved. For an extended discussion on this, see Maddala, *op. cit.*, Chap. 6.

<sup>40</sup>It is not clear whether the computed  $d$  in the first-difference regression can be interpreted in the same way as it was in the original, level form regression. However, applying the runs test, it can be seen that there is no evidence of autocorrelation in the residuals of the first-difference regression.



discussed this topic, we put the restriction that  $|\rho| < 1$ . But it is clear from (12.2.1) that if the autocorrelation coefficient is in fact 1, then (12.2.1) becomes

$$u_t = u_{t-1} + \varepsilon_t$$

or

$$(u_t - u_{t-1}) = \Delta u_t = \varepsilon_t \quad (12.9.10)$$

That is, it is the first-differenced  $u_t$  that becomes stationary, for it is equal to  $\varepsilon_t$ , which is a white noise error term.

The point of the preceding discussion is that if the original time series are nonstationary, very often their first differences become stationary. And, therefore, first-difference transformation serves a dual purpose in that it might get rid of (first-order) autocorrelation and also render the time series stationary. We will revisit this topic in **Part V**, where we discuss the econometrics of time series analysis in some depth.

We mentioned that the first-difference transformation may be appropriate if  $\rho$  is high or  $d$  is low. Strictly speaking, the first-difference transformation is valid only if  $\rho = 1$ . As a matter of fact, there is a test, called the **Berenblutt-Webb test**,<sup>41</sup> to test the hypothesis that  $\rho = 1$ . The test statistic they use is called the ***g* statistic**, which is defined as follows:

$$g = \frac{\sum_2^n \hat{e}_t^2}{\sum_1^n \hat{u}_t^2} \quad (12.9.11)$$

where  $\hat{u}_t$  are the OLS residuals from the original (i.e., level form) regression and  $e_t$  are the OLS residuals from the first-difference regression. Keep in mind that in the first-difference form there is no intercept.

To test the significance of the  $g$  statistic, assuming that the level form regression contains the intercept term, we can use the Durbin-Watson tables except that now the null hypothesis is that  $\rho = 1$  rather than the Durbin-Watson hypothesis that  $\rho = 0$ .

Revisiting our wages-productivity regression, for the original regression (12.5.1) we obtain  $\sum \hat{u}_t^2 = 272.0220$  and for the first regression (12.7.11) we obtain  $\sum \hat{e}_t^2 = 0.334270$ . Putting these values into the  $g$  statistic given in (12.9.11), we obtain

$$g = \frac{0.334270}{272.0220} = 0.0012 \quad (12.9.12)$$

Consulting the Durbin-Watson table for 39 observations and 1 explanatory variable, we find that  $d_L = 1.435$  and  $d_U = 1.540$  (5 percent level). Since the observed  $g$  lies below the lower limit of  $d$ , we do not reject the hypothesis that true  $\rho = 1$ . *Keep in mind that although we use the same Durbin-Watson*

<sup>41</sup>I. I. Berenblutt and G. I. Webb, "A New Test for Autocorrelated Errors in the Linear Regression Model," *Journal of the Royal Statistical Society*, Series B, vol. 35, No.1, 1973, pp. 33-50.



tables, now the null hypothesis is that  $\rho = 1$  and not that  $\rho = 0$ . In view of this finding, the results given in (12.9.9) may be acceptable.

**$\rho$  Based on Durbin–Watson  $d$  Statistic.** If we cannot use the first difference transformation because  $\rho$  is not sufficiently close to unity, we have an easy method of estimating it from the relationship between  $d$  and  $\rho$  established previously in (12.6.10), from which we can estimate  $\rho$  as follows:

$$\hat{\rho} \approx 1 - \frac{d}{2} \quad (12.9.13)$$

Thus, in reasonably large samples one can obtain rho from (12.9.13) and use it to transform the data as shown in the generalized difference equation (12.9.5). Keep in mind that the relationship between  $\rho$  and  $d$  given in (12.9.13) may not hold true in small samples, for which Theil and Nagar have proposed a modification, which is given in exercise 12.6.

In our wages–productivity regression (12.5.1), we obtain a  $d$  value of 0.1229. Using this value in (12.9.13), we obtain  $\hat{\rho} \approx 0.9386$ . Using this estimated rho value, we can estimate regression (12.9.5). All we have to do is subtract 0.9386 times the previous value of  $Y$  from its current value and similarly subtract 0.9386 times the previous value of  $X$  from its current value and run the OLS regression on the variables thus transformed as in (12.9.6), where  $Y_t^* = (Y_t - 0.9386Y_{t-1})$  and  $X_t^* = (X_t - 0.9386X_{t-1})$ .

**$\rho$  Estimated from the Residuals.** If the AR(1) scheme  $u_t = \rho u_{t-1} + \varepsilon_t$  is valid, a simple way to estimate rho is to regress the residuals  $\hat{u}_t$  on  $\hat{u}_{t-1}$ , for the  $\hat{u}_t$  are consistent estimators of the true  $u_t$ , as noted previously. That is, we run the following regression:

$$\hat{u}_t = \rho \cdot \hat{u}_{t-1} + v_t \quad (12.9.14)$$

where  $\hat{u}_t$  are the residuals obtained from the original (level form) regression and where  $v_t$  are the error term of this regression. Note that there is no need to introduce the intercept term in (12.9.14), for we know the OLS residuals sum to zero.

The residuals from our wages–productivity regression given in (12.5.1) are already shown in Table 12.5. Using these residuals, the following regression results were obtained:

$$\begin{aligned} \hat{u}_t &= 0.9142\hat{u}_{t-1} \\ t &= (16.2281) \quad r^2 = 0.8736 \end{aligned} \quad (12.9.15)$$

As this regression shows,  $\hat{\rho} = 0.9142$ . Using this estimate, one can transform the original model as per (12.9.6). Since the rho estimated by this procedure is about the same as that obtained from the Durbin–Watson  $d$ , the

regression results using the rho of (12.9.15) should not be very different from those obtained from the rho estimated from the Durbin–Watson  $d$ . We leave it to the reader to verify this.

**Iterative Methods of Estimating  $\rho$ .** All the methods of estimating  $\rho$  discussed previously provide us with only a single estimate of  $\rho$ . But there are the so-called **iterative methods** that estimate  $\rho$  iteratively, that is, by successive approximation, starting with some initial value of  $\rho$ . Among these methods the following may be mentioned: the **Cochrane–Orcutt iterative procedure**, the **Cochrane–Orcutt two-step procedure**, the **Durbin two-step procedure**, and the **Hildreth–Lu scanning or search procedure**. Of these, the most popular is the Cochrane–Orcutt iterative method. To save space, the iterative methods are discussed by way of exercises. Remember that the ultimate objective of these methods is to provide an estimate of  $\rho$  that may be used to obtain GLS estimates of the parameters. One advantage of the Cochrane–Orcutt iterative method is that it can be used to estimate not only an AR(1) scheme, but also higher-order autoregressive schemes, such as  $\hat{u}_t = \hat{\rho}_1 \hat{u}_{t-1} + \hat{\rho}_2 \hat{u}_{t-2} + v_t$ , which is AR(2). Having obtained the two rhos, one can easily extend the generalized difference equation (12.9.6). Of course, the computer can now do all this.

Returning to our wages–productivity regression, and assuming an AR(1) scheme, we use the Cochrane–Orcutt iterative method, which gives the following estimates of rho: 0.9142, 0.9052, 0.8992, 0.8956, 0.8935, 0.8924, and 0.8919. The last value of 0.8919 can now be used to transform the original model as in (12.9.6) and estimate it by OLS. Of course, OLS on the transformed model is simply the GLS. The results are as follows:

**Dropping the First Observation** Since the first observation has no antecedent, in estimating (12.9.6), we drop the first observation. The regression results are as follows:

$$\begin{aligned} \hat{Y}_t^* &= 45.105 + 0.5503X_t^* \\ \text{se} &= (6.190) \quad (0.0652) && \text{(12.9.16)} \\ t &= (7.287) \quad (8.433) && r^2 = 0.9959 \end{aligned}$$

Comparing the results of this regression with the original regression given in (12.5.1), we see that the slope coefficient has dropped dramatically. Notice two things about (12.9.16). First, the intercept coefficient in (12.9.16) is  $\beta_1(1 - \rho)$ , from which the original  $\beta_1$  can be easily retrieved, since we know that  $\rho = 0.8913$ . Secondly, the  $r^2$ 's of the transformed model (12.9.16) and the original model (12.5.1) cannot be directly compared, since the dependent variables in the two models are different.

**Retaining the First Observation à la Prais–Winsten.** We cautioned earlier that in small samples keeping the first observation or omitting it can

make a substantial difference in small samples, although in large samples the difference may be inconsequential.

Retaining the first observation à la Prais–Winsten, we obtain the following regression results<sup>42</sup>:

$$\begin{aligned} \hat{Y}_t^* &= 26.454 + 0.7245X_t^* \\ \text{se} &= (5.4520) \quad (0.0612) \\ t &= (4.8521) \quad (11.8382) \quad r^2 = 0.9949 \end{aligned} \tag{12.9.17}$$

The difference between (12.9.16) and (12.9.17) tells us that the inclusion or exclusion of the first observation can make a substantial difference in the regression results. Also, note that the slope coefficient in (12.9.17) is approximately the same as that in (12.5.1).

**General Comments.** There are several points about correcting for autocorrelation using the various methods discussed above.

*First*, since the OLS estimators are consistent despite autocorrelation, in large samples, it makes little difference whether we estimate  $\rho$  from the Durbin–Watson  $d$ , or from the regression of the residuals in the current period on the residuals in the previous period, or from the Cochrane–Orcutt iterative procedure because they all provide consistent estimates of the true  $\rho$ . *Second*, the various methods discussed above are basically two-step methods. In step 1 we obtain an estimate of the unknown  $\rho$  and in step 2 we use that estimate to transform the variables to estimate the generalized difference equation, which is basically GLS. But since we use  $\hat{\rho}$  instead of the true  $\rho$ , all these methods of estimation are known in the literature as **feasible GLS (FGLS)** or **estimated GLS (EGLS)** methods.

*Third*, it is important to note that whenever we use an **FGLS** or **EGLS** method to estimate the parameters of the transformed model, the estimated coefficients will not necessarily have the usual optimum properties of the classical model, such as BLUE, especially in small samples. Without going into complex technicalities, it may be stated as a *general principle that whenever we use an estimator in place of its true value, the estimated OLS coefficients may have the usual optimum properties asymptotically, that is, in large samples. Also, the conventional hypothesis testing procedures are, strictly speaking, valid asymptotically. In small samples, therefore, one has to be careful in interpreting the estimated results.*

*Fourth*, in using EGLS, if we do not include the first observation (as was originally the case with the Cochrane–Orcutt procedure), not only the

<sup>42</sup>Including the first observation, the iterated values of rho are: 0.9142, 9.9462, 0.9556, 0.9591, 0.9605, and 0.9610. The last value was used in transforming the data to form the generalized difference equation.

numerical values but also the efficiency of the estimators can be adversely affected, especially if the sample size is small and if the regressors are not strictly speaking nonstochastic.<sup>43</sup> Therefore, in small samples it is important to keep the first observation *à la* Prais–Winsten. Of course, if the sample size is reasonably large, EGLS, with or without the first observation, gives similar results. Incidentally, in the literature EGLS with Prais–Winsten transformation is known as the **full EGLS**, or **FEGLS**, for short.

### 12.10 THE NEWEY–WEST METHOD OF CORRECTING THE OLS STANDARD ERRORS

Instead of using the FGLS methods discussed in the previous section, we can still use OLS but correct the standard errors for autocorrelation by a procedure developed by Newey and West.<sup>44</sup> This is an extension of White's heteroscedasticity-consistent standard errors that we discussed in the previous chapter. The corrected standard errors are known as **HAC (heteroscedasticity- and autocorrelation-consistent) standard errors** or simply as **Newey–West standard errors**. We will not present the mathematics behind the Newey–West procedure, for it is involved.<sup>45</sup> But most modern computer packages now calculate the Newey–West standard errors. But it is important to point out that the Newey–West procedure is *strictly speaking valid in large samples* and may not be appropriate in small samples. But in large samples we now have a method that produces autocorrelation-corrected standard errors so that we do not have to worry about the EGLS transformations discussed in the previous chapter. Therefore, if a sample is reasonably large, one should use the Newey–West procedure to correct OLS standard errors not only in situations of autocorrelation only but also in cases of heteroscedasticity, for the HAC method can handle both, unlike the White method, which was designed specifically for heteroscedasticity.

Once again let us return to our wages–productivity regression (12.5.1). We know that this regression suffers from autocorrelation. Our sample of 40 observations is reasonably large, so we can use the HAC procedure. Using *Eviews 4*, we obtain the following regression results:

$$\begin{aligned} \hat{Y}_t &= 29.5192 + 0.7136\hat{X}_t \\ \text{se} &= (4.1180)^* \quad (0.0512)^* && \text{(12.10.1)} \\ & && r^2 = 0.9584 \quad d = 0.1229 \end{aligned}$$

where \* denotes HAC standard errors.

<sup>43</sup>This is especially so if the regressors exhibit a trend, which is quite common in economic data.

<sup>44</sup>W. K. Newey, and K. West, "A Simple Positive Semi-Definite Heteroscedasticity and Autocorrelation Consistent Covariance Matrix," *Econometrica*, vol. 55, 1987, pp. 703–708.

<sup>45</sup>If you can handle matrix algebra, the method is discussed in Greene, op. cit, 4th ed., pp. 462–463.

Comparing this regression with (12.5.1), we find that in both the equations the estimated coefficients and the  $r^2$  value are the same. But, importantly, note that the HAC standard errors are much greater than the OLS standard errors and therefore the HAC  $t$  ratios are much smaller than the OLS  $t$  ratios. This shows that OLS had in fact underestimated the true standard errors. Curiously, the  $d$  statistics in both (12.5.1) and (12.10.1) is the same. But don't worry, for the HAC procedure has already taken this into account in correcting the OLS standard errors.

### 12.11 OLS VERSUS FGLS AND HAC

The practical problem facing the researcher is this: In the presence of autocorrelation, OLS estimators, although unbiased, consistent, and asymptotically normally distributed, are not efficient. Therefore, the usual inference procedure based on the  $t$ ,  $F$ , and  $\chi^2$  tests is no longer appropriate. On the other hand, FGLS and HAC produce estimators that are efficient, but the finite, or small-sample, properties of these estimators are not well documented. This means in small samples the FGLS and HAC might actually do worse than OLS. As a matter of fact, in a Monte Carlo study Griliches and Rao<sup>46</sup> found that if the sample is relatively small and the coefficient of autocorrelation,  $\rho$ , is less than 0.3, OLS is as good or better than FGLS. As a practical matter, then, one may use OLS in small samples in which the estimated  $\rho$  is, say, less than 0.3. Of course, what is a large and what is a small sample are relative questions, and one has to use some practical judgment. If you have only 15 to 20 observations, the sample may be small, but if you have, say, 50 or more observations, the sample may be reasonably large.

### 12.12 FORECASTING WITH AUTOCORRELATED ERROR TERMS

In Section 5.10, we introduced the basics of forecasting in the context of the two-variable regression model using the classical framework. How do these basics change if there is autocorrelation? Although this topic is generally covered in a course in economic forecasting, we can provide a glimpse of it here. To be specific, we will continue with the two-variable model and assume an AR(1) scheme. Thus,

$$Y_t = \beta_1 + \beta_2 X_t + u_t \quad (12.12.1)$$

$$u_t = \rho u_{t-1} + \varepsilon_t \quad -1 < \rho < 1 \quad (12.12.2)$$

where  $\varepsilon_t$  is a white noise error term.

Substituting (12.12.2) into (12.12.1), we obtain

$$Y_t = \beta_1 + \beta_2 X_t + \rho u_{t-1} + \varepsilon_t \quad (12.12.3)$$

<sup>46</sup>Z. Griliches, and P. Rao, "Small Sample Properties of Several Two-stage Regression Methods in the Context of Autocorrelated Errors," *Journal of the American Statistical Association*, vol. 64, 1969, pp. 253-272.

If we want to forecast  $Y$  for the next time period ( $t + 1$ ), we obtain

$$Y_{t+1} = \beta_1 + \beta_2 X_{t+1} + \rho u_t + \varepsilon_{t+1} \quad (12.12.4)$$

Thus, the forecast for the next period consists of three parts: (1) its expected value  $= (\beta_1 + \beta_2 X_{t+1})$ , (2)  $\rho$  times the preceding error term, and (3) a purely white noise term, whose expected value is zero. Given the value of  $X_{t+1}$ , we estimate (1) by  $\hat{\beta}_1 + \hat{\beta}_2 X_{t+1}$ , where the OLS estimators are obtained from a given sample, and we estimate (2) as  $\hat{\rho} \hat{u}_t$ , where  $\hat{\rho}$  is estimated by one of the methods discussed in Section 12.9. At time ( $t + 1$ ), the value of  $\hat{u}_t$  is already known. Therefore, the *estimated* value of  $Y_{t+1}$  in (12.1.4) is:

$$\hat{Y}_{t+1} = \hat{\beta}_1 + \hat{\beta}_2 X_{t+1} + \hat{\rho} \hat{u}_t \quad (12.12.5)$$

Following this logic,

$$\hat{Y}_{t+2} = \hat{\beta}_1 + \hat{\beta}_2 X_{t+2} + \hat{\rho}^2 \hat{u}_t \quad (12.12.6)$$

for the second period, and so on.

The forecasting that we did in Section 5.10 is called **statistic forecasting**, whereas that represented by (12.12.5) and (12.12.6) is called **dynamic forecasting**, for in making these forecasts we are taking into account the errors made in the past forecasts.

As in Section 5.10, we will need to compute the forecast (standard) errors of (12.12.5) and (12.12.6). But the formulas become complicated.<sup>47</sup> Since most modern econometrics packages, such as Microfit, Eviews, and Shazam, produce the standard errors of forecast, there is no need to present the computing formulas here.

As an illustration, let us fall back on our wages–productivity regression. Recall that our sample data is for the period 1959–1998. We reestimated this model using the data for 1959–1996 only, saving the last two observations for forecasting purposes. Using Microfit 4.1, we obtained the following forecast values of  $Y$  for 1997 and 1998, both static and dynamic, using the estimated regression for 1959–1996.

	Year 1997	Year 1998
Actual $Y$ value	101.1	105.1
Static forecast of $Y$	107.24 (2.64)	109.45 (2.67)
Static forecast error	–6.14	–4.35
Dynamic forecast	100.75 (1.08)	101.95 (1.64)
Dynamic forecast error	0.35	3.14

*Note:* Figures in parentheses are the estimated standard errors of forecast values.

<sup>47</sup>For further discussion, see, Robert S. Pindyck and Daniel L. Rubinfeld, *Econometric Models and Economic Forecasts*, McGraw-Hill, 4th ed., 1998, pp. 214–217.

As you can see from the preceding exercise, the dynamic forecasts are closer to their actual values than the static forecasts and the standard errors of dynamic forecasts are smaller than their static counterpart. So, it may be profitable to incorporate the AR(1) scheme (or higher-order schemes) for the purpose of forecasting. However, note that for both types of forecasts the standard errors of forecast for 1998 are greater than that for 1997, which suggests, not surprisingly, that forecasting into the distant future may be hazardous.

## 12.13 ADDITIONAL ASPECTS OF AUTOCORRELATION

### Dummy Variables and Autocorrelation

In Chapter 9 we considered dummy variable regression models. In particular, recall the U.S. savings–income regression model for 1970–1995 that we presented in (9.5.1), which for convenience is reproduced below:

$$Y_t = \alpha_1 + \alpha_2 + \beta_1 X_t + \beta_2 (D_t X_t) + u_t \quad (12.13.1)$$

where  $Y$  = savings

$X$  = income

$D = 1$  for observations in period 1982–1995

$D = 0$  for observations in period 1970–1981

The regression results based on this model are given in (9.5.4). Of course, this model was estimated with the usual OLS assumptions.

But now suppose that  $u_t$  follows a first-order autoregressive, AR(1), scheme. That is,  $u_t = \rho u_{t-1} + \varepsilon_t$ . Ordinarily, if  $\rho$  is known or can be estimated by one of the methods discussed above, we can use the generalized difference method to estimate the parameters of the model that is free from (first-order) autocorrelation. However, the presence of the dummy variable  $D$  poses a special problem: Note that the dummy variable simply classifies an observation as belonging to the first or second period. How do we transform it? One can follow the following procedure.<sup>48</sup>

1. In (12.13.1), values of  $D$  are zero for all observations in the first period; in period 2 the value of  $D$  for the **first** observation is  $1/(1 - \rho)$  instead of 1, and 1 for all other observations.

2. The variable  $X_t$  is transformed as  $(X_t - \rho X_{t-1})$ . Note that we lose one observation in this transformation, unless one resorts to **Prais–Winsten transformation** for the first observation, as noted earlier.

3. The value of  $D_t X_t$  is zero for all observations in the first period (*note*:  $D_t$  is zero in the first period); in the second period the first observation takes the value of  $D_t X_t = X_t$  and the remaining observations in the second period are set to  $(D_t X_t - D_t \rho X_{t-1}) = (X_t - \rho X_{t-1})$ . (*Note*: the value of  $D_t$  in the second period is 1.)

<sup>48</sup>See Maddala, *op. cit.*, pp. 321–322.



As the preceding discussion points out, the *critical observation* is the first observation in the second period. If this is taken care of in the manner just suggested, there should be no problem in estimating regressions like (12.13.1) subject to AR(1) autocorrelation. In exercise 12.37, the reader is asked to carry such a transformation for the data on U.S. savings and income given in Chapter 9.

### ARCH and GARCH Models

Just as the error term  $u$  at time  $t$  can be correlated with the error term at time  $(t - 1)$  in an AR(1) scheme or with various lagged error terms in a general AR( $p$ ) scheme, can there be autocorrelation in the variance  $\sigma^2$  at time  $t$  with its values lagged one or more periods? Such an autocorrelation has been observed by researchers engaged in forecasting financial time series, such as stock prices, inflation rates, and foreign exchange rates. Such autocorrelation is given the rather daunting names **autoregressive conditional heteroscedasticity (ARCH)** if the error variance is related to the squared error term in the previous term and **generalized autoregressive conditional heteroscedasticity (GARH)** if the error variance is related to squared error terms several periods in the past. Since this topic belongs in the general area of time series econometrics, we will discuss it in some depth in the chapters on time series econometrics. Our objective here is to point out that autocorrelation is not confined to relationships between current and past error terms but also with current and past error variances.

### Coexistence of Autocorrelation and Heteroscedasticity

What happens if a regression model suffers from both heteroscedasticity and autocorrelation? Can we solve the problem sequentially, that is, take care of heteroscedasticity first and then autocorrelation? As a matter of fact, one author contends that “Autoregression can only be detected after the heteroscedasticity is controlled for.”<sup>49</sup> But can we develop an omnipotent test that can solve these and other problems (e.g., model specification) simultaneously? Yes, such tests exist, but their discussion will take us far afield. It is better to leave them for references.<sup>50</sup>

### 12.14 SUMMARY AND CONCLUSIONS

1. If the assumption of the classical linear regression model—that the errors or disturbances  $u_t$  entering into the population regression function (PRF) are random or uncorrelated—is violated, the problem of serial or autocorrelation arises.

<sup>49</sup>Lois W. Sayrs, *Pooled Time Series Analysis*, Sage Publications, California, 1989, p. 19.

<sup>50</sup>See Jeffrey M. Wooldridge, op. cit., pp. 402–403, and A. K. Bera and C. M. Jarque, “Efficient Tests for Normality, Homoscedasticity and Serial Independence of Regression Residuals: Monte Carlo Evidence,” *Economic Letters*, vol. 7, 1981, pp. 313–318.



2. Autocorrelation can arise for several reasons, such as inertia or sluggishness of economic time series, specification bias resulting from excluding important variables from the model or using incorrect functional form, the cobweb phenomenon, data massaging, and data transformation. As a result, it is useful to distinguish between pure autocorrelation and “induced” autocorrelation because of one or more factors just discussed.

3. Although in the presence of autocorrelation the OLS estimators remain unbiased, consistent, and asymptotically normally distributed, they are no longer efficient. As a consequence, the usual  $t$ ,  $F$ , and  $\chi^2$  tests cannot be legitimately applied. Hence, remedial results may be called for.

4. The remedy depends on the nature of the interdependence among the disturbances  $u_t$ . But since the disturbances are unobservable, the common practice is to assume that they are generated by some mechanism.

5. The mechanism that is commonly assumed is the Markov first-order autoregressive scheme, which assumes that the disturbance in the current time period is linearly related to the disturbance term in the previous time period, the coefficient of autocorrelation  $\rho$  providing the extent of the interdependence. This mechanism is known as the AR(1) scheme.

6. If the AR(1) scheme is valid and the coefficient of autocorrelation is known, the serial correlation problem can be easily attacked by transforming the data following the generalized difference procedure. The AR(1) scheme can be easily generalized to an AR( $p$ ). One can also assume a moving average (MA) mechanism or a mixture of AR and MA schemes, known as ARMA. This topic will be discussed in the chapters on time series econometrics.

7. Even if we use an AR(1) scheme, the coefficient of autocorrelation is not known a priori. We considered several methods of estimating  $\rho$ , such as the Durbin–Watson  $d$ , Theil–Nagar modified  $d$ , Cochrane–Orcutt (C–O) iterative procedure, C–O two-step method, and the Durbin two-step procedure. In large samples, these methods generally yield similar estimates of  $\rho$ , although in small samples they perform differently. In practice, the C–O iterative method has become quite popular.

8. Using any of the methods just discussed, we can use the generalized difference method to estimate the parameters of the transformed model by OLS, which essentially amounts to GLS. But since we estimate  $\rho$  ( $= \hat{\rho}$ ), we call the method of estimation as feasible, or estimated, GLS, or FGLS or EGLS for short.

9. In using EGLS, one has to be careful in dropping the first observation, for in small samples the inclusion or exclusion of the first observation can make a dramatic difference in the results. Therefore, in small samples it is advisable to transform the first observation according to the Prais–Winsten procedure. In large samples, however, it makes little difference if the first observation is included or not.

10. It is very important to note that the method of EGLS has the usual optimum statistical properties only in large samples. In small samples, OLS may actually do better than EGLS, especially if  $\rho < 0.3$ .

**11.** Instead of using ECLS, we can still use OLS but correct the standard errors for autocorrelation by the Newey–West HAC procedure. Strictly speaking, this procedure is valid in large samples. One advantage of the HAC procedure is that it not only corrects for autocorrelation but also for heteroscedasticity, if it is present.

**12.** Of course, before remediation comes detection of autocorrelation. There are formal and informal methods of detection. Among the informal methods, one can simply plot the actual or standardized residuals, or plot current residuals against past residuals. Among formal methods, one can use the runs test, Durbin–Watson  $d$  test, asymptotic normality test, Berenblutt–Webb test, and Breusch–Godfrey (BG) test. Of these, the most popular and routinely used is the Durbin–Watson  $d$  test. Despite its hoary past, this test has severe limitations. It is better to use the BG test, for it is much more general in that it allows for both AR and MA error structures as well as the presence of lagged regressand as an explanatory variable. But keep in mind that it is a large sample test.

**13.** In this chapter we also discussed very briefly the detection of autocorrelation in the presence of dummy regressors, the use of autocorrelated errors for forecasting purposes, and the topic of ARCH and GARCH.

## EXERCISES

### Questions

- 12.1.** State whether the following statements are true or false. Briefly justify your answer.
- When autocorrelation is present, OLS estimators are biased as well as inefficient.
  - The Durbin–Watson  $d$  test assumes that the variance of the error term  $u_t$  is homoscedastic.
  - The first-difference transformation to eliminate autocorrelation assumes that the coefficient of autocorrelation  $\rho$  is  $-1$ .
  - The  $R^2$  values of two models, one involving regression in the first-difference form and another in the level form, are not directly comparable.
  - A significant Durbin–Watson  $d$  does not necessarily mean there is autocorrelation of the first order.
  - In the presence of autocorrelation, the conventionally computed variances and standard errors of forecast values are inefficient.
  - The exclusion of an important variable(s) from a regression model may give a significant  $d$  value.
  - In the AR(1) scheme, a test of the hypothesis that  $\rho = 1$  can be made by the Berenblutt–Webb  $g$  statistic as well as the Durbin–Watson  $d$  statistic.
  - In the regression of the first difference of  $Y$  on the first differences of  $X$ , if there is a constant term and a linear trend term, it means in the original model there is a linear as well as a quadratic trend term.

- 12.2.** Given a sample of 50 observations and 4 explanatory variables, what can you say about autocorrelation if (a)  $d = 1.05$ ? (b)  $d = 1.40$ ? (c)  $d = 2.50$ ? (d)  $d = 3.97$ ?
- 12.3.** In studying the movement in the production workers' share in the value added (i.e., labor's share), the following models were considered by Gujarati\*:

$$\text{Model A: } Y_t = \beta_0 + \beta_1 t + u_t$$

$$\text{Model B: } Y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + u_t$$

where  $Y$  = labor's share and  $t$  = time. Based on annual data for 1949–1964, the following results were obtained for the primary metal industry:

$$\text{Model A: } \hat{Y}_t = 0.4529 - 0.0041t \quad R^2 = 0.5284 \quad d = 0.8252 \\ (-3.9608)$$

$$\text{Model B: } \hat{Y}_t = 0.4786 - 0.0127t + 0.0005t^2 \\ (-3.2724) \quad (2.7777) \\ R^2 = 0.6629 \quad d = 1.82$$

where the figures in the parentheses are  $t$  ratios.

- Is there serial correlation in model A? In model B?
  - What accounts for the serial correlation?
  - How would you distinguish between "pure" autocorrelation and specification bias?
- 12.4.** *Detecting autocorrelation: von Neumann ratio test.*<sup>†</sup> Assuming that the residual  $\hat{u}_t$  are random drawings from normal distribution, von Neumann has shown that for large  $n$ , the ratio

$$\frac{\delta^2}{s^2} = \frac{\sum(\hat{u}_i - \hat{u}_{i-1})^2 / (n-1)}{\sum(\hat{u}_i - \bar{\hat{u}})^2 / n} \quad \text{Note: } \bar{\hat{u}} = 0 \text{ in OLS}$$

called the *von Neumann ratio*, is approximately normally distributed with mean

$$E \frac{\delta^2}{s^2} = \frac{2n}{n-1}$$

and variance

$$\text{var} \frac{\delta^2}{s^2} = 4n^2 \frac{n-2}{(n+1)(n-1)^3}$$

- If  $n$  is sufficiently large, how would you use the von Neumann ratio to test for autocorrelation?
- What is the relationship between the Durbin-Watson  $d$  and the ratio?

\*Damodar Gujarati, "Labor's Share in Manufacturing Industries," *Industrial and Labor Relations Review*, vol. 23, no. 1, October 1969, pp. 65–75.

†J. von Neumann, "Distribution of the Ratio of the Mean Square Successive Difference to the Variance," *Annals of Mathematical Statistics*, vol. 12, 1941, pp. 367–395.

- c. The  $d$  statistic lies between 0 and 4. What are the corresponding limits for the von Neumann ratio?
- d. Since the ratio depends on the assumption that the  $\hat{u}$ 's are random drawings from normal distribution, how valid is this assumption for the OLS residuals?
- e. Suppose in an application the ratio was found to be 2.88 with 100 observations. Test the hypothesis that there is no serial correlation in the data.  
*Note:* B. I. Hart has tabulated the critical values of the von Neumann ratio for sample sizes of up to 60 observations.\*
- 12.5. In a sequence of 17 residuals, 11 positive and 6 negative, the number of runs was 3. Is there evidence of autocorrelation? Would the answer change if there were 14 runs?
- 12.6. **Theil-Nagar  $\rho$  estimate based on  $d$  statistic.** Theil and Nagar have suggested that in small samples instead of estimating  $\rho$  as  $(1 - d/2)$ , it be estimated as

$$\hat{\rho} = \frac{n^2(1 - d/2) + k^2}{n^2 - k^2}$$

where  $n$  = total number of observations,  $d$  = Durbin-Watson  $d$ , and  $k$  = number of coefficients (including the intercept) to be estimated.

Show that for large  $n$ , this estimate of  $\rho$  is equal to the one obtained by the simpler formula  $(1 - d/2)$ .

- 12.7. **Estimating  $\rho$ : The Hildreth-Lu scanning or search procedure.**<sup>†</sup> Since in the first-order autoregressive scheme

$$u_t = \rho u_{t-1} + \varepsilon_t$$

$\rho$  is expected to lie between  $-1$  and  $+1$ , Hildreth and Lu suggest a systematic "scanning" or search procedure to locate it. They recommend selecting  $\rho$  between  $-1$  and  $+1$  using, say, 0.1 unit intervals and transforming the data by the generalized difference equation (12.6.5). Thus, one may choose  $\rho$  from  $-0.9, -0.8, \dots, 0.8, 0.9$ . For each chosen  $\rho$  we run the generalized difference equation and obtain the associated RSS:  $\sum \hat{u}_t^2$ . Hildreth and Lu suggest choosing that  $\rho$  which minimizes the RSS (hence maximizing the  $R^2$ ). If further refinement is needed, they suggest using smaller unit intervals, say, 0.01 unit such as  $-0.99, -0.98, \dots, 0.90, 0.91$ , and so on.

- a. What are the advantages of the Hildreth-Lu procedure?
- b. How does one know that the  $\rho$  value ultimately chosen to transform the data will, in fact, guarantee minimum  $\sum \hat{u}_t^2$ ?
- 12.8. **Estimating  $\rho$ : The Cochrane-Orcutt (C-O) iterative procedure.**<sup>‡</sup> As an illustration of this procedure, consider the two-variable model:

$$Y_t = \beta_1 + \beta_2 X_t + u_t \quad (1)$$

\*The table may be found in Johnston, op. cit., 3d ed., p. 559.

<sup>†</sup>G. Hildreth and J. Y. Lu, "Demand Relations with Autocorrelated Disturbances," Michigan State University, *Agricultural Experiment Station*, Tech. Bull. 276, November 1960.

<sup>‡</sup>D. Cochrane and G. H. Orcutt, "Applications of Least-Squares Regressions to Relationships Containing Autocorrelated Error Terms," *Journal of the American Statistical Association*, vol. 44, 1949, pp 32-61.

and the AR(1) scheme

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad -1 < \rho < 1 \quad (2)$$

Cochrane and Orcutt then recommend the following steps to estimate  $\rho$ .

1. Estimate (1) by the usual OLS routine and obtain the residuals,  $\hat{u}_t$ . Incidentally, note that you can have more than one  $X$  variable in the model.
2. Using the residuals obtained in step 1, run the following regression:

$$\hat{u}_t = \hat{\rho} \hat{u}_{t-1} + v_t \quad (3)$$

which is the empirical counterpart of (2).\*

3. Using  $\hat{\rho}$  obtained in (3), estimate the generalized difference equation (12.9.6).
4. Since a priori it is not known if the  $\hat{\rho}$  obtained from (3) is the best estimate of  $\rho$ , substitute the values of  $\hat{\beta}_1^*$  and  $\hat{\beta}_2^*$  obtained in step (3) in the original regression (1) and obtain the new residuals, say,  $\hat{u}_t^*$  as

$$\hat{u}_t^* = Y_t - \hat{\beta}_1^* - \hat{\beta}_2^* X_t \quad (4)$$

which can be easily computed since  $Y_t$ ,  $X_t$ ,  $\hat{\beta}_1^*$ , and  $\hat{\beta}_2^*$  are all known.

5. Now estimate the following regression:

$$\hat{u}_t^* = \hat{\rho}^* \hat{u}_{t-1}^* + w_t \quad (5)$$

which is similar to (3) and thus provides the second round estimate of  $\rho$

Since we do not know whether this second-round estimate of  $\rho$  is the best estimate of the true  $\rho$ , we go into the third-round estimate, and so on. That is why the C–O procedure is called an iterative procedure. But how long should we go on this (merry) go-round? The general recommendation is to stop carrying out iterations when the successive estimates of  $\rho$  differ by a small amount, say, by less than 0.01 or 0.005. In our wages–productivity example, it took about seven iterations before we stopped.

- a. Using software of your choice, verify that the estimated  $\rho$  value of 0.8919 for Eq. (12.9.16) and 0.9610 for Eq. (12.9.17) are approximately correct.
- b. Does the rho value obtained by the C–O procedure guarantee the global minimum or just the local minimum?
- c. *Optional:* Apply the C–O method to the log–linear wages–productivity regression given in (12.5.2), retaining the first observation as well as dropping it. Compare your results with those of regression (12.5.1).

**12.9. Estimating  $\rho$ : The Cochrane–Orcutt two-step procedure.** This is a shortened version of the C–O iterative procedure. In step 1, we estimate  $\rho$  from the first iteration, that is from Eq. (3) in the preceding exercise,

\*Note that  $\hat{\rho} = \sum \hat{u}_t \hat{u}_{t-1} / \sum \hat{u}_t^2$  (why?). Although biased,  $\hat{\rho}$  is a consistent estimator of the true  $\rho$ .

and in step 2 we use that estimate of  $\rho$  to run the generalized difference equation, as in Eq. (4) in the preceding exercise. Sometimes in practice, this two-step method gives results quite similar to those obtained from the more elaborate C–O iterative procedure.

Apply the C–O two-step method to the illustrative wages–productivity regression given in the text and compare your results with those obtained from the iterative method. Pay special attention to the first observation in the transformation.

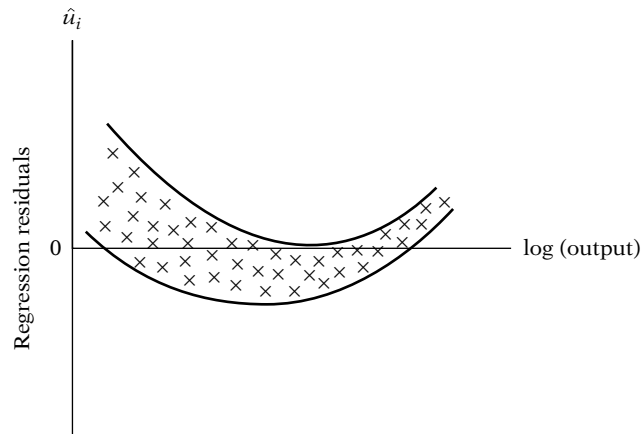
- 12.10.** *Estimating  $\rho$ : Durbin’s two-step method.*\* To explain this method, we can write the generalized difference equation (12.9.5) equivalently as follows:

$$Y_t = \beta_1(1 - \rho) + \beta_2 X_t - \beta_2 \rho X_{t-1} + \rho Y_{t-1} + \varepsilon_t \quad (1)$$

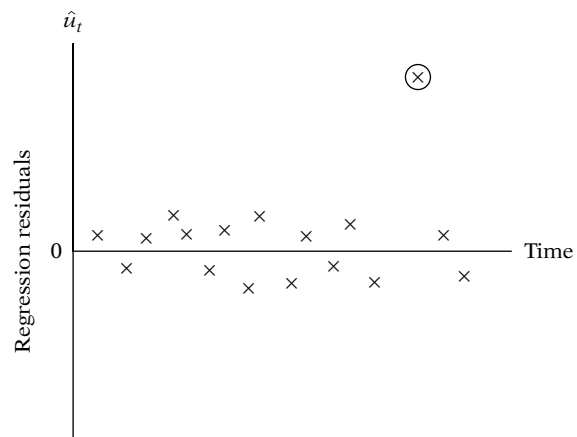
Durbin suggests the following two-step procedure to estimate  $\rho$ . *First*, treat (1) as a multiple regression model, regressing  $Y_t$  on  $X_t$ ,  $X_{t-1}$ , and  $Y_{t-1}$  and treat the estimated value of the regression coefficient of  $Y_{t-1}$  ( $= \hat{\rho}$ ) as an estimate of  $\rho$ . *Second*, having obtained  $\hat{\rho}$ , use it to estimate the parameters of the generalized difference equation (12.9.5) or its equivalent (12.9.6).

- a. Apply the Durbin two-step method to the wages–productivity example discussed in the text and compare your results with those obtained from the Cochrane–Orcutt iterative procedure and the C–O two-step method. And comment on the “quality” of your results.
  - b. If you examine Eq. (1) above, you will observe that the coefficient of  $X_{t-1}$  ( $= -\rho\beta_2$ ) is equal to minus 1 times the product of the coefficient of  $X_t$  ( $= \beta_2$ ) and the coefficient of  $Y_{t-1}$  ( $= \rho$ ). How would you test that coefficients obey the preceding restriction?
- 12.11.** In measuring returns to scale in electricity supply, Nerlove used cross-sectional data of 145 privately owned utilities in the United States for the period 1955 and regressed the log of total cost on the logs of output, wage rate, price of capital, and price of fuel. He found that the residuals estimated from this regression exhibited “serial” correlation, as judged by the Durbin–Watson  $d$ . To seek a remedy, he plotted the estimated residuals on the log of output and obtained Figure 12.11.
- a. What does Figure 12.11 show?
  - b. How can you get rid of “serial” correlation in the preceding situation?
- 12.12.** The residuals from a regression when plotted against time gave the scattergram in Figure 12.12. The encircled “extreme” residual is called an *outlier*. An outlier is an observation whose value exceeds the values of other observations in the sample by a large amount, perhaps three or four standard deviations away from the mean value of all the observations.
- a. What are the reasons for the existence of the outlier(s)?
  - b. If there is an outlier(s), should that observation(s) be discarded and the regression run on the remaining observations?
  - c. Is the Durbin–Watson  $d$  applicable in the presence of the outlier(s)?

\*J. Durbin, “Estimation of Parameters in Time-Series Regression Models,” *Journal of the Royal Statistical Society*, series B, vol. 22, 1960, p. 139–153.



**FIGURE 12.11** Regression residuals from the Nerlove study. (Adapted from Marc Nerlove, “Return to Scale in Electric Supply,” in Carl F. Christ et al., *Measurement in Economics*, Stanford University Press, Stanford, Calif., 1963.)



**FIGURE 12.12** Hypothetical regression residuals plotted against time.

**12.13.** Based on the Durbin–Watson  $d$  statistic, how would you distinguish “pure” autocorrelation from specification bias?

**12.14.** Suppose in the model

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

the  $u$ 's are in fact serially independent. What would happen in this situation if, assuming that  $u_t = \rho u_{t-1} + \varepsilon_t$ , we use the generalized difference regression

$$Y_t - \rho Y_{t-1} = \beta_1(1 - \rho) + \beta_2 X_t - \rho \beta_2 X_{t-1} + \varepsilon_t$$

Discuss in particular the properties of the disturbance term  $\varepsilon_t$ .

- 12.15.** In a study of the determination of prices of final output at factor cost in the United Kingdom, the following results were obtained on the basis of annual data for the period 1951–1969:

$$\widehat{PF}_t = 2.033 + 0.273W_t - 0.521X_t + 0.256M_t + 0.028M_{t-1} + 0.121PF_{t-1}$$

$$\text{se} = (0.992) \quad (0.127) \quad (0.099) \quad (0.024) \quad (0.039) \quad (0.119)$$

$$R^2 = 0.984 \quad d = 2.54$$

where PF = prices of final output at factor cost,  $W$  = wages and salaries per employee,  $X$  = gross domestic product per person employed,  $M$  = import prices,  $M_{t-1}$  = import prices lagged 1 year, and  $PF_{t-1}$  = prices of final output at factor cost in the previous year.\*

“Since for 18 observations and 5 explanatory variables, the 5 percent lower and upper  $d$  values are 0.71 and 2.06, the estimated  $d$  value of 2.54 indicates that there is no positive autocorrelation.” Comment.

- 12.16.** Give circumstances under which each of the following methods of estimating the first-order coefficient of autocorrelation  $\rho$  may be appropriate:
- First-difference regression
  - Moving average regression
  - Theil–Nagar transform
  - Cochrane and Orcutt iterative procedure
  - Hildreth–Lu scanning procedure
  - Durbin two-step procedure
- 12.17.** Consider the model:

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

where

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \varepsilon_t$$

that is, the error term follows an AR(2) scheme, and where  $\varepsilon_t$  is a white noise error term. Outline the steps you would take to estimate the model taking into account the second-order autoregression.

- 12.18.** Including the correction factor  $C$ , the formula for  $\hat{\beta}_2^{\text{GLS}}$  given in (12.3.1) is

$$\hat{\beta}_2^{\text{GLS}} = \frac{(1 - \rho^2)x_1 y_1 + \sum_{t=2}^n (x_t - \rho x_{t-1})(y_t - \rho y_{t-1})}{(1 - \rho^2)x_1^2 + \sum_{t=2}^n (x_t - \rho x_{t-1})^2}$$

Given this formula and (12.3.1), find the expression for the correction factor  $C$ .

- 12.19.** Show that estimating (12.9.5) is equivalent to estimating the GLS discussed in Section 12.3, excluding the first observation on  $Y$  and  $X$ .

\*Source: *Prices and Earnings in 1951–1969: An Econometric Assessment*, Department of Employment, Her Majesty’s Stationery Office, 1971, Table C, p. 37, Eq. 63.



**12.20.** For regression (12.9.9), the estimated residuals have the following signs, which for ease of exposition are bracketed.

(++++)(-)(+++++)(-)(++++)(--)(+)(--)(+)(--)(++)(-)  
(+)(-----)(+)

On the basis of the runs test, do you reject the null hypothesis that there is no autocorrelation in the residuals?

\***12.21. Testing for higher-order serial correlation.** Suppose we have time series data on a quarterly basis. In regression models involving quarterly data, instead of using the AR(1) scheme given in (12.2.1), it may be more appropriate to assume an AR(4) scheme as follows:

$$u_t = \rho_4 u_{t-4} + \varepsilon_t$$

that is, to assume that the current disturbance term is correlated with that of the same quarter in the previous year rather than that of the preceding quarter.

To test the hypothesis that  $\rho_4 = 0$ , Wallis<sup>†</sup> suggests the following modified Durbin–Watson  $d$  test:

$$d_4 = \frac{\sum_{t=5}^n (\hat{u}_t - \hat{u}_{t-4})^2}{\sum_{t=1}^n \hat{u}_t^2}$$

The testing procedure follows the usual  $d$  test routine discussed in the text.

Wallis has prepared  $d_4$  tables, which may be found in his original article.

Suppose now we have monthly data. Could the Durbin–Watson test be generalized to take into account such data? If so, write down the appropriate  $d_{12}$  formula.

**12.22.** Suppose you estimate the following regression:

$$\Delta \ln \text{output}_t = \beta_1 + \beta_2 \Delta \ln L_t + \beta_3 \Delta \ln K_t + u_t$$

where  $Y$  is output,  $L$  is labor input, and  $K$  is capital input, and  $\Delta$  is the first-difference operator. How would you interpret  $\beta_1$  in this model? Could it be regarded as an estimate of technological change? Justify your answer.

**12.23.** As noted in the text, Maddala has suggested that if the Durbin–Watson  $d$  is smaller than  $R^2$ , one may run the regression in the first-difference form. What is the logic behind this suggestion?

\*Optional.

<sup>†</sup>Kenneth Wallis, “Testing for Fourth Order Autocorrelation in Quarterly Regression Equations,” *Econometrica*, vol. 40, 1972, pp. 617–636. Tables of  $d_4$  can also be found in J. Johnston, *op. cit.*, 3d ed., p. 558.

- 12.24.** Refer to Eq. (12.4.1). Assume  $r = 0$  but  $\rho \neq 0$ . What is the effect on  $E(\hat{\sigma}^2)$  if (a)  $0 < \rho < 1$  and (b)  $-1 < \rho < 0$ ? When will the bias in  $\hat{\sigma}^2$  be reasonably small?
- 12.25.** The residuals from the wages–productivity regression given in (12.5.1) were regressed on lagged residuals going back six periods [i.e., AR(6)], yielding the following results:

Dependent Variable: RES1  
Method: Least Squares  
Sample (adjusted): 1965–1998  
Included Observations: 34 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	5.590462	1.963603	2.847043	0.0085
X	-0.066605	0.023469	-2.838058	0.0087
RES1(-1)	0.814971	0.216231	3.768978	0.0009
RES1(-2)	-0.268651	0.273887	-0.980882	0.3357
RES1(-3)	-0.106017	0.272780	-0.388652	0.7007
RES1(-4)	0.305630	0.273258	1.118467	0.2736
RES1(-5)	-0.064375	0.280577	-0.229438	0.8203
RES1(-6)	0.216156	0.222160	0.972976	0.3395
$R^2 = 0.8920$		Durbin-Watson $d$ stat		1.7589
$\bar{R}^2 = 0.8629$				

- a.** From the preceding results, what can you say about the nature of autocorrelation in the wages–productivity data?
- b.** If you think that an AR(1) mechanism characterizes autocorrelation in our data, would you use the first-difference transformation to get rid of autocorrelation? Justify your answer.

### Problems

- 12.26.** Refer to the data on the copper industry given in Table 12.7.
- a.** From these data estimate the following regression model:

$$\ln C_t = \beta_1 + \beta_2 \ln I_t + \beta_3 \ln L_t + \beta_4 \ln H_t + \beta_5 \ln A_t + u_t$$

Interpret the results.

- b.** Obtain the residuals and standardized residuals from the preceding regression and plot them. What can you surmise about the presence of autocorrelation in these residuals?
- c.** Estimate the Durbin–Watson  $d$  statistic and comment on the nature of autocorrelation present in the data.
- d.** Carry out the runs test and see if your answer differs from that just given in **c.**
- e.** How would you find out if an AR( $p$ ) process better describes autocorrelation than an AR(1) process?
- Note:* Save the data for further analysis. (See exercise 12.28.)
- 12.27.** You are given the data in Table 12.8.
- a.** Verify that Durbin–Watson  $d = 0.4148$ .
- b.** Is there positive serial correlation in the disturbances?

**TABLE 12.7** DETERMINANTS OF U.S. DOMESTIC PRICE OF COPPER, 1951–1980

Year	<i>C</i>	<i>G</i>	<i>I</i>	<i>L</i>	<i>H</i>	<i>A</i>
1951	21.89	330.2	45.1	220.4	1,491.0	19.00
52	22.29	347.2	50.9	259.5	1,504.0	19.41
53	19.63	366.1	53.3	256.3	1,438.0	20.93
54	22.85	366.3	53.6	249.3	1,551.0	21.78
55	33.77	399.3	54.6	352.3	1,646.0	23.68
56	39.18	420.7	61.1	329.1	1,349.0	26.01
57	30.58	442.0	61.9	219.6	1,224.0	27.52
58	26.30	447.0	57.9	234.8	1,382.0	26.89
59	30.70	483.0	64.8	237.4	1,553.7	26.85
60	32.10	506.0	66.2	245.8	1,296.1	27.23
61	30.00	523.3	66.7	229.2	1,365.0	25.46
62	30.80	563.8	72.2	233.9	1,492.5	23.88
63	30.80	594.7	76.5	234.2	1,634.9	22.62
64	32.60	635.7	81.7	347.0	1,561.0	23.72
65	35.40	688.1	89.8	468.1	1,509.7	24.50
66	36.60	753.0	97.8	555.0	1,195.8	24.50
67	38.60	796.3	100.0	418.0	1,321.9	24.98
68	42.20	868.5	106.3	525.2	1,545.4	25.58
69	47.90	935.5	111.1	620.7	1,499.5	27.18
70	58.20	982.4	107.8	588.6	1,469.0	28.72
71	52.00	1,063.4	109.6	444.4	2,084.5	29.00
72	51.20	1,171.1	119.7	427.8	2,378.5	26.67
73	59.50	1,306.6	129.8	727.1	2,057.5	25.33
74	77.30	1,412.9	129.3	877.6	1,352.5	34.06
75	64.20	1,528.8	117.8	556.6	1,171.4	39.79
76	69.60	1,700.1	129.8	780.6	1,547.6	44.49
77	66.80	1,887.2	137.1	750.7	1,989.8	51.23
78	66.50	2,127.6	145.2	709.8	2,023.3	54.42
79	98.30	2,628.8	152.5	935.7	1,749.2	61.01
80	101.40	2,633.1	147.1	940.9	1,298.5	70.87

*Note:* The data were collected by Gary R. Smith from sources such as *American Metal Market*, *Metals Week*, and U.S. Department of Commerce publications.

*C* = 12-month average U.S. domestic price of copper (cents per pound)

*G* = annual gross national product (\$, billions)

*I* = 12-month average index of industrial production

*L* = 12-month average London Metal Exchange price of copper (pounds sterling)

*H* = number of housing starts per year (thousands of units)

*A* = 12-month average price of aluminum (cents per pound)

- c. If so, estimate  $\rho$  by the
    - i. Theil–Nagar method
    - ii. Durbin two-step procedure
    - iii. Cochrane–Orcutt method
  - d. Use the Theil–Nagar method to transform the data and run the regression on the transformed data.
  - e. Does the regression estimated in **d** exhibit autocorrelation? If so, how would you get rid of it?
- 12.28.** Refer to exercise 12.26 and the data given in Table 12.7. If the results of this exercise show serial correlation,

TABLE 12.8

Y, personal consumption expenditure, billions of 1958 dollars	X, time	$\hat{Y}$ , estimated $Y^*$	$\hat{u}$ , residuals
281.4	1 (= 1956)	261.4208	19.9791
288.1	2	276.6026	11.4973
290.0	3	291.7844	-1.7844
307.3	4	306.9661	0.3338
316.1	5	322.1479	-6.0479
322.5	6	337.3297	-14.8297
338.4	7	352.5115	-14.1115
353.3	8	367.6933	-14.3933
373.7	9	382.8751	-9.1751
397.7	10	398.0569	-0.3569
418.1	11	413.2386	4.8613
430.1	12	428.4206	1.6795
452.7	13	443.6022	9.0977
469.1	14	458.7840	10.3159
476.9	15 (= 1970)	473.9658	2.9341

\*Obtained from the regression  $Y_t = \beta_0 + \beta_1 X_t + u_t$ .

- a. Use the Cochrane–Orcutt two-stage procedure and obtain the estimates of the feasible GLS or the generalized difference regression and compare your results.
  - b. If the  $\rho$  estimated from the Cochrane–Orcutt method in **a** differs substantially from that estimated from the  $d$  statistic, which method of estimating  $\rho$  would you choose and why?
- 12.29.** Refer to Example 7.4. Omitting the variables  $X^2$  and  $X^3$ , run the regression and examine the residuals for “serial” correlation. If serial correlation is found, how would you rationalize it? What remedial measures would you suggest?
- 12.30.** Refer to exercise 7.21. A priori autocorrelation is expected in such data. Therefore, it is suggested that you regress the log of real money supply on the logs of real national income and long-term interest rate in the first-difference form. Run this regression, and then rerun the regression in the original form. Is the assumption underlying the first-difference transformation satisfied? If not, what kinds of biases are likely to result from such a transformation? Illustrate with the data at hand.
- 12.31. The use of Durbin–Watson  $d$  for testing nonlinearity.** Continue with exercise 12.29. Arrange the residuals obtained in that regression according to increasing values of  $X$ . Using the formula given in (12.6.5), estimate  $d$  from the rearranged residuals. If the computed  $d$  value indicates autocorrelation, this would imply that the linear model was incorrect and that the full model should include  $X_i^2$  and  $X_i^3$  terms. Can you give an intuitive justification for such a procedure? See if your answer agrees with that given by Henri Theil.\*

\*Henri Theil, *Introduction to Econometrics*, Prentice Hall, Englewood Cliffs, N.J., 1978, pp. 307–308.

**12.32.** Refer to exercise 11.22. Obtain the residuals and find out if there is autocorrelation in the residuals. How would you transform the data in case serial correlation is detected? What is the meaning of serial correlation in the present instance?

**12.33. Monte Carlo experiment.** Refer to Tables 12.1 and 12.2. Using  $\varepsilon_t$  and  $X_t$  data given there, generate a sample of 10  $Y$  values from the model

$$Y_t = 3.0 + 0.5X_t + u_t$$

where  $u_t = 0.9u_{t-1} + \varepsilon_t$ . Assume  $u_0 = 10$ .

**a.** Estimate the equation and comment on your results.

**b.** Now assume  $u_0 = 17$ . Repeat this exercise 10 times and comment on the results.

**c.** Keep the preceding setup intact except now let  $\rho = 0.3$  instead of  $\rho = 0.9$  and compare your results with those given in **b.**

**12.34.** Using the data given in Table 12.9, estimate the model

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

where  $Y$  = inventories and  $X$  = sales, both measured in billions of dollars.

**TABLE 12.9** INVENTORIES AND SALES IN U.S. MANUFACTURING, 1950–1991  
(Millions of Dollars)

Year	Sales*	Inventories†	Year	Sales*	Inventories†
1950	38,596	59,822	1970	108,352	178,594
1951	43,356	70,242	1971	117,023	188,991
1952	44,840	72,377	1972	131,227	203,227
1953	47,987	76,122	1973	153,881	234,406
1954	46,443	73,175	1974	178,201	287,144
1955	51,694	79,516	1975	182,412	288,992
1956	54,063	87,304	1976	204,386	318,345
1957	55,879	89,052	1977	229,786	350,706
1958	54,201	87,055	1978	260,755	400,929
1959	59,729	92,097	1979	298,328	452,636
1960	60,827	94,719	1980	328,112	510,124
1961	61,159	95,580	1981	356,909	547,169
1962	65,662	101,049	1982	348,771	575,486
1963	68,995	105,463	1983	370,501	591,858
1964	73,682	111,504	1984	411,427	651,527
1965	80,283	120,929	1985	423,940	665,837
1966	87,187	136,824	1986	431,786	664,654
1967	90,918	145,681	1987	459,107	711,745
1968	98,794	156,611	1988	496,334	767,387
1969	105,812	170,400	1989	522,344	813,018
			1990	540,788	835,985
			1991	533,838	828,184

Source: *Economic Report of the President*, 1993, Table B-53, p. 408.

\*Annual data are averages of monthly, not seasonally adjusted, figures.

†Seasonally adjusted, end of period figures beginning 1982 are not comparable with earlier period.

- a. Estimate the preceding regression.
- b. From the estimated residuals find out if there is positive autocorrelation using (i) the Durbin–Watson test and (ii) the large-sample normality test given in (12.6.13).
- c. If  $\rho$  is positive, apply the Berenblutt–Webb test to test the hypothesis that  $\rho = 1$ .
- d. If you suspect that the autoregressive error structure is of order  $p$ , use the Breusch–Godfrey test to verify this. How would you choose the order of  $p$ ?
- e. On the basis of the results of this test, how would you transform the data to remove autocorrelation? Show all your calculations.
- f. Repeat the preceding steps using the following model:

$$\ln Y_t = \beta_1 + \beta_2 \ln X_t + u_t$$

- g. How would you decide between the linear and log–linear specifications? Show explicitly the test(s) you use.
- 12.35.** Table 12.10 gives data on real rate of return on common stocks at time ( $RR_t$ ), output growth in period ( $t + 1$ ), ( $OG_{t+1}$ ) and inflation in period

**TABLE 12.10** RATE OF RETURN, OUTPUT GROWTH AND INFLATION, UNITED STATES, 1965–81

Observation	RR	Growth	Inflation
1954	53.0	6.7	−0.4
1955	31.2	2.1	0.4
1956	3.7	1.8	2.9
1957	−13.8	−0.4	3.0
1958	41.7	6.0	1.7
1959	10.5	2.1	1.5
1960	−1.3	2.6	1.8
1961	26.1	5.8	0.8
1962	−10.5	4.0	1.8
1963	21.2	5.3	1.6
1964	15.5	6.0	1.0
1965	10.2	6.0	2.3
1966	−13.3	2.7	3.2
1967	21.3	4.6	2.7
1968	6.8	2.8	4.3
1969	−13.5	−0.2	5.0
1970	−0.4	3.4	4.4
1971	10.5	5.7	3.8
1972	15.4	5.8	3.6
1973	−22.6	−0.6	7.9
1974	−37.3	−1.2	10.8
1975	31.2	5.4	6.0
1976	19.1	5.5	4.7
1977	−13.1	5.0	5.9
1978	−1.3	2.8	7.9
1979	8.6	−0.3	9.8
1980	−22.2	2.6	10.2
1981	−12.2	−1.9	7.3

$t(\text{Inf}_t)$ , all in percent form, for the U.S. economy for the period 1954–1981.

- a. Regress  $\text{RR}_t$  on inflation.
- b. Regress  $\text{RR}_t$  on  $\text{OG}_{t+1}$  and  $\text{Inf}_t$ .
- c. Comment on the two regression results in view of Eugene Fama's observation that "the negative simple correlation between real stock returns and inflation is spurious because it is the result of two structural relationships: a positive relation between current real stock returns and expected output growth [measured by  $\text{OG}_{t+1}$ ], and a negative relationship between expected output growth and current inflation."
- d. Would you expect autocorrelation in either of the regressions in **a** and **b**? Why or why not? If you do, take the appropriate corrective action and present the revised results.

**13.36. The Durbin  $h$  statistic:** Consider the following model of wage determination:

$$Y_t = \beta_1 + \beta_2 X_t + \beta_3 Y_{t-1} + u_t$$

where  $Y$  = wages = index of real compensation per hour

$X$  = productivity = index of output per hour.

- a. Using the data in Table 12.4, estimate the above model and interpret your results.
- b. Since the model contains lagged regressand as a regressor, the Durbin–Watson  $d$  is not appropriate to find out if there is serial correlation in the data. For such models, called autoregressive models, Durbin has developed the so-called  **$h$  statistic** to test for first-order autocorrelation, which is defined as\*:

$$h = \hat{\rho} \sqrt{\frac{n}{1 - n[\text{var}(\hat{\beta}_3)]}}$$

where  $n$  = sample size,  $\text{var}(\hat{\beta}_3)$  = variance of the coefficient of the lagged  $Y_{t-1}$ , and  $\hat{\rho}$  = estimate of the first-order serial correlation.

For large sample size (technically, asymptotic), Durbin has shown that, under the null hypothesis that  $\rho = 0$ ,

$$h \sim N(0, 1)$$

that is, the  $h$  statistic follows the standard normal distribution. From the properties of the normal distribution we know that the probability of  $|h| > 1.96$  is about 5 percent. Therefore, if in an application  $|h| > 1.96$ , we can reject the null hypothesis that  $\rho = 0$ , that is, there is evidence of first-order autocorrelation in the autoregressive model given above.

To apply the test, we proceed as follows: *First*, estimate the above model by OLS (don't worry about any estimation problems at this stage). *Second*, note  $\text{var}(\hat{\beta}_3)$  in this model as well as the routinely

\*J. Durbin, "Testing for Serial Correlation in Least-squares Regression When Some of the Regressors Are Lagged Dependent Variables," *Econometrica*, vol. 38, pp. 410–421.

computed  $d$  statistic. *Third*, using the  $d$  value, obtain  $\hat{\rho} \approx (1 - d/2)$ . It is interesting to note that although we cannot use the  $d$  value to test for serial correlation in this model, we can use it to obtain an estimate of  $\rho$ . *Fourth*, now compute the  $h$  statistic. *Fifth*, if the sample size is reasonably large and if the computed  $|h|$  exceeds 1.96, we can conclude that there is evidence of first-order autocorrelation. Of course, you can use any level of significance you want.

Apply the  $h$  test to the autoregressive wage determination model given earlier and draw appropriate conclusions and compare your results with those given in regression (12.5.1).

- 12.37.** *Dummy variables and autocorrelation.* Refer to the savings–income regression discussed in Chapter 9. Using the data given in Table 9.2, and assuming an AR(1) scheme, reestimate the savings–income regression, taking into account autocorrelation. Pay close attention to the transformation of the dummy variable. Compare your results with those presented in Chapter 9.
- 12.38.** Using the wages–productivity data given in Table 12.4, estimate model (12.9.8) and compare your results with those given in regression (12.9.9). What conclusion(s) do you draw?

## APPENDIX 12A

### 12A.1 PROOF THAT THE ERROR TERM $v_t$ IN (12.1.11) IS AUTOCORRELATED

Since  $v_t = u_t - u_{t-1}$ , it is easy to show that  $E(v_t) = E(u_t - u_{t-1}) = E(u_t) - E(u_{t-1}) = 0$ , since  $E(u) = 0$ , for each  $t$ . Now,  $\text{var}(v_t) = \text{var}(u_t - u_{t-1}) = \text{var}(u_t) + \text{var}(u_{t-1}) = 2\sigma^2$ , since the variance of each  $u_t$  is  $\sigma^2$  and the  $u$ 's are independently distributed. Hence,  $v_t$  is homoscedastic. But

$$\begin{aligned}\text{cov}(v_t, v_{t-1}) &= E(v_t v_{t-1}) = E[(u_t - u_{t-1})(u_{t-1} - u_{t-2})] \\ &= -\sigma^2\end{aligned}$$

which is obviously nonzero. Therefore, although the  $u$ 's are not autocorrelated, the  $v$ 's are.

### 12A.2 PROOF OF EQUATIONS (12.2.3), (12.2.4), AND (12.2.5)

Under AR(1),

$$u_t = \rho u_{t-1} + \varepsilon_t \quad (1)$$

Therefore,

$$E(u_t) = \rho E(u_{t-1}) + E(\varepsilon_t) = 0 \quad (2)$$

So,

$$\text{var}(u_t) = \rho^2 \text{var}(u_{t-1}) + \text{var}(\varepsilon_t) \quad (3)$$

because the  $u$ 's and  $\varepsilon$ 's are uncorrelated.



Since  $\text{var}(u_t) = \text{var}(u_{t-1}) = \sigma^2$  and  $\text{var}(\varepsilon_t) = \sigma_\varepsilon^2$ , we get

$$\text{var}(u_t) = \frac{\sigma_\varepsilon^2}{1 - \rho^2} \quad (4)$$

Now multiply (1) by  $u_{t-1}$  and take expectations on both sides to obtain:

$$\text{cov}(u_t, u_{t-1}) = E(u_t u_{t-1}) = E[\rho u_{t-1}^2 + u_{t-1} \varepsilon_t] = \rho E(u_{t-1}^2)$$

Noting that the covariance between  $u_{t-1}$  and  $\varepsilon_t$  is zero (why?) and that  $\text{var}(u_t) = \text{var}(u_{t-1}) = \sigma_\varepsilon^2 / (1 - \rho^2)$ , we obtain

$$\text{cov}(u_t, u_{t-1}) = \rho \frac{\sigma_\varepsilon^2}{(1 - \rho^2)} \quad (5)$$

Continuing in this fashion,

$$\text{cov}(u_t, u_{t-2}) = \rho^2 \frac{\sigma_\varepsilon^2}{(1 - \rho^2)}$$

$$\text{cov}(u_t, u_{t-3}) = \rho^3 \frac{\sigma_\varepsilon^2}{(1 - \rho^2)}$$

and so on. Now the correlation coefficient is the ratio of covariance to variance. Hence,

$$\text{cor}(u_t, u_{t-1}) = \rho \quad \text{cov}(u_t, u_{t-2}) = \rho^2 \quad \text{and so on}$$

# 13

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## ECONOMETRIC MODELING: MODEL SPECIFICATION AND DIAGNOSTIC TESTING

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Applied econometrics cannot be done mechanically; it needs understanding, intuition and skill.<sup>1</sup>

. . . we generally drive across bridges without worrying about the soundness of their construction because we are reasonably sure that someone rigorously checked their engineering principles and practice. Economists must do likewise with models or else attach the warning ‘not responsible if attempted use leads to collapse’.<sup>2</sup>

Economists’ search for “truth” has over the years given rise to the view that economists are people searching in a dark room for a non-existent black cat; econometricians are regularly accused of finding one.<sup>3</sup>

One of the assumptions of the classical linear regression model (CLRM), Assumption 9, is that the regression model used in the analysis is “correctly” specified: If the model is not “correctly” specified, we encounter the problem of **model specification error** or **model specification bias**. In this chapter we take a close and critical look at this assumption, because searching for the correct model is like searching for the Holy Grail. In particular we examine the following questions:

1. How does one go about finding the “correct” model? In other words, what are the criteria in choosing a model for empirical analysis?

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<sup>1</sup>Keith Cuthbertson, Stephen G. Hall, and Mark P. Taylor, *Applied Econometrics Techniques*, Michigan University Press, 1992, p. X.

<sup>2</sup>David F. Hendry, *Dynamic Econometrics*, Oxford University Press, U.K., 1995, p. 68.

<sup>3</sup>Peter Kennedy, *A Guide to Econometrics*, 3d ed., The MIT Press, Cambridge, Mass., 1992, p. 82.

2. What types of model specification errors is one likely to encounter in practice?
3. What are the consequences of specification errors?
4. How does one detect specification errors? In other words, what are some of the diagnostic tools that one can use?
5. Having detected specification errors, what remedies can one adopt and with what benefits?
6. How does one evaluate the performance of competing models?

The topic of model specification and evaluation is vast, and very extensive empirical work has been done in this area. Not only that, but there are philosophical differences on this topic. Although we cannot do full justice to this topic in one chapter, we hope to bring out some of the essential issues involved in model specification and model evaluation.

### 13.1 MODEL SELECTION CRITERIA

According to Hendry and Richard, a model chosen for empirical analysis should satisfy the following criteria<sup>4</sup>:

1. *Be data admissible*; that is, predictions made from the model must be logically possible.
2. *Be consistent with theory*; that is, it must make good economic sense. For example, if Milton Friedman's **permanent income hypothesis** holds, the intercept value in the regression of permanent consumption on permanent income is expected to be zero.
3. *Have weakly exogenous regressors*; that is, the explanatory variables, or regressors, must be uncorrelated with the error term.
4. *Exhibit parameter constancy*; that is, the values of the parameters should be stable. Otherwise, forecasting will be difficult. As Friedman notes, "The only relevant test of the validity of a hypothesis [model] is comparison of its predictions with experience."<sup>5</sup> In the absence of parameter constancy, such predictions will not be reliable.
5. *Exhibit data coherency*; that is, the residuals estimated from the model must be purely random (technically, white noise). In other words, if the regression model is adequate, the residuals from this model must be white noise. If that is not the case, there is some specification error in the model. Shortly, we will explore the nature of specification error(s).
6. *Be encompassing*; that is, the model should *encompass* or include all the rival models in the sense that it is capable of explaining their results. In short, other models cannot be an improvement over the chosen model.

<sup>4</sup>D. F. Hendry and J. F. Richard, "The Econometric Analysis of Economic Time Series," *International Statistical Review*, vol. 51, 1983, pp. 3–33.

<sup>5</sup>Milton Friedman, "The Methodology of Positive Economics," in *Essays in Positive Economics*, University of Chicago Press, Chicago, 1953, p. 7.

It is one thing to list criteria of a “good” model and quite another to actually develop it, for in practice one is likely to commit various model specification errors, which we discuss in the next section.

### 13.2 TYPES OF SPECIFICATION ERRORS

Assume that on the basis of the criteria just listed we arrive at a model that we accept as a good model. To be concrete, let this model be

$$Y_i = \beta_1 + \beta_2 X_i + \beta_3 X_i^2 + \beta_4 X_i^3 + u_{1i} \quad (13.2.1)$$

where  $Y$  = total cost of production and  $X$  = output. Equation (13.2.1) is the familiar textbook example of the cubic total cost function.

But suppose for some reason (say, laziness in plotting the scattergram) a researcher decides to use the following model:

$$Y_i = \alpha_1 + \alpha_2 X_i + \alpha_3 X_i^2 + u_{2i} \quad (13.2.2)$$

Note that we have changed the notation to distinguish this model from the true model.

Since (13.2.1) is assumed true, adopting (13.2.2) would constitute a specification error; the error consisting in **omitting a relevant variable** ( $X_i^3$ ). Therefore, the error term  $u_{2i}$  in (13.2.2) is in fact

$$u_{2i} = u_{1i} + \beta_4 X_i^3 \quad (13.2.3)$$

We shall see shortly the importance of this relationship.

Now suppose that another researcher uses the following model:

$$Y_i = \lambda_1 + \lambda_2 X_i + \lambda_3 X_i^2 + \lambda_4 X_i^3 + \lambda_5 X_i^4 + u_{3i} \quad (13.2.4)$$

If (13.2.1) is the “truth,” (13.2.4) also constitutes a specification error, the error here consisting in **including an unnecessary or irrelevant variable** in the sense that the true model assumes  $\lambda_5$  to be zero. The new error term is in fact

$$\begin{aligned} u_{3i} &= u_{1i} - \lambda_5 X_i^4 \\ &= u_{1i} \quad \text{since } \lambda_5 = 0 \text{ in the true model} \quad (\text{Why?}) \end{aligned} \quad (13.2.5)$$

Now assume that yet another researcher postulates the following model:

$$\ln Y_i = \gamma_1 + \gamma_2 X_i + \gamma_3 X_i^2 + \gamma_4 X_i^3 + u_{4i} \quad (13.2.6)$$

In relation to the true model, (13.2.6) would also constitute a specification bias, the bias here being the use of the **wrong functional form**: In (13.2.1)  $Y$  appears linearly, whereas in (13.2.6) it appears log-linearly.

Finally, consider the researcher who uses the following model:

$$Y_i^* = \beta_1^* + \beta_2^* X_i^* + \beta_3^* X_i^{*2} + \beta_4^* X_i^{*3} + u_i^* \quad (13.2.7)$$

where  $Y_i^* = Y_i + \varepsilon_i$  and  $X_i^* = X_i + w_i$ ,  $\varepsilon_i$  and  $w_i$  being the errors of measurement. What (13.2.7) states is that instead of using the true  $Y_i$  and  $X_i$  we use their proxies,  $Y_i^*$  and  $X_i^*$ , which may contain errors of measurement. Therefore, in (13.2.7) we commit the **errors of measurement bias**. In applied work data are plagued by errors of approximations or errors of incomplete coverage or simply errors of omitting some observations. In the social sciences we often depend on secondary data and usually have no way of knowing the types of errors, if any, made by the primary data-collecting agency.

Another type of specification error relates to the way the stochastic error  $u_i$  (or  $u_i$ ) enters the regression model. Consider for instance, the following bivariate regression model without the intercept term:

$$Y_i = \beta X_i u_i \quad (13.2.8)$$

where the stochastic error term enters multiplicatively with the property that  $\ln u_i$  satisfies the assumptions of the CLRM, against the following model

$$Y_i = \alpha X_i + u_i \quad (13.2.9)$$

where the error term enters additively. Although the variables are the same in the two models, we have denoted the slope coefficient in (13.2.8) by  $\beta$  and the slope coefficient in (13.2.9) by  $\alpha$ . Now if (13.2.8) is the “correct” or “true” model, would the estimated  $\alpha$  provide an unbiased estimate of the true  $\beta$ ? That is, will  $E(\hat{\alpha}) = \beta$ ? If that is not the case, improper stochastic specification of the error term will constitute another source of specification error.

To sum up, in developing an empirical model, one is likely to commit one or more of the following specification errors:

1. Omission of a relevant variable(s)
2. Inclusion of an unnecessary variable(s)
3. Adopting the wrong functional form
4. Errors of measurement
5. Incorrect specification of the stochastic error term

Before turning to an examination of these specification errors in some detail, it may be fruitful to distinguish between **model specification errors** and **model mis-specification errors**. The first four types of error discussed above are essentially in the nature of model specification errors in that we have in mind a “true” model but somehow we do not estimate the correct model. In model mis-specification errors, we do not know what the true model is to begin with. In this context one may recall the controversy

between the Keynesians and the monetarists. The monetarists give primacy to money in explaining changes in GDP, whereas the Keynesians emphasize the role of government expenditure to explain changes in GDP. So to speak, there are two competing models.

In what follows, we will first consider model specification errors and then examine model mis-specification errors.

### 13.3 CONSEQUENCES OF MODEL SPECIFICATION ERRORS

Whatever the sources of specification errors, what are the consequences? To keep the discussion simple, we will answer this question in the context of the three-variable model and consider in this section the first two types of specification errors discussed earlier, namely, (1) **underfitting a model**, that is, omitting relevant variables, and (2) **overfitting a model**, that is, including unnecessary variables. Our discussion here can be easily generalized to more than two regressors, but with tedious algebra<sup>6</sup>; matrix algebra becomes almost a necessity once we go beyond the three-variable case.

#### Underfitting a Model (Omitting a Relevant Variable)

Suppose the true model is:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i \quad (13.3.1)$$

but for some reason we fit the following model:

$$Y_i = \alpha_1 + \alpha_2 X_{2i} + v_i \quad (13.3.2)$$

The consequences of omitting variable  $X_3$  are as follows:

1. If the left-out, or omitted, variable  $X_3$  is correlated with the included variable  $X_2$ , that is,  $r_{23}$ , the correlation coefficient between the two variables, is *nonzero*,  $\hat{\alpha}_1$  and  $\hat{\alpha}_2$  are *biased as well as inconsistent*. That is,  $E(\hat{\alpha}_1) \neq \beta_1$  and  $E(\hat{\alpha}_2) \neq \beta_2$ , and the bias does not disappear as the sample size gets larger.

2. Even if  $X_2$  and  $X_3$  are not correlated,  $\hat{\alpha}_1$  is biased, although  $\hat{\alpha}_2$  is now unbiased.

3. The disturbance variance  $\sigma^2$  is incorrectly estimated.

4. The conventionally measured variance of  $\hat{\alpha}_2 (= \sigma^2 / \sum x_{2i}^2)$  is a *biased* estimator of the variance of the true estimator  $\hat{\beta}_2$ .

5. In consequence, the usual confidence interval and hypothesis-testing procedures are likely to give misleading conclusions about the statistical significance of the estimated parameters.

<sup>6</sup>But see exercise 13.32.

6. As another consequence, the forecasts based on the incorrect model and the forecast (confidence) intervals will be unreliable.

Although proofs of each of the above statements will take us far afield,<sup>7</sup> it is shown in Appendix 13A, Section 13A.1, that

$$E(\hat{\alpha}_2) = \beta_2 + \beta_3 b_{32} \quad (13.3.3)$$

where  $b_{32}$  is the slope in the regression of the excluded variable  $X_3$  on the included variable  $X_2$  ( $b_{32} = \sum x_{3i}x_{2i} / \sum x_{2i}^2$ ). As (13.3.3) shows,  $\hat{\alpha}_2$  is biased, unless  $\beta_3$  or  $b_{32}$  or both are zero. We rule out  $\beta_3$  being zero, because in that case we do not have specification error to begin with. The coefficient  $b_{32}$  will be zero if  $X_2$  and  $X_3$  are uncorrelated, which is unlikely in most economic data.

Generally, however, the extent of the bias will depend on the *bias term*  $\beta_3 b_{32}$ . If, for instance,  $\beta_3$  is positive (i.e.,  $X_3$  has a positive effect on  $Y$ ) and  $b_{32}$  is positive (i.e.,  $X_2$  and  $X_3$  are positively correlated),  $\hat{\alpha}_2$ , on average, will overestimate the true  $\beta_2$  (i.e., positive bias). But this result should not be surprising, for  $X_2$  represents not only its *direct effect* on  $Y$  but also its *indirect effect* (via  $X_3$ ) on  $Y$ . In short,  $X_2$  gets credit for the influence that is rightly attributable to  $X_3$ , the latter prevented from showing its effect explicitly because it is not “allowed” to enter the model. As a concrete example, consider the example discussed in Chapter 7.

#### ILLUSTRATIVE EXAMPLE: CHILD MORTALITY REVISITED

Regressing child mortality (CM) on per capita GNP (PGNP) and female literacy rate (FLR), we obtained the regression results shown in Eq. (7.6.2), giving the partial slope coefficient values of the two variables as  $-0.0056$  and  $-2.2316$ , respectively. But if we now drop the FLR variable, we obtain the results shown in Eq. (7.7.2). If we regard (7.6.2) as the correct model, then (7.7.2) is a misspecified model in that it omits the relevant variable FLR. Now you can see that in the correct model the coefficient of the PGNP variable was  $-0.0056$ , whereas in the “incorrect” model (7.7.2) it is now  $-0.0114$ .

In absolute terms, now PGNP has a greater impact on CM as compared with the true model. But if we

regress FLR on PGNP (regression of the excluded variable on the included variable), the slope coefficient in this regression [ $b_{32}$  in terms of Eq. (13.3.3)] is  $0.00256$ .<sup>8</sup> This suggests that as PGNP increases by a unit, on average, FLR goes up by  $0.00256$  units. But if FLR goes up by these units, its effect on CM will be  $(-2.2316)(0.00256) = \hat{\beta}_3 b_{32} = -0.00543$ .

Therefore, from (13.3.3) we finally have  $(\hat{\beta}_2 + \hat{\beta}_3 b_{32}) = [-0.0056 + (-2.2316)(0.00256)] \approx -0.0111$ , which is about the value of the PGNP coefficient obtained in the incorrect model (7.7.2).<sup>9</sup> As this example illustrates, the true impact of PGNP on CM is much less ( $-0.0056$ ) than that suggested by the incorrect model (7.7.2), namely,  $(-0.0114)$ .

<sup>7</sup>For an algebraic treatment, see Jan Kmenta, *Elements of Econometrics*, Macmillan, New York, 1971, pp. 391–399. Those with a matrix algebra background may want to consult J. Johnston, *Econometrics Methods*, 4th ed., McGraw-Hill, New York, 1997, pp. 119–112.

<sup>8</sup>The regression results are:

$$\widehat{\text{FLR}} = 47.5971 + 0.00256\text{PGNP}$$

$$\text{se} = (3.5553) \quad (0.0011) \quad r^2 = 0.0721$$

<sup>9</sup>Note that in the true model  $\hat{\beta}_2$  and  $\hat{\beta}_3$  are unbiased estimates of their true values.

Now let us examine the variances of  $\hat{\alpha}_2$  and  $\hat{\beta}_2$

$$\text{var}(\hat{\alpha}_2) = \frac{\sigma^2}{\sum x_{2i}^2} \quad (13.3.4)$$

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2(1 - r_{23}^2)} = \frac{\sigma^2}{\sum x_{2i}^2} \text{VIF} \quad (13.3.5)$$

where VIF (a measure of collinearity) is the variance inflation factor [ $= 1/(1 - r_{23}^2)$ ] discussed in Chapter 10 and  $r_{23}$  is the correlation coefficient between variables  $X_2$  and  $X_3$ ; Eqs. (13.3.4) and (13.3.5) are familiar to us from Chapters 3 and 7.

As formulas (13.3.4) and (13.3.5) are not the same, in general,  $\text{var}(\hat{\alpha}_2)$  will be different from  $\text{var}(\hat{\beta}_2)$ . But we know that  $\text{var}(\hat{\beta}_2)$  is unbiased (why?). Therefore,  $\text{var}(\hat{\alpha}_2)$  is biased, thus substantiating the statement made in point 4 earlier. Since  $0 < r_{23}^2 < 1$ , it would *seem* that in the present case  $\text{var}(\hat{\alpha}_2) < \text{var}(\hat{\beta}_2)$ . Now we face a dilemma: Although  $\hat{\alpha}_2$  is biased, its variance is smaller than the variance of the unbiased estimator  $\hat{\beta}_2$  (of course, we are ruling out the case where  $r_{23} = 0$ , since in practice there is some correlation between regressors). So, there is a tradeoff involved here.<sup>10</sup>

The story is not complete yet, however, for the  $\sigma^2$  estimated from model (13.3.2) and that estimated from the true model (13.3.1) are not the same because the RSS of the two models as well as their degrees of freedom (df) are different. You may recall that we obtain an estimate of  $\sigma^2$  as  $\hat{\sigma}^2 = \text{RSS}/\text{df}$ , which depends on the number of regressors included in the model as well as the df ( $= n$ , number of parameters estimated). Now if we add variables to the model, the RSS generally decreases (recall that as more variables are added to the model, the  $R^2$  increases), but the degrees of freedom also decrease because more parameters are estimated. The net outcome depends on whether the RSS decreases sufficiently to offset the loss of degrees of freedom due to the addition of regressors. It is quite possible that if a regressor has a strong impact on the regressand—for example, it may reduce RSS more than the loss in degrees of freedom as a result of its addition to the model—inclusion of such variables will not only reduce the bias but will also increase precision (i.e., reduce standard errors) of the estimators.

On the other hand, if the relevant variables have only a marginal impact on the regressand, and if they are highly correlated (i.e., VIF is larger), we may reduce the bias in the coefficients of the variables already included in the model, but increase their standard errors (i.e., make them less efficient). Indeed, the tradeoff in this situation between bias and precision can be substantial. As you can see from this discussion, the tradeoff will depend on the relative importance of the various regressors.

<sup>10</sup>To bypass the tradeoff between bias and efficiency, one could choose to minimize the mean square error (MSE), since it accounts for both bias and efficiency. On MSE, see the statistical appendix, **App. A**. See also exercise 13.6.



To conclude this discussion, let us consider the special case where  $r_{23} = 0$ , that is,  $X_2$  and  $X_3$  are uncorrelated. This will result in  $b_{32}$  being zero (why?). Therefore, it can be seen from (13.3.3) that  $\hat{\alpha}_2$  is now unbiased.<sup>11</sup> Also, it seems from (13.3.4) and (13.3.5) that the variances of  $\hat{\alpha}_2$  and  $\hat{\beta}_2$  are the same. Is there no harm in dropping the variable  $X_3$  from the model even though it may be relevant theoretically? The answer generally is no, for in this case, as noted earlier,  $\text{var}(\hat{\alpha}_2)$  estimated from (13.3.4) is still biased and therefore our hypothesis-testing procedures are likely to remain suspect.<sup>12</sup> Besides, in most economic research  $X_2$  and  $X_3$  will be correlated, thus creating the problems discussed previously. **The point is clear: Once a model is formulated on the basis of the relevant theory, one is ill-advised to drop a variable from such a model.**

#### Inclusion of an Irrelevant Variable (Overfitting a Model)

Now let us assume that

$$Y_i = \beta_1 + \beta_2 X_{2i} + u_i \quad (13.3.6)$$

is the truth, but we fit the following model:

$$Y_i = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + v_i \quad (13.3.7)$$

and thus commit the specification error of including an unnecessary variable in the model.

The consequences of this specification error are as follows:

1. The OLS estimators of the parameters of the “incorrect” model are all *unbiased and consistent*, that is,  $E(\hat{\alpha}_1) = \beta_1$ ,  $E(\hat{\alpha}_2) = \beta_2$ , and  $E(\hat{\alpha}_3) = \beta_3 = 0$ .
2. The error variance  $\sigma^2$  is correctly estimated.
3. The usual confidence interval and hypothesis-testing procedures remain valid.
4. However, the estimated  $\alpha$ 's will be generally inefficient, that is, their variances will be generally larger than those of the  $\hat{\beta}$ 's of the true model. The proofs of some of these statements can be found in Appendix 13A, Section 13A.2. The point of interest here is the relative inefficiency of the  $\hat{\alpha}$ 's. This can be shown easily.

From the usual OLS formula we know that

$$\text{var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum x_{2i}^2} \quad (13.3.8)$$

<sup>11</sup>Note, though,  $\hat{\alpha}_1$  is still biased, which can be seen intuitively as follows: We know that  $\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}_2 - \hat{\beta}_3 \bar{X}_3$ , whereas  $\hat{\alpha}_1 = \bar{Y} - \hat{\alpha}_2 \bar{X}_2$ , and even if  $\hat{\alpha}_2 = \hat{\beta}_2$ , the two intercept estimators will not be the same.

<sup>12</sup>For details, see Adrian C. Darnell, *A Dictionary of Econometrics*, Edward Elgar Publisher, 1994, pp. 371–372.

and

$$\text{var}(\hat{\alpha}_2) = \frac{\sigma^2}{\sum x_{2i}^2(1 - r_{23}^2)} \quad (13.3.9)$$

Therefore,

$$\frac{\text{var}(\hat{\alpha}_2)}{\text{var}(\hat{\beta}_2)} = \frac{1}{1 - r_{23}^2} \quad (13.3.10)$$

Since  $0 \leq r_{23}^2 \leq 1$ , it follows that  $\text{var}(\hat{\alpha}_2) \geq \text{var}(\hat{\beta}_2)$ ; that is, the variance of  $\hat{\alpha}_2$  is generally greater than the variance of  $\hat{\beta}_2$  even though, on average,  $\hat{\alpha}_2 = \beta_2$  [i.e.,  $E(\hat{\alpha}_2) = \beta_2$ ].

The implication of this finding is that the inclusion of the unnecessary variable  $X_3$  makes the variance of  $\hat{\alpha}_2$  larger than necessary, thereby making  $\hat{\alpha}_2$  less precise. This is also true of  $\hat{\alpha}_1$ .

Notice the **asymmetry** in the two types of specification biases we have considered. If we exclude a relevant variable, the coefficients of the variables retained in the model are generally biased as well as inconsistent, the error variance is incorrectly estimated, and the usual hypothesis-testing procedures become invalid. On the other hand, including an irrelevant variable in the model still gives us unbiased and consistent estimates of the coefficients in the true model, the error variance is correctly estimated, and the conventional hypothesis-testing methods are still valid; the only penalty we pay for the inclusion of the superfluous variable is that the estimated variances of the coefficients are larger, and as a result our probability inferences about the parameters are less precise. An unwanted conclusion here would be that it is better to include irrelevant variables than to omit the relevant ones. But this philosophy is not to be espoused because addition of unnecessary variables will lead to loss in efficiency of the estimators and may also lead to the problem of multicollinearity (why?), not to mention the loss of degrees of freedom. Therefore,

In general, the best approach is to include only explanatory variables that, on theoretical grounds, *directly* influence the dependent variable and that are not accounted for by other included variables.<sup>13</sup>

### 13.4 TESTS OF SPECIFICATION ERRORS

Knowing the consequences of specification errors is one thing but finding out whether one has committed such errors is quite another, for we do not deliberately set out to commit such errors. Very often specification biases arise inadvertently, perhaps from our inability to formulate the model as

<sup>13</sup>Michael D. Intriligator, *Econometric Models, Techniques and Applications*, Prentice Hall, Englewood Cliffs, N.J., 1978, p. 189. Recall the Occam's razor principle.

precisely as possible because the underlying theory is weak or because we do not have the right kind of data to test the model. As Davidson notes, “Because of the non-experimental nature of economics, we are never sure how the observed data were generated. The test of any hypothesis in economics always turns out to depend on additional assumptions necessary to specify a reasonably parsimonious model, which may or may not be justified.”<sup>14</sup>

The practical question then is not why specification errors are made, for they generally are, but how to detect them. Once it is found that specification errors have been made, the remedies often suggest themselves. If, for example, it can be shown that a variable is inappropriately omitted from a model, the obvious remedy is to include that variable in the analysis, assuming, of course, the data on that variable are available.

In this section we discuss some tests that one may use to detect specification errors.

### Detecting the Presence of Unnecessary Variables (Overfitting a Model)

Suppose we develop a  $k$ -variable model to explain a phenomenon:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i \quad (13.4.1)$$

However, we are not totally sure that, say, the variable  $X_k$  really belongs in the model. One simple way to find this out is to test the significance of the estimated  $\beta_k$  with the usual  $t$  test:  $t = \hat{\beta}_k / \text{se}(\hat{\beta}_k)$ . But suppose that we are not sure whether, say,  $X_3$  and  $X_4$  legitimately belong in the model. This can be easily ascertained by the  $F$  test discussed in Chapter 8. Thus, detecting the presence of an irrelevant variable (or variables) is not a difficult task.

It is, however, very important to remember that in carrying out these tests of significance we have a specific model in mind. We accept that model as the **maintained hypothesis** or the “truth,” however tentative it may be. Given that model, then, we can find out whether one or more regressors are really relevant by the usual  $t$  and  $F$  tests. But note carefully that we should not use the  $t$  and  $F$  tests to build a model *iteratively*, that is, we should not say that initially  $Y$  is related to  $X_2$  only because  $\hat{\beta}_2$  is statistically significant and then expand the model to include  $X_3$  and decide to keep that variable in the model if  $\hat{\beta}_3$  turns out to be statistically significant, and so on. This strategy of building a model is called the **bottom-up approach** (starting with a smaller model and expanding it as one goes along) or by the somewhat pejorative term, **data mining** (other names are **regression fishing**, **data grubbing**, **data snooping**, and **number crunching**).

<sup>14</sup>James Davidson, *Econometric Theory*, Blackwell Publishers, Oxford, U.K., 2000, p. 153.

The primary objective of data mining is to develop the “best” model after several diagnostic tests so that the model finally chosen is a “good” model in the sense that all the estimated coefficients have the “right” signs, they are statistically significant on the basis of the  $t$  and  $F$  tests, the  $R^2$  value is reasonably high and the Durbin–Watson  $d$  has acceptable value (around 2), etc. The purists in the profession look down on the practice of data mining. In the words of William Pool, “. . . making an empirical regularity the foundation, rather than an implication of economic theory, is always dangerous.”<sup>15</sup> One reason for “condemning” data mining is as follows.

**Nominal versus True Level of Significance in the Presence of Data Mining.** A danger of data mining that the unwary researcher faces is that the conventional levels of significance ( $\alpha$ ) such as 1, 5, or 10 percent are *not the true levels of significance*. Lovell has suggested that if there are  $c$  candidate regressors out of which  $k$  are finally selected ( $k \leq c$ ) on the basis of data mining, then the true level of significance ( $\alpha^*$ ) is related to the nominal level of significance ( $\alpha$ ) as follows:<sup>16</sup>

$$\alpha^* = 1 - (1 - \alpha)^{c/k} \quad (13.4.2)$$

or approximately as

$$\alpha^* \approx (c/k)\alpha \quad (13.4.3)$$

For example, if  $c = 15$ ,  $k = 5$ , and  $\alpha = 5$  percent, from (13.4.3) the true level of significance is  $(15/5)(5) = 15$  percent. Therefore, if a researcher data-mines and selects 5 out of 15 regressors and reports only the results of the condensed model at the nominal 5 percent level of significance and declares that the results are statistically significant, one should take this conclusion with a big grain of salt, for we know the (true) level of significance is in fact 15 percent. It should be noted that if  $c = k$ , that is, there is no data mining, the true and nominal levels of significance are the same. Of course, in practice most researchers report only the results of their “final” regression without necessarily telling about all the data mining, or **pretesting**, that has gone before.<sup>17</sup>

Despite some of its obvious drawbacks, there is increasing recognition, especially among applied econometricians, that the purist (i.e., non-data mining) approach to model building is not tenable. As Zaman notes:

Unfortunately, experience with real data sets shows that such a [purist approach] is neither feasible nor desirable. It is not feasible because it is a rare economic

<sup>15</sup>William Pool, “Is Inflation Too Low,” the *Cato Journal*, vol. 18, no. 3, Winter 1999, p. 456.

<sup>16</sup>M. Lovell, “Data Mining,” *Review of Economics and Statistics*, vol. 65, 1983, pp. 1–12.

<sup>17</sup>For a detailed discussion of pretesting and the biases it can lead to, see Wallace, T. D., “Pretest Estimation in Regression: A Survey,” *American Journal of Agricultural Economics*, vol. 59, 1977, pp. 431–443.

theory which leads to a unique model. It is not desirable because a crucial aspect of learning from the data is learning what types of models are and are not supported by data. Even if, by rare luck, the initial model shows a good fit, it is frequently important to explore and learn the types of the models the data does or does not agree with.<sup>18</sup>

A similar view is expressed by Kerry Patterson who maintains that:

This [data mining] approach suggests that economic theory and empirical specification interact rather than be kept in separate compartments.<sup>19</sup>

Instead of getting caught in the data mining versus the purist approach to model-building controversy, one can endorse the view expressed by Peter Kennedy:

[that model specification] needs to be a well-thought-out combination of theory and data, and that testing procedures used in specification searches should be designed to minimize the costs of data mining. Examples of such procedures are setting aside data for out-of-sample prediction tests, adjusting significance levels [a la Lovell], and avoiding questionable criteria such as maximizing  $R^2$ .<sup>20</sup>

If we look at data mining in a broader perspective as a process of discovering empirical regularities that might suggest errors and/or omissions in (existing) theoretical models, it has a very useful role to play. To quote Kennedy again, "The art of the applied econometrician is to allow for data-driven theory while avoiding the considerable dangers in data mining."<sup>21</sup>

### Tests for Omitted Variables and Incorrect Functional Form

In practice we are never sure that the model adopted for empirical testing is "the truth, the whole truth and nothing but the truth." On the basis of theory or introspection and prior empirical work, we develop a model that we believe captures the essence of the subject under study. We then subject the model to empirical testing. After we obtain the results, we begin the post-mortem, keeping in mind the criteria of a good model discussed earlier. It is at this stage that we come to know if the chosen model is adequate. In determining model adequacy, we look at some broad features of the results, such as the  $\bar{R}^2$  value, the estimated  $t$  ratios, the signs of the estimated coefficients in relation to their prior expectations, the Durbin-Watson statistic, and the like. If these diagnostics are reasonably good, we proclaim that the

<sup>18</sup>Asad Zaman, *Statistical Foundations for Econometric Techniques*, Academic Press, New York, 1996, p. 226.

<sup>19</sup>Kerry Patterson, *An Introduction to Applied Econometrics*, St. Martin's Press, New York, 2000, p. 10.

<sup>20</sup>Peter Kennedy, "Sinning in the Basement: What Are the Rules? The Ten Commandments of Applied Econometrics," unpublished manuscript.

<sup>21</sup>Kennedy, op. cit., p. 13.

chosen model is a fair representation of reality. By the same token, if the results do not look encouraging because the  $\bar{R}^2$  value is too low or because very few coefficients are statistically significant or have the correct signs or because the Durbin–Watson  $d$  is too low, then we begin to worry about model adequacy and look for remedies: Maybe we have omitted an important variable, or have used the wrong functional form, or have not first-differenced the time series (to remove serial correlation), and so on. To aid us in determining whether model inadequacy is on account of one or more of these problems, we can use some of the following methods.

**Examination of Residuals.** As noted in Chapter 12, examination of the residuals is a good visual diagnostic to detect autocorrelation or heteroscedasticity. But these residuals can also be examined, especially in cross-sectional data, for model specification errors, such as omission of an important variable or incorrect functional form. If in fact there are such errors, a plot of the residuals will exhibit distinct patterns.

To illustrate, let us reconsider the cubic total cost of production function first considered in Chapter 7. Assume that the true total cost function is described as follows, where  $Y$  = total cost and  $X$  = output:

$$Y_i = \beta_1 + \beta_2 X_i + \beta_3 X_i^2 + \beta_4 X_i^3 + u_i \quad (13.4.4)$$

but a researcher fits the following quadratic function:

$$Y_i = \alpha_1 + \alpha_2 X_i + \alpha_3 X_i^2 + u_{2i} \quad (13.4.5)$$

and another researcher fits the following linear function:

$$Y_i = \lambda_1 + \lambda_2 X_i + u_{3i} \quad (13.4.6)$$

Although we know that both researchers have made specification errors, for pedagogical purposes let us see how the estimated residuals look in the three models. (The cost-output data are given in Table 7.4.) Figure 13.1 speaks for itself: As we move from left to right, that is, as we approach the truth, not only are the residuals smaller (in absolute value) but also they do not exhibit the pronounced cyclical swings associated with the misfitted models.

The utility of examining the residual plot is thus clear: If there are specification errors, the residuals will exhibit noticeable patterns.

**The Durbin–Watson  $d$  Statistic Once Again.** If we examine the routinely calculated Durbin–Watson  $d$  in Table 13.1, we see that for the linear cost function the estimated  $d$  is 0.716, suggesting that there is positive “correlation” in the estimated residuals: for  $n = 10$  and  $k' = 1$ , the 5 percent

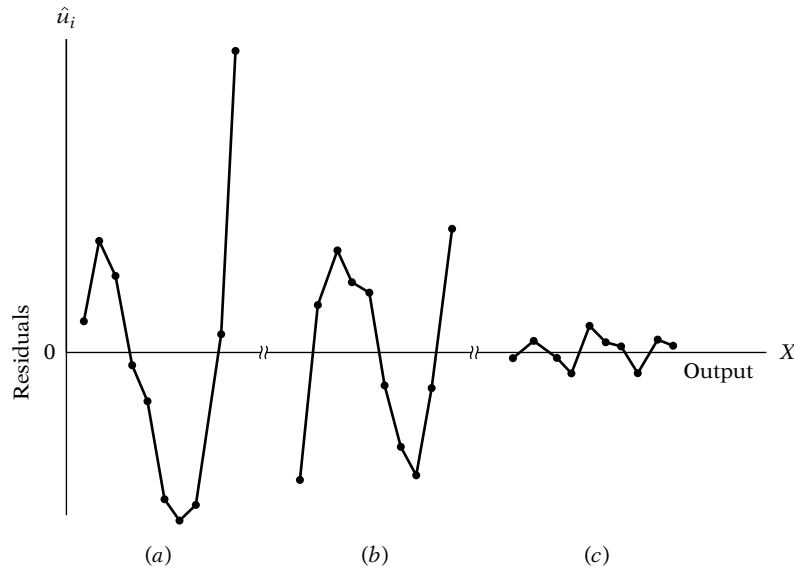


FIGURE 13.1 Residuals  $\hat{u}_i$  from (a) linear, (b) quadratic, and (c) cubic total cost functions.

TABLE 13.1 ESTIMATED RESIDUALS FROM THE LINEAR, QUADRATIC, AND CUBIC TOTAL COST FUNCTIONS

Observation number	$\hat{u}_i$ linear model*	$\hat{u}_i$ quadratic model†	$\hat{u}_i$ cubic model**
1	6.600	-23.900	-0.222
2	19.667	9.500	1.607
3	13.733	18.817	-0.915
4	-2.200	13.050	-4.426
5	-9.133	11.200	4.435
6	-26.067	-5.733	1.032
7	-32.000	-16.750	0.726
8	-28.933	-23.850	-4.119
9	4.133	-6.033	1.859
10	54.200	23.700	0.022

* $\hat{Y}_i = 166.467 + 19.933X_i$ (19.021) (3.066) (8.752) (6.502)		$R^2 = 0.8409$ $\bar{R}^2 = 0.8210$ $d = 0.716$
† $\hat{Y}_i = 222.383 - 8.0250X_i + 2.542X_i^2$ (23.488) (9.809) (0.869) (9.468) (-0.818) (2.925)		$R^2 = 0.9284$ $\bar{R}^2 = 0.9079$ $d = 1.038$
** $\hat{Y}_i = 141.767 + 63.478X_i - 12.962X_i^2 + 0.939X_i^3$ (6.375) (4.778) (0.9856) (0.0592) (22.238) (13.285) (-13.151) (15.861)		$R^2 = 0.9983$ $\bar{R}^2 = 0.9975$ $d = 2.70$



critical  $d$  values are  $d_L = 0.879$  and  $d_U = 1.320$ . Likewise, the computed  $d$  value for the quadratic cost function is 1.038, whereas the 5 percent critical values are  $d_L = 0.697$  and  $d_U = 1.641$ , indicating indecision. But if we use the modified  $d$  test (see Chapter 12), we can say that there is positive “correlation” in the residuals, for the computed  $d$  is less than  $d_U$ . For the cubic cost function, the true specification, the estimated  $d$  value does not indicate any positive “correlation” in the residuals.<sup>22</sup>

The observed positive “correlation” in the residuals when we fit the linear or quadratic model is not a measure of (first-order) serial correlation but of (model) specification error(s). The observed correlation simply reflects the fact that some variable(s) that belong in the model are included in the error term and need to be culled out from it and introduced in their own right as explanatory variables: If we exclude the  $X_i^3$  from the cost function, then as (13.2.3) shows, the error term in the mis-specified model (13.2.2) is in fact  $(u_{1i} + \beta_4 X_i^3)$  and it will exhibit a systematic pattern (e.g., positive autocorrelation) if  $X_i^3$  in fact affects  $Y$  significantly.

To use the Durbin–Watson test for detecting model specification error(s), we proceed as follows:

1. From the assumed model, obtain the OLS residuals.
2. If it is believed that the assumed model is mis-specified because it excludes a relevant explanatory variable, say,  $Z$  from the model, order the residuals obtained in Step 1 according to increasing values of  $Z$ . *Note:* The  $Z$  variable could be one of the  $X$  variables included in the assumed model or it could be some function of that variable, such as  $X^2$  or  $X^3$ .
3. Compute the  $d$  statistic from the residuals thus ordered by the usual  $d$  formula, namely,

$$d = \frac{\sum_{t=2}^n (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^n \hat{u}_t^2}$$

*Note:* The subscript  $t$  is the index of observation here and does not necessarily mean that the data are time series.

4. From the Durbin–Watson tables, if the estimated  $d$  value is significant, then one can accept the hypothesis of model mis-specification. If that turns out to be the case, the remedial measures will naturally suggest themselves.

In our cost example, the  $Z (= X)$  variable (output) was already ordered.<sup>23</sup> Therefore, we do not have to compute the  $d$  statistic afresh. As we have seen, the  $d$  statistic for both the linear and quadratic cost functions suggests

<sup>22</sup>In the present context, a value of  $d = 2$  will mean no specification error. (Why?)

<sup>23</sup>It does not matter if we order  $\hat{u}_i$  according to  $X_i^2$  or  $X_i^3$  since these are functions of  $X_i$ , which is already ordered.

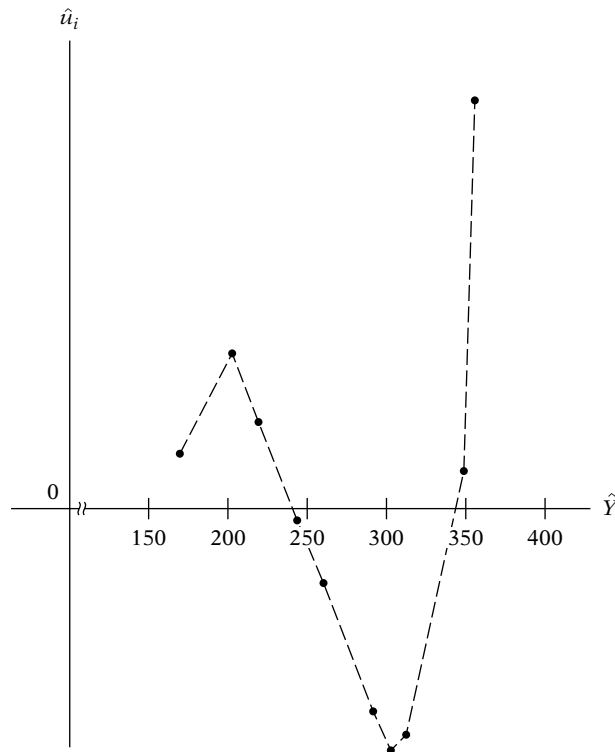


specification errors. The remedies are clear: Introduce the quadratic and cubic terms in the linear cost function and the cubic term in the quadratic cost function. In short, run the cubic cost model.

**Ramsey's RESET Test.** Ramsey has proposed a general test of specification error called RESET (regression specification error test).<sup>24</sup> Here we will illustrate only the simplest version of the test. To fix ideas, let us continue with our cost-output example and assume that the cost function is linear in output as

$$Y_i = \lambda_1 + \lambda_2 X_i + u_{3i} \quad (13.4.6)$$

where  $Y$  = total cost and  $X$  = output. Now if we plot the residuals  $\hat{u}_i$  obtained from this regression against  $\hat{Y}_i$ , the estimated  $Y_i$  from this model, we get the picture shown in Figure 13.2. Although  $\sum \hat{u}_i$  and  $\sum \hat{u}_i \hat{Y}_i$  are necessarily zero



**FIGURE 13.2** Residuals  $\hat{u}_i$  and estimated  $Y$  from the linear cost function:  $Y_i = \lambda_1 + \lambda_2 X_i + u_i$ .

<sup>24</sup>J. B. Ramsey, "Tests for Specification Errors in Classical Linear Least Squares Regression Analysis," *Journal of the Royal Statistical Society*, series B, vol. 31, 1969, pp. 350–371.

(why? see Chapter 3), the residuals in this figure show a pattern in which their mean changes systematically with  $\hat{Y}_i$ . This would suggest that if we introduce  $\hat{Y}_i$  in some form as regressor(s) in (13.4.6), it should increase  $R^2$ . And if the increase in  $R^2$  is statistically significant (on the basis of the  $F$  test discussed in Chapter 8), it would suggest that the linear cost function (13.4.6) was mis-specified. This is essentially the idea behind RESET. **The steps involved in RESET are as follows:**

1. From the chosen model, e.g., (13.4.6), obtain the estimated  $Y_i$ , that is,  $\hat{Y}_i$ .

2. Rerun (13.4.6) introducing  $\hat{Y}_i$  in some form as an additional regressor(s). From Figure 13.2, we observe that there is a curvilinear relationship between  $\hat{u}_i$  and  $\hat{Y}_i$ , suggesting that one can introduce  $\hat{Y}_i^2$  and  $\hat{Y}_i^3$  as additional regressors. Thus, we run

$$Y_i = \beta_1 + \beta_2 X_i + \beta_3 \hat{Y}_i^2 + \beta_4 \hat{Y}_i^3 + u_i \quad (13.4.7)$$

3. Let the  $R^2$  obtained from (13.4.7) be  $R_{\text{new}}^2$  and that obtained from (13.4.6) be  $R_{\text{old}}^2$ . Then we can use the  $F$  test first introduced in (8.5.18), namely,

$$F = \frac{(R_{\text{new}}^2 - R_{\text{old}}^2)/\text{number of new regressors}}{(1 - R_{\text{new}}^2)/(n - \text{number of parameters in the new model})} \quad (8.5.18)$$

to find out if the increase in  $R^2$  from using (13.4.7) is statistically significant.

4. If the computed  $F$  value is significant, say, at the 5 percent level, one can accept the hypothesis that the model (13.4.6) is mis-specified.

Returning to our illustrative example, we have the following results (standard errors in parentheses):

$$\hat{Y}_i = 166.467 + 19.933X_i \quad (19.021) \quad (3.066) \quad R^2 = 0.8409 \quad (13.4.8)$$

$$\hat{Y}_i = 2140.7223 + 476.6557X_i - 0.09187\hat{Y}_i^2 + 0.000119\hat{Y}_i^3 \quad (132.0044) \quad (33.3951) \quad (0.00620) \quad (0.0000074) \quad R^2 = 0.9983 \quad (13.4.9)$$

Note:  $\hat{Y}_i^2$  and  $\hat{Y}_i^3$  in (13.4.9) are obtained from (13.4.8).

Now applying the  $F$  test we find

$$F = \frac{(0.9983 - 0.8409)/2}{(1 - 0.9983)/(10 - 4)} = 284.4035 \quad (13.4.10)$$

The reader can easily verify that this  $F$  value is highly significant, indicating that the model (13.4.8) is mis-specified. Of course, we have reached the same conclusion on the basis of the visual examination of the residuals as well as the Durbin–Watson  $d$  value.

One advantage of RESET is that it is easy to apply, for it does not require one to specify what the alternative model is. But that is also its disadvantage because knowing that a model is mis-specified does not help us necessarily in choosing a better alternative.

**Lagrange Multiplier (LM) Test for Adding Variables.** This is an alternative to Ramsey’s RESET test. To illustrate this test, we will continue with the preceding illustrative example.

If we compare the linear cost function (13.4.6) with the cubic cost function (13.4.4), the former is a *restricted version* of the latter (recall our discussion of **restricted least-squares** from Chapter 8). The restricted regression (13.4.6) assumes that the coefficients of the squared and cubed output terms are equal to zero. To test this, the LM test proceeds as follows:

1. Estimate the restricted regression (13.4.6) by OLS and obtain the residuals,  $\hat{u}_i$ .
2. If in fact the unrestricted regression (13.4.4) is the true regression, the residuals obtained in (13.4.6) should be related to the squared and cubed output terms, that is,  $X_i^2$  and  $X_i^3$ .
3. This suggests that we regress the  $\hat{u}_i$  obtained in Step 1 on all the regressors (including those in the restricted regression), which in the present case means

$$\hat{u}_i = \alpha_1 + \alpha_2 X_i + \alpha_3 X_i^2 + \alpha_4 X_i^3 + v_i \quad (13.4.11)$$

where  $v$  is an error term with the usual properties.

4. For large-sample size, Engle has shown that  $n$  (the sample size) times the  $R^2$  estimated from the (auxiliary) regression (13.4.11) follows the chi-square distribution with df equal to the number of restrictions imposed by the restricted regression, two in the present example since the terms  $X_i^2$  and  $X_i^3$  are dropped from the model.<sup>25</sup> Symbolically, we write

$$nR^2 \underset{\text{asy}}{\sim} \chi^2_{(\text{number of restrictions})} \quad (13.4.12)$$

where asy means asymptotically, that is, in large samples.

5. If the chi-square value obtained from (13.4.12) exceeds the critical chi-square value at the chosen level of significance, we reject the restricted regression. Otherwise, we do not reject it.

<sup>25</sup>R. F. Engle, “A General Approach to Lagrangian Multiplier Model Diagnostics,” *Journal of Econometrics*, vol. 20, 1982, pp. 83–104.

For our example, the regression results are as follows:

$$\hat{Y}_i = 166.467 + 19.333X_i \quad (13.4.13)$$

where  $Y$  is total cost and  $X$  is output. The standard errors for this regression are already given in Table 13.1.

When the residuals from (13.4.13) are regressed as just suggested in Step 3, we obtain the following results:

$$\begin{aligned} \hat{u}_i &= -24.7 + 43.5443X_i - 12.9615X_i^2 + 0.9396X_i^3 \\ \text{se} &= (6.375) \quad (4.779) \quad (0.986) \quad (0.059) \quad (13.4.14) \\ R^2 &= 0.9896 \end{aligned}$$

Although our sample size of 10 is by no means large, just to illustrate the LM mechanism, we obtain  $nR^2 = (10)(0.9896) = 9.896$ . From the chi-square table we observe that for 2 df the 1 percent critical chi-square value is about 9.21. Therefore, the observed value of 9.896 is significant at the 1 percent level, and our conclusion would be to reject the restricted regression (i.e., the linear cost function). We reached the similar conclusion on the basis of Ramsey's RESET test.

### 13.5 ERRORS OF MEASUREMENT

All along we have assumed implicitly that the dependent variable  $Y$  and the explanatory variables, the  $X$ 's, are measured without any errors. Thus, in the regression of consumption expenditure on income and wealth of households, we assume that the data on these variables are "accurate"; they are not *guess estimates*, extrapolated, interpolated, or rounded off in any systematic manner, such as to the nearest hundredth dollar, and so on. Unfortunately, this ideal is not met in practice for a variety of reasons, such as nonresponse errors, reporting errors, and computing errors. Whatever the reasons, error of measurement is a potentially troublesome problem, for it constitutes yet another example of specification bias with the consequences noted below.

#### Errors of Measurement in the Dependent Variable $Y$

Consider the following model:

$$Y_i^* = \alpha + \beta X_i + u_i \quad (13.5.1)$$

where  $Y_i^*$  = permanent consumption expenditure<sup>26</sup>

$X_i$  = current income

$u_i$  = stochastic disturbance term

<sup>26</sup>This phrase is due to Milton Friedman. See also exercise 13.8.

Since  $Y_i^*$  is not directly measurable, we may use an observable expenditure variable  $Y_i$  such that

$$Y_i = Y_i^* + \varepsilon_i \quad (13.5.2)$$

where  $\varepsilon_i$  denote errors of measurement in  $Y_i^*$ . Therefore, instead of estimating (13.5.1), we estimate

$$\begin{aligned} Y_i &= (\alpha + \beta X_i + u_i) + \varepsilon_i \\ &= \alpha + \beta X_i + (u_i + \varepsilon_i) \\ &= \alpha + \beta X_i + v_i \end{aligned} \quad (13.5.3)$$

where  $v_i = u_i + \varepsilon_i$  is a composite error term, containing the population disturbance term (which may be called the *equation error term*) and the measurement error term.

For simplicity assume that  $E(u_i) = E(\varepsilon_i) = 0$ ,  $\text{cov}(X_i, u_i) = 0$  (which is the assumption of the classical linear regression), and  $\text{cov}(X_i, \varepsilon_i) = 0$ ; that is, the errors of measurement in  $Y_i^*$  are uncorrelated with  $X_i$ , and  $\text{cov}(u_i, \varepsilon_i) = 0$ ; that is, the equation error and the measurement error are uncorrelated. With these assumptions, it can be seen that  $\beta$  estimated from either (13.5.1) or (13.5.3) will be an unbiased estimator of the true  $\beta$  (see exercise 13.7); that is, the errors of measurement in the dependent variable  $Y$  do not destroy the unbiasedness property of the OLS estimators. However, the variances and standard errors of  $\beta$  estimated from (13.5.1) and (13.5.3) will be different because, employing the usual formulas (see Chapter 3), we obtain

$$\text{Model (13.5.1):} \quad \text{var}(\hat{\beta}) = \frac{\sigma_u^2}{\sum x_i^2} \quad (13.5.4)$$

$$\begin{aligned} \text{Model (13.5.3):} \quad \text{var}(\hat{\beta}) &= \frac{\sigma_v^2}{\sum x_i^2} \\ &= \frac{\sigma_u^2 + \sigma_\varepsilon^2}{\sum x_i^2} \end{aligned} \quad (13.5.5)$$

Obviously, the latter variance is larger than the former.<sup>27</sup> Therefore, **although the errors of measurement in the dependent variable still give unbiased estimates of the parameters and their variances, the estimated variances are now larger than in the case where there are no such errors of measurement.**

<sup>27</sup>But note that this variance is still unbiased because under the stated conditions the composite error term  $v_i = u_i + \varepsilon_i$  still satisfies the assumptions underlying the method of least squares.

**Errors of Measurement in the Explanatory Variable  $X$** 

Now assume that instead of (13.5.1), we have the following model:

$$Y_i = \alpha + \beta X_i^* + u_i \quad (13.5.6)$$

where  $Y_i$  = current consumption expenditure  
 $X_i^*$  = permanent income  
 $u_i$  = disturbance term (equation error)

Suppose instead of observing  $X_i^*$ , we observe

$$X_i = X_i^* + w_i \quad (13.5.7)$$

where  $w_i$  represents errors of measurement in  $X_i^*$ . Therefore, instead of estimating (13.5.6), we estimate

$$\begin{aligned} Y_i &= \alpha + \beta(X_i - w_i) + u_i \\ &= \alpha + \beta X_i + (u_i - \beta w_i) \\ &= \alpha + \beta X_i + z_i \end{aligned} \quad (13.5.8)$$

where  $z_i = u_i - \beta w_i$ , a compound of equation and measurement errors.

Now even if we assume that  $w_i$  has zero mean, is serially independent, and is uncorrelated with  $u_i$ , we can no longer assume that the composite error term  $z_i$  is independent of the explanatory variable  $X_i$  because [assuming  $E(z_i) = 0$ ]

$$\begin{aligned} \text{cov}(z_i, X_i) &= E[z_i - E(z_i)][X_i - E(X_i)] \\ &= E(u_i - \beta w_i)(w_i) \quad \text{using (13.5.7)} \\ &= E(-\beta w_i^2) \\ &= -\beta \sigma_w^2 \end{aligned} \quad (13.5.9)$$

Thus, the explanatory variable and the error term in (13.5.8) are correlated, which violates the crucial assumption of the classical linear regression model that the explanatory variable is uncorrelated with the stochastic disturbance term. If this assumption is violated, it can be shown that the *OLS estimators are not only biased but also inconsistent, that is, they remain biased even if the sample size  $n$  increases indefinitely.*<sup>28</sup>

<sup>28</sup>As shown in **App. A**,  $\hat{\beta}$  is a consistent estimator of  $\beta$  if, as  $n$  increases indefinitely, the sampling distribution of  $\hat{\beta}$  will ultimately collapse to the true  $\beta$ . Technically, this is stated as  $\text{plim}_{n \rightarrow \infty} \hat{\beta} = \beta$ . As noted in **App. A**, consistency is a large-sample property and is often used to study the behavior of an estimator when its finite or small-sample properties (e.g., unbiasedness) cannot be determined.

For model (13.5.8), it is shown in Appendix 13A, Section 13A.3 that

$$\text{plim } \hat{\beta} = \beta \left[ \frac{1}{1 + \sigma_w^2 / \sigma_{X^*}^2} \right] \quad (13.5.10)$$

where  $\sigma_w^2$  and  $\sigma_{X^*}^2$  are variances of  $w_i$  and  $X^*$ , respectively, and where  $\text{plim } \hat{\beta}$  means the probability limit of  $\hat{\beta}$ .

Since the term inside the brackets is expected to be less than 1 (why?), (13.5.10) shows that even if the sample size increases indefinitely,  $\hat{\beta}$  will not converge to  $\beta$ . Actually, if  $\beta$  is assumed positive,  $\hat{\beta}$  will underestimate  $\beta$ , that is, it is biased toward zero. Of course, if there are no measurement errors in  $X$  (i.e.,  $\sigma_w^2 = 0$ ),  $\hat{\beta}$  will provide a consistent estimator of  $\beta$ .

Therefore, measurement errors pose a serious problem when they are present in the explanatory variable(s) because they make consistent estimation of the parameters impossible. Of course, as we saw, if they are present only in the dependent variable, the estimators remain unbiased and hence they are consistent too. If errors of measurement are present in the explanatory variable(s), what is the solution? The answer is not easy. At one extreme, we can assume that if  $\sigma_w^2$  is small compared to  $\sigma_{X^*}^2$ , for all practical purposes we can “assume away” the problem and proceed with the usual OLS estimation. Of course, the rub here is that we cannot readily observe or measure  $\sigma_w^2$  and  $\sigma_{X^*}^2$  and therefore there is no way to judge their relative magnitudes.

One other suggested remedy is the use of **instrumental** or **proxy variables** that, although highly correlated with the original  $X$  variables, are uncorrelated with the equation and measurement error terms (i.e.,  $u_i$  and  $w_i$ ). If such proxy variables can be found, then one can obtain a consistent estimate of  $\beta$ . But this task is much easier said than done. In practice it is not easy to find good proxies; we are often in the situation of complaining about the bad weather without being able to do much about it. Besides, it is not easy to find out if the selected instrumental variable is in fact independent of the error terms  $u_i$  and  $w_i$ .

In the literature there are other suggestions to solve the problem.<sup>29</sup> But most of them are specific to the given situation and are based on restrictive assumptions. There is really no satisfactory answer to the measurement errors problem. That is why it is so crucial to measure the data as accurately as possible.

<sup>29</sup>See Thomas B. Fomby, R. Carter Hill, and Stanley R. Johnson, *Advanced Econometric Methods*, Springer-Verlag, New York, 1984, pp. 273–277. See also Kennedy, op. cit., pp. 138–140, for a discussion of weighted regression as well as instrumental variables.

**AN EXAMPLE**

We conclude this section with an example constructed to highlight the preceding points.

Table 13.2 gives hypothetical data on true consumption expenditure  $Y^*$ , true income  $X^*$ , measured consumption  $Y$ , and measured income  $X$ . The table also explains how these variables were measured.<sup>30</sup>

**Measurement Errors in the Dependent Variable  $Y$  Only**

Based on the given data, the true consumption function is

$$\begin{aligned} \hat{Y}_i^* &= 25.00 + 0.6000X_i^* \\ &\quad (10.477) \quad (0.0584) \\ t &= (2.3861) \quad (10.276) \\ R^2 &= 0.9296 \end{aligned} \tag{13.5.11}$$

whereas, if we use  $Y_i$  instead of  $Y_i^*$ , we obtain

$$\begin{aligned} \hat{Y}_i &= 25.00 + 0.6000X_i^* \\ &\quad (12.218) \quad (0.0681) \\ t &= (2.0461) \quad (8.8118) \\ R^2 &= 0.9066 \end{aligned} \tag{13.5.12}$$

As these results show, and according to the theory, the estimated coefficients remain the same. The only effect of errors of measurement in the dependent variable is that the estimated standard errors of the coefficients

tend to be larger [see (13.5.5)], which is clearly seen in (13.5.12). In passing, note that the regression coefficients in (13.5.11) and (13.5.12) are the same because the sample was generated to match the assumptions of the measurement error model.

**Errors of Measurement in  $X$**

We know that the true regression is (13.5.11). Suppose now that instead of using  $X_i^*$ , we use  $X_i$ . (Note: In reality  $X_i^*$  is rarely observable.) The regression results are as follows:

$$\begin{aligned} \hat{Y}_i^* &= 25.992 + 0.5942X_i \\ &\quad (11.0810) \quad (0.0617) \\ t &= (2.3457) \quad (9.6270) \\ R^2 &= 0.9205 \end{aligned} \tag{13.5.13}$$

These results are in accord with the theory—when there are measurement errors in the explanatory variable(s), the estimated coefficients are biased. Fortunately, in this example the bias is rather small—from (13.5.10) it is evident that the bias depends on  $\sigma_w^2/\sigma_{X^*}^2$ , and in generating the data it was assumed that  $\sigma_w^2 = 36$  and  $\sigma_{X^*}^2 = 3667$ , thus making the bias factor rather small, about 0.98 percent ( $= 36/3667$ ).

We leave it to the reader to find out what happens when there are errors of measurement in both  $Y$  and  $X$ , that is, if we regress  $Y_i$  on  $X_i$  rather than  $Y_i^*$  on  $X_i^*$  (see exercise 13.23).

**TABLE 13.2**  
HYPOTHETICAL DATA ON  $Y^*$  (TRUE CONSUMPTION EXPENDITURE),  $X^*$  (TRUE INCOME),  $Y$  (MEASURED CONSUMPTION EXPENDITURE), AND  $X$  (MEASURED INCOME); ALL DATA IN DOLLARS

$Y^*$	$X^*$	$Y$	$X$	$\varepsilon$	$w$	$u$
75.4666	80.00	67.6011	80.0940	-7.8655	0.0940	2.4666
74.9801	100.00	75.4438	91.5721	0.4636	-8.4279	-10.0199
102.8242	120.00	109.6956	112.1406	6.8714	2.1406	5.8242
125.7651	140.00	129.4159	145.5969	3.6509	5.5969	16.7651
106.5035	160.00	104.2388	168.5579	-2.2647	8.5579	-14.4965
131.4318	180.00	125.8319	171.4793	-5.5999	-8.5207	-1.5682
149.3693	200.00	153.9926	203.5366	4.6233	3.5366	4.3693
143.8628	220.00	152.9208	222.8533	9.0579	2.8533	-13.1372
177.5218	240.00	176.3344	232.9879	-1.1874	-7.0120	8.5218
182.2748	260.00	174.5252	261.1813	-7.7496	1.1813	1.2748

Note: The data on  $X^*$  are assumed to be given. In deriving the other variables the assumptions made were as follows: (1)  $E(u) = E(\varepsilon) = E(w) = 0$ ; (2)  $\text{cov}(X, u) = \text{cov}(X, \varepsilon) = \text{cov}(u, \varepsilon) = \text{cov}(w, u) = \text{cov}(\varepsilon, w) = 0$ ; (3)  $\sigma_u^2 = 100$ ,  $\sigma_\varepsilon^2 = 36$ , and  $\sigma_w^2 = 36$ ; and (4)  $Y_i^* = 25 + 0.6X_i^* + u_i$ ,  $Y_i = Y_i^* + \varepsilon_i$ , and  $X_i = X_i^* + w_i$ .

<sup>30</sup>I am indebted to Kenneth J. White for constructing this example. See his *Computer Handbook Using SHAZAM*, for use with Damodar Gujarati, *Basic Econometrics*, September 1985, pp. 117-121.



### 13.6 INCORRECT SPECIFICATION OF THE STOCHASTIC ERROR TERM

A common problem facing a researcher is the specification of the error term  $u_i$  that enters the regression model. Since the error term is not directly observable, there is no easy way to determine the form in which it enters the model. To see this, let us return to the models given in (13.2.8) and (13.2.9). For simplicity of exposition, we have assumed that there is no intercept in the model. We further assume that  $u_i$  in (13.2.8) is such that  $\ln u_i$  satisfies the usual OLS assumptions.

If we assume that (13.2.8) is the “correct” model but estimate (13.2.9), what are the consequences? It is shown in Appendix 13.A, Section 13A.4, that if  $\ln u_i \sim N(0, \sigma^2)$ , then

$$u_i \sim \text{log normal} [e^{\sigma^2/2}, e^{\sigma^2}(e^{\sigma^2} - 1)] \quad (13.6.1)$$

as a result:

$$E(\hat{\alpha}) = \beta e^{\sigma^2/2} \quad (13.6.2)$$

where  $e$  is the base of the natural logarithm.

As you can see,  $\hat{\alpha}$  is a biased estimator, as its average value is not equal to the true  $\beta$ .

We will have more to say about the specification of the stochastic error term in the chapter on nonlinear-in-the-parameter regression models.

### 13.7 NESTED VERSUS NON-NESTED MODELS

In carrying out specification testing, it is useful to distinguish between **nested and non-nested models**. To distinguish between the two, consider the following models:

$$\text{Model A: } Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \beta_5 X_{5i} + u_i$$

$$\text{Model B: } Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

We say that Model B is nested in Model A because it is a special case of Model A: If we estimate Model A and test the hypothesis that  $\beta_4 = \beta_5 = 0$  and do not reject it on the basis of, say, the  $F$  test,<sup>31</sup> Model A reduces to Model B. If we add variable  $X_4$  to Model B, then Model A will reduce to Model B if  $\beta_5$  is zero; here we will use the  $t$  test to test the hypothesis that the coefficient of  $X_5$  is zero.

Without calling them such, the specification error tests that we have discussed previously and the restricted  $F$  test that we discussed in Chapter 8 are essentially tests of nested hypothesis.

<sup>31</sup>More generally, one can use the likelihood ratio test, or the Wald test or the Lagrange Multiplier test, which were discussed briefly in Chap. 8.

Now consider the following models:

$$\text{Model C: } Y_i = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + u_i$$

$$\text{Model D: } Y_i = \beta_1 + \beta_2 Z_{2i} + \beta_3 Z_{3i} + v_i$$

where the  $X$ 's and  $Z$ 's are different variables. We say that Models C and D are **non-nested** because one cannot be derived as a special case of the other. In economics, as in other sciences, more than one competing theory may explain a phenomenon. Thus, the monetarists would emphasize the role of money in explaining changes in GDP, whereas the Keynesians may explain them by changes in government expenditure.

It may be noted here that one can allow Models C and D to contain regressors that are common to both. For example,  $X_3$  could be included in Model D and  $Z_2$  could be included in Model C. Even then these are non-nested models, because Model C does not contain  $Z_3$  and Model D does not contain  $X_2$ .

Even if the same variables enter the model, the functional form may make two models non-nested. For example, consider the model:

$$\text{Model E: } Y_i = \beta_1 + \beta_2 \ln Z_{2i} + \beta_3 \ln Z_{3i} + w_i$$

Models D and E are non-nested, as one cannot be derived as a special case of the other.

Since we already have looked at tests of nested models ( $t$  and  $F$  tests), in the following section we discuss some of the tests of non-nested models, which earlier we called model mis-specification errors.

### 13.8 TESTS OF NON-NESTED HYPOTHESES

According to Harvey,<sup>32</sup> there are two approaches to testing non-nested hypotheses: (1) the **discrimination approach**, where given two or more competing models, one chooses a model based on some criteria of goodness of fit, and (2) the **discerning approach** (my terminology) where, in investigating one model, we take into account information provided by other models. We consider these approaches briefly.

#### The Discrimination Approach

Consider Models C and D above. Since both models involve the same dependent variable, we can choose between two (or more) models based on some goodness-of-fit criterion, such as  $R^2$  or adjusted  $R^2$ , which we have already discussed. But keep in mind that in comparing two or more models,

<sup>32</sup>Andrew Harvey, *The Econometric Analysis of Time Series*, 2d ed., The MIT Press, Cambridge, Mass., 1990, Chap. 5.

the regressand must be the same. Besides these criteria, there are other criteria that are also used. These include **Akaike's information criterion (AIC)**, **Schwarz's information criterion (SIC)**, and **Mallows's  $C_p$  criterion**. We discuss these criteria in Section 13.9. Most modern statistical software packages have one or more of these criteria built into their regression routines. In the last section of this chapter, we will illustrate these criteria using an extended example. On the basis of one or more of these criteria a model is finally selected that has the highest  $\bar{R}^2$  or the lowest value of AIC or SIC, etc.

### The Discerning Approach

**The Non-Nested  $F$  Test or Encompassing  $F$  Test.** Consider Models C and D introduced earlier. How do we choose between the two models? For this purpose suppose we estimate the following nested or *hybrid* model:

$$\text{Model F: } Y_i = \lambda_1 + \lambda_2 X_{2i} + \lambda_3 X_{3i} + \lambda_4 Z_{2i} + \lambda_5 Z_{3i} + u_i$$

Notice that Model F *neests or encompasses* models C and D. But note that C is not nested in D and D is not nested in C, so they are non-nested models.

Now if Model C is correct,  $\lambda_4 = \lambda_5 = 0$ , whereas Model D is correct if  $\lambda_2 = \lambda_3 = 0$ . This testing can be done by the usual  $F$  test, hence the name non-nested  $F$  test.

However, there are problems with this testing procedure. *First*, if the  $X$ 's and the  $Z$ 's are highly correlated, then, as noted in the chapter on multicollinearity, it is quite likely that one or more of the  $\lambda$ 's are individually statistically insignificant, although on the basis of the  $F$  test one can reject the hypothesis that all the slope coefficients are simultaneously zero. In this case, we have no way of deciding whether Model C or Model D is the correct model. *Second*, there is another problem. Suppose we choose Model C as the *reference hypothesis* or model, and find that all its coefficients are significant. Now we add  $Z_2$  or  $Z_3$  or both to the model and find, using the  $F$  test, that their incremental contribution to the explained sum of squares (ESS) is statistically insignificant. Therefore, we decide to choose Model C.

But suppose we had instead chosen Model D as the reference model and found that all its coefficients were statistically significant. But when we add  $X_2$  or  $X_3$  or both to this model, we find, again using the  $F$  test, that their incremental contribution to ESS is insignificant. Therefore, we would have chosen model D as the correct model. Hence, "the choice of the reference hypothesis could determine the outcome of the choice model,"<sup>33</sup> especially if severe multicollinearity is present in the competing regressors. *Finally*, the artificially nested model  $F$  may not have any economic meaning.

<sup>33</sup>Thomas B. Fomby, R. Carter Hill, and Stanley R. Johnson, *Advanced Econometric Methods*, Springer Verlag, New York, 1984, p. 416.

AN ILLUSTRATIVE EXAMPLE: THE ST. LOUIS MODEL

To determine whether changes in nominal GNP can be explained by changes in the money supply (monetarism) or by changes in government expenditure (Keynesianism), we consider the following models:

$$\begin{aligned} \dot{Y}_t &= \alpha + \beta_0 \dot{M}_t + \beta_1 \dot{M}_{t-1} + \beta_2 \dot{M}_{t-2} + \beta_3 \dot{M}_{t-3} + \beta_4 \dot{M}_{t-4} + u_{1t} \\ &= \alpha + \sum_{i=0}^4 \beta_i \dot{M}_{t-i} + u_{1t} \end{aligned} \quad (13.8.1)$$

$$\begin{aligned} \dot{Y}_t &= \gamma + \lambda_0 \dot{E}_t + \lambda_1 \dot{E}_{t-1} + \lambda_2 \dot{E}_{t-2} + \lambda_3 \dot{E}_{t-3} + \lambda_4 \dot{E}_{t-4} + u_{2t} \\ &= \gamma + \sum_{i=0}^4 \lambda_i \dot{E}_{t-i} + u_{2t} \end{aligned} \quad (13.8.2)$$

where  $\dot{Y}_t$  = rate of growth in nominal GNP at time  $t$

$\dot{M}_t$  = rate of growth in the money supply ( $M_1$  version) at time  $t$

$\dot{E}_t$  = rate of growth in full, or high, employment government expenditure at time  $t$

In passing, note that (13.8.1) and (13.8.2) are examples of **distributed lag models**, a topic thoroughly discussed in Chapter 17. For the time being, simply note that the effect of a unit change in the money supply or government expenditure on GNP is distributed over a period of time and is not instantaneous.

Since a priori it may be difficult to decide between the two competing models, let us enmesh the two models as shown below:

$$\dot{Y}_t = \text{constant} + \sum_{i=0}^4 \beta_i \dot{M}_{t-i} + \sum_{i=0}^4 \lambda_i \dot{E}_{t-i} + u_{3t} \quad (13.8.3)$$

This nested model is one form in which the famous (Federal Reserve Bank of) St. Louis model, a pro-monetary-school bank, has been expressed and estimated. The results of this model for the period 1953–I to 1976–IV for the United States are as follows ( $t$  ratios in parentheses).<sup>34</sup>

Coefficient	Estimate	Coefficient	Estimate	
$\beta_0$	0.40 (2.96)	$\lambda_0$	0.08 (2.26)	
$\beta_1$	0.41 (5.26)	$\lambda_1$	0.06 (2.52)	
$\beta_2$	0.25 (2.14)	$\lambda_2$	0.00 (0.02)	
$\beta_3$	0.06 (0.71)	$\lambda_3$	-0.06 (-2.20)	(13.8.4)
$\beta_4$	-0.05 (-0.37)	$\lambda_4$	-0.07 (-1.83)	
$\sum_{i=0}^4 \beta_i$	1.06 (5.59)	$\sum_{i=0}^4 \lambda_i$	0.03 (0.40)	$R^2 = 0.40$
				$d = 1.78$

What do these results suggest about the superiority of one model over the other? If we consider the cumulative effect of a unit change in  $\dot{M}$  and  $\dot{E}$  on  $\dot{Y}$ , we obtain, respectively,  $\sum_{i=0}^4 \beta_i = 1.06$  and  $\sum_{i=0}^4 \lambda_i = 0.03$ , the former being statistically significant and the latter not. This comparison would tend to support the monetarist claim that it is changes in the money supply that determine changes in the (nominal) GNP. It is left as an exercise for the reader to evaluate critically this claim.

<sup>34</sup>See Keith M. Carlson, "Does the St. Louis Equation Now Believe in Fiscal Policy?" *Review, Federal Reserve Bank of St. Louis*, vol. 60, no. 2, February 1978, p. 17, table IV.

**Davidson–MacKinnon *J* Test.**<sup>35</sup> Because of the problems just listed in the non-nested *F* testing procedure, alternatives have been suggested. One is the *Davidson–MacKinnon J* test. To illustrate this test, suppose we want to compare hypothesis or Model C with hypothesis or Model D. The ***J* test** proceeds as follows:

1. We estimate Model D and from it we obtain the estimated *Y* values,  $\hat{Y}_i^D$ .
2. We add the predicted *Y* value in Step 1 as an additional regressor to Model C and estimate the following model:

$$Y_i = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + \alpha_4 \hat{Y}_i^D + u_i \quad (13.8.5)$$

where the  $\hat{Y}_i^D$  values are obtained from Step 1. This model is an example of the **encompassing principle**, as in the Hendry methodology.

3. Using the *t* test, test the hypothesis that  $\alpha_4 = 0$ .
4. If the hypothesis that  $\alpha_4 = 0$  is not rejected, we can accept (i.e., not reject) Model C as the true model because  $\hat{Y}_i^D$  included in (13.8.5), which represent the influence of variables not included in Model C, have no additional explanatory power beyond that contributed by Model C. In other words, Model C *encompasses* Model D in the sense that the latter model does not contain any additional information that will improve the performance of Model C. By the same token, if the null hypothesis is rejected, Model C cannot be the true model (why?).
5. Now we reverse the roles of hypotheses, or Models C and D. We now estimate Model C first, use the estimated *Y* values from this model as regressor in (13.8.5), repeat Step 4, and decide whether to accept Model D over Model C. More specifically, we estimate the following model:

$$Y_i = \beta_1 + \beta_2 Z_{2i} + \beta_3 Z_{3i} + \beta_4 \hat{Y}_i^C + u_i \quad (13.8.6)$$

where  $\hat{Y}_i^C$  are the estimated *Y* values from Model C. We now test the hypothesis that  $\beta_4 = 0$ . If this hypothesis is not rejected, we choose Model D over C. If the hypothesis that  $\beta_4 = 0$  is rejected, choose C over D, as the latter does not improve over the performance of C.

Although it is intuitively appealing, the *J* test has some problems. Since the tests given in (13.8.5) and (13.8.6) are performed independently, we have the following likely outcomes:

Hypothesis: $\alpha_4 = 0$		
Hypothesis: $\beta_4 = 0$	Do not reject	Reject
Do not reject	Accept both C and D	Accept D, reject C
Reject	Accept C, reject D	Reject both C and D

<sup>35</sup>R. Davidson and J. G. MacKinnon, "Several Tests for Model Specification in the Presence of Alternative Hypotheses," *Econometrica*, vol. 49, 1981, pp. 781–793.

As this table shows, we will not be able to get a clear answer if the  $J$  testing procedure leads to the acceptance or rejection of both models. In case both models are rejected, neither model helps us to explain the behavior of  $Y$ . Similarly, if both models are accepted, as Kmenta notes, “the data are apparently not rich enough to discriminate between the two hypotheses [models].”<sup>36</sup>

Another problem with the  $J$  test is that when we use the  $t$  statistic to test the significance of the estimated  $Y$  variable in models (13.8.5) and (13.8.6), the  $t$  statistic has the standard normal distribution only asymptotically, that is, in large samples. Therefore, the  $J$  test may not be very powerful (in the statistical sense) in small samples because it tends to reject the true hypothesis or model more frequently than it ought to.

AN ILLUSTRATIVE EXAMPLE

To illustrate the  $J$  test, consider the data given in Table 13.3. This table gives data on per capita personal consumption expenditure (PPCE) and per capita disposable personal income (PDPI), both measured in 1987 dollars, for the United States for the period 1970–1991. Now consider the following rival models:

$$\text{Model A: } PPCE_t = \alpha_1 + \alpha_2 PDPI_t + \alpha_3 PDPI_{t-1} + u_t \quad (13.8.7)$$

$$\text{Model B: } PPCE_t = \beta_1 + \beta_2 PDPI_t + \beta_3 PPCE_{t-1} + u_t \quad (13.8.8)$$

Model A states that PPCE depends on PDPI in the current and previous time period; this model is an example of what is known as the **distributed lag model** (see Chapter 17). Model B postulates that PPCE depends on current PDPI as well as PPCE in the previous time period; this model represents what is known as the **autoregressive model** (see Chapter 17). The

TABLE 13.3

PER CAPITA PERSONAL CONSUMPTION EXPENDITURE (PPCE) AND PER CAPITA PERSONAL DISPOSABLE INCOME (PDPI), 1987 DOLLARS, U.S., 1970–1991

Year	PPCE	PDPI	Year	PPCE	PDPI
1970	8,842	9,875	1981	10,770	12,156
1971	9,022	10,111	1982	10,782	12,146
1972	9,425	10,414	1983	11,179	12,349
1973	9,752	11,013	1984	11,617	13,029
1974	9,602	10,832	1985	12,015	13,258
1975	9,711	10,906	1986	12,336	13,552
1976	10,121	11,192	1987	12,568	13,545
1977	10,425	11,406	1988	12,903	13,890
1978	10,744	11,851	1989	13,029	14,005
1979	10,876	12,039	1990	13,044	14,068
1980	10,746	12,005	1991	12,824	13,886

Source: *Economic Report of the President, 1993*, Table B-5, p. 355.

(Continued)

<sup>36</sup>Jan Kmenta, op. cit., p. 597.

## AN ILLUSTRATIVE EXAMPLE (Continued)

reason for introducing the lagged value of PPCE in this model is to reflect inertia or habit persistence.

The results of estimating these models separately were as follows:

$$\begin{aligned} \text{Model A: } \widehat{PPCE}_t &= -1,299.0536 + 0.9204 PDPI_t + 0.0931 PDPI_{t-1} \\ t &= \quad (-4.0378) \quad (6.0178) \quad (0.6308) \quad (13.8.9) \\ & \quad \quad \quad \quad \quad \quad R^2 = 0.9888 \quad d = 0.8092 \end{aligned}$$

$$\begin{aligned} \text{Model B: } \widehat{PPCE}_t &= -841.8568 + 0.7117 PDPI_t + 0.2954 PPCE_{t-1} \\ t &= \quad (-2.4137) \quad (5.4634) \quad (2.3681) \quad (13.8.10) \\ & \quad \quad \quad \quad \quad \quad R^2 = 0.9912 \quad d = 1.0144 \end{aligned}$$

If one were to choose between these two models on the basis of the discrimination approach, using, say, the highest  $R^2$  criterion, one would choose (13.8.10); besides, in (13.8.10) both variables seem to be individually statistically significant, whereas in (13.8.9) only the current PDPI is statistically significant (but beware of the collinearity problem!).

But choosing (13.8.10) over (13.8.9) may not be appropriate because for predictive purposes there is not much difference in the two estimated  $R^2$  values.

To apply the  $J$  test, suppose we assume Model A is the null hypothesis, that is, the maintained model, and Model B is the alternative hypothesis. Now following the  $J$  test steps discussed earlier we use the estimated PPCE values from model (13.8.10) as an additional regressor in Model A, giving the following outcome:

$$\begin{aligned} \widehat{PPCE}_t &= 1,322.7958 - 0.7061 PDPI_t - 0.4357 PDPI_{t-1} + 2.1335 \widehat{PPCE}_t^B \\ t &= \quad (1.5896) \quad (-1.3958) \quad (-2.1926) \quad (3.3141) \quad (13.8.11) \\ & \quad \quad \quad \quad \quad \quad R^2 = 0.9932 \quad d = 1.7115 \end{aligned}$$

where  $\widehat{PPCE}_t^B$  on the right side of (13.8.11) are the estimated PPCE values from model B, (13.8.10). Since the coefficient of this variable is statistically significant (at the two-tail 0.004 level), following the  $J$  test procedure, we have to reject Model A in favor of Model B.

Now assuming Model B as the maintained hypothesis and Model A as the alternative hypothesis, and following exactly the same procedure as before, we obtain the following results:

$$\begin{aligned} \widehat{PPCE}_t &= -6,549.8659 + 5.1176 PDPI_t + 0.6302 PPCE_{t-1} - 4.6776 \widehat{PPCE}_t^A \\ t &= \quad (-2.4976) \quad (2.5424) \quad (3.4141) \quad (-2.1926) \quad (13.8.12) \\ & \quad \quad \quad \quad \quad \quad R^2 = 0.9920 \quad d = 1.7115 \end{aligned}$$

where  $\widehat{PPCE}_t^A$  on the right side of (13.8.12) is obtained from the Model A, (13.8.9). But in this regression, the coefficient of  $\widehat{PPCE}_t^A$  on the right side is also statistically significant (at the two-tail 0.0425 level). This result would suggest that we should now reject Model B in favor of Model A!

All this tells us is that neither model is particularly useful in explaining the behavior of per capita personal consumption expenditure in the United States over the period 1970–1991.

Of course, we have considered only two competing models. In reality, there may be more than two models. The  $J$  test procedure can be extended to multiple model comparisons, although the analysis can become quickly complex.

This example shows very vividly why the CLRM assumes that the regression model used in the analysis is correctly specified. Obviously it is very crucial in developing a model to pay very careful attention to the phenomenon being modeled.



**Other Tests of Model Selection.** The  $J$  test just discussed is only one of a group of tests of model selection. There is the **Cox test**, the **JA test**, the **P test**, **Mizon–Richard encompassing test**, and variants of these tests. Obviously, we cannot hope to discuss these specialized tests, for which the reader may want to consult the references cited in the various footnotes.<sup>37</sup>

### 13.9 MODEL SELECTION CRITERIA

In this section we discuss several criteria that have been used to choose among competing models and/or to compare models for forecasting purposes. Here we distinguish between **in-sample** forecasting and **out-of-sample** forecasting. In-sample forecasting essentially tells us how the chosen model fits the data in a given sample. Out-of-sample forecasting is concerned with determining how a fitted model forecasts future values of the regressand, given the values of the regressors.

Several criteria are used for this purpose. In particular, we discuss these criteria: (1)  $R^2$ , (2) adjusted  $R^2 (= \bar{R}^2)$ , (3) Akaike information criterion (AIC), (4) Schwarz Information criterion (SIC), (5) Mallows's  $C_p$  criterion, and (6) forecast  $\chi^2$  (chi-square). All these criteria aim at minimizing the residual sum of squares (RSS) (or increasing the  $R^2$  value). However, except for the first criterion, criteria (2), (3), (4), and (5) impose a penalty for including an increasingly large number of regressors. Thus there is a tradeoff between goodness of fit of the model and its complexity (as judged by the number of regressors).

#### The $R^2$ Criterion

We know that one of the measures of goodness of fit of a regression model is  $R^2$ , which, as we know, is defined as:

$$R^2 = \frac{\text{ESS}}{\text{TSS}} = 1 - \frac{\text{RSS}}{\text{TSS}} \quad (13.9.1)$$

$R^2$ , thus defined, of necessity lies between 0 and 1. The closer it is to 1, the better is the fit. But there are problems with  $R^2$ . *First*, it measures *in-sample* goodness of fit in the sense of how close an estimated  $Y$  value is to its actual value in the given sample. There is no guarantee that it will forecast well *out-of-sample* observations. *Second*, in comparing two or more  $R^2$ 's, the dependent variable, or regressand, must be the same. *Third*, and more importantly, an  $R^2$  cannot fall when more variables are added to the model. Therefore, there is every temptation to play the game of “maximizing the  $R^2$ ” by simply adding more variables to the model. Of course, adding more variables to the model may increase  $R^2$  but it may also increase the variance of forecast error.

<sup>37</sup>See also Badi H. Baltagi, *Econometrics*, Springer, New York, 1998, pp. 209–222.



**Adjusted  $R^2$** 

As a penalty for adding regressors to increase the  $R^2$  value, Henry Theil developed the adjusted  $R^2$ , denoted by  $\bar{R}^2$ , which we studied in Chapter 7. Recall that

$$\bar{R}^2 = 1 - \frac{\text{RSS}/(n-k)}{\text{TSS}/(n-1)} = 1 - (1 - R^2) \frac{n-1}{n-k} \quad (13.9.2)$$

As you can see from this formula,  $\bar{R}^2 \leq R^2$ , showing how the adjusted  $R^2$  penalizes for adding more regressors. As we noted in Chapter 8, unlike  $R^2$ , the adjusted  $R^2$  will increase only if the absolute  $t$  value of the added variable is greater than 1. For comparative purposes, therefore,  $\bar{R}^2$  is a better measure than  $R^2$ . But again keep in mind that the regressand must be the same for the comparison to be valid.

**Akaike Information Criterion (AIC)**

The idea of imposing a penalty for adding regressors to the model has been carried further in the AIC criterion, which is defined as:

$$\text{AIC} = e^{2k/n} \frac{\sum \hat{u}_i^2}{n} = e^{2k/n} \frac{\text{RSS}}{n} \quad (13.9.3)$$

where  $k$  is the number of regressors (including the intercept) and  $n$  is the number of observations. For mathematical convenience, (13.9.3) is written as

$$\ln \text{AIC} = \left( \frac{2k}{n} \right) + \ln \left( \frac{\text{RSS}}{n} \right) \quad (13.9.4)$$

where  $\ln \text{AIC}$  = natural log of AIC and  $2k/n$  = penalty factor. Some textbooks and software packages define AIC only in terms of its log transform so there is no need to put  $\ln$  before AIC. As you see from this formula, AIC imposes a harsher penalty than  $\bar{R}^2$  for adding more regressors. In comparing two or more models, the model with the lowest value of AIC is preferred. One advantage of AIC is that it is useful for not only in-sample but also out-of-sample forecasting performance of a regression model. Also, it is useful for both nested and non-nested models. It has been also used to determine the lag length in an  $\text{AR}(p)$  model.

**Schwarz Information Criterion (SIC)**

Similar in spirit to the AIC, the SIC criterion is defined as:

$$\text{SIC} = n^{k/n} \frac{\sum \hat{u}_i^2}{n} = n^{k/n} \frac{\text{RSS}}{n} \quad (13.9.5)$$

or in log-form:

$$\ln \text{SIC} = \frac{k}{n} \ln n + \ln \left( \frac{\text{RSS}}{n} \right) \quad (13.9.6)$$

where  $[(k/n) \ln n]$  is the penalty factor. SIC imposes a harsher penalty than AIC, as is obvious from comparing (13.9.6) to (13.9.4). Like AIC, the lower the value of SIC, the better the model. Again, like AIC, SIC can be used to compare in-sample or out-of-sample forecasting performance of a model.

### Mallows's $C_p$ Criterion

Suppose we have a model consisting of  $k$  regressors, including the intercept. Let  $\hat{\sigma}^2$  as usual be the estimator of the true  $\sigma^2$ . But suppose that we only choose  $p$  regressors ( $p \leq k$ ) and obtain the RSS from the regression using these  $p$  regressors. Let  $\text{RSS}_p$  denote the residual sum of squares using the  $p$  regressors. Now C. P. Mallows has developed the following criterion for model selection, known as the  $C_p$  criterion:

$$C_p = \frac{\text{RSS}_p}{\hat{\sigma}^2} - (n - 2p) \quad (13.9.7)$$

where  $n$  is the number of observations.

We know that  $E(\hat{\sigma}^2)$  is an unbiased estimator of the true  $\sigma^2$ . Now, if the model with  $p$  regressors is adequate in that it does not suffer from lack of fit, it can be shown<sup>38</sup> that  $E(\text{RSS}_p) = (n - p)\sigma^2$ . In consequence, it is true *approximately* that

$$E(C_p) \approx \frac{(n - p)\sigma^2}{\sigma^2} - (n - 2p) \approx p \quad (13.9.8)$$

In choosing a model according to the  $C_p$  criterion, we would look for a model that has a low  $C_p$  value, about equal to  $p$ . In other words, following the principle of parsimony, we will choose a model with  $p$  regressors ( $p < k$ ) that gives a fairly good fit to the data.

In practice, one usually plots  $C_p$  computed from (13.9.7) against  $p$ . An "adequate" model will show up as a point close to the  $C_p = p$  line, as can be seen from Figure 13.3. As this figure shows, Model A may be preferable to Model B, as it is closer to the  $C_p = p$  line than Model B.

### A Word of Caution about Model Selection Criteria

We have discussed several model selection criteria. But one should look at these criteria as an adjunct to the various specification tests we have

<sup>38</sup>Norman D. Draper and Harry Smith, *Applied Regression Analysis*, 3d ed., John Wiley & Sons, New York, 1998, p. 332. See this book for some worked examples of  $C_p$ .

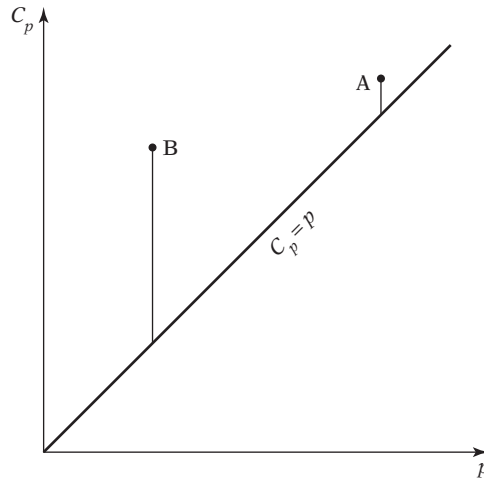


FIGURE 13.3 Mallows's  $C_p$  plot.

discussed in this chapter. Some of the criteria discussed above are purely descriptive and may not have strong theoretical properties. Some of them may even be open to the charge of data mining. Nonetheless, they are so frequently used by the practitioner that the reader should be aware of them. No one of these criteria is necessarily superior to the others.<sup>39</sup> Most modern software packages now include  $R^2$ , adjusted  $R^2$ , AIC, and SIC. Mallows's  $C_p$  is not routinely given, although it can be easily computed from its definition.

### Forecast Chi-Square ( $\chi^2$ )

Suppose we have a regression model based on  $n$  observations and suppose we want to use it to forecast the (mean) values of the regressand for an additional  $t$  observations. As noted elsewhere, it is a good idea to save part of the sample data to see how the estimated model forecasts the observations not included in the sample, the postsample period.

Now the forecast  $\chi^2$  test is defined as follows:

$$\text{Forecast, } \chi^2 = \frac{\sum_{n+1}^{n+t} \hat{u}_i^2}{\hat{\sigma}^2} \quad (13.9.9)$$

where  $\hat{u}_i$  is the forecast error made for period  $i$  ( $= n + 1, n + 2, \dots, n + t$ ), using the parameters obtained from the fitted regression and the values of the regressors in the postsample period.  $\hat{\sigma}^2$  is the usual OLS estimator of  $\sigma^2$  based on the fitted regression.

<sup>39</sup>For a useful discussion on this topic, see Francis X. Diebold, *Elements of Forecasting*, 2d ed., South Western Publishing, 2001, pp. 83–89. On balance, Diebold recommends the SIC criterion.

If we hypothesize that the parameter values have not changed between the sample and postsample periods, it can be shown that the statistic given in (13.9.9) follows the chi-square distribution with  $t$  degrees of freedom, where  $t$  is the number of periods for which the forecast is made. As Charemza and Deadman note, the forecast  $\chi^2$  test has *weak statistical power*, meaning that the probability that the test will correctly reject a false null hypothesis is low and therefore the test should be used as a signal rather than a definitive test.<sup>40</sup>

### 13.10 ADDITIONAL TOPICS IN ECONOMETRIC MODELING

As noted in the introduction to this chapter, the topic of econometric modeling and diagnostic testing is so vast and evolving that specialized books are written on this topic. In the previous section we have touched on some major themes in this area. In this section we consider a few additional features that researchers may find useful in practice. In particular, we consider these topics: (1) **outliers, leverage, and influence**; (2) **recursive least squares**, and (3) **Chow's prediction failure test**. Of necessity the discussion of each of these topics will be brief.

#### Outliers, Leverage, and Influence<sup>41</sup>

Recall that, in minimizing the residual sum of squares (RSS), OLS gives equal weight to every observation in the sample. But every observation may not have equal impact on the regression results because of the presence of three types of special data points called **outliers, leverage points, and influence points**. It is important that we know what they are and how they influence regression analysis.

In the regression context, an **outlier** may be defined as an observation with a "large residual." Recall that  $\hat{u}_i = (Y_i - \hat{Y}_i)$ , that is, the residual represents the difference (positive or negative) between the actual value of the regressand and its value estimated from the regression model. When we say that a residual is large, it is in comparison with the other residuals and very often such a large residual catches our attention immediately because of its rather large vertical distance from the estimated regression line. Note that in a data set there may be more than one outlier. We have already encountered an example of this in exercise 11.22, where you were asked to regress percent change in stock prices ( $Y$ ) on percent change in consumer prices ( $X$ ) for a sample of 20 countries. One observation, that relating to Chile, was an outlier.

<sup>40</sup>Wojciech W. Charemza and Derek F. Deadman, *New Directions in Econometric Practice: A General to Specific Modelling, Cointegration and Vector Autoregression*, 2d ed., Edward Elgar Publishers, 1997, p. 30. See also pp. 250–252 for their views on various model selection criteria.

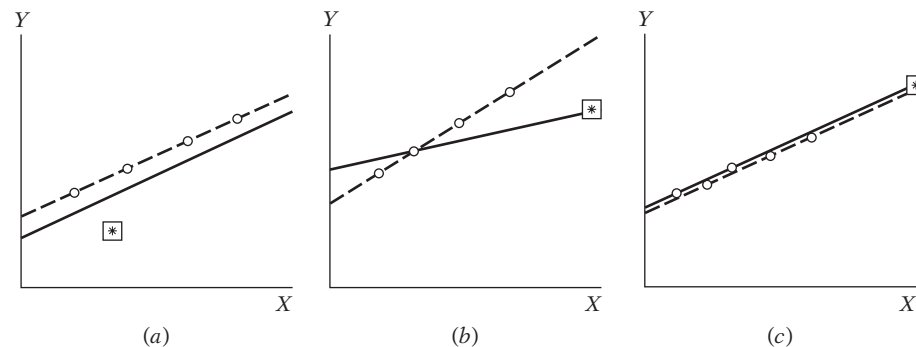
<sup>41</sup>The following discussion is influenced by Chandan Mukherjee, Howard White, and Marc Wyuts, *Econometrics and Data Analysis for Developing Countries*, Routledge, New York, 1998, pp. 137–148.

A data point is said to exert (high) **leverage** if it is disproportionately distant from the bulk of the values of a regressor(s). Why does a leverage point matter? It matters because it is capable of pulling the regression line toward itself, thus distorting the slope of the regression line. If this actually happens, then we call such a leverage (data) point an **influential point**. The removal of such a data point from the sample can dramatically affect the regression line. Returning to exercise 11.22, you will see that if you regress  $Y$  on  $X$  including the observation for Chile, the slope coefficient is positive and “highly statistically significant.” But if you drop the observation for Chile, the slope coefficient is practically zero. Thus the Chilean observation has leverage and is also an influential observation.

To further clarify the nature of outliers, leverage and influence points, consider the diagram in Figure 13.4, which is self-explanatory.<sup>42</sup>

How do we handle such data points? Should we just drop them and confine our attention to the remaining data points? According to Draper and Smith:

Automatic rejection of outliers is not always a wise procedure. Sometimes the outlier is providing information that other data points cannot due to the fact that it arises from an unusual combination of circumstances which may be of vital interest and requires further investigation rather than rejection. As a general rule, outliers should be rejected out of hand only if they can be traced to causes such as errors of recording the observations or setting up the apparatus [in a physical experiment]. Otherwise, careful investigation is in order.<sup>43</sup>



**FIGURE 13.4** In each subfigure, the solid line gives the OLS line for all the data and the broken line gives the OLS line with the outlier, denoted by an  $\square$ , omitted. In (a), the outlier is near the mean value of  $X$  and has low leverage and little influence on the regression coefficients. In (b), the outlier is far away from the mean value of  $X$  and has high leverage as well as substantial influence on the regression coefficients. In (c), the outlier has high leverage but low influence on the regression coefficients because it is in line with the rest of the observations.

Source: Adapted from John Fox, op. cit., p. 268.

<sup>42</sup>Adapted from John Fox, *Applied Regression Analysis, Linear Models, and Related Methods*, Sage Publications, California, 1997, p. 268.

<sup>43</sup>Norman R. Draper and Harry Smith, op. cit., p. 76.

What are some of the tests that one can use to detect outliers and leverage points? There are several tests discussed in the literature, but we will not discuss them here because that will take us far afield.<sup>44</sup> Software packages such as Shazam and Microfit have routines to detect outliers, leverage, and influential points.

### Recursive Least Squares

In Chapter 8 we examined the question of the structural stability of a regression model involving time series data and showed how the **Chow test** can be used for this purpose. Specifically, you may recall that in that chapter we discussed a simple savings function (savings as a function of income) for the United States for the period 1970–1995. There we saw that the savings income relationship probably changed around 1982. Knowing the point of the structural break we were able to confirm it with the Chow test.

But what happens if we do not know the point of the structural break (or breaks)? This is where one can use **recursive least squares (RELS)**. The basic idea behind RELS is very simple and can be explained with the savings–income regression.

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

where  $Y$  = savings and  $X$  = income and where the sample is for the period 1970–1995. (See the data in Table 8.9.)

Suppose we first use the data for 1970–1974 and estimate the savings function, obtaining the estimates of  $\beta_1$  and  $\beta_2$ . Then we use the data for 1970–1975 and again estimate the savings function and obtain the estimates of the two parameters. Then we use the data for 1970–1976 and re-estimate the savings model. In this fashion we go on adding an additional data point on  $Y$  and  $X$  until we exhaust the entire sample. As you can imagine, each regression run will give you a new set of estimates of  $\beta_1$  and  $\beta_2$ . If you plot the estimated values of these parameters against each iteration, you will see how the values of estimated parameters change. If the model under consideration is structurally stable, the changes in the estimated values of the two parameters will be small and essentially random. However, if the estimated values of the parameters change significantly, it would indicate a structural break. RELS is thus a useful routine with time series data since time is ordered chronologically. It is also a useful diagnostic tool in cross-sectional data where the data are ordered by some “size” or “scale” variable, such as

<sup>44</sup>Here are some accessible sources: Alvin C. Rencher, *Linear Models in Statistics*, John Wiley & Sons, New York, 2000, pp. 219–224; A. C. Atkinson, *Plots, Transformations and Regression: An Introduction to Graphical Methods of Diagnostic Regression Analysis*, Oxford University Press, New York, 1985, Chap. 3; Ashis Sen and Muni Srivastava, *Regression Analysis: Theory, Methods, and Applications*, Springer-Verlag, New York, 1990, Chap. 8; and John Fox, op. cit., Chap. 11.

the employment or asset size of the firm. In exercise 13.30 you are asked to apply RELS to the savings data given in Table 8.9.

Software packages such as Shazam, Eviews, and Microfit now do recursive least-squares estimates routinely. RELS also generates **recursive residuals** on which several diagnostic tests have been based.<sup>45</sup>

### Chow's Prediction Failure Test

We have already discussed Chow's test of structural stability in Chapter 8. Chow has shown that his test can be modified to test the predictive power of a regression model. Again, we will revert to the U.S. savings-income regression for the period 1970–1995.

Suppose we estimate the savings-income regression for the period 1970–1981, obtaining  $\hat{\beta}_{1,70-81}$  and  $\hat{\beta}_{2,70-81}$ , which are the estimated intercept and slope coefficients based on the data for 1970–1981. Now using the actual values of income for period 1982–1995 and the intercept and slope values for the period 1970–1981, we predict the values of savings for each of 1982–1995 years. The logic here is that if there is no serious structural change in the parameter values, the values of savings estimated for 1982–1995 based on the parameter estimates for the earlier period, should not be very different from the actual values of savings prevailing in the latter period. Of course, if there is a vast difference between the actual and predicted values of savings for the latter period, it will cast doubts on the stability of the savings-income relation for the entire data period.

Whether the difference between the actual and estimated savings value is large or small can be tested by the  $F$  test as follows:

$$F = \frac{(\sum \hat{u}_t^{*2} - \sum \hat{u}_t^2)/n_2}{(\sum \hat{u}_t^2)/(n_1 - k)} \quad (13.10.1)$$

where  $n_1$  = number of observations in the first period (1970–1981) on which the initial regression is based,  $n_2$  = number of observations in the second or forecast period,  $\sum \hat{u}_t^{*2}$  = RSS when the equation estimated for all the observations ( $n_1 + n_2$ ), and  $\sum \hat{u}_t^2$  = RSS when the equation is estimated for the first  $n_1$  observations and  $k$  is the number of parameters estimated (two in the present instance). If the errors are independent, and identically, normally distributed, the  $F$  statistic given in (13.10.1) follows the  $F$  distribution with  $n_2$  and  $n_1$  df, respectively. In exercise 13.31 you are asked to apply Chow's predictive failure test to find out if the savings-income relation has in fact changed. In passing, note the similarity between this test and the forecast  $\chi^2$  test discussed previously.

<sup>45</sup>For details, see Jack Johnston and John DiNardo, *Econometric Methods*, 4th ed., McGraw-Hill, New York, 1997, pp. 117–121.



**13.11 A CONCLUDING EXAMPLE****EXAMPLE: A MODEL OF HOURLY WAGE DETERMINATION**

To determine what factors determine hourly wages, the following model was considered:

$$\begin{aligned} \text{Hwage} = & \beta_1 + \beta_2 \text{Edu}_i + \beta_3 \text{Gender}_i + \beta_4 \text{Hispanic}_i + \beta_5 \text{Lfxp}_i \\ & + \beta_6 \text{Mstatus}_i + \beta_7 \text{Race}_i + \beta_8 \text{Region}_i + \beta_9 \text{Union}_i + u_i \end{aligned} \quad (13.11.1)$$

where

Hwage = hourly wage (\$)

Edu = education in years

Gender = 1 if female, 0 otherwise

Hispanic = 1 if Hispanic, 0 otherwise

Race = 1 if nonwhite and non-Hispanic, 0 otherwise

Lfxp = potential labor market experience in years

Mstatus = marital status, 1 if married, 0 otherwise

Region = region of residence, 1 if south, 0 otherwise

Union = union status, 1 if in union job and 0 otherwise

The origin of the wage function (13.11.1) can be traced to Jacob Mincer.<sup>46</sup> As you can see, the wage function includes quantitative as well as qualitative or dummy variables. A priori, all these variables seem logical. Notice that the race variable has three categories: Hispanic, non-Hispanic whites, and non-Hispanic nonwhites (largely black or African-American); hence, there are two dummies. The left-out or reference category thus is non-Hispanic whites.

The data consist of 528 persons interviewed in 1985 as a part of the current population survey (CPS) periodically conducted by the U.S. Census Bureau. These data were originally collected by Berndt and were adapted by Arthur Goldberg. We have already discussed this source in Chapter 2. Keep in mind that the data are cross sectional.

A priori, hourly wage is expected to be positively related to education, life experience, marital status and union status and negatively related to Hispanic, race, gender, and region; again note that all comparisons are in relation to non-Hispanic whites. Consult any book on labor economics to learn more about the various determinants of hourly wages.<sup>47</sup>

Using the data, I asked my students to estimate the model (13.11.1). The regression results are given in Table 13.4. As you can see, all the variables in (13.11.1) have the expected signs, although not all variables are individually statistically significant. The  $R^2$  value of about 0.2826 might seem low, but such low  $R^2$  values are typically observed in cross-sectional data with a large number of observations. But this  $R^2$  value is statistically significant, since the computed  $F$  value of about 25.56 is highly significant, as its  $p$  value is almost zero: Remember that the  $F$  statistic tests the hypothesis that all the

<sup>46</sup>See J. Mincer, *School, Experience and Earnings*, Columbia University Press, New York, 1974.

<sup>47</sup>See, for example, George Borjas, *Labor Economics*, 2d. ed., McGraw-Hill, New York, 2000.



**TABLE 13.4** REGRESSION RESULTS BASED ON (13.11.1)

Dependent Variable: HWAGE Sample: 1 528				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-4.182714	1.275908	-3.278227	0.0011
EDUCATION	0.937130	0.082625	11.34194	0.0000
GENDER	-2.140661	0.391546	-5.467200	0.0000
HISPANIC	-0.512385	0.911056	-0.562408	0.5741
LFEXP	0.098486	0.017494	5.629597	0.0000
MSTATUS	0.485134	0.418881	1.158167	0.2473
RACE	-0.942389	0.583578	-1.614849	0.1070
REGION	-0.771424	0.430173	-1.793287	0.0735
UNION	1.468088	0.512735	2.863248	0.0044
R-squared	0.282693	Mean dependent var		9.047538
Adjusted R-squared	0.271636	S.D. dependent var		5.144082
S.E. of regression	4.390177	Akaike info criterion		5.813515
Sum squared resid	10003.03	Schwarz criterion		5.886283
Log likelihood	-1525.768	F-statistic		25.56745
Durbin-Watson stat	1.857457	Prob(F-statistic)		0.000000

slope coefficients are simultaneously zero; that is, all the explanatory values jointly have no impact on the regressand.

Noting the individual statistical insignificance of the variables Hispanic, marital status, and race, but noting that the region variable is “reasonably” statistically significant, some of my students dropped the first three of these variables and obtained the results shown in Table 13.5. Now all the variables are individually statistically significant at a 5 percent or better level (i.e., the  $p$  value less than 5 percent). The interpretation of the various coefficients is

**TABLE 13.5**

Dependent Variable: HWAGE Sample: 1 528				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-4.289796	1.258229	-3.409392	0.0007
EDUCATION	0.953006	0.082184	11.59596	0.0000
GENDER	-2.134171	0.391740	-5.447929	0.0000
LFEXP	0.104037	0.016888	6.160545	0.0000
REGION	-0.840832	0.427621	-1.966303	0.0498
UNION	1.427421	0.509978	2.798988	0.0053
R-squared	0.276707	Mean dependent var		9.047538
Adjusted R-squared	0.269779	S.D. dependent var		5.144082
S.E. of regression	4.395772	Akaike info criterion		5.810462
Sum squared resid	10086.51	Schwarz criterion		5.858974
Log likelihood	-1527.962	F-statistic		39.93978
Durbin-Watson stat	1.858629	Prob(F-statistic)		0.000000

straightforward. For example, the value of  $-0.8408$  of the region dummy suggests that holding all the other variables constant, on average, workers in the South earn about 84 cents less per hour than their counterparts elsewhere, perhaps because of the low cost of living in the South and/or the fact the South is less unionized. Similarly, on average, women earn less than their male counterparts, by about \$2.13, holding all other factors constant. Whether this amounts to gender discrimination cannot be told from the statistical analysis alone.

As expected, the “short” regression (omitting Hispanic, marital status, and race variables) has a lower adjusted  $R^2$  than the “long” regression (i.e., the regression that includes all the variables), as one would expect. But notice the **Akaike** and **Schwarz** statistics: They are both lower for the short regression compared to the long regression, showing how they penalize for introducing more regressors in the model. Since the values of both statistics are so close that one can choose either of the statistics, the Durbin–Watson  $d$  value in both models is sufficiently close to 2 to suggest any “autocorrelation” or specification errors.

Since the data underlying regression (13.11.1) are given in the data disk, you may want to “experiment” with the data. It is quite possible that there might be some interaction between the gender and education dummies or gender and marital status dummies. It is also possible that the relationship between hourly wage and labor market experience is nonlinear, necessitating the introduction of the squared education term in the regression model. As you can see, even with a given data set, there are several possibilities. This might sound like data mining, but we have already noted that data mining may have some role to play in econometric modeling. Of course, you should keep in mind the true level of significance in carrying out data mining.

### 13.12 A WORD TO THE PRACTITIONER

We have covered a lot of ground in this chapter. There is no question that model building is an art as well as a science. A practical researcher may be bewildered by theoretical niceties and an array of diagnostic tools. But it is well to keep in mind Martin Feldstein’s caution that “The applied econometrician, like the theorist, soon discovers from experience that a useful model is not one that is ‘true’ or ‘realistic’ but one that is parsimonious, plausible and informative.”<sup>48</sup>

Peter Kennedy of Simon Fraser University in Canada advocates the following “Ten Commandments of Applied Econometrics”<sup>49</sup>:

1. Thou shalt use common sense and economic theory.
2. Thou shalt ask the right questions (i.e., put relevance before mathematical elegance).

<sup>48</sup>Martin S. Feldstein, “Inflation, Tax Rules and Investment: Some Econometric Evidence,” *Econometrica*, vol. 30, 1982, p. 829.

<sup>49</sup>Peter Kennedy, *op. cit.*, pp. 17–18.

3. Thou shalt know the context (do not perform ignorant statistical analysis).
4. Thou shalt inspect the data.
5. Thou shalt not worship complexity. Use the **KISS principle**, that is, *keep it stochastically simple*.
6. Thou shalt look long and hard at thy results.
7. Thou shalt beware the costs of data mining.
8. Thou shalt be willing to compromise (do not worship textbook prescriptions).
9. Thou shalt not confuse significance with substance (do not confuse statistical significance with practical significance).
10. Thou shalt confess in the presence of sensitivity (that is, anticipate criticism).

You may want to read Kennedy's paper fully to appreciate the conviction with which he advocates the above ten commandments. Some of these commandments may sound tongue-in-cheek, but there may be a grain of truth in each.

### 13.13 SUMMARY AND CONCLUSIONS

1. The assumption of the CLRM that the econometric model used in analysis is correctly specified has two meanings. One, there are no **equation specification errors**, and two, there are no **model specification errors**. In this chapter the major focus was on equation specification errors.

2. The equation specification errors discussed in this chapter were (1) omission of important variable(s), (2) inclusion of superfluous variable(s), (3) adoption of the wrong function form, (4) incorrect specification of the error term  $u_i$ , and (5) errors of measurement in the regressand and regressors.

3. When legitimate variables are omitted from a model, the consequences can be very serious: The OLS estimators of the variables retained in the model not only are biased but are inconsistent as well. Additionally, the variances and standard errors of these coefficients are incorrectly estimated, thereby vitiating the usual hypothesis-testing procedures.

4. The consequences of including irrelevant variables in the model are fortunately less serious: The estimators of the coefficients of the relevant as well as "irrelevant" variables remain unbiased as well as consistent, and the error variance  $\sigma^2$  remains correctly estimated. The only problem is that the estimated variances tend to be larger than necessary, thereby making for less precise estimation of the parameters. That is, the confidence intervals tend to be larger than necessary.

5. To detect equation specification errors, we considered several tests, such as (1) examination of residuals, (2) the Durbin-Watson  $d$  statistic, (3) Ramsey's RESET test, and (4) the Lagrange multiplier test.

6. A special kind of specification error is errors of measurement in the values of the regressand and regressors. If there are errors of measurement in the regressand only, the OLS estimators are unbiased as well as consistent but they are less efficient. If there are errors of measurement in the regressors, the OLS estimators are biased as well as inconsistent.

7. Even if errors of measurement are detected or suspected, the remedies are often not easy. The use of instrumental or proxy variables is theoretically attractive but not always practical. Thus it is very important in practice that the researcher be careful in stating the sources of his/her data, how they were collected, what definitions were used, etc. Data collected by official agencies often come with several footnotes and the researcher should bring those to the attention of the reader.

8. Model mis-specification errors can be as serious as equation specification errors. In particular, we distinguished between nested and non-nested models. To decide on the appropriate model we discussed the non-nested, or encompassing,  $F$  test and the Davidson–MacKinnon  $J$  test and pointed out the limitation of each test.

9. In choosing an empirical model in practice researchers have used a variety of criteria. We discussed some of these, such as the Akaike and Schwarz information criteria, Mallows's  $C_p$  criterion, and forecast  $\chi^2$  criterion. We discussed the advantages and disadvantages of these criteria and also warned the reader that these criteria are not absolute but are adjunct to a careful specification analysis.

10. We also discussed these additional topics: (1) outliers, leverage, and influence; (2) recursive least squares; and (3) Chow's prediction failure test. We discussed the role of each in applied work.

11. We concluded this chapter by discussing Peter Kennedy's "ten commandments of applied econometrics." The point of these commandments is to ask the researcher to look beyond the purely technical aspects of econometrics.

## EXERCISES

### Questions

13.1. Refer to the demand function for chicken estimated in Eq. (8.7.23). Considering the attributes of a good model discussed in Section 13.1, could you say that this demand function is "correctly" specified?

13.2. Suppose that the true model is

$$Y_i = \beta_1 X_i + u_i \quad (1)$$

but instead of fitting this regression through the origin you routinely fit the usual intercept-present model:

$$Y_i = \alpha_0 + \alpha_1 X_i + v_i \quad (2)$$

Assess the consequences of this specification error.

- 13.3.** Continue with exercise 13.2 but assume that it is model (2) that is the truth. Discuss the consequences of fitting the mis-specified model (1).
- 13.4.** Suppose that the “true” model is

$$Y_i = \beta_1 + \beta_2 X_{2i} + u_i \quad (1)$$

but we add an “irrelevant” variable  $X_3$  to the model (irrelevant in the sense that the true  $\beta_3$  coefficient attached to the variable  $X_3$  is zero) and estimate

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + v_i \quad (2)$$

- Would the  $R^2$  and the adjusted  $R^2$  for model (2) be larger than that for model (1)?
  - Are the estimates of  $\beta_1$  and  $\beta_2$  obtained from (2) unbiased?
  - Does the inclusion of the “irrelevant” variable  $X_3$  affect the variances of  $\hat{\beta}_1$  and  $\hat{\beta}_2$ ?
- 13.5.** Consider the following “true” (Cobb–Douglas) production function:

$$\ln Y_i = \alpha_0 + \alpha_1 \ln L_{1i} + \alpha_2 \ln L_{2i} + \alpha_3 \ln K_i + u_i$$

where  $Y$  = output

$L_1$  = production labor

$L_2$  = nonproduction labor

$K$  = capital

But suppose the regression actually used in empirical investigation is

$$\ln Y_i = \beta_0 + \beta_1 \ln L_{1i} + \beta_2 \ln K_i + u_i$$

On the assumption that you have cross-sectional data on the relevant variables,

- Will  $E(\hat{\beta}_1) = \alpha_1$  and  $E(\hat{\beta}_2) = \alpha_3$ ?
  - Will the answer in **a** hold if it is known that  $L_2$  is an *irrelevant* input in the production function? Show the necessary derivations.
- 13.6.** Refer to Eqs. (13.3.4) and (13.3.5). As you can see,  $\hat{\alpha}_2$ , although biased, has a smaller variance than  $\hat{\beta}_2$ , which is unbiased. How would you decide on the tradeoff between bias and smaller variance? *Hint:* The MSE (mean-square error) for the two estimators is expressed as

$$\begin{aligned} \text{MSE}(\hat{\alpha}_2) &= \left( \sigma^2 / \sum x_{2i}^2 \right) + \beta_3^2 b_{32}^2 \\ &= \text{sampling variance} + \text{square of bias} \end{aligned}$$

$$\text{MSE}(\hat{\beta}_2) = \sigma^2 / \sum x_2^2 (1 - r_{23}^2)$$

On MSE, see **Appendix A**.

- 13.7.** Show that  $\beta$  estimated from either (13.5.1) or (13.5.3) provides an unbiased estimate of true  $\beta$ .
- 13.8.** Following Friedman’s permanent income hypothesis, we may write

$$Y_i^* = \alpha + \beta X_i^* \quad (1)$$

where  $Y_i^*$  = “permanent” consumption expenditure and  $X_i^*$  = “permanent” income. Instead of observing the “permanent” variables, we observe

$$\begin{aligned} Y_i &= Y_i^* + u_i \\ X_i &= X_i^* + v_i \end{aligned}$$

where  $Y_i$  and  $X_i$  are the quantities that can be observed or measured and where  $u_i$  and  $v_i$  are measurement errors in  $Y^*$  and  $X^*$ , respectively.

Using the observable quantities, we can write the consumption function as

$$\begin{aligned} Y_i &= \alpha + \beta(X_i - v_i) + u_i \\ &= \alpha + \beta X_i + (u_i - \beta v_i) \end{aligned} \quad (2)$$

Assuming that (1)  $E(u_i) = E(v_i) = 0$ , (2)  $\text{var}(u_i) = \sigma_u^2$  and  $\text{var}(v_i) = \sigma_v^2$ , (3)  $\text{cov}(Y_i^*, u_i) = 0$ ,  $\text{cov}(X_i^*, v_i) = 0$ , and (4)  $\text{cov}(u_i, X_i^*) = \text{cov}(v_i, Y_i^*) = \text{cov}(u_i, v_i) = 0$ , show that in large samples  $\beta$  estimated from (2) can be expressed as

$$\text{plim}(\hat{\beta}) = \frac{\beta}{1 + (\sigma_v^2 / \sigma_{X^*}^2)}$$

- a. What can you say about the nature of the bias in  $\hat{\beta}$ ?
- b. If the sample size increases indefinitely, will the estimated  $\beta$  tend to equality with the true  $\beta$ ?

**13.9. Capital asset pricing model.** The capital asset pricing model (CAPM) of modern investment theory postulates the following relationship between the average rate of return of a security (common stock), measured over a certain period, and the volatility of the security, called the *beta coefficient* (volatility is measure of risk):

$$\bar{R}_i = \alpha_1 + \alpha_2(\beta_i) + u_i \quad (1)$$

where  $\bar{R}_i$  = average rate of return of security  $i$   
 $\beta_i$  = true beta coefficient of security  $i$   
 $u_i$  = stochastic disturbance term

The true  $\beta_i$  is not directly observable but is measured as follows:

$$r_{it} = \alpha_1 + \beta^* r_{m_t} + e_t \quad (2)$$

where  $r_{it}$  = rate of return of security  $i$  for time  $t$   
 $r_{m_t}$  = market rate of return for time  $t$  (this rate is the rate of return on some broad market index, such as the S&P index of industrial securities)  
 $e_t$  = residual term

and where  $\beta^*$  is an estimate of the “true” beta coefficient. In practice, therefore, instead of estimating (1), one estimates

$$\bar{R}_i = \alpha_1 + \alpha_2(\beta_i^*) + u_i \quad (3)$$

where  $\beta_i^*$  are obtained from the regression (2). But since  $\beta_i^*$  are estimated, the relationship between true  $\beta$  and  $\beta^*$  can be written as

$$\beta_i^* = \beta_i + v_i \quad (4)$$

where  $v_i$  can be called the *error of measurement*.

- a. What will be the effect of this error of measurement on the estimate of  $\alpha_2$ ?
- b. Will the  $\alpha_2$  estimated from (3) provide an unbiased estimate of true  $\alpha_2$ ? If not, is it a consistent estimate of  $\alpha_2$ ? If not, what remedial measures do you suggest?

**13.10.** Consider the model

$$Y_i = \beta_1 + \beta_2 X_{2i} + u_i \quad (1)$$

To find out whether this model is mis-specified because it omits the variable  $X_3$  from the model, you decide to regress the residuals obtained from model (1) on the variable  $X_3$  only (*Note:* There is an intercept in this regression). The Lagrange multiplier (LM) test, however, requires you to regress the residuals from (1) on both  $X_2$  and  $X_3$  and a constant. Why is your procedure likely to be inappropriate?\*

**13.11.** Consider the model

$$Y_i = \beta_1 + \beta_2 X_i^* + u_i$$

In practice we measure  $X_i^*$  by  $X_i$  such that

- a.  $X_i = X_i^* + 5$
- b.  $X_i = 3X_i^*$
- c.  $X_i = (X_i^* + \varepsilon_i)$ , where  $\varepsilon_i$  is a purely random term with the usual properties

What will be the effect of these measurement errors on estimates of true  $\beta_1$  and  $\beta_2$ ?

**13.12.** Refer to the regression Eqs. (13.3.1) and (13.3.2). In a manner similar to (13.3.3) show that

$$E(\hat{\alpha}_1) = \beta_1 + \beta_3(\bar{X}_3 - b_{32}\bar{X}_2)$$

where  $b_{32}$  is the slope coefficient in the regression of the omitted variable  $X_3$  on the included variable  $X_2$ .

**13.13.** Critically evaluate the following view expressed by Leamer<sup>†</sup>:

My interest in metastatistics [i.e., theory of inference actually drawn from data] stems from my observations of economists at work. The opinion that econometric theory is irrelevant is held by an embarrassingly large share of the economic profession. The wide gap between econometric theory and econometric practice might be expected to cause professional tension. In fact, a calm equilibrium permeates our

\*See Maddala, op. cit., p. 477.

<sup>†</sup>Edward E. Leamer, *Specification Searches: Ad Hoc Inference with Nonexperimental Data*, John Wiley & Sons, New York, 1978, p. vi.

journals and our [professional] meetings. We comfortably divide ourselves into a celibate priesthood of statistical theorists, on the one hand, and a legion of inveterate sinner-data analysts, on the other. The priests are empowered to draw up lists of sins and are revered for the special talents they display. Sinners are not expected to avoid sins; they need only confess their errors openly.

**13.14.** Evaluate the following statement made by Henry Theil\*:

Given the present state of the art, the most sensible procedure is to interpret confidence coefficients and significance limits liberally when confidence intervals and test statistics are computed from the final regression of a regression strategy in the conventional way. That is, a 95 percent confidence coefficient may actually be an 80 percent confidence coefficient and a 1 percent significance level may actually be a 10 percent level.

**13.15.** Commenting on the econometric methodology practiced in the 1950s and early 1960s, Blaug stated†:

... much of it [i.e., empirical research] is like playing tennis with the net down: instead of attempting to refute testable predictions, modern economists all too frequently are satisfied to demonstrate that the real world conforms to their predictions, thus replacing falsification [à la Popper], which is difficult, with verification, which is easy.

Do you agree with this view? You may want to peruse Blaug's book to learn more about his views.

**13.16.** According to Blaug, "There is no logic of proof but there is logic of disproof."‡ What does he mean by this?

**13.17.** Refer to the St. Louis model discussed in the text. Keeping in mind the problems associated with the nested  $F$  test, critically evaluate the results presented in regression (13.8.4).

**13.18.** Suppose the true model is

$$Y_i = \beta_1 + \beta_2 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + u_i$$

but you estimate

$$Y_i = \alpha_1 + \alpha_2 X_i + v_i$$

If you use observations of  $Y$  at  $X = -3, -2, -1, 0, 1, 2, 3$ , and estimate the "incorrect" model, what bias will result in these estimates?§

**13.19.** To see if the variable  $X_i^2$  belongs in the model  $Y_i = \beta_1 + \beta_2 X_i + u_i$ , Ramsey's RESET test would estimate the linear model, obtaining the estimated  $Y_i$  values from this model [i.e.,  $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$ ] and then

\*Henry Theil, *Principles of Econometrics*, John Wiley & Sons, New York, 1971, pp. 605–606.

†M. Blaug, *The Methodology of Economics. Or How Economists Explain*, Cambridge University Press, New York, 1980, p. 256.

‡Ibid., p. 14.

§Adapted from G. A. F., *Linear Regression Analysis*, John Wiley & Sons, New York, 1977, p. 176.



estimating the model  $Y_i = \alpha_1 + \alpha_2 X_i + \alpha_3 \hat{Y}_i^2 + v_i$  and testing the significance of  $\alpha_3$ . Prove that, if  $\hat{\alpha}_3$  turns out to be statistically significant in the preceding (RESET) equation, it is the same thing as estimating the following model directly:  $Y_i = \beta_1 + \beta_2 X_i + \beta_3 X_i^2 + u_i$ . (*Hint: Substitute for  $\hat{Y}_i$  in the RESET regression*\*)).

- 13.20.** State with reason whether the following statements are true or false.<sup>†</sup>
- An observation can be influential but not an outlier.
  - An observation can be an outlier but not influential.
  - An observation can be both influential and an outlier.
  - If in the model  $Y_i = \beta_1 + \beta_2 X_i + \beta_3 X_i^2 + u_i$   $\hat{\beta}_3$  turns out to be statistically significant, we should retain the linear term  $X_i$  even if  $\hat{\beta}_2$  is statistically insignificant.
  - If you estimate the model  $Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$  or  $Y_i = \alpha_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + u_i$  by OLS, the estimated regression line is the same, where  $x_{2i} = (X_{2i} - \bar{X}_2)$  and  $x_{3i} = (X_{3i} - \bar{X}_3)$ .

## Problems

- 13.21.** Use the data for the demand for chicken given in exercise 7.19. Suppose you are told that the true demand function is

$$\ln Y_t = \beta_1 + \beta_2 \ln X_{2t} + \beta_3 \ln X_{3t} + \beta_6 \ln X_{6t} + u_t \quad (1)$$

but you think differently and estimate the following demand function:

$$\ln Y_t = \alpha_1 + \alpha_2 \ln X_{2t} + \alpha_3 \ln X_{3t} + v_t \quad (2)$$

where  $Y$  = per capita consumption of chickens (lb)

$X_2$  = real disposable per capita income

$X_3$  = real retail price of chickens

$X_6$  = composite real price of chicken substitutes

- Carry out RESET and LM tests of specification errors, assuming the demand function (1) just given is the truth.
  - Suppose  $\hat{\beta}_6$  in (1) turns out to be statistically insignificant. Does that mean there is no specification error if we fit (2) to the data?
  - If  $\hat{\beta}_6$  turns out to be insignificant, does that mean one should not introduce the price of a substitute product(s) as an argument in the demand function?
- 13.22.** Continue with exercise 13.21. Strictly for pedagogical purposes, assume that model (2) is the true demand function.
- If we now estimate model (1), what type of specification error is committed in this instance?
  - What are the theoretical consequences of this specification error? Illustrate with the data at hand.
- 13.23.** The true model is

$$Y_i^* = \beta_1 + \beta_2 X_i^* + u_i \quad (1)$$

\*Adapted from Kerry Peterson, op. cit., pp. 184–185.

†Adapted from Norman R. Draper and Harry Smith, op. cit., pp. 606–607.

but because of errors of measurement you estimate

$$Y_i = \alpha_1 + \alpha_2 X_i + v_i \quad (2)$$

where  $Y_i = Y_i^* + \varepsilon_i$  and  $X_i = X_i^* + w_i$ , where  $\varepsilon_i$  and  $w_i$  are measurement errors.

Using the data given in Table 13.2, document the consequences of estimating (2) instead of the true model (1).

- 13.24.** In exercise 6.14 you were asked to estimate the elasticity of substitution between labor and capital using the CES (constant elasticity of substitution) production function. But the function shown there is based on the assumption that there is perfect competition in the labor market. If competition is imperfect, the correct formulation of the model is

$$\log\left(\frac{V}{L}\right) = \log \beta_1 + \beta_2 \log W + \beta_3 \log\left(1 + \frac{1}{E}\right)$$

where  $(V/L)$  = value added per unit of labor

$L$  = labor input

$W$  = real wage rate

$E$  = elasticity of supply of labor

- a. What kind of specification error is involved in the original CES estimation of the elasticity of substitution if in fact the labor market is imperfect?
  - b. What are the theoretical consequences of this error for  $\beta_2$ , the elasticity of substitution parameter?
  - c. Assume that the labor supply elasticities in the industries shown in exercise 6.23 were as follows: 2.0, 1.8, 2.5, 2.3, 1.9, 2.1, 1.7, 2.7, 2.2, 2.1, 2.9, 2.8, 3.2, 2.9, and 3.1. Using these data along with those given in exercise 6.14, estimate the foregoing model and comment on your results in light of the theory of specification errors.
- 13.25. Monte Carlo experiment\*:** Ten individuals had weekly permanent income as follows: \$200, 220, 240, 260, 280, 300, 320, 340, 380, and 400. Permanent consumption ( $Y_i^*$ ) was related to permanent income  $X_i^*$  as

$$Y_i^* = 0.8X_i^* \quad (1)$$

Each of these individuals had transitory income equal to 100 times a random number  $u_i$  drawn from a normal population with mean = 0 and  $\sigma^2 = 1$  (i.e., standard normal variable). Assume that there is no transitory component in consumption. Thus, measured consumption and permanent consumption are the same.

- a. Draw 10 random numbers from a normal population with zero mean and unit variance and obtain 10 numbers for measured income  $X_i$  ( $= X_i^* + 100u_i$ ).
- b. Regress permanent (= measured) consumption on measured income using the data obtained in a and compare your results with those

\*Adapted from Christopher Dougherty, *Introduction to Econometrics*, Oxford University Press, New York, 1992, pp. 253–256.

shown in (1). A priori, the intercept should be zero (why?). Is that the case? Why or why not?

- c. Repeat **a** 100 times and obtain 100 regressions as shown in **b** and compare your results with the true regression (1). What general conclusions do you draw?

- 13.26.** Refer to exercise 8.26. With the definitions of the variables given there, consider the following two models to explain  $Y$ :

$$\text{Model A: } Y_t = \alpha_1 + \alpha_2 X_{3t} + \alpha_3 X_{4t} + \alpha_4 X_{6t} + u_t$$

$$\text{Model B: } Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{5t} + \beta_4 X_{6t} + u_t$$

Using the nested  $F$  test, how will you choose between the two models?

- 13.27.** Continue with exercise 13.26. Using the  $J$  test, how would you decide between the two models?

- 13.28.** Refer to exercise 7.19, which is concerned with the demand for chicken in the United States. There you were given five models.

- a. What is the difference between model 1 and model 2? If model 2 is correct and you estimate model 1, what kind of error is committed? Which test would you apply—equation specification error or model selection error? Show the necessary calculations.

- b. Between models 1 and 5, which would you choose? Which test(s) do you use and why?

- 13.29.** Refer to Table 8.9, which gives data on personal savings ( $Y$ ) and personal disposable income ( $X$ ) for the period 1970–1995. Now consider the following models:

$$\text{Model A: } Y_t = \alpha_1 + \alpha_2 X_t + \alpha_3 X_{t-1} + u_t$$

$$\text{Model B: } Y_t = \beta_1 + \beta_2 X_t + \beta_3 Y_{t-1} + u_t$$

How would you choose between these two models? State clearly the test procedure(s) you use and show all the calculations. Suppose someone contends that the interest rate variable belongs in the savings function. How would you test this? Collect data on 3-month treasury bill rate as a proxy for the interest and demonstrate your answer.

- 13.30.** Use the data in exercise 13.29. To familiarize yourself with recursive least squares, estimate the savings functions for 1970–1981, 1970–1985, 1970–1990, and 1970–1995. Comment on the stability of estimated coefficients in the savings functions.

- 13.31.** Continue with exercise 13.30. Suppose you estimate the savings function for 1970–1981. Using the parameters thus estimated and the personal disposable income data from 1982–1995, estimate the predicted savings for the latter period and use Chow's prediction failure test to find out if it rejects the hypothesis that the savings function between the two time periods has not changed.

- 13.32.** *Omission of a variable in the  $K$ -variable regression model.* Refer to Eq. (13.3.3), which shows the bias in omitting the variable  $X_3$  from the model  $Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$ . This can be generalized as follows: In the  $k$ -variable model  $Y_i = \beta_1 + \beta_2 X_{2i} + \dots + \beta_k X_{ki} + u_i$ , suppose we omit the variable  $X_k$ . Then it can be shown that the omitted variable bias

of the slope coefficient of included variable  $X_j$  is:

$$E(\hat{\beta}_j) = \beta_j + \beta_k b_{kj} \quad j = 2, 3, \dots, (k-1)$$

where  $b_{kj}$  is the (partial) slope coefficient of  $X_j$  in the auxiliary regression of the excluded variable  $X_k$  on all the explanatory variables included in the model.\*

Refer to exercise 13.21. Find out the bias of the coefficients in Eq. (1) if we excluded the variable  $\ln X_6$  from the model. Is this exclusion serious? Show the necessary calculations.

## APPENDIX 13A

### 13A.1 THE PROOF THAT $E(b_{12}) = \beta_2 + \beta_3 b_{32}$ [EQUATION (13.3.3)]

In the deviation form the three-variable population regression model can be written as

$$y_i = \beta_2 x_{2i} + \beta_3 x_{3i} + (u_i - \bar{u}) \quad (1)$$

First multiplying by  $x_2$  and then by  $x_3$ , the usual normal equations are

$$\sum y_i x_{2i} = \beta_2 \sum x_{2i}^2 + \beta_3 \sum x_{2i} x_{3i} + \sum x_{2i} (u_i - \bar{u}) \quad (2)$$

$$\sum y_i x_{3i} = \beta_2 \sum x_{2i} x_{3i} + \beta_3 \sum x_{3i}^2 + \sum x_{3i} (u_i - \bar{u}) \quad (3)$$

Dividing (2) by  $\sum x_{2i}^2$  on both sides, we obtain

$$\frac{\sum y_i x_{2i}}{\sum x_{2i}^2} = \beta_2 + \beta_3 \frac{\sum x_{2i} x_{3i}}{\sum x_{2i}^2} + \frac{\sum x_{2i} (u_i - \bar{u})}{\sum x_{2i}^2} \quad (4)$$

Now recalling that

$$b_{12} = \frac{\sum y_i x_{2i}}{\sum x_{2i}^2}$$

$$b_{32} = \frac{\sum x_{2i} x_{3i}}{\sum x_{2i}^2}$$

Eq. (4) can be written as

$$b_{12} = \beta_2 + \beta_3 b_{32} + \frac{\sum x_{2i} (u_i - \bar{u})}{\sum x_{2i}^2} \quad (5)$$

\*This can be generalized to the case where more than one relevant  $X$  variable is excluded from the model. On this, see Chandan Mukherjee et al., op. cit., p. 215.

Taking the expected value of (5) on both sides, we finally obtain

$$E(b_{12}) = \beta_2 + \beta_3 b_{32} \quad (6)$$

where use is made of the facts that (a) for a given sample,  $b_{32}$  is a known fixed quantity, (b)  $\beta_2$  and  $\beta_3$  are constants, and (c)  $u_i$  is uncorrelated with  $X_{2i}$  (as well as  $X_{3i}$ ).

### 13A.2 THE CONSEQUENCES OF INCLUDING AN IRRELEVANT VARIABLE: THE UNBIASEDNESS PROPERTY

For the true model (13.3.6), we have

$$\hat{\beta}_2 = \frac{\sum yx_2}{\sum x_2^2} \quad (1)$$

and we know that it is unbiased.

For the model (13.3.7), we obtain

$$\hat{\alpha}_2 = \frac{(\sum yx_2)(\sum x_3^2) - (\sum yx_3)(\sum x_2x_3)}{\sum x_2^2 \sum x_3^2 - (\sum x_2x_3)^2} \quad (2)$$

Now the true model in deviation form is

$$y_i = \beta_2 x_{2i} + (u_i - \bar{u}) \quad (3)$$

Substituting for  $y_i$  from (3) into (2) and simplifying, we obtain

$$\begin{aligned} E(\hat{\alpha}_2) &= \beta_2 \frac{\sum x_2^2 \sum x_3^2 - (\sum x_2x_3)^2}{\sum x_2^2 \sum x_3^2 - (\sum x_2x_3)^2} \\ &= \beta_2 \end{aligned} \quad (4)$$

that is,  $\hat{\alpha}_2$  remains unbiased.

We also obtain

$$\hat{\alpha}_3 = \frac{(\sum yx_3)(\sum x_2^2) - (\sum yx_2)(\sum x_2x_3)}{\sum x_2^2 \sum x_3^2 - (\sum x_2x_3)^2} \quad (5)$$

Substituting for  $y_i$  from (3) into (5) and simplifying, we obtain

$$\begin{aligned} E(\hat{\alpha}_3) &= \beta_2 \frac{[(\sum x_2x_3)(\sum x_2^2) - (\sum x_2x_3)(\sum x_2^2)]}{\sum x_2^2 \sum x_3^2 - (\sum x_2x_3)^2} \\ &= 0 \end{aligned} \quad (6)$$

which is its value in the true model since  $X_3$  is absent from the true model.

**13A.3 THE PROOF OF EQUATION (13.5.10)**

We have

$$Y = \alpha + \beta X_i^* + u_i \quad (1)$$

$$X_i = X_i^* + w_i \quad (2)$$

Therefore, in deviation form we obtain

$$y_i = \beta x_i^* + (u_i - \bar{u}) \quad (3)$$

$$x_i = x_i^* + (w_i - \bar{w}) \quad (4)$$

Now when we use

$$Y_i = \alpha + \beta X_i + u_i \quad (5)$$

we obtain

$$\begin{aligned} \hat{\beta} &= \frac{\sum yx}{\sum x^2} \\ &= \frac{\sum [\beta x^* + (u - \bar{u})][x^* + (w - \bar{w})]}{\sum [x^* + (w - \bar{w})]^2} \quad \text{using (3) and (4)} \\ &= \frac{\beta \sum x^{*2} + \beta \sum x^*(w - \bar{w}) + \sum x^*(u - \bar{u}) + \sum (u - \bar{u})(w - \bar{w})}{\sum x^{*2} + 2 \sum x^*(w - \bar{w}) + \sum (w - \bar{w})^2} \end{aligned}$$

Since we cannot take expectation of this expression because the expectation of the ratio of two variables is not equal to the ratio of their expectations (*note*: the expectations operator  $E$  is a linear operator), first we divide each term of the numerator and the denominator by  $n$  and take the probability limit, plim (see **Appendix A** for details of plim), of

$$\hat{\beta} = \frac{(1/n) [\beta \sum x^{*2} + \beta \sum x^*(w - \bar{w}) + \sum x^*(u - \bar{u}) + \sum (u - \bar{u})(w - \bar{w})]}{(1/n) [\sum x^{*2} + 2 \sum x^*(w - \bar{w}) + \sum (w - \bar{w})^2]}$$

Now the probability limit of the ratio of two variables is the ratio of their probability limits. Applying this rule and taking plim of each term, we obtain

$$\text{plim } \hat{\beta} = \frac{\beta \sigma_{X^*}^2}{\sigma_{X^*}^2 + \sigma_w^2}$$

where  $\sigma_{X^*}^2$  and  $\sigma_w^2$  are variances of  $X^*$  and  $w$  as sample size increases indefinitely and where we have used the fact that as the sample size increases indefinitely there is no correlation between the errors  $u$  and  $w$  as well as

between them and the true  $X^*$ . From the preceding expression, we finally obtain

$$\text{plim } \hat{\beta} = \beta \left[ \frac{1}{1 + (\sigma_w^2 / \sigma_{X^*}^2)} \right]$$

which is the required result.

#### 13A.4 THE PROOF OF EQUATION (13.6.2)

Since there is no intercept in the model, the estimate of  $\alpha$ , according to the formula for the regression through the origin, is as follows:

$$\hat{\alpha} = \frac{\sum X_i Y_i}{\sum X_i^2} \quad (1)$$

Substituting for  $Y$  from the true model (13.2.8), we obtain

$$\hat{\alpha} = \frac{\sum X_i (\beta X_i u_i)}{\sum X_i^2} = \beta \frac{\sum X_i^2 u_i}{\sum X_i^2} \quad (2)$$

Statistical theory shows that if  $\ln u_i \sim N(0, \sigma^2)$  then

$$u_i = \log \text{ normal } [e^{\sigma^2/2}, e^{\sigma^2}(e^{\sigma^2}-1)] \quad (3)$$

Therefore,

$$\begin{aligned} E(\hat{\alpha}) &= \beta E\left(\frac{\sum X_i^2 u_i}{\sum X_i^2}\right) \\ &= \beta \left( E\left(\frac{X_1^2 u_1 + X_2^2 u_2 + \cdots + X_n^2 u_n}{\sum X_i^2}\right) \right) \\ &= \beta e^{\sigma^2/2} \left( \frac{\sum X_i^2}{\sum X_i^2} \right) = \beta e^{\sigma^2/2} \end{aligned}$$

where use is made of the fact that the  $X$ 's are nonstochastic and each  $u_i$  has an expected value of  $e^{\sigma^2/2}$ .

Since  $E(\hat{\alpha}) \neq \beta$ ,  $\hat{\alpha}$  is a biased estimator of  $\beta$ .

# PART THREE

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## TOPICS IN ECONOMETRICS

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In **Part I** we introduced the classical linear regression model with all its assumptions. In **Part II** we examined in detail the consequences that ensue when one or more of the assumptions are not satisfied and what can be done about them. In **Part III** we study some selected but commonly encountered econometric techniques. In particular, we discuss these topics: (1) nonlinear-in-the-parameter regression models, (2) qualitative response regression models, (3) panel data regression models, and (4) dynamic econometric models.

In Chapter 14, we consider models that are *intrinsically* nonlinear in the parameters. With the ready availability of software packages, it is no longer a big challenge to estimate such models. Although the underlying mathematics may elude some readers, the basic ideas of nonlinear-in-the-parameter regression models can be explained intuitively. With suitable examples, this chapter shows how such models are estimated and interpreted.

In Chapter 15, we consider regression models in which the dependent variable is qualitative in nature. This chapter therefore complements Chapter 9, where we discussed models in which the explanatory variables were qualitative in nature. The basic thrust of this chapter is on developing models in which the regressand is of the yes or no type. Since OLS poses several problems in estimating such models, several alternatives have been developed. In this chapter we consider two such alternatives, namely, the **logit model** and the **probit model**. This chapter also discusses several variants of the qualitative response models, such as the **Tobit model** and the **Poisson regression model**. Several extensions of the qualitative response models are also briefly discussed, such as the **ordered probit**, **ordered logit**, and **multinomial logit**.



In Chapter 16 we discuss **panel data regression models**. Such models combine time series and cross-section observations. Although by combining such observations we increase the sample size, panel data regression models pose several estimation challenges. In this chapter we discuss only the essentials of such models and guide the reader to the appropriate resources for further study.

In Chapter 17, we consider regression models that include current as well as past, or lagged, values of the explanatory variables in addition to models that include lagged value(s) of the dependent variable as one of the explanatory variables. These models are called, respectively, the **distributed lag** and **autoregressive models**. Although such models are extremely useful in empirical econometrics, they pose some special estimating problems because they violate one or more assumptions of the classical regression model. We consider these special problems in the context of the Koyck, the adaptive-expectations (AE), and the partial-adjustment models. We also note the criticism leveled against the AE model by the advocates of the so-called rational expectations (RE) school.

# 14

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## NONLINEAR REGRESSION MODELS

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The major emphasis of this book is on linear regression models, that is, models that are linear in the parameters and/or models that can be transformed so that they are linear in the parameters. On occasions, however, for theoretical or empirical reasons we have to consider models that are nonlinear in the parameters.<sup>1</sup> In this chapter we take a look at such models and study their special features.

### 14.1 INTRINSICALLY LINEAR AND INTRINSICALLY NONLINEAR REGRESSION MODELS

When we started our discussion of linear regression models in Chapter 2, we stated that our concern in this book is basically with models that are linear in the parameters; they may or may not be linear in the variables. If you refer to Table 2.3, you will see that a model that is linear in the parameters as well as the variables is a linear regression model and so is a model that is linear in the parameters but nonlinear in the variables. On the other hand, if a model is nonlinear in the parameters it is a nonlinear (in-the-parameter) regression model whether the variables of such a model are linear or not.

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<sup>1</sup>We noted in Chap. 4 that under the assumption of normally distributed error term, the OLS estimators are not only BLUE but are BUE (best unbiased estimator) in the entire class of estimators, linear or not. But if we drop the assumption of normality, as Davidson and MacKinnon note, it is possible to obtain nonlinear and/or biased estimators that may perform better than the OLS estimators. See Russell Davidson and James G. MacKinnon, *Estimation and Inference in Econometrics*, Oxford University Press, New York, 1993, p. 161.

However, one has to be careful here, for some models may look nonlinear in the parameters but are **inherently** or **intrinsically** linear because with suitable transformation they can be made linear-in-the-parameter regression models. But if such models cannot be linearized in the parameters, they are called **intrinsically nonlinear regression models**. *From now on when we talk about a nonlinear regression model, we mean that it is intrinsically nonlinear.* For brevity, we will call them **NLRM**.

To drive home the distinction between the two, let us revisit exercises 2.6 and 2.7. In exercise 2.6, Models **a**, **b**, **c**, and **e** are linear regression models because they are all linear in the parameters. Model **d** is a mixed bag, for  $\beta_2$  is linear but not  $\ln \beta_1$ . But if we let  $\alpha = \ln \beta_1$ , then this model is linear in  $\alpha$  and  $\beta_2$ .

In exercise 2.7, Models **d** and **e** are intrinsically nonlinear because there is no simple way to linearize them. Model **c** is obviously a linear regression model. What about Models **a** and **b**? Taking the logarithms on both sides of **a**, we obtain  $\ln Y_i = \beta_1 + \beta_2 X_i + u_i$ , which is linear in the parameters. Hence Model **a** is *intrinsically* a linear regression model. Model **b** is an example of the **logistic (probability) distribution function**, and we will study this in Chapter 15. On the surface, it seems that this is a nonlinear regression model. But a simple mathematical trick will render it a linear regression model, namely,

$$\ln \left( \frac{1 - Y_i}{Y_i} \right) = \beta_1 + \beta_2 X_i + u_i \quad (14.1.1)$$

Therefore, Model **b** is intrinsically linear. We will see the utility of models like (14.1.1) in the next chapter.

Consider now the famous **Cobb–Douglas (C–D) production function**. Letting  $Y$  = output,  $X_2$  = labor input, and  $X_3$  = capital input, we will write this function in three different ways:

$$Y_i = \beta_1 X_{2i}^{\beta_2} X_{3i}^{\beta_3} e^{u_i} \quad (14.1.2)$$

or,

$$\ln Y_i = \alpha + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + u_i \quad (14.1.2a)$$

where  $\alpha = \ln \beta_1$ . Thus in this format the C–D function is intrinsically linear.

Now consider this version of the C–D function:

$$Y_i = \beta_1 X_{2i}^{\beta_2} X_{3i}^{\beta_3} u_i \quad (14.1.3)$$

or,

$$\ln Y_i = \alpha + \beta_2 \ln X_{2i} + \beta_3 \ln X_{3i} + \ln u_i \quad (14.1.3a)$$

where  $\alpha = \ln \beta_1$ . This model too is linear in the parameters.

But now consider the following version of the C–D function:

$$Y_i = \beta_1 X_{2i}^{\beta_2} X_{3i}^{\beta_3} + u_i \quad (14.1.4)$$

As we just noted, C–D versions (14.1.2a) and (14.1.3a) are intrinsically linear (in the parameter) regression models, but there is no way to transform (14.1.4) so that the transformed model can be made linear in the parameters.<sup>2</sup> Therefore, (14.1.4) is intrinsically a nonlinear regression model.

Another well-known but intrinsically nonlinear function is the **constant elasticity of substitution (CES)** production function of which the Cobb–Douglas production is a special case. The CES production takes the following form:

$$Y_i = A[\delta K_i^{-\beta} + (1 - \delta)L_i^{-\beta}]^{-1/\beta} \quad (14.1.5)$$

where  $Y$  = output,  $K$  = capital input,  $L$  = labor input,  $A$  = scale parameter,  $\delta$  = distribution parameter ( $0 < \delta < 1$ ), and  $\beta$  = substitution parameter ( $\beta \geq -1$ ).<sup>3</sup> No matter in what form you enter the stochastic error term  $u_i$  in this production function, there is no way to make it a linear (in parameter) regression model. It is intrinsically a nonlinear regression model.

## 14.2 ESTIMATION OF LINEAR AND NONLINEAR REGRESSION MODELS

To see the difference in estimating linear and nonlinear regression models, consider the following two models:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (14.2.1)$$

$$Y_i = \beta_1 e^{\beta_2 X_i} + u_i \quad (14.2.2)$$

By now you know that (14.2.1) is a linear regression model, whereas (14.2.2) is a nonlinear regression model. Regression (14.2.2) is known as the **exponential regression model** and is often used to measure the growth of a variable, such as population, GDP, or money supply.

<sup>2</sup>If you try to log-transform the model, it will not work because  $\ln(A + B) \neq \ln A + \ln B$ .

<sup>3</sup>For properties of the CES production function, see Michael D. Intriligator, Ronald Bodkin, and Cheng Hsiao, *Econometric Models, Techniques, and Applications*, 2d ed., Prentice Hall, 1996, pp. 294–295.

Suppose we consider estimating the parameters of the two models by OLS. In OLS we minimize the residual sum of squares (RSS), which for model (14.2.1) is:

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2 \quad (14.2.3)$$

where as usual  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are the OLS estimators of the true  $\beta$ 's. Differentiating the preceding expression with respect to the two unknowns, we obtain the **normal equations** shown in (3.1.4) and (3.1.5). Solving these equations simultaneously, we obtain the OLS estimators given in Eqs. (3.1.6) and (3.1.7). Observe very carefully that in these equations the unknowns ( $\beta$ 's) are on the left-hand side and the knowns ( $X$  and  $Y$ ) are on the right-hand side. As a result we get explicit solutions of the two unknowns in terms of our data.

Now see what happens if we try to minimize the RSS of (14.2.2). As shown in Appendix 14A, Section 14A.1, the normal equations corresponding to (3.1.4) and (3.1.5) are as follows:

$$\sum Y_i e^{\hat{\beta}_2 X_i} = \hat{\beta}_1 \sum e^{2\hat{\beta}_2 X_i} \quad (14.2.4)$$

$$\sum Y_i X_i e^{\hat{\beta}_2 X_i} = \hat{\beta}_1 \sum X_i e^{2\hat{\beta}_2 X_i} \quad (14.2.5)$$

Unlike the normal equations in the case of the linear regression model, the normal equations for nonlinear regression have the unknowns (the  $\hat{\beta}$ 's) both on the left- and right-hand sides of the equations. As a consequence, we *cannot obtain explicit solutions* of the unknowns in terms of the known quantities. To put it differently, the unknowns are expressed in terms of themselves and the data! Therefore, although we can apply the method of least squares to estimate the parameters of the nonlinear regression models, we cannot obtain explicit solutions of the unknowns. Incidentally, OLS applied to a nonlinear regression model is called **nonlinear least squares (NLLS)**. So, what is the solution? We take this question up next.

### 14.3 ESTIMATING NONLINEAR REGRESSION MODELS: THE TRIAL-AND-ERROR METHOD

To set the stage, let us consider a concrete example. The data in Table 14.1 relates to the management fees that a leading mutual fund in the United States pays to its investment advisors to manage its assets. The fees paid depend on the net asset value of the fund. As you can see, the higher the net asset value of the fund, the lower are the advisory fees, which can be seen clearly from Figure 14.1.

To see how the exponential regression model in (14.2.2) fits the data given in Table 14.1, we can proceed by trial and error. Suppose we assume that

TABLE 14.1 ADVISORY FEES CHARGED AND ASSET SIZE

	Fee, %	Asset*
1	0.520	0.5
2	0.508	5.0
3	0.484	10
4	0.46	15
5	0.4398	20
6	0.4238	25
7	0.4115	30
8	0.402	35
9	0.3944	40
10	0.388	45
11	0.3825	55
12	0.3738	60

\*Asset represents net asset value, billions of dollars.

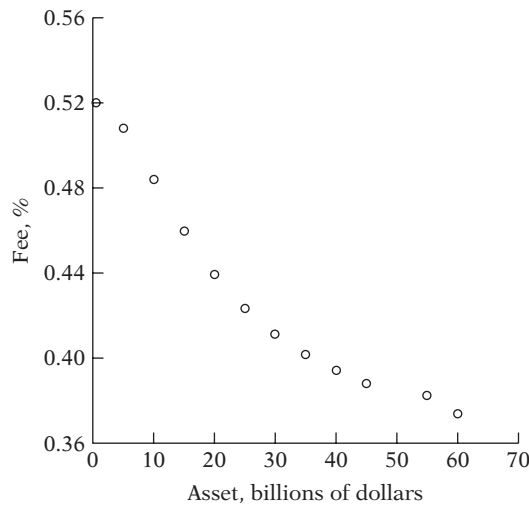


FIGURE 14.1 Relationship of advisory fees to fund assets.

initially  $\beta_1 = 0.45$  and  $\beta_2 = 0.01$ . These are pure guesses, sometimes based on prior experience or prior empirical work or obtained by just fitting a linear regression model even though it may not be appropriate. At this stage do not worry about how these values are obtained.

Since we know the values of  $\beta_1$  and  $\beta_2$ , we can write (14.2.2) as:

$$u_i = Y_i - \beta_1 e^{\beta_2 X_i} = Y_i - 0.45 e^{0.01 X_i} \quad (14.3.1)$$

Therefore,

$$\sum u_i^2 = \sum (Y_i - 0.45e^{0.01X_i})^2 \quad (14.3.2)$$

Since  $Y$ ,  $X$ ,  $\beta_1$ , and  $\beta_2$  are known, we can easily find the *error sum of squares* in (14.3.2).<sup>4</sup> Remember that in OLS our objective is to find those values of the unknown parameters that will make the error sum of squares as small as possible. This will happen if the estimated  $Y$  values from the model are as close as possible to the actual  $Y$  values. With the given values, we obtain  $\sum u_i^2 = 0.3044$ . But how do we know that this is the least possible error sum of squares that we can obtain? What happens if you choose another value for  $\beta_1$  and  $\beta_2$ , say, 0.50 and  $-0.01$ , respectively? Repeating the procedure just laid down, we find that we now obtain  $\sum u_i^2 = 0.0073$ . Obviously, this error sum of squares is much smaller than the one obtained before, namely, 0.3044. But how do we know that we have reached the lowest possible error sum of squares, for by choosing yet another set of values for the  $\beta$ 's, we will obtain yet another error sum of squares?

As you can see, such a trial-and-error, or **iterative**, process can be easily implemented. And if one has infinite time and infinite patience, the trial-and-error process *may* ultimately produce values of  $\beta_1$  and  $\beta_2$  that may guarantee the lowest possible error sum of squares. But you might ask, how did we go from  $(\beta_1 = 0.45; \beta_2 = 0.01)$  to  $(\beta_1 = 0.50; \beta_2 = -0.1)$ ? Clearly, we need some kind of *algorithm* that will tell us how we go from one set of values of the unknowns to another set before we stop. Fortunately such algorithms are available, and we discuss them in the next section.

#### 14.4 APPROACHES TO ESTIMATING NONLINEAR REGRESSION MODELS

There are several approaches, or algorithms, to NLRMs: (1) direct search or trial and error, (2) direct optimization, and (3) iterative linearization.<sup>5</sup>

##### Direct Search or Trial-and-Error or Derivative-Free Method

In the previous section we showed how this method works. Although intuitively appealing because it does not require the use of calculus methods as the other methods do, this method is generally not used. *First*, if an NLRM

<sup>4</sup>Note that we call  $\sum u_i^2$  the error sum of squares and not the usual residual sum of squares because the values of the parameters are assumed to be known.

<sup>5</sup>The following discussion leans heavily on these sources: Robert S. Pindyck and Daniel L. Rubinfeld, *Econometric Models and Economic Forecasts*, 4th ed., McGraw-Hill, 1998, Chap. 10; Norman R. Draper and Harry Smith, *Applied Regression Analysis*, 3d ed., John Wiley & Sons, 1998, Chap. 24; Arthur S. Goldberger, *A Course in Econometrics*, Harvard University Press, 1991, Chap. 29; Russell Davidson and James MacKinnon, op. cit., pp. 201–207; John Fox, *Applied Regression Analysis, Linear Models, and Related Methods*, Sage Publications, 1997, pp. 393–400; and Ronald Gallant, *Nonlinear Statistical Models*, John Wiley and Sons, 1987.

involves several parameters, the method becomes very cumbersome and computationally expensive. For example, if an NLRM involves 5 parameters and 25 alternative values for each parameter are considered, you will have to compute the error sum of squares  $(25)^5 = 9,765,625$  times! *Second*, there is no guarantee that the final set of parameter values you have selected will necessarily give you the absolute minimum error sum of squares. In the language of calculus, you may obtain a local and not an absolute minimum. In fact, no method guarantees a global minimum.

### Direct Optimization

In direct optimization we differentiate the error sum of squares with respect to each unknown coefficient, or parameter, set the resulting equation to zero, and solve the resulting normal equations simultaneously. We have already seen this in Eqs. (14.2.4) and (14.2.5). But as you can see from these equations, they cannot be solved explicitly or *analytically*. Some iterative routine is therefore called for. One routine is called the **method of steepest descent**. We will not discuss the technical details of this method as they are somewhat involved, but the reader can find the details in the references. Like the method of trial and error, the method of steepest descent also involves selecting initial trial values of the unknown parameters but then it proceeds more systematically than the hit-or-miss or trial-and-error method. One disadvantage of this method is that it may converge to the final values of the parameters extremely slowly.

### Iterative Linearization Method

In this method we linearize a nonlinear equation around some initial values of the parameters. The linearized equation is then estimated by OLS and the initially chosen values are adjusted. These adjusted values are used to *relinearize* the model, and again we estimate it by OLS and readjust the estimated values. This process is continued until there is no substantial change in the estimated values from the last couple of iterations. The main technique used in linearizing a nonlinear equation is the **Taylor series expansion** from calculus. Rudimentary details of this method are given in Appendix 14A, Section 14A.2. Estimating NLRM using Taylor series expansion is systematized in two algorithms, known as the **Gauss-Newton iterative method** and the **Newton-Raphson iterative method**. Since one or both of these methods are now incorporated in several computer packages, and since a discussion of their technical details will take us far beyond the scope of this book, there is no need to dwell on them here.<sup>6</sup> In the next section we discuss some examples using these methods.

<sup>6</sup>There is another method that is sometimes used, called the **Marquard method**, which is a compromise between the method of steepest descent and the linearization (or Taylor series) method. The interested reader may consult the references for the details of this method.



## 14.5 ILLUSTRATIVE EXAMPLES

**EXAMPLE 14.1****MUTUAL FUND ADVISORY FEES**

Refer to the data given in Table 14.1 and the NLRM (14.2.2). Using the Eviews 4 nonlinear regression routine, which uses the linearization method,<sup>7</sup> we obtained the following regression results; the coefficients, their standard errors, and their  $t$  values are given in a *tabular form*:

Variable	Coefficient	Std. error	$t$ value	$p$ value
Intercept	0.5089	0.0074	68.2246	0.0000
Asset	-0.0059	0.00048	-12.3150	0.0000

$$R^2 = 0.9385 \quad d = 0.3493$$

From these results, we can write the estimated model as:

$$\widehat{Fee}_i = 0.5089 \text{ Asset}^{-0.0059} \quad (14.5.1)$$

Before we discuss these results, it may be noted that if you do not supply the initial values of the parameters to start the linearization process, Eviews will do it on its own. It took Eviews five iterations to obtain the results shown in (14.5.1). However, you can supply your own initial values to start the process. To demonstrate, we chose the initial value of  $\beta_1 = 0.45$  and  $\beta_2 = 0.01$ . We obtained the same results as in (14.5.1) but it took eight iterations. *It is important to note that fewer iterations will be required if your initial values are not very far from the final values.* In some cases you can choose the initial values of the parameters by simply running an OLS regression of the regressand on the regressor(s), simply ignoring the nonlinearities. For instance, using the data in Table 14.1, if you were to regress fee on assets, the OLS estimate of  $\beta_1$  is 0.5028 and that of  $\beta_2$  is -0.002, which are much closer to the final values given in (14.5.1). (For the technical details, see Appendix 14A, Section 14A.3.)

Now about the properties of NLLS estimators. You may recall that, in the case of linear regression models with normally distributed error terms, we were able to develop exact inference procedures (i.e., test hypotheses) using the  $t$ ,  $F$ , and  $\chi^2$  tests in small as well as large samples. Unfortunately, this is not the case with NLRMs, even with normally distributed error terms. *The NLLS estimators are not normally distributed, are not unbiased, and do not have minimum variance* in finite, or small, samples. As a result, we cannot use the  $t$  test (to test the significance of an individual coefficient) or the  $F$  test (to test the overall significance of the estimated regression) because we cannot obtain an unbiased estimate of the error variance  $\sigma^2$  from the estimated residuals. Furthermore, the residuals (the difference between the actual  $Y$  values and the estimated  $Y$  values from the NLRM) do not necessarily sum to zero, ESS and RSS do not necessarily add up to the TSS, and therefore  $R^2 = \text{ESS}/\text{TSS}$  may not be a meaningful descriptive statistic for such models. However, we can compute  $R^2$  as:

$$R^2 = 1 - \frac{\sum \hat{u}_i^2}{\sum (Y_i - \bar{Y})^2} \quad (14.5.2)$$

where  $Y$  = regressand and  $\hat{u}_i = Y_i - \hat{Y}_i$ , where  $\hat{Y}_i$  are the estimated  $Y$  values from the (fitted) NLRM.

(Continued)

<sup>7</sup>Eviews provides three options: quadratic hill climbing, Newton-Raphson, and Berndt-Hall-Hall-Hausman. The default option is quadratic hill climbing, which is a variation of the Newton-Raphson method.

**EXAMPLE 14.1** (Continued)

Consequently, inferences about the regression parameters in nonlinear regression are usually based on large-sample theory. This theory tells us that the least-squares and maximum likelihood estimators for nonlinear regression models with normal error terms, when the sample size is large, are approximately normally distributed and almost unbiased, and have almost minimum variance. This large-sample theory also applies when the error terms are not normally distributed.<sup>8</sup>

In short, then, all inference procedures in NLRM are large sample, or asymptotic. Returning to Example 14.1, the *t* statistics given in (14.5.1) are meaningful only if interpreted in the large-sample context. In that sense, we can say that estimated coefficients shown in Eq. (14.5.1) are individually statistically significant. Of course, our sample in the present instance is rather small.

Returning to Eq. (14.5.1), how do we find out the rate of change of *Y* (= fee) with respect to *X* (asset size)? Using the basic rules of derivatives, the reader can see that the rate of change of *Y* with respect to *X* is:

$$\frac{dY}{dX} = \beta_1 \beta_2 e^{\beta_2 X} = (-0.0059)(0.5089)e^{-0.0059X} \quad (14.5.3)$$

As can be seen, the rate of change of fee depends on the value of the assets. For example, if *X* = 20 (million), the expected rate of change in the fees charged can be seen from (14.5.3) to be about -0.0031 percent. Of course, this answer will change depending on the *X* value used in the computation. Judged by the *R*<sup>2</sup> as computed from (14.5.2), the *R*<sup>2</sup> value of 0.9385 suggests that the chosen NLRM fits the data in Table 14.1 quite well. The estimated Durbin-Watson value of 0.3493 may suggest that there is autocorrelation or possibly model specification error. Although there are procedures to take care of these problems as well as the problem of heteroscedasticity in NLRM, we will not pursue these topics here. The interested reader may consult the references.

**EXAMPLE 14.2**

THE COBB-DOUGLAS PRODUCTION OF THE MEXICAN ECONOMY

Refer to the data given in exercise 14.9. These data refer to the Mexican economy for years 1955–1974. We will see if the NLRM given in (14.1.4) fits the data, noting that *Y* = output, *X*<sub>2</sub> = labor input, and *X*<sub>3</sub> = capital input. Using Eviews 4, we obtained the following regression results, after 32 iterations.

Variable	Coefficient	Std. error	<i>t</i> value	<i>p</i> value
Intercept	0.5292	0.2712	1.9511	0.0677
Labor	0.1810	0.1412	1.2814	0.2173
Capital	0.8827	0.0708	12.4658	0.0000

$$R^2 = 0.9942 \quad d = 0.2899$$

(Continued)

<sup>8</sup>John Neter, Michael H. Kutner, Christopher J. Nachtsheim, and William Wasserman, *Applied Regression Analysis*, 3d ed., Irwin, 1996, pp. 548–549.

**EXAMPLE 14.2** (Continued)

Therefore, the estimated Cobb–Douglas function is:

$$\widehat{\text{GDP}}_t = 0.5292 \text{Labor}_t^{0.1810} \text{Capital}_t^{0.8827} \quad (14.5.2)$$

Interpreted asymptotically, the equation shows that only the coefficient of the capital input is significant in this model. In exercise 14.9 you are asked to compare these results with those obtained from the multiplicative Cobb–Douglas production function as given in (14.1.2).

**EXAMPLE 14.3**

GROWTH OF U.S. POPULATION, 1970–1999

The table in exercise 14.8 gives you data on total U.S. population for the period 1970–1999. A **logistic growth model** of the following type is often used to measure the growth of a population:

$$Y_t = \frac{\beta_1}{1 + e^{(\beta_2 + \beta_3 t)}} + u_t \quad (14.5.4)$$

where  $Y$  = population;  $t$  = time, measured chronologically; and the  $\beta$ 's are the parameters. Notice an interesting thing about this model. Although there are only two variables, population and time, there are three unknowns, which shows that in a NLRM there can be more parameters than variables.

Sample: 1970–1999

Included observations: 30

Convergence achieved after one iteration

	Coefficient	Std. error	$t$ statistic	$p$ value
$\beta_1$	1432.738	508.0113	2.8202	0.0089
$\beta_2$	1.7986	0.4124	4.3613	0.0002
$\beta_3$	−0.0117	0.0008	−14.0658	0.0000

$$R^2 = 0.9997 \quad d = 0.3345$$

The estimated model, therefore, is:

$$\hat{Y}_t = \frac{1432.739}{1 + e^{1.7986 - 0.0117t}} \quad (14.5.5)$$

Since we have a reasonably large sample, asymptotically all the estimated coefficients are statistically significant. The low Durbin–Watson statistic suggests that the error term is probably autocorrelated. In exercise 14.8 you are asked to compare the preceding model with the semilog model:  $\ln Y_t = \beta_1 + \beta_2 \text{time} + u_t$  and compute the underlying growth rate of population for both models.

## 14.6 SUMMARY AND CONCLUSIONS

The main points discussed in this chapter can be summarized as follows:

1. Although linear regression models predominate theory and practice, there are occasions where nonlinear-in-the-parameter regression models (NLRM) are useful.

2. The mathematics underlying linear regression models is comparatively simple in that one can obtain explicit, or analytical, solutions of the coefficients of such models. The small-sample and large-sample theory of inference of such models is well established.

3. In contrast, for intrinsically nonlinear regression models, parameter values cannot be obtained explicitly. They have to be estimated numerically, that is, by iterative procedures.

4. There are several methods of obtaining estimates of NLRMs, such as (1) trial and error, (2) nonlinear least squares (NLLS), and (3) linearization through Taylor series expansion.

5. Computer packages now have built-in routines, such as Gauss-Newton, Newton-Raphson, and Marquard. These are all iterative routines.

6. NLLS estimators do not possess optimal properties in finite samples, but in large samples they do have such properties. Therefore, the results of NLLS in small samples must be interpreted carefully.

7. Autocorrelation, heteroscedasticity, and model specification problems can plague NLRM, as they do linear regression models.

8. We illustrated the NLLS with several examples. With the ready availability of user-friendly software packages, estimation of NLRM should no longer be a mystery. Therefore, the reader should not shy away from such models whenever theoretical or practical reasons dictate their use. As a matter of fact, if you refer to exercise 12.10, you will see from Eq. (1) that there is intrinsically a nonlinear regression model that should be estimated as such.

## EXERCISES

### Questions

- 14.1. What is meant by intrinsically linear and intrinsically nonlinear regression models? Give some examples.
- 14.2. Since the error term in the Cobb-Douglas production function can be entered multiplicatively or additively, how would you decide between the two?
- 14.3. What is the difference between OLS and nonlinear least-squares (NLLS) estimation?
- 14.4. The relationship between pressure and temperature in saturated steam can be expressed as<sup>9</sup>:

$$Y = \beta_1(10)^{\beta_2 t / (\gamma + t)} + u_t$$

<sup>9</sup>Adapted from Draper and Smith, op. cit., p. 554.

where  $Y$  = pressure and  $t$  = temperature. Using the method of nonlinear least squares (NLLS), obtain the *normal equations* for this model.

- 14.5.** State whether the following statements are true or false. Give your reasoning.
- a.** Statistical inference in NLLS regression cannot be made on the basis of the usual  $t$ ,  $F$ , and  $\chi^2$  tests even if the error term is assumed to be normally distributed.
  - b.** The coefficient of determination ( $R^2$ ) is not a particularly meaningful number for an NLRM.
- 14.6.** How would you linearize the CES production function discussed in the chapter? Show the necessary steps.
- 14.7.** Models that describe the behavior of a variable over time are called growth models. Such models are used in a variety of fields, such as economics, biology, botany, ecology, and demography. Growth models can take a variety of forms, both linear and nonlinear. Consider the following models, where  $Y$  is the variable whose growth we want to measure;  $t$  is time, measured chronologically; and  $u_t$  is the stochastic error term.
- a.**  $Y_t = \beta_1 + \beta_2 t + u_t$
  - b.**  $\ln Y_t = \beta_1 + \beta_2 t + u_t$
  - c.** Logistic growth model:  $Y_t = \frac{\beta_1}{1 + \beta_2 e^{-\beta_3 t}} + u_t$
  - d.** Gompertz growth model:  $Y_t = \beta_1 e^{-\beta_2 e^{-\beta_3 t}} + u_t$
- Find out the properties of these models by considering the growth of  $Y$  in relation to time.

**Problems**

- 14.8.** The data in Table 14.2 gives U.S. population, in millions of persons, for the period 1970–1999. Fit the growth models given in exercise 14.7 and decide which model gives a better fit. Interpret the parameters of the model.

**TABLE 14.2**

Observation	U.S. population	Time	Observation	U.S. population	Time
1970	205.052	1	1985	238.466	16
1971	207.661	2	1986	240.651	17
1972	209.896	3	1987	242.804	18
1973	211.909	4	1988	245.021	19
1974	213.854	5	1989	247.342	20
1975	215.973	6	1990	249.948	21
1976	218.035	7	1991	252.639	22
1977	220.239	8	1992	255.374	23
1978	222.585	9	1993	258.083	24
1979	225.055	10	1994	260.599	25
1980	227.726	11	1995	263.044	26
1981	229.966	12	1996	265.463	27
1982	232.188	13	1997	268.008	28
1983	234.307	14	1998	270.561	29
1984	236.348	15	1999	273.131	30

Source: *Economic Report of the President*, 2000.

**TABLE 14.3** PRODUCTION FUNCTION DATA FOR THE MEXICAN ECONOMY

Observation	GDP	Labor	Capital	Observation	GDP	Labor	Capital
1955	114,043	8,310	182,113	1965	212,323	11,746	315,715
1956	120,410	8,529	193,749	1966	226,977	11,521	337,642
1957	129,187	8,738	205,192	1967	241,194	11,540	363,599
1958	134,705	8,952	215,130	1968	260,881	12,066	391,847
1959	139,960	9,171	225,021	1969	277,498	12,297	422,382
1960	150,511	9,569	237,026	1970	296,530	12,955	455,049
1961	157,897	9,527	248,897	1971	306,712	13,338	484,677
1962	165,286	9,662	260,661	1972	329,030	13,738	520,553
1963	178,491	10,334	275,466	1973	354,057	15,924	561,531
1964	199,457	10,981	295,378	1974	374,977	14,154	609,825

*Source:* Victor J. Elias, *Sources of Growth: A Study of Seven Latin American Economies*, International Center for Economic Growth, ICS Press, San Francisco, 1992, Tables E-5, E-12, E-14.

*Notes:* GDP is in millions of 1960 pesos.  
Labor is in thousands of people.  
Capital is in millions of 1960 pesos.

**14.9.** Table 14.3 gives you data on real GDP, labor, and real fixed capital for Mexico for the period 1955–1974. See if the additive Cobb–Douglas production function given in Eq. (14.1.4) fits these data. Compare your results with those obtained from fitting the multiplicative Cobb–Douglas production function given in (14.1.2). Which is a better fit?

## APPENDIX 14A

### 14A.1 DERIVATION OF EQUATIONS (14.2.4) AND (14.2.5)

Write Eq. (14.2.2) as

$$u_i = Y_i - \beta_1 e^{\beta_2 X_i} \quad (1)$$

Therefore,

$$\sum u_i^2 = \sum (Y_i - \beta_1 e^{\beta_2 X_i})^2 \quad (2)$$

The error sum of squares is thus a function of  $\beta_1$  and  $\beta_2$ , since the values of  $Y$  and  $X$  are known. Therefore, to minimize the error sum of squares, we have to partially differentiate it with respect to the two unknowns, which gives:

$$\frac{\partial \sum u_i^2}{\partial \beta_1} = 2 \sum (Y_i - \beta_1 e^{\beta_2 X_i})(-1 e^{\beta_2 X_i}) \quad (3)$$

$$\frac{\partial \sum u_i^2}{\partial \beta_2} = 2 \sum (Y_i - \beta_1 e^{\beta_2 X_i})(-\beta_1 e^{\beta_2 X_i} X_i) \quad (4)$$

By the first-order condition of optimization, setting the preceding equations to zero and solving them simultaneously, we obtain Eqs. (14.2.4) and (14.2.5). Note that in differentiating the error sum of squares we have used the chain rule.

#### 14A.2 THE LINEARIZATION METHOD

Students familiar with calculus will recall **Taylor's theorem**, which states that any arbitrary function  $f(X)$  that is continuous and has a continuous  $n$ th-order derivative can be approximated around point  $X = X_0$  by a polynomial function and a remainder as follows:

$$f(X) = \frac{f(X_0)}{0!} + \frac{f'(X_0)(X - X_0)}{1!} + \frac{f''(X_0)(X - X_0)^2}{2!} + \dots \\ + \frac{f^n(X_0)(X - X_0)^n}{n!} + R \quad (1)$$

where  $f'(X_0)$  is the first derivative of  $f(X)$  evaluated at  $X = X_0$ ,  $f''(X_0)$  is the second derivative of  $f(X)$  evaluated at  $X = X_0$  and so on, where  $n!$  (read  $n$  factorial) stands for  $n(n-1)(n-2)\dots 1$  with the convention that  $0! = 1$ , and  $R$  stands for the remainder. If we take  $n = 1$ , we get a linear approximation; choosing  $n = 2$ , we get a second-degree polynomial approximation. As you can expect, the higher the order of the polynomial, the better the approximation to the original function. The series given in (1) is called **Taylor's series expansion of  $f(X)$  around the point  $X = X_0$** . As an example, consider the function:

$$Y = f(X) = \alpha_1 + \alpha_2 X + \alpha_3 X^2 + \alpha_4 X^3$$

Suppose we want to approximate it at  $X = 0$ . We now obtain:

$$f(0) = \alpha_1 \quad f'(0) = \alpha_2 \quad f''(0) = 2\alpha_3 \quad f'''(0) = 6\alpha_4$$

Hence we can obtain the following approximations:

$$\text{First order: } Y = \alpha_1 + \frac{f'(0)}{1!} X = \alpha_1 + \alpha_2 X + \text{remainder} (= \alpha_3 X^2 + \alpha_4 X^3)$$

$$\text{Second order: } Y = f(0) + \frac{f'(0)}{1!} X + \frac{f''(0)}{2!} X^2 \\ = \alpha_1 + \alpha_2 X + \alpha_3 X^2 + \text{remainder} (= \alpha_4 X^3)$$

$$\text{Third order: } Y = \alpha_1 + \alpha_2 X + \alpha_3 X^2 + \alpha_4 X^3$$

The third-order approximation reproduces the original equation exactly.

The objective of Taylor series approximation is usually to choose a lower-order polynomial in the hope that the remainder term will be inconsequential. It is often used to approximate a nonlinear function by a linear function, by dropping the higher-order terms.

The Taylor series approximation can be easily extended to a function containing more than one  $X$ . For example, consider the following function:

$$Y = f(X, Z) \quad (2)$$

and suppose we want to expand it around  $X = a$  and  $Z = b$ . Taylor's theorem shows that

$$\begin{aligned} f(x, z) = & f(a, b) + f_x(a, b)(x - a) \\ & + f_z(a, b)f(z - b) + \frac{1}{2!}[f_{xx}(a, b)(x - a)^2 \\ & - 2f_{xz}(a, b)(x - a)(z - b) + f_{zz}(a, b)(z - b)^2] + \dots \end{aligned} \quad (3)$$

where  $f_x$  = partial derivative of the function with respect to (w.r.t)  $X$ ,  $f_{xx}$  = second partial derivative of the function w.r.t.  $X$  and similarly for the variable  $Z$ . If we want a linear approximation to the function, we will use the first two terms in (3), if we want a quadratic, or second-degree, approximation, we will use the first three terms in (3), and so on.

### 14A.3 LINEAR APPROXIMATION OF THE EXPONENTIAL FUNCTION GIVEN IN (14.2.2)

The function under consideration is:

$$Y = f(\beta_1, \beta_2) = \beta_1 e^{\beta_2 X} \quad (1)$$

*Note:* For ease of manipulation, we have dropped the observation subscript.

Remember that in this function the unknowns are the  $\beta$  coefficients. Let us linearize this function at  $\beta_1 = \beta_1^*$  and  $\beta_2 = \beta_2^*$ , where the starred quantities are *given fixed values*. To *linearize* this, we proceed as follows:

$$Y = f(\beta_1, \beta_2) = f(\beta_1^*, \beta_2^*) + f_{\beta_1}(\beta_1^*, \beta_2^*)(\beta_1 - \beta_1^*) + f_{\beta_2}(\beta_1^*, \beta_2^*)(\beta_2 - \beta_2^*) \quad (2)$$

where  $f_{\beta_1}$  and  $f_{\beta_2}$  are the partial derivatives of the function (1) with respect to the unknowns and these derivatives will be evaluated at the (assumed) starred values of the unknown parameters. Note that we are using only the first derivatives in the preceding expression, since we are linearizing the function. Now assume that  $\beta_1^* = 0.45$  and  $\beta_2^* = 0.01$ , which are pure guess-estimates of the true coefficients. Now

$$\begin{aligned} f(\beta_1^* = 0.45, \beta_2^* = 0.01) &= 0.45e^{0.01X_i} \\ f_{\beta_1} &= e^{\beta_2 X_i} \quad \text{and} \quad f_{\beta_2} = \beta_1 X_i e^{\beta_2 X_i} \end{aligned} \quad (3)$$



by the standard rules of differentiation. Evaluating these derivatives at the given values and reverting to (2), we obtain:

$$Y_i = 0.45e^{0.01X_i} + e^{0.01X_i}(\beta_1 - 0.45) + (0.45)X_i e^{0.01X_i}(\beta_2 - 0.01) \quad (4)$$

which we write as:

$$(Y_i - 0.45e^{0.01X_i}) = e^{0.01X_i}\alpha_1 + 0.45X_i e^{0.01X_i}\alpha_2 \quad (5)$$

where

$$\alpha_1 = (\beta_1 - 0.45) \quad \text{and} \quad \alpha_2 = (\beta_2 - 0.01) \quad (6)$$

Now let  $Y_i^* = (Y_i - 0.45e^{0.01X_i})$ ,  $X_{1i} = e^{0.01X_i}$ , and  $X_{2i} = 0.45X_i e^{0.01X_i}$ . Using these definitions and adding the error term  $u_i$ , we can finally write (5) as:

$$Y_i^* = \alpha_1 X_{1i} + \alpha_2 X_{2i} + u_i \quad (7)$$

Lo and behold, we now have a linear regression model. Since  $Y_i^*$ ,  $X_{1i}$ , and  $X_{2i}$  can be readily computed from the data, we can easily estimate (7) by OLS and obtain the values of  $\alpha_1$  and  $\alpha_2$ . Then, from (6), we obtain:

$$\beta_1 = \hat{\alpha}_1 + 0.45 \quad \text{and} \quad \beta_2 = \hat{\alpha}_2 + 0.01 \quad (8)$$

Call these values  $\beta_1^{**}$  and  $\beta_2^{**}$ , respectively. Using these (revised) values, we can start the iterative process given in (2), obtaining yet another set of values of the  $\beta$  coefficients. We can go on iterating (or linearizing) in this fashion until there is no substantial change in the values of the  $\beta$  coefficients. In Example 14.1, it took five iterations, but for the Mexican Cobb–Douglas example it took 32 iterations. But the underlying logic behind these iterations is the procedure just illustrated.

For the mutual fund fee structure, the  $Y^*$ ,  $X_1$ , and  $X_2$  as given in (6) are as shown in Table 14.4; the basic data are given in Table 14.1. From these values, the regression results corresponding to (7) are:

Dependent variable:  $Y^*$   
Method: least squares

Variable	Coefficient	Std. error	t statistic	Probability
$X_1$	0.039775	0.006229	6.3856	0.0001
$X_2$	0.001303	0.000157	8.3095	0.0000

$$R^2 = 0.948378 \quad \text{Durbin-Watson } d = 0.58337$$

Now using (8), the reader can verify that

$$\beta_1^{**} = 0.4897 \quad \text{and} \quad \beta_2^{**} = 0.0113 \quad (9)$$

TABLE 14.4

$\gamma^*$	$X_1$	$X_2$
0.02249	0.45225	0.22612
0.03238	0.47307	2.3653
0.03158	0.49732	4.9732
0.02964	0.52282	7.8423
0.03043	0.54963	10.9926
0.03439	0.57781	14.4452
0.04109	0.60743	18.2230
0.04965	0.63858	22.3503
0.05923	0.67132	26.8528
0.06918	0.70574	31.7583
0.09402	0.77996	42.8980
0.09939	0.81995	49.1972

Contrast these numbers with the initial guesses of 0.45 and 0.01, respectively, for the two parameters. Using the new estimates given in (9), you can start the iterative procedure once more and go on iterating until there is “convergence” in the sense that the final round of the estimates does not differ much from the round before that. Of course, you will require fewer iterations if your initial guess is closer to the final values. Also, notice that we have used only the linear term in Taylor’s series expansion. If you were to use the quadratic or higher-order terms in the expansion, perhaps you would reach the final values much quicker. But in many applications the linear approximation has proved to be quite good.

# 15

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## QUALITATIVE RESPONSE REGRESSION MODELS

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In all the regression models that we have considered so far, we have implicitly assumed that the regressand, the dependent variable, or the *response* variable  $Y$  is quantitative, whereas the explanatory variables are either quantitative, qualitative (or dummy), or a mixture thereof. In fact, in Chapter 9, on dummy variables, we saw how the dummy regressors are introduced in a regression model and what role they play in specific situations.

In this chapter we consider several models in which the regressand itself is qualitative in nature. Although increasingly used in various areas of social sciences and medical research, qualitative response regression models pose interesting estimation and interpretation challenges. In this chapter we only touch on some of the major themes in this area, leaving the details to more specialized books.<sup>1</sup>

### 15.1 THE NATURE OF QUALITATIVE RESPONSE MODELS

Suppose we want to study the labor force participation (LFP) decision of adult males. Since an adult is either in the labor force or not, LFP is a *yes* or *no* decision. Hence, the response variable, or regressand, can take only two

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<sup>1</sup>At the introductory level, the reader may find the following sources very useful. Daniel A. Powers and Yu Xie, *Statistical Methods for Categorical Data Analysis*, Academic Press, 2000; John H. Aldrich and Forrest Nelson, *Linear Probability, Logit, and Probit Models*, Sage Publications, 1984; Tim Futing Liao, *Interpreting Probability Models: Logit, Probit and Other Generalized Linear Models*, Sage Publications, 1994. For a very comprehensive review of the literature, see G. S. Maddala, *Limited-Dependent and Qualitative Variables in Econometrics*, Cambridge University Press, 1983.

values, say, 1 if the person is in the labor force and 0 if he or she is not. In other words, the regressand is a **binary, or dichotomous, variable**. Labor economics research suggests that the LFP decision is a function of the unemployment rate, average wage rate, education, family income, etc.

As another example, consider U.S. presidential elections. Assume that there are two political parties, Democratic and Republican. The dependent variable here is vote choice between the two political parties. Suppose we let  $Y = 1$ , if the vote is for a Democratic candidate, and  $Y = 0$ , if the vote is for a Republican candidate. A considerable amount of research on this topic has been done by the economist Ray Fair of Yale University and several political scientists.<sup>2</sup> Some of the variables used in the vote choice are growth rate of GDP, unemployment and inflation rates, whether the candidate is running for reelection, etc. For the present purposes, the important thing to note is that the regressand is a qualitative variable.

One can think of several other examples where the regressand is qualitative in nature. Thus, a family either owns a house or it does not, it has disability insurance or it does not, both husband and wife are in the labor force or only one spouse is. Similarly, a certain drug is effective in curing an illness or it is not. A firm decides to declare a stock dividend or not, a senator decides to vote for a tax cut or not, a U.S. President decides to veto a bill or accept it, etc.

We do not have to restrict our response variable to yes/no or dichotomous categories only. Returning to our presidential elections example, suppose there are three parties, Democratic, Republican, and Independent. The response variable here is **trichotomous**. In general, we can have a **polychotomous** (or **multiple-category**) response variable.

What we plan to do is to first consider the dichotomous regressand and then consider various extensions of the basic model. But before we do that, it is important to note a fundamental difference between a regression model where the regressand  $Y$  is quantitative and a model where it is qualitative.

In a model where  $Y$  is quantitative, our objective is to estimate its expected, or mean, value given the values of the regressors. In terms of Chapter 2, what we want is  $E(Y_i | X_{1i}, X_{2i}, \dots, X_{ki})$ , where the  $X$ 's are regressors, both quantitative and qualitative. In models where  $Y$  is qualitative, our objective is to find the probability of something happening, such as voting for a Democratic candidate, or owning a house, or belonging to a union, or participating in a sport etc. Hence, qualitative response regression models are often known as *probability models*.

In the rest of this chapter, we seek answers to the following questions:

1. How do we estimate qualitative response regression models? Can we simply estimate them with the usual OLS procedures?

<sup>2</sup>See, for example, Ray Fair, "Econometrics and Presidential Elections," *Journal of Economic Perspective*, Summer 1996, pp. 89–102, and Machael S. Lewis-Beck, *Economics and Elections: The Major Western Democracies*, University of Michigan Press, Ann Arbor, 1980.

2. Are there special inference problems? In other words, is the hypothesis testing procedure any different from the ones we have learned so far?

3. If a regressand is qualitative, how can we measure the goodness of fit of such models? Is the conventionally computed  $R^2$  of any value in such models?

4. Once we go beyond the dichotomous regressand case, how do we estimate and interpret the polychotomous regression models? Also, how do we handle models in which the regressand is **ordinal**, that is, an ordered categorical variable, such as schooling (less than 8 years, 8 to 11 years, 12 years, and 13 or more years), or the regressand is **nominal** where there is no inherent ordering, such as ethnicity (Black, White, Hispanic, Asian, and other)?

5. How do we model phenomena, such as the number of visits to one's physician per year, the number of patents received by a firm in a given year, the number of articles published by a college professor in a year, the number of telephone calls received in a span of 5 minutes, or the number of cars passing through a toll booth in a span of 5 minutes? Such phenomena, called **count data**, or **rare event data**, are an example of the **Poisson** (probability) process.

In this chapter we provide answers to some of these questions at the elementary level, for some of the topics are quite advanced and require more background in mathematics and statistics than assumed in this book. References cited in the various footnotes may be consulted for further details.

We start our study of qualitative response models by first considering the **binary response** regression model. There are three approaches to developing a probability model for a binary response variable:

1. The **linear probability model (LPM)**
2. The **logit model**
3. The **probit model**

Because of its comparative simplicity, and because it can be estimated by OLS, we will first consider the LPM, leaving the other two models for subsequent sections.

## 15.2 THE LINEAR PROBABILITY MODEL (LPM)

To fix ideas, consider the following regression model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \quad (15.2.1)$$

where  $X$  = family income and  $Y = 1$  if the family owns a house and 0 if it does not own a house.

Model (15.2.1) looks like a typical linear regression model but because the regressand is binary, or dichotomous, it is called a **linear probability model (LPM)**. This is because the conditional expectation of  $Y_i$  given

$X_i$ ,  $E(Y_i | X_i)$ , can be interpreted as the *conditional probability* that the event will occur given  $X_i$ , that is,  $\Pr(Y_i = 1 | X_i)$ . Thus, in our example,  $E(Y_i | X_i)$  gives the probability of a family owning a house and whose income is the given amount  $X_i$ .

The justification of the name LPM for models like (15.2.1) can be seen as follows: Assuming  $E(u_i) = 0$ , as usual (to obtain unbiased estimators), we obtain

$$E(Y_i | X_i) = \beta_1 + \beta_2 X_i \quad (15.2.2)$$

Now, if  $P_i = \text{probability that } Y_i = 1$  (that is, the event occurs), and  $(1 - P_i) = \text{probability that } Y_i = 0$  (that is, that the event does not occur), the variable  $Y_i$  has the following (probability) distribution.

$Y_i$	Probability
0	$1 - P_i$
1	$P_i$
Total	1

That is,  $Y_i$  follows the **Bernoulli probability distribution**.

Now, by the definition of mathematical expectation, we obtain:

$$E(Y_i) = 0(1 - P_i) + 1(P_i) = P_i \quad (15.2.3)$$

Comparing (15.2.2) with (15.2.3), we can equate

$$E(Y_i | X_i) = \beta_1 + \beta_2 X_i = P_i \quad (15.2.4)$$

that is, the conditional expectation of the model (15.2.1) can, in fact, be interpreted as the conditional probability of  $Y_i$ . In general, the expectation of a Bernoulli random variable is the probability that the random variable equals 1. In passing note that if there are  $n$  independent trials, each with a probability  $p$  of success and probability  $(1 - p)$  of failure, and  $X$  of these trials represent the number of successes, then  $X$  is said to follow the **binomial distribution**. The mean of the binomial distribution is  $np$  and its variance is  $np(1 - p)$ . The term *success* is defined in the context of the problem.

Since the probability  $P_i$  must lie between 0 and 1, we have the restriction

$$0 \leq E(Y_i | X_i) \leq 1 \quad (15.2.5)$$

that is, the conditional expectation (or conditional probability) must lie between 0 and 1.

From the preceding discussion it would seem that OLS can be easily extended to binary dependent variable regression models. So, perhaps there

is nothing new here. Unfortunately, this is not the case, for the LPM poses several problems, which are as follows:

### Non-Normality of the Disturbances $u_i$

Although OLS does not require the disturbances ( $u_i$ ) to be normally distributed, we assumed them to be so distributed for the purpose of statistical inference.<sup>3</sup> But the assumption of normality for  $u_i$  is not tenable for the LPMs because, like  $Y_i$ , the disturbances  $u_i$  also take only two values; that is, they also follow the Bernoulli distribution. This can be seen clearly if we write (15.2.1) as

$$u_i = Y_i - \beta_1 - \beta_2 X_i \quad (15.2.6)$$

The probability distribution of  $u_i$  is

	$u_i$	Probability
When $Y_i = 1$	$1 - \beta_1 - \beta_2 X_i$	$P_i$
When $Y_i = 0$	$-\beta_1 - \beta_2 X_i$	$(1 - P_i)$

(15.2.7)

Obviously,  $u_i$  cannot be assumed to be normally distributed; they follow the Bernoulli distribution.

But the nonfulfillment of the normality assumption may not be so critical as it appears because we know that the OLS point estimates still remain unbiased (recall that, if the objective is point estimation, the normality assumption is not necessary). Besides, as the sample size increases indefinitely, statistical theory shows that the OLS estimators tend to be normally distributed generally.<sup>4</sup> As a result, in large samples the statistical inference of the LPM will follow the usual OLS procedure under the normality assumption.

### Heteroscedastic Variances of the Disturbances

Even if  $E(u_i) = 0$  and  $\text{cov}(u_i, u_j) = 0$  for  $i \neq j$  (i.e., no serial correlation), it can no longer be maintained that in the LPM the disturbances are

<sup>3</sup>Recall that we have recommended that the normality assumption be checked in an application by suitable normality tests, such as the Jarque-Bera test.

<sup>4</sup>The proof is based on the central limit theorem and may be found in E. Malinvaud, *Statistical Methods of Econometrics*, Rand McNally, Chicago, 1966, pp. 195–197. If the regressors are deemed stochastic and are jointly normally distributed, the  $F$  and  $t$  tests can still be used even though the disturbances are non-normal. Also keep in mind that as the sample size increases indefinitely, the binomial distribution converges to the normal distribution.

homoscedastic. This is, however, not surprising. As statistical theory shows, for a Bernoulli distribution the theoretical mean and variance are, respectively,  $p$  and  $p(1 - p)$ , where  $p$  is the probability of success (i.e., something happening), showing that the variance is a function of the mean. Hence the error variance is heteroscedastic.

For the distribution of the error term given in (15.2.7), applying the definition of variance, the reader should verify that (see exercise 15.10)

$$\text{var}(u_i) = P_i(1 - P_i) \quad (15.2.8)$$

That is, the variance of the error term in the LPM is heteroscedastic. Since  $P_i = E(Y_i | X_i) = \beta_1 + \beta_2 X_i$ , the variance of  $u_i$  ultimately depends on the values of  $X$  and hence is not homoscedastic.

We already know that, in the presence of heteroscedasticity, the OLS estimators, although unbiased, are not efficient; that is, they do not have minimum variance. But the problem of heteroscedasticity, like the problem of non-normality, is not insurmountable. In Chapter 11 we discussed several methods of handling the heteroscedasticity problem. Since the variance of  $u_i$  depends on  $E(Y_i | X_i)$ , one way to resolve the heteroscedasticity problem is to transform the model (15.2.1) by dividing it through by

$$\sqrt{E(Y_i | X_i)[1 - E(Y_i | X_i)]} = \sqrt{P_i(1 - P_i)} = \text{say } \sqrt{w_i}$$

that is,

$$\frac{Y_i}{\sqrt{w_i}} = \frac{\beta_1}{\sqrt{w_i}} + \beta_2 \frac{X_i}{\sqrt{w_i}} + \frac{u_i}{\sqrt{w_i}} \quad (15.2.9)$$

As you can readily verify, the transformed error term in (15.2.9) is homoscedastic. Therefore, after estimating (15.2.1), we can now estimate (15.2.9) by OLS, which is nothing but the *weighted least squares* (WLS) with  $w_i$  serving as the weights.

In theory, what we have just described is fine. But in practice the true  $E(Y_i | X_i)$  is unknown; hence the weights  $w_i$  are unknown. To estimate  $w_i$ , we can use the following two-step procedure<sup>5</sup>:

**Step 1.** Run the OLS regression (15.2.1) despite the heteroscedasticity problem and obtain  $\hat{Y}_i =$  estimate of the true  $E(Y_i | X_i)$ . Then obtain  $\hat{w}_i = \hat{Y}_i(1 - \hat{Y}_i)$ , the estimate of  $w_i$ .

<sup>5</sup>For the justification of this procedure, see Arthur S. Goldberger, *Econometric Theory*, John Wiley & Sons, New York, 1964, pp. 249–250. The justification is basically a large-sample one that we discussed under the topic of feasible or estimated generalized least squares in the chapter on heteroscedasticity (see Sec. 11.6).



**Step 2.** Use the estimated  $w_i$  to transform the data as shown in (15.2.9) and estimate the transformed equation by OLS (i.e., weighted least squares).

We will illustrate this procedure for our example shortly. But there is another problem with LPM that we need to address first.

### Nonfulfillment of $0 \leq E(Y_i | X) \leq 1$

Since  $E(Y_i | X)$  in the linear probability models measures the conditional probability of the event  $Y$  occurring given  $X$ , it must necessarily lie between 0 and 1. Although this is true a priori, there is no guarantee that  $\hat{Y}_i$ , the estimators of  $E(Y_i | X_i)$ , will necessarily fulfill this restriction, *and this is the real problem with the OLS estimation of the LPM*. There are two ways of finding out whether the estimated  $\hat{Y}_i$  lie between 0 and 1. One is to estimate the LPM by the usual OLS method and find out whether the estimated  $\hat{Y}_i$  lie between 0 and 1. If some are less than 0 (that is, negative),  $\hat{Y}_i$  is assumed to be zero for those cases; if they are greater than 1, they are assumed to be 1. The second procedure is to devise an estimating technique that will guarantee that the estimated conditional probabilities  $\hat{Y}_i$  will lie between 0 and 1. The logit and probit models discussed later will guarantee that the estimated probabilities will indeed lie between the logical limits 0 and 1.

### Questionable Value of $R^2$ as a Measure of Goodness of Fit

The conventionally computed  $R^2$  is of limited value in the dichotomous response models. To see why, consider the following figure. Corresponding to a given  $X$ ,  $Y$  is either 0 or 1. Therefore, all the  $Y$  values will either lie along the  $X$  axis or along the line corresponding to 1. Therefore, generally no LPM is expected to fit such a scatter well, whether it is the *unconstrained LPM* (Figure 15.1a) or the *truncated* or *constrained LPM* (Figure 15.1b), an LPM estimated in such a way that it will not fall outside the logical band 0–1. As a result, the conventionally computed  $R^2$  is likely to be much lower than 1 for such models. In most practical applications the  $R^2$  ranges between 0.2 to 0.6.  $R^2$  in such models will be high, say, in excess of 0.8 only when the actual scatter is very closely clustered around points A and B (Figure 15.1c), for in that case it is easy to fix the straight line by joining the two points A and B. In this case the predicted  $Y_i$  will be very close to either 0 or 1.

For these reasons John Aldrich and Forrest Nelson contend that “use of the coefficient of determination as a summary statistic should be avoided in models with qualitative dependent variable.”<sup>6</sup>

<sup>6</sup>Aldrich and Nelson, op. cit., p. 15. For other measures of goodness of fit in models involving dummy regressands, see T. Amemiya, “Qualitative Response Models,” *Journal of Economic Literature*, vol. 19, 1981, pp. 331–354.

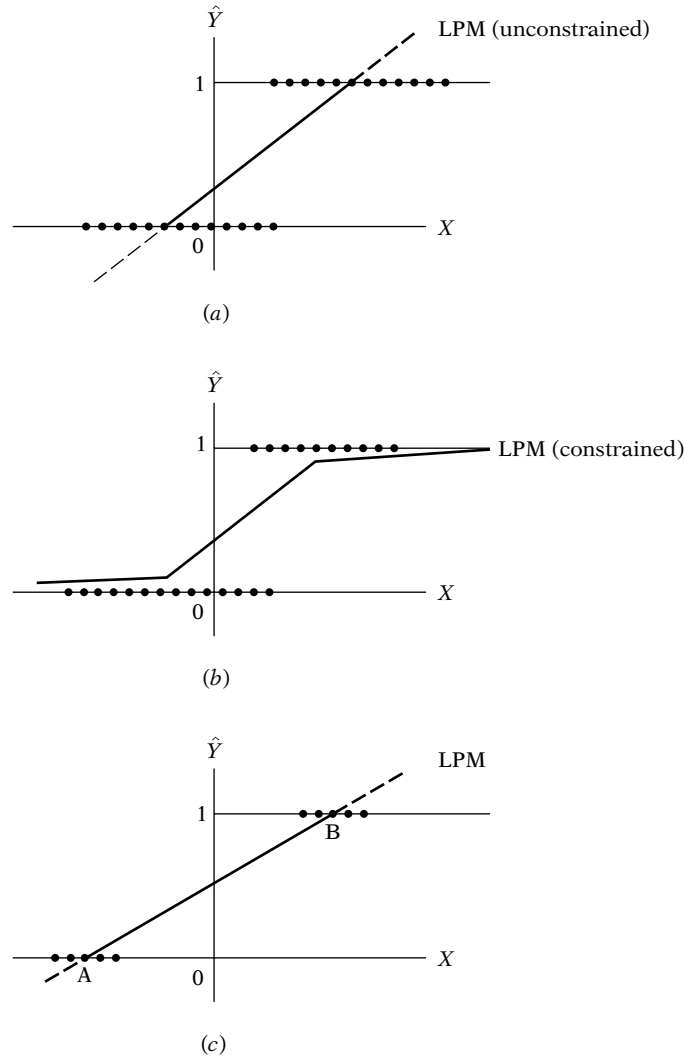


FIGURE 15.1 Linear probability models.

LPM: A NUMERICAL EXAMPLE

To illustrate some of the points made about the LPM in the preceding section, we present a numerical example. Table 15.1 gives invented data on home ownership  $Y$  ( $1 =$  owns a house,  $0 =$  does not own a house) and family income  $X$  (thousands of dollars) for 40 families.

From these data the LPM estimated by OLS was as follows:

$$\hat{Y}_i = -0.9457 + 0.1021X_i$$

(0.1228) (0.0082) **(15.2.10)**

$$t = (-7.6984) (12.515) \quad R^2 = 0.8048$$

(Continued)

LPM: A NUMERICAL EXAMPLE (Continued)

**TABLE 15.1**  
HYPOTHETICAL DATA ON HOME OWNERSHIP ( $Y = 1$  IF OWNS HOME, 0 OTHERWISE)  
AND INCOME  $X$  (THOUSANDS OF DOLLARS)

Family	Y	X	Family	Y	X
1	0	8	21	1	22
2	1	16	22	1	16
3	1	18	23	0	12
4	0	11	24	0	11
5	0	12	25	1	16
6	1	19	26	0	11
7	1	20	27	1	20
8	0	13	28	1	18
9	0	9	29	0	11
10	0	10	30	0	10
11	1	17	31	1	17
12	1	18	32	0	13
13	0	14	33	1	21
14	1	20	34	1	20
15	0	6	35	0	11
16	1	19	36	0	8
17	1	16	37	1	17
18	0	10	38	1	16
19	0	8	39	0	7
20	1	18	40	1	17

First, let us interpret this regression. The intercept of  $-0.9457$  gives the “probability” that a family with zero income will own a house. Since this value is negative, and since probability cannot be negative, we treat this value as zero, which is sensible in the present instance.<sup>7</sup> The slope value of  $0.1021$  means that for a unit change in income (here \$1000), on the average the probability of owning a house increases by  $0.1021$  or about 10 percent. Of course, given a particular level of income, we can estimate the actual probability of owning a house from (15.2.10). Thus, for  $X = 12$  (\$12,000), the estimated probability of owning a house is

$$\begin{aligned} (\hat{Y}_i | X = 12) &= -0.9457 + 12(0.1021) \\ &= 0.2795 \end{aligned}$$

That is, the probability that a family with an income of \$12,000 will own a house is about 28 percent. Table 15.2 shows the estimated probabilities,  $\hat{Y}_i$ , for the various income levels listed in the table. The most

noticeable feature of this table is that six estimated values are negative and six values are in excess of 1, demonstrating clearly the point made earlier that, although  $E(Y_i | X)$  is positive and less than 1, their estimators,  $\hat{Y}_i$ , need not be necessarily positive or less than 1. This is one reason that the LPM is not the recommended model when the dependent variable is dichotomous.

Even if the estimated  $Y_i$  were all positive and less than 1, the LPM still suffers from the problem of heteroscedasticity, which can be seen readily from (15.2.8). As a consequence, we cannot trust the estimated standard errors reported in (15.12.10). (Why?) But we can use the weighted least-squares (WLS) procedure discussed earlier to obtain more efficient estimates of the standard errors. The necessary weights,  $\hat{w}_i$ , required for the application of WLS are also shown in Table 15.2. But note that since some  $Y_i$  are negative and some are in excess of one, the  $\hat{w}_i$  corresponding to these values will be negative. Thus, we cannot use these observations in WLS (why?), thereby reducing the number of

(Continued)

<sup>7</sup>One can loosely interpret the highly negative value as near improbability of owning a house when income is zero.

LPM: A NUMERICAL EXAMPLE (Continued)

**TABLE 15.2**  
ACTUAL  $Y_i$ , ESTIMATED  $\hat{Y}_i$ , AND WEIGHTS  $w_i$  FOR THE HOME OWNERSHIP EXAMPLE

$Y_i$	$\hat{Y}_i$	$\hat{w}_i^\ddagger$	$\sqrt{\hat{w}_i}$	$Y_i$	$\hat{Y}_i$	$\hat{w}_i^\ddagger$	$\sqrt{\hat{w}_i}$
0	-0.129*			1	1.301†		
1	0.688	0.2146	0.4633	1	0.688	0.2147	0.4633
1	0.893	0.0956	0.3091	0	0.280	0.2016	0.4990
0	0.178	0.1463	0.3825	0	0.178	0.1463	0.3825
0	0.280	0.2016	0.4490	1	0.688	0.2147	0.4633
1	0.995	0.00498	0.0705	0	0.178	0.1463	0.3825
1	1.098†			1	1.097†		
0	0.382	0.2361	0.4859	1	0.893	0.0956	0.3091
0	-0.0265*			0	0.178	0.1463	0.3825
0	0.076	0.0702	0.2650	0	0.076	0.0702	0.2650
1	0.791	0.1653	0.4066	1	0.791	0.1653	0.4055
1	0.893	0.0956	0.3091	0	0.382	0.2361	0.4859
0	0.484	0.2497	0.4997	1	1.199†		
1	1.097†			1	1.097†		
0	-0.333*			0	0.178	0.1463	0.3825
1	0.995	0.00498	0.0705	0	-0.129*		
1	0.688	0.2147	0.4633	1	0.791	0.1653	0.4066
0	0.076	0.0702	0.2650	1	0.688	0.2147	0.4633
0	-0.129*			0	-0.231*		
1	0.893	0.0956	0.3091	1	0.791	0.1653	0.4066

\*Treated as zero to avoid probabilities being negative.  
†Treated as unity to avoid probabilities exceeding one.  
‡ $\hat{Y}_i(1 - \hat{Y}_i)$ .

observations, from 40 to 28 in the present example.<sup>8</sup> Omitting these observations, the WLS regression is

$$\frac{\hat{Y}_i}{\sqrt{\hat{w}_i}} = -1.2456 \frac{1}{\sqrt{\hat{w}_i}} + 0.1196 \frac{X_i}{\sqrt{\hat{w}_i}} \quad (15.2.11)$$

$t = (-10.332) \quad (17.454) \quad R^2 = 0.9214$

These results show that, compared with (15.12.10), the estimated standard errors are smaller and, correspondingly, the estimated  $t$  ratios (in absolute value) larger. But one should take this result with a grain of salt since in estimating (15.12.11) we had to drop 12 observations. Also, since  $w_i$  are estimated, the usual statistical hypothesis-testing procedures are, strictly speaking, valid in the large samples (see Chapter 11).

### 15.3 APPLICATIONS OF LPM

Until the availability of readily accessible computer packages to estimate the logit and probit models (to be discussed shortly), the LPM was used quite extensively because of its simplicity. We now illustrate some of these applications.

<sup>8</sup>To avoid the loss of the degrees of freedom, we could let  $\hat{Y}_i = 0.01$  when the estimated  $Y_i$  are negative and  $\hat{Y}_i = 0.99$  when they are in excess of or equal to 1. See exercise 15.1.

**EXAMPLE 15.1**COHEN-REA-LERMAN STUDY<sup>9</sup>

In a study prepared for the U.S. Department of Labor, Cohen, Rea, and Lerman were interested in examining the labor-force participation of various categories of labor as a function of several socioeconomic-demographic variables. In all their regressions, the dependent variable was a dummy, taking a value of 1 if a person is in the labor force, 0 if he or she is not. In Table 15.3 we reproduce one of their several dummy-dependent variable regressions.

Before interpreting the results, note these features: The preceding regression was estimated by using the OLS. To correct for heteroscedasticity, the authors used the two-step procedure outlined previously in some of their regressions but found that the standard errors of the estimates thus obtained did not differ materially from those obtained without correction for heteroscedasticity. Perhaps this result is due to the sheer size of the sample, namely, about 25,000. Because of this large sample size, the estimated  $t$  values may be tested for statistical significance by the usual OLS procedure even though the error term takes dichotomous values. The estimated  $R^2$  of 0.175 may seem rather low, but in view of the large sample size, this  $R^2$  is still significant on the basis of the  $F$  test given in Section 8.5. Finally, notice how the authors have blended quantitative and qualitative variables and how they have taken into account the interaction effects.

Turning to the interpretations of the findings, we see that each slope coefficient gives the rate of change in the conditional probability of the event occurring for a given unit change in the value of the explanatory variable. For instance, the coefficient of  $-0.2753$  attached to the variable "age 65 and over" means, holding all other factors constant, the probability of participation in the labor force by women in this age group is smaller by about 27 percent (as compared with the base category of women aged 22 to 54). By the same token, the coefficient of  $0.3061$  attached to the variable "16 or more years of schooling" means, holding all other factors constant, the probability of women with this much education participating in the labor force is higher by about 31 percent (as compared with women with less than 5 years of schooling, the base category).

Now consider the **interaction term** marital status and age. The table shows that the labor-force participation probability is higher by some 29 percent for those women who were never married (as compared with the base category) and smaller by about 28 percent for those women who are 65 and over (again in relation to the base category). But the probability of participation of women who were never married and are 65 or over is smaller by about 20 percent as compared with the base category. This implies that women aged 65 and over but never married are likely to participate in the labor force more than those who are aged 65 and over and are married or fall into the "other" category.

Following this procedure, the reader can easily interpret the rest of the coefficients given in Table 15.3. From the given information, it is easy to obtain the estimates of the conditional probabilities of labor-force participation of the various categories. Thus, if we want to find the probability for married women (other), aged 22 to 54, with 12 to 15 years of schooling, with an unemployment rate of 2.5 to 3.4 percent, employment change of 3.5 to 6.49 percent, relative employment opportunities of 74 percent and over, and with FILOW of \$7500 and over, we obtain

$$0.4368 + 0.1523 + 0.2231 - 0.0213 + 0.0301 + 0.0571 - 0.2455 = 0.6326$$

In other words, the probability of labor-force participation by women with the preceding characteristics is estimated to be about 63 percent.

(Continued)

<sup>9</sup>Malcolm S. Cohen, Samuel A. Rea, Jr., and Robert I. Lerman, *A Micro Model of Labor Supply*, BLS Staff Paper 4, U.S. Department of Labor, 1970.

**EXAMPLE 15.1** (Continued)

**TABLE 15.3** LABOR-FORCE PARTICIPATION

Regression of women, age 22 and over, living in largest 96 standard metropolitan statistical areas (SMSA) (dependent variable: in or out of labor force during 1966)

Explanatory variable	Coefficient	<i>t</i> ratio
Constant	0.4368	15.4
Marital status		
Married, spouse present	—	—
Married, other	0.1523	13.8
Never married	0.2915	22.0
Age		
22–54	—	—
55–64	–0.0594	–5.7
65 and over	–0.2753	–9.0
Years of schooling		
0–4	—	—
5–8	0.1255	5.8
9–11	0.1704	7.9
12–15	0.2231	10.6
16 and over	0.3061	13.3
Unemployment rate (1966), %		
Under 2.5	—	—
2.5–3.4	–0.0213	–1.6
3.5–4.0	–0.0269	–2.0
4.1–5.0	–0.0291	–2.2
5.1 and over	–0.0311	–2.4
Employment change (1965–1966), %		
Under 3.5	—	—
3.5–6.49	0.0301	3.2
6.5 and over	0.0529	5.1
Relative employment opportunities, %		
Under 62	—	—
62–73.9	0.0381	3.2
74 and over	0.0571	3.2
FILOW, \$		
Less than 1,500 and negative	—	—
1,500–7,499	–0.1451	–15.4
7,500 and over	–0.2455	–24.4
Interaction (marital status and age)		
Marital status      Age		
Other            55–64	–0.0406	–2.1
Other            65 and over	–0.1391	–7.4
Never married    55–64	–0.1104	–3.3
Never married    65 and over	–0.2045	–6.4
Interaction (age and years of schooling completed)		
Age                      Years of schooling		
65 and over      5–8	–0.0885	–2.8
65 and over      9–11	–0.0848	–2.4
65 and over      12–15	–0.1288	–4.0
65 and over      16 and over	–0.1628	–3.6

$R^2 = 0.175$

No. of observations = 25,153

Note: — indicates the base or omitted category.

FILOW: family income less own wage and salary income.

Source: Malcolm S. Cohen, Samuel A. Rea, Jr., and Robert I. Lerman, *A Micro Model of Labor Supply*, BLS Staff Paper 4, U.S. Department of Labor, 1970, Table F-6, pp. 212–213.

**EXAMPLE 15.2**

**PREDICTING A BOND RATING**

Based on a pooled time series and cross-sectional data of 200 Aa (high-quality) and Baa (medium-quality) bonds over the period 1961–1966, Joseph Cappelleri estimated the following bond rating prediction model.<sup>10</sup>

$$Y_i = \beta_1 + \beta_2 X_{2i}^2 + \beta_3 X_{3i} + \beta_4 X_{4i} + \beta_5 X_{5i} + u_i$$

where  $Y_i = 1$  if the bond rating is Aa (Moody's rating)  
 $= 0$  if the bond rating is Baa (Moody's rating)  
 $X_2 =$  debt capitalization ratio, a measure of leverage  
 $= \frac{\text{dollar value of long-term debt}}{\text{dollar value of total capitalization}} \cdot 100$

$X_3 =$  profit rate  
 $= \frac{\text{dollar value of after-tax income}}{\text{dollar value of net total assets}} \cdot 100$

$X_4 =$  standard deviation of the profit rate, a measure of profit rate variability  
 $X_5 =$  net total assets (thousands of dollars), a measure of size

A priori,  $\beta_2$  and  $\beta_4$  are expected to be negative (why?) and  $\beta_3$  and  $\beta_5$  are expected to be positive.

After correcting for heteroscedasticity and first-order autocorrelation, Cappelleri obtained the following results<sup>11</sup>:

$$\hat{Y}_i = 0.6860 - 0.0179X_{2i}^2 + 0.0486X_{3i} + 0.0572X_{4i} + 0.378(E-7)X_5$$

(0.1775) (0.0024) (0.0486) (0.0178) (0.039)(E-8) **(15.3.1)**

$R^2 = 0.6933$

*Note:* 0.378 E-7 means 0.000000378, etc.

All but the coefficient of  $X_4$  have the correct signs. It is left to finance students to rationalize why the profit rate variability coefficient has a positive sign, for one would expect that the greater the variability in profits, the less likely it is Moody's would give an Aa rating, other things remaining the same.

The interpretation of the regression is straightforward. For example, 0.0486 attached to  $X_3$  means that, other things being the same, a 1 percentage point increase in the profit rate will lead on average to about a 0.05 increase in the probability of a bond getting the Aa rating. Similarly, the higher the squared leveraged ratio, the lower by 0.02 is the probability of a bond being classified as an Aa bond per unit increase in this ratio.

<sup>10</sup>Joseph Cappelleri, "Predicting a Bond Rating," unpublished term paper, C.U.N.Y. The model used in the paper is a modification of the model used by Thomas F. Pogue and Robert M. Soldofsky, "What Is in a Bond Rating?" *Journal of Financial and Quantitative Analysis*, June 1969, pp. 201–228.

<sup>11</sup>Some of the estimated probabilities before correcting for heteroscedasticity were negative and some were in excess of 1; in these cases they were assumed to be 0.01 and 0.99, respectively, to facilitate the computation of the weights  $w_i$ .

**EXAMPLE 15.3****PREDICTING BOND DEFAULTS**

To predict the probability of default on their bond obligations, Daniel Rubinfeld studied a sample of 35 municipalities in Massachusetts for the year 1930, several of which did in fact default. The LPM model he chose and estimated was as follows<sup>12</sup>:

$$\hat{P} = 1.96 - 0.029 \text{ TAX} - 4.86 \text{ INT} + 0.063 \text{ AV} + 0.007 \text{ DAV} - 0.48 \text{ WELF} \quad (15.3.2)$$

(0.29) (0.009) (2.13) (0.028) (0.003) (0.88)

$R^2 = 0.36$

Where  $P = 0$  if the municipality defaulted and 1 otherwise, TAX = average of 1929, 1930, and 1931 tax rates; INT = percentage of current budget allocated to interest payments in 1930; AV = percentage growth in assessed property valuation from 1925 to 1930; DAV = ratio of total direct net debt to total assessed valuation in 1930; and WELF = percentage of 1930 budget allocated to charities, pensions, and soldiers' benefits.

The interpretation (15.3.2) is again fairly straightforward. Thus, other things being the same, an increase in the tax rate of \$1 per thousand will raise the probability of default by about 0.03, or 3 percent. The  $R^2$  value is rather low but, as noted previously, in LPMs the  $R^2$  values generally tend to be lower and are of limited use in judging the goodness of fit of the model.

**15.4 ALTERNATIVES TO LPM**

As we have seen, the LPM is plagued by several problems, such as (1) non-normality of  $u_i$ , (2) heteroscedasticity of  $u_i$ , (3) possibility of  $\hat{Y}_i$  lying outside the 0–1 range, and (4) the generally lower  $R^2$  values. But these problems are surmountable. For example, we can use WLS to resolve the heteroscedasticity problem or increase the sample size to minimize the non-normality problem. By resorting to restricted least-squares or mathematical programming techniques we can even make the estimated probabilities lie in the 0–1 interval.

But even then the fundamental problem with the LPM is that it is not logically a very attractive model because it assumes that  $P_i = E(Y = 1 | X)$  increases linearly with  $X$ , that is, the marginal or incremental effect of  $X$  remains constant throughout. Thus, in our home ownership example we found that as  $X$  increases by a unit (\$1000), the probability of owning a house increases by the same constant amount of 0.10. This is so whether the income level is \$8000, \$10,000, \$18,000, or \$22,000. This seems patently unrealistic. In reality one would expect that  $P_i$  is nonlinearly related to  $X_i$ :

<sup>12</sup>D. Rubinfeld, "An Econometric Analysis of the Market for General Municipal Bonds," unpublished doctoral dissertation, Massachusetts Institute of Technology, 1972. The results given in this example are reproduced from Robert S. Pindyck and Daniel L. Rubinfeld, *Econometric Models and Economic Forecasts*, 2d ed., McGraw-Hill, New York, 1981, p. 279.



At very low income a family will not own a house but at a sufficiently high level of income, say,  $X^*$ , it most likely will own a house. Any increase in income beyond  $X^*$  will have little effect on the probability of owning a house. Thus, at both ends of the income distribution, the probability of owning a house will be virtually unaffected by a small increase in  $X$ .

Therefore, what we need is a (probability) model that has these two features: (1) As  $X_i$  increases,  $P_i = E(Y = 1 | X)$  increases but never steps outside the 0–1 interval, and (2) the relationship between  $P_i$  and  $X_i$  is nonlinear, that is, “one which approaches zero at slower and slower rates as  $X_i$  gets small and approaches one at slower and slower rates as  $X_i$  gets very large.”<sup>13</sup>

Geometrically, the model we want would look something like Figure 15.2. Notice in this model that the probability lies between 0 and 1 and that it varies nonlinearly with  $X$ .

The reader will realize that the sigmoid, or S-shaped, curve in the figure very much resembles the **cumulative distribution function** (CDF) of a random variable.<sup>14</sup> Therefore, one can easily use the CDF to model regressions where the response variable is dichotomous, taking 0–1 values. The practical question now is, which CDF? For although all CDFs are S shaped, for each random variable there is a unique CDF. For historical as well as practical reasons, the CDFs commonly chosen to represent the 0–1 response

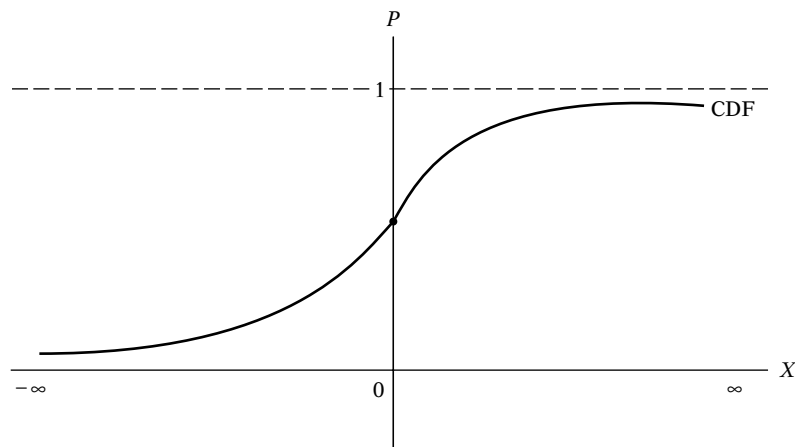


FIGURE 15.2 A cumulative distribution function (CDF).

<sup>13</sup>John Aldrich and Forrest Nelson, op. cit., p. 26.

<sup>14</sup>As discussed in **App. A**, the CDF of a random variable  $X$  is simply the probability that it takes a value less than or equal to  $x_0$ , where  $x_0$  is some specified numerical value of  $X$ . In short,  $F(X)$ , the CDF of  $X$ , is  $F(X = x_0) = P(X \leq x_0)$ .

models are (1) the logistic and (2) the normal, the former giving rise to the **logit** model and the latter to the **probit** (or **normit**) model.

Although a detailed discussion of the logit and probit models is beyond the scope of this book, we will indicate somewhat informally how one estimates such models and how one interprets them.

### 15.5 THE LOGIT MODEL

We will continue with our home ownership example to explain the basic ideas underlying the logit model. Recall that in explaining home ownership in relation to income, the LPM was

$$P_i = E(Y = 1 | X_i) = \beta_1 + \beta_2 X_i \quad (15.5.1)$$

where  $X$  is income and  $Y = 1$  means the family owns a house. But now consider the following representation of home ownership:

$$P_i = E(Y = 1 | X_i) = \frac{1}{1 + e^{-(\beta_1 + \beta_2 X_i)}} \quad (15.5.2)$$

For ease of exposition, we write (15.5.2) as

$$P_i = \frac{1}{1 + e^{-Z_i}} = \frac{e^z}{1 + e^z} \quad (15.5.3)$$

where  $Z_i = \beta_1 + \beta_2 X_i$ .

Equation (15.5.3) represents what is known as the (cumulative) **logistic distribution function**.<sup>15</sup>

It is easy to verify that as  $Z_i$  ranges from  $-\infty$  to  $+\infty$ ,  $P_i$  ranges between 0 and 1 and that  $P_i$  is nonlinearly related to  $Z_i$  (i.e.,  $X_i$ ), thus satisfying the two requirements considered earlier.<sup>16</sup> But it seems that in satisfying these requirements, we have created an estimation problem because  $P_i$  is nonlinear not only in  $X$  but also in the  $\beta$ 's as can be seen clearly from (15.5.2). This means that we cannot use the familiar OLS procedure to estimate the parameters.<sup>17</sup> But this problem is more apparent than real because (15.5.2) can be linearized, which can be shown as follows.

<sup>15</sup>The logistic model has been used extensively in analyzing growth phenomena, such as population, GNP, money supply, etc. For theoretical and practical details of logit and probit models, see J. S. Kramer, *The Logit Model for Economists*, Edward Arnold Publishers, London, 1991; and G. S. Maddala, *op. cit.*

<sup>16</sup>Note that as  $Z_i \rightarrow +\infty$ ,  $e^{-Z_i}$  tends to zero and as  $Z_i \rightarrow -\infty$ ,  $e^{-Z_i}$  increases indefinitely. Recall that  $e = 2.71828$ .

<sup>17</sup>Of course, one could use nonlinear estimation techniques discussed in Chap. 14. See also Sec. 15.8.

If  $P_i$ , the probability of owning a house, is given by (15.5.3), then  $(1 - P_i)$ , the probability of not owning a house, is

$$1 - P_i = \frac{1}{1 + e^{Z_i}} \quad (15.5.4)$$

Therefore, we can write

$$\frac{P_i}{1 - P_i} = \frac{1 + e^{Z_i}}{1 + e^{-Z_i}} = e^{Z_i} \quad (15.5.5)$$

Now  $P_i/(1 - P_i)$  is simply the **odds ratio** in favor of owning a house—the ratio of the probability that a family will own a house to the probability that it will not own a house. Thus, if  $P_i = 0.8$ , it means that odds are 4 to 1 in favor of the family owning a house.

Now if we take the natural log of (15.5.5), we obtain a very interesting result, namely,

$$\begin{aligned} L_i &= \ln\left(\frac{P_i}{1 - P_i}\right) = Z_i \\ &= \beta_1 + \beta_2 X_i \end{aligned} \quad (15.5.6)$$

that is,  $L$ , the log of the odds ratio, is not only linear in  $X$ , but also (from the estimation viewpoint) linear in the parameters.<sup>18</sup>  $L$  is called the **logit**, and hence the name **logit model** for models like (15.5.6).

Notice these features of the logit model.

1. As  $P$  goes from 0 to 1 (i.e., as  $Z$  varies from  $-\infty$  to  $+\infty$ ), the logit  $L$  goes from  $-\infty$  to  $+\infty$ . That is, although the probabilities (of necessity) lie between 0 and 1, the logits are not so bounded.

2. Although  $L$  is linear in  $X$ , the probabilities themselves are not. This property is in contrast with the LPM model (15.5.1) where the probabilities increase linearly with  $X$ .<sup>19</sup>

3. Although we have included only a single  $X$  variable, or regressor, in the preceding model, one can add as many regressors as may be dictated by the underlying theory.

4. If  $L$ , the logit, is positive, it means that when the value of the regressor(s) increases, the odds that the regressand equals 1 (meaning some event of interest happens) increases. If  $L$  is negative, the odds that the regressand equals 1 decreases as the value of  $X$  increases. To put it differently, the logit

<sup>18</sup>Recall that the linearity assumption of OLS does not require that the  $X$  variable be necessarily linear. So we can have  $X^2$ ,  $X^3$ , etc., as regressors in the model. For our purpose, it is linearity in the parameters that is crucial.

<sup>19</sup>Using calculus, it can be shown that  $dP/dX = \beta_2 P(1 - P)$ , which shows that the rate of change in probability with respect to  $X$  involves not only  $\beta_2$  but also the level of probability from which the change is measured (but more on this in Sec. 15.7). In passing, note that the effect of a unit change in  $X_i$  on  $P$  is greatest when  $P = 0.5$  and least when  $P$  is close to 0 or 1.

becomes negative and increasingly large in magnitude as the odds ratio decreases from 1 to 0 and becomes increasingly large and positive as the odds ratio increases from 1 to infinity.<sup>20</sup>

5. More formally, the interpretation of the logit model given in (15.5.6) is as follows:  $\beta_2$ , the slope, measures the change in  $L$  for a unit change in  $X$ , that is, it tells how the log-odds in favor of owning a house change as income changes by a unit, say, \$1000. The intercept  $\beta_1$  is the value of the log-odds in favor of owning a house if income is zero. Like most interpretations of intercepts, this interpretation may not have any physical meaning.

6. Given a certain level of income, say,  $X^*$ , if we actually want to estimate not the odds in favor of owning a house but the probability of owning a house itself, this can be done directly from (15.5.3) once the estimates of  $\beta_1 + \beta_2$  are available. This, however, raises the most important question: How do we estimate  $\beta_1$  and  $\beta_2$  in the first place? The answer is given in the next section.

7. Whereas the LPM assumes that  $P_i$  is linearly related to  $X_i$ , the logit model assumes that the log of the odds ratio is linearly related to  $X_i$ .

## 15.6 ESTIMATION OF THE LOGIT MODEL

For estimation purposes, we write (15.5.6) as follows:

$$L_i = \ln \left( \frac{P_i}{1 - P_i} \right) = \beta_1 + \beta_2 X_i + u_i \quad (15.6.1)$$

We will discuss the properties of the stochastic error term  $u_i$  shortly.

To estimate (15.6.1), we need, apart from  $X_i$ , the values of the regressand, or logit,  $L_i$ . This depends on the type of data we have for analysis. We distinguish two types of data: (1) *data at the individual, or micro, level*, and (2) *grouped or replicated data*.

### Data at the Individual Level

If we have data on individual families, as in the case of Table 15.1, OLS estimation of (15.6.1) is infeasible. This is easy to see. In terms of the data given in Table 15.1,  $P_i = 1$  if a family owns a house and  $P_i = 0$  if it does not own a house. But if we put these values directly into the logit  $L_i$ , we obtain:

$$L_i = \ln \left( \frac{1}{0} \right) \quad \text{if a family own a house}$$

$$L_i = \ln \left( \frac{0}{1} \right) \quad \text{if a family does not own a house}$$

Obviously, these expressions are meaningless. Therefore, if we have data at the micro, or individual, level, we cannot estimate (15.6.1) by the standard

<sup>20</sup>This point is due to David Garson.

OLS routine. In this situation we may have to resort to the **maximum-likelihood (ML)** method to estimate the parameters. Although the rudiments of this method were discussed in the appendix to Chapter 4, its application in the present context will be discussed in Appendix 15A, Section 15A.1, for the benefit of readers who would like to learn more about it.<sup>21</sup> Software packages, such as Microfit, Eviews, Limdep, Shazam, PcGive, and Minitab, have built-in routines to estimate the logit model at the individual level. We will illustrate the use of the ML method later in the chapter.

### Grouped or Replicated Data

Now consider the data given in Table 15.4. This table gives data on several families *grouped* or *replicated* (repeat observations) according to income level and the number of families owning a house at each income level. Corresponding to each income level  $X_i$ , there are  $N_i$  families,  $n_i$  among whom are home owners ( $n_i \leq N_i$ ). Therefore, if we compute

$$\hat{P}_i = \frac{n_i}{N_i} \tag{15.6.2}$$

that is, the *relative frequency*, we can use it as an estimate of the true  $P_i$  corresponding to each  $X_i$ . If  $N_i$  is fairly large,  $\hat{P}_i$  will be a reasonably good estimate of  $P_i$ .<sup>22</sup> Using the estimated  $P_i$ , we can obtain the estimated logit as

$$\hat{L}_i = \ln \left( \frac{\hat{P}_i}{1 - \hat{P}_i} \right) = \hat{\beta}_1 + \hat{\beta}_2 X_i \tag{15.6.3}$$

**TABLE 15.4** HYPOTHETICAL DATA ON  $X_i$  (INCOME),  $N_i$  (NUMBER OF FAMILIES AT INCOME  $X_i$ ), AND  $n_i$  (NUMBER OF FAMILIES OWNING A HOUSE)

$X$ (thousands of dollars)	$N_i$	$n_i$
6	40	8
8	50	12
10	60	18
13	80	28
15	100	45
20	70	36
25	65	39
30	50	33
35	40	30
40	25	20

<sup>21</sup>For a comparatively simple discussion of maximum likelihood in the context of the logit model, see John Aldrich and Forrest Nelson, *op. cit.*, pp. 49–54. See also, Alfred Demarsi, *Logit Modeling: Practical Applications*, Sage Publications, Newbury Park, Calif., 1992.

<sup>22</sup>From elementary statistics recall that the probability of an event is the limit of the relative frequency as the sample size becomes infinitely large.

which will be a fairly good estimate of the true logit  $L_i$  if the number of observations  $N_i$  at each  $X_i$  is reasonably large.

In short, given the *grouped* or *replicated* data, such as Table 15.4, one can obtain the data on the dependent variable, the logits, to estimate the model (15.6.1). Can we then apply OLS to (15.6.3) and estimate the parameters in the usual fashion? The answer is, not quite, since we have not yet said anything about the properties of the stochastic disturbance term. It can be shown that if  $N_i$  is fairly large and if each observation in a given income class  $X_i$  is distributed independently as a binomial variable, then

$$u_i \sim N \left[ 0, \frac{1}{N_i P_i (1 - P_i)} \right] \quad (15.6.4)$$

that is  $u_i$  follows the normal distribution with zero mean and variance equal to  $1/[N_i P_i (1 - P_i)]$ .<sup>23</sup>

Therefore, as in the case of the LPM, the disturbance term in the logit model is heteroscedastic. Thus, instead of using OLS we will have to use the weighted least squares (WLS). For empirical purposes, however, we will replace the unknown  $P_i$  by  $\hat{P}_i$  and use

$$\hat{\sigma}^2 = \frac{1}{N_i \hat{P}_i (1 - \hat{P}_i)} \quad (15.6.5)$$

as estimator of  $\sigma^2$ .

We now describe the various steps in estimating the logit regression (15.6.1):

1. For each income level  $X$ , compute the probability of owning a house as  $\hat{P}_i = n_i/N_i$ .
2. For each  $X_i$ , obtain the logit as<sup>24</sup>

$$\hat{L}_i = \ln[\hat{P}_i/(1 - \hat{P}_i)]$$

3. To resolve the problem of heteroscedasticity, transform (15.6.1) as follows<sup>25</sup>:

$$\sqrt{w_i} L_i = \beta_1 \sqrt{w_i} + \beta_2 \sqrt{w_i} X_i + \sqrt{w_i} u_i \quad (15.6.6)$$

<sup>23</sup>As shown in elementary probability theory,  $\hat{P}_i$ , the proportion of successes (here, owning a house), follows the binomial distribution with mean equal to true  $P_i$  and variance equal to  $P_i(1 - P_i)/N_i$ ; and as  $N_i$  increases indefinitely the binomial distribution approximates the normal distribution. The distributional properties of  $u_i$  given in (15.6.4) follow from this basic theory. For details, see Henry Theil, "On the Relationships Involving Qualitative Variables," *American Journal of Sociology*, vol. 76, July 1970, pp. 103–154.

<sup>24</sup>Since  $\hat{P}_i = n_i/N_i$ ,  $L_i$  can be alternatively expressed as  $\hat{L}_i = \ln n_i/(N_i - n_i)$ . In passing it should be noted that to avoid  $\hat{P}_i$  taking the value of 0 or 1, in practice  $\hat{L}_i$  is measured as  $\hat{L}_i = \ln(n_i + \frac{1}{2})/(N_i - n_i + \frac{1}{2}) = \ln(\hat{P}_i + 1/2N_i)/(1 - \hat{P}_i + 1/2N_i)$ . It is recommended as a rule of thumb that  $N_i$  be at least 5 at each value of  $X_i$ . For additional details, see D. R. Cox, *Analysis of Binary Data*, Methuen, London, 1970, p. 33.

<sup>25</sup>If we estimate (15.6.1) disregarding heteroscedasticity, the estimators, although unbiased, will not be efficient, as we know from Chap. 11.

which we write as

$$L_i^* = \beta_1 \sqrt{w_i} + \beta_2 X_i^* + v_i \quad (15.6.7)$$

where the weights  $w_i = N_i \hat{P}_i(1 - \hat{P}_i)$ ;  $L_i^*$  = transformed or weighted  $L_i$ ;  $X_i^*$  = transformed or weighted  $X_i$ ; and  $v_i$  = transformed error term. It is easy to verify that the transformed error term  $v_i$  is homoscedastic, keeping in mind that the original error variance is  $\sigma_u^2 = 1/[N_i P_i(1 - P_i)]$ .

4. Estimate (15.6.6) by OLS—recall that WLS is OLS on the transformed data. Notice that in (15.6.6) there is no intercept term introduced explicitly (why?). Therefore, one will have to use the regression through the origin routine to estimate (15.6.6).

5. Establish confidence intervals and/or test hypotheses in the usual OLS framework, *but keep in mind that all the conclusions will be valid strictly speaking if the sample is reasonably large* (why?). Therefore, in small samples, the estimated results should be interpreted carefully.

### 15.7 THE GROUPED LOGIT (GLOGIT) MODEL: A NUMERICAL EXAMPLE

To illustrate the theory just discussed, we will use the data given in Table 15.4. Since the data in the table are grouped, the logit model based on this data will be called a grouped logit model, glogit, for short. The necessary raw data and other relevant calculations necessary to implement glogit are given in Table 15.5. The results of the weighted least-squares regression (15.6.7) based on the data given in Table 15.5 are as follows: Note that there is no intercept in (15.6.7); hence the regression-through-origin procedure is appropriate here.

$$\begin{aligned} \hat{L}_i^* &= -1.59474\sqrt{w_i} + 0.07862X_i^* \\ \text{se} &= (0.11046) \quad (0.00539) \\ t &= (-14.43619) \quad (14.56675) \quad R^2 = 0.9642 \end{aligned} \quad (15.7.1)$$

The  $R^2$  is the squared correlation coefficient between actual and estimated  $L_i^*$ .  $L_i^*$  and  $X_i^*$  are weighted  $L_i$  and  $X_i$ , as shown in (15.6.6).

#### Interpretation of the Estimated Logit Model

How do we interpret (15.7.1)? There are various ways, some intuitive and some not:

**Logit Interpretation.** As (15.7.1) shows, the estimated slope coefficient suggests that for a unit (\$1000) increase in weighted income, the weighted log of the odds in favor of owning a house goes up by 0.08 units. This mechanical interpretation, however, is not very appealing.

TABLE 15.5 DATA TO ESTIMATE THE LOGIT MODEL OF OWNERSHIP

X (thousands of dollars) (1)	$N_i$ (2)	$n_i$ (3)	$\hat{P}_i$ (4) = (3) ÷ (2)	$1 - \hat{P}_i$ (5)	$\frac{\hat{P}_i}{1 - \hat{P}_i}$ (6)	$\hat{L}_i = \ln\left(\frac{\hat{P}_i}{1 - \hat{P}_i}\right)$ (7)	$N_i \hat{P}_i (1 - \hat{P}_i)$ = $w_i$ (8)	$\sqrt{w_i} =$ $\sqrt{N_i \hat{P}_i (1 - \hat{P}_i)}$ (9) = $\sqrt{(8)}$	$\hat{L}_i^* =$ $\hat{L}_i / \sqrt{w_i}$ (10) = (7)/(9)	$\hat{X}_i^* =$ $X_i / \sqrt{w_i}$ (11) = (1)/(9)
6	40	8	0.20	0.80	0.25	-1.3863	6.40	2.5298	-3.5071	15.1788
8	50	12	0.24	0.76	0.32	-1.1526	9.12	3.0199	-3.4807	24.1592
10	60	18	0.30	0.70	0.43	-0.8472	12.60	3.5496	-3.0072	35.4960
13	80	28	0.35	0.65	0.54	-0.6190	18.20	4.2661	-2.6407	55.4593
15	100	45	0.45	0.55	0.82	-0.2007	24.75	4.9749	-0.9985	74.6235
20	70	36	0.51	0.49	1.04	0.0400	17.49	4.1825	0.1673	83.6506
25	65	39	0.60	0.40	1.50	0.4054	15.60	3.9497	1.6012	98.7425
30	50	33	0.66	0.34	1.94	0.6633	11.20	3.3496	2.2218	100.4880
35	40	30	0.75	0.25	3.0	1.0986	7.50	2.7386	3.0086	95.8405
40	25	20	0.80	0.20	4.0	1.3863	4.00	2.000	2.7726	80.0000



**Odds Interpretation.** Remember that  $L_i = \ln [P_i/(1 - P_i)]$ . Therefore, taking the antilog of the estimated logit, we get  $P_i/(1 - P_i)$ , that is, the odds ratio. Hence, taking the antilog of (15.7.1), we obtain:

$$\begin{aligned} \frac{\hat{P}_i}{1 - \hat{P}_i} &= e^{-1.59474\sqrt{w_i} + 0.07862X_i^*} \\ &= e^{-1.59474\sqrt{w_i}} \cdot e^{0.07862X_i^*} \end{aligned} \tag{15.7.2}$$

Using a calculator, you can easily verify that  $e^{0.07862} = 1.0817$ . This means that for a unit increase in weighted income, the (weighted) odds in favor of owing a house increases by 1.0817 or about 8.17%. *In general, if you take the antilog of the  $j$ th slope coefficient (in case there is more than one regressor in the model), subtract 1 from it, and multiply the result by 100, you will get the percent change in the odds for a unit increase in the  $j$ th regressor.*

Incidentally, if you want to carry the analysis in terms of unweighted logit, all you have to do is to divide the estimated  $L_i^*$  by  $\sqrt{w_i}$ . Table 15.6 gives the estimated weighted and unweighted logits for each observation and some other data, which we will discuss shortly.

**Computing Probabilities.** Since the language of logit and odds ratio may be unfamiliar to some, we can always compute the probability of owning a house at a certain level of income. Suppose we want to compute this probability at  $X = 20$  (\$20,000). Plugging this value in (15.7.1), we obtain:  $\hat{L}_i^* = -0.09311$  and dividing this by  $\sqrt{w_i} = 4.2661$  (see Table 15.5), we obtain  $\hat{L}_i = -0.02226$ . Therefore, at the income level of \$20,000, we have

$$-0.02226 = \ln \left( \frac{\hat{P}_i}{1 - \hat{P}_i} \right)$$

**TABLE 15.6** LSTAR, XSTAR, ESTIMATED LSTAR, PROBABILITY, AND CHANGE IN PROBABILITY\*

Lstar	Xstar	ELstar	Logit	Probability, $\hat{P}$	Change in probability <sup>†</sup>
-3.50710	15.1788	-2.84096	-1.12299	0.24545	0.01456
-3.48070	24.15920	-2.91648	-0.96575	0.27572	0.01570
-3.48070	35.49600	-2.86988	-0.80850	0.30821	0.01676
-2.64070	55.45930	-2.44293	-0.57263	0.36063	0.01813
-0.99850	74.62350	-2.06652	-0.41538	0.39762	0.01883
0.16730	83.65060	-0.09311	-0.02226	0.49443	0.01965
1.60120	98.74250	1.46472	0.37984	0.59166	0.01899
2.22118	100.48800	2.55896	0.76396	0.68221	0.01704
3.00860	95.84050	3.16794	1.15677	0.76074	0.01431
2.77260	80.00000	3.10038	1.55019	0.82494	0.01135

\*Lstar and Xstar are from Table 15.5. ELstar is the estimated Lstar. Logit is the unweighted logit. Probability is the estimated probability of owning a house. Change in probability is the change per unit change in income.

<sup>†</sup>Computed from  $\beta_2 \hat{P}(1 - \hat{P}) = 0.07862 \hat{P}(1 - \hat{P})$ .

Therefore,

$$\frac{\hat{P}}{1 - \hat{P}_i} = e^{-0.02226} = 1.0225$$

Solving this for

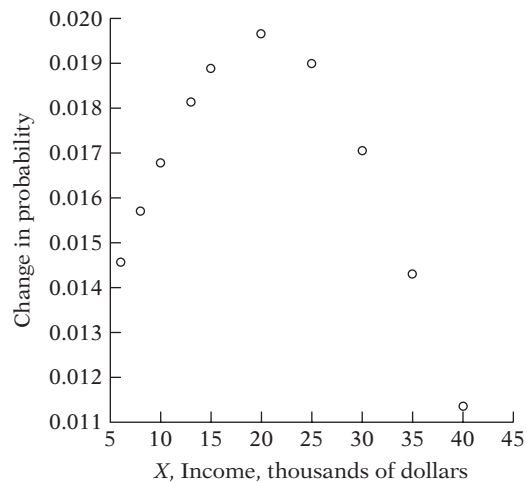
$$\hat{P}_i = \frac{e^{-0.02226}}{1 + e^{-0.02226}}$$

the reader can see that the estimated probability is 0.4944. That is, given the income of \$20,000, the probability of a family owning a house is about 49 percent. Table 15.6 shows the probabilities thus computed at various income levels. As this table shows, the probability of house ownership increases with income, but not linearly as with the LPM model.

**Computing the Rate of Change of Probability.** As you can gather from Table 15.6, the probability of owning a house depends on the income level. How can we compute the rate of change of probabilities as income varies? As noted in footnote 19, that depends not only on the estimated slope coefficient  $\beta_2$  but also on the level of the probability from which the change is measured; the latter of course depends on the income level at which the probability is computed.

To illustrate, suppose we want to measure the change in the probability of owning a house at the income level \$20,000. Then, from footnote 19 the change in probability for a unit increase in income from the level 20 (thousand) is:  $\hat{\beta}(1 - \hat{P})\hat{P} = 0.07862(0.5056)(0.4944) = 0.01965$ .

It is left as an exercise for the reader to show that at income level \$40,000, the change in probability is 0.01135. Table 15.6 shows the change in probability of owning a house at various income levels; these probabilities are also depicted in Figure 15.3.



**FIGURE 15.3** Change in probability in relation to income.

To conclude our discussion of the logit model, we present the results based on OLS, or unweighted regression, for the home ownership example:

$$\begin{aligned} \hat{L}_i &= -1.6587 + 0.0792X_i \\ \text{se} &= (0.0958) \quad (0.0041) \\ t &= (-17.32) \quad (19.11) \quad r^2 = 0.9786 \end{aligned} \tag{15.7.3}$$

We leave it to the reader to compare this regression with the weighted least-squares regression given by (15.7.1).

### 15.8 THE LOGIT MODEL FOR UNGROUPED OR INDIVIDUAL DATA

To set the stage, consider the data given in Table 15.7. Letting  $Y = 1$  if a student's final grade in an intermediate microeconomics course was A and  $Y = 0$  if the final grade was a B or a C, Spector and Mazzeo used grade point average (GPA), TUCE, and Personalized System of Instruction (PSI) as the

**TABLE 15.7** DATA ON THE EFFECT OF PERSONALIZED SYSTEM OF INSTRUCTION (PSI) ON COURSE GRADES

Observation	GPA grade	TUCE grade	PSI	Grade	Letter grade	Observation	GPA grade	TUCE grade	PSI	Grade	Letter grade
1	2.66	20	0	0	C	17	2.75	25	0	0	C
2	2.89	22	0	0	B	18	2.83	19	0	0	C
3	3.28	24	0	0	B	19	3.12	23	1	0	B
4	2.92	12	0	0	B	20	3.16	25	1	1	A
5	4.00	21	0	1	A	21	2.06	22	1	0	C
6	2.86	17	0	0	B	22	3.62	28	1	1	A
7	2.76	17	0	0	B	23	2.89	14	1	0	C
8	2.87	21	0	0	B	24	3.51	26	1	0	B
9	3.03	25	0	0	C	25	3.54	24	1	1	A
10	3.92	29	0	1	A	26	2.83	27	1	1	A
11	2.63	20	0	0	C	27	3.39	17	1	1	A
12	3.32	23	0	0	B	28	2.67	24	1	0	B
13	3.57	23	0	0	B	29	3.65	21	1	1	A
14	3.26	25	0	1	A	30	4.00	23	1	1	A
15	3.53	26	0	0	B	31	3.10	21	1	0	C
16	2.74	19	0	0	B	32	2.39	19	1	1	A

Notes: Grade  $Y = 1$  if the final grade is A  
 $= 0$  if the final grade is B or C

TUCE = score on an examination given at the beginning of the term to test entering knowledge of macroeconomics

PSI = 1 if the new teaching method is used  
 $= 0$  otherwise

GPA = the entering grade point average

Source: L. Spector and M. Mazzeo, "Probit Analysis and Economic Education," *Journal of Economic Education*, vol. 11, 1980, pp. 37-44.

grade predictors. The logit model here can be written as:

$$L_i = \left( \frac{P_i}{1 - P_i} \right) = \beta_1 + \beta_2 \text{GPA}_i + \beta_3 \text{TUCE}_i + \beta_4 \text{PSI}_i + u_i \quad (15.8.1)$$

As we noted in Section 15.6, we cannot simply put  $P_i = 1$  if a family owns a house, and zero if it does not own a house. Here neither OLS nor weighted least squares (WLS) is helpful. We have to resort to nonlinear estimating procedures using the method of maximum likelihood. The details of this method are given in Appendix 15A, Section 15A.1. Since most modern statistical packages have routines to estimate logit models on the basis of ungrouped data, we will present the results of model (15.8.1) using the data given in Table 15.7 and show how to interpret the results. The results are given in Table 15.8 in tabular form and are obtained by using Eviews 4. Before interpreting these results, some general observations are in order.

1. Since we are using the method of maximum likelihood, which is generally a large-sample method, the estimated standard errors are *asymptotic*.
2. As a result, instead of using the  $t$  statistic to evaluate the statistical significance of a coefficient, we use the (standard normal)  $Z$  statistic. So inferences are based on the normal table. Recall that if the sample size is reasonably large, the  $t$  distribution converges to the normal distribution.
3. As noted earlier, the conventional measure of goodness of fit,  $R^2$ , is not particularly meaningful in binary regressand models. Measures similar to  $R^2$ , called **pseudo  $R^2$** , are available, and there are a variety of them.<sup>26</sup> Eviews presents one such measure, the McFadden  $R^2$ , denoted by  $R^2_{\text{McF}}$ , whose

**TABLE 15.8** REGRESSION RESULTS OF (15.8.1)

Dependent Variable: Grade				
Method: ML-Binary Logit				
Convergence achieved after 5 iterations				
Variable	Coefficient	Std. error	Z statistic	Probability
C	-13.0213	4.931	-2.6405	0.0082
GPA	2.8261	1.2629	2.2377	0.0252
TUCE	0.0951	0.1415	0.67223	0.5014
PSI	2.3786	1.0645	2.2345	0.0255
McFadden $R^2 = 0.3740$		LR statistic (3 df) = 15.40419		

<sup>26</sup>For an accessible discussion, see J. Scott Long, *Regression Models for Categorical and Limited Dependent Variables*, Sage Publications, Newbury Park, California, 1997, pp. 102–113.

value in our example is 0.3740.<sup>27</sup> Like  $R^2$ ,  $R^2_{\text{MCF}}$  also ranges between 0 and 1. Another comparatively simple measure of goodness of fit is the **count  $R^2$** , which is defined as:

$$\text{Count } R^2 = \frac{\text{number of correct predictions}}{\text{total number of observations}} \quad (15.8.2)$$

Since the regressand in the logit model takes a value of 1 or zero, if the predicted probability is greater than 0.5, we classify that as 1, but if it is less than 0.5, we classify that as 0. We then count the number of correct predictions and compute the  $R^2$  as given in (15.7.2). We will illustrate this shortly.

It should be noted, however, that in binary regressand models, goodness of fit is of secondary importance. What matters is the expected signs of the regression coefficients and their statistical and/or practical significance.

4. To test the null hypothesis that all the slope coefficients are simultaneously equal to zero, the equivalent of the  $F$  test in the linear regression model is the **likelihood ratio (LR) statistic**. Given the null hypothesis, the LR statistic follows the  $\chi^2$  distribution with df equal to the number of explanatory variables, three in the present example. (*Note:* Exclude the intercept term in computing the df).

Now let us interpret the regression results given in (15.8.1). Each slope coefficient in this equation is a *partial slope* coefficient and measures the change in the estimated logit for a unit change in the value of the given regressor (holding other regressors constant). Thus, the GPA coefficient of 2.8261 means, with other variables held constant, that if GPA increases by a unit, on average the estimated logit increases by about 2.83 units, suggesting a positive relationship between the two. As you can see, all the other regressors have a positive effect on the logit, although statistically the effect of TUCE is not significant. However, together all the regressors have a significant impact on the final grade, as the LR statistic is 15.40, whose  $p$  value is about 0.0015, which is very small.

As noted previously, a more meaningful interpretation is in terms of odds, which are obtained by taking the antilog of the various slope coefficients. Thus, if you take the antilog of the PSI coefficient of 2.3786 you will get 10.7897 ( $\approx e^{2.3786}$ ). This suggests that students who are exposed to the new method of teaching are more than 10 times likely to get an A than students who are not exposed to it, other things remaining the same.

Suppose we want to compute the actual probability of a student getting an A grade. Consider student number 10 in Table 15.7. Putting the actual data for this student in the estimated logit model given in Table 15.8, the reader can check that the estimated logit value for this student is 0.8178.

<sup>27</sup>Technically, this is defined as:  $1 - (\text{LLF}_{\text{ur}}/\text{LLF}_r)$ , where  $\text{LLF}_{\text{ur}}$  is the unrestricted log likelihood function where all regressors are included in the model and  $\text{LLF}_r$  is the restricted log likelihood function where only the intercept is included in the model. Conceptually,  $\text{LLF}_{\text{ur}}$  is equivalent to RSS and  $\text{LLF}_r$  is the equivalent to TSS of the linear regression model.

Using Eq. (15.5.2), the reader can easily check that the estimated probability is 0.69351. Since this student's actual final grade was an A, and since our logit model assigns a probability of 1 to a student who gets an A, the estimated probability of 0.69351 is not exactly 1 but close to it.

Recall the count  $R^2$  defined earlier. Table 15.9 gives you the actual and predicted values of the regressand for our illustrative example. From this table you can observe that, out of 32 observations, there were 6 incorrect predictions (students 14, 19, 24, 26, 31, and 32). Hence the count  $R^2$  value is  $26/32 = 0.8125$ , whereas the McFadden  $R^2$  value is 0.3740. Although these two values are not directly comparable, they give you some idea about the orders of magnitude. Besides, one should not overplay the importance of goodness of fit in models where the regressand is dichotomous.

**TABLE 15.9** ACTUAL AND FITTED VALUES BASED ON REGRESSION IN TABLE 15.8

Observation	Actual	Fitted	Residual	Residual plot
1	0	0.02658	-0.02658	
2	0	0.05950	-0.05950	
3	0	0.18726	-0.18726	
4	0	0.02590	-0.02590	
5	1	0.56989	0.43011	
6	0	0.03486	-0.03486	
7	0	0.02650	-0.02650	
8	0	0.05156	-0.05156	
9	0	0.11113	-0.11113	
10	1	0.69351	0.30649	
11	0	0.02447	-0.02447	
12	0	0.19000	-0.19000	
13	0	0.32224	-0.32224	
*14	1	0.19321	0.80679	
15	0	0.36099	-0.36099	
16	0	0.03018	-0.03018	
17	0	0.05363	-0.05363	
18	0	0.03859	-0.03859	
*19	0	0.58987	-0.58987	
20	1	0.66079	0.33921	
21	0	0.06138	-0.06138	
22	1	0.90485	0.09515	
23	0	0.24177	-0.24177	
*24	0	0.85209	-0.85209	
25	1	0.83829	0.16171	
*26	1	0.48113	0.51887	
27	1	0.63542	0.36458	
28	0	0.30722	-0.30722	
29	1	0.84170	0.15830	
30	1	0.94534	0.05466	
*31	0	0.52912	-0.52912	
*32	1	0.11103	0.88897	

\*Incorrect predictions.

## 15.9 THE PROBIT MODEL

As we have noted, to explain the behavior of a dichotomous dependent variable we will have to use a suitably chosen CDF. The logit model uses the cumulative logistic function, as shown in (15.5.2). But this is not the only CDF that one can use. In some applications, the normal CDF has been found useful. The estimating model that emerges from the normal CDF<sup>28</sup> is popularly known as the **probit model**, although sometimes it is also known as the **normit model**. In principle one could substitute the normal CDF in place of the logistic CDF in (15.5.2) and proceed as in Section 16.5. Instead of following this route, we will present the probit model based on utility theory, or rational choice perspective on behavior, as developed by McFadden.<sup>29</sup>

To motivate the probit model, assume that in our home ownership example the decision of the  $i$ th family to own a house or not depends on an *unobservable utility index*  $I_i$  (also known as a **latent variable**), that is determined by one or more explanatory variables, say income  $X_i$ , in such a way that the larger the value of the index  $I_i$ , the greater the probability of a family owning a house. We express the index  $I_i$  as

$$I_i = \beta_1 + \beta_2 X_i \quad (15.9.1)$$

where  $X_i$  is the income of the  $i$ th family.

How is the (unobservable) index related to the actual decision to own a house? As before, let  $Y = 1$  if the family owns a house and  $Y = 0$  if it does not. Now it is reasonable to assume that there is a **critical** or **threshold level** of the index, call it  $I_i^*$ , such that if  $I_i$  exceeds  $I_i^*$ , the family will own a house, otherwise it will not. The threshold  $I_i^*$ , like  $I_i$ , is not observable, but if we assume that it is normally distributed with the same mean and variance, it is possible not only to estimate the parameters of the index given in (15.9.1) but also to get some information about the unobservable index itself. This calculation is as follows.

Given the assumption of normality, the probability that  $I_i^*$  is less than or equal to  $I_i$  can be computed from the standardized normal CDF as<sup>30</sup>:

$$P_i = P(Y = 1 | X) = P(I_i^* \leq I_i) = P(Z_i \leq \beta_1 + \beta_2 X_i) = F(\beta_1 + \beta_2 X_i) \quad (15.9.2)$$

<sup>28</sup>See **App. A** for a discussion of the normal CDF. Briefly, if a variable  $X$  follows the normal distribution with mean  $\mu$  and variance  $\sigma^2$ , its PDF is

$$f(X) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-(X-\mu)^2/2\sigma^2}$$

and its CDF is

$$F(X) = \int_{-\infty}^{X_0} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-(X-\mu)^2/2\sigma^2}$$

where  $X_0$  is some specified value of  $X$ .

<sup>29</sup>D. McFadden, "Conditional Logit Analysis of Qualitative Choice Behavior," in P. Zarembka (ed.), *Frontiers in Econometrics*, Academic Press, New York, 1973.

<sup>30</sup>A normal distribution with zero mean and unit (= 1) variance is known as a standard or standardized normal variable (see **App. A**).

where  $P(Y = 1 | X)$  means the probability that an event occurs given the value(s) of the  $X$ , or explanatory, variable(s) and where  $Z_i$  is the standard normal variable, i.e.,  $Z \sim N(0, \sigma^2)$ .  $F$  is the standard normal CDF, which written explicitly in the present context is:

$$\begin{aligned} F(I_i) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{I_i} e^{-z^2/2} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta_1 + \beta_2 X_i} e^{-z^2/2} dz \end{aligned} \tag{15.9.3}$$

Since  $P$  represents the probability that an event will occur, here the probability of owning a house, it is measured by the area of the standard normal curve from  $-\infty$  to  $I_i$  as shown in Figure 15.4a.

Now to obtain information on  $I_i$ , the utility index, as well as on  $\beta_1$  and  $\beta_2$ , we take the inverse of (15.9.2) to obtain:

$$\begin{aligned} I_i &= F^{-1}(I_i) = F^{-1}(P_i) \\ &= \beta_1 + \beta_2 X_i \end{aligned} \tag{15.9.4}$$

where  $F^{-1}$  is the inverse of the normal CDF. What all this means can be made clear from Figure 15.4. In panel *a* of this figure we obtain from the ordinate the (cumulative) probability of owning a house given  $I_i^* \leq I_i$ , whereas in panel *b* we obtain from the abscissa the value of  $I_i$  given the value of  $P_i$ , which is simply the reverse of the former.

But how do we actually go about obtaining the index  $I_i$  as well as estimating  $\beta_1$  and  $\beta_2$ ? As in the case of the logit model, the answer depends on whether we have grouped data or ungrouped data. We consider the two cases individually.

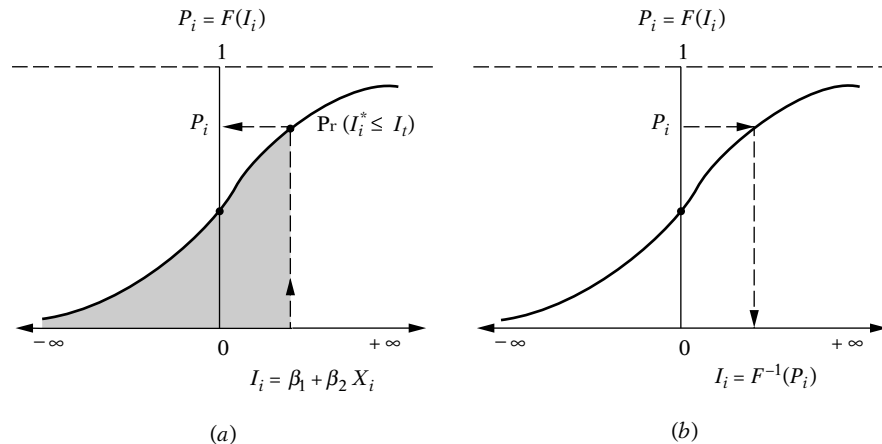


FIGURE 15.4 Probit model: (a) given  $I_i$ , read  $P_i$  from the ordinate; (b) given  $P_i$ , read  $I_i$  from the abscissa.



**Probit Estimation with Grouped Data: gprobit**

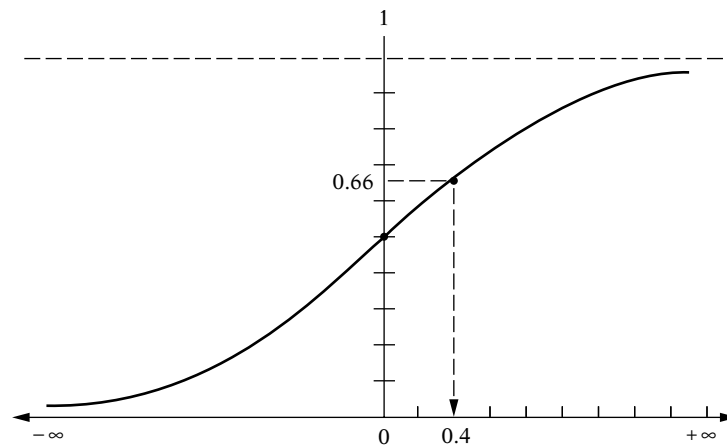
We will use the same data that we used for glogit, which is given in Table 15.4. Since we already have  $\hat{P}_i$ , the relative frequency (the empirical measure of probability) of owning a house at various income levels as shown in Table 15.5, we can use it to obtain  $I_i$  from the normal CDF as shown in Table 15.10, or from Figure 15.5.

Once we have the estimated  $I_i$ , estimating  $\beta_1$  and  $\beta_2$  is relatively straightforward, as we show shortly. In passing, note that in the language of probit analysis the unobservable utility index  $I_i$  is known as the **normal equivalent deviate** (n.e.d.) or simply **normit**. Since the n.e.d. or  $I_i$  will be negative whenever  $P_i < 0.5$ , in practice the number 5 is added to the n.e.d. and the result is called a probit.

**TABLE 15.10** ESTIMATING THE INDEX  $I_i$  FROM THE STANDARD NORMAL CDF

$\hat{P}_i$	$I_i = F^{-1}(\hat{P}_i)$
0.20	-0.8416
0.24	-0.7063
0.30	-0.5244
0.35	-0.3853
0.45	-0.1257
0.51	0.0251
0.60	0.2533
0.66	0.4125
0.75	0.6745
0.80	0.8416

Notes: (1)  $\hat{P}_i$  are from Table 15.5; (2)  $I_i$  are estimated from the standard normal CDF.



**FIGURE 15.5** Normal CDF.

ILLUSTRATION OF GPROBIT USING HOUSING EXAMPLE

Let us continue with our housing example. We have already presented the results of the logit model for this example. The grouped probit (gprobit) results of the same data are as follows:

Using the n.e.d. (=  $I$ ) given in Table 15.10, the regression results are as shown in Table 15.11.<sup>31</sup> The regression results based on the probits (= n.e.d. + 5) are as shown in Table 15.12.

Except for the intercept term, these results are identical with those given in the previous table. But this should not be surprising. (Why?)

**TABLE 15.11**

Dependent Variable:  $I$

Variable	Coefficient	Std. error	t statistic	Probability
C	-1.0166	0.0572	-17.7473	1.0397E-07
Income	0.04846	0.00247	19.5585	4.8547E-08
$R^2 = 0.97951$		Durbin-Watson statistic = 0.91384		

**TABLE 15.12**

Dependent Variable: Probit

Variable	Coefficient	Std. error	t statistic	Probability
C	3.9833	0.05728	69.5336	2.03737E-12
Income	0.04846	0.00247	19.5585	4.8547E-08
$R^2 = 0.9795$		Durbin-Watson statistic = 0.9138		

*Note:* These results are not corrected for heteroscedasticity (see exercise 15.12).

**Interpretation of the Probit Estimates in Table 15.11.** How do we interpret the preceding results? Suppose we want to find out the effect of a unit change in  $X$  (income measured in thousands of dollars) on the probability that  $Y = 1$ , that is, a family purchases a house. To do this, look at Eq. (15.9.2). We want to take the derivative of this function with respect to  $X$  (that is, the rate of change of the probability with respect to income). It turns out that this derivative is:

$$\frac{dP_i}{dX_i} = f(\beta_1 + \beta_2 X_i) \beta_2 \quad (15.9.5)^{32}$$

<sup>31</sup>The following results are not corrected for heteroscedasticity. See exercise 15.12 for the appropriate procedure to correct heteroscedasticity.

<sup>32</sup>We use the chain rule of derivatives:

$$\frac{dP_i}{dX_i} = \frac{dF(t)}{dt} \cdot \frac{dt}{dX}$$

where  $t = \beta_1 + \beta_2 X_i$ .

where  $f(\beta_1 + \beta_2 X_i)$  is the standard normal probability density function evaluated at  $\beta_1 + \beta_2 X_i$ . As you will realize, this evaluation will depend on the particular value of the  $X$  variables. Let us take a value of  $X$  from Table 15.5, say,  $X = 6$  (thousand dollars). Using the estimated values of the parameters given in Table 15.11, we thus want to find the normal density function at  $f[-1.0166 + 0.04846(6)] = f(-0.72548)$ . If you refer to the normal distribution tables, you will find that for  $Z = -0.72548$ , the normal density is about 0.3066.<sup>33</sup> Now multiplying this value by the estimated slope coefficient of 0.04846, we obtain 0.01485. This means that starting with an income level of \$6000, if the income goes up by \$1000, the probability of a family purchasing a house goes up by about 1.4 percent. (Compare this result with that given in Table 15.6.)

As you can see from the preceding discussion, compared with the LPM and logit models, the computation of changes in probability using the probit model is a bit tedious.

Instead of computing changes in probability, suppose you want to find the estimated probabilities from the fitted gprobit model. This can be done easily. Using the data in Table 15.11 and inserting the values of  $X$  from Table 15.5, the reader can check that the estimated n.i.d. values (to two digits) are as follows:

$X$	6	8	10	13	15	20	25	30	35	40
Estimated n.i.d.	-0.72	-0.63	-0.53	-0.39	-0.29	-0.05	0.19	0.43	0.68	0.92

Now statistical packages such as Minitab can easily compute the (cumulative) probabilities associated with the various n.i.d.'s. For example, corresponding to an n.i.d. value  $-0.63$ , the estimated probability is 0.2647 and, corresponding to an n.i.d. value of 0.43, the estimated probability is 0.6691. If you compare these estimates with the actual values given in Table 15.5, you will find that the two are fairly close, suggesting that the fitted model is quite good. Graphically, what we have just done is already shown in Figure 15.4.

### The Probit Model for Ungrouped or Individual Data

Let us revisit Table 15.7, which gives data on 32 individuals about their final grade in intermediate microeconomics examination in relation to the variables GPA, TUCE, and PSI. The results of the logit regression are given in Table 15.8. Let us see what the probit results look like. Notice that as in the case of the logit model for individual data, we will have to use a nonlinear estimating procedure based on the method of maximum likelihood. The regression results calculated by Eviews 4 are given in Table 15.13.

<sup>33</sup>Note that the standard normal  $Z$  can range from  $-\infty$  to  $+\infty$ , but the density function  $f(Z)$  is always positive.

**TABLE 15.13** Dependent Variable: grade  
Method: ML–binary probit  
Convergence achieved after 5 iterations

Variable	Coefficient	Std. error	Z statistic	Probability
C	-7.4523	2.5424	-2.9311	0.0033
GPA	1.6258	0.6938	2.3430	0.0191
TUCE	0.0517	0.0838	0.6166	0.5374
PSI	1.4263	5950	2.3970	0.0165
LR statistic (3 df) = 15.5458			McFadden $R^2$ = 0.3774	
Probability (LR stat) = 0.0014				

**TABLE 15.14** Dependent Variable: grade

Variable	Coefficient	Std. error	t statistic	Probability
C	-1.4980	0.5238	-2.8594	0.0079
GPA	0.4638	0.1619	2.8640	0.0078
TUCE	0.0104	0.0194	0.5386	0.5943
PSI	0.3785	0.1391	2.7200	0.0110
$R^2$ = 0.4159		Durbin-Watson $d$ = 2.3464		$F$ statistic = 6.6456

“Qualitatively,” the results of the probit model are comparable with those obtained from the logit model in that GPA and PSI are individually statistically significant. Collectively, all the coefficients are statistically significant, since the value of the LR statistic is 15.5458 with a  $p$  value of 0.0014. For reasons discussed in the next sections, we cannot directly compare the logit and probit regression coefficients.

For comparative purposes, we present the results based on the linear probability model (LPM) for the grade data in Table 15.14. Again, qualitatively, the LPM results are similar to the logit and probit models in that GPA and PSI are individually statistically significant but TUCE is not. Also, together the explanatory variables have a significant impact on grade, as the  $F$  value of 6.6456 is statistically significant because its  $p$  value is only 0.0015.

### The Marginal Effect of a Unit Change in the Value of a Regressor in the Various Regression Models

In the *linear regression model*, the slope coefficient measures the change in the average value of the regressand for a unit change in the value of a regressor, with all other variables held constant.

In the *LPM*, the slope coefficient measures directly the change in the probability of an event occurring as the result of a unit change in the value of a regressor, with the effect of all other variables held constant.

In the *logit model* the slope coefficient of a variable gives the change in the log of the odds associated with a unit change in that variable, again holding all other variables constant. But as noted previously, for the logit model the rate of change in the probability of an event happening is given by  $\beta_j P_i(1 - P_i)$ , where  $\beta_j$  is the (partial regression) coefficient of the  $j$ th regressor. But in evaluating  $P_i$ , all the variables included in the analysis are involved.

In the *probit model*, as we saw earlier, the rate of change in the probability is somewhat complicated and is given by  $\beta_j f(Z_i)$ , where  $f(Z_i)$  is the density function of the standard normal variable and  $Z_i = \beta_1 + \beta_2 X_{2i} + \dots + \beta_k X_{ki}$ , that is, the regression model used in the analysis.

Thus, in both the logit and probit models all the regressors are involved in computing the changes in probability, whereas in the LPM only the  $j$ th regressor is involved. This difference may be one reason for the early popularity of the LPM model.

### 15.10 LOGIT AND PROBIT MODELS

Although for our grade example LPM, logit, and probit give qualitatively similar results, we will confine our attention to logit and probit models because of the problems with the LPM noted earlier. Between logit and probit, which model is preferable? In most applications the models are quite similar, the main difference being that the logistic distribution has slightly fatter tails, which can be seen from Figure 15.6. That is to say, the conditional probability  $P_i$  approaches zero or one at a slower rate in logit than in probit. This can be seen more clearly from Table 15.15. Therefore, there is no compelling reason to choose one over the other. In practice many researchers choose the logit model because of its comparative mathematical simplicity.

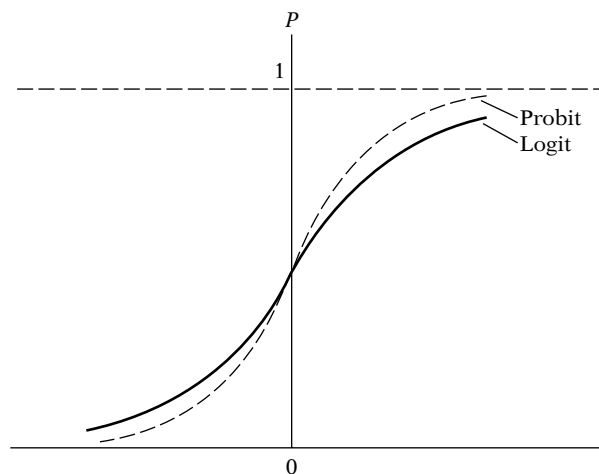


FIGURE 15.6 Logit and probit cumulative distributions.

TABLE 15.15 VALUES OF CUMULATIVE PROBABILITY FUNCTIONS

Z	Cumulative normal	Cumulative logistic
	$P_1(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^Z e^{-s^2/2} ds$	$P_2(Z) = \frac{1}{1 + e^{-Z}}$
-3.0	0.0013	0.0474
-2.0	0.0228	0.1192
-1.5	0.0668	0.1824
-1.0	0.1587	0.2689
-0.5	0.3085	0.3775
0	0.5000	0.5000
0.5	0.6915	0.6225
1.0	0.8413	0.7311
1.5	0.9332	0.8176
2.0	0.9772	0.8808
3.0	0.9987	0.9526

Though the models are similar, one has to be careful in interpreting the coefficients estimated by the two models. For example, for our grade example, the coefficient of GPA of 1.6258 of the probit model and 2.8261 of the logit model are not directly comparable. The reason is that, although the standard logistic (the basis of logit) and the standard normal distributions (the basis of probit) both have a mean value of zero, their variances are different; 1 for the standard normal (as we already know) and  $\pi^2/3$  for the logistic distribution, where  $\pi \approx 22/7$ . Therefore, if you multiply the probit coefficient by about 1.81 (which is approximately  $= \pi/\sqrt{3}$ ), you will get approximately the logit coefficient. For our example, the probit coefficient of GPA is 1.6258. Multiplying this by 1.81, we obtain 2.94, which is close to the logit coefficient. Alternatively, if you multiply a logit coefficient by 0.55 ( $= 1/1.81$ ), you will get the probit coefficient. Amemiya, however, suggests multiplying a logit estimate by 0.625 to get a better estimate of the corresponding probit estimate.<sup>34</sup> Conversely, multiplying a probit coefficient by 1.6 ( $= 1/0.625$ ) gives the corresponding logit coefficient.

Incidentally, Amemiya has also shown that the coefficients of LPM and logit models are related as follows:

$$\beta_{LPM} = 0.25\beta_{logit} \quad \text{except for intercept}$$

and

$$\beta_{LPM} = 0.25\beta_{logit} + 0.5 \quad \text{for intercept}$$

We leave it to the reader to find out if these approximations hold for our grade example.

<sup>34</sup>T. Amemiya, "Qualitative Response Model: A Survey," *Journal of Economic Literature*, vol. 19, 1981, pp. 481-536.

**15.11 THE TOBIT MODEL**

An extension of the probit model is the **tobit model** originally developed by James Tobin, the Nobel laureate economist. To explain this model, we continue with our home ownership example. In the probit model our concern was with estimating the probability of owning a house as a function of some socioeconomic variables. In the tobit model our interest is in finding out the amount of money a person or family spends on a house in relation to socioeconomic variables. Now we face a dilemma here: If a consumer does not purchase a house, obviously we have no data on housing expenditure for such consumers; we have such data only on consumers who actually purchase a house.

Thus consumers are divided into two groups, one consisting of, say,  $n_1$  consumers about whom we have information on the regressors (say, income, mortgage interest rate, number of people in the family, etc.) as well as the regressand (amount of expenditure on housing) and another consisting of  $n_2$  consumers about whom we have information only on the regressors but not on the regressand. A sample in which information on the regressand is available only for some observations is known as a **censored sample**.<sup>35</sup> Therefore, the tobit model is also known as a censored regression model. Some authors call such models **limited dependent variable regression models** because of the restriction put on the values taken by the regressand.

Statistically, we can express the tobit model as

$$\begin{aligned} Y_i &= \beta_1 + \beta_2 X_i + u_i && \text{if RHS} > 0 \\ &= 0 && \text{otherwise} \end{aligned} \quad (15.11.1)$$

where RHS = right-hand side. *Note:* Additional  $X$  variables can be easily added to the model.

Can we estimate regression (15.11.1) using only  $n_1$  observations and not worry about the remaining  $n_2$  observations? The answer is no, for the OLS estimates of the parameters obtained from the subset of  $n_1$  observations will be *biased as well as inconsistent*; that is, they are biased even asymptotically.<sup>36</sup>

To see this, consider Figure 15.7. As the figure shows, if  $Y$  is not observed (because of censoring), all such observations ( $= n_2$ ), denoted by crosses, will

<sup>35</sup>A censored sample should be distinguished from a **truncated sample** in which information on the regressors is available only if the regressand is observed. We will not pursue this topic here, but the interested reader may consult William H. Greene, *Econometric Analysis*, Prentice Hall, 4th ed., Englewood Cliffs, N.J., Chap. 19. For an intuitive discussion, see Peter Kennedy, *A Guide to Econometrics*, The MIT Press, Cambridge, Mass., 4th ed., 1998, Chap. 16.

<sup>36</sup>The bias arises from the fact that if we consider only the  $n_1$  observations and omit the others, there is no guarantee that  $E(u_i)$  will be necessarily zero. And without  $E(u_i) = 0$  we cannot guarantee that the OLS estimates will be unbiased. This bias can be readily seen from the discussion in App. 3A, Eqs. (4) and (5).

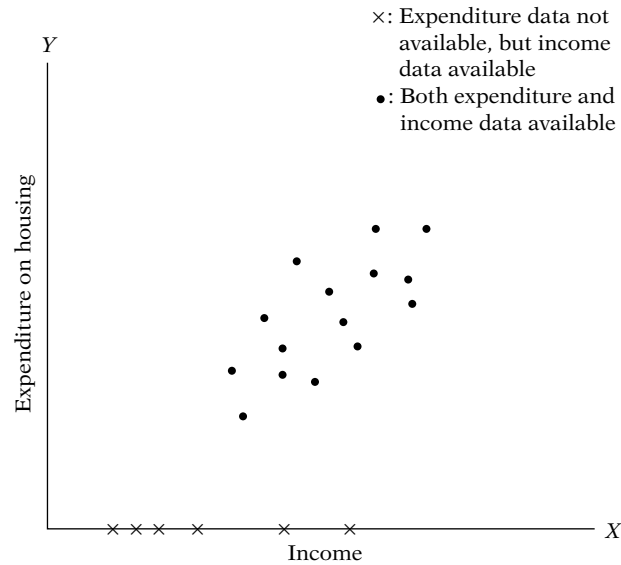


FIGURE 15.7 Plot of amount of money consumer spends in buying a house versus income.

lie on the horizontal axis. If  $Y$  is observed, the observations ( $= n_1$ ), denoted by dots, will lie in the  $X$ - $Y$  plane. It is intuitively clear that if we estimate a regression line based on the  $n_1$  observations only, the resulting intercept and slope coefficients are bound to be different than if all the  $(n_1 + n_2)$  observations were taken into account.

How then does one estimate tobit, or censored regression, models, such as (15.11.1)? The actual mechanics involves the method of maximum likelihood, which is rather involved and is beyond the scope of this book. But the reader can get more information about the ML method from the references.<sup>37</sup>

James Heckman has proposed an alternative to the ML method, which is comparatively simple.<sup>38</sup> This alternative consists of a two-step estimating procedure. In step 1, we first estimate the probability of a consumer owning a house, which is done on the basis of the probit model. In step 2, we estimate the model (15.11.1) by adding to it a variable (called the **inverse Mills ratio or the hazard rate**) that is derived from the probit estimate. For the actual mechanics, see the Heckman article. The Heckman procedure yields consistent estimates of the parameters of (15.11.1), but they are not as efficient as the ML estimates. Since most modern statistical software packages

<sup>37</sup>See Greene, op. cit. A somewhat less technical discussion can be found in Richard Breen, *Regression Models: Censored, Sample Selected or Truncated Data*, Sage Publications, Newbury Park, California, 1996.

<sup>38</sup>J. J. Heckman, "Sample Selection Bias as a Specification Error," *Econometrica*, vol. 47, pp. 153-161.



have the ML routine, it may be preferable to use these packages rather than use the Heckman two-step procedure.

### Illustration of the Tobit Model: Ray Fair's Model of Extramarital Affairs<sup>39</sup>

In an interesting and theoretically innovative article, Ray Fair collected a sample of 601 men and women then married for the first time and analyzed their responses to a question about extramarital affairs.<sup>40</sup> The variables used in this study are defined as follows:

$Y$  = number of affairs in the past year, 0, 1, 2, 3, 4–10 (coded as 7)

$Z_1$  = 0 for female and 1 for male

$Z_2$  = number of years married

$Z_4$  = children: 0 if no children and 1 if children

$Z_5$  = religiousness on a scale of 1 to 5, 1 being antireligion

$Z_6$  = education, years: grade school = 9; high school = 12, Ph.D. or other = 20

$Z_7$  = occupation, "Hollingshead" scale, 1–7

$Z_8$  = self-rating of marriage, 1 = very unhappy, 5 = very happy

Of the 601 responses, 451 individuals had no extramarital affairs, and 150 individuals had one or more affairs.

In terms of Figure 15.7, if we plot the number of affairs on the vertical axis and, say, education on the horizontal axis, there will be 451 observations lying along the horizontal axis. Thus, we have a censored sample, and a tobit model may be appropriate.

Table 15.16 gives estimates of the preceding model using both (the inappropriate) OLS and (the appropriate) ML procedures. As you can see, OLS includes 451 individuals who had no affairs and 150 who had one or more affairs. The ML method takes this into account explicitly but the OLS method does not, thus the difference between the two estimates. For reasons already discussed, one should rely on the ML and not the OLS estimates. The coefficients in the two models can be interpreted like any other regression coefficients. The negative coefficient of  $Z_8$  (marital happiness) means that the higher the marital happiness, the lower is the incidence of extramarital affairs, perhaps an unsurprising finding.

In passing, note that if we are interested in the probability of extramarital affairs and not in the number of such affairs, we can use the probit model

<sup>39</sup>Ray Fair, "A Theory of Extramarital Affairs," *Journal of Political Economy*, vol. 86, 1978, pp. 45–61. For the article and the data, see <http://fairmodel.econ.yale.edu/rayfair/pdf/1978DAT.ZIP>.

<sup>40</sup>In 1969 *Psychology Today* published a 101-question survey on sex and asked its readers to mail in their answers. In the July 1970 issue of the magazine the survey results were discussed on the basis of about 2000 replies that were collected in electronic form. Ray Fair extracted the sample of 601 from these replies.

**TABLE 15.16** OLS AND TOBIT ESTIMATES OF EXTRAMARITAL AFFAIRS

Explanatory variable	OLS estimate	Tobit estimate
Intercept	5.8720 (5.1622)*	7.6084 (1.9479)†
Z <sub>1</sub>	0.0540 (0.1799)	0.9457 (0.8898)
Z <sub>2</sub>	-0.0509 (-2.2536)	-0.1926 (-2.3799)
Z <sub>3</sub>	0.1694 (4.1109)	0.5331 (3.6368)
Z <sub>4</sub>	-0.1426 (-0.4072)	1.0191 (0.7965)
Z <sub>5</sub>	-0.4776 (-4.2747)	-1.6990 (-4.1906)
Z <sub>6</sub>	-0.0137 (-0.2143)	0.0253 (0.1113)
Z <sub>7</sub>	0.1049 (1.1803)	0.2129 (0.6631)
Z <sub>8</sub>	-0.7118 (-5.9319)	-2.2732 (-5.4724)
R <sup>2</sup>	0.1317	0.1515

\*The figures in the parentheses are the *t* values.

†The figures in the parentheses are the *Z* (standard normal) values.

Note: In all there are 601 observations, of which 451 have zero values for the dependent variable (number of extramarital affairs) and 150 have nonzero values.

**TABLE 15.17** Dependent Variable: YSTAR  
Method: ML-binary probit  
Sample: 1-601  
Included observations: 601  
Convergence achieved after 5 iterations

Variable	Coefficient	Std. error	Z statistic	Probability
C	0.779402	0.512549	1.520638	0.1284
Z <sub>1</sub>	0.173457	0.137991	1.257015	0.2087
Z <sub>2</sub>	-0.024584	0.010418	-2.359844	0.0183
Z <sub>3</sub>	0.054343	0.018809	2.889278	0.0039
Z <sub>4</sub>	0.216644	0.165168	1.311657	0.1896
Z <sub>5</sub>	-0.185468	0.051626	-3.592551	0.0003
Z <sub>6</sub>	0.011262	0.029517	0.381556	0.7028
Z <sub>7</sub>	0.013669	0.041404	0.330129	0.7413
Z <sub>8</sub>	-0.271791	0.053475	-5.082608	0.0000
Mean dependent var	0.249584	S.D. dependent var	0.433133	
S.E. of regression	0.410279	Akaike info criterion	1.045584	
Sum squared resid	99.65088	Schwarz criterion	1.111453	
Log likelihood	-305.1980	Hannan-Quinn criter.	1.071224	
Restr. log likelihood	-337.6885	Avg. log likelihood	-0.507817	
LR statistic (8 df)	64.98107	McFadden R-squared	0.096215	
Probability (LR stat)	4.87E-11			
Obs with Dep = 0	451	Total obs	601	
Obs with Dep = 1	150			

assigning  $Y = 0$  for individuals who did not have any affairs and  $Y = 1$  for those who had such affairs, giving the results shown in Table 15.17. With the knowledge of probit modeling, readers should be able to interpret the probit results given in this table on their own.

**15.12 MODELING COUNT DATA: THE POISSON  
REGRESSION MODEL**

There are many phenomena where the regressand is of the **count type**, such as the number of vacations taken by a family per year, the number of patents received by a firm per year, the number of visits to a dentist or a doctor per year, the number of visits to a grocery store per week, the number of parking or speeding tickets received per year, the number of days stayed in a hospital in a given period, the number of cars passing through a toll booth in a span of, say, 5 minutes, and so on. The underlying variable in each case is discrete, taking only a finite number of values. Sometimes count data can also refer to *rare*, or *infrequent*, occurrences such as getting hit by lightning in a span of a week, winning more than one lottery within 2 weeks, or having two or more heart attacks in a span of 4 weeks. How do we model such phenomena?

Just as the Bernoulli distribution was chosen to model the yes/no decision in the linear probability model, the probability distribution that is specifically suited for count data is the **Poisson** probability distribution. The pdf of the Poisson distribution is given by<sup>41</sup>:

$$f(Y_i) = \frac{\mu^Y e^{-\mu}}{Y!} \quad Y = 0, 1, 2, \dots \quad (15.12.1)$$

where  $f(Y)$  denotes the probability that the variable  $Y$  takes non-negative integer values, and where  $Y!$  (read  $Y$  factorial) stands for  $Y! = Y \times (Y - 1) \times (Y - 2) \times \dots \times 1$ . It can be proved that

$$E(Y) = \mu \quad (15.12.2)$$

$$\text{var}(Y) = \mu \quad (15.12.3)$$

Notice an interesting feature of the Poisson distribution: *Its variance is the same as its mean value.*

The Poisson regression model may be written as:

$$Y_i = E(Y_i) + u_i = \mu_i + u_i \quad (15.12.4)$$

where the  $Y$ 's are independently distributed as Poisson random variables with mean  $\mu_i$  for each individual expressed as

$$\mu_i = E(Y_i) = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \dots + \beta_k X_{ki} \quad (15.12.5)$$

where the  $X$ 's are some of the variables that might affect the mean value. For example, if our count variable is the number of visits to the Metropolitan Museum of Art in New York in a given year, this number will depend on

<sup>41</sup>See any standard book on statistics for the details of this distribution.

variables such as income of the consumer, admission price, distance from the museum, and parking fees.

For estimation purposes, we write the model as:

$$Y_i = \frac{\mu^Y e^{-\mu}}{Y!} + u_i \quad (15.12.6)$$

with  $\mu$  replaced by (5.12.5). As you can readily see, the resulting regression model will be nonlinear in the parameters, necessitating nonlinear regression estimation discussed in the previous chapter. Let us consider a concrete example to see how all this works out.

**AN ILLUSTRATIVE EXAMPLE: GERIATRIC STUDY  
OF FREQUENCY OF FALLS**

The data used here were collected by Netere et al.<sup>42</sup> The data relate to 100 individuals 65 years of age and older. The objective of the study was to record the number of falls ( $= Y$ ) suffered by these individuals in relation to gender ( $X_2 = 0$  female and 1 for male), a balance index ( $X_3$ ), and a strength index ( $X_4$ ). The higher the balance index, the more stable is the subject, and the higher the strength index, the stronger is the subject. To find out if education or education plus aerobic exercise has any effect on the number of falls, the authors introduced an additional variable ( $X_1$ ), called the *intervention* variable, such that  $X_1 = 0$  if only education and  $X_1 = 1$  if education plus aerobic exercise training. The subjects were randomly assigned to the two intervention methods.

Using Eviews 4, we obtained the output in Table 15.18.

**TABLE 15.18**

Dependent Variable: Y

Sample: 1-100

Convergence achieved after 7 iterations

Y=EXP(C(0)+C(1)\*X1+C(2)\*X2+C(3)\*X3+C(4)\*X4)

	Coefficient	Std. error	t statistic	Probability
C(0)	0.37020	0.3459	1.0701	0.2873
C(1)	-1.10036	0.1705	-6.4525	0.0000
C(2)	-0.02194	0.1105	-0.1985	0.8430
C(3)	0.01066	0.0027	3.9483	0.0001
C(4)	0.00927	0.00414	2.2380	0.0275

$R^2 = 0.4857$     Adjusted  $R^2 = 0.4640$

Log likelihood = -197.2096    Durbin-Watson statistic = 1.7358

Note: EXP( ) means e (the base of natural logarithm) raised by the expression in ( ).

(Continued)

<sup>42</sup>John Neter, Michael H. Kutner, Christopher J. Nachtsheim, and William Wasserman, *Applied Regression Models*, Irwin, 3d ed., Chicago, 1996. The data were obtained from the data disk included in the book and refer to exercise 14.28.

AN ILLUSTRATIVE EXAMPLE: (Continued)

### Interpretation of Results

Keep in mind that what we have obtained in Table 15.18 is the estimated mean value for the  $i$ th individual,  $\hat{\mu}_i$ ; that is, what we have estimated is:

$$\hat{\mu}_i = e^{0.3702 - 1.100366X_{1i} - 0.02194X_{2i} + 0.0106X_{3i} + 0.00927X_{4i}} \quad (15.12.7)$$

To find the actual mean value for the  $i$ th subject, we need to put the values of the various  $X$  variables for that subject. For example, subject 99 had these values:  $Y = 4$ ,  $X_1 = 0$ ,  $X_2 = 1$ ,  $X_3 = 50$ , and  $X_4 = 56$ . Putting these values in (15.12.7), we obtain  $\hat{\mu}_{99} = 3.3538$  as the estimated mean value for the 99th subject. The actual  $Y$  value for this individual was 4.

Now if we want to find out the probability that a subject similar to subject 99 has less than 5 falls per year, we can obtain it as follows:

$$\begin{aligned} P(Y < 5) &= P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) \\ &= \frac{(3.3538)^0 e^{-3.3538}}{0!} + \frac{(3.3538)^1 e^{-3.3538}}{1!} + \frac{(3.3538)^2 e^{-3.3538}}{2!} \\ &\quad + \frac{(3.3538)^3 e^{-3.3538}}{3!} + \frac{(3.3538)^4 e^{-3.3538}}{4!} \\ &= 0.7491 \end{aligned}$$

We can also find out the marginal, or partial, effect of a regressor on the mean value of  $Y$  as follows. In terms of our illustrative example, suppose we want to find out the effect of a unit increase in the strength index ( $X_4$ ) on mean  $Y$ . Since

$$\mu = e^{C_0 + C_1 X_{1i} + C_2 X_{2i} + C_3 X_{3i} + C_4 X_{4i}} \quad (15.12.8)$$

we want to find  $\partial \mu / \partial X_4$ . Using the chain rule of calculus, it can be easily shown that this is equal to

$$\frac{\partial \mu}{\partial X_4} = C_4 e^{C_0 + C_1 X_{1i} + C_2 X_{2i} + C_3 X_{3i} + C_4 X_{4i}} = C_4 \mu \quad (15.12.9)$$

That is, the rate of change of the mean value with respect to a regressor is equal to the coefficient of that regressor times the mean value. Of course, the mean value  $\mu$  will depend on the values taken by all the regressors in the model. This is similar to the logit and probit models we discussed earlier, where the marginal contribution of a variable also depended on the values taken by all the variables in the model.

Returning to the statistical significance of the individual coefficients, we see that the intercept and variable  $X_2$  are individually statistically insignificant. But note that the standard errors given in the table are asymptotic and hence the  $t$  values are to be interpreted asymptotically. As noted previously, generally the results of all nonlinear iterative estimating procedures have validity in large samples only.

In concluding our discussion of the Poisson regression model, it may be noted that the model makes restrictive assumptions in that the mean and the variance of the Poisson process are the same and that the probability of an occurrence is constant at any point in time.

### 15.13 FURTHER TOPICS IN QUALITATIVE RESPONSE REGRESSION MODELS

As noted at the outset, the topic of qualitative response regression models is vast. What we have presented in this chapter are some of the basic models in this area. For those who want to pursue this topic further, we discuss below very briefly some other models in this area. We will not pursue them here, for that would take us far away from the scope of this book.

#### Ordinal Logit and Probit Models

In the bivariate logit and probit models we were interested in modeling a yes or no response variable. But often the response variable, or regressand, can have more than two outcomes and very often these outcomes are **ordinal** in nature; that is, they cannot be expressed on an interval scale. Frequently, in survey-type research the responses are on a Likert-type scale, such as “strongly agree,” “somewhat agree,” or “strongly disagree.” Or the responses in an educational survey may be “less than high school,” “high school,” “college,” or “professional degrees.” Very often these responses are coded as 0 (less than high school), 1 (high school), 2 (college), 3 (postgraduate). These are ordinal scales in that there is clear ranking among the categories but we cannot say that 2 (college education) is twice 1 (high school education) or 3 (postgraduate education) is three times 1 (high school education).

To study phenomena such as the preceding, one can extend the bivariate logit and probit models to take into account multiple ranked categories. The arithmetic gets quite involved as we have to use multistage normal and logistic probability distributions to allow for the various ranked categories. For the underlying mathematics and some of the applications, the reader may consult the Greene and Maddala texts cited earlier. At a comparatively intuitive level, the reader may consult the Liao monograph.<sup>43</sup> Software packages such as Limdep, Eviews, and Shazam have routines to estimate ordered logit and probit models.

#### Multinomial Logit and Probit Models

In the ordered probit and logit models the response variable has more than two ordered, or ranked, categories. But there are situations where the regressand is unordered. Take, for example, the choice of transportation mode to work. The choices may be bicycle, motorbike, car, bus, or train. Although these are categorical responses, there is no ranking or order here; they are essentially nominal in character. For another example, consider occupational classifications, such as unskilled, semiskilled, and highly skilled. Again, there is no order here. Similarly, occupational choices such as self-employed, working for a private firm, working for a local government, and working for the federal government are essentially nominal in character.

<sup>43</sup>Tim Futing Liao, *op. cit.*

The techniques of multinomial logit or probit models can be employed to study such nominal categories. Again, the mathematics gets a little involved. The references cited previously will give the essentials of these techniques. And the statistical packages cited earlier can be used to implement such models, if their use is required in specific cases.

### Duration Models

Consider questions such as these: (1) What determines the duration of unemployment spells? (2) What determines the life of a light bulb? (3) What factors determine the duration of a strike? (4) What determines the survival time of a HIV-positive patient?

Subjects such as these are the topic of duration models, popularly known as **survival analysis** or **time-to-event data analysis**. In each of the examples cited above, the key variable is the length of time or spell length, which is modeled as a random variable. Again the mathematics involves the CDFs and PDFs of appropriate probability distributions. Although the technical details can be tedious, there are accessible books on this subject.<sup>44</sup> Statistical packages such as Stata and Limdep can easily estimate such duration models. These packages have worked examples to aid the researcher in the use of such models.

### 15.14 SUMMARY AND CONCLUSIONS

1. Qualitative response regression models refer to models in which the response, or regressand, variable is not quantitative or an interval scale.

2. The simplest possible qualitative response regression model is the binary model in which the regressand is of the yes/no or presence/absence type.

3. The simplest possible binary regression model is the linear probability model (LPM) in which the binary response variable is regressed on the relevant explanatory variables by using the standard OLS methodology. Simplicity may not be a virtue here, for the LPM suffers from several estimation problems. Even if some of the estimation problems can be overcome, the fundamental weakness of the LPM is that it assumes that the probability of something happening increases linearly with the level of the regressor. This very restrictive assumption can be avoided if we use the logit and probit models.

4. In the logit model the dependent variable is the log of the odds ratio, which is a linear function of the regressors. The probability function that underlies the logit model is the logistic distribution. If the data are available in grouped form, we can use OLS to estimate the parameters of the logit model, provided we take into account explicitly the heteroscedastic nature of the error term. If the data are available at the individual, or micro, level, nonlinear-in-the-parameter estimating procedures are called for.

<sup>44</sup>See, for example, David W. Hosmer, Jr., and Stanley Lemeshow, *Applied Survival Analysis*, John Wiley & Sons, New York, 1999.

5. If we choose the normal distribution as the appropriate probability distribution, then we can use the probit model. This model is mathematically a bit difficult as it involves integrals. But for all practical purposes, both logit and probit models give similar results. In practice, the choice therefore depends on the ease of computation, which is not a serious problem with sophisticated statistical packages that are now readily available.

6. If the response variable is of the count type, the model that is most frequently used in applied work is the Poisson regression model, which is based on the Poisson probability distribution.

7. A model that is closely related to the probit model is the tobit model, also known as a censored regression model. In this model, the response variable is observed only if certain condition(s) are met. Thus, the question of how much one spends on a car is meaningful only if one decides to buy a car to begin with. However, Maddala notes that the tobit model is “applicable only in those cases where the latent variable [i.e., the basic variable underlying a phenomenon] can, in principle, take negative values and the observed zero values are a consequence of censoring and nonobservability.”<sup>45</sup>

8. There are various extensions of the binary response regression models. These include ordered probit and logit and nominal probit and logit models. The philosophy underlying these models is the same as the simpler logit and probit models, although the mathematics gets rather complicated.

9. Finally, we considered briefly the so-called duration models in which the duration of a phenomenon, such as unemployment or sickness, depends on several factors. In such models, the length, or the spell of duration, becomes the variable of research interest.

## EXERCISES

### Questions

- 15.1. Refer to the data given in Table 15.2. If  $\hat{Y}_i$  is negative, assume it to be equal to 0.01 and if it is greater than 1, assume it to be equal to 0.99. Recalculate the weights  $w_i$  and estimate the LPM using WLS. Compare your results with those given in (15.2.11) and comment.
- 15.2. For the home ownership data given in Table 15.1, the maximum likelihood estimates of the logit model are as follows:

$$\hat{L}_i = \ln \left( \frac{\hat{P}_i}{1 - \hat{P}_i} \right) = -493.54 + 32.96 \text{ income} \\ t = (-0.000008)(0.000008)$$

Comment on these results, bearing in mind that all values of income above 16 (thousand dollars) correspond to  $Y = 1$  and all values of income below 16 correspond to  $Y = 0$ . A priori, what would you expect in such a situation?

<sup>45</sup>G. S. Maddala, *Introduction to Econometrics*, 2d ed., Macmillan, New York, 1992, p. 342.



- 15.3.** In studying the purchase of durable goods  $Y$  ( $Y = 1$  if purchased,  $Y = 0$  if no purchase) as a function of several variables for a total of 762 households, Janet A. Fisher\* obtained the following LPM results:

Explanatory variable	Coefficient	Standard error
Constant	0.1411	—
1957 disposable income, $X_1$	0.0251	0.0118
(Disposable income = $X_1$ ) <sup>2</sup> , $X_2$	-0.0004	0.0004
Checking accounts, $X_3$	-0.0051	0.0108
Savings accounts, $X_4$	0.0013	0.0047
U.S. Savings Bonds, $X_5$	-0.0079	0.0067
Housing status: rent, $X_6$	-0.0469	0.0937
Housing status: own, $X_7$	0.0136	0.0712
Monthly rent, $X_8$	-0.7540	1.0983
Monthly mortgage payments, $X_9$	-0.9809	0.5162
Personal noninstallment debt, $X_{10}$	-0.0367	0.0326
Age, $X_{11}$	0.0046	0.0084
Age squared, $X_{12}$	-0.0001	0.0001
Marital status, $X_{13}$ (1 = married)	0.1760	0.0501
Number of children, $X_{14}$	0.0398	0.0358
(Number of children = $X_{14}$ ) <sup>2</sup> , $X_{15}$	-0.0036	0.0072
Purchase plans, $X_{16}$ (1 = planned; 0 otherwise)	0.1760	0.0384
$R^2 = 0.1336$		

Notes: All financial variables are in thousands of dollars.

Housing status: Rent (1 if rents; 0 otherwise)

Housing status: Own (1 if owns; 0 otherwise)

Source: Janet A. Fisher, "An Analysis of Consumer Good Expenditure," *The Review of Economics and Statistics*, vol. 64, no. 1, Table 1, 1962, p. 67.

- Comment generally on the fit of the equation.
  - How would you interpret the coefficient of  $-0.0051$  attached to checking account variable? How would you rationalize the negative sign for this variable?
  - What is the rationale behind introducing the age-squared and number of children-squared variables? Why is the sign negative in both cases?
  - Assuming values of zero for all but the income variable, find out the conditional probability of a household whose income is \$20,000 purchasing a durable good.
  - Estimate the conditional probability of owning durable good(s), given:  $X_1 = \$15,000$ ,  $X_3 = \$3000$ ,  $X_4 = \$5000$ ,  $X_6 = 0$ ,  $X_7 = 1$ ,  $X_8 = \$500$ ,  $X_9 = \$300$ ,  $X_{10} = 0$ ,  $X_{11} = 35$ ,  $X_{13} = 1$ ,  $X_{14} = 2$ ,  $X_{16} = 0$ .
- 15.4.** The  $R^2$  value in the labor-force participation regression given in Table 15.3 is 0.175, which is rather low. Can you test this value for statistical significance? Which test do you use and why? Comment in general on the value of  $R^2$  in such models.
- 15.5.** Estimate the probabilities of owning a house at the various income levels underlying the regression (15.7.1). Plot them against income and comment on the resulting relationship.

\*"An Analysis of Consumer Good Expenditure," *The Review of Economics and Statistics*, vol. 64, no. 1, 1962, pp. 64-71.

- \*15.6. In the probit regression given in Table 15.11 show that the intercept is equal to  $-\mu_x/\sigma_x$  and the slope is equal to  $1/\sigma_x$ , where  $\mu_x$  and  $\sigma_x$  are the mean and standard deviation of  $X$ .
- 15.7. From data for 54 standard metropolitan statistical areas (SMSA), Demaris estimated the following logit model to explain high murder rate versus low murder rate<sup>†</sup>:

$$\ln \hat{O}_i = 1.1387 + 0.0014P_i + 0.0561C_i - 0.4050R_i$$

$$\text{se} = \quad (0.0009) \quad (0.0227) \quad (0.1568)$$

where  $O$  = the odds of a high murder rate,  $P$  = 1980 population size in thousands,  $C$  = population growth rate from 1970 to 1980,  $R$  = reading quotient, and the se are the asymptotic standard errors.

- How would you interpret the various coefficients?
  - Which of the coefficients are individually statistically significant?
  - What is the effect of a unit increase in the reading quotient on the odds of having a higher murder rate?
  - What is the effect of a percentage point increase in the population growth rate on the odds of having a higher murder rate?
- 15.8. Compare and comment on the OLS and WLS regressions (15.7.3) and (15.7.1).

### Problems

- 15.9. From the household budget survey of 1980 of the Dutch Central Bureau of Statistics, J. S. Cramer obtained the following logit model based on a sample of 2820 households. (The results given here are based on the method of maximum likelihood and are after the third iteration.)\*\* The purpose of the logit model was to determine car ownership as a function of (logarithm of) income. Car ownership was a binary variable:  $Y = 1$  if a household owns a car, zero otherwise.

$$\hat{L}_i = -2.77231 + 0.347582 \ln \text{Income}$$

$$t = (-3.35) \quad (4.05)$$

$$\chi^2(1 \text{ df}) = 16.681 \quad (p \text{ value} = 0.0000)$$

where  $\hat{L}_i$  = estimated logit and where  $\ln \text{Income}$  is the logarithm of income. The  $\chi^2$  measures the goodness of fit of the model.

- Interpret the estimated logit model.
- From the estimated logit model, how would you obtain the expression for the probability of car ownership?
- What is the probability that a household with an income of 20,000 will own a car? And at an income level of 25,000? What is the rate of change of probability at the income level of 20,000?
- Comment on the statistical significance of the estimated logit model.

\*Optional.

<sup>†</sup>Demaris, op. cit., p. 46.

\*\*J. S. Cramer, *An Introduction to the Logit Model for Economists*, 2d ed., published and distributed by Timberlake Consultants Ltd., 2001, p. 33. These results are reproduced from the statistical package PcGive 10 published by Timberlake Consultants, p. 51.

**15.10.** Establish Eq. (15.2.8).

**15.11.** In an important study of college graduation rates of all high school matriculants and Black-only matriculants, Bowen and Bok obtained the results in Table 15.19, based on the logit model.\*

**TABLE 15.19** LOGISTIC REGRESSION MODEL PREDICTING GRADUATION RATES, 1989 ENTERING COHORT

Variable	All matriculants			Black only		
	Parameter estimate	Standard error	Odds ratio	Parameter estimate	Standard error	Odds ratio
Intercept	0.957	0.052	—	0.455	0.112	—
Female	<b>0.280</b>	0.031	1.323	<b>0.265</b>	0.101	1.303
Black	<b>-0.513</b>	0.056	0.599			
Hispanic	<b>-0.350</b>	0.080	0.705			
Asian	<b>0.122</b>	0.055	1.130			
Other race	<b>-0.330</b>	0.104	0.719			
SAT > 1299	<b>0.331</b>	0.059	1.393	0.128	0.248	1.137
SAT 1200–1299	<b>0.253</b>	0.055	1.288	0.232	0.179	1.261
SAT 1100–1199	<b>0.350</b>	0.053	1.420	0.308	0.149	1.361
SAT 1000–1099	<b>0.192</b>	0.054	1.211	0.141	0.136	1.151
SAT not available	<b>-0.330</b>	0.127	0.719	0.048	0.349	1.050
Top 10% of high school class	<b>0.342</b>	0.036	1.407	<b>0.315</b>	0.117	1.370
High school class rank not available	-0.065	0.046	0.937	-0.065	0.148	0.937
High socioeconomic status (SES)	<b>0.283</b>	0.036	1.327	<b>0.557</b>	0.175	1.746
Low SES	<b>-0.385</b>	0.079	0.680	<b>-0.305</b>	0.143	0.737
SES not available	<b>0.110</b>	0.050	1.116	0.031	0.172	1.031
SEL-1	<b>1.092</b>	0.058	2.979	<b>0.712</b>	0.161	2.038
SEL-2	<b>0.193</b>	0.036	1.212	<b>0.280</b>	0.119	1.323
Women's college	<b>-0.299</b>	0.069	0.742	0.158	0.269	1.171
Number of observations	32,524			2,354		
-2 log likelihood						
Restricted	31,553			2,667		
Unrestricted	30,160			2,569		
Chi square	1,393 with 18 d.f.			98 with 14 d.f.		

Notes: Bold coefficients are significant at the .05 level; other coefficients are not. The omitted categories in the model are White, male, SAT < 1000, bottom 90% of high school class, middle SES, SEL-3, coed institution. Graduation rates are 6-year, first-school graduation rates, as defined in the notes to Appendix Table D.3.1. Institutional selectivity categories are as defined in the notes to Appendix Table D.3.1. See **Appendix B** for definition of socioeconomic status (SES).

SEL-1 = institutions with mean combined SAT scores of 1300 and above.

SEL-2 = institutions with mean combined SAT scores between 1150 and 1299.

SEL-3 = institutions with mean combined SAT scores below 1150.

Source: Bowen and Bok, op. cit., p. 381.

\*William G. Bowen and Derek Bok, *The Shape of the River: Long Term Consequences of Considering Race in College and University Admissions*, Princeton University Press, Princeton, N.J., 1998, p. 381.

- a. What general conclusion do you draw about graduation rates of all matriculants and black-only matriculants?
  - b. The *odds ratio* is the ratio of two odds. Compare two groups of all matriculants, one with a SAT score of greater than 1299 and the other with a SAT score of less than 1000 (the base category). The odds ratio of 1.393 means the odds of matriculants in the first category graduating from college are 39 percent higher than those in the latter category. Do the various odds ratios shown in the table accord with a priori expectations?
  - c. What can you say about the statistical significance of the estimated parameters? What about the overall significance of the estimated model?
- 15.12.** In the probit model given in Table 15.11 the disturbance  $u_i$  has this variance:

$$\sigma_u^2 = \frac{P_i(1 - P_i)}{N_i f_i^2}$$

where  $f_i$  is the standard normal density function evaluated at  $F^{-1}(P_i)$ .

- a. Given the preceding variance of  $u_i$ , how would you transform the model in Table 15.10 to make the resulting error term homoscedastic?
  - b. Use the data in Table 15.10 to show the transformed data.
  - c. Estimate the probit model based on the transformed data and compare the results with those based on the original data.
- 15.13.** Since  $R^2$  as a measure of goodness of fit is not particularly well suited for the dichotomous dependent variable models, one suggested alternative is the  $\chi^2$  test described below:

$$\chi^2 = \sum_{i=1}^G \frac{N_i(\hat{P}_i - P_i^*)^2}{P_i^*(1 - P_i^*)}$$

where  $N_i$  = number of observations in the  $i$ th cell

$\hat{P}_i$  = actual probability of the event occurring ( $= n_i/N_i$ )

$P_i^*$  = estimated probability

$G$  = number of cells (i.e., the number of levels at which  $X_i$  is measured, e.g., 10 in Table 15.4)

It can be shown that, for large samples,  $\chi^2$  is distributed according to the  $\chi^2$  distribution with  $(G - k)$  df, where  $k$  is the number of parameters in the estimating model ( $k < G$ ).

Apply the preceding  $\chi^2$  test to regression (15.7.1) and comment on the resulting goodness of fit and compare it with the reported  $R^2$  value.

- 15.14.** Table 15.20 gives data on the results of spraying rotenone of different concentrations on the chrysanthemum aphid in batches of approximately fifty. Develop a suitable model to express the probability of death as a function of the log of  $X$ , the log of dosage, and comment on the results. Also compute the  $\chi^2$  test of fit discussed in exercise 15.13.
- 15.15.** Fourteen applicants to a graduate program had quantitative and verbal scores on the GRE as listed in Table 15.21. Six students were admitted to the program.

**TABLE 15.20** TOXICITY STUDY AND ROTENONE ON CHRYSANTHEMUM APHIS

Concentration, milligrams per liter		Total, $N_i$	Death, $n_i$	$\hat{P}_i = n_i / N_i$
$X$	$\log(X)$			
2.6	0.4150	50	6	0.120
3.8	0.5797	48	16	0.333
5.1	0.7076	46	24	0.522
7.7	0.8865	49	42	0.857
10.2	1.0086	50	44	0.880

Source: D. J. Fennel, *Probit Analysis*, Cambridge University Press, London, 1964.

**TABLE 15.21**

Student number	GRE aptitude test scores		Admitted to graduate program (Yes = 1, No = 0)
	Quantitative, $Q$	Verbal, $V$	
1	760	550	1
2	600	350	0
3	720	320	0
4	710	630	1
5	530	430	0
6	650	570	0
7	800	500	1
8	650	680	1
9	520	660	0
10	800	250	0
11	670	480	0
12	670	520	1
13	780	710	1

Source: Donald F. Morrison, *Applied Linear Statistical Methods*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1983, p. 279 (adapted).

- a. Use the LPM model to predict the probability of admission to the program based on quantitative and verbal scores in the GRE.
  - b. Is this a satisfactory model? If not, what alternative(s) do you suggest?
- 15.16.** To study the effectiveness of a price discount coupon on a six-pack of a two-liter soft drink, Douglas Montgomery and Elizabeth Peck collected the data shown in Table 15.22. A sample of 5500 consumers was randomly assigned to the eleven discount categories shown in the table, 500 per category. The response variable is whether or not consumers redeemed the coupon within one month.
- a. See if the logit model fits the data, treating the redemption rate as the dependent variable and price discount as the explanatory variable.
  - b. See if the probit model does as well as the logit model.
  - c. What is the predicted redemption rate if the price discount was 17 cents?
  - d. Estimate the price discount for which 70 percent of the coupons will be redeemed.

TABLE 15.22

Price discount $X, \text{¢}$	Sample size $N_i$	Number of coupons redeemed $n_i$
5	500	100
7	500	122
9	500	147
11	500	176
13	500	211
15	500	244
17	500	277
19	500	310
21	500	343
23	500	372
25	500	391

Source: Douglas C. Montgomery and Elizabeth A. Peck, *Introduction to Linear Regression Analysis*, John Wiley & Sons, New York, 1982, p. 243 (notation changed).

**15.17.** To find out who has a bank account (checking, savings, etc.) and who doesn't, John Caskey and Andrew Peterson estimated a probit model for the years 1977 and 1989, using data on U.S. households. The results are given in Table 15.23. The values of the slope coefficients given in the table measure the implied effect of a unit change in a regressor on the probability that a household has a bank account, these marginal effects being calculated at the mean values of the regressors included in the model.

- For 1977, what is the effect of marital status on ownership of a bank account? And for 1989? Do these results make economic sense?
- Why is the coefficient for the *minority* variable negative for both 1977 and 1989?
- How can you rationalize the negative sign for the number of children variable?
- What does the chi-square statistic given in the table suggest? (*Hint*: exercise 15.13.)

**15.18. Monte Carlo study:** As an aid to understanding the probit model, William Becker and Donald Waldman assumed the following\*:

$$E(Y | X) = -1 + 3X$$

Then, letting  $Y_i = -1 + 3X + \varepsilon_i$ , where  $\varepsilon_i$  is assumed standard normal (i.e., zero mean and unit variance), they generated a sample of 35 observations as shown in Table 15.24.

- From the data on  $Y$  and  $X$  given in this table, can you estimate an LPM? Remember that the true  $E(Y | X) = -1 + 3X$ .
- Given  $X = 0.48$ , estimate  $E(Y | X = 0.48)$  and compare it with the true  $E(Y | X = 0.48)$ . Note  $\bar{X} = 0.48$ .

\*William E. Becker and Donald M. Waldman, "A Graphical Interpretation of Probit Coefficients," *Journal of Economic Education*, vol. 20, no. 4, Fall 1989, pp. 371–378.

**TABLE 15.23** PROBIT REGRESSIONS WHERE DEPENDENT VARIABLE IS OWNERSHIP OF A DEPOSIT ACCOUNT

	1977 data		1989 data	
	Coefficients	Implied slope	Coefficients	Implied slope
Constant	-1.06 (3.3)*		-2.20 (6.8)*	
Income (thousands 1991 \$)	0.030 (6.9)	0.002	0.025 (6.8)	0.002
Married	0.127 (0.8)	0.008	0.235 (1.7)	0.023
Number of children	-0.131 (3.6)	-0.009	-0.084 (2.0)	-0.008
Age of head of household (HH)	0.006 (1.7)	0.0004	0.021 (6.3)	0.002
Education of HH	0.121 (7.4)	0.008	0.128 (7.7)	0.012
Male HH	-0.078 (0.5)	-0.005	-0.144 (0.9)	-0.011
Minority	-0.750 (6.8)	-0.050	-0.600 (6.5)	-0.058
Employed	0.186 (1.6)	0.012	0.402 (3.6)	0.039
Homeowner	0.520 (4.7)	0.035	0.522 (5.3)	0.051
Log likelihood	-430.7		-526.0	
Chi-square statistic ( $H_0$ : All coefficients except constant equal zero)	408		602	
Number of observations	2,025		2,091	
Percentage in sample with correct predictions	91		90	

\*Numbers in parentheses are  $t$  statistics.

Source: John P. Caskey and Andrew Peterson, "Who Has a Bank Account and Who Doesn't: 1977 and 1989," Research Working Paper 93-10, Federal Reserve Bank of Kansas City, October 1993.

- c. Using the data on  $Y^*$  and  $X$  given in Table 15.24, estimate a probit model. You may use any statistical package you want. The authors' estimated probit model is the following:

$$\hat{Y}_i^* = -0.969 + 2.764X_i$$

Find out the  $P(Y^* = 1 | X = 0.48)$ , that is,  $P(Y_1 > 0 | X = 0.48)$ . See if your answer agrees with the authors' answer of 0.64.

- d. The sample standard deviation of the  $X$  values given in Table 15.24 is 0.31. What is the predicted change in probability if  $X$  is one standard deviation above the mean value, that is, what is  $P(Y^* = 1 | X = 0.79)$ ? The authors' answer is 0.25.

**TABLE 15.24** HYPOTHETICAL DATA SET GENERATED BY THE MODEL  $Y = -1 + 3X + \varepsilon$   
AND  $Y^* = 1$  IF  $Y > 0$

$Y$	$Y^*$	$X$	$Y$	$Y^*$	$X$
-0.3786	0	0.29	-0.3753	0	0.56
1.1974	1	0.59	1.9701	1	0.61
-0.4648	0	0.14	-0.4054	0	0.17
1.1400	1	0.81	2.4416	1	0.89
0.3188	1	0.35	0.8150	1	0.65
2.2013	1	1.00	-0.1223	0	0.23
2.4473	1	0.80	0.1428	1	0.26
0.1153	1	0.40	-0.6681	0	0.64
0.4110	1	0.07	1.8286	1	0.67
2.6950	1	0.87	-0.6459	0	0.26
2.2009	1	0.98	2.9784	1	0.63
0.6389	1	0.28	-2.3326	0	0.09
4.3192	1	0.99	0.8056	1	0.54
-1.9906	0	0.04	-0.8983	0	0.74
-0.9021	0	0.37	-0.2355	0	0.17
0.9433	1	0.94	1.1429	1	0.57
-3.2235	0	0.04	-0.2965	0	0.18
0.1690	1	0.07			

Source: William E. Becker and Donald M. Waldman, "A Graphical Interpretation of Probit Coefficients," *Journal of Economic Education*, Fall 1989, Table 1, p. 373.

## APPENDIX 15A

### 15A.1 MAXIMUM LIKELIHOOD ESTIMATION OF THE LOGIT AND PROBIT MODELS FOR INDIVIDUAL (UNGROUPED) DATA\*

As in the text, assume that we are interested in estimating the probability that an individual owns a house, given the individual's income  $X$ . We assume that this probability can be expressed by the logistic function (15.5.2), which is reproduced below for convenience.

$$P_i = \frac{1}{1 + e^{-(\beta_1 + \beta_2 X_i)}} \quad (1)$$

We do not actually observe  $P_i$ , but only observe the outcome  $Y = 1$ , if an individual owns a house, and  $Y = 0$ , if the individual does not own a house.

Since each  $Y_i$  is a Bernoulli random variable, we can write

$$\Pr(Y_i = 1) = P_i \quad (2)$$

$$\Pr(Y_i = 0) = (1 - P_i) \quad (3)$$

\*The following discussion leans heavily on John Neter, Michael H. Kutner, Christopher J. Nachstein, and William Wasserman, *Applied Linear Statistical Models*, 4th ed., Irwin, 1996, pp. 573–574.



Suppose we have a *random sample* of  $n$  observations. Letting  $f_i(Y_i)$  denote the probability that  $Y_i = 1$  or  $0$ , the joint probability of observing the  $n$   $Y$  values, i.e.,  $f(Y_1, Y_2, \dots, Y_n)$  is given as:

$$f(Y_1, Y_2, \dots, Y_n) = \prod_1^n f_i(Y_i) = \prod_1^n P_i^{Y_i} (1 - P_i)^{1 - Y_i} \quad (4)$$

where  $\prod$  is the product operator. Note that we can write the joint probability density function as a product of individual density functions because each  $Y_i$  is drawn independently and each  $Y_i$  has the same (logistic) density function. The joint probability given in Eq. (4) is known as the **likelihood function (LF)**.

Equation (4) is a little awkward to manipulate. But if we take its natural logarithm, we obtain what is called the **log likelihood function (LLF)**:

$$\begin{aligned} \ln f(Y_1, Y_2, \dots, Y_n) &= \sum_1^n [Y_i \ln P_i + (1 - Y_i) \ln (1 - P_i)] \\ &= \sum_1^n [Y_i \ln P_i - Y_i \ln (1 - P_i) + \ln (1 - P_i)] \\ &= \sum_1^n \left[ Y_i \ln \left( \frac{P_i}{1 - P_i} \right) \right] + \sum_1^n \ln (1 - P_i) \end{aligned} \quad (5)$$

From (1) it is easy to verify that

$$(1 - P_i) = \frac{1}{1 + e^{\beta_1 + \beta_2 X_i}} \quad (6)$$

as well as

$$\ln \left( \frac{P_i}{1 - P_i} \right) = \beta_1 + \beta_2 X_i \quad (7)$$

Using (6) and (7), we can write the LLF (5) as:

$$\ln f(Y_1, Y_2, \dots, Y_n) = \sum_1^n Y_i (\beta_1 + \beta_2 X_i) - \sum_1^n \ln [1 + e^{(\beta_1 + \beta_2 X_i)}] \quad (8)$$

As you can see from (8), the log likelihood function is a function of the parameters  $\beta_1$  and  $\beta_2$ , since the  $X_i$  are known.

In ML our objective is to maximize the LF (or LLF), that is, to obtain the values of the unknown parameters in such a manner that the probability of observing the given  $Y$ 's is as high (maximum) as possible. For this purpose, we differentiate (8) partially with respect to each unknown, set the resulting expressions to zero and solve the resulting expressions. One can then apply the second-order condition of maximization to verify that the values of the parameters we have obtained do in fact maximize the LF.

So, you have to differentiate (8) with respect to  $\beta_1$  and  $\beta_2$  and proceed as indicated. As you will quickly realize, the resulting expressions become highly nonlinear in the parameters and no explicit solutions can be obtained. That is why we will have to use one of the methods of nonlinear estimation discussed in the previous chapter to obtain numerical solutions. Once the numerical values of  $\beta_1$  and  $\beta_2$  are obtained, we can easily estimate (1).

The ML procedure for the probit model is similar to that for the logit model, except that in (1) we use the normal CDF rather than the logistic CDF. The resulting expression becomes rather complicated, but the general idea is the same. So, we will not pursue it any further.

# 16

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## PANEL DATA REGRESSION MODELS

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In Chapter 1 we discussed briefly the types of data that are generally available for empirical analysis, namely, **time series, cross section, and panel**. In time series data we observe the values of one or more variables over a period of time (e.g., GDP for several quarters or years). In cross-section data, values of one or more variables are collected for several sample units, or entities, at the same point in time (e.g., crime rates for 50 states in the United States for a given year). *In panel data the same cross-sectional unit (say a family or a firm or a state) is surveyed over time.* In short, panel data have *space as well as time dimensions*.

We have already seen an example of this in Table 1.1, which gives data on eggs produced and their prices for 50 states in the United States for years 1990 and 1991. For any given year, the data on eggs and their prices represent a cross-sectional sample. For any given state, there are two time series observations on eggs and their prices. Thus, we have in all  $(50 \times 2) = 100$  (*pooled*) observations on eggs produced and their prices.

There are other names for panel data, such as **pooled data** (pooling of time series and cross-sectional observations), **combination of time series and cross-section data, micropanel data, longitudinal data** (a study over time of a variable or group of subjects), **event history analysis** (e.g., studying the movement over time of subjects through successive states or conditions), **cohort analysis** (e.g., following the career path of 1965 graduates of a business school). Although there are subtle variations, all these names *essentially connote movement over time of cross-sectional units*. We will therefore use the term panel data in a generic sense to include one or more of these terms. *And we will call regression models based on such data panel data regression models.*

Panel data are now being increasingly used in economic research. Some of the well-known panel data sets are:

1. The **Panel Study of Income Dynamics (PSID)** conducted by the Institute of Social Research at the University of Michigan. Started in 1968, each year the Institute collects data on some 5000 families about various socioeconomic and demographic variables.

2. The Bureau of the Census of the Department of Commerce conducts a survey similar to PSID, called the **Survey of Income and Program Participation (SIPP)**. Four times a year, the respondents are interviewed about their economic condition.

There are also many other surveys that are conducted by various governmental agencies.

At the outset a warning is in order. The topic of panel data regressions is vast, and some of the mathematics and statistics involved is quite complicated. We only hope to touch on some of the essentials of the panel data regression models, leaving the details for the references.<sup>1</sup> But be forewarned that some of these references are highly technical. Fortunately, user-friendly software packages such as Limdep, PcGive, SAS, STATA, Shazam, and Eviews, among others, have made the task of actually implementing panel data regressions quite easy.

## 16.1 WHY PANEL DATA?

What are the advantages of panel data over cross-section or time series data? Baltagi lists the following advantages of panel data<sup>2</sup>:

1. Since panel data relate to individuals, firms, states, countries, etc., over time, there is bound to be heterogeneity in these units. The techniques of panel data estimation can take such heterogeneity explicitly into account by allowing for individual-specific variables, as we shall show shortly. We use the term *individual* in a generic sense to include microunits such as individuals, firms, states, and countries.

2. By combining time series of cross-section observations, panel data give “more informative data, more variability, less collinearity among variables, more degrees of freedom and more efficiency.”

<sup>1</sup>Some of the references are G. Chamberlain, “Panel Data,” in *Handbook of Econometrics*, vol. II, Z. Griliches and M. D. Intriligator, eds., North-Holland Publishers, 1984, Chap. 22.; C. Hsiao, *Analysis of Panel Data*, Cambridge University Press, 1986; G. G. Judge, R. C. Hill, W. E. Griffiths, H. Lutkepohl, and T. C. Lee, *Introduction to the Theory and Practice of Econometrics*, 2d ed., John Wiley & Sons, New York, 1985, Chap. 11; W. H. Greene, *Econometric Analysis*, 4th ed., Prentice-Hall, Englewood Cliffs, N.J., 2000, Chap. 14; Badi H. Baltagi, *Econometric Analysis of Panel Data*, John Wiley and Sons, New York, 1995; and J. M. Wooldridge, *Econometric Analysis of Cross Section and Panel Data*, MIT Press, Cambridge, Mass., 1999.

<sup>2</sup>Baltagi, op. cit., pp. 3–6.

3. By studying the repeated cross section of observations, panel data are better suited to study the *dynamics of change*. Spells of unemployment, job turnover, and labor mobility are better studied with panel data.

4. Panel data can better detect and measure effects that simply cannot be observed in pure cross-section or pure time series data. For example, the effects of minimum wage laws on employment and earnings can be better studied if we include successive waves of minimum wage increases in the federal and/or state minimum wages.

5. Panel data enables us to study more complicated behavioral models. For example, phenomena such as economies of scale and technological change can be better handled by panel data than by pure cross-section or pure time series data.

6. By making data available for several thousand units, panel data can minimize the bias that might result if we aggregate individuals or firms into broad aggregates.

In short, panel data can enrich empirical analysis in ways that may not be possible if we use only cross-section or time series data. This is not to suggest that there are no problems with panel data modeling. We will discuss them after we cover some theory and discuss an example.

## 16.2 PANEL DATA: AN ILLUSTRATIVE EXAMPLE

To set the stage, let us consider a concrete example. Consider the data given in Table 16.1, which are taken from a famous study of investment theory proposed by Y. Grunfeld.<sup>3</sup>

Grunfeld was interested in finding out how real gross investment ( $Y$ ) depends on the real value of the firm ( $X_2$ ) and real capital stock ( $X_3$ ). Although the original study covered several companies, for illustrative purposes we have obtained data on four companies, General Electric (GE), General Motor (GM), U.S. Steel (US), and Westinghouse. Data for each company on the preceding three variables are available for the period 1935–1954. Thus, there are four cross-sectional units and 20 time periods. In all, therefore, we have 80 observations. A priori,  $Y$  is expected to be positively related to  $X_2$  and  $X_3$ .

In principle, we could run four time series regressions, one for each company or we could run 20 cross-sectional regressions, one for each year, although in the latter case we will have to worry about the degrees of freedom.<sup>4</sup>

<sup>3</sup>Y. Grunfeld, "The Determinants of Corporate Investment," unpublished Ph.D. thesis, Department of Economics, University of Chicago, 1958. The data are reproduced in several books. We have taken them from H. D. Vinod and Aman Ullha, *Recent Advances in Regression Methods*, Marcel Dekker, New York, 1981, pp. 259–261. The Grunfeld study has become a favorite of textbook writers as the data is manageable for illustration purposes.

<sup>4</sup>For each year, we have only four observations on the regressand and the regressors. If we also allow for the intercept, we will have to estimate three parameters, leaving only a single degree of freedom. Obviously, such a regression may not be meaningful.

**TABLE 16.1** INVESTMENT DATA FOR FOUR COMPANIES, 1935–1954

Observation	$I$	$F_{-1}$	$C_{-1}$	Observation	$I$	$F_{-1}$	$C_{-1}$
GE				US			
1935	33.1	1170.6	97.8	1935	209.9	1362.4	53.8
1936	45.0	2015.8	104.4	1936	355.3	1807.1	50.5
1937	77.2	2803.3	118.0	1937	469.9	2673.3	118.1
1938	44.6	2039.7	156.2	1938	262.3	1801.9	260.2
1939	48.1	2256.2	172.6	1939	230.4	1957.3	312.7
1940	74.4	2132.2	186.6	1940	361.6	2202.9	254.2
1941	113.0	1834.1	220.9	1941	472.8	2380.5	261.4
1942	91.9	1588.0	287.8	1942	445.6	2168.6	298.7
1943	61.3	1749.4	319.9	1943	361.6	1985.1	301.8
1944	56.8	1687.2	321.3	1944	288.2	1813.9	279.1
1945	93.6	2007.7	319.6	1945	258.7	1850.2	213.8
1946	159.9	2208.3	346.0	1946	420.3	2067.7	232.6
1947	147.2	1656.7	456.4	1947	420.5	1796.7	264.8
1948	146.3	1604.4	543.4	1948	494.5	1625.8	306.9
1949	98.3	1431.8	618.3	1949	405.1	1667.0	351.1
1950	93.5	1610.5	647.4	1950	418.8	1677.4	357.8
1951	135.2	1819.4	671.3	1951	588.2	2289.5	341.1
1952	157.3	2079.7	726.1	1952	645.2	2159.4	444.2
1953	179.5	2371.6	800.3	1953	641.0	2031.3	623.6
1954	189.6	2759.9	888.9	1954	459.3	2115.5	669.7
GM				WEST			
1935	317.6	3078.5	2.8	1935	12.93	191.5	1.8
1936	391.8	4661.7	52.6	1936	25.90	516.0	0.8
1937	410.6	5387.1	156.9	1937	35.05	729.0	7.4
1938	257.7	2792.2	209.2	1938	22.89	560.4	18.1
1939	330.8	4313.2	203.4	1939	18.84	519.9	23.5
1940	461.2	4643.9	207.2	1940	28.57	628.5	26.5
1941	512.0	4551.2	255.2	1941	48.51	537.1	36.2
1942	448.0	3244.1	303.7	1942	43.34	561.2	60.8
1943	499.6	4053.7	264.1	1943	37.02	617.2	84.4
1944	547.5	4379.3	201.6	1944	37.81	626.7	91.2
1945	561.2	4840.9	265.0	1945	39.27	737.2	92.4
1946	688.1	4900.0	402.2	1946	53.46	760.5	86.0
1947	568.9	3526.5	761.5	1947	55.56	581.4	111.1
1948	529.2	3245.7	922.4	1948	49.56	662.3	130.6
1949	555.1	3700.2	1020.1	1949	32.04	583.8	141.8
1950	642.9	3755.6	1099.0	1950	32.24	635.2	136.7
1951	755.9	4833.0	1207.7	1951	54.38	732.8	129.7
1952	891.2	4924.9	1430.5	1952	71.78	864.1	145.5
1953	1304.4	6241.7	1777.3	1953	90.08	1193.5	174.8
1954	1486.7	5593.6	2226.3	1954	68.60	1188.9	213.5

Notes:  $Y = I$  = gross investment = additions to plant and equipment plus maintenance and repairs, in millions of dollars deflated by  $P_1$   
 $X_2 = F$  = value of the firm = price of common and preferred shares at Dec. 31 (or average price of Dec. 31 and Jan. 31 of the following year) times number of common and preferred shares outstanding plus total book value of debt at Dec. 31, in millions of dollars deflated by  $P_2$   
 $X_3 = C$  = stock of plant and equipment = accumulated sum of net additions to plant and equipment deflated by  $P_1$  minus depreciation allowance deflated by  $P_3$  in these definitions  
 $P_1$  = implicit price deflator of producers' durable equipment (1947 = 100)  
 $P_2$  = implicit price deflator of GNP (1947 = 100)  
 $P_3$  = depreciation expense deflator = 10-year moving average of wholesale price index of metals and metal products (1947 = 100)

Source: Reproduced from H. D. Vinod and Aman Ullah, *Recent Advances in Regression Methods*, Marcel Dekker, New York, 1981, pp. 259–261.

Pooling, or combining, all the 80 observations, we can write the Grunfeld investment function as:

$$\begin{aligned} Y_{it} &= \beta_1 + \beta_2 X_{2it} + \beta_3 X_{3it} + u_{it} \\ i &= 1, 2, 3, 4 \\ t &= 1, 2, \dots, 20 \end{aligned} \quad (16.2.1)$$

where  $i$  stands for the  $i$ th cross-sectional unit and  $t$  for the  $t$ th time period. As a matter of convention, we will let  $i$  denote the cross-section identifier and  $t$  the time identifier. It is assumed that there are a maximum of  $N$  cross-sectional units or observations and a maximum of  $T$  time periods. If each cross-sectional unit has the same number of time series observations, then such a panel (data) is called a **balanced panel**. In the present example we have a balanced panel, as each company in the sample has 20 observations. If the number of observations differs among panel members, we call such a panel an **unbalanced panel**. In this chapter we will largely be concerned with a balanced panel.

Initially, we assume that the  $X$ 's are nonstochastic and that the error term follows the classical assumptions, namely,  $E(u_{it}) \sim N(0, \sigma^2)$ .

Notice carefully the double and triple subscripted notation, which should be self-explanatory.

How do we estimate (16.2.1)? The answer follows.

### 16.3 ESTIMATION OF PANEL DATA REGRESSION MODELS: THE FIXED EFFECTS APPROACH

Estimation of (16.2.1) depends on the assumptions we make about the intercept, the slope coefficients, and the error term,  $u_{it}$ . There are several possibilities<sup>5</sup>:

1. Assume that the intercept and slope coefficients are constant across time and space and the error term captures differences over time and individuals.

2. The slope coefficients are constant but the intercept varies over individuals.

3. The slope coefficients are constant but the intercept varies over individuals and time.

4. All coefficients (the intercept as well as slope coefficients) vary over individuals.

5. The intercept as well as slope coefficients vary over individuals and time.

<sup>5</sup>This discussion is influenced by Judge et al., op. cit., and Hsiao, op. cit., pp. 9–10.

As you can see, each of these cases introduces increasing complexity (and perhaps more reality) in estimating panel data regression models, such as (16.2.1). Of course, the complexity will increase if we add more regressors to the model because of the possibility of collinearity among the regressors.

To cover each of the preceding categories in depth will require a separate book, and there are already several ones on the market.<sup>6</sup> In what follows, we will cover some of the main features of the various possibilities, especially the first four. Our discussion is nontechnical.

### 1. All Coefficients Constant across Time and Individuals

The simplest, and possibly naive, approach is to disregard the space and time dimensions of the pooled data and just estimate the usual OLS regression. That is, stack the 20 observations for each company one on top of the other, thus giving in all 80 observations for each of the variables in the model. The OLS results are as follows

$$\begin{aligned} \hat{Y} &= -63.3041 + 0.1101X_2 + 0.3034X_3 \\ \text{se} &= (29.6124) \quad (0.0137) \quad (0.0493) \\ t &= (-2.1376) \quad (8.0188) \quad (6.1545) \quad \quad \quad \mathbf{(16.3.1)} \\ R^2 &= 0.7565 \quad \text{Durbin-Watson} = 0.2187 \\ n &= 80 \quad \text{df} = 77 \end{aligned}$$

If you examine the results of the **pooled regression**, and applying the conventional criteria, you will see that all the coefficients are individually statistically significant, the slope coefficients have the expected positive signs and the  $R^2$  value is reasonably high. As expected,  $Y$  is positively related to  $X_2$  and  $X_3$ . The “only” fly in the ointment is that the estimated Durbin–Watson statistic is quite low, suggesting that perhaps there is autocorrelation in the data. Of course, as we know, a low Durbin–Watson value could be due to specification errors also. For instance, the estimated model assumes that the intercept value of GE, GM, US, and Westinghouse are the same. It also assumes that the slope coefficients of the two  $X$  variables are all identical for all the four firms. Obviously, these are highly restricted assumptions. Therefore, despite its simplicity, the pooled regression (16.2.1) may distort the true picture of the relationship between  $Y$  and the  $X$ ’s across the four companies. What we need to do is find some way to take into account the specific nature of the four companies. How this can be done is explained next.

<sup>6</sup>Besides the books mentioned in footnote 1, see Terry E. Dielman, *Pooled Cross-sectional and Time Series Data Analysis*, Marcel Dekker, New York, 1989, and Lois W. Sayrs, *Pooled Time Series Analysis*, Sage Publications, Newbury Park, California, 1989.



## 2. Slope Coefficients Constant but the Intercept Varies across Individuals: The Fixed Effects or Least-Squares Dummy Variable (LSDV) Regression Model

One way to take into account the “individuality” of each company or each cross-sectional unit is to let the intercept vary for each company but still assume that the slope coefficients are constant across firms. To see this, we write model (16.2.1) as:

$$Y_{it} = \beta_{1i} + \beta_2 X_{2it} + \beta_3 X_{3it} + u_{it} \quad (16.3.2)$$

Notice that we have put the subscript  $i$  on the intercept term to suggest that the intercepts of the four firms may be different; the differences may be due to special features of each company, such as managerial style or managerial philosophy.

In the literature, model (16.3.2) is known as the **fixed effects** (regression) model (**FEM**). The term “fixed effects” is due to the fact that, although the intercept may differ across individuals (here the four companies), each individual’s intercept does not vary over time; that is, it is *time invariant*. Notice that if we were to write the intercept as  $\beta_{1it}$ , it will suggest that the intercept of each company or individual is *time variant*. It may be noted that the FEM given in (16.3.2) assumes that the (slope) coefficients of the regressors do not vary across individuals or over time.

How do we actually allow for the (fixed effect) intercept to vary between companies? We can easily do that by the dummy variable technique that we learned in Chapter 9, particularly, the **differential intercept dummies**. Therefore, we write (16.3.2) as:

$$Y_{it} = \alpha_1 + \alpha_2 D_{2i} + \alpha_3 D_{3i} + \alpha_4 D_{4i} + \beta_2 X_{2it} + \beta_3 X_{3it} + u_{it} \quad (16.3.3)$$

where  $D_{2i} = 1$  if the observation belongs to GM, 0 otherwise;  $D_{3i} = 1$  if the observation belongs to US, 0 otherwise; and  $D_{4i} = 1$  if the observation belongs to WEST, 0 otherwise. Since we have four companies, we have used only three dummies to avoid falling into the **dummy-variable trap** (i.e., the situation of perfect collinearity). Here there is no dummy for GE. In other words,  $\alpha_1$  represents the intercept of GE and  $\alpha_2$ ,  $\alpha_3$ , and  $\alpha_4$ , the *differential intercept* coefficients, tell by how much the intercepts of GM, US, and WEST differ from the intercept of GE. In short, GE becomes the comparison company. Of course, you are free to choose any company as the comparison company.

Incidentally, if you want explicit intercept values for each company, you can introduce four dummy variables provided you run your regression through the origin, that is, drop the common intercept in (16.3.3); if you do not do this, you will fall into the dummy variable trap.

Since we are using dummies to estimate the fixed effects, in the literature the model (16.3.3) is also known as the **least-squares dummy variable (LSDV) model**. So, the terms fixed effects and LSDV can be used inter-

changeably. In passing, note that the LSDV model (16.3.3) is also known as the **covariance model** and  $X_2$  and  $X_3$  are known as *covariates*.

The results based on (16.3.3) are as follows:

$$\begin{aligned} \hat{Y}_{it} &= -245.7924 + 161.5722D_{2i} + 339.6328D_{3i} + 186.5666D_{3i} + 0.1079X_{2i} + 0.3461X_{3i} \\ \text{se} &= (35.8112) \quad (46.4563) \quad (23.9863) \quad (31.5068) \quad (0.0175) \quad (0.0266) \\ t &= (-6.8635) \quad (3.4779) \quad (14.1594) \quad (5.9214) \quad (6.1653) \quad (12.9821) \\ R^2 &= 0.9345 \quad d = 1.1076 \quad \text{df} = 74 \quad \mathbf{(16.3.4)} \end{aligned}$$

Compare this regression with (16.3.1). In (16.3.4) all the estimated coefficients are individually highly significant, as the  $p$  values of the estimated  $t$  coefficients are extremely small. The intercept values of the four companies are statistically different; being  $-245.7924$  for GE,  $-84.220 (= -245.7924 + 161.5722)$  for GM,  $93.8774 (= -245.7924 + 339.6328)$  for US, and  $-59.2258 (= -245.7924 + 186.5666)$  for WEST. These differences in the intercepts may be due to unique features of each company, such as differences in management style or managerial talent.

Which model is better—(16.3.1) or (16.3.4)? The answer should be obvious, judged by the statistical significance of the estimated coefficients, and the fact that the  $R^2$  value has increased substantially and the fact that the Durbin–Watson  $d$  value is much higher, suggesting that model (16.3.1) was mis-specified. The increased  $R^2$  value, however, should not be surprising as we have more variables in model (16.3.4).

We can also provide a formal test of the two models. In relation to (16.3.4), model (16.3.1) is a restricted model in that it imposes a common intercept on all the companies. Therefore, we can use the **restricted  $F$  test** discussed in Chapter 8. Using formula (8.7.10), the reader can easily check that in the present instance the  $F$  value is:

$$F \frac{(R_{UR}^2 - R_R^2)/3}{(1 - R_{UR}^2)/74} = \frac{(0.9345 - 0.7565)/3}{(1 - 0.9345)/74} = 66.9980 \quad \mathbf{(16.3.5)}$$

where the restricted  $R^2$  value is from (16.3.1) and the unrestricted  $R^2$  is from (16.3.4) and where the number of restrictions is 3, since model (16.3.1) assumes that the intercepts of the GE, GM, US, and WEST are the same.

Clearly, the  $F$  value of 66.9980 (for 3 numerator df and 74 denominator df) is highly significant and, therefore, the restricted regression (16.3.1) seems to be invalid.

**The Time Effect.** Just as we used the dummy variables to account for individual (company) effect, we can allow for *time effect* in the sense that the Grunfeld investment function shifts over time because of factors such as technological changes, changes in government regulatory and/or tax policies, and external effects such as wars or other conflicts. Such time effects

can be easily accounted for if we introduce time dummies, one for each year. Since we have data for 20 years, from 1935 to 1954, we can introduce 19 time dummies (why?), and write the model (16.3.3) as:

$$Y_{it} = \lambda_0 + \lambda_1 \text{Dum35} + \lambda_2 \text{Dum36} + \cdots + \lambda_{19} \text{Dum53} + \beta_2 X_{2it} + \beta_3 X_{3it} + u_{it} \quad (16.3.6)$$

where Dum35 takes a value of 1 for observation in year 1935 and 0 otherwise, etc. We are treating the year 1954 as the base year, whose intercept value is given by  $\lambda_0$  (why?)

We are not presenting the regression results based on (16.3.6), for none of the individual time dummies were individually statistically significant. The  $R^2$  value of (16.3.6) was 0.7697, whereas that of (16.3.1) was 0.7565, an increment of only 0.0132. It is left as an exercise for the reader to show that, on the basis of the restricted  $F$  test, this increment is not significant, which probably suggests that the year or time effect is not significant. This might suggest that perhaps the investment function has not changed much over time.

We have already seen that the individual company effects were statistically significant, but the individual year effects were not. Could it be that our model is mis-specified in that we have not taken into account both individual and time effects together? Let us consider this possibility.

### 3. Slope Coefficients Constant but the Intercept Varies over Individuals As Well As Time

To consider this possibility, we can combine (16.3.4) and (16.3.6), as follows:

$$Y_{it} = \alpha_1 + \alpha_2 D_{GM_i} + \alpha_3 D_{US_i} + \alpha_4 D_{WEST_i} + \lambda_0 + \lambda_1 \text{Dum35} + \cdots + \lambda_{19} \text{Dum53} + \beta_2 X_{2i} + \beta_3 X_{3i} + u_{it} \quad (16.3.7)$$

When we run this regression, we find the company dummies as well as the coefficients of the  $X$  are individually statistically significant, but none of the time dummies are. Essentially, we are back to (16.3.4).

The overall conclusion that emerges is that perhaps there is pronounced individual company effect but no time effect. In other words, the investment functions for the four companies are the same except for their intercepts. In all the cases we have considered, the  $X$  variables had a strong impact on  $Y$ .

### 4. All Coefficients Vary across Individuals

Here we assume that the intercepts and the slope coefficients are different for all individual, or cross-section, units. This is to say that the investment functions of GE, GM, US, and WEST are all different. We can easily extend our LSDV model to take care of this situation. Reconsider (16.3.4). There we introduced the individual dummies in an *additive* manner. But in Chapter 9

on dummy variables, we showed how *interactive*, or *differential*, *slope dummies*, can account for differences in slope coefficients. To do this in the context of the Grunfeld investment function, what we have to do is multiply each of the company dummies by each of the  $X$  variables [this will add six more variables to (16.3.4)]. That is, we estimate the following model:

$$\begin{aligned}
 Y_{it} = & \alpha_1 + \alpha_2 D_{2i} + \alpha_3 D_{3i} + \alpha_4 D_{4i} + \beta_2 X_{2it} + \beta_3 X_{3it} + \gamma_1 (D_{2i} X_{2it}) + \gamma_2 (D_{2i} X_{3it}) \\
 & + \gamma_3 (D_{3i} X_{2it}) + \gamma_4 (D_{3i} X_{3it}) + \gamma_5 (D_{4i} X_{2it}) + \gamma_6 (D_{4i} X_{3it}) + u_{it} \quad \mathbf{(16.3.8)}
 \end{aligned}$$

You will notice that the  $\gamma$ 's are the *differential slope coefficients*, just as  $\alpha_2$ ,  $\alpha_3$ , and  $\alpha_4$  are the *differential intercepts*. If one or more of the  $\gamma$  coefficients are statistically significant, it will tell us that one or more slope coefficients are different from the base group. For example, say  $\beta_2$  and  $\gamma_1$  are statistically significant. In this case  $(\beta_2 + \gamma_1)$  will give the value of the slope coefficient of  $X_2$  for General Motors, suggesting that the GM slope coefficient of  $X_2$  is different from that of General Electric, which is our comparison company.

If all the differential intercept and all the differential slope coefficients are statistically significant, we can conclude that the investment functions of General Motors, United States Steel, and Westinghouse are different from that of General Electric. If this is in fact the case, there may be little point in estimating the pooled regression (16.3.1).

Let us examine the regression results based on (16.3.8). For ease of reading, the regression results of (16.3.8) are given in tabular form in Table 16.2.

As these results reveal,  $Y$  is significantly related to  $X_2$  and  $X_3$ . However, several differential slope coefficients are statistically significant. For instance, the slope coefficient of  $X_2$  is 0.0902 for GE, but 0.1828 (0.0902 + 0.092) for GM. Interestingly, none of the differential intercepts are statistically significant.

**TABLE 16.2** RESULTS OF REGRESSION (16.3.8)

Variable	Coefficient	Std. error	$t$ value	$p$ value
Intercept	-9.9563	76.3518	-0.1304	0.8966
$D_{2i}$	-139.5104	109.2808	-1.2766	0.2061
$D_{3i}$	-40.1217	129.2343	-0.3104	0.7572
$D_{4i}$	9.3759	93.1172	0.1006	0.9201
$X_{2i}$	0.0926	0.0424	2.1844	0.0324
$X_{3i}$	0.1516	0.0625	2.4250	0.0180
$D_{2i}X_{2i}$	0.0926	0.0424	2.1844	0.0324
$D_{2i}X_{3i}$	0.2198	0.0682	3.2190	0.0020
$D_{3i}X_{2i}$	0.1448	0.0646	2.2409	0.0283
$D_{3i}X_{3i}$	0.2570	0.1204	2.1333	0.0365
$D_{4i}X_{2i}$	0.0265	0.1114	0.2384	0.8122
$D_{4i}X_{3i}$	-0.0600	0.3785	-0.1584	0.8745

$R^2 = 0.9511$       $d = 1.0896$

All in all, it seems that the investment functions of the four companies are different. This might suggest that the data of the four companies are not “poolable,” in which case one can estimate the investment functions for each company separately. (See exercise 16.13.) This is a reminder that panel data regression models may not be appropriate in each situation, despite the availability of both time series and cross-sectional data.

**A Caution on the Use of the Fixed Effects, or LSDV, Model.** Although easy to use, the LSDV model has some problems that need to be borne in mind.

*First*, if you introduce too many dummy variables, as in the case of model (16.3.7), you will run up against the degrees of freedom problem. In the case of (16.3.7), we have 80 observations, but only 55 degrees of freedom—we lose 3 df for the three company dummies, 19 df for the 19 year dummies, 2 for the two slope coefficients, and 1 for the common intercept.

*Second*, with so many variables in the model, there is always the possibility of multicollinearity, which might make precise estimation of one or more parameters difficult.

*Third*, suppose in the FEM (16.3.1) we also include variables such as sex, color, and ethnicity, which are time invariant too because an individual's sex, color, or ethnicity does not change over time. Hence, the LSDV approach may not be able to identify the impact of such time-invariant variables.

*Fourth*, we have to think carefully about the error term  $u_{it}$ . All the results we have presented so far are based on the assumption that the error term follows the classical assumptions, namely,  $u_{it} \sim N(0, \sigma^2)$ . Since the  $i$  index refers to cross-sectional observations and  $t$  to time series observations, the classical assumption for  $u_{it}$  may have to be modified. There are several possibilities.

1. We can assume that the error variance is the same for all cross-section units or we can assume that the error variance is heteroscedastic.

2. For each individual we can assume that there is no autocorrelation over time. Thus, for example, we can assume that the error term of the investment function for General Motors is nonautocorrelated. Or we could assume that it is autocorrelated, say, of the AR(1) type.

3. For a given time, it is possible that the error term for General Motors is correlated with the error term for, say, U.S. Steel or both U.S. Steel and Westinghouse.<sup>7</sup> Or, we could assume that there is no such correlation.

4. We can think of other permutations and combinations of the error term. As you will quickly realize, allowing for one or more of these possibilities will make the analysis that much more complicated. Space and mathematical demands preclude us from considering all the possibilities. A somewhat accessible discussion of the various possibilities can be found in

<sup>7</sup>This leads to the so-called **seemingly unrelated regression (SURE) modeling**, originally proposed by Arnold Zellner. For a discussion of this model, see Terry E. Dielman, *op. cit.*

Dielman, Sayers, and Kmenta.<sup>8</sup> However, some of the problems *may* be alleviated if we resort to the so-called **random effects model**, which we discuss next.

#### 16.4 ESTIMATION OF PANEL DATA REGRESSION MODELS: THE RANDOM EFFECTS APPROACH

Although straightforward to apply, fixed effects, or LSDV, modeling can be expensive in terms of degrees of freedom if we have several cross-sectional units. Besides, as Kmenta notes:

An obvious question in connection with the covariance [i.e., LSDV] model is whether the inclusion of the dummy variables—and the consequent loss of the number of degrees of freedom—is really necessary. The reasoning underlying the covariance model is that in specifying the regression model we have failed to include relevant explanatory variables that do not change over time (and possibly others that do change over time but have the same value for all cross-sectional units), and that the inclusion of dummy variables is a *cover up of our ignorance* [emphasis added].<sup>9</sup>

If the dummy variables do in fact represent a lack of knowledge about the (true) model, why not express this ignorance through the disturbance term  $u_{it}$ ? This is precisely the approach suggested by the proponents of the so-called **error components model (ECM) or random effects model (REM)**.

The basic idea is to start with (16.3.2):

$$Y_{it} = \beta_{1i} + \beta_2 X_{2it} + \beta_3 X_{3it} + u_{it} \quad (16.4.1)$$

Instead of treating  $\beta_{1i}$  as fixed, we assume that it is a random variable with a mean value of  $\beta_1$  (no subscript  $i$  here). And the intercept value for an individual company can be expressed as

$$\beta_{1i} = \beta_1 + \varepsilon_i \quad i = 1, 2, \dots, N \quad (16.4.2)$$

where  $\varepsilon_i$  is a random error term with a mean value of zero and variance of  $\sigma_\varepsilon^2$ .

What we are essentially saying is that the four firms included in our sample are a drawing from a much larger universe of such companies and that they have a common mean value for the intercept ( $= \beta_1$ ) and the individual differences in the intercept values of each company are reflected in the error term  $\varepsilon_i$ .

Substituting (16.4.2) into (16.4.1), we obtain:

$$\begin{aligned} Y_{it} &= \beta_1 + \beta_2 X_{2it} + \beta_3 X_{3it} + \varepsilon_i + u_{it} \\ &= \beta_1 + \beta_2 X_{2it} + \beta_3 X_{3it} + w_{it} \end{aligned} \quad (16.4.3)$$

<sup>8</sup>Dielman, op. cit., Sayers, op. cit., Jan Kmenta, *Elements of Econometrics*, 2d ed., Macmillan, New York, 1986, Chap. 12.

<sup>9</sup>Kmenta, op. cit., p. 633.

where

$$w_{it} = \varepsilon_i + u_{it} \quad (16.4.4)$$

The composite error term  $w_{it}$  consists of two components,  $\varepsilon_i$ , which is the cross-section, or individual-specific, error component, and  $u_{it}$ , which is the combined time series and cross-section error component. The term *error components model* derives its name because the composite error term  $w_{it}$  consists of two (or more) error components.

The usual assumptions made by ECM are that

$$\begin{aligned} \varepsilon_i &\sim N(0, \sigma_\varepsilon^2) \\ u_{it} &\sim N(0, \sigma_u^2) \\ E(\varepsilon_i u_{it}) &= 0 \quad E(\varepsilon_i \varepsilon_j) = 0 \quad (i \neq j) \\ E(u_{it} u_{is}) &= E(u_{it} u_{jt}) = E(u_{it} u_{js}) = 0 \quad (i \neq j; t \neq s). \end{aligned} \quad (16.4.5)$$

that is, the individual error components are not correlated with each other and are not autocorrelated across both cross-section and time series units.

Notice carefully the difference between FEM and ECM. In FEM each cross-sectional unit has its own (fixed) intercept value, in all  $N$  such values for  $N$  cross-sectional units. In ECM, on the other hand, the intercept  $\beta_1$  represents the mean value of all the (cross-sectional) intercepts and the error component  $\varepsilon_i$  represents the (random) deviation of individual intercept from this mean value. However, keep in mind that  $\varepsilon_i$  is not directly observable; it is what is known as an **unobservable**, or **latent, variable**.

As a result of the assumptions stated in (16.4.5), it follows that

$$E(w_{it}) = 0 \quad (16.4.6)$$

$$\text{var}(w_{it}) = \sigma_\varepsilon^2 + \sigma_u^2 \quad (16.4.7)$$

Now if  $\sigma_\varepsilon^2 = 0$ , there is no difference between models (16.2.1) and (16.4.3), in which case we can simply pool all the (cross-sectional and time series) observations and just run the pooled regression, as we did in (16.3.1).

As (16.4.7) shows, the error term  $w_{it}$  is homoscedastic. However, it can be shown that  $w_{it}$  and  $w_{is}$  ( $t \neq s$ ) are correlated; that is, the error terms of a given cross-sectional unit at two different points in time are correlated. The correlation coefficient,  $\text{corr}(w_{it}, w_{is})$ , is as follows:

$$\text{corr}(w_{it}, w_{is}) = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_u^2} \quad (16.4.8)$$

Notice two special features of the preceding correlation coefficient. *First*, for any given cross-sectional unit, the value of the correlation between error terms at two different times remains the same no matter how far apart the



two time periods are, as is clear from (16.4.8). This is in strong contrast to the first-order [AR(1)] scheme that we discussed in Chapter 12, where we found that the correlation between time periods declines over time. *Second*, the correlation structure given in (16.4.8) remains the same for all cross-sectional units; that is, it is identical for all individuals.

If we do not take this correlation structure into account, and estimate (16.4.3) by OLS, the resulting estimators will be inefficient. The most appropriate method here is the method of *generalized least squares* (GLS).

We will not discuss the mathematics of GLS in the present context because of its complexity.<sup>10</sup> Since most modern statistical software packages now have routines to estimate ECM (as well as FEM), we will only present the results for our investment example. But before we do that, it may be noted that we can easily extend (16.4.4) to allow for a random error component to take into account variation over time (see exercise 16.6).

The results of ECM estimation of the Grunfeld investment function are presented in Table 16.3. Several aspects of this regression should be noted. *First*, if you sum the random effect values given for the four companies, it will be zero, as it should (why?). *Second*, the mean value of the random error component,  $\varepsilon_i$ , is the common intercept value of  $-73.0353$ . The random effect value of GE of  $-169.9282$  tells us by how much the random error component of GE differs from the common intercept value. Similar interpretation applies to the other three values of the random effects. *Third*, the  $R^2$  value is obtained from the transformed GLS regression.

If you compare the results of the ECM model given in Table 16.3 with those obtained from FEM, you will see that generally the coefficient values of the two  $X$  variables do not seem to differ much, except for those given in Table 16.2, where we allowed the slope coefficients of the two variables to differ across cross-sectional units.

**TABLE 16.3** ECM ESTIMATION OF THE GRUNFELD INVESTMENT FUNCTION

Variable	Coefficient	Std. error	t statistic	p value
Intercept	-73.0353	83.9495	-0.8699	0.3870
$X_2$	0.1076	0.0168	6.4016	0.0000
$X_3$	0.3457	0.0168	13.0235	0.0000
Random effect:				
GE	-169.9282			
GM	-9.5078			
USS	165.5613			
Westinghouse	13.87475			
$R^2 = 0.9323$ (GLS)				

<sup>10</sup>The interested reader may refer to Kmenta, op. cit., pp. 625–630 for an accessible discussion.



## 16.5 FIXED EFFECTS (LSDV) VERSUS RANDOM EFFECTS MODEL

The challenge facing a researcher is: Which model is better, FEM or ECM? The answer to this question hinges around the assumption one makes about the likely correlation between the individual, or cross-section specific, error component  $\varepsilon_i$  and the  $X$  regressors.

If it is assumed that  $\varepsilon_i$  and the  $X$ 's are *uncorrelated*, ECM may be appropriate, whereas if  $\varepsilon_i$  and the  $X$ 's are *correlated*, FEM may be appropriate.

Why would one expect correlation between the individual error component  $\varepsilon_i$  and one or more regressors? Consider an example. Suppose we have a random sample of a large number of individuals and we want to model their wage, or earnings, function. Suppose earnings are a function of education, work experience, etc. Now if we let  $\varepsilon_i$  stand for innate ability, family background, etc., then when we model the earnings function including  $\varepsilon_i$  it is very likely to be correlated with education, for innate ability and family background are often crucial determinants of education. As Wooldridge contends, "In many applications, the whole reason for using panel data is to allow the unobserved effect [i.e.,  $\varepsilon_i$ ] to be correlated with the explanatory variables."<sup>11</sup>

The assumptions underlying ECM is that the  $\varepsilon_i$  are a random drawing from a much larger population. But sometimes this may not be so. For example, suppose we want to study the crime rate across the 50 states in the United States. Obviously, in this case, the assumption that the 50 states are a random sample is not tenable.

Keeping this fundamental difference in the two approaches in mind, what more can we say about the choice between FEM and ECM? Here the observations made by Judge et al. may be helpful<sup>12</sup>:

1. If  $T$  (the number of time series data) is large and  $N$  (the number of cross-sectional units) is small, there is likely to be little difference in the values of the parameters estimated by FEM and ECM. Hence the choice here is based on computational convenience. On this score, FEM may be preferable.

2. When  $N$  is large and  $T$  is small, the estimates obtained by the two methods can differ significantly. Recall that in ECM  $\beta_{1i} = \beta_1 + \varepsilon_i$ , where  $\varepsilon_i$  is the cross-sectional random component, whereas in FEM we treat  $\beta_{1i}$  as fixed and not random. In the latter case, statistical inference is conditional on the observed cross-sectional units in the sample. This is appropriate if we strongly believe that the individual, or cross-sectional, units in our sample are not random drawings from a larger sample. In that case, FEM is appropriate. However, if the cross-sectional units in the sample are regarded as random drawings, then ECM is appropriate, for in that case statistical inference is unconditional.

3. If the individual error component  $\varepsilon_i$  and one or more regressors are correlated, then the ECM estimators are biased, whereas those obtained from FEM are unbiased.

<sup>11</sup>Wooldridge, op. cit., p. 450.

<sup>12</sup>Judge et al., op. cit., pp. 489–491.

4. If  $N$  is large and  $T$  is small, and if the assumptions underlying ECM hold, ECM estimators are more efficient than FEM estimators.<sup>13</sup>

Is there a formal test that will help us to choose between FEM and ECM? Yes, a test was developed by Hausman in 1978.<sup>14</sup> We will not discuss the details of this test, for they are beyond the scope of this book.<sup>15</sup> The null hypothesis underlying the Hausman test is that the FEM and ECM estimators do not differ substantially. The test statistic developed by Hausman has an asymptotic  $\chi^2$  distribution. If the null hypothesis is rejected, the conclusion is that ECM is not appropriate and that we may be better off using FEM, in which case statistical inferences will be conditional on the  $\varepsilon_i$  in the sample.

Despite the Hausman test, it is important to keep in mind the warning sounded by Johnston and DiNardo. In deciding between fixed effects or random effects models, they argue that, “. . . there is no simple rule to help the researcher navigate past the Scylla of fixed effects and the Charybdis of measurement error and dynamic selection. Although they are an improvement over cross-section data, panel data do not provide a cure-all for all of an econometrician’s problems.”<sup>16</sup>

## 16.6 PANEL DATA REGRESSIONS: SOME CONCLUDING COMMENTS

As noted at the outset, the topic of panel data modeling is vast and complex. We have barely scratched the surface. Among the topics that we have not discussed, the following may be mentioned.

1. Hypothesis testing with panel data.
2. Heteroscedasticity and autocorrelation in ECM.
3. Unbalanced panel data.
4. Dynamic panel data models in which the lagged value(s) of the regressand ( $Y_{it}$ ) appears as an explanatory variable.
5. Simultaneous equations involving panel data.
6. Qualitative dependent variables and panel data.

One or more of these topics can be found in the references cited in this chapter, and the reader is urged to consult them to learn more about this topic. These references also cite several empirical studies in various areas of business and economics that have used panel data regression models. The beginner is well advised to read some of these applications to get a feel about how researchers have actually implemented such models.

<sup>13</sup>Taylor has shown that for  $T \geq 3$  and  $(N - K) \geq 9$ , where  $K$  is the number of regressors, the statement holds. See W. E. Taylor, “Small Sample Considerations in Estimation from Panel Data,” *Journal of Econometrics*, vol. 13, 1980, pp. 203–223.

<sup>14</sup>J. A. Hausman, “Specification Tests in Econometrics,” *Econometrica*, vol. 46, 1978, pp. 1251–1271.

<sup>15</sup>For the details, see Baltagi, op. cit., pp. 68–73.

<sup>16</sup>Jack Johnson and John DiNardo, *Econometric Methods*, 4th ed., McGraw-Hill, 1997, p. 403.

## 16.7 SUMMARY AND CONCLUSIONS

1. Panel regression models are based on panel data. Panel data consist of observations on the same cross-sectional, or individual, units over several time periods.

2. There are several advantages to using panel data. *First*, they increase the sample size considerably. *Second*, by studying repeated cross-section observations, panel data are better suited to study the dynamics of change. *Third*, panel data enable us to study more complicated behavioral models.

3. Despite their substantial advantages, panel data pose several estimation and inference problems. Since such data involve both cross-section and time dimensions, problems that plague cross-sectional data (e.g., heteroscedasticity) and time series data (e.g., autocorrelation) need to be addressed. There are some additional problems, such as cross-correlation in individual units at the same point in time.

4. There are several estimation techniques to address one or more of these problems. The two most prominent are (1) the fixed effects model (FEM) and (2) the random effects model (REM) or error components model (ECM).

5. In FEM the intercept in the regression model is allowed to differ among individuals in recognition of the fact each individual, or cross-sectional, unit may have some special characteristics of its own. To take into account the differing intercepts, one can use dummy variables. The FEM using dummy variables is known as the least-squares dummy variable (LSDV) model. FEM is appropriate in situations where the individual-specific intercept may be correlated with one or more regressors. A disadvantage of LSDV is that it consumes a lot of degrees of freedom when the number of cross-sectional units,  $N$ , is very large, in which case we will have to introduce  $N$  dummies (but suppress the common intercept term).

6. An alternative to FEM is ECM. In ECM it is assumed that the intercept of an individual unit is a random drawing from a much larger population with a constant mean value. The individual intercept is then expressed as a deviation from this constant mean value. One advantage of ECM over FEM is that it is economical in degrees of freedom, as we do not have to estimate  $N$  cross-sectional intercepts. We need only to estimate the mean value of the intercept and its variance. ECM is appropriate in situations where the (random) intercept of each cross-sectional unit is uncorrelated with the regressors.

7. The Hausman test can be used to decide between FEM and ECM.

8. Despite its increasing popularity in applied research, and despite increasing availability of such data, panel data regressions may not be appropriate in every situation. One has to use some practical judgment in each case.

## EXERCISES

### Questions

- 16.1. What are the special features of (a) cross-section data, (b) time series data, and (c) panel data?

- 16.2. What is meant by a fixed effects model (FEM)? Since panel data have both time and space dimensions, how does FEM allow for both dimensions?
- 16.3. What is meant by an error components model (ECM)? How does it differ from FEM? When is ECM appropriate? And when is FEM appropriate?
- 16.4. Is there a difference in FEM, least-squares dummy variable (LSDV) model, and covariance model?
- 16.5. When are panel data regression models inappropriate? Give examples.
- 16.6. How would you extend model (16.4.4) to allow for a time error component? In that case what will happen to formulas (16.3.6), (16.3.7), and (16.3.8)?
- 16.7. Refer to the eggs and their price data given in Table 1.1. Which model may be appropriate here, FEM or ECM? And why?
- 16.8. In the regression results in (16.3.4), what are the fixed effects intercepts of the four companies? Are these effects statistically different?
- 16.9. For the investment example discussed in the chapter, Table 16.3 gives the results based on the ECM. If you compare these results with those given in (16.3.4), what general conclusions would you draw?
- 16.10. Based on the Michigan Income Dynamics Study, Hausman attempted to estimate a wage, or earnings, model using a sample of 629 high school graduates, who were followed for a period of 6 years, thus giving in all 3774 observations. The dependent variable in this study was logarithm of wage, and the explanatory variables were age (divided into several age groups), unemployment in the previous year, poor health in the previous year, self-employment, region of residence (South = 1; 0 otherwise), area of residence (rural = 1; 0 otherwise). Hausman used both FEM and ECM. The results are given in Table 16.4 (standard errors in parentheses):
  - a. Do the results make economic sense?
  - b. Is there a vast difference in the results produced by the two models? If so, what might account for these differences?

**TABLE 16.4** WAGE EQUATIONS (DEPENDENT VARIABLE: LOG WAGE\*)

Variable	Fixed effects	Random effects
1. Age 1 (20–35)	0.0557 (0.0042)	0.0393 (0.0033)
2. Age 2 (35–45)	0.0351 (0.0051)	0.0092 (0.0036)
3. Age 3 (45–55)	0.0209 (0.0055)	–0.0007 (0.0042)
4. Age 4 (55–65)	0.0209 (0.0078)	–0.0097 (0.0060)
5. Age 5 (65–)	–0.0171 (0.0155)	–0.0423 (0.0121)
6. Unemployed previous year	–0.0042 (0.0153)	–0.0277 (0.0151)
7. Poor health previous year	–0.0204 (0.0221)	–0.0250 (0.0215)
8. Self-employment	–0.2190 (0.0297)	–0.2670 (0.0263)
9. South	–0.1569 (0.0656)	–0.0324 (0.0333)
10. Rural	–0.0101 (0.0317)	–0.1215 (0.0237)
11. Constant	— —	0.8499 (0.0433)
$s^2$	0.0567	0.0694
Degrees of freedom	3,135	3,763

\*3774 observations; standard errors are in parentheses.

Reproduced from Cheng Hsiao, *Analysis of Panel Data*, Cambridge University Press, 1986, p. 42. *Original source*: J. A. Hausman, "Specification Tests in Econometrics," *Econometrica*, vol. 46, 1978, pp. 1251–1271.

- c. On the basis of the data given in the table, which model, if any, would you choose?

### Problems

- 16.11.** Refer to the data in Table 1.1.
- Let  $Y$  = eggs produced (in millions) and  $X$  = price of eggs (cents per dozen). Estimate the model:  $Y_i = \beta_1 + \beta_2 X_i + u_i$  for the years 1990 and 1991 separately.
  - Pool the observations for the 2 years and estimate the pooled regression. What assumptions are you making in pooling the data?
  - Use the fixed effects model, distinguishing the 2 years, and present the regression results.
  - Can you use the fixed effects model, distinguishing the 50 states? Why or why not?
  - Would it make sense to distinguish both the state effect and the year effect? If so, how many dummy variables would you have to introduce?
  - Would the error components model be appropriate to model the production of eggs? Why or why not? See if you can estimate such a model using, say, Eviews.
- 16.12.** Continue with exercise 16.11. Before deciding to run the pooled regression, you want to find out whether the data are “poolable.” For this purpose you decide to use the Chow test discussed in Chapter 8. Show the necessary calculations involved and determine if the pooled regression makes any sense.
- 16.13.** Return to the Grunfeld investment function discussed in Section 16.2.
- Estimate the Grunfeld investment function for GE, GM, U.S. Steel, and Westinghouse individually. The results of pooling all the 80 observations are already given in (16.3.1).
  - To determine whether the pooled regression (16.3.1) is appropriate, you decide to conduct the Chow test discussed in Chapter 8. Carry out this test. *Hint:* Get the RSS from the pooled regression, get the RSS from each of the four investment functions, and then apply the Chow test.
  - From the Chow test, what conclusions do you draw? If your conclusion is not to pool the data, what then can you say about the utility of panel data regression techniques?
- 16.14.** Table 16.5 gives data on the civilian unemployment rate  $Y$  (%) and manufacturing hourly compensation in U.S. dollars  $X$  (index, 1992 = 100) for Canada, the United Kingdom, and the United States for the period 1980–1999. Consider the model:

$$Y_{it} = \beta_1 + \beta_2 X_{it} + u_{it} \quad (1)$$

- A priori, what is the expected relationship between  $Y$  and  $X$ ? Why?
- Estimate the model given in (1) for each country.
- Estimate the model, pooling all the 60 observations.
- Estimate the fixed effects model.
- Estimate the error components model.
- Which is a better model, FEM or ECM? Justify your answer.

**TABLE 16.5** UNEMPLOYMENT RATE AND HOURLY COMPENSATION IN MANUFACTURING, IN THE UNITED STATES, CANADA, AND THE UNITED KINGDOM, 1980–1999

Observation	United States		Canada		United Kingdom	
	Compensation, \$/hour	Unemployment, %	Compensation, \$/hour	Unemployment, %	Compensation, \$/hour	Unemployment, %
1980	55.6	7.1	49.0	7.2	43.7	7.0
1981	61.1	7.6	54.1	7.3	44.1	10.5
1982	67.0	9.7	59.6	10.6	42.2	11.3
1983	68.8	9.6	63.9	11.5	39.0	11.8
1984	71.2	7.5	64.3	10.9	37.2	11.7
1985	75.1	7.2	63.5	10.2	39.0	11.2
1986	78.5	7.0	63.3	9.2	47.8	11.2
1987	80.7	6.2	68.0	8.4	60.2	10.3
1988	84.0	5.5	76.0	7.3	68.3	8.6
1989	86.6	5.3	84.1	7.0	67.7	7.2
1990	90.8	5.6	91.5	7.7	81.7	6.9
1991	95.6	6.8	100.1	9.8	90.5	8.8
1992	100.0	7.5	100.0	10.6	100.0	10.1
1993	102.7	6.9	95.5	10.7	88.7	10.5
1994	105.6	6.1	91.7	9.4	92.3	9.7
1995	107.9	5.6	93.3	8.5	95.9	8.7
1996	109.3	5.4	93.1	8.7	95.6	8.2
1997	111.4	4.9	94.4	8.2	103.3	7.0
1998	117.3	4.5	90.6	7.5	109.8	6.3
1999	123.2	4.0	91.9	5.7	112.2	6.1

Hourly compensation is in U.S. dollars, index 1992 = 100.

Source: *Economic Report of the President*, January 2001, Table B109, p. 399.

# 17

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## DYNAMIC ECONOMETRIC MODELS: AUTOREGRESSIVE AND DISTRIBUTED-LAG MODELS

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In regression analysis involving time series data, if the regression model includes not only the current but also the lagged (past) values of the explanatory variables (the  $X$ 's), it is called a **distributed-lag model**. If the model includes one or more lagged values of the dependent variable among its explanatory variables, it is called an **autoregressive model**. Thus,

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + u_t$$

represents a distributed-lag model, whereas

$$Y_t = \alpha + \beta X_t + \gamma Y_{t-1} + u_t$$

is an example of an autoregressive model. The latter are also known as **dynamic models** since they portray the time path of the dependent variable in relation to its past value(s).

Autoregressive and distributed-lag models are used extensively in econometric analysis, and in this chapter we take a close look at such models with a view to finding out the following:

1. What is the role of lags in economics?
2. What are the reasons for the lags?
3. Is there any theoretical justification for the commonly used lagged models in empirical econometrics?
4. What is the relationship, if any, between autoregressive and distributed-lag models? Can one be derived from the other?

5. What are some of the statistical problems involved in estimating such models?
6. Does a lead-lag relationship between variables imply causality? If so, how does one measure it?

### 17.1 THE ROLE OF "TIME," OR "LAG," IN ECONOMICS

In economics the dependence of a variable  $Y$  (the dependent variable) on another variable(s)  $X$  (the explanatory variable) is rarely instantaneous. Very often,  $Y$  responds to  $X$  with a lapse of time. Such a lapse of time is called a *lag*. To illustrate the nature of the lag, we consider several examples.

#### EXAMPLE 17.1

##### THE CONSUMPTION FUNCTION

Suppose a person receives a salary increase of \$2000 in annual pay, and suppose that this is a "permanent" increase in the sense that the increase in salary is maintained. What will be the effect of this increase in income on the person's annual consumption expenditure?

Following such a gain in income, people usually do not rush to spend all the increase immediately. Thus, our recipient may decide to increase consumption expenditure by \$800 in the first year following the salary increase in income, by another \$600 in the next year, and by another \$400 in the following year, saving the remainder.

By the end of the third year, the person's annual consumption expenditure will be increased by \$1800. We can thus write the consumption function as

$$Y_t = \text{constant} + 0.4X_t + 0.3X_{t-1} + 0.2X_{t-2} + u_t \quad (17.1.1)$$

where  $Y$  is consumption expenditure and  $X$  is income.

Equation (17.1.1) shows that the effect of an increase in income of \$2000 is spread, or distributed, over a period of 3 years. Models such as (17.1.1) are therefore called **distributed-lag models** because the effect of a given cause (income) is spread over a number of time periods. Geometrically, the distributed-lag model (17.1.1) is shown in Figure 17.1, or alternatively, in Figure 17.2.

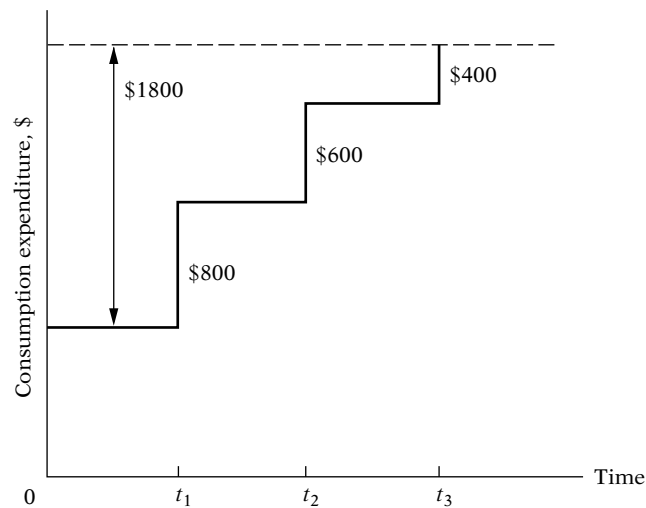
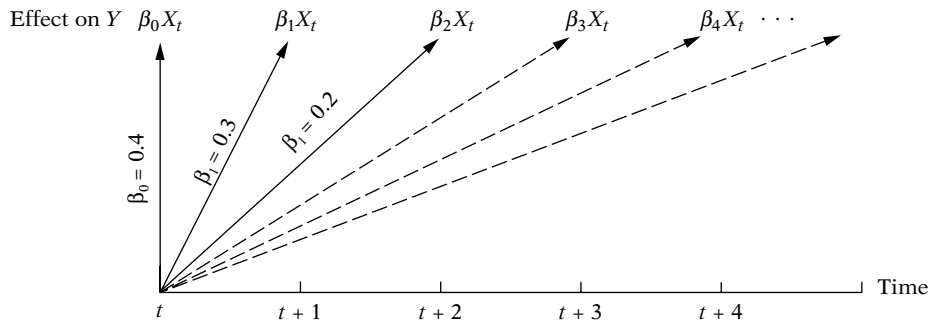


FIGURE 17.1 Example of distributed lags.

(Continued)



**EXAMPLE 17.1** (Continued)



**FIGURE 17.2**  
The effect of a unit change in  $X$  at time  $t$  on  $Y$  at time  $t$  and subsequent time periods.

More generally we may write

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \cdots + \beta_k X_{t-k} + u_t \quad (17.1.2)$$

which is a distributed-lag model with a finite lag of  $k$  time periods. The coefficient  $\beta_0$  is known as the **short-run**, or **impact, multiplier** because it gives the change in the mean value of  $Y$  following a unit change in  $X$  in the same time period.<sup>1</sup> If the change in  $X$  is maintained at the same level thereafter, then,  $(\beta_0 + \beta_1)$  gives the change in (the mean value of)  $Y$  in the next period,  $(\beta_0 + \beta_1 + \beta_2)$  in the following period, and so on. These partial sums are called **interim**, or **intermediate, multipliers**. Finally, after  $k$  periods we obtain

$$\sum_{i=0}^k \beta_i = \beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k = \beta \quad (17.1.3)$$

which is known as the **long-run**, or **total, distributed-lag multiplier**, provided the sum  $\beta$  exists (to be discussed elsewhere).

If we define

$$\beta_i^* = \frac{\beta_i}{\sum \beta_i} = \frac{\beta_i}{\beta} \quad (17.1.4)$$

we obtain “standardized”  $\beta_i$ . Partial sums of the standardized  $\beta_i$  then give the proportion of the long-run, or total, impact felt by a certain time period.

Returning to the consumption regression (17.1.1), we see that the short-run multiplier, which is nothing but the short-run marginal propensity to consume (MPC), is 0.4, whereas the long-run multiplier, which is the

<sup>1</sup>Technically,  $\beta_0$  is the partial derivative of  $Y$  with respect to  $X_t$ ,  $\beta_1$  that with respect to  $X_{t-1}$ ,  $\beta_2$  that with respect to  $X_{t-2}$ , and so forth. Symbolically,  $\partial Y_t / \partial X_{t-k} = \beta_k$ .

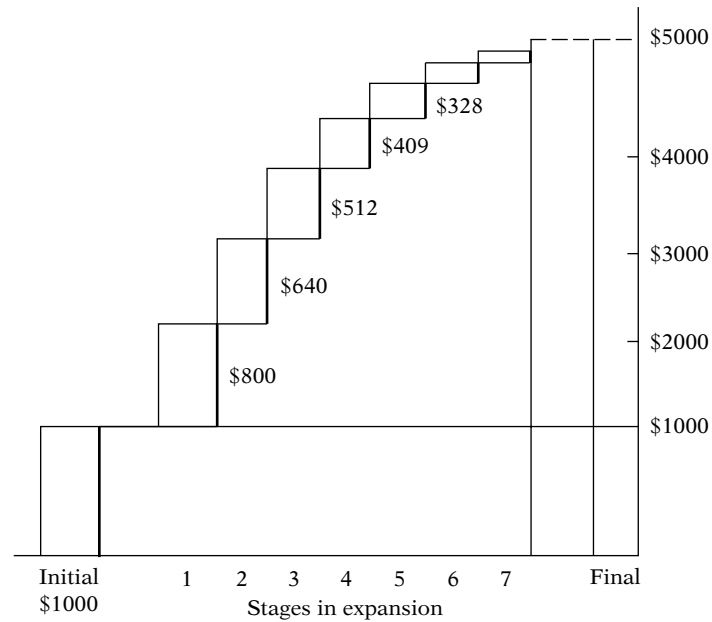
long-run marginal propensity to consume, is  $0.4 + 0.3 + 0.2 = 0.9$ . That is, following a \$1 increase in income, the consumer will increase his or her level of consumption by about 40 cents in the year of increase, by another 30 cents in the next year, and by yet another 20 cents in the following year. The long-run impact of an increase of \$1 in income is thus 90 cents. If we divide each  $\beta_i$  by 0.9, we obtain, respectively, 0.44, 0.33, and 0.23, which indicate that 44 percent of the total impact of a unit change in  $X$  on  $Y$  is felt immediately, 77 percent after one year, and 100 percent by the end of the second year.

**EXAMPLE 17.2**

CREATION OF BANK MONEY (DEMAND DEPOSITS)

Suppose the Federal Reserve System pours \$1000 of new money into the banking system by buying government securities. What will be the total amount of bank money, or demand deposits, that will be generated ultimately?

Following the fractional reserve system, if we assume that the law requires banks to keep a 20 percent reserve backing for the deposits they create, then by the well-known multiplier process the total amount of demand deposits that will be generated will be equal to  $\$1000[1/(1 - 0.8)] = \$5000$ . Of course, \$5000 in demand deposits will not be created overnight. The process takes time, which can be shown schematically in Figure 17.3.



**FIGURE 17.3**  
Cumulative expansion in bank deposits (initial reserve \$1000 and 20 percent reserve requirement).

**EXAMPLE 17.3**

LINK BETWEEN MONEY AND PRICES

According to the monetarists, inflation is essentially a monetary phenomenon in the sense that a continuous increase in the general price level is due to the rate of expansion in money supply far in excess of the amount of money actually demanded by the economic units. Of course, this link between inflation and changes in money supply is not instantaneous. Studies have shown that the lag between the two is anywhere from 3 to about 20 quarters. The results of one such study are shown in Table 17.1,<sup>2</sup> where we see the effect of a 1 percent change in the M1B money supply (= currency + checkable deposits at financial institutions) is felt over a period of 20 quarters. The long-run impact of a 1 percent change in the money supply on inflation is about 1 ( $= \sum m_i$ ), which is statistically significant, whereas the short-run impact is about 0.04, which is not significant, although the intermediate multipliers seem to be generally significant. Incidentally, note that since  $P$  and  $M$  are both in percent forms, the  $m_i$  ( $\beta_i$  in our usual notation) give the elasticity of  $P$  with respect to  $M_i$ , that is, the percent response of prices to a 1 percent increase in the money supply. Thus,  $m_0 = 0.041$  means that for a 1 percent increase in the money supply the short-run elasticity of prices is about 0.04 percent. The long-term elasticity is 1.03 percent, implying that in the long run a 1 percent increase in the money supply is reflected by just about the same percentage increase in the prices. In short, a 1 percent increase in the money supply is accompanied in the long run by a 1 percent increase in the inflation rate.

**TABLE 17.1** ESTIMATE OF MONEY-PRICE EQUATION: ORIGINAL SPECIFICATION

Sample period: 1955–I to 1969–IV:  $m_{21} = 0$

$$\dot{P} = -0.146 + \sum_{i=0}^{20} m_i \dot{M}_{-i}$$

(0.395)

	Coeff.	t		Coeff.	t		Coeff.	t
$m_0$	0.041	1.276	$m_8$	0.048	3.249	$m_{16}$	0.069	3.943
$m_1$	0.034	1.538	$m_9$	0.054	3.783	$m_{17}$	0.062	3.712
$m_2$	0.030	1.903	$m_{10}$	0.059	4.305	$m_{18}$	0.053	3.511
$m_3$	0.029	2.171	$m_{11}$	0.065	4.673	$m_{19}$	0.039	3.338
$m_4$	0.030	2.235	$m_{12}$	0.069	4.795	$m_{20}$	0.022	3.191
$m_5$	0.033	2.294	$m_{13}$	0.072	4.694	$\sum m_i$	1.031	7.870
$m_6$	0.037	2.475	$m_{14}$	0.073	4.468	<b>Mean lag</b>	10.959	5.634
$m_7$	0.042	2.798	$m_{15}$	0.072	4.202			
$\bar{R}^2$	0.525							
se	1.066							
D.W.	2.00							

Notation:  $\dot{P}$  = compounded annual rate of change of GNP deflator  
 $\dot{M}$  = compounded annual rate of change of M1B

Source: Keith M. Carlson, "The Lag from Money to Prices," *Review*, Federal Reserve Bank of St. Louis, October 1980, Table 1, p. 4.

<sup>2</sup>Keith M. Carlson, "The Lag from Money to Prices," *Review*, Federal Reserve Bank of St. Louis, October, 1980, Table 1, p. 4.

**EXAMPLE 17.4**

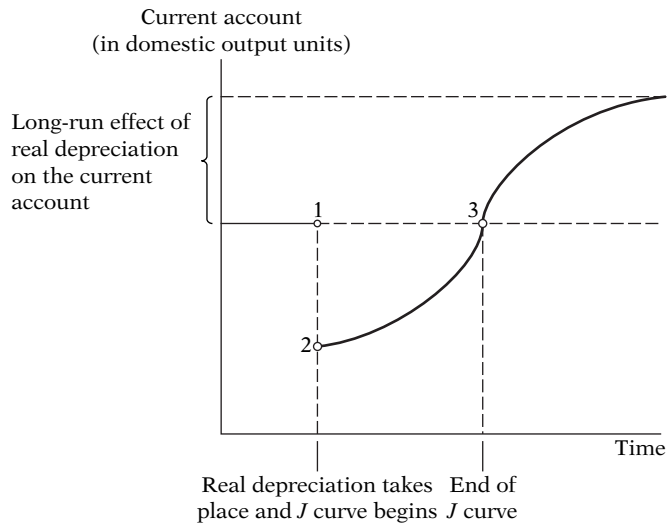
**LAG BETWEEN R&D EXPENDITURE AND PRODUCTIVITY**

The decision to invest in research and development (R&D) expenditure and its ultimate pay-off in terms of increased productivity involve considerable lag, actually several lags, such as, "... the lag between the investment of funds and the time inventions actually begin to appear, the lag between the invention of an idea or device and its development up to a commercially applicable stage, and the lag which is introduced by the process of diffusion: it takes time before all the old machines are replaced by the better new ones."<sup>3</sup>

**EXAMPLE 17.5**

**THE J CURVE OF INTERNATIONAL ECONOMICS**

Students of international economics are familiar with what is called the *J curve*, which shows the relationship between trade balance and depreciation of currency. Following depreciation of a country's currency (e.g., due to devaluation), initially the trade balance deteriorates but eventually it improves, assuming other things are the same. The curve is as shown in Figure 17.4.



**FIGURE 17.4** The *J* curve.

Source: Paul R. Krugman and Maurice Obstfeld, *International Economics: Theory and Practice*, 3d ed., Harper Collins, New York, 1994, p. 465.

<sup>3</sup>Zvi Griliches, "Distributed Lags: A Survey," *Econometrica*, vol. 36, no. 1, January 1967, pp. 16-49.

**EXAMPLE 17.6**

## THE ACCELERATOR MODEL OF INVESTMENT

In its simplest form, the acceleration principle of investment theory states that investment is proportional to changes in output. Symbolically,

$$I_t = \beta(X_t - X_{t-1}) \quad \beta > 0 \quad (17.1.5)$$

where  $I_t$  is investment at time  $t$ ,  $X_t$  is output at time  $t$ , and  $X_{t-1}$  is output at time  $(t - 1)$ .

The preceding examples are only a sample of the use of lag in economics. Undoubtedly, the reader can produce several examples from his or her own experience.

**17.2 THE REASONS FOR LAGS<sup>4</sup>**

Although the examples cited in Section 17.1 point out the nature of lagged phenomena, they do not fully explain why lags occur. There are three main reasons:

**1. Psychological reasons.** As a result of the force of habit (inertia), people do not change their consumption habits immediately following a price decrease or an income increase perhaps because the process of change may involve some immediate disutility. Thus, those who become instant millionaires by winning lotteries may not change the lifestyles to which they were accustomed for a long time because they may not know how to react to such a windfall gain immediately. Of course, given reasonable time, they may learn to live with their newly acquired fortune. Also, people may not know whether a change is “permanent” or “transitory.” Thus, my reaction to an increase in my income will depend on whether or not the increase is permanent. If it is only a nonrecurring increase and in succeeding periods my income returns to its previous level, I may save the entire increase, whereas someone else in my position might decide to “live it up.”

**2. Technological reasons.** Suppose the price of capital relative to labor declines, making substitution of capital for labor economically feasible. Of course, addition of capital takes time (the gestation period). Moreover, if the drop in price is expected to be temporary, firms may not rush to substitute capital for labor, especially if they expect that after the temporary drop the price of capital may increase beyond its previous level. Sometimes, imperfect knowledge also accounts for lags. At present the market for personal computers is glutted with all kinds of computers with varying features and prices. Moreover, since their introduction in the late 1970s, the prices of most personal computers have dropped dramatically. As a result, prospective consumers for the personal computer may hesitate to buy until they have

<sup>4</sup>This section leans heavily on Marc Nerlove, *Distributed Lags and Demand Analysis for Agricultural and Other Commodities*, Agricultural Handbook No. 141, U.S. Department of Agriculture, June 1958.

had time to look into the features and prices of all the competing brands. Moreover, they may hesitate to buy in the expectation of further decline in price or innovations.

**3. Institutional reasons.** These reasons also contribute to lags. For example, contractual obligations may prevent firms from switching from one source of labor or raw material to another. As another example, those who have placed funds in long-term savings accounts for fixed durations such as 1 year, 3 years, or 7 years, are essentially “locked in” even though money market conditions may be such that higher yields are available elsewhere. Similarly, employers often give their employees a choice among several health insurance plans, but once a choice is made, an employee may not switch to another plan for at least 1 year. Although this may be done for administrative convenience, the employee is locked in for 1 year.

For the reasons just discussed, lag occupies a central role in economics. This is clearly reflected in the short-run–long-run methodology of economics. It is for this reason we say that short-run price or income elasticities are generally smaller (in absolute value) than the corresponding long-run elasticities or that short-run marginal propensity to consume is generally smaller than long-run marginal propensity to consume.

### 17.3 ESTIMATION OF DISTRIBUTED-LAG MODELS

Granted that distributed-lag models play a highly useful role in economics, how does one estimate such models? Specifically, suppose we have the following distributed-lag model in one explanatory variable:<sup>5</sup>

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \cdots + u_t \quad (17.3.1)$$

where we have not defined the length of the lag, that is, how far back into the past we want to go. Such a model is called an **infinite (lag) model**, whereas a model of the type (17.1.2) is called a **finite (lag) distributed-lag model** because the length of the lag  $k$  is specified. We shall continue to use (17.3.1) because it is easy to handle mathematically, as we shall see.<sup>6</sup>

How do we estimate the  $\alpha$  and  $\beta$ 's of (17.3.1)? We may adopt two approaches: (1) ad hoc estimation and (2) a priori restrictions on the  $\beta$ 's by assuming that the  $\beta$ 's follow some systematic pattern. We shall consider ad hoc estimation in this section and the other approach in Section 17.4.

#### Ad Hoc Estimation of Distributed-Lag Models

Since the explanatory variable  $X_t$  is assumed to be nonstochastic (or at least uncorrelated with the disturbance term  $u_t$ ),  $X_{t-1}$ ,  $X_{t-2}$ , and so on, are nonstochastic, too. Therefore, in principle, the ordinary least squares (OLS) can

<sup>5</sup>If there is more than one explanatory variable in the model, each variable may have a lagged effect on  $Y$ . For simplicity only, we assume one explanatory variable.

<sup>6</sup>In practice, however, the coefficients of the distant  $X$  values are expected to have negligible effect on  $Y$ .

be applied to (17.3.1). This is the approach taken by Alt<sup>7</sup> and Tinbergen.<sup>8</sup> They suggest that to estimate (17.3.1) one may proceed *sequentially*; that is, first regress  $Y_t$  on  $X_t$ , then regress  $Y_t$  on  $X_t$  and  $X_{t-1}$ , then regress  $Y_t$  on  $X_t$ ,  $X_{t-1}$ , and  $X_{t-2}$ , and so on. This sequential procedure stops when the regression coefficients of the lagged variables start becoming statistically insignificant and/or the coefficient of at least one of the variables changes signs from positive to negative or vice versa. Following this precept, Alt regressed fuel oil consumption  $Y$  on new orders  $X$ . Based on the quarterly data for the period 1930–1939, the results were as follows:

$$\hat{Y}_t = 8.37 + 0.171X_t$$

$$\hat{Y}_t = 8.27 + 0.111X_t + 0.064X_{t-1}$$

$$\hat{Y}_t = 8.27 + 0.109X_t + 0.071X_{t-1} - 0.055X_{t-2}$$

$$\hat{Y}_t = 8.32 + 0.108X_t + 0.063X_{t-1} + 0.022X_{t-2} - 0.020X_{t-3}$$

Alt chose the second regression as the “best” one because in the last two equations the sign of  $X_{t-2}$  was not stable and in the last equation the sign of  $X_{t-3}$  was negative, which may be difficult to interpret economically.

Although seemingly straightforward, ad hoc estimation suffers from many drawbacks, such as the following:

1. There is no a priori guide as to what is the maximum length of the lag.<sup>9</sup>
2. As one estimates successive lags, there are fewer degrees of freedom left, making statistical inference somewhat shaky. Economists are not usually that lucky to have a long series of data so that they can go on estimating numerous lags.
3. More importantly, in economic time series data, successive values (lags) tend to be highly correlated; hence multicollinearity rears its ugly head. As noted in Chapter 10, multicollinearity leads to imprecise estimation; that is, the standard errors tend to be large in relation to the estimated coefficients. As a result, based on the routinely computed  $t$  ratios, we may tend to declare (erroneously), that a lagged coefficient(s) is statistically insignificant.
4. The sequential search for the lag length opens the researcher to the charge of **data mining**. Also, as we noted in Section 13.4, the nominal and true level of significance to test statistical hypotheses becomes an important issue in such sequential searches [see Eq. (13.4.2)].

In view of the preceding problems, the ad hoc estimation procedure has very little to recommend it. Clearly, some prior or theoretical considerations must be brought to bear upon the various  $\beta$ 's if we are to make headway with the estimation problem.

<sup>7</sup>F. F. Alt, “Distributed Lags,” *Econometrica*, vol. 10, 1942, pp. 113–128.

<sup>8</sup>J. Tinbergen, “Long-Term Foreign Trade Elasticities,” *Metroeconomica*, vol. 1, 1949, pp. 174–185.

<sup>9</sup>If the lag length,  $k$ , is incorrectly specified, we will have to contend with the problem of misspecification errors discussed in Chap. 13. Also keep in mind the warning about **data mining**.

### 17.4 THE KOYCK APPROACH TO DISTRIBUTED-LAG MODELS

Koyck has proposed an ingenious method of estimating distributed-lag models. Suppose we start with the infinite lag distributed-lag model (17.3.1). Assuming that the  $\beta$ 's are all of the same sign, Koyck assumes that they decline geometrically as follows.<sup>10</sup>

$$\beta_k = \beta_0 \lambda^k \quad k = 0, 1, \dots \quad (17.4.1)^{11}$$

where  $\lambda$ , such that  $0 < \lambda < 1$ , is known as the *rate of decline*, or *decay*, of the distributed lag and where  $1 - \lambda$  is known as the *speed of adjustment*.

What (17.4.1) postulates is that each successive  $\beta$  coefficient is numerically less than each preceding  $\beta$  (this statement follows since  $\lambda < 1$ ), implying that as one goes back into the distant past, the effect of that lag on  $Y_t$  becomes progressively smaller; a quite plausible assumption. After all, current and recent past incomes are expected to affect current consumption expenditure more heavily than income in the distant past. Geometrically, the Koyck scheme is depicted in Figure 17.5.

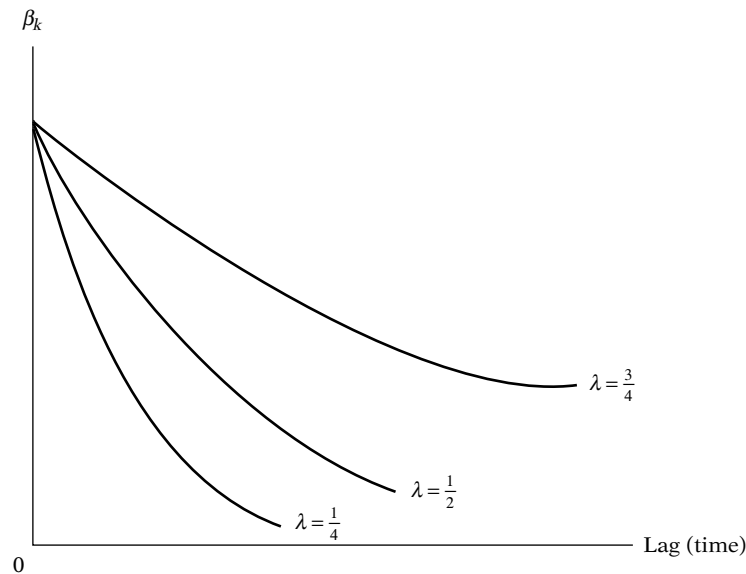


FIGURE 17.5 Koyck scheme (declining geometric distribution).

<sup>10</sup>L. M. Koyck, *Distributed Lags and Investment Analysis*, North Holland Publishing Company, Amsterdam, 1954.

<sup>11</sup>Sometimes this is also written as

$$\beta_k = \beta_0 (1 - \lambda) \lambda^k \quad k = 0, 1, \dots$$

for reasons given in footnote 12.



As this figure shows, the value of the lag coefficient  $\beta_k$  depends, apart from the common  $\beta_0$ ; on the value of  $\lambda$ . The closer  $\lambda$  is to 1, the slower the rate of decline in  $\beta_k$ , whereas the closer it is to zero, the more rapid the decline in  $\beta_k$ . In the former case, distant past values of  $X$  will exert sizable impact on  $Y_t$ , whereas in the latter case their influence on  $Y_t$  will peter out quickly. This pattern can be seen clearly from the following illustration:

$\lambda$	$\beta_0$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$	$\beta_5$	...	$\beta_{10}$
0.75	$\beta_0$	$0.75\beta_0$	$0.56\beta_0$	$0.42\beta_0$	$0.32\beta_0$	$0.24\beta_0$	...	$0.06\beta_0$
0.25	$\beta_0$	$0.25\beta_0$	$0.06\beta_0$	$0.02\beta_0$	$0.004\beta_0$	$0.001\beta_0$	...	0.0

Note these features of the Koyck scheme: (1) By assuming nonnegative values for  $\lambda$ , Koyck rules out the  $\beta$ 's from changing sign; (2) by assuming  $\lambda < 1$ , he gives lesser weight to the distant  $\beta$ 's than the current ones; and (3) he ensures that the sum of the  $\beta$ 's, which gives the long-run multiplier, is finite, namely,

$$\sum_{k=0}^{\infty} \beta_k = \beta_0 \left( \frac{1}{1 - \lambda} \right) \quad (17.4.2)^{12}$$

As a result of (17.4.1), the infinite lag model (17.3.1) may be written as

$$Y_t = \alpha + \beta_0 X_t + \beta_0 \lambda X_{t-1} + \beta_0 \lambda^2 X_{t-2} + \dots + u_t \quad (17.4.3)$$

As it stands, the model is still not amenable to easy estimation since a large (literally infinite) number of parameters remain to be estimated and the parameter  $\lambda$  enters in a highly nonlinear form: Strictly speaking, the method of linear (in the parameters) regression analysis cannot be applied to such a model. But now Koyck suggests an ingenious way out. He lags (17.4.3) by one period to obtain

$$Y_{t-1} = \alpha + \beta_0 X_{t-1} + \beta_0 \lambda X_{t-2} + \beta_0 \lambda^2 X_{t-3} + \dots + u_{t-1} \quad (17.4.4)$$

He then multiplies (17.4.4) by  $\lambda$  to obtain

$$\lambda Y_{t-1} = \lambda \alpha + \lambda \beta_0 X_{t-1} + \beta_0 \lambda^2 X_{t-2} + \beta_0 \lambda^3 X_{t-3} + \dots + \lambda u_{t-1} \quad (17.4.5)$$

<sup>12</sup>This is because

$$\sum \beta_k = \beta_0 (1 + \lambda + \lambda^2 + \lambda^3 + \dots) = \beta_0 \left( \frac{1}{1 - \lambda} \right)$$

since the expression in the parentheses on the right side is an infinite geometric series whose sum is  $1/(1 - \lambda)$  provided  $0 < \lambda < 1$ . In passing, note that if  $\beta_k$  is as defined in footnote 11,  $\sum \beta_k = \beta_0(1 - \lambda)/(1 - \lambda) = \beta_0$  thus ensuring that the weights  $(1 - \lambda)\lambda^k$  sum to one.

Subtracting (17.4.5) from (17.4.3), he gets

$$Y_t - \lambda Y_{t-1} = \alpha(1 - \lambda) + \beta_0 X_t + (u_t - \lambda u_{t-1}) \quad (17.4.6)$$

or, rearranging,

$$Y_t = \alpha(1 - \lambda) + \beta_0 X_t + \lambda Y_{t-1} + v_t \quad (17.4.7)$$

where  $v_t = (u_t - \lambda u_{t-1})$ , a moving average of  $u_t$  and  $u_{t-1}$ .

The procedure just described is known as the **Koyck transformation**. Comparing (17.4.7) with (17.3.1), we see the tremendous simplification accomplished by Koyck. Whereas before we had to estimate  $\alpha$  and an infinite number of  $\beta$ 's, we now have to estimate only three unknowns:  $\alpha$ ,  $\beta_0$ , and  $\lambda$ . Now there is no reason to expect multicollinearity. In a sense multicollinearity is resolved by replacing  $X_{t-1}$ ,  $X_{t-2}$ ,  $\dots$ , by a single variable, namely,  $Y_{t-1}$ . But note the following features of the Koyck transformation:

1. We started with a distributed-lag model but ended up with an autoregressive model because  $Y_{t-1}$  appears as one of the explanatory variables. This transformation shows how one can “convert” a distributed-lag model into an autoregressive model.

2. The appearance of  $Y_{t-1}$  is likely to create some statistical problems.  $Y_{t-1}$ , like  $Y_t$ , is stochastic, which means that we have a stochastic explanatory variable in the model. Recall that the classical least-squares theory is predicated on the assumption that the explanatory variables either are non-stochastic or, if stochastic, are distributed independently of the stochastic disturbance term. Hence, we must find out if  $Y_{t-1}$  satisfies this assumption. (We shall return to this point in Section 17.8.)

3. In the original model (17.3.1) the disturbance term was  $u_t$ , whereas in the transformed model it is  $v_t = (u_t - \lambda u_{t-1})$ . The statistical properties of  $v_t$  depend on what is assumed about the statistical properties of  $u_t$ , for, as shown later, if the original  $u_t$ 's are serially uncorrelated, the  $v_t$ 's are serially correlated. Therefore, we may have to face up to the serial correlation problem in addition to the stochastic explanatory variable  $Y_{t-1}$ . We shall do that in Section 17.8.

4. The presence of lagged  $Y$  violates one of the assumptions underlying the Durbin–Watson  $d$  test. Therefore, we will have to develop an alternative to test for serial correlation in the presence of lagged  $Y$ . One alternative is the **Durbin  $h$  test**, which is discussed in Section 17.10.

As we saw in (17.1.4), the partial sums of the standardized  $\beta_i$  tell us the proportion of the long-run, or total, impact felt by a certain time period. In

practice, though, the **mean** or **median lag** is often used to characterize the nature of the lag structure of a distributed lag model.

### The Median Lag

The median lag is the time required for the first half, or 50 percent, of the total change in  $Y$  following a unit sustained change in  $X$ . For the Koyck model, the median lag is as follows (see exercise 17.6):

$$\text{Koyck model: Median lag} = -\frac{\log 2}{\log \lambda} \quad (17.4.8)$$

Thus, if  $\lambda = 0.2$  the median lag is 0.4306, but if  $\lambda = 0.8$  the median lag is 3.1067. Verbally, in the former case 50 percent of the total change in  $Y$  is accomplished in less than half a period, whereas in the latter case it takes more than 3 periods to accomplish the 50 percent change. But this contrast should not be surprising, for as we know, the higher the value of  $\lambda$  the lower the speed of adjustment, and the lower the value of  $\lambda$  the greater the speed of adjustment.

### The Mean Lag

Provided all  $\beta_k$  are positive, the mean, or average, lag is defined as

$$\text{Mean lag} = \frac{\sum_0^{\infty} k\beta_k}{\sum_0^{\infty} \beta_k} \quad (17.4.9)$$

which is simply the weighted average of all the lags involved, with the respective  $\beta$  coefficients serving as weights. In short, it is a **lag-weighted average** of time. For the Koyck model the mean lag is (see exercise 17.7)

$$\text{Koyck model: Mean lag} = \frac{\lambda}{1 - \lambda} \quad (17.4.10)$$

Thus, if  $\lambda = \frac{1}{2}$ , the mean lag is 1.

From the preceding discussion it is clear that the median and mean lags serve as a summary measure of the speed with which  $Y$  responds to  $X$ . In the example given in Table 17.1 the mean lag is about 11 quarters, showing that it takes quite some time, on the average, for the effect of changes in the money supply to be felt on price changes.

**EXAMPLE 17.7**

PER CAPITA PERSONAL CONSUMPTION

This example examines per capita personal consumption expenditure (PPCE) in relation to per capita disposable income (PPDI) in the United States for the period 1970–1999, all data in chained 1996 dollars. As an illustration of the Koyck model, consider the data given in Table 17.2. Regression of PPCE on PPDI and lagged PPCE gave the following results:

$$\widehat{PPCE}_t = -1242.169 + 0.6033PPDI_t + 0.4106PPCE_{t-1}$$

$$se = (402.5784) \quad (0.1502) \quad (0.1546) \quad (17.4.11)$$

$$t = (-3.0855) \quad (4.0155) \quad (2.6561)$$

$$R^2 = 0.9926 \quad d = 1.0056 \quad \text{Durbin } h = 5.119$$

*Note:* The calculation of Durbin  $h$  is discussed in Section 17.10.

If we assume that this model resulted from a Koyck-type transformation,  $\lambda$  is 0.4106. The median lag is:

$$-\frac{\log(2)}{\log \lambda} = -\frac{\log(2)}{\log(0.4106)} = 0.7786$$

and the mean lag is:

$$\frac{\lambda}{1 - \lambda} = \frac{0.4106}{0.5894} = 0.6966$$

In words, it seems that PPCE adjusts to PPDI within a relatively short time.

**TABLE 17.2** PPCE AND PPDI, 1970–1999

Observation	PPCE	PPDI	Observation	PPCE	PPDI
1970	11,300	12,823	1985	16,020	18,229
1971	11,581	13,218	1986	16,541	18,641
1972	12,149	13,692	1987	16,398	18,870
1973	12,626	14,496	1988	17,463	19,522
1974	12,407	14,268	1989	17,760	19,833
1975	12,551	14,393	1990	17,899	20,058
1976	13,155	14,873	1991	17,677	19,919
1977	13,583	15,256	1992	17,989	20,318
1978	14,035	15,845	1993	18,399	20,384
1979	14,230	16,120	1994	18,910	20,709
1980	14,021	16,063	1995	19,294	21,055
1981	14,069	16,265	1996	19,727	21,385
1982	14,105	16,328	1997	20,232	21,838
1983	14,741	16,673	1998	20,989	22,672
1984	15,401	17,799	1999	21,901	23,191

*Notes:* PPCE = per capita personal consumption expenditure, in 1996 dollars.

PPDI = per capita personal disposable income, in 1996 dollars.

*Source:* *Economic Report of the President, 2001, Table B-31, p. 311.*

**17.5 RATIONALIZATION OF THE KOYCK MODEL:  
THE ADAPTIVE EXPECTATIONS MODEL**

Although very neat, the Koyck model (17.4.7) is ad hoc since it was obtained by a purely algebraic process; it is devoid of any theoretical underpinning. But this gap can be filled if we start from a different perspective. Suppose we postulate the following model:

$$Y_t = \beta_0 + \beta_1 X_t^* + u_t \quad (17.5.1)$$

where  $Y$  = demand for money (real cash balances)

$X^*$  = equilibrium, optimum, expected long-run or normal rate of interest

$u$  = error term

Equation (17.5.1) postulates that the demand for money is a function of *expected* (in the sense of anticipation) rate of interest.

Since the expectational variable  $X^*$  is not directly observable, let us propose the following hypothesis about how expectations are formed:

$$X_t^* - X_{t-1}^* = \gamma(X_t - X_{t-1}^*) \quad (17.5.2)^{13}$$

where  $\gamma$ , such that  $0 < \gamma \leq 1$ , is known as the **coefficient of expectation**. Hypothesis (17.5.2) is known as the **adaptive expectation, progressive expectation**, or **error learning** hypothesis, popularized by Cagan<sup>14</sup> and Friedman.<sup>15</sup>

What (17.5.2) implies is that “economic agents will adapt their expectations in the light of past experience and that in particular they will learn from their mistakes.”<sup>16</sup> More specifically, (17.5.2) states that expectations are revised each period by a fraction  $\gamma$  of the gap between the current value of the variable and its previous expected value. Thus, for our model this would mean that expectations about interest rates are revised each period by a fraction  $\gamma$  of the discrepancy between the rate of interest observed in the current period and what its anticipated value had been in the previous period. Another way of stating this would be to write (17.5.2) as

$$X_t^* = \gamma X_t + (1 - \gamma)X_{t-1}^* \quad (17.5.3)$$

<sup>13</sup>Sometimes the model is expressed as

$$X_t^* - X_{t-1}^* = \gamma(X_{t-1} - X_{t-1}^*)$$

<sup>14</sup>P. Cagan, “The Monetary Dynamics of Hyperinflations,” in M. Friedman (ed.), *Studies in the Quantity Theory of Money*, University of Chicago Press, Chicago, 1956.

<sup>15</sup>Milton Friedman, *A Theory of the Consumption Function*, National Bureau of Economic Research, Princeton University Press, Princeton, N.J., 1957.

<sup>16</sup>G. K. Shaw, *Rational Expectations: An Elementary Exposition*, St. Martin's Press, New York, 1984, p. 25.

which shows that the expected value of the rate of interest at time  $t$  is a weighted average of the actual value of the interest rate at time  $t$  and its value expected in the previous period, with weights of  $\gamma$  and  $1 - \gamma$ , respectively. If  $\gamma = 1$ ,  $X_t^* = X_t$ , meaning that expectations are realized immediately and fully, that is, in the same time period. If, on the other hand,  $\gamma = 0$ ,  $X_t^* = X_{t-1}^*$ , meaning that expectations are static, that is, “conditions prevailing today will be maintained in all subsequent periods. Expected future values then become identified with current values.”<sup>17</sup>

Substituting (17.5.3) into (17.5.1), we obtain

$$\begin{aligned} Y_t &= \beta_0 + \beta_1[\gamma X_t + (1 - \gamma)X_{t-1}^*] + u_t \\ &= \beta_0 + \beta_1\gamma X_t + \beta_1(1 - \gamma)X_{t-1}^* + u_t \end{aligned} \quad (17.5.4)$$

Now lag (17.5.1) one period, multiply it by  $1 - \gamma$ , and subtract the product from (17.5.4). After simple algebraic manipulations, we obtain

$$\begin{aligned} Y_t &= \gamma\beta_0 + \gamma\beta_1 X_t + (1 - \gamma)Y_{t-1} + u_t - (1 - \gamma)u_{t-1} \\ &= \gamma\beta_0 + \gamma\beta_1 X_t + (1 - \gamma)Y_{t-1} + v_t \end{aligned} \quad (17.5.5)$$

where  $v_t = u_t - (1 - \gamma)u_{t-1}$ .

Before proceeding any further, let us note the difference between (17.5.1) and (17.5.5). In the former,  $\beta_1$  measures the average response of  $Y$  to a unit change in  $X^*$ , the equilibrium or long-run value of  $X$ . In (17.5.5), on the other hand,  $\gamma\beta_1$  measures the average response of  $Y$  to a unit change in the actual or observed value of  $X$ . These responses will not be the same unless, of course,  $\gamma = 1$ , that is, the current and long-run values of  $X$  are the same. In practice, we first estimate (17.5.5). Once an estimate of  $\gamma$  is obtained from the coefficient of lagged  $Y$ , we can easily compute  $\beta_1$  by simply dividing the coefficient of  $X_t$  ( $= \gamma\beta_1$ ) by  $\gamma$ .

The similarity between the adaptive expectation model (17.5.5) and the Koyck model (17.4.7) should be readily apparent although the interpretations of the coefficients in the two models are different. Note that like the Koyck model, the adaptive expectations model is autoregressive and its error term is similar to the Koyck error term. We shall return to the estimation of the adaptive expectations model in Section 17.8 and to some examples in Section 17.12. Now that we have sketched the adaptive expectations (AE) model, how realistic is it? It is true that it is more appealing than the purely algebraic Koyck approach, but is the AE hypothesis reasonable? In favor of the AE hypothesis one can say the following:

It provides a fairly simple means of modelling expectations in economic theory whilst postulating a mode of behaviour upon the part of economic agents which

<sup>17</sup>Ibid., pp. 19–20.

seems eminently sensible. The belief that people learn from experience is obviously a more sensible starting point than the implicit assumption that they are totally devoid of memory, characteristic of static expectations thesis. Moreover, the assertion that more distant experiences exert a lesser effect than more recent experience would accord with common sense and would appear to be amply confirmed by simple observation.<sup>18</sup>

Until the advent of the **rational expectations (RE) hypothesis**, initially put forward by J. Muth and later propagated by Robert Lucas and Thomas Sargent, the AE hypothesis was quite popular in empirical economics. The proponents of the RE hypothesis contend that the AE hypothesis is inadequate because it relies solely on the past values of a variable in formulating expectations,<sup>19</sup> whereas the RE hypothesis assumes, “that individual economic agents use *current available* and *relevant* information in forming their expectations and do not rely purely upon past experience.”<sup>20</sup> In short, the RE hypothesis contends that “expectations are ‘rational’ in the sense that they efficiently incorporate *all* information available at the time the expectation is formulated”<sup>21</sup> and not just the past information.

The criticism directed by the RE proponents against the AE hypothesis is well-taken, although there are many critics of the RE hypothesis itself.<sup>22</sup> This is not the place to get bogged down with this rather heady material. Perhaps one could agree with Stephen McNees that, “At best, the adaptive expectations assumption can be defended only as a ‘working hypothesis’ proxying for a more complex, perhaps changing expectations formulation mechanism.”<sup>23</sup>

#### EXAMPLE 17.8

##### EXAMPLE 17.7 REVISITED

If we consider the model given in Eq. (17.4.11), as generated by the adaptive expectations mechanism (i.e., PPCE as a function of expected PPDI), then  $\gamma$ , the expectations coefficient can be obtained from (17.5.5) as:  $\gamma = 1 - 0.4106 = 0.5894$ . Then, following the preceding discussion about the AE model, we can say that about 59 percent of the discrepancy between actual and expected PPCE is eliminated within a year.

<sup>18</sup>Ibid., p. 27.

<sup>19</sup>Like the Koyck model, it can be shown that, under AE, expectations of a variable are an exponentially weighted average of past values of that variable.

<sup>20</sup>G. K. Shaw, op. cit., p. 47. For additional details of the RE hypothesis, see Steven M. Sheffrin, *Rational Expectations*, Cambridge University Press, New York, 1983.

<sup>21</sup>Stephen K. McNees, “The Phillips Curve: Forward- or Backward-Looking?” *New England Economic Review*, July–August 1979, p. 50.

<sup>22</sup>For a recent critical appraisal of the RE hypothesis, see Michael C. Lovell, “Test of the Rational Expectations Hypothesis,” *American Economic Review*, March 1966, pp. 110–124.

<sup>23</sup>Stephen K. McNees, op. cit., p. 50.

### 17.6 ANOTHER RATIONALIZATION OF THE KOYCK MODEL: THE STOCK ADJUSTMENT, OR PARTIAL ADJUSTMENT, MODEL

The adaptive expectation model is one way of rationalizing the Koyck model. Another rationalization is provided by Marc Nerlove in the so-called **stock adjustment** or **partial adjustment model (PAM)**.<sup>24</sup> To illustrate this model, consider the **flexible accelerator model** of economic theory, which assumes that there is an *equilibrium, optimal, desired, or long-run* amount of capital stock needed to produce a given output under the given state of technology, rate of interest, etc. For simplicity assume that this desired level of capital  $Y_t^*$  is a linear function of output  $X$  as follows:

$$Y_t^* = \beta_0 + \beta_1 X_t + u_t \quad (17.6.1)$$

Since the desired level of capital is not directly observable, Nerlove postulates the following hypothesis, known as the **partial adjustment, or stock adjustment, hypothesis**:

$$Y_t - Y_{t-1} = \delta(Y_t^* - Y_{t-1}) \quad (17.6.2)^{25}$$

where  $\delta$ , such that  $0 < \delta \leq 1$ , is known as the **coefficient of adjustment** and where  $Y_t - Y_{t-1}$  = actual change and  $(Y_t^* - Y_{t-1})$  = desired change.

Since  $Y_t - Y_{t-1}$ , the change in capital stock between two periods, is nothing but investment, (17.6.2) can alternatively be written as

$$I_t = \delta(Y_t^* - Y_{t-1}) \quad (17.6.3)$$

where  $I_t$  = investment in time period  $t$ .

Equation (17.6.2) postulates that the actual change in capital stock (investment) in any given time period  $t$  is some fraction  $\delta$  of the desired change for that period. If  $\delta = 1$ , it means that the actual stock of capital is equal to the desired stock; that is, actual stock adjusts to the desired stock instantaneously (in the same time period). However, if  $\delta = 0$ , it means that nothing changes since actual stock at time  $t$  is the same as that observed in the previous time period. Typically,  $\delta$  is expected to lie between these extremes since adjustment to the desired stock of capital is likely to be

<sup>24</sup>Marc Nerlove, *Distributed Lags and Demand Analysis for Agricultural and Other Commodities*, op. cit.

<sup>25</sup>Some authors do not add the stochastic disturbance term  $u_t$  to the relation (17.6.1) but add it to this relation, believing that if the former is truly an equilibrium relation, there is no scope for the error term, whereas the adjustment mechanism can be imperfect and may require the disturbance term. In passing, note that (17.6.2) is sometimes also written as

$$Y_t - Y_{t-1} = \delta(Y_{t-1}^* - Y_{t-1})$$



incomplete because of rigidity, inertia, contractual obligations, etc.—hence the name **partial adjustment model**. Note that the adjustment mechanism (17.6.2) alternatively can be written as

$$Y_t = \delta Y_t^* + (1 - \delta)Y_{t-1} \quad (17.6.4)$$

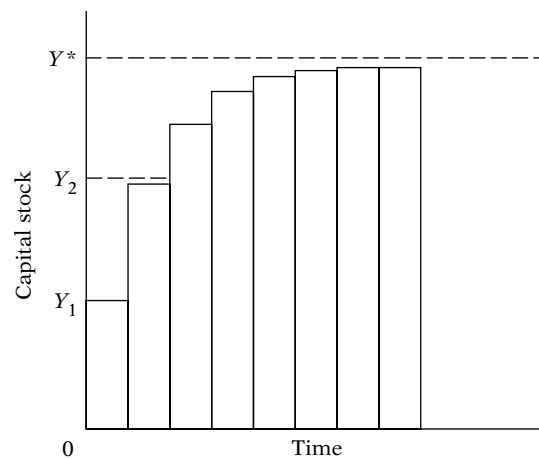
showing that the observed capital stock at time  $t$  is a weighted average of the desired capital stock at that time and the capital stock existing in the previous time period,  $\delta$  and  $(1 - \delta)$  being the weights. Now substitution of (17.6.1) into (17.6.4) gives

$$\begin{aligned} Y_t &= \delta(\beta_0 + \beta_1 X_t + u_t) + (1 - \delta)Y_{t-1} \\ &= \delta\beta_0 + \delta\beta_1 X_t + (1 - \delta)Y_{t-1} + \delta u_t \end{aligned} \quad (17.6.5)$$

This model is called the **partial adjustment model (PAM)**.

Since (17.6.1) represents the long-run, or equilibrium, demand for capital stock, (17.6.5) can be called the *short-run* demand function for capital stock since in the short run the existing capital stock may not necessarily be equal to its long-run level. Once we estimate the short-run function (17.6.5) and obtain the estimate of the adjustment coefficient  $\delta$  (from the coefficient of  $Y_{t-1}$ ), we can easily derive the long-run function by simply dividing  $\delta\beta_0$  and  $\delta\beta_1$  by  $\delta$  and omitting the lagged  $Y$  term, which will then give (17.6.1).

Geometrically, the partial adjustment model can be shown as in Figure 17.6.<sup>26</sup> In this figure  $Y^*$  is the desired capital stock and  $Y_1$  the current actual capital stock. For illustrative purposes assume that  $\delta = 0.5$ . This implies that the firm plans to close half the gap between the actual and the



**FIGURE 17.6** The gradual adjustment of the capital stock.

<sup>26</sup>This is adapted from Figure 7.4 from Rudiger Dornbusch and Stanley Fischer, *Macroeconomics*, 3d ed., McGraw-Hill, New York, 1984, p. 216.

desired stock of capital each period. Thus, in the first period it moves to  $Y_2$ , with investment equal to  $(Y_2 - Y_1)$ , which in turn is equal to half of  $(Y^* - Y_1)$ . In each subsequent period it closes half the gap between the capital stock at the beginning of the period and the desired capital stock  $Y^*$ .

The partial adjustment model resembles both the Koyck and adaptive expectation models in that it is autoregressive. But it has a much simpler disturbance term: the original disturbance term  $u_t$  multiplied by a constant  $\delta$ . But bear in mind that although similar in appearance, the adaptive expectation and partial adjustment models are conceptually very different. The former is based on uncertainty (about the future course of prices, interest rates, etc.), whereas the latter is due to technical or institutional rigidities, inertia, cost of change, etc. However, both of these models are theoretically much sounder than the Koyck model.

Since in appearance the adaptive expectations and partial adjustment models are indistinguishable, the  $\gamma$  coefficient of 0.5894 of the adaptive expectations model can also be interpreted as the  $\delta$  coefficient of the stock adjustment model if we assume that the latter model is operative in the present case (i.e., it is the desired or expected PPCE that is linearly related to the current PDPI).

The important point to keep in mind is that since Koyck, adaptive expectations, and stock adjustment models—apart from the difference in the appearance of the error term—yield the same final estimating model, one must be extremely careful in telling the reader which model the researcher is using and why. Thus, researchers must specify the theoretical underpinning of their model.

### \*17.7 COMBINATION OF ADAPTIVE EXPECTATIONS AND PARTIAL ADJUSTMENT MODELS

Consider the following model:

$$Y_t^* = \beta_0 + \beta_1 X_t^* + u_t \quad (17.7.1)$$

where  $Y_t^*$  = desired stock of capital and  $X_t^*$  = expected level of output.

Since both  $Y_t^*$  and  $X_t^*$  are not directly observable, one could use the partial adjustment mechanism for  $Y_t^*$  and the adaptive expectations model for  $X_t^*$  to arrive at the following estimating equation (see exercise 17.2):

$$\begin{aligned} Y_t &= \beta_0 \delta \gamma + \beta_1 \delta \gamma X_t + [(1 - \gamma) + (1 - \delta)]Y_{t-1} \\ &\quad - (1 - \delta)(1 - \gamma)Y_{t-2} + [\delta u_t - \delta(1 - \gamma)u_{t-1}] \\ &= \alpha_0 + \alpha_1 X_t + \alpha_2 Y_{t-1} + \alpha_3 Y_{t-2} + v_t \end{aligned} \quad (17.7.2)$$

\*Optional.

where  $v_t = \delta[u_t - (1 - \gamma)u_{t-1}]$ . This model too is autoregressive, the only difference from the purely adaptive expectations model being that  $Y_{t-2}$  appears along with  $Y_{t-1}$  as an explanatory variable. Like Koyck and the AE models, the error term in (17.7.2) follows a moving average process. Another feature of this model is that although the model is linear in the  $\alpha$ 's, it is nonlinear in the original parameters.

A celebrated application of (17.7.1) has been Friedman's permanent income hypothesis, which states that "permanent" or long-run consumption is a function of "permanent" or long-run income.<sup>27</sup>

The estimation of (17.7.2) presents the same estimation problems as the Koyck's or the AE model in that all these models are autoregressive with similar error structures. In addition, (17.7.2) involves some nonlinear estimation problems that we consider briefly in exercise 17.10, but do not delve into in this book.

## 17.8 ESTIMATION OF AUTOREGRESSIVE MODELS

From our discussion thus far we have the following three models:

*Koyck*

$$Y_t = \alpha(1 - \lambda) + \beta_0 X_t + \lambda Y_{t-1} + (u_t - \lambda u_{t-1}) \quad (17.4.7)$$

*Adaptive expectation*

$$Y_t = \gamma \beta_0 + \gamma \beta_1 X_t + (1 - \gamma) Y_{t-1} + [u_t - (1 - \gamma) u_{t-1}] \quad (17.5.5)$$

*Partial adjustment*

$$Y_t = \delta \beta_0 + \delta \beta_1 X_t + (1 - \delta) Y_{t-1} + \delta u_t \quad (17.6.5)$$

All these models have the following common form:

$$Y_t = \alpha_0 + \alpha_1 X_t + \alpha_2 Y_{t-1} + v_t \quad (17.8.1)$$

that is, they are all autoregressive in nature. Therefore, we must now look at the estimation problem of such models, because the classical least-squares may not be directly applicable to them. **The reason is twofold: the presence of stochastic explanatory variables and the possibility of serial correlation.**

Now, as noted previously, for the application of the classical least-squares theory, it must be shown that the stochastic explanatory variable  $Y_{t-1}$  is

<sup>27</sup>Milton Friedman, *A Theory of Consumption Function*, Princeton University Press, Princeton, N.J., 1957.

distributed independently of the disturbance term  $v_t$ . To determine whether this is so, it is essential to know the properties of  $v_t$ . If we assume that the original disturbance term  $u_t$  satisfies all the classical assumptions, such as  $E(u_t) = 0$ ,  $\text{var}(u_t) = \sigma^2$  (the assumption of homoscedasticity), and  $\text{cov}(u_t, u_{t+s}) = 0$  for  $s \neq 0$  (the assumption of no autocorrelation),  $v_t$  may not inherit all these properties. Consider, for example, the error term in the Koyck model, which is  $v_t = (u_t - \lambda u_{t-1})$ . Given the assumptions about  $u_t$ , we can easily show that  $v_t$  is serially correlated because

$$E(v_t v_{t-1}) = -\lambda \sigma^2 \quad (17.8.2)^{28}$$

which is nonzero (unless  $\lambda$  happens to be zero). And since  $Y_{t-1}$  appears in the Koyck model as an explanatory variable, it is bound to be correlated with  $v_t$  (via the presence of  $u_{t-1}$  in it). As a matter of fact, it can be shown that

$$\text{cov}[Y_{t-1}, (u_t - \lambda u_{t-1})] = -\lambda \sigma^2 \quad (17.8.3)$$

which is the same as (17.8.2). The reader can verify that the same holds true of the adaptive expectations model.

What is the implication of the finding that in the Koyck model as well as the adaptive expectations model the stochastic explanatory variable  $Y_{t-1}$  is correlated with the error term  $v_t$ ? As noted previously, **if an explanatory variable in a regression model is correlated with the stochastic disturbance term, the OLS estimators are not only biased but also not even consistent; that is, even if the sample size is increased indefinitely, the estimators do not approximate their true population values.**<sup>29</sup> Therefore, estimation of the Koyck and adaptive expectation models by the usual OLS procedure may yield seriously misleading results.

The partial adjustment model is different, however. In this model  $v_t = \delta u_t$ , where  $0 < \delta \leq 1$ . Therefore, if  $u_t$  satisfies the assumptions of the classical linear regression model given previously, so will  $\delta u_t$ . Thus, OLS estimation of the partial adjustment model will yield consistent estimates although the estimates tend to be biased (in finite or small samples).<sup>30</sup> Intuitively, the reason for consistency is this: Although  $Y_{t-1}$  depends on  $u_{t-1}$  and all the

<sup>28</sup> $E(v_t v_{t-1}) = E(u_t - \lambda u_{t-1})(u_{t-1} - \lambda u_{t-2})$   
 $= -\lambda E(u_{t-1})^2$  since covariances between  $u$ 's are zero by assumption  
 $= -\lambda \sigma^2$

<sup>29</sup>The proof is beyond the scope of this book and may be found in Griliches, op. cit., pp. 36–38. However, see Chap. 18 for an outline of the proof in another context. See also Asatoshi Maeshiro, "Teaching Regressions with a Lagged Dependent Variable and Autocorrelated Disturbances," *The Journal of Economic Education*, Winter 1996, vol. 27, no. 1, pp. 72–84.

<sup>30</sup>For proof, see J. Johnston, *Econometric Methods*, 3d ed., McGraw-Hill, New York, 1984, pp. 360–362. See also H. E. Doran and J. W. B. Guise, *Single Equation Methods in Econometrics: Applied Regression Analysis*, University of New England Teaching Monograph Series 3, Armidale, NSW, Australia, 1984, pp. 236–244.

previous disturbance terms, it is not related to the current error term  $u_t$ . Therefore, as long as  $u_t$  is serially independent,  $Y_{t-1}$  will also be independent or at least uncorrelated with  $u_t$ , thereby satisfying an important assumption of OLS, namely, noncorrelation between the explanatory variable(s) and the stochastic disturbance term.

Although OLS estimation of the stock, or partial, adjustment model provides consistent estimation because of the simple structure of the error term in such a model, one should not assume that it applies rather than the Koyck or adaptive expectations model.<sup>31</sup> The reader is strongly advised against doing so. A model should be chosen on the basis of strong theoretical considerations, not simply because it leads to easy statistical estimation. Every model should be considered on its own merit, paying due attention to the stochastic disturbances appearing therein. If in models such as the Koyck or adaptive expectations model OLS cannot be straightforwardly applied, methods need to be devised to resolve the estimation problem. Several alternative estimation methods are available although some of them may be computationally tedious. In the following section we consider one such method.

### 17.9 THE METHOD OF INSTRUMENTAL VARIABLES (IV)

The reason why OLS cannot be applied to the Koyck or adaptive expectations model is that the explanatory variable  $Y_{t-1}$  tends to be correlated with the error term  $v_t$ . If somehow this correlation can be removed, one can apply OLS to obtain consistent estimates, as noted previously. (*Note:* There will be some small sample bias.) How can this be accomplished? Liviatan has proposed the following solution.<sup>32</sup>

Let us suppose that we find a *proxy* for  $Y_{t-1}$  that is highly correlated with  $Y_{t-1}$  but is uncorrelated with  $v_t$ , where  $v_t$  is the error term appearing in the Koyck or adaptive expectations model. Such a proxy is called an **instrumental variable (IV)**.<sup>33</sup> Liviatan suggests  $X_{t-1}$  as the instrumental variable for  $Y_{t-1}$  and further suggests that the parameters of the regression (17.8.1) can be obtained by solving the following normal equations:

$$\begin{aligned}\sum Y_t &= n\hat{\alpha}_0 + \hat{\alpha}_1 \sum X_t + \hat{\alpha}_2 \sum Y_{t-1} \\ \sum Y_t X_t &= \hat{\alpha}_0 \sum X_t + \hat{\alpha}_1 \sum X_t^2 + \hat{\alpha}_2 \sum Y_{t-1} X_t \\ \sum Y_t X_{t-1} &= \hat{\alpha}_0 \sum X_{t-1} + \hat{\alpha}_1 \sum X_t X_{t-1} + \hat{\alpha}_2 \sum Y_{t-1} X_{t-1}\end{aligned}\quad (17.9.1)$$

<sup>31</sup>Also, as J. Johnston notes (op. cit., p. 350), “[the] pattern of adjustment [suggested by the partial adjustment model] . . . may sometimes be implausible.”

<sup>32</sup>N. Liviatan, “Consistent Estimation of Distributed Lags,” *International Economic Review*, vol. 4, January 1963, pp. 44–52.

<sup>33</sup>Such instrumental variables are used frequently in simultaneous equation models (see Chap. 20).

Notice that if we were to apply OLS directly to (17.8.1), the usual OLS normal equations would be (see Section 7.4)

$$\begin{aligned}\sum Y_t &= n\hat{\alpha}_0 + \hat{\alpha}_1 \sum X_t + \hat{\alpha}_2 \sum Y_{t-1} \\ \sum Y_t X_t &= \hat{\alpha}_0 \sum X_t + \hat{\alpha}_1 \sum X_t^2 + \hat{\alpha}_2 \sum Y_{t-1} X_t \\ \sum Y_t Y_{t-1} &= \hat{\alpha}_0 \sum Y_{t-1} + \hat{\alpha}_1 \sum X_t Y_{t-1} + \hat{\alpha}_2 \sum Y_{t-1}^2\end{aligned}\quad (17.9.2)$$

The difference between the two sets of normal equations should be readily apparent. Liviatan has shown that the  $\alpha$ 's estimated from (17.9.1) are consistent, whereas those estimated from (17.9.2) may not be consistent because  $Y_{t-1}$  and  $v_t$  [ $= u_t - \lambda u_{t-1}$  or  $u_t - (1 - \gamma)u_{t-1}$ ] may be correlated whereas  $X_t$  and  $X_{t-1}$  are uncorrelated with  $v_t$ . (Why?)

Although easy to apply in practice once a suitable proxy is found, the Liviatan technique is likely to suffer from the multicollinearity problem because  $X_t$  and  $X_{t-1}$ , which enter in the normal equations of (17.9.1), are likely to be highly correlated (as noted in Chapter 12, most economic time series typically exhibit a high degree of correlation between successive values). The implication, then, is that although the Liviatan procedure yields consistent estimates, the estimators are likely to be inefficient.<sup>34</sup>

Before we move on, the obvious question is: How does one find a "good" proxy for  $Y_{t-1}$  in such a way that, although highly correlated with  $Y_{t-1}$ , it is uncorrelated with  $v_t$ ? There are some suggestions in the literature, which we take up by way of an exercise (see exercise 17.5). But it must be stated that finding good proxies is not always easy, in which case the IV method is of little practical use and one may have to resort to maximum likelihood estimation techniques, which are beyond the scope of this book.<sup>35</sup>

Is there a test one can use to find out if the chosen instrument(s) is valid? Dennis Sargan has developed a test, dubbed the **SARG test**, for this purpose. The test is described in Appendix 17A, Section 17A.1.

### 17.10 DETECTING AUTOCORRELATION IN AUTOREGRESSIVE MODELS: DURBIN $h$ TEST

As we have seen, the likely serial correlation in the errors  $v_t$  make the estimation problem in the autoregressive model rather complex: In the stock adjustment model the error term  $v_t$  did not have (first-order) serial correlation if the error term  $u_t$  in the original model was serially uncorrelated, whereas in the Koyck and adaptive expectation models  $v_t$  was serially

<sup>34</sup>To see how the efficiency of the estimators can be improved, consult Lawrence R. Klien, *A Textbook of Econometrics*, 2d ed., Prentice-Hall, Englewood Cliffs, N.J., 1974, p. 99. See also William H. Greene, *Econometric Analysis*, Macmillan, 2d ed., New York, 1993, pp. 535–538.

<sup>35</sup>For a condensed discussion of the ML methods, see J. Johnston, op. cit., pp. 366–371, as well as App. 4A and App. 15A.

correlated even if  $u_t$  was serially independent. The question, then, is: How does one know if there is serial correlation in the error term appearing in the autoregressive models?

As noted in Chapter 12, the Durbin–Watson  $d$  statistic may not be used to detect (first-order) serial correlation in autoregressive models, because the computed  $d$  value in such models generally tends toward 2, which is the value of  $d$  expected in a truly random sequence. In other words, if we routinely compute the  $d$  statistic for such models, there is a built-in bias against discovering (first-order) serial correlation. Despite this, many researchers compute the  $d$  value for want of anything better. Recently, however, Durbin himself has proposed a *large-sample* test of first-order serial correlation in autoregressive models.<sup>36</sup> This test is called the ***h* statistic**.

We have already discussed the Durbin  $h$  test in exercise 12.36. For convenience, we reproduce the  $h$  statistic (with a slight change in notation):

$$h = \hat{\rho} \sqrt{\frac{n}{1 - n[\text{var}(\hat{\alpha}_2)]}} \quad (17.10.1)$$

where  $n$  is the sample size,  $\text{var}(\hat{\alpha}_2)$  is the variance of the lagged  $Y_t (= Y_{t-1})$  coefficient in (17.8.1), and  $\hat{\rho}$  is an estimate of the first-order serial correlation  $\rho$ , first discussed in Chapter 12.

As noted in exercise 12.36, for large sample, Durbin has shown that, under the null hypothesis that  $\rho = 0$ , the  $h$  statistic of (17.10.1) follows the standard normal distribution. That is,

$$h_{\text{asy}} \sim N(0, 1) \quad (17.10.2)$$

where asy means asymptotically.

In practice, as noted in Chapter 12, one can estimate  $\rho$  as

$$\hat{\rho} \approx 1 - \frac{d}{2} \quad (17.10.3)$$

It is interesting to observe that although we cannot use the Durbin  $d$  to test for autocorrelation in autoregressive models, we can use it as an input in computing the  $h$  statistic.

Let us illustrate the use of the  $h$  statistic with our Example 17.7. In this example,  $n = 30$ ,  $\hat{\rho} \approx (1 - d/2) = 0.4972$  (note:  $d = 1.0056$ ), and  $\text{var}(\hat{\alpha}_2) = \text{var}(\text{PPCE}_{t-1}) = (0.1546)^2 = 0.0239$ . Putting these values in (17.10.1), we obtain:

$$h = 0.4972 \sqrt{\frac{30}{1 - 30(0.0239)}} = 5.1191 \quad (17.10.4)$$

<sup>36</sup>J. Durbin, "Testing for Serial Correlation in Least-Squares Regression When Some of the Regressors Are Lagged Dependent Variables," *Econometrica*, vol. 38, 1970, pp. 410–421.



Since this  $h$  value has the standard normal distribution under the null hypothesis, the probability of obtaining such a high  $h$  value is very small. Recall that the probability that a standard normal variate exceeds the value of  $\pm 3$  is extremely small. In the present example our conclusion, then, is that there is (positive) autocorrelation. Of course, bear in mind that  $h$  follows the standard normal distribution asymptotically. Our sample of 30 observations may not be necessarily large.

Note these features of the  $h$  statistic.

1. It does not matter how many  $X$  variables or how many lagged values of  $Y$  are included in the regression model. To compute  $h$ , we need consider only the variance of the coefficient of lagged  $Y_{t-1}$ .

2. The test is not applicable if  $[n \text{ var}(\hat{\alpha}_2)]$  exceeds 1. (Why?) In practice, though, this does not usually happen.

3. Since the test is a large-sample test, its application in small samples is not strictly justified, as shown by Inder<sup>37</sup> and Kiviet.<sup>38</sup> It has been suggested that the Breusch–Godfrey (BG) test, also known as the Lagrange multiplier test, discussed in Chapter 12 is statistically more powerful not only in the large samples but also in finite, or small, samples and is therefore preferable to the  $h$  test.<sup>39</sup>

### 17.11 A NUMERICAL EXAMPLE: THE DEMAND FOR MONEY IN CANADA, 1979–I TO 1988–IV

To illustrate the use of the models we have discussed thus far, consider one of the earlier empirical applications, namely, the demand for money (or real cash balances). In particular, consider the following model.<sup>40</sup>

$$M_t^* = \beta_0 R_t^{\beta_1} Y_t^{\beta_2} e^{u_t} \quad (17.11.1)$$

where  $M_t^*$  = desired, or long-run, demand for money (real cash balances)

$R_t$  = long-term interest rate, %

$Y_t$  = aggregate real national income

For statistical estimation, (17.11.1) may be expressed conveniently in log form as

$$\ln M_t^* = \ln \beta_0 + \beta_1 \ln R_t + \beta_2 \ln Y_t + u_t \quad (17.11.2)$$

<sup>37</sup>B. Inder, "An Approximation to the Null Distribution of the Durbin–Watson Statistic in Models Containing Lagged Dependent Variables," *Econometric Theory*, vol. 2, no. 3, 1986, pp. 413–428.

<sup>38</sup>J. F. Kiviet, "On the Vigour of Some Misspecification Tests for Modelling Dynamic Relationships," *Review of Economic Studies*, vol. 53, no. 173, 1986, pp. 241–262.

<sup>39</sup>Gabor Korosi, Laszlo Matyas, and Istvan P. Szekely, *Practical Econometrics*, Ashgate Publishing Company, Brookfield, Vermont, 1992, p. 92.

<sup>40</sup>For a similar model, see Gregory C. Chow, "On the Long-Run and Short-Run Demand for Money," *Journal of Political Economy*, vol. 74, no. 2, 1966, pp. 111–131. Note that one advantage of the multiplicative function is that the exponents of the variables give direct estimates of elasticities (see Chap. 6).



Since the desired demand variable is not directly observable, let us assume the stock adjustment hypothesis, namely,

$$\frac{M_t}{M_{t-1}} = \left( \frac{M_t^*}{M_{t-1}} \right)^\delta \quad 0 < \delta \leq 1 \quad (17.11.3)$$

Equation (17.11.3) states that a constant percentage (why?) of the discrepancy between the actual and desired real cash balances is eliminated within a single period (year). In log form, Eq. (17.11.3) may be expressed as

$$\ln M_t - \ln M_{t-1} = \delta(\ln M_t^* - \ln M_{t-1}) \quad (17.11.4)$$

Substituting  $\ln M_t^*$  from (17.11.2) into Eq. (17.11.4) and rearranging, we obtain

$$\ln M_t = \delta \ln \beta_0 + \beta_1 \delta \ln R_t + \beta_2 \delta \ln Y_t + (1 - \delta) \ln M_{t-1} + \delta u_t \quad (17.11.5)^{41}$$

which may be called the *short-run demand function* for money. (Why?)

As an illustration of the short-term and long-term demand for real cash balances, consider the data given in Table 17.3. These quarterly data pertain to Canada for the period 1979 to 1988. The variables are defined as follows:  $M$  [as defined by M1 money supply, Canadian dollars (C\$), millions],  $P$  (implicit price deflator, 1981 = 100), GDP at constant 1981 prices (C\$, millions) and  $R$  (90-day prime corporate rate of interest, %).<sup>42</sup> M1 was deflated by  $P$  to obtain figures for real cash balances. A priori, real money demand is expected to be positively related to GDP (positive income effect) and negatively related to  $R$  (the higher the interest rate, the higher the opportunity cost of holding money, as M1 money pays very little interest, if any).

The regression results were as follows<sup>43</sup>:

$$\begin{aligned} \widehat{\ln M_t} &= 0.8561 - 0.0634 \ln R_t - 0.0237 \ln \text{GDP}_t + 0.9607 \ln M_{t-1} \\ \text{se} &= (0.5101) \quad (0.0131) \quad (0.0366) \quad (0.0414) \\ t &= (1.6782) \quad (-4.8134) \quad (-0.6466) \quad (23.1972) \\ R^2 &= 0.9482 \quad d = 2.4582 \quad F = 213.7234 \quad (17.11.6)^{43} \end{aligned}$$

<sup>41</sup>In passing, note that this model is essentially nonlinear in the parameters. Therefore, although OLS may give an unbiased estimate of, say,  $\beta_1 \delta$  taken together, it may not give unbiased estimates of  $\beta_1$  and  $\delta$  individually, especially if the sample is small.

<sup>42</sup>These data are obtained from B. Bhaskar Rao, ed., *Cointegration for the Applied Economist*, St. Martin's Press, New York, 1994, pp. 210–213. The original data is from 1956-I to 1988-IV, but for illustration purposes we begin our analysis from the first quarter of 1979.

<sup>43</sup>Note this feature of the estimated standard errors. The standard error of, say, the coefficient of  $\ln R_t$  refers to the standard error of  $\widehat{\beta_1 \delta}$ , an estimator of  $\beta_1 \delta$ . There is no simple way to obtain the standard errors of  $\hat{\beta}_1$  and  $\hat{\delta}$  individually from the standard error of  $\widehat{\beta_1 \delta}$ , especially if the sample is relatively small. For large samples, however, individual standard errors of  $\hat{\beta}_1$  and  $\hat{\delta}$  can be obtained approximately, but the computations are involved. See Jan Kmenta, *Elements of Econometrics*, Macmillan, New York, 1971, p. 444.

TABLE 17.3 MONEY, INTEREST RATE, PRICE INDEX, AND GDP, CANADA

Observation	M1	<i>R</i>	<i>P</i>	GDP
1979-1	22,175.00	11.13333	0.77947	334,800
1979-2	22,841.00	11.16667	0.80861	336,708
1979-3	23,461.00	11.80000	0.82649	340,096
1979-4	23,427.00	14.18333	0.84863	341,844
1980-1	23,811.00	14.38333	0.86693	342,776
1980-2	23,612.33	12.98333	0.88950	342,264
1980-3	24,543.00	10.71667	0.91553	340,716
1980-4	25,638.66	14.53333	0.93743	347,780
1981-1	25,316.00	17.13333	0.96523	354,836
1981-2	25,501.33	18.56667	0.98774	359,352
1981-3	25,382.33	21.01666	1.01314	356,152
1981-4	24,753.00	16.61665	1.03410	353,636
1982-1	25,094.33	15.35000	1.05743	349,568
1982-2	25,253.66	16.04999	1.07748	345,284
1982-3	24,936.66	14.31667	1.09666	343,028
1982-4	25,553.00	10.88333	1.11641	340,292
1983-1	26,755.33	9.616670	1.12303	346,072
1983-2	27,412.00	9.316670	1.13395	353,860
1983-3	28,403.33	9.333330	1.14721	359,544
1983-4	28,402.33	9.550000	1.16059	362,304
1984-1	28,715.66	10.08333	1.17117	368,280
1984-2	28,996.33	11.45000	1.17406	376,768
1984-3	28,479.33	12.45000	1.17795	381,016
1984-4	28,669.00	10.76667	1.18438	385,396
1985-1	29,018.66	10.51667	1.18990	390,240
1985-2	29,398.66	9.666670	1.20625	391,580
1985-3	30,203.66	9.033330	1.21492	396,384
1985-4	31,059.33	9.016670	1.21805	405,308
1986-1	30,745.33	11.03333	1.22408	405,680
1986-2	30,477.66	8.733330	1.22856	408,116
1986-3	31,563.66	8.466670	1.23916	409,160
1986-4	32,800.66	8.400000	1.25368	409,616
1987-1	33,958.33	7.250000	1.27117	416,484
1987-2	35,795.66	8.300000	1.28429	422,916
1987-3	35,878.66	9.300000	1.29599	429,980
1987-4	36,336.00	8.700000	1.31001	436,264
1988-1	36,480.33	8.616670	1.32325	440,592
1988-2	37,108.66	9.133330	1.33219	446,680
1988-3	38,423.00	10.05000	1.35065	450,328
1988-4	38,480.66	10.83333	1.36648	453,516

Notes: M1 = C\$, millions

*P* = implicit price deflator (1981 = 100)*R* = 90-day prime corporate interest rate, %

GDP = C\$, millions (1981 prices)

Source: Rao, op. cit., pp. 210-213.

The estimated short-run demand function shows that the short-run interest elasticity has the correct sign and that it is statistically quite significant, as its  $p$  value is almost zero. The short-run income elasticity is surprisingly negative, although statistically it is not different from zero. The coefficient of adjustment is  $\delta = (1 - 0.9607) = 0.0393$ , implying that only about 4 percent of the discrepancy between the desired and actual real cash balances is eliminated in a quarter, a rather slow adjustment.

To get back to the long-run demand function (17.11.2), all that needs to be done is to divide the short-run demand function through by  $\delta$  (why?) and drop the  $\ln M_{t-1}$  term. The results are:

$$\widehat{\ln M_t^*} = 21.7888 - 1.6132 \ln R_t - 0.6030 \ln \text{GDP} \quad (17.11.7)^{44}$$

As can be seen, the long-run interest elasticity of demand for money is substantially greater (in absolute terms) than the corresponding short-run elasticity, which is also true of the income elasticity, although in the present instance its economic and statistical significance is dubious.

Note that the estimated Durbin-Watson  $d$  is 2.4582, which is close to 2. This substantiates our previous remark that in the autoregressive models the computed  $d$  is generally close to 2. Therefore, we should not trust the computed  $d$  to find out whether there was serial correlation in our data. The sample size in our case is 40 observations, which may be reasonably large to apply the  $h$  test. In the present case, the reader can verify that the estimated  $h$  value is  $-1.5008$ , which is not significant at the 5 percent level, perhaps suggesting that there is no first-order autocorrelation in the error term.

## 17.12 ILLUSTRATIVE EXAMPLES

In this section we present a few examples of distributed lag models to show how researchers have used them in empirical studies.

### EXAMPLE 17.9

#### THE FED AND THE REAL RATE OF INTEREST

To assess the effect of M1 (currency + checkable deposits) growth on Aaa bond real interest rate measure, G. J. Santoni and Courtenay C. Stone<sup>45</sup> estimated, using monthly data, the following distributed lag model for the United States.

$$r_t = \text{constant} + \sum_{i=0}^{11} a_i \dot{M}_{t-i} + u_i \quad (17.12.1)$$

where  $r_t$  = Moody's Index of Aaa bond yield minus the average annual rate of change in the seasonally adjusted consumer price index over the prior 36 months, which is used as the measure of real interest rate, and  $\dot{M}_t$  = monthly  $M_1$  growth.

According to the "neutrality of money doctrine," which states that real economic variables—such as output, employment, economic growth and the real rate of interest—are not influenced permanently by money growth and, therefore, are essentially unaffected by monetary policy. . . . Given this argument, the Federal

(Continued)

<sup>44</sup>Note that we have not presented the standard errors of the estimated coefficients for reasons discussed in footnote 43.

<sup>45</sup>"The Fed and the Real Rate of Interest," *Review*, Federal Reserve Bank of St. Louis, December 1982, pp. 8–18.

**EXAMPLE 17.9** (Continued)

Reserve has no permanent influence over the real rate of interest whatsoever.<sup>46</sup>

If this doctrine is valid, then one should expect the distributed lag coefficients  $a_i$  as well as their sum to be statistically indifferent from zero. To find out whether this is the case, the authors estimated (17.12.1) for two different time periods, February 1951 to September 1979 and October 1979 to November 1982, the latter to take into account the change in the Fed's monetary policy, which since October 1979 has paid more attention to the rate of growth of the money supply than to the rate of

interest, which was the policy in the earlier period. Their regression results are presented in Table 17.4. The results seem to support the "neutrality of money doctrine," since for the period February 1951 to September 1979 the current as well as lagged money growth had no statistically significant effect on the real interest rate measure. For the latter period, too, the neutrality doctrine seems to hold since  $\sum a_i$  is not statistically different from zero; only the coefficient  $a_1$  is significant, but it has the wrong sign. (Why?)

**TABLE 17.4**  
INFLUENCE OF MONTHLY M1 GROWTH ON AN AAA BOND REAL INTEREST RATE MEASURE:  
FEBRUARY 1951 TO NOVEMBER 1982

$$r = \text{constant} + \sum_{i=0}^{11} a_i M_{1,t-i}$$

	February 1951 to September 1979		October 1979 to November 1982	
	Coefficient	t *	Coefficient	t
Constant	1.4885 <sup>†</sup>	2.068	1.0360	0.801
$a_0$	-0.00088	0.388	0.00840	1.014
$a_1$	0.00171	0.510	0.03960 <sup>†</sup>	3.419
$a_2$	0.00170	0.423	0.03112	2.003
$a_3$	0.00233	0.542	0.02719	1.502
$a_4$	-0.00249	0.553	0.00901	0.423
$a_5$	-0.00160	0.348	0.01940	0.863
$a_6$	0.00292	0.631	0.02411	1.056
$a_7$	0.00253	0.556	0.01446	0.666
$a_8$	0.00000	0.001	-0.00036	0.019
$a_9$	0.00074	0.181	-0.00499	0.301
$a_{10}$	0.00016	0.045	-0.01126	0.888
$a_{11}$	0.00025	0.107	-0.00178	0.211
$\sum a_i$	0.00737	0.221	0.1549	0.926
$\bar{R}^2$	0.9826		0.8662	
D-W	2.07		2.04	
RH01	1.27 <sup>†</sup>	24.536	1.40 <sup>†</sup>	9.838
RH02	-0.28 <sup>†</sup>	5.410	-0.48 <sup>†</sup>	3.373
NOB	344.		38.	
SER (= RSS)	0.1548		0.3899	

\*|t| = absolute t value.

<sup>†</sup>Significantly different from zero at the 0.05 level.

Source: G. J. Santoni and Courtenay C. Stone, "The Fed and the Real Rate of Interest," *Review*, Federal Reserve Bank of St. Louis, December 1982, p. 16.

<sup>46</sup>"The Fed and the Real Rate of Interest," *Review*, Federal Reserve Bank of St. Louis, December 1982, p. 15.

**EXAMPLE 17.10**

THE SHORT- AND LONG-RUN AGGREGATE  
CONSUMPTION FOR SRI LANKA, 1967–1993

Suppose consumption  $C$  is linearly related to permanent  
income  $X^*$ :

$$C_t = \beta_1 + \beta_2 X_t^* + u_t \quad (17.12.2)$$

Since  $X_t^*$  is not directly observable, we need to specify  
the mechanism that generates permanent income. Sup-  
pose we adopt the adaptive expectations hypothesis  
specified in (17.5.2). Using (17.5.2) and simplifying, we  
obtain the following estimating equation (cf. 17.5.5):

$$C_t = \alpha_1 + \alpha_2 X_t + \alpha_3 C_{t-1} + v_t \quad (17.12.3)$$

where  $\alpha_1 = \gamma\beta_1$

$$\alpha_2 = \gamma\beta_2$$

$$\alpha_3 = (1 - \gamma)$$

$$v_t = [u_t - (1 - \gamma)u_{t-1}]$$

As we know,  $\beta_2$  gives the mean response of con-  
sumption to, say, a \$1 increase in permanent income,  
whereas  $\alpha_2$  gives the mean response of consumption to  
a \$1 increase in current income.

From annual data for Sri Lanka for the period  
1967–1993 given in Table 17.5, the following regression  
results were obtained<sup>47</sup>:

$$\begin{aligned} \hat{C} &= 1038.403 + 0.4043X_t + 0.5009C_{t-1} \\ \text{se} &= (2501.455) \quad (0.0919) \quad (0.1213) \quad (17.12.4) \\ t &= (0.4151) \quad (4.3979) \quad (4.1293) \\ R^2 &= 0.9912 \quad d = 1.4162 \quad F = 1298.466 \end{aligned}$$

where  $C$  = private consumption expenditure, and  $X$  =  
GDP, both at constant prices. We also introduced real  
interest rate in the model, but it was not statistically  
significant.

The results show that the short-run marginal propen-  
sity to consume (MPC) is 0.4043, suggesting that a  
1 rupee increase in the current or observed real income  
(as measured by real GDP) would increase mean con-  
sumption by about 0.40 rupee. But if the increase in  
income is sustained, then eventually the MPC out of  
the permanent income will be  $\beta_2 = \gamma\beta_2/\gamma = 0.4043/$   
 $0.4991 = 0.8100$  or about 0.81 rupee. In other words,  
when consumers have had time to adjust to the 1 rupee  
change in income, they will increase their consumption  
ultimately by about 0.81 rupee.

**TABLE 17.5** PRIVATE CONSUMPTION EXPENDITURE AND GDP, SRI LANKA

Observation	PCON	GDP	Observation	PCON	GDP
1967	61,284	78,221	1981	120,477	152,846
1968	68,814	83,326	1982	133,868	164,318
1969	76,766	90,490	1983	148,004	172,414
1970	73,576	92,692	1984	149,735	178,433
1971	73,256	94,814	1985	155,200	185,753
1972	67,502	92,590	1986	154,165	192,059
1973	78,832	101,419	1987	155,445	191,288
1974	80,240	105,267	1988	157,199	196,055
1975	84,477	112,149	1989	158,576	202,477
1976	86,038	116,078	1990	169,238	223,225
1977	96,275	122,040	1991	179,001	233,231
1978	101,292	128,578	1992	183,687	242,762
1979	105,448	136,851	1993	198,273	259,555
1980	114,570	144,734			

Notes: PCON = private consumption expenditure.

GDP = gross domestic product.

Source: See footnote 47.

(Continued)

<sup>47</sup>The data are obtained from the data disk in Chandan Mukherjee, Howard White, and Marc Wuyts, *Econometrics and Data Analysis for Developing Countries*, Routledge, New York, 1998. The original data is from World Bank's World Tables.

**EXAMPLE 17.10** (Continued)

Now suppose that our consumption function were

$$C_t^* = \beta_1 + \beta_2 X_t + u_t \quad (17.12.5)$$

In this formulation permanent or long-run consumption  $C_t^*$  is a linear function of the current or observed income. Since  $C_t^*$  is not directly observable, let us invoke the partial adjustment model (17.6.2). Using this model, and after algebraic manipulations, we obtain

$$\begin{aligned} C_t &= \delta\beta_1 + \delta\beta_2 X_t + (1 - \delta)C_{t-1} + \delta u_t \\ &= \alpha_1 + \alpha_2 X_t + \alpha_3 C_{t-1} + v_t \end{aligned} \quad (17.12.6)$$

In appearance, this model is indistinguishable from the adaptive expectations model (17.12.3). Therefore, the regression results given in (17.12.4) are equally applicable here. However, there is a major difference in the interpretation of the two models, not to mention the estimation problem associated with the autoregressive and possibly serially correlated model (17.12.3). The model (17.12.5) is the long-run, or equilibrium, consumption function, whereas (17.12.6) is the short-run consumption function.  $\beta_2$  measures the long-run MPC, whereas  $\alpha_2 (= \delta\beta_2)$  gives the short-run MPC; the former can be obtained from the latter by dividing it by  $\delta$ , the coefficient of adjustment.

Returning to (17.12.4), we can now interpret 0.4043 as the short-run MPC. Since  $\delta = 0.4991$ , the long-run MPC is 0.81. Note that the adjustment coefficient of about 0.50 suggests that in any given time period consumers only adjust their consumption one-half of the way toward its desired or long-run level.

This example brings out the crucial point that in appearance the adaptive expectations and the partial adjustment models, or the Koyck model for that matter, are so similar that by just looking at the estimated regression, such as (17.12.4), one cannot tell which is the correct specification. That is why it is so vital that one specify the theoretical underpinning of the model chosen for empirical analysis and then proceed appropriately. If habit or inertia characterizes consumption behavior, then the partial adjustment model is appropriate. On the other hand, if consumption behavior is forward-looking in the sense that it is based on expected future income, then the adaptive expectations model is appropriate. If it is the latter, then, one will have to pay close attention to the estimation problem to obtain consistent estimators. In the former case, the OLS will provide consistent estimators, provided the usual OLS assumptions are fulfilled.

### 17.13 THE ALMON APPROACH TO DISTRIBUTED-LAG MODELS: THE ALMON OR POLYNOMIAL DISTRIBUTED LAG (PDL)<sup>48</sup>

Although used extensively in practice, the Koyck distributed-lag model is based on the assumption that the  $\beta$  coefficients decline geometrically as the lag lengthens (see Figure 17.5). This assumption may be too restrictive in some situations. Consider, for example, Figure 17.7.

In Figure 17.7a it is assumed that the  $\beta$ 's increase at first and then decrease, whereas in Figure 17.7c it is assumed that they follow a cyclical pattern. Obviously, the Koyck scheme of distributed-lag models will not work in these cases. However, after looking at Figure 17.7a and c, it seems that one can express  $\beta_i$  as a function of  $i$ , the length of the lag (time), and fit suitable curves to reflect the functional relationship between the two, as indicated in Figure 17.7b and d. This approach is precisely the one suggested by Shirley Almon. To illustrate her technique, let us revert to the finite distributed-lag model considered previously, namely,

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \cdots + \beta_k X_{t-k} + u_t \quad (17.1.2)$$

<sup>48</sup>Shirley Almon, "The Distributed Lag between Capital Appropriations and Expenditures," *Econometrica*, vol. 33, January 1965, pp. 178–196.

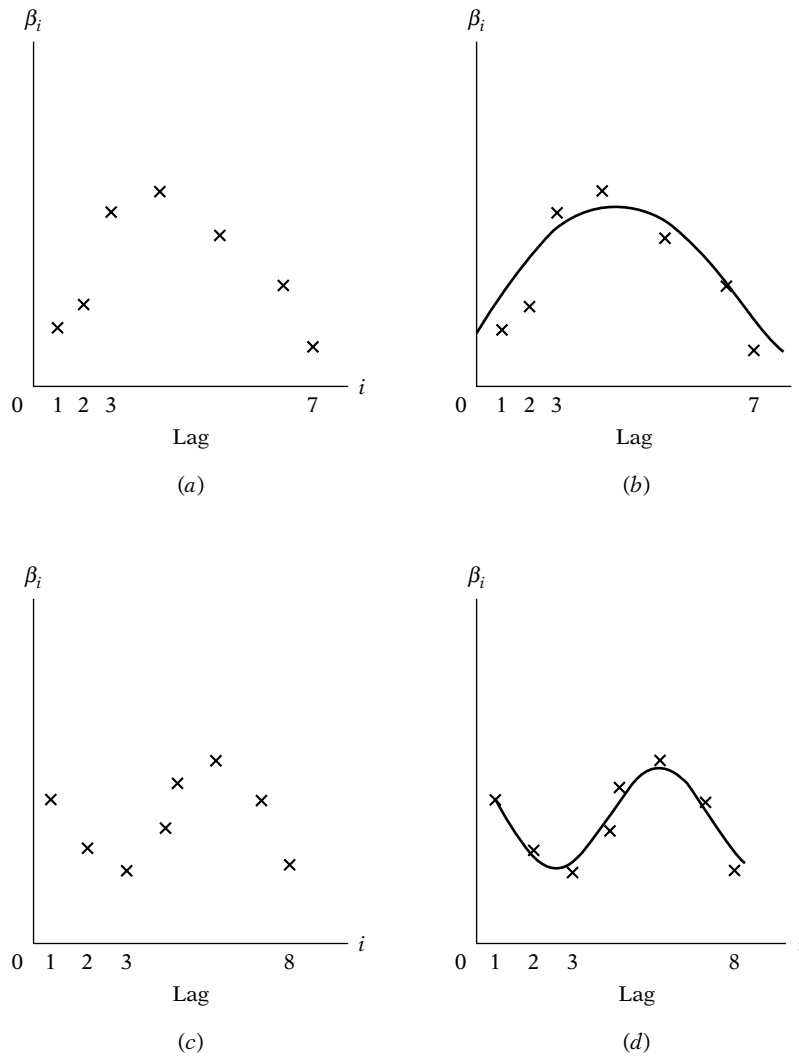


FIGURE 17.7 Almon polynomial-lag scheme.

which may be written more compactly as

$$Y_t = \alpha + \sum_{i=0}^k \beta_i X_{t-i} + u_t \quad (17.13.1)$$

Following a theorem in mathematics known as **Weierstrass' theorem**, Almon assumes that  $\beta_i$  can be approximated by a suitable-degree polynomial in  $i$ , the length of the lag.<sup>49</sup> For instance, if the lag scheme shown in

<sup>49</sup>Broadly speaking, the theorem states that on a finite closed interval any continuous function may be approximated uniformly by a polynomial of a suitable degree.

Figure 17.7a applies, we can write

$$\beta_i = a_0 + a_1i + a_2i^2 \quad (17.13.2)$$

which is a quadratic, or second-degree, polynomial in  $i$  (see Figure 17.7b). However, if the  $\beta$ 's follow the pattern of Figure 17.7c, we can write

$$\beta_i = a_0 + a_1i + a_2i^2 + a_3i^3 \quad (17.13.3)$$

which is a third-degree polynomial in  $i$  (see Figure 17.7d). More generally, we may write

$$\beta_i = a_0 + a_1i + a_2i^2 + \cdots + a_mi^m \quad (17.13.4)$$

which is an  $m$ th-degree polynomial in  $i$ . It is assumed that  $m$  (the degree of the polynomial) is less than  $k$  (the maximum length of the lag).

To explain how the Almon scheme works, let us assume that the  $\beta$ 's follow the pattern shown in Figure 17.7a and, therefore, the second-degree polynomial approximation is appropriate. Substituting (17.13.2) into (17.13.1), we obtain

$$\begin{aligned} Y_t &= \alpha + \sum_{i=0}^k (a_0 + a_1i + a_2i^2)X_{t-i} + u_t \\ &= \alpha + a_0 \sum_{i=0}^k X_{t-i} + a_1 \sum_{i=0}^k iX_{t-i} + a_2 \sum_{i=0}^k i^2X_{t-i} + u_t \end{aligned} \quad (17.13.5)$$

Defining

$$\begin{aligned} Z_{0t} &= \sum_{i=0}^k X_{t-i} \\ Z_{1t} &= \sum_{i=0}^k iX_{t-i} \\ Z_{2t} &= \sum_{i=0}^k i^2X_{t-i} \end{aligned} \quad (17.13.6)$$

we may write (17.13.5) as

$$Y_t = \alpha + a_0Z_{0t} + a_1Z_{1t} + a_2Z_{2t} + u_t \quad (17.13.7)$$

In the Almon scheme  $Y$  is regressed on the constructed variables  $Z$ , not the original  $X$  variables. Note that (17.13.7) can be estimated by the usual OLS procedure. The estimates of  $\alpha$  and  $a_i$  thus obtained will have all the desirable statistical properties provided the stochastic disturbance term  $u$  satisfies the assumptions of the classical linear regression model. In this respect, the Almon technique has a distinct advantage over the Koyck method because, as we have seen, the latter has some serious estimation problems that result from the presence of the stochastic explanatory variable  $Y_{t-1}$  and its likely correlation with the disturbance term.



Once the  $a$ 's are estimated from (17.13.7), the original  $\beta$ 's can be estimated from (17.13.2) [or more generally from (17.13.4)] as follows:

$$\begin{aligned} \hat{\beta}_0 &= \hat{a}_0 \\ \hat{\beta}_1 &= \hat{a}_0 + \hat{a}_1 + \hat{a}_2 \\ \hat{\beta}_2 &= \hat{a}_0 + 2\hat{a}_1 + 4\hat{a}_2 \\ \hat{\beta}_3 &= \hat{a}_0 + 3\hat{a}_1 + 9\hat{a}_2 \\ &\dots\dots\dots \\ \hat{\beta}_k &= \hat{a}_0 + k\hat{a}_1 + k^2\hat{a}_2 \end{aligned} \tag{17.13.8}$$

Before we apply the Almon technique, we must resolve the following practical problems.

1. The maximum length of the lag  $k$  must be specified in advance. Here perhaps one can follow the advice of Davidson and MacKinnon:

The best approach is probably to settle the question of lag length first, by starting with a very large value of  $q$  [the lag length] and then seeing whether the fit of the model deteriorates significantly when it is reduced without imposing any restrictions on the shape of the distributed lag.<sup>50</sup>

This advice is in the spirit of Hendry's top-down approach discussed in Chapter 13. Remember that if there is some "true" lag length, choosing fewer lags will lead to the "omission of relevant variable bias," whose consequences, as we saw in Chapter 13, can be very serious. On the other hand, choosing more lags than necessary will lead to the "inclusion of irrelevant variable bias," whose consequences are less serious; the coefficients can be consistently estimated by OLS, although their variances may be less efficient.

One can use the *Akaike* or *Schwarz information criterion* discussed in Chapter 13 to choose the appropriate lag length. These criteria can also be used to discuss the appropriate degree of the polynomial in addition to the discussion in point 2.

2. Having specified  $k$ , we must also specify the degree of the polynomial  $m$ . Generally, the degree of the polynomial should be at least one more than the number of turning points in the curve relating  $\beta_i$  to  $i$ . Thus, in Figure 17.7a there is only one turning point; hence a second-degree polynomial will be a good approximation. In Figure 17.7c there are two turning points; hence a third-degree polynomial will provide a good approximation. A priori, however, one may not know the number of turning points, and therefore, the choice of  $m$  is largely subjective. However, theory may suggest a particular shape in some cases. In practice, one hopes that a fairly low-degree polynomial (say,  $m = 2$  or  $3$ ) will give good results. Having chosen a particular value of  $m$ , if we want to find out whether a higher-degree polynomial will give a better fit, we can proceed as follows.

<sup>50</sup>Russell Davidson and James G. MacKinnon, *Estimation and Inference in Econometrics*, Oxford University Press, New York, 1993, pp. 675–676.

Suppose we must decide between the second- and third-degree polynomials. For the second-degree polynomial the estimating equation is as given by (17.13.7). For the third-degree polynomial the corresponding equation is

$$Y_t = \alpha + a_0 Z_{0t} + a_1 Z_{1t} + a_2 Z_{2t} + a_3 Z_{3t} + u_t \quad (17.13.9)$$

where  $Z_{3t} = \sum_{i=0}^k i^3 X_{t-i}$ . After running regression (17.13.9), if we find that  $a_2$  is statistically significant but  $a_3$  is not, we may assume that the second-degree polynomial provides a reasonably good approximation.

Alternatively, as Davidson and MacKinnon suggest, "After  $q$  [the lag length] is determined, one can then attempt to determine  $d$  [the degree of the polynomial] once again starting with a large value and then reducing it."

However, we must beware of the problem of multicollinearity, which is likely to arise because of the way the  $Z$ 's are constructed from the  $X$ 's, as shown in (17.13.6) [see also (17.13.10)]. As shown in Chapter 10, in cases of serious multicollinearity,  $\hat{a}_3$  may turn out to be statistically insignificant, not because the true  $a_3$  is zero, but simply because the sample at hand does not allow us to assess the separate impact of  $Z_3$  on  $Y$ . Therefore, in our illustration, before we accept the conclusion that the third-degree polynomial is not the correct choice, we must make sure that the multicollinearity problem is not serious enough, which can be done by applying the techniques discussed in Chapter 10.

3. Once  $m$  and  $k$  are specified, the  $Z$ 's can be readily constructed. For instance, if  $m = 2$  and  $k = 5$ , the  $Z$ 's are

$$\begin{aligned} Z_{0t} &= \sum_{i=0}^5 X_{t-i} = (X_t + X_{t-1} + X_{t-2} + X_{t-3} + X_{t-4} + X_{t-5}) \\ Z_{1t} &= \sum_{i=0}^5 i X_{t-i} = (X_{t-1} + 2X_{t-2} + 3X_{t-3} + 4X_{t-4} + 5X_{t-5}) \\ Z_{2t} &= \sum_{i=0}^5 i^2 X_{t-i} = (X_{t-1} + 4X_{t-2} + 9X_{t-3} + 16X_{t-4} + 25X_{t-5}) \end{aligned} \quad (17.13.10)$$

Notice that the  $Z$ 's are linear combinations of the original  $X$ 's. Also notice why the  $Z$ 's are likely to exhibit multicollinearity.

Before proceeding to a numerical example, note the advantages of the Almon method. First, it provides a flexible method of incorporating a variety of lag structures (see exercise 17.17). The Koyck technique, on the other hand, is quite rigid in that it assumes that the  $\beta$ 's decline geometrically. Second, unlike the Koyck technique, in the Almon method we do not have to worry about the presence of the lagged dependent variable as an explanatory variable in the model and the problems it creates for estimation. Finally, if a sufficiently low-degree polynomial can be fitted, the number of coefficients to be estimated (the  $a$ 's) is considerably smaller than the original number of coefficients (the  $\beta$ 's).

But let us re-emphasize the problems with the Almon technique. First, the degree of the polynomial as well as the maximum value of the lag is largely a subjective decision. Second, for reasons noted previously, the  $Z$  variables are likely to exhibit multicollinearity. Therefore, in models like (17.13.9) the estimated  $a$ 's are likely to show large standard errors (relative to the values of these coefficients), thereby rendering one or more such coefficients statistically insignificant on the basis of the conventional  $t$  test. But this does not necessarily mean that one or more of the original  $\hat{\beta}$  coefficients will also be statistically insignificant. (The proof of this statement is slightly involved but is suggested in exercise 17.18.) As a result, the multicollinearity problem may not be as serious as one might think. Besides, as we know, in cases of multicollinearity even if we cannot estimate an individual coefficient precisely, a linear combination of such coefficients (the **estimable function**) can be estimated more precisely.

**EXAMPLE 17.11**

ILLUSTRATION OF THE ALMON DISTRIBUTED-LAG MODEL

To illustrate the Almon technique, Table 17.6 gives data on inventories  $Y$  and sales  $X$  for the United States for the period 1954–1999.

For illustrative purposes, assume that inventories depend on sales in the current year and in the preceding 3 years as follows:

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \beta_3 X_{t-3} + u_t \quad (17.13.11)$$

Furthermore, assume that  $\beta_i$  can be approximated by a second-degree polynomial as shown in (17.13.2). Then, following (17.13.5), we may write

$$Y_t = \alpha + a_0 Z_{0t} + a_1 Z_{1t} + a_2 Z_{2t} + u_t \quad (17.13.12)$$

where

$$\begin{aligned} Z_{0t} &= \sum_{i=0}^3 X_{t-i} = (X_t + X_{t-1} + X_{t-2} + X_{t-3}) \\ Z_{1t} &= \sum_{i=0}^3 i X_{t-i} = (X_{t-1} + 2X_{t-2} + 3X_{t-3}) \\ Z_{2t} &= \sum_{i=0}^3 i^2 X_{t-i} = (X_{t-1} + 4X_{t-2} + 9X_{t-3}) \end{aligned} \quad (17.13.13)$$

The  $Z$  variables thus constructed are shown in Table 17.6. Using the data on  $Y$  and the  $Z$ 's, we obtain the following regression:

$$\begin{aligned} \hat{Y}_t &= 25,845.06 + 1.1149Z_{0t} - 0.3713Z_{1t} - 0.0600Z_{2t} \\ \text{se} &= (6596.998) \quad (0.5381) \quad (1.3743) \quad (0.4549) \\ t &= (3.9177) \quad (2.0718) \quad (-0.2702) \quad (-0.1319) \\ R^2 &= 0.9755 \quad d = 0.1643 \quad F = 517.7656 \end{aligned} \quad (17.13.14)$$

*Note:* Since we are using a 3-year lag, the total number of observations has been reduced from 46 to 43.

(Continued)

**EXAMPLE 17.11** (Continued)**TABLE 17.6**  
INVENTORIES  $Y$  AND SALES  $X$ , U.S. MANUFACTURING, AND CONSTRUCTED  $Z$ 'S

Observation	Inventory	Sales	$Z_0$	$Z_1$	$Z_2$
1954	41,612	23,355	NA	NA	NA
1955	45,069	26,480	NA	NA	NA
1956	50,642	27,740	NA	NA	NA
1957	51,871	28,736	106,311	150,765	343,855
1958	50,203	27,248	110,204	163,656	378,016
1959	52,913	30,286	114,010	167,940	391,852
1960	53,786	30,878	117,148	170,990	397,902
1961	54,871	30,922	119,334	173,194	397,254
1962	58,172	33,358	125,444	183,536	427,008
1963	60,029	35,058	130,216	187,836	434,948
1964	63,410	37,331	136,669	194,540	446,788
1965	68,207	40,995	146,742	207,521	477,785
1966	77,986	44,870	158,254	220,831	505,841
1967	84,646	46,486	169,682	238,853	544,829
1968	90,560	50,229	182,580	259,211	594,921
1969	98,145	53,501	195,086	277,811	640,003
1970	101,599	52,805	203,021	293,417	672,791
1971	102,567	55,906	212,441	310,494	718,870
1972	108,121	63,027	225,239	322,019	748,635
1973	124,499	72,931	244,669	333,254	761,896
1974	157,625	84,790	276,654	366,703	828,193
1975	159,708	86,589	307,337	419,733	943,757
1976	174,636	98,797	343,107	474,962	1,082,128
1977	188,378	113,201	383,377	526,345	1,208,263
1978	211,691	126,905	425,492	570,562	1,287,690
1979	242,157	143,936	482,839	649,698	1,468,882
1980	265,215	154,391	538,433	737,349	1,670,365
1981	283,413	168,129	593,361	822,978	1,872,280
1982	311,852	163,351	629,807	908,719	2,081,117
1983	312,379	172,547	658,418	962,782	2,225,386
1984	339,516	190,682	694,709	1,003,636	2,339,112
1985	334,749	194,538	721,118	1,025,829	2,351,029
1986	322,654	194,657	752,424	1,093,543	2,510,189
1987	338,109	206,326	786,203	1,155,779	2,688,947
1988	369,374	224,619	820,140	1,179,254	2,735,796
1989	391,212	236,698	862,300	1,221,242	2,801,836
1990	405,073	242,686	910,329	1,304,914	2,992,108
1991	390,905	239,847	943,850	1,389,939	3,211,049
1992	382,510	250,394	969,625	1,435,313	3,340,873
1993	384,039	260,635	993,562	1,458,146	3,393,956
1994	404,877	279,002	1,029,878	1,480,964	3,420,834
1995	430,985	299,555	1,089,586	1,551,454	3,575,088
1996	436,729	309,622	1,148,814	1,639,464	3,761,278
1997	456,133	327,452	1,215,631	1,745,738	4,018,860
1998	466,798	337,687	1,274,316	1,845,361	4,261,935
1999	470,377	354,961	1,329,722	1,921,457	4,434,093

Note:  $Y$  and  $X$  are in millions of dollars, seasonally adjusted.

Source: *Economic Report of the President, 2001*, Table B-57, p. 340. The  $Z$ 's are as shown in (17.13.13).

(Continued)

**EXAMPLE 17.11** (Continued)

A brief comment on the preceding results. Of the three  $Z$  variables, only  $Z_0$  is individually statistical significant at the 5 percent level, but the others are not, yet the  $F$  value is so high that we can reject the null hypothesis that collectively the  $Z$ 's have no effect on  $Y$ . As you can suspect, this might very well be due to multicollinearity. Also, note that the computed  $d$  value is very low. This does not necessarily mean that the residuals suffer from autocorrelation. More likely, the low  $d$  value suggests that the model we have used is probably mis-specified. We will comment on this shortly.

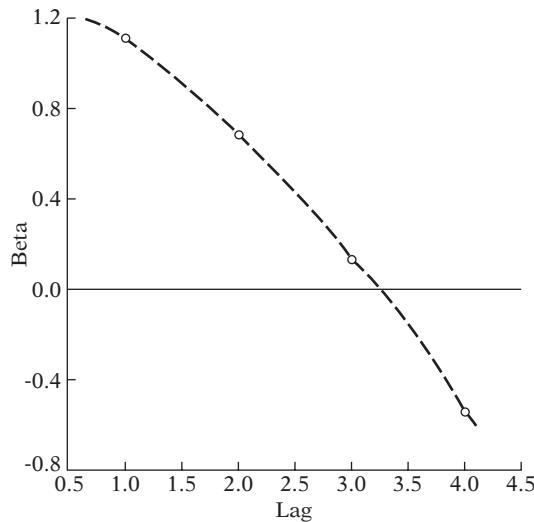
From the estimated  $a$ 's given in (17.13.3), we can easily estimate the original  $\beta$ 's easily, as shown in (17.13.8). In the present example, the results are as follows:

$$\begin{aligned}\hat{\beta}_0 &= \hat{a}_0 = 1.1149 \\ \hat{\beta}_1 &= (\hat{a}_0 + \hat{a}_1 + \hat{a}_2) = 0.6836 \\ \hat{\beta}_2 &= (\hat{a}_0 + 2\hat{a}_1 + 4\hat{a}_2) = 0.1321 \\ \hat{\beta}_3 &= (\hat{a}_0 + 3\hat{a}_1 + 9\hat{a}_2) = -0.5394\end{aligned}\tag{17.13.15}$$

Thus, the estimated distributed-lag model corresponding to (17.13.11) is:

$$\begin{aligned}\hat{Y}_t &= 25,845.0 & + & 1.1150X_0 & + & 0.6836X_{t-1} & + & 0.1321X_{t-2} & - & 0.5394X_{t-3} \\ \text{se} &= (6596.99) & & (0.5381) & & (0.4672) & & (0.4656) & & (0.5656) \\ t &= (3.9177) & & (2.0718) & & (1.4630) & & (0.2837) & & (-0.9537)\end{aligned}\tag{17.13.16}$$

Geometrically, the estimated  $\beta_j$  is as shown in Figure 17.8.



**FIGURE 17.8** Lag structure of the illustrative example.

Our illustrative example may be used to point out a few additional features of the Almon lag procedure:

1. The standard errors of the  $a$  coefficients are directly obtainable from the OLS regression (17.13.14), but the standard errors of some of the  $\hat{\beta}$

coefficients, the objective of primary interest, cannot be so obtained. But they can be obtained from the standard errors of the estimated  $a$  coefficients by using a well-known formula from statistics, which is given in exercise 17.18. Of course, there is no need to do this manually, for most statistical packages can do this routinely. The standard errors given in (17.13.15) were obtained from Eviews 4.

2. The  $\hat{\beta}$ 's obtained in (17.13.16) are called *unrestricted estimates* in the sense that no a priori restrictions are placed on them. In some situations, however, one may want to impose the so-called **endpoint restrictions** on the  $\beta$ 's by assuming that  $\beta_0$  and  $\beta_k$  (the current and  $k$ th lagged coefficient) are zero. Because of psychological, institutional, or technical reasons, the value of the explanatory variable in the current period may not have any impact on the current value of the regressand, thereby justifying the zero value for  $\beta_0$ . By the same token, beyond a certain time the  $k$ th lagged coefficient may not have any impact on the regressand, thus supporting the assumption that  $\beta_k$  is zero. In our inventory example, the coefficient of  $X_{t-3}$  had a negative sign, which may not make economic sense. Hence, one may want to constrain that coefficient to zero.<sup>51</sup> Of course, you do not have to constrain both ends; you could put restriction only on the first coefficient, called near-end restriction or on the last coefficient, called far-end restriction. For our inventory example, this is illustrated in exercise 17.28. Sometimes the  $\beta$ 's are estimated with the restriction that their sum is one. But one should not put such restrictions mindlessly because such restrictions also affect the values of the other (unconstrained) lagged coefficients.

3. Since the choice of the number of lagged coefficients as well as the degree of the polynomial is at the discretion of the modeler, some trial and error is inevitable, the charge of data mining notwithstanding. Here is where the **Akaike and Schwarz information criteria** discussed in Chapter 13 may come in handy.

4. Since we estimated (17.13.16) using three lags and the second-degree polynomial, it is a *restricted least-squares* model. Suppose, we decide to use three lags but do not use the Almon polynomial approach. That is, we estimate (17.13.11) by OLS. What then? Let us first see the results:

$$\begin{aligned} \hat{Y}_t &= 26,008.60 + 0.9771X_t + 1.0139X_{t-1} - 0.2022X_{t-2} - 0.3935X_{t-3} \\ \text{se} &= (6691.12) \quad (0.6820) \quad (1.0920) \quad (1.1021) \quad (0.7186) \\ t &= (3.8870) \quad (1.4327) \quad (0.9284) \quad (-0.1835) \quad (-0.5476) \\ R^2 &= 0.9755 \quad d = 0.1571 \quad F = 379.51 \quad (17.13.17) \end{aligned}$$

If you compare these results with those given in (17.13.16), you will see that the overall  $R^2$  is practically the same, although the lagged pattern in (17.13.17) shows more of a humped shape than that exhibited by (17.13.16).

<sup>51</sup>For a concrete application, see D. B. Batten and Daniel Thornton, "Polynomial Distributed Lags and the Estimation of the St. Louis Equation," *Review*, Federal Reserve Bank of St. Louis, April 1983, pp. 13–25.

As this example illustrates, one has to be careful in using the Almon distributed lag technique, as the results might be sensitive to the choice of the degree of the polynomial and/or the number of lagged coefficients.

### 17.14 CAUSALITY IN ECONOMICS: THE GRANGER CAUSALITY TEST<sup>52</sup>

Back in Section 1.4 we noted that, although regression analysis deals with the dependence of one variable on other variables, it does not necessarily imply causation. In other words, the existence of a relationship between variables does not prove causality or the direction of influence. But in regressions involving time series data, the situation may be somewhat different because, as one author puts it,

... time does not run backward. That is, if event *A* happens before event *B*, then it is *possible* that *A* is causing *B*. However, it is not *possible* that *B* is causing *A*. In other words, events in the past can cause events to happen today. Future events cannot.<sup>53</sup> (Emphasis added.)

This is roughly the idea behind the so-called Granger causality test.<sup>54</sup> But it should be noted clearly that the question of causality is deeply philosophical with all kinds of controversies. At one extreme are people who believe that “everything causes everything,” and at the other extreme are people who deny the existence of causation whatsoever.<sup>55</sup> The econometrician Edward Leamer prefers the term **precedence** over causality. Francis Diebold prefers the term **predictive causality**. As he writes:

... the statement “ $y_i$  causes  $y_j$ ” is just shorthand for the more precise, but long-winded, statement, “ $y_i$  contains useful information for predicting  $y_j$  (in the linear least squares sense), over and above the past histories of the other variables in the system.” To save space, we simply say that  $y_i$  causes  $y_j$ .<sup>56</sup>

#### The Granger Test

To explain the Granger test, we will consider the often asked question in macroeconomics: Is it GDP that “causes” the money supply  $M$  ( $GDP \rightarrow M$ )

<sup>52</sup>There is another test of causality that is sometimes used, the so-called **Sims test of causality**. We discuss it by way of an exercise.

<sup>53</sup>Gary Koop, *Analysis of Economic Data*, John Wiley & Sons, New York, 2000, p. 175.

<sup>54</sup>C. W. J. Granger, “Investigating Causal Relations by Econometric Models and Cross-Spectral Methods,” *Econometrica*, July 1969, pp. 424–438. Although popularly known as the Granger causality test, it is appropriate to call it the **Wiener–Granger causality test**, for it was earlier suggested by Wiener. See N. Wiener, “The Theory of Prediction,” in E. F. Beckenback, ed., *Modern Mathematics for Engineers*, McGraw-Hill, New York, 1956, pp. 165–190.

<sup>55</sup>For an excellent discussion of this topic, see Arnold Zellner, “Causality and Econometrics,” *Carnegie-Rochester Conference Series*, 10, K. Brunner and A. H. Meltzer, eds., North Holland Publishing Company, Amsterdam, 1979, pp. 9–50.

<sup>56</sup>Francis X. Diebold, *Elements of Forecasting*, South Western Publishing, 2d ed., 2001, p. 254.



or is it the money supply  $M$  that causes GDP ( $M \rightarrow \text{GDP}$ ), where the arrow points to the direction of causality. The Granger causality test assumes that the information relevant to the prediction of the respective variables, GDP and  $M$ , is contained solely in the time series data on these variables. The test involves estimating the following pair of regressions:

$$\text{GDP}_t = \sum_{i=1}^n \alpha_i M_{t-i} + \sum_{j=1}^n \beta_j \text{GDP}_{t-j} + u_{1t} \quad (17.14.1)$$

$$M_t = \sum_{i=1}^n \lambda_i M_{t-i} + \sum_{j=1}^n \delta_j \text{GDP}_{t-j} + u_{2t} \quad (17.14.2)$$

where it is assumed that the disturbances  $u_{1t}$  and  $u_{2t}$  are uncorrelated. In passing, note that, since we have two variables, we are dealing with **bilateral causality**. In the chapters on time series econometrics, we will extend this to multivariable causality through the technique of **vector autoregression (VAR)**.

Equation (17.14.1) postulates that current GDP is related to past values of itself as well as that of  $M$ , and (17.14.2) postulates a similar behavior for  $M$ . Note that these regressions can be cast in growth forms,  $\dot{\text{GDP}}$  and  $\dot{M}$ , where a dot over a variable indicates its growth rate. We now distinguish four cases:

**1. Unidirectional causality from  $M$  to GDP** is indicated if the estimated coefficients on the lagged  $M$  in (17.14.1) are statistically different from zero as a group (i.e.,  $\sum \alpha_i \neq 0$ ) and the set of estimated coefficients on the lagged GDP in (17.14.2) is not statistically different from zero (i.e.,  $\sum \delta_j = 0$ ).

**2. Conversely, unidirectional causality from GDP to  $M$**  exists if the set of lagged  $M$  coefficients in (17.14.1) is not statistically different from zero (i.e.,  $\sum \alpha_i = 0$ ) and the set of the lagged GDP coefficients in (17.14.2) is statistically different from zero (i.e.,  $\sum \delta_j \neq 0$ ).

**3. Feedback, or bilateral causality**, is suggested when the sets of  $M$  and GDP coefficients are statistically significantly different from zero in both regressions.

**4. Finally, independence** is suggested when the sets of  $M$  and GDP coefficients are not statistically significant in both the regressions.

More generally, since the future cannot predict the past, if variable  $X$  (Granger) causes variable  $Y$ , then changes in  $X$  should *precede* changes in  $Y$ . Therefore, in a regression of  $Y$  on other variables (including its own past values) if we include past or lagged values of  $X$  and it significantly improves the prediction of  $Y$ , then we can say that  $X$  (Granger) causes  $Y$ . A similar definition applies if  $Y$  (Granger) causes  $X$ .

The steps involved in implementing the Granger causality test are as follows. We illustrate these steps with the GDP-money example given in Eq. (17.14.1).



1. Regress current GDP on all lagged GDP terms and other variables, if any, but *do not* include the lagged  $M$  variables in this regression. As per Chapter 8, this is the restricted regression. From this regression obtain the restricted residual sum of squares,  $RSS_R$ .

2. Now run the regression including the lagged  $M$  terms. In the language of Chapter 8, this is the unrestricted regression. From this regression obtain the unrestricted residual sum of squares,  $RSS_{UR}$ .

3. The null hypothesis is  $H_0: \sum \alpha_i = 0$ , that is, lagged  $M$  terms do not belong in the regression.

4. To test this hypothesis, we apply the  $F$  test given by (8.7.9), namely,

$$F = \frac{(RSS_R - RSS_{UR})/m}{RSS_{UR}/(n - k)} \quad (8.7.9)$$

which follows the  $F$  distribution with  $m$  and  $(n - k)$  df. In the present case  $m$  is equal to the number of lagged  $M$  terms and  $k$  is the number of parameters estimated in the unrestricted regression.

5. If the computed  $F$  value exceeds the critical  $F$  value at the chosen level of significance, we reject the null hypothesis, in which case the lagged  $M$  terms belong in the regression. This is another way of saying that  $M$  causes GDP.

6. Steps 1 to 5 can be repeated to test the model (17.14.2), that is, whether GDP causes  $M$ .

Before we illustrate the Granger causality test, there are several things that need to be noted:

1. It is assumed that the two variables, GDP and  $M$ , are *stationary*. We have already discussed the concept of stationarity in intuitive terms before and will discuss it more formally in Chapter 21. Sometimes taking the first differences of the variables makes them stationary, if they are not already stationary in the level form.

2. The number of lagged terms to be introduced in the causality tests is an important practical question. As in the case of the distributed lag models, we may have to use the *Akaike or Schwarz information criterion* to make the choice. But it should be added that the *direction of causality may depend critically on the number of lagged terms included*.

3. We have assumed that the error terms entering the causality test are uncorrelated. If this is not the case, appropriate transformation, as discussed in Chapter 12, may have to be taken.<sup>57</sup>

4. Since our interest is in testing for causality, one need not present the estimated coefficients of models (17.14.1) and (17.14.2) explicitly (to save space); just the results of the  $F$  test given in (8.7.9) will suffice.

<sup>57</sup>For further details, see Wojciech W. Charemza and Derek F. Deadman, *New Directions in Econometric Practice: General to Specific Modelling, Cointegration and Vector Autoregression*, 3d ed., Edward Elgar Publisher, 1997, Chap. 6.

**EXAMPLE 17.12**

CAUSALITY BETWEEN MONEY AND INCOME

R. W. Hafer used the Granger test to find out the nature of causality between GNP (rather than GDP) and  $M$  for the United States for the period 1960-I to 1980-IV. Instead of using the gross values of these variables, he used their growth rates,  $\dot{GNP}$  and  $\dot{M}$ , and used four lags of each variable in the two regressions given previously. The results were as follows<sup>58</sup>: The null hypothesis in each case is that the variable under consideration does not “Granger-cause” the other variable.

Direction of causality	$F$ value	Decision
$\dot{M} \rightarrow \dot{GNP}$	2.68	Reject
$\dot{GNP} \rightarrow \dot{M}$	0.56	Do not reject

These results suggest that the direction of causality is from money growth to GNP growth since the estimated  $F$  is significant at the 5 percent level; the critical  $F$  value is 2.50 (for 4 and 71 df). On the other hand, there is no “reverse causation” from GNP growth to money growth, since the  $F$  value is statistically insignificant.

**EXAMPLE 17.13**

CAUSALITY BETWEEN MONEY AND INTEREST RATE IN CANADA

Refer to the Canadian data given in Table 17.3. Suppose we want to find out if there is any causality between money supply and interest rate in Canada for the quarterly periods of 1979–1988. To show that the Granger causality test depends critically on the number of lagged terms introduced in the model, we present below the results of the  $F$  test using several (quarterly) lags. In each case, the null hypothesis is that interest rate does *not* (Granger) cause money supply and vice versa.

Direction of causality	Number of lags	$F$ value	Decision
$R \rightarrow M$	2	12.92	Reject
$M \rightarrow R$	2	3.22	Reject
$R \rightarrow M$	4	5.59	Reject
$M \rightarrow R$	4	2.45	Reject (at 7%)
$R \rightarrow M$	6	3.5163	Reject
$M \rightarrow R$	6	2.71	Reject
$R \rightarrow M$	8	1.40	Do not reject
$M \rightarrow R$	8	1.62	Do not reject

Note these features of the preceding results of the  $F$  test: Up to six lags, there is bilateral causality between money supply and interest rate. However, at eight lags, there is no statistically discernible relationship between the two variables. This reinforces the point made earlier that the outcome of the Granger test is sensitive to the number of lags introduced in the model.

<sup>58</sup>R. W. Hafer, “The Role of Fiscal Policy in the St. Louis Equation,” *Review*, Federal Reserve Bank of St. Louis, January 1982, pp. 17–22. See his footnote 12 for the details of the procedure.

**EXAMPLE 17.14****CAUSALITY BETWEEN GDP GROWTH RATE AND GROSS SAVING RATE IN NINE EAST ASIAN COUNTRIES**

A study of the bilateral causality between GDP growth rate ( $g$ ) and gross savings rate ( $s$ ) showed the results given in Table 17.7.<sup>59</sup> For comparison, the results for the United States are also presented in the table. By and large, the results presented in Table 17.7 show that for most East Asian countries the causality runs from the GDP growth rate to the gross savings rate. By contrast, for the United States for the period 1950–1988 up to lag 3, causality ran in both directions, but for lags 4 and 5,

the causality ran from the GDP growth rate to savings rate but not the other way round.

To conclude our discussion of Granger causality, keep in mind that the question we are examining is whether *statistically* one can detect the direction of causality when *temporally* there is a lead-lag relationship between two variables. If causality is established, it suggests that one can use a variable to better predict the other variable than simply the past history of that other variable. In the case of the East Asian economies, it seems that we can better predict the gross savings rate by considering the lagged values of the GDP growth rate than merely the lagged values of the gross savings rate.

**TABLE 17.7**

TESTS OF BIVARIATE GRANGER CAUSALITY BETWEEN THE REAL PER CAPITA GDP GROWTH RATE AND THE GROSS SAVINGS RATE

Economy, years	Years of lags	Lagged right-hand side variable		Economy, years	Years of lags	Lagged right-hand side variable	
		savings	Growth			savings	Growth
Hong Kong, 1960–88	1	Sig	Sig	Philippines, 1950–88	1	NS	Sig
	2	Sig	Sig		2	NS	Sig
	3	Sig	Sig		3	NS	Sig
	4	Sig	Sig		4	NS	Sig
	5	Sig	Sig		5	NS	Sig
Indonesia, 1965	1	Sig	Sig	Singapore, 1960–88	1	NS	NS
	2	NS	Sig		2	NS	NS
	3	NS	Sig		3	NS	NS
	4	NS	Sig		4	Sig	NS
	5	NS	Sig		5	Sig	NS
Japan, 1950–88	1	NS	Sig	Taiwan, China, 1950–88	1	Sig	Sig
	2	NS	Sig		2	NS	Sig
	3	NS	Sig		3	NS	Sig
	4	NS	Sig		4	NS	Sig
	5	NS	Sig		5	NS	Sig
Korea, Rep. of, 1955–88	1	Sig	Sig	Thailand, 1950–88	1	NS	Sig
	2	NS	Sig		2	NS	Sig
	3	NS	Sig		3	NS	Sig
	4	NS	Sig		4	NS	Sig
	5	NS	Sig		5	NS	Sig
Malaysia, 1955–88	1	Sig	Sig	United States, 1950–88	1	Sig	Sig
	2	Sig	Sig		2	Sig	Sig
	3	NS	NS		3	Sig	Sig
	4	NS	NS		4	NS	Sig
	5	NS	Sig		5	NS	Sig

Sig: Significant; NS: Not significant.

Note: Growth is real per capita GDP growth at 1985 international prices.

Source: World Bank, *The East Asian Miracle: Economic Growth and Public Policy*, Oxford University Press, New York, 1993, p. 244, (Table A5-2). The original source is Robert Summers and Alan Heston, "The Penn World Tables (Mark 5): An Expanded Set of International Comparisons, 1950–88," *Quarterly Journal of Economics*, vol. 105, no. 2, 1991.

<sup>59</sup>These results are obtained from *The East Asian Miracle: Economic Growth and Public Policy*, published for the World Bank by Oxford University Press, 1993, p. 244.

**\*A Note on Causality and Exogeneity**

As we will study in the chapters on simultaneous equation model in **Part IV** of this text, economic variables are often classified into two broad categories, **endogenous** and **exogenous**. Loosely speaking, endogenous variables are the equivalent of the dependent variable in the single-equation regression model and exogenous variables are the equivalent of the  $X$  variables, or regressors, in such a model, provided the  $X$  variables are uncorrelated with the error term in that equation.<sup>60</sup>

Now we raise an interesting question: Suppose in a Granger causality test we find that an  $X$  variable (Granger) causes a  $Y$  variable without being caused by the latter (i.e., no bilateral causality). Can we then treat the  $X$  variable as exogenous? In other words, can we use Granger causality (or non-causality) to establish exogeneity?

To answer this question, we need to distinguish three types of exogeneity: (1) weak, (2) strong, and (3) super. To keep the exposition simple, suppose we consider only two variables,  $Y_t$  and  $X_t$ , and further suppose we regress  $Y_t$  on  $X_t$ . We say that  $X_t$  is *weakly exogenous* if  $Y_t$  also does not explain  $X_t$ . In this case estimation and testing of the regression model can be done, conditional on the values of  $X_t$ . As a matter of fact, going back to Chapter 2, you will realize that our regression modeling was conditional on the values of the  $X$  variables.  $X_t$  is said to be *strongly exogenous* if current and lagged  $Y$  values do not explain it (i.e., no feedback relationship). And  $X_t$  is *super-exogenous* if the parameters in the regression of  $Y$  and  $X$  do not change even if the  $X$  values change; that is, the parameter values are invariant to changes in the value(s) of  $X$ . If that is in fact the case, then, the famous “Lucas critique” may lose its force.<sup>61</sup>

The reason for distinguishing the three types of exogeneity is that, “In general, weak exogeneity is all that is needed for estimating and testing, strong exogeneity is necessary for forecasting and super exogeneity for policy analysis.”<sup>62</sup>

Returning to Granger causality, if a variable, say  $Y$ , does not cause another variable, say  $X$ , can we then assume that the latter is exogenous? Unfortunately, the answer is not straightforward. If we are talking about weak exogeneity, it can be shown that *Granger causality is neither necessary nor sufficient to establish exogeneity*. On the other hand, Granger causality is necessary (but not sufficient) for strong exogeneity. The proofs of these

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\*Optional.

<sup>60</sup>Of course, if the explanatory variables include one or more lagged terms of the endogenous variable, this requirement may not be fulfilled.

<sup>61</sup>The Nobel laureate Robert Lucas put forth the proposition that existing relations between economic variables may change when policy changes, in which case the estimated parameters from a regression model will be of little value for prediction. On this, see Oliver Blanchard, *Macroeconomics*, Prentice Hall, 1997, pp. 371–372.

<sup>62</sup>Keith Cuthbertson, Stephen G. Hall, and Mark P. Taylor, *Applied Econometric Techniques*, University of Michigan Press, 1992, p. 100.

statements are beyond the scope of this book.<sup>63</sup> For our purpose, then, it is better to keep the concepts of Granger causality and exogeneity separate and treat the former as a useful descriptive tool for time series data. In Chapter 19 we will discuss a test to find out if a variable can be treated as exogenous.

### 17.15 SUMMARY AND CONCLUSIONS

1. For psychological, technological, and institutional reasons, a regressand may respond to a regressor(s) with a time lag. Regression models that take into account time lags are known as **dynamic** or **lagged regression models**.

2. There are two types of lagged models: **distributed-lag** and **autoregressive**. In the former, the current and lagged values of regressors are explanatory variables. In the latter, the lagged value(s) of the regressand appear as explanatory variables.

3. A purely distributed-lag model can be estimated by OLS, but in that case there is the problem of multicollinearity since successive lagged values of a regressor tend to be correlated.

4. As a result, some shortcut methods have been devised. These include the Koyck, the adaptive expectations, and partial adjustment mechanisms, the first being a purely algebraic approach and the other two being based on economic principles.

5. But a unique feature of the **Koyck, adaptive expectations, and partial adjustment models** is that they all are autoregressive in nature in that the lagged value(s) of the regressand appear as one of the explanatory variables.

6. Autoregressiveness poses estimation challenges; if the lagged regressand is correlated with the error term, OLS estimators of such models are not only biased but also are inconsistent. Bias and inconsistency are the case with the Koyck and the adaptive expectations models; the partial adjustment model is different in that it can be consistently estimated by OLS despite the presence of the lagged regressand.

7. To estimate the Koyck and adaptive expectations models consistently, the most popular method is the **method of instrumental variable**. The instrumental variable is a proxy variable for the lagged regressand but with the property that it is uncorrelated with the error term.

8. An alternative to the lagged regression models just discussed is the **Almon polynomial distributed-lag model**, which avoids the estimation problems associated with the autoregressive models. The major problem with the Almon approach, however, is that one must *prespecify* both the lag length and the degree of the polynomial. There are both formal and informal methods of resolving the choice of the lag length and the degree of the polynomial.

<sup>63</sup>For a comparatively simple discussion, see G. S. Maddala, *Introduction to Econometrics*, 2d ed., Macmillan, New York, 1992, pp. 394–395, and also David F. Hendry, *Dynamic Econometrics*, Oxford University Press, New York, 1995, Chap. 5.

9. Despite the estimation problems, which can be surmounted, the distributed and autoregressive models have proved extremely useful in empirical economics because they make the otherwise static economic theory a dynamic one by taking into account explicitly the role of time. Such models help us to distinguish between short- and long-run response of the dependent variable to a unit change in the value of the explanatory variable(s). Thus, for estimating short- and long-run price, income, substitution, and other elasticities these models have proved to be highly useful.<sup>64</sup>

10. Because of the lags involved, distributed and or autoregressive models raise the topic of causality in economic variables. In applied work, the **Granger causality** modeling has received considerable attention. But one has to exercise great caution in using the Granger methodology because it is very sensitive to the lag length used in the model.

11. Even if a variable ( $X$ ) “Granger-causes” another variable ( $Y$ ), it does not mean that  $X$  is exogenous. We distinguished three types of exogeneity—weak, strong, and super—and pointed out the importance of the distinction.

## EXERCISES

### Questions

- 17.1. Explain with a brief reason whether the following statements are true, false, or uncertain:
- All econometric models are essentially dynamic.
  - The Koyck model will not make much sense if some of the distributed-lag coefficients are positive and some negative.
  - If the Koyck and adaptive expectations models are estimated by OLS, the estimators will be biased but consistent.
  - In the partial adjustment model, OLS estimators are biased in finite samples.
  - In the presence of a stochastic regressor(s) and an autocorrelated error term, the method of instrumental variables will produce unbiased as well as consistent estimates.
  - In the presence of a lagged regressand as a regressor, the Durbin-Watson  $d$  statistic to detect autocorrelation is practically useless.
  - The Durbin  $h$  test is valid in both large and small samples.
  - The Granger test is a test of precedence rather than a test of causality.
- 17.2. Establish Eq. (17.7.2).
- 17.3. Prove Eq. (17.8.3).
- 17.4. Assume that prices are formed according to the following adaptive expectations hypothesis:

$$P_t^* = \gamma P_{t-1} + (1 - \gamma)P_{t-1}^*$$

where  $P^*$  is the expected price and  $P$  the actual price.

<sup>64</sup>For applications of these models, see Arnold C. Harberger, ed., *The Demand for Durable Goods*, University of Chicago Press, Chicago, 1960.

Complete the following table, assuming  $\gamma = 0.5^*$ :

Period	$P^*$	$P$
$t - 3$	100	110
$t - 2$		125
$t - 1$		155
$t$		185
$t + 1$		—

17.5. Consider the model

$$Y_t = \alpha + \beta_1 X_{1t} + \beta_2 X_{2t} + \beta_3 Y_{t-1} + v_t$$

Suppose  $Y_{t-1}$  and  $v_t$  are correlated. To remove the correlation, suppose we use the following instrumental variable approach: First regress  $Y_t$  on  $X_{1t}$  and  $X_{2t}$  and obtain the estimated  $\hat{Y}_t$  from this regression. Then regress

$$Y_t = \alpha + \beta_1 X_{1t} + \beta_2 X_{2t} + \beta_3 \hat{Y}_{t-1} + v_t$$

where  $\hat{Y}_{t-1}$  are estimated from the first-stage regression.

- a. How does this procedure remove the correlation between  $Y_{t-1}$  and  $v_t$  in the original model?
  - b. What are the advantages of the recommended procedure over the Liviatan approach?
- †17.6. a. Establish (17.4.8).  
 b. Evaluate the median lag for  $\lambda = 0.2, 0.4, 0.6, 0.8$ .  
 c. Is there any systematic relationship between the value of  $\lambda$  and the value of the median lag?
- 17.7. a. Prove that for the Koyck model, the mean lag is as shown in (17.4.10).  
 b. If  $\lambda$  is relatively large, what are its implications?
- 17.8. Using the formula for the mean lag given in (17.4.9), verify the mean lag of 10.959 quarters reported in the illustration of Table 17.1.
- 17.9. Suppose

$$M_t = \alpha + \beta_1 Y_t^* + \beta_2 R_t^* + u_t$$

where  $M$  = demand for real cash balances,  $Y^*$  = expected real income, and  $R^*$  = expected interest rate. Assume that expectations are formulated as follows:

$$Y_t^* = \gamma_1 Y_t + (1 - \gamma_1) Y_{t-1}^*$$

$$R_t^* = \gamma_2 R_t + (1 - \gamma_2) R_{t-1}^*$$

where  $\gamma_1$  and  $\gamma_2$  are coefficients of expectation, both lying between 0 and 1.

- a. How would you express  $M_t$  in terms of the observable quantities?
- b. What estimation problems do you foresee?

\*Adapted from G. K. Shaw, op. cit., p. 26.

†Optional.



\***17.10.** If you estimate (17.7.2) by OLS, can you derive estimates of the original parameters? What problems do you foresee? (For details, see Roger N. Waud.<sup>†</sup>)

**17.11. Serial correlation model.** Consider the following model:

$$Y_t = \alpha + \beta X_t + u_t$$

Assume that  $u_t$  follows the Markov first-order autoregressive scheme given in Chapter 12, namely,

$$u_t = \rho u_{t-1} + \varepsilon_t$$

where  $\rho$  is the coefficient of (first-order) autocorrelation and where  $\varepsilon_t$  satisfies all the assumptions of the classical OLS. Then, as shown in Chapter 12, the model

$$Y_t = \alpha(1 - \rho) + \beta(X_t - \rho X_{t-1}) + \rho Y_{t-1} + \varepsilon_t$$

will have a serially independent error term, making OLS estimation possible. But this model, called the **serial correlation model**, very much resembles the Koyck, adaptive expectation, and partial adjustment models. How would you know in any given situation which of the preceding models is appropriate?<sup>‡</sup>

**17.12.** Consider the Koyck (or for that matter the adaptive expectation) model given in (17.4.7), namely,

$$Y_t = \alpha(1 - \lambda) + \beta_0 X_t + \lambda Y_{t-1} + (u_t - \lambda u_{t-1})$$

Suppose in the original model  $u_t$  follows the first-order autoregressive scheme  $u_t - \rho u_{t-1} = \varepsilon_t$ , where  $\rho$  is the coefficient of autocorrelation and where  $\varepsilon_t$  satisfies all the classical OLS assumptions.

**a.** If  $\rho = \lambda$ , can the Koyck model be estimated by OLS?

**b.** Will the estimates thus obtained be unbiased? Consistent? Why or why not?

**c.** How reasonable is it to assume that  $\rho = \lambda$ ?

**17.13. Triangular, or arithmetic, distributed-lag model.<sup>§</sup>** This model assumes that the stimulus (explanatory variable) exerts its greatest impact in the current time period and then declines by equal decrements to zero as one goes into the distant past. Geometrically, it is shown in Figure 17.9. Following this distribution, suppose we run the following

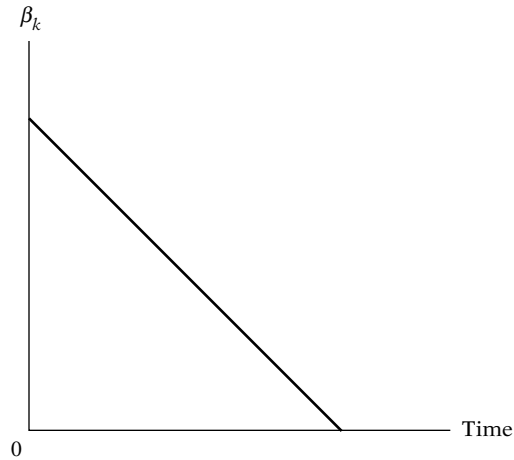
\*Optional.

<sup>†</sup>"Misspecification in the 'Partial Adjustment' and 'Adaptive Expectations' Models," *International Economic Review*, vol. 9, no. 2, June 1968, pp. 204–217.

<sup>‡</sup>For a discussion of the serial correlation model, see Zvi Griliches, "Distributed Lags: A Survey," *Econometrica*, vol. 35, no. 1, January 1967, p. 34.

<sup>§</sup>This model was proposed by Irving Fisher in "Note on a Short-Cut Method for Calculating Distributed Lags," *International Statistical Bulletin*, 1937, pp. 323–328.





**FIGURE 17.9** Triangular or arithmetic lag scheme (Fisher's).

succession of regressions:

$$Y_t = \alpha + \beta \left( \frac{2X_t + X_{t-1}}{3} \right)$$

$$Y_t = \alpha + \beta \left( \frac{3X_t + 2X_{t-1} + X_{t-2}}{6} \right)$$

$$Y_t = \alpha + \beta \left( \frac{4X_t + 3X_{t-1} + 2X_{t-2} + X_{t-3}}{10} \right)$$

etc., and choose the regression that gives the highest  $R^2$  as the "best" regression. Comment on this strategy.

- 17.14.** From the quarterly data for the period 1950–1960, F. P. R. Brechling obtained the following demand function for labor for the British economy (the figures in parentheses are standard errors)\*:

$$\hat{E}_t = 14.22 + 0.172Q_t - 0.028t - 0.0007t^2 - 0.297E_{t-1}$$

(2.61) (0.014) (0.015) (0.0002) (0.033)

$$\bar{R}^2 = 0.76 \quad d = 1.37$$

where  $\dot{E}_t = (E_t - E_{t-1})$   
 $Q$  = output  
 $t$  = time

The preceding equation was based on the assumption that the desired level of employment  $E_t^*$  is a function of output, time, and time squared and on the hypothesis that  $E_t - E_{t-1} = \delta(E_t^* - E_{t-1})$ , where  $\delta$ , the coefficient of adjustment, lies between 0 and 1.

\*F. P. R. Brechling, "The Relationship between Output and Employment in British Manufacturing Industries," *Review of Economic Studies*, vol. 32, July 1965.

- a. Interpret the preceding regression.  
 b. What is the value of  $\delta$ ?  
 c. Derive the long-run demand function for labor from the estimated short-run demand function.  
 d. How would you test for serial correlation in the preceding model?
- 17.15. In studying the farm demand for tractors, Griliches used the following model\*:

$$T_t^* = \alpha X_{1,t-1}^{\beta_1} X_{2,t-1}^{\beta_2}$$

where  $T^*$  = desired stock of tractors  
 $X_1$  = relative price of tractors  
 $X_2$  = interest rate

Using the stock adjustment model, he obtained the following results for the period 1921–1957:

$$\widehat{\log T_t} = \text{constant} - 0.218 \log X_{1,t-1} - 0.855 \log X_{2,t-1} + 0.864 \log T_{t-1}$$

(0.051)                      (0.170)                      (0.035)

$R^2 = 0.987$

where the figures in the parentheses are the estimated standard errors.

- a. What is the estimated coefficient of adjustment?  
 b. What are the short- and long-run price elasticities?  
 c. What are the corresponding interest elasticities?  
 d. What are the reasons for high or low rate of adjustment in the present model?
- 17.16. Whenever the lagged dependent variable appears as an explanatory variable, the  $R^2$  is usually much higher than when it is not included. What are the reasons for this observation?
- 17.17. Consider the lag patterns in Figure 17.10. What degree polynomials would you fit to the lag structures and why?
- 17.18. Consider the Eq. (17.13.4)

$$\beta_i = a_0 + a_1 i + a_2 i^2 + \cdots + a_m i^m$$

To obtain the variance of  $\hat{\beta}_i$  from the variances of  $\hat{a}_i$ , we use the following formula:

$$\begin{aligned} \text{var}(\hat{\beta}_i) &= \text{var}(\hat{a}_0 + \hat{a}_1 i + \hat{a}_2 i^2 + \cdots + \hat{a}_m i^m) \\ &= \sum_{j=0}^m i^{2j} \text{var}(\hat{a}_j) + 2 \sum_{j < p} i^{(j+p)} \text{cov}(\hat{a}_j \hat{a}_p) \end{aligned}$$

\*Zvi Griliches, "The Demand for a Durable Input: Farm Tractors in the United States, 1921–1957," in Arnold C. Harberger, ed., *The Demand for Durable Goods*, University of Chicago Press, Chicago, 1960.

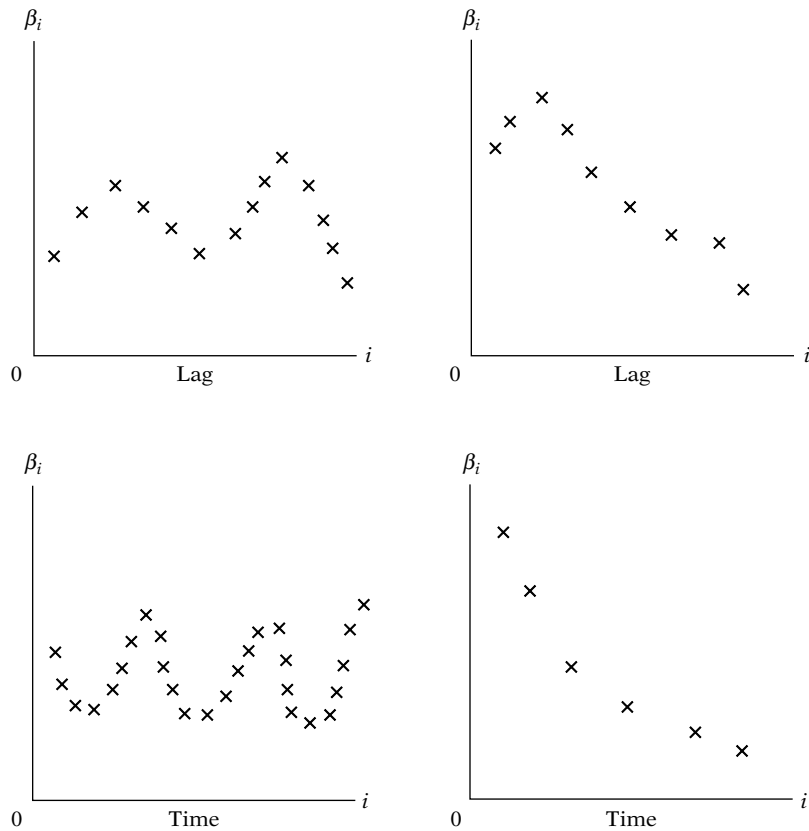


FIGURE 17.10 Hypothetical lag structures.

a. Using the preceding formula, find the variance of  $\hat{\beta}_i$  expressed as

$$\hat{\beta}_i = \hat{a}_0 + \hat{a}_1 i + \hat{a}_2 i^2$$

$$\hat{\beta}_i = \hat{a}_0 + \hat{a}_1 i + \hat{a}_2 i^2 + \hat{a}_3 i^3$$

b. If the variances of  $\hat{a}_i$  are large relative to themselves, will the variance of  $\hat{\beta}_i$  be large also? Why or why not?

17.19. Consider the following distributed-lag model:

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \beta_3 X_{t-3} + \beta_4 X_{t-4} + u_t$$

Assume that  $\beta_i$  can be adequately expressed by the second-degree polynomial as follows:

$$\beta_i = a_0 + a_1 i + a_2 i^2$$

How would you estimate the  $\beta$ 's if we want to impose the restriction that  $\beta_0 = \beta_4 = 0$ ?

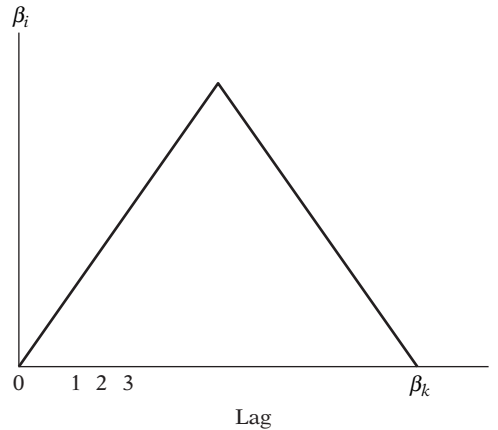


FIGURE 17.11 Inverted V distributed-lag model.

**17.20. The inverted V distributed-lag model.** Consider the  $k$ -period finite distributed-lag model

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \cdots + \beta_k X_{t-k} + u_t$$

F. DeLeeuw has proposed the structure for the  $\beta$ 's as in Figure 17.11, where the  $\beta$ 's follow the inverted V shape. Assuming for simplicity that  $k$  (the maximum length of the lag) is an even number, and further assuming that  $\beta_0$  and  $\beta_k$  are zero, DeLeeuw suggests the following scheme for the  $\beta$ 's\*:

$$\begin{aligned} \beta_i &= i\beta & 0 \leq i \leq \frac{k}{2} \\ &= (k-i)\beta & \frac{k}{2} \leq i < k \end{aligned}$$

How would you use the DeLeeuw scheme to estimate the parameters of the preceding  $k$ -period distributed-lag model?

**17.21.** Refer to exercise 12.15. Since the  $d$  value shown there is of little use in detecting (first-order) autocorrelation (why?), how would you test for autocorrelation in this case?

### Problems

**17.22.** Consider the following model:

$$Y_t^* = \alpha + \beta_0 X_t + u_t$$

where  $Y^*$  = desired, or long-run, business expenditure for new plant and equipment,  $X_t$  = sales, and  $t$  = time. Using the stock adjustment model, estimate the parameters of the long- and short-run demand function for expenditure on new plant and equipment given in Table 17.8.

How would you find out if there is serial correlation in the data?

\*See his article, "The Demand for Capital Goods by Manufacturers: A Study of Quarterly Time Series," *Econometrica*, vol. 30, no. 3, July 1962, pp. 407-423.

**TABLE 17.8** INVESTMENT IN FIXED PLANT AND EQUIPMENT IN MANUFACTURING YAND  
MANUFACTURING SALES  $X_2$  IN BILLIONS OF DOLLARS, SEASONALLY ADJUSTED,  
UNITED STATES, 1970–1991

Year	Plant expenditure, $Y$	Sales, $X_2$	Year	Plant expenditure, $Y$	Sales, $X_2$
1970	36.99	52.805	1981	128.68	168.129
1971	33.60	55.906	1982	123.97	163.351
1972	35.42	63.027	1983	117.35	172.547
1973	42.35	72.931	1984	139.61	190.682
1974	52.48	84.790	1985	152.88	194.538
1975	53.66	86.589	1986	137.95	194.657
1976	58.53	98.797	1987	141.06	206.326
1977	67.48	113.201	1988	163.45	223.541
1978	78.13	126.905	1989	183.80	232.724
1979	95.13	143.936	1990	192.61	239.459
1980	112.60	154.391	1991	182.81	235.142

Source: *Economic Report of the President*, 1993. Data on  $Y$  from Table B-52, p. 407; data on  $X_2$  from Table 8-53, p. 408.

**17.23.** Use the data of exercise 17.22 but consider the following model:

$$Y_t^* = \beta_0 X_t^{\beta_1} e^{u_t}$$

Using the stock adjustment model (why?), estimate the short- and long-run elasticities of expenditure on new plant and equipment with respect to sales. Compare your results with those for exercise 17.22. Which model would you choose and why? Is there serial correlation in the data? How do you know?

**17.24.** Use the data of exercise 17.22 but assume that

$$Y_t = \alpha + \beta X_t^* + u_t$$

where  $X_t^*$  are the desired sales. Estimate the parameters of this model and compare the results with those obtained in exercise 17.22. How would you decide which is the appropriate model? On the basis of the  $h$  statistic, would you conclude there is serial correlation in the data?

**17.25.** Suppose someone convinces you that the relationship between business expenditure for new plant and equipment and sales is as follows:

$$Y_t^* = \alpha + \beta X_t^* + u_t$$

where  $Y^*$  is desired expenditure and  $X^*$  is desired or expected sales. Use the data given in exercise 17.22 to estimate this model and comment on your results.

**17.26.** Using the data given in exercise 17.22, determine whether plant expenditure Granger-causes sales or sales Granger-causes plant expenditure. Use up to six lags and comment on your results. What important conclusion do you draw from this exercise?

- 17.27.** Assume that sales in Exercise 17.22 has a distributed-lag effect on expenditure on plant and equipment. Fit a suitable Almon lag model to the data.
- 17.28.** Reestimate Eq. (17.13.16) imposing (1) near-end restriction, (2) far-end restriction, and (3) both end restrictions and compare your results given in Eq. (17.13.16). What general conclusion do you draw?
- 17.29.** Table 17.9 gives data on private fixed investment in information processing and equipment ( $Y$ , in billions of dollars), sales in total manufacturing and trade ( $X_2$ , in millions of dollars), and interest rate ( $X_3$ , Moody's triple A corporate bond rate, percent); data on  $Y$  and  $X_2$  are seasonally adjusted.
- Test for bilateral causality between  $Y$  and  $X_2$ , paying careful attention to the lag length.
  - Test for bilateral causality between  $Y$  and  $X_3$ , again paying careful attention to the lag length.
  - To allow for the distributed lag effect of sales on investment, suppose you decide to use the Almon lag technique. Show the estimated model, after paying due attention to the length of the lag as well as the degree of the polynomial.
- 17.30.** Table 17.10 gives data on indexes of real compensation per hour ( $Y$ ) and output per hour ( $X_2$ ), with both indexes to base 1992 = 100, in the business sector of the U.S. economy for the period 1960–1999, as well as the civilian unemployment rate ( $X_3$ ) for the same period.

**TABLE 17.9** INVESTMENTS, SALES, AND INTEREST RATE, UNITED STATES, 1960–1999

Observation	Investment	Sales	Interest	Observation	Investment	Sales	Interest
1960	4.9	60,827	4.41	1980	69.6	327,233	11.94
1961	5.2	61,159	4.35	1981	82.4	355,822	14.17
1962	5.7	65,662	4.33	1982	88.9	347,625	13.79
1963	6.5	68,995	4.26	1983	100.8	369,286	12.04
1964	7.3	73,682	4.40	1984	121.7	410,124	12.71
1965	8.5	80,283	4.49	1985	130.8	422,583	11.37
1966	10.6	87,187	5.13	1986	137.6	430,419	9.02
1967	11.2	90,820	5.51	1987	141.9	457,735	9.38
1968	11.9	96,685	6.18	1988	155.9	497,157	9.71
1969	14.6	105,690	7.03	1989	173.0	527,039	9.26
1970	16.7	108,221	8.04	1990	176.1	545,909	9.32
1971	17.3	116,895	7.39	1991	181.4	542,815	8.77
1972	19.3	131,081	7.21	1992	197.5	567,176	8.14
1973	23.0	153,677	7.44	1993	215.0	595,628	7.22
1974	26.8	177,912	8.57	1994	233.7	639,163	7.96
1975	28.2	182,198	8.83	1995	262.0	684,982	7.59
1976	32.4	204,150	8.43	1996	287.3	718,113	7.37
1977	38.6	229,513	8.02	1997	325.2	753,445	7.26
1978	48.3	260,320	8.73	1998	367.4	779,413	6.53
1979	58.6	297,701	9.63	1999	433.0	833,079	7.04

Notes: Investment = private fixed investment in information processing equipment and software, billions of dollars, seasonally adjusted.

Sales = sales in total manufacturing and trade, millions of dollars, seasonally adjusted.

Interest = Moody's triple A corporate bond rate, %.

Source: *Economic Report of the President*, 2001, Tables B-18, B-57, and B-73.

**TABLE 17.10** COMPENSATION, PRODUCTIVITY AND UNEMPLOYMENT RATE, UNITED STATES, 1960–1999

Observation	COMP	PRODUCT	UNRate	Observation	COMP	PRODUCT	UNRate
1960	60.0	48.8	5.5	1980	89.5	80.4	7.1
1961	61.8	50.6	6.7	1981	89.5	82.0	7.6
1962	63.9	52.9	5.5	1982	90.9	81.7	9.7
1963	65.4	55.0	5.7	1983	91.0	84.6	9.6
1964	67.9	57.5	5.2	1984	91.3	87.0	7.5
1965	69.4	59.6	4.5	1985	92.7	88.7	7.2
1966	71.9	62.0	3.8	1986	95.8	91.4	7.0
1967	73.8	63.4	3.8	1987	96.3	91.9	6.2
1968	76.3	65.4	3.6	1988	97.3	93.0	5.5
1969	77.4	65.7	3.5	1989	95.9	93.9	5.3
1970	78.9	67.0	4.9	1990	96.5	95.2	5.6
1971	80.4	69.9	5.9	1991	97.5	96.3	6.8
1972	82.7	72.2	5.6	1992	100.0	100.0	7.5
1973	84.5	74.5	4.9	1993	99.9	100.5	6.9
1974	83.5	73.2	5.6	1994	99.7	101.9	6.1
1975	84.4	75.8	8.5	1995	99.3	102.6	5.6
1976	86.8	78.5	7.7	1996	99.7	105.4	5.4
1977	87.9	79.8	7.1	1997	100.4	107.6	4.9
1978	89.5	80.7	6.1	1998	104.3	110.5	4.5
1979	89.7	80.7	5.8	1999	107.3	114.0	4.2

Notes: COMP = index of real compensation per hour (1992 = 100).

PRODUCT = index of output per hour (1992 = 100).

UNRate = civilian unemployment rate, %.

Source: *Economic Report of the President*, 2001, Table B-49, p. 332.

- a. How would you decide whether it is wage compensation that determines labor productivity or the other way round?
  - b. Develop a suitable model to test your conjecture in **a**, providing the usual statistics.
  - c. Do you think the unemployment rate has any effect on wage compensation, and if so, how would you take that into account? Show the necessary statistical analysis.
- 17.31. Sims' test of causality.\*** In a twist of Granger causality, Sims exploits the fact that the future cannot cause the present. Suppose we want to find out if  $X$  causes  $Y$ . Now consider the following model:

$$Y_t = \alpha + \beta_k X_{t-k} + \beta_{k-1} X_{t-k-1} + \cdots + \beta_1 X_{t-1} + \beta_0 X_t \\ + \lambda_1 X_{t+1} + \lambda_2 X_{t+2} + \cdots + \lambda_m X_{t+m} + u_t$$

This regression includes the lagged, current, and future, or **lead**, values of the regressor  $X$ ; terms such as  $X_{t+1}$  and  $X_{t+2}$  are called **lead terms**. In the preceding regression, there are  $k$  lagged and  $m$  lead terms. If  $X$  is to

\*C. A. Sims, "Money, Income, and Causality," *American Economic Review*, vol. 62, 1972, pp. 540–552.

(Granger) cause  $Y$ , the sum of the coefficients of the lead  $X$  terms must be statistically equal to zero.\*

Apply the Sims' test of causality to the data given in exercise 17.22 to determine whether sales (Granger) cause investment expenditure. Decide for yourself the appropriate lead and lag values of the regressor.

## APPENDIX 17A

### 17A.1 THE SARGAN TEST FOR THE VALIDITY OF INSTRUMENTS

Suppose we use an instrumental variable(s) to replace an explanatory variable(s) that is correlated with the error term. How valid is the instrumental variable(s), that is, how do we know that the instruments chosen are independent of the error term? Sargan has developed a statistic, dubbed SARG, to test the validity of the instruments used in instrumental variable(s) (IV).† The steps involved in SARG are as follows‡:

1. Divide the variables included in a regression equation into two groups, those that are independent of the error term (say,  $X_1, X_2, \dots, X_p$ ) and those that are not independent of the error term (say,  $Z_1, Z_2, \dots, Z_q$ ).
2. Let  $W_1, W_2, \dots, W_s$  be the instruments chosen for the  $Z$  variables in 1, where  $s \geq q$ .
3. Estimate the original regression, replacing the  $Z$ 's by the  $W$ 's, that is, estimate the original regression by IV and obtain the residuals, say,  $\hat{u}$ .
4. Regress  $\hat{u}$  on a constant, all the  $X$  variables and all the  $W$  variables but exclude all the  $Z$  variables. Obtain  $R^2$  from this regression.
5. Now compute the SARG statistic, defined as:

$$\text{SARG} = (n - k)R^2 \quad (17A.1.1)$$

where  $n$  = the number of observations and  $k$  = the number of coefficients in the original regression equation. Sargent has shown that (17A.1.1) follows the  $\chi^2$  distribution with  $r$  df, where  $r = s - q$ .

6. The null hypothesis is that all ( $w$ ) instruments are valid. If the computed chi-square exceeds the critical chi-square value, we reject the null hypothesis, which means that at least one instrument is correlated with the error term and therefore the IV estimates based on the chosen instruments are not valid.

\*The choice between Granger and Sims causality tests is not clear. For further discussion of these tests, see G. Chamberlain, "The General Equivalence of Granger and Sims Causality," *Econometrica*, vol. 50, 1982, pp. 569–582.

†Sargan, J. D., "Wages and Prices in the United Kingdom: A Study in Econometric Methodology," in P. E. Hart, G. Mills, and J. K. Whitaker (eds.) *Econometric Analysis for National Economic Planning*, Butterworths, London, 1964.

‡The following discussion leans on H. R. Seddighi, K. A. Lawler and A. V. Katos, *Econometrics: A Practical Approach*, Routledge, New York, 2000, pp. 155–156.



## PART FOUR

SIMULTANEOUS-EQUATION  
MODELS

A casual look at the published empirical work in business and economics will reveal that many economic relationships are of the single-equation type. That is why we devoted the first three parts of this book to the discussion of single-equation regression models. In such models, one variable (the dependent variable  $Y$ ) is expressed as a linear function of one or more other variables (the explanatory variables, the  $X$ 's). In such models an implicit assumption is that the cause-and-effect relationship, if any, between  $Y$  and the  $X$ 's is unidirectional: The explanatory variables are the *cause* and the dependent variable is the *effect*.

However, there are situations where there is a two-way flow of influence among economic variables; that is, one economic variable affects another economic variable(s) and is, in turn, affected by it (them). Thus, in the regression of money  $M$  on the rate of interest  $r$ , the single-equation methodology assumes implicitly that the rate of interest is fixed (say, by the Federal Reserve System) and tries to find out the response of money demanded to the changes in the level of the interest rate. But what happens if the rate of interest depends on the demand for money? In this case, the conditional regression analysis made in this book thus far may not be appropriate because now  $M$  depends on  $r$  and  $r$  depends on  $M$ . Thus, we need to consider two equations, one relating  $M$  to  $r$  and another relating  $r$  to  $M$ . And this leads us to consider simultaneous-equation models, models in which there is more than one regression equation, one for each interdependent variable.

In **Part IV** we present a very elementary and often heuristic introduction to the complex subject of **simultaneous-equation models**, the details being left for the references.

In Chapter 18, we provide several examples of simultaneous-equation models and show why the method of ordinary least squares considered previously is generally inapplicable to estimate the parameters of each of the equations in the model.

In Chapter 19, we consider the so-called **identification problem**. If in a system of simultaneous equations containing two or more equations it is not possible to obtain numerical values of each parameter in each equation because the equations are *observationally indistinguishable*, or look too much like one another, then we have the identification problem. Thus, in the regression of quantity  $Q$  on price  $P$ , is the resulting equation a demand function or a supply function, for  $Q$  and  $P$  enter into both functions? Therefore, if we have data on  $Q$  and  $P$  only and no other information, it will be difficult if not impossible to identify the regression as the demand or supply function. It is essential to resolve the identification problem before we proceed to estimation because if we do not know what we are estimating, estimation per se is meaningless. In Chapter 19 we offer various methods of solving the identification problem.

In Chapter 20, we consider several estimation methods that are designed specifically for estimating the simultaneous-equation models and consider their merits and limitations.

# 18

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## SIMULTANEOUS-EQUATION MODELS

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In this and the following two chapters we discuss the simultaneous-equation models. In particular, we discuss their special features, their estimation, and some of the statistical problems associated with them.

### 18.1 THE NATURE OF SIMULTANEOUS-EQUATION MODELS

In **Parts I to III** of this text we were concerned exclusively with single-equation models, i.e., models in which there was a single dependent variable  $Y$  and one or more explanatory variables, the  $X$ 's. In such models the emphasis was on estimating and/or predicting the average value of  $Y$  conditional upon the fixed values of the  $X$  variables. The cause-and-effect relationship, if any, in such models therefore ran from the  $X$ 's to the  $Y$ .

But in many situations, such a one-way or unidirectional cause-and-effect relationship is not meaningful. This occurs if  $Y$  is determined by the  $X$ 's, and some of the  $X$ 's are, in turn, determined by  $Y$ . In short, there is a two-way, or simultaneous, relationship between  $Y$  and (some of) the  $X$ 's, which makes the distinction between *dependent* and *explanatory* variables of dubious value. It is better to lump together a set of variables that can be determined simultaneously by the remaining set of variables—precisely what is done in simultaneous-equation models. In such models there is more than one equation—one for each of the *mutually*, or *jointly*, dependent or **endogenous variables**.<sup>1</sup> And unlike the single-equation models, in the

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<sup>1</sup>In the context of the simultaneous-equation models, the jointly dependent variables are called **endogenous variables** and the variables that are truly nonstochastic or can be so regarded are called the **exogenous**, or **predetermined, variables**. (More on this in Chap. 19.)

simultaneous-equation models one may not estimate the parameters of a single equation without taking into account information provided by other equations in the system.

What happens if the parameters of each equation are estimated by applying, say, the method of OLS, disregarding other equations in the system? Recall that one of the crucial assumptions of the method of OLS is that the explanatory  $X$  variables are either nonstochastic or, if stochastic (random), are distributed independently of the stochastic disturbance term. If neither of these conditions is met, then, as shown later, the least-squares estimators are not only biased but also inconsistent; that is, as the sample size increases indefinitely, the estimators do not converge to their true (population) values. Thus, in the following hypothetical system of equations,<sup>2</sup>

$$Y_{1i} = \beta_{10} + \beta_{12}Y_{2i} + \gamma_{11}X_{1i} + u_{1i} \quad (18.1.1)$$

$$Y_{2i} = \beta_{20} + \beta_{21}Y_{1i} + \gamma_{21}X_{1i} + u_{2i} \quad (18.1.2)$$

where  $Y_1$  and  $Y_2$  are mutually dependent, or endogenous, variables and  $X_1$  is an exogenous variable and where  $u_1$  and  $u_2$  are the stochastic disturbance terms, the variables  $Y_1$  and  $Y_2$  are both stochastic. Therefore, unless it can be shown that the stochastic explanatory variable  $Y_2$  in (18.1.1) is distributed independently of  $u_1$  and the stochastic explanatory variable  $Y_1$  in (18.1.2) is distributed independently of  $u_2$ , application of the classical OLS to these equations individually will lead to inconsistent estimates.

In the remainder of this chapter we give a few examples of simultaneous-equation models and show the bias involved in the direct application of the least-squares method to such models. After discussing the so-called identification problem in Chapter 19, in Chapter 20 we discuss some of the special methods developed to handle the simultaneous-equation models.

## 18.2 EXAMPLES OF SIMULTANEOUS-EQUATION MODELS

### EXAMPLE 18.1

#### DEMAND-AND-SUPPLY MODEL

As is well known, the price  $P$  of a commodity and the quantity  $Q$  sold are determined by the intersection of the demand-and-supply curves for that commodity. Thus, assuming for simplicity that the demand-and-supply curves are linear and adding the stochastic disturbance terms  $u_1$  and  $u_2$ , we may write the empirical demand-and-supply functions as

$$\text{Demand function:} \quad Q_t^d = \alpha_0 + \alpha_1 P_t + u_{1t} \quad \alpha_1 < 0 \quad (18.2.1)$$

$$\text{Supply function:} \quad Q_t^s = \beta_0 + \beta_1 P_t + u_{2t} \quad \beta_1 > 0 \quad (18.2.2)$$

$$\text{Equilibrium condition:} \quad Q_t^d = Q_t^s$$

(Continued)

<sup>2</sup>These economical but self-explanatory notations will be generalized to more than two equations in Chap. 19.

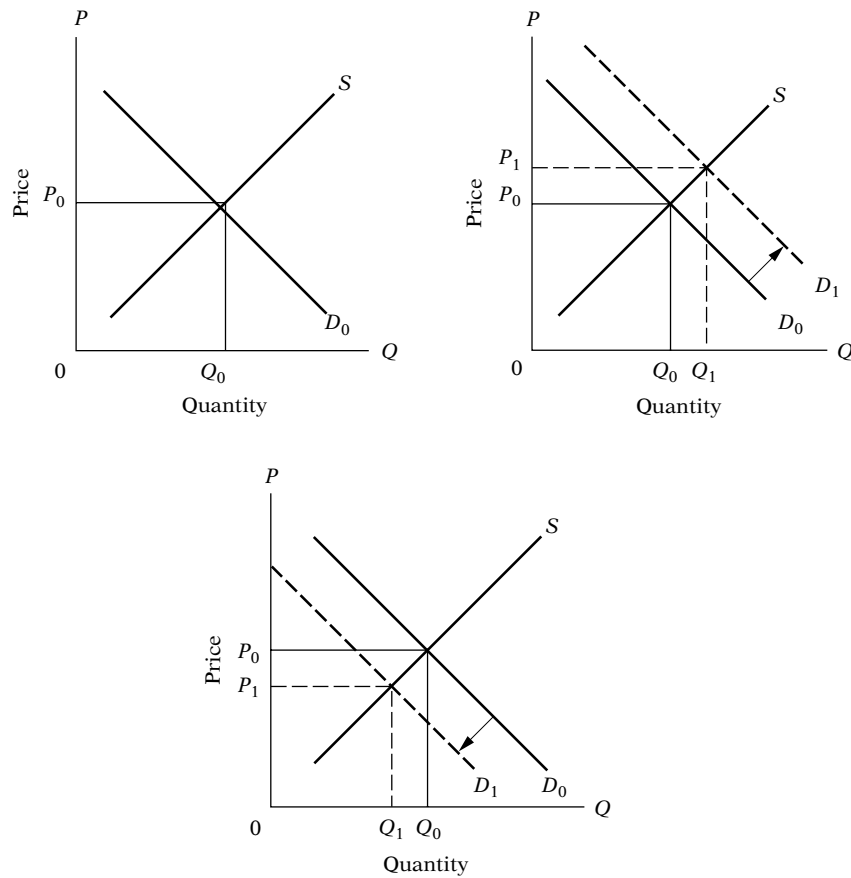
**EXAMPLE 18.1** (Continued)

where  $Q^d$  = quantity demanded  
 $Q^s$  = quantity supplied  
 $t$  = time

and the  $\alpha$ 's and  $\beta$ 's are the parameters. A priori,  $\alpha_1$  is expected to be negative (downward-sloping demand curve), and  $\beta_1$  is expected to be positive (upward-sloping supply curve).

Now it is not too difficult to see that  $P$  and  $Q$  are jointly dependent variables. If, for example,  $u_{1t}$  in (18.2.1) changes because of changes in other variables affecting  $Q_t^d$  (such as income, wealth, and tastes), the demand curve will shift upward if  $u_{1t}$  is positive and downward if  $u_{1t}$  is negative. These shifts are shown in Figure 18.1.

As the figure shows, a shift in the demand curve changes both  $P$  and  $Q$ . Similarly, a change in  $u_{2t}$  (because of strikes, weather, import or export restrictions, etc.) will shift the supply curve, again affecting both  $P$  and  $Q$ . Because of this simultaneous dependence between  $Q$  and  $P$ ,  $u_{1t}$  and  $P_t$  in (18.2.1) and  $u_{2t}$  and  $P_t$  in (18.2.2) cannot be independent. Therefore, a regression of  $Q$  on  $P$  as in (18.2.1) would violate an important assumption of the classical linear regression model, namely, the assumption of no correlation between the explanatory variable(s) and the disturbance term.



**FIGURE 18.1** Interdependence of price and quantity.

**EXAMPLE 18.2**

**KEYNESIAN MODEL OF INCOME DETERMINATION**

Consider the simple Keynesian model of income determination:

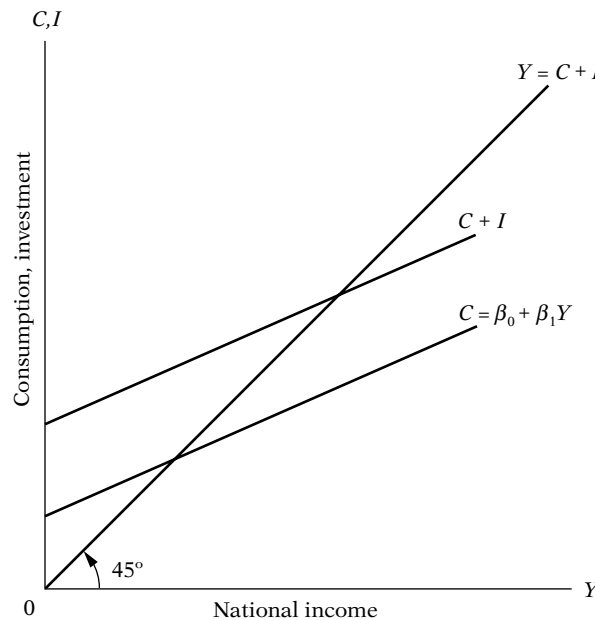
$$\text{Consumption function: } C_t = \beta_0 + \beta_1 Y_t + u_t \quad 0 < \beta_1 < 1 \quad (18.2.3)$$

$$\text{Income identity: } Y_t = C_t + I_t (= S_t) \quad (18.2.4)$$

where  $C$  = consumption expenditure  
 $Y$  = income  
 $I$  = investment (assumed exogenous)  
 $S$  = savings  
 $t$  = time  
 $u$  = stochastic disturbance term  
 $\beta_0$  and  $\beta_1$  = parameters

The parameter  $\beta_1$  is known as the *marginal propensity to consume* (MPC) (the amount of extra consumption expenditure resulting from an extra dollar of income). From economic theory,  $\beta_1$  is expected to lie between 0 and 1. Equation (18.2.3) is the (stochastic) consumption function; and (18.2.4) is the national income identity, signifying that total income is equal to total consumption expenditure plus total investment expenditure, it being understood that total investment expenditure is equal to total savings. Diagrammatically, we have Figure 18.2.

From the postulated consumption function and Figure 18.2 it is clear that  $C$  and  $Y$  are interdependent and that  $Y_t$  in (18.2.3) is not expected to be independent of the disturbance term because when  $u_t$  shifts (because of a variety of factors subsumed in the error term), then the consumption function also shifts, which, in turn, affects  $Y_t$ . Therefore, once again the classical least-squares method is inapplicable to (18.2.3). If applied, the estimators thus obtained will be inconsistent, as we shall show later.



**FIGURE 18.2** Keynesian model of income determination.

**EXAMPLE 18.3**

WAGE-PRICE MODELS

Consider the following Phillips-type model of money-wage and price determination:

$$\dot{W}_t = \alpha_0 + \alpha_1 UN_t + \alpha_2 \dot{P}_t + u_{1t} \quad (18.2.5)$$

$$\dot{P}_t = \beta_0 + \beta_1 \dot{W}_t + \beta_2 \dot{R}_t + \beta_3 \dot{M}_t + u_{2t} \quad (18.2.6)$$

where  $\dot{W}$  = rate of change of money wages  
 $UN$  = unemployment rate, %  
 $\dot{P}$  = rate of change of prices  
 $\dot{R}$  = rate of change of cost of capital  
 $\dot{M}$  = rate of change of price of imported raw material  
 $t$  = time  
 $u_1, u_2$  = stochastic disturbances

Since the price variable  $\dot{P}$  enters into the wage equation and the wage variable  $\dot{W}$  enters into the price equation, the two variables are jointly dependent. Therefore, these stochastic explanatory variables are expected to be correlated with the relevant stochastic disturbances, once again rendering the classical OLS method inapplicable to estimate the parameters of the two equations individually.

**EXAMPLE 18.4**

THE IS MODEL OF MACROECONOMICS

The celebrated IS, or goods market equilibrium, model of macroeconomics<sup>3</sup> in its non-stochastic form can be expressed as

$$\text{Consumption function: } C_t = \beta_0 + \beta_1 Y_{dt} \quad 0 < \beta_1 < 1 \quad (18.2.7)$$

$$\text{Tax function: } T_t = \alpha_0 + \alpha_1 Y_t \quad 0 < \alpha_1 < 1 \quad (18.2.8)$$

$$\text{Investment function: } I_t = \gamma_0 + \gamma_1 r_t \quad (18.2.9)$$

$$\text{Definition: } Y_{dt} = Y_t - T_t \quad (18.2.10)$$

$$\text{Government expenditure: } G_t = \bar{G} \quad (18.2.11)$$

$$\text{National income identity: } Y_t = C_t + I_t + G_t \quad (18.2.12)$$

where  $Y$  = national income  
 $C$  = consumption spending  
 $I$  = planned or desired net investment  
 $\bar{G}$  = given level of government expenditure  
 $T$  = taxes  
 $Y_d$  = disposable income  
 $r$  = interest rate

(Continued)

<sup>3</sup>The goods market equilibrium schedule, or IS schedule, shows combinations of interest rates and levels of output such that planned spending equals income." See Rudiger Dornbusch and Stanley Fischer, *Macroeconomics*, 3d ed., McGraw-Hill, New York, 1984, p. 102. Note that for simplicity we have assumed away the foreign trade sector.

**EXAMPLE 18.4** (Continued)

If you substitute (18.2.10) and (18.2.8) into (18.2.7) and substitute the resulting equation for  $C$  and Eq. (18.2.9) and (18.2.11) into (18.2.12), you should obtain

$$\text{IS equation: } Y_t = \pi_0 + \pi_1 r_t \quad (18.2.13)$$

where

$$\pi_0 = \frac{\beta_0 - \alpha_0 \beta_1 + \gamma_0 + \bar{G}}{1 - \beta_1(1 - \alpha_1)} \quad (18.2.14)$$

$$\pi_1 = \frac{1}{1 - \beta_1(1 - \alpha_1)}$$

Equation (18.2.13) is the equation of the IS, or goods market equilibrium, that is, it gives the combinations of the interest rate and level of income such that the goods market clears or is in equilibrium. Geometrically, the IS curve is shown in Figure 18.3.

What would happen if we were to estimate, say, the consumption function (18.2.7) in isolation? Could we obtain unbiased and/or consistent estimates of  $\beta_0$  and  $\beta_1$ ? Such a result is unlikely because consumption depends on disposable income, which depends on national income  $Y$ , but the latter depends on  $r$  and  $\bar{G}$  as well as the other parameters entering in  $\pi_0$ . Therefore, unless we take into account all these influences, a simple regression of  $C$  on  $Y_d$  is bound to give biased and/or inconsistent estimates of  $\beta_0$  and  $\beta_1$ .



**FIGURE 18.3** The IS curve.

**EXAMPLE 18.5**

THE LM MODEL

The other half of the famous IS-LM paradigm is the LM, or money market equilibrium, relation, which gives the combinations of the interest rate and level of income such that the money market is cleared, that is, the demand for money is equal to its supply. Algebraically, the model, in the nonstochastic form, may be expressed as:

$$\text{Money demand function: } M_t^d = a + bY_t - cr_t \quad (18.2.15)$$

$$\text{Money supply function: } M_t^s = \bar{M} \quad (18.2.16)$$

$$\text{Equilibrium condition: } M_t^d = M_t^s \quad (18.2.17)$$

(Continued)



**EXAMPLE 18.5** (Continued)

where  $Y$  = income,  $r$  = interest rate, and  $\bar{M}$  = assumed level of money supply, say, determined by the Fed.

Equating the money demand and supply functions and simplifying, we obtain:

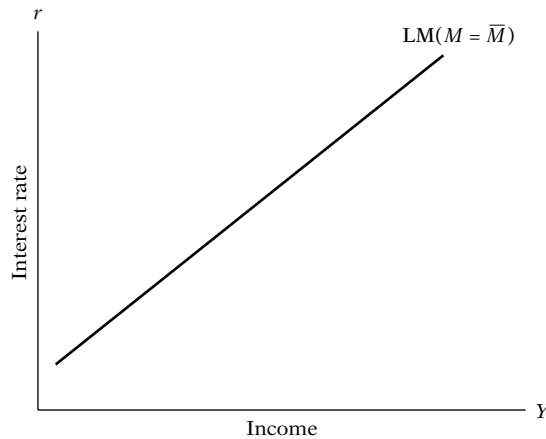
$$LM \text{ equation: } Y_t = \lambda_0 + \lambda_1 \bar{M} + \lambda_2 r_t \quad (18.2.18)$$

where

$$\begin{aligned} \lambda_0 &= -a/b \\ \lambda_1 &= 1/b \\ \lambda_2 &= c/b \end{aligned} \quad (18.2.19)$$

For a given  $M = \bar{M}$ , the LM curve representing the relation (18.2.18) is as shown in Figure 18.4.

The IS and LM curves show, respectively, that a whole array of interest rates is consistent with goods market equilibrium and a whole array of interest rates is compatible with equilibrium in the money market. Of course, only one interest rate and one level of income will be consistent simultaneously with the two equilibria. To obtain these, all that needs to be done is to equate (18.2.13) and (18.2.18). In exercise 18.4 you are asked to show the level of the interest rate and income that is simultaneously compatible with the goods and money market equilibrium.



**FIGURE 18.4** The LM curve.

**EXAMPLE 18.6**

ECONOMETRIC MODELS

An extensive use of simultaneous-equation models has been made in the econometric models built by several econometricians. An early pioneer in this field was Professor Lawrence Klein of the Wharton School of the University of Pennsylvania. His initial model, known as **Klein's model I**, is as follows:

$$\text{Consumption function: } C_t = \beta_0 + \beta_1 P_t + \beta_2 (W + W')_t + \beta_3 P_{t-1} + u_{1t} \quad (\text{Continued})$$

**EXAMPLE 18.6** (Continued)

$$\begin{aligned}
 \text{Investment function:} \quad & I_t = \beta_4 + \beta_5 P_t + \beta_6 P_{t-1} + \beta_7 K_{t-1} + u_{2t} \\
 \text{Demand for labor:} \quad & W_t = \beta_8 + \beta_9(Y + T - W')_t \\
 & \quad \quad \quad + \beta_{10}(Y + T - W')_{t-1} + \beta_{11}t + u_{3t} \quad (18.2.20) \\
 \text{Identity:} \quad & Y_t + T_t = C_t + I_t + G_t \\
 \text{Identity:} \quad & Y_t = W_t^i + W_t + P_t \\
 \text{Identity:} \quad & K_t = K_{t-1} + I_t
 \end{aligned}$$

where

- $C$  = consumption expenditure
- $I$  = investment expenditure
- $G$  = government expenditure
- $P$  = profits
- $W$  = private wage bill
- $W'$  = government wage bill
- $K$  = capital stock
- $T$  = taxes
- $Y$  = income after tax
- $t$  = time
- $u_1, u_2,$  and  $u_3$  = stochastic disturbances<sup>4</sup>

In the preceding model the variables  $C, I, W, Y, P,$  and  $K$  are treated as jointly dependent, or endogenous, variables and the variables  $P_{t-1}, K_{t-1},$  and  $Y_{t-1}$  are treated as predetermined.<sup>5</sup> In all, there are six equations (including the three identities) to study the interdependence of six endogenous variables.

In Chapter 20 we shall see how such econometric models are estimated. For the time being, note that because of the interdependence among the endogenous variables, in general they are not independent of the stochastic disturbance terms, which therefore makes it inappropriate to apply the method of OLS to an individual equation in the system. As shown in Section 18.3, the estimators thus obtained are inconsistent; they do not converge to their true population values even when the sample size is very large.

### 18.3 THE SIMULTANEOUS-EQUATION BIAS: INCONSISTENCY OF OLS ESTIMATORS

As stated previously, the method of least squares may not be applied to estimate a single equation embedded in a system of simultaneous equations if one or more of the explanatory variables are correlated with the disturbance term in that equation because the estimators thus obtained are inconsistent. To show this, let us revert to the simple Keynesian model of income determination given in Example 18.2. Suppose that we want to

<sup>4</sup>L. R. Klein, *Economic Fluctuations in the United States, 1921–1941*, John Wiley & Sons, New York, 1950.

<sup>5</sup>The model builder will have to specify which of the variables in a model are endogenous and which are predetermined.  $K_{t-1}$  and  $Y_{t-1}$  are predetermined because at time  $t$  their values are known. (More on this in Chap. 19.)

estimate the parameters of the consumption function (18.2.3). Assuming that  $E(u_t) = 0$ ,  $E(u_t^2) = \sigma^2$ ,  $E(u_t u_{t+j}) = 0$  (for  $j \neq 0$ ), and  $\text{cov}(I_t, u_t) = 0$ , which are the assumptions of the classical linear regression model, we first show that  $Y_t$  and  $u_t$  in (18.2.3) are correlated and then prove that  $\hat{\beta}_1$  is an inconsistent estimator of  $\beta_1$ .

To prove that  $Y_t$  and  $u_t$  are correlated, we proceed as follows. Substitute (18.2.3) into (18.2.4) to obtain

$$Y_t = \beta_0 + \beta_1 Y_t + u_t + I_t$$

that is,

$$Y_t = \frac{\beta_0}{1 - \beta_1} + \frac{1}{1 - \beta_1} I_t + \frac{1}{1 - \beta_1} u_t \quad (18.3.1)$$

Now

$$E(Y_t) = \frac{\beta_0}{1 - \beta_1} + \frac{1}{1 - \beta_1} I_t \quad (18.3.2)$$

where use is made of the fact that  $E(u_t) = 0$  and that  $I_t$  being exogenous, or predetermined (because it is fixed in advance), has as its expected value  $I_t$ .

Therefore, subtracting (18.3.2) from (18.3.1) results in

$$Y_t - E(Y_t) = \frac{u_t}{1 - \beta_1} \quad (18.3.3)$$

Moreover,

$$u_t - E(u_t) = u_t \quad (\text{Why?}) \quad (18.3.4)$$

whence

$$\begin{aligned} \text{cov}(Y_t, u_t) &= E[Y_t - E(Y_t)][u_t - E(u_t)] \\ &= \frac{E(u_t^2)}{1 - \beta_1} \quad \text{from (18.3.3) and (18.3.4)} \quad (18.3.5) \\ &= \frac{\sigma^2}{1 - \beta_1} \end{aligned}$$

Since  $\sigma^2$  is positive by assumption (why?), the covariance between  $Y$  and  $u$  given in (18.3.5) is bound to be different from zero.<sup>6</sup> As a result,  $Y_t$  and  $u_t$  in (18.2.3) are expected to be correlated, which violates the assumption of the classical linear regression model that the disturbances are independent or at least uncorrelated with the explanatory variables. As noted previously, the OLS estimators in this situation are inconsistent.

<sup>6</sup>It will be greater than zero as long as  $\beta_1$ , the MPC, lies between 0 and 1, and it will be negative if  $\beta_1$  is greater than unity. Of course, a value of MPC greater than unity would not make much economic sense. In reality therefore the covariance between  $Y_t$  and  $u_t$  is expected to be positive.

To show that the OLS estimator  $\hat{\beta}_1$  is an inconsistent estimator of  $\beta_1$  because of correlation between  $Y_t$  and  $u_t$ , we proceed as follows:

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum(C_t - \bar{C})(Y_t - \bar{Y})}{\sum(Y_t - \bar{Y})^2} \\ &= \frac{\sum c_t y_t}{\sum y_t^2} \\ &= \frac{\sum C_t y_t}{\sum y_t^2}\end{aligned}\quad (18.3.6)$$

where the lowercase letters, as usual, indicate deviations from the (sample) mean values. Substituting for  $C_t$  from (18.2.3), we obtain

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum(\beta_0 + \beta_1 Y_t + u_t)y_t}{\sum y_t^2} \\ &= \beta_1 + \frac{\sum y_t u_t}{\sum y_t^2}\end{aligned}\quad (18.3.7)$$

where in the last step use is made of the fact that  $\sum y_t = 0$  and  $(\sum Y_t y_t / \sum y_t^2) = 1$  (why?).

If we take the expectation of (18.3.7) on both sides, we obtain

$$E(\hat{\beta}_1) = \beta_1 + E\left[\frac{\sum y_t u_t}{\sum y_t^2}\right]\quad (18.3.8)$$

Unfortunately, we cannot evaluate  $E(\sum y_t u_t / \sum y_t^2)$  since the expectations operator is a linear operator. [Note:  $E(A/B) \neq E(A)/E(B)$ .] But intuitively it should be clear that unless the term  $(\sum y_t u_t / \sum y_t^2)$  is zero,  $\hat{\beta}_1$  is a biased estimator of  $\beta_1$ . But have we not shown in (18.3.5) that the covariance between  $Y$  and  $u$  is nonzero and therefore would  $\hat{\beta}_1$  not be biased? The answer is, not quite, since  $\text{cov}(Y_t, u_t)$ , a population concept, is not quite  $\sum y_t u_t$ , which is a sample measure, although as the sample size increases indefinitely the latter will tend toward the former. But if the sample size increases indefinitely, then we can resort to the concept of consistent estimator and find out what happens to  $\hat{\beta}_1$  as  $n$ , the sample size, increases indefinitely. In short, when we cannot explicitly evaluate the expected value of an estimator, as in (18.3.8), we can turn our attention to its behavior in the large sample.

Now an estimator is said to be consistent if its **probability limit**,<sup>7</sup> or **plim** for short, is equal to its true (population) value. Therefore, to show that  $\hat{\beta}_1$  of (18.3.7) is inconsistent, we must show that its plim is not equal to the true  $\beta_1$ .

<sup>7</sup>See **App. A** for the definition of probability limit.

Applying the rules of probability limit to (18.3.7), we obtain<sup>8</sup>

$$\begin{aligned}\text{plim}(\hat{\beta}_1) &= \text{plim}(\beta_1) + \text{plim}\left(\frac{\sum y_i u_i}{\sum y_i^2}\right) \\ &= \text{plim}(\beta_1) + \text{plim}\left(\frac{\sum y_i u_i/n}{\sum y_i^2/n}\right) \\ &= \beta_1 + \frac{\text{plim}(\sum y_i u_i/n)}{\text{plim}(\sum y_i^2/n)}\end{aligned}\quad (18.3.9)$$

where in the second step we have divided  $\sum y_i u_i$  and  $\sum y_i^2$  by the total number of observations in the sample  $n$  so that the quantities in the parentheses are now the sample covariance between  $Y$  and  $u$  and the sample variance of  $Y$ , respectively.

In words, (18.3.9) states that the probability limit of  $\hat{\beta}_1$  is equal to true  $\beta_1$  plus the ratio of the plim of the sample covariance between  $Y$  and  $u$  to the plim of the sample variance of  $Y$ . Now as the sample size  $n$  increases indefinitely, one would expect the sample covariance between  $Y$  and  $u$  to approximate the true population covariance  $E[Y_i - E(Y_i)][u_i - E(u_i)]$ , which from (18.3.5) is equal to  $[\sigma^2/(1 - \beta_1)]$ . Similarly, as  $n$  tends to infinity, the sample variance of  $Y$  will approximate its population variance, say  $\sigma_Y^2$ . Therefore, Eq. (18.3.9) may be written as

$$\begin{aligned}\text{plim}(\hat{\beta}_1) &= \beta_1 + \frac{\sigma^2/(1 - \beta_1)}{\sigma_Y^2} \\ &= \beta_1 + \frac{1}{1 - \beta_1} \left(\frac{\sigma^2}{\sigma_Y^2}\right)\end{aligned}\quad (18.3.10)$$

Given that  $0 < \beta_1 < 1$  and that  $\sigma^2$  and  $\sigma_Y^2$  are both positive, it is obvious from Eq. (18.3.10) that  $\text{plim}(\hat{\beta}_1)$  will always be greater than  $\beta_1$ ; that is,  $\hat{\beta}_1$  will overestimate the true  $\beta_1$ .<sup>9</sup> In other words,  $\hat{\beta}_1$  is a biased estimator, and the bias will not disappear no matter how large the sample size.

#### 18.4 THE SIMULTANEOUS-EQUATION BIAS: A NUMERICAL EXAMPLE

To demonstrate some of the points made in the preceding section, let us return to the simple Keynesian model of income determination given in Example 18.2 and carry out the following **Monte Carlo** study.<sup>10</sup> Assume that

<sup>8</sup>As stated in **App. A**, the plim of a constant (for example,  $\beta_1$ ) is the same constant and the plim of  $(A/B) = \text{plim}(A)/\text{plim}(B)$ . Note, however, that  $E(A/B) \neq E(A)/E(B)$ .

<sup>9</sup>In general, however, the direction of the bias will depend on the structure of the particular model and the true values of the regression coefficients.

<sup>10</sup>This is borrowed from Kenneth J. White, Nancy G. Horsman, and Justin B. Wyatt, *SHAZAM: Computer Handbook for Econometrics for Use with Basic Econometrics*, McGraw-Hill, New York, 1985, pp. 131–134.

TABLE 18.1

$Y_t$ (1)	$C_t$ (2)	$I_t$ (3)	$u_t$ (4)
18.15697	16.15697	2.0	-0.3686055
19.59980	17.59980	2.0	-0.8004084E-01
21.93468	19.73468	2.2	0.1869357
21.55145	19.35145	2.2	0.1102906
21.88427	19.48427	2.4	-0.2314535E-01
22.42648	20.02648	2.4	0.8529544E-01
25.40940	22.80940	2.6	0.4818807
22.69523	20.09523	2.6	-0.6095481E-01
24.36465	21.56465	2.8	0.7292983E-01
24.39334	21.59334	2.8	0.7866819E-01
24.09215	21.09215	3.0	-0.1815703
24.87450	21.87450	3.0	-0.2509900E-01
25.31580	22.11580	3.2	-0.1368398
26.30465	23.10465	3.2	0.6092946E-01
25.78235	22.38235	3.4	-0.2435298
26.08018	22.68018	3.4	-0.1839638
27.24440	23.64440	3.6	-0.1511200
28.00963	24.40963	3.6	0.1926739E-02
30.89301	27.09301	3.8	0.3786015
28.98706	25.18706	3.8	-0.2588852E-02

Source: Kenneth J. White, Nancy G. Horsman, and Justin B. Wyatt, *SHAZAM Computer Handbook for Econometrics for Use with Damodar Gujarati: Basic Econometrics*, September 1985, p. 132.

the values of investment  $I$  are as shown in column 3 of Table 18.1. Further assume that

$$E(u_t) = 0$$

$$E(u_t u_{t+j}) = 0 \quad (j \neq 0)$$

$$\text{var}(u_t) = \sigma^2 = 0.04$$

$$\text{cov}(u_t, I_t) = 0$$

The  $u_t$  thus generated are shown in column (4).

For the consumption function (18.2.3) assume that the values of the true parameters are known and are  $\beta_0 = 2$  and  $\beta_1 = 0.8$ .

From the assumed values of  $\beta_0$  and  $\beta_1$  and the generated values of  $u_t$  we can generate the values of income  $Y_t$  from (18.3.1), which are shown in column 1 of Table 18.1. Once  $Y_t$  are known, and knowing  $\beta_0$ ,  $\beta_1$ , and  $u_t$ , one can easily generate the values of consumption  $C_t$  from (18.2.3). The  $C_t$ 's thus generated are given in column 2.

Since the true  $\beta_0$  and  $\beta_1$  are known, and since our sample errors are exactly the same as the "true" errors (because of the way we designed the Monte Carlo study), if we use the data of Table 18.1 to regress  $C_t$  on  $Y_t$  we should obtain  $\beta_0 = 2$  and  $\beta_1 = 0.8$ , if OLS were unbiased. But from (18.3.7) we know that this will not be the case if the regressor  $Y_t$  and the disturbance  $u_t$  are correlated. Now it is not too difficult to verify from our data that the (sample) covariance between  $Y_t$  and  $u_t$  is  $\sum y_t u_t = 3.8$  and that  $\sum y_t^2 = 184$ .

Then, as (18.3.7) shows, we should have

$$\begin{aligned}\hat{\beta}_1 &= \beta_1 + \frac{\sum y_t u_t}{\sum y_t^2} \\ &= 0.8 + \frac{3.8}{184} \\ &= 0.82065\end{aligned}\tag{18.4.1}$$

That is,  $\hat{\beta}_1$  is upward-biased by 0.02065.

Now let us regress  $C_t$  on  $Y_t$ , using the data given in Table 18.1. The regression results are

$$\begin{aligned}\hat{C}_t &= 1.4940 + 0.82065Y_t \\ \text{se} &= (0.35413) \quad (0.01434) \\ t &= (4.2188) \quad (57.209) \quad R^2 = 0.9945\end{aligned}\tag{18.4.2}$$

As expected, the estimated  $\beta_1$  is precisely the one predicted by (18.4.1). In passing, note that the estimated  $\beta_0$  too is biased.

In general the amount of the bias in  $\hat{\beta}_1$  depends on  $\beta_1$ ,  $\sigma^2$  and  $\text{var}(Y)$  and, in particular, on the degree of covariance between  $Y$  and  $u$ .<sup>11</sup> As Kenneth White et al. note, "This is what simultaneous equation bias is all about. In contrast to single equation models, we can no longer assume that variables on the right hand side of the equation are uncorrelated with the error term."<sup>12</sup> Bear in mind that this bias remains even in large samples.

In view of the potentially serious consequences of applying OLS in simultaneous-equation models, is there a test of simultaneity that can tell us whether in a given instance we have the simultaneity problem? One version of the **Hausman specification test** can be used for this purpose, which we discuss in Chapter 19.

## 18.5 SUMMARY AND CONCLUSIONS

**1.** In contrast to single-equation models, in simultaneous-equation models more than one dependent, or **endogenous**, variable is involved, necessitating as many equations as the number of endogenous variables.

**2.** A unique feature of simultaneous-equation models is that the endogenous variable (i.e., regressand) in one equation may appear as an explanatory variable (i.e., regressor) in another equation of the system.

**3.** As a consequence, such an **endogenous explanatory variable** becomes stochastic and is usually correlated with the disturbance term of the equation in which it appears as an explanatory variable.

**4.** In this situation the classical OLS method may not be applied because the estimators thus obtained are not consistent, that is, they do not converge to their true population values no matter how large the sample size.

<sup>11</sup>See Eq. (18.3.5).

<sup>12</sup>Op. cit., pp. 133–134.

5. The Monte Carlo example presented in the text shows the nature of the bias involved in applying OLS to estimate the parameters of a regression equation in which the regressor is correlated with the disturbance term, which is typically the case in simultaneous-equation models.

6. Since simultaneous-equation models are used frequently, especially in econometric models, alternative estimating techniques have been developed by various authors. These are discussed in Chapter 20, after the topic of the **identification problem** is considered in Chapter 19, a topic logically prior to estimation.

## EXERCISES

### Questions

- 18.1. Develop a simultaneous-equation model for the supply of and demand for dentists in the United States. Specify the endogenous and exogenous variables in the model.
- 18.2. Develop a simple model of the demand for and supply of money in the United States and compare your model with those developed by K. Brunner and A. H. Meltzer\* and R. Tiegen.†
- 18.3. a. For the demand-and-supply model of Example 18.1, obtain the expression for the probability limit of  $\hat{\alpha}_1$ .  
b. Under what conditions will this probability limit be equal to the true  $\alpha_1$ ?
- 18.4. For the IS-LM model discussed in the text, find the level of interest rate and income that is simultaneously compatible with the goods and money market equilibrium.
- 18.5. To study the relationship between inflation and yield on common stock, Bruno Oudet‡ used the following model:

$$R_{bt} = \alpha_1 + \alpha_2 R_{st} + \alpha_3 R_{bt-1} + \alpha_4 L_t + \alpha_5 Y_t + \alpha_6 \text{NIS}_t + \alpha_7 I_t + u_{1t}$$

$$R_{st} = \beta_1 + \beta_2 R_{bt} + \beta_3 R_{st-1} + \beta_4 L_t + \beta_5 Y_t + \beta_6 \text{NIS}_t + \beta_7 E_t + u_{2t}$$

where  $L$  = real per capita monetary base  
 $Y$  = real per capita income  
 $I$  = the expected rate of inflation  
 $\text{NIS}$  = a new issue variable  
 $E$  = expected end-of-period stock returns, proxied by lagged stock price ratios  
 $R_{bt}$  = bond yield  
 $R_{st}$  = common stock returns

\*"Some Further Evidence on Supply and Demand Functions for Money," *Journal of Finance*, vol. 19, May 1964, pp. 240–283.

†"Demand and Supply Functions for Money in the United States," *Econometrica*, vol. 32, no. 4, October 1964, pp. 476–509.

‡Bruno A. Oudet, "The Variation of the Return on Stocks in Periods of Inflation," *Journal of Financial and Quantitative Analysis*, vol. 8, no. 2, March 1973, pp. 247–258.



- a. Offer a theoretical justification for this model and see if your reasoning agrees with that of Oudet.
- b. Which are the endogenous variables in the model? And the exogenous variables?
- c. How would you treat the lagged  $R_{bt}$ —endogenous or exogenous?
- 18.6.** In their article, “A Model of the Distribution of Branded Personal Products in Jamaica,”\* John U. Farley and Harold J. Levitt developed the following model (the personal products considered were shaving cream, skin cream, sanitary napkins, and toothpaste):

$$Y_{1i} = \alpha_1 + \beta_1 Y_{2i} + \beta_2 Y_{3i} + \beta_3 Y_{4i} + u_{1i}$$

$$Y_{2i} = \alpha_2 + \beta_4 Y_{1i} + \beta_5 Y_{5i} + \gamma_1 X_{1i} + \gamma_2 X_{2i} + u_{2i}$$

$$Y_{3i} = \alpha_3 + \beta_6 Y_{2i} + \gamma_3 X_{3i} + u_{3i}$$

$$Y_{4i} = \alpha_4 + \beta_7 Y_{2i} + \gamma_4 X_{4i} + u_{4i}$$

$$Y_{5i} = \alpha_5 + \beta_8 Y_{2i} + \beta_9 Y_{3i} + \beta_{10} Y_{4i} + u_{5i}$$

where  $Y_1$  = percent of stores stocking the product

$Y_2$  = sales in units per month

$Y_3$  = index of direct contact with importer and manufacturer for the product

$Y_4$  = index of wholesale activity in the area

$Y_5$  = index of depth of brand stocking for the product (i.e., average number of brands of the product stocked by stores carrying the product)

$X_1$  = target population for the product

$X_2$  = income per capita in the parish where the area is

$X_3$  = distance from the population center of gravity to Kingston

$X_4$  = distance from population center to nearest wholesale town

- a. Can you identify the endogenous and exogenous variables in the preceding model?
- b. Can one or more equations in the model be estimated by the method of least squares? Why or why not?
- 18.7.** To study the relationship between advertising expenditure and sales of cigarettes, Frank Bass used the following model†:

$$Y_{1t} = \alpha_1 + \beta_1 Y_{3t} + \beta_2 Y_{4t} + \gamma_1 X_{1t} + \gamma_2 X_{2t} + u_{1t}$$

$$Y_{2t} = \alpha_2 + \beta_3 Y_{3t} + \beta_4 Y_{4t} + \gamma_3 X_{1t} + \gamma_4 X_{2t} + u_{2t}$$

$$Y_{3t} = \alpha_3 + \beta_5 Y_{1t} + \beta_6 Y_{2t} + u_{3t}$$

$$Y_{4t} = \alpha_4 + \beta_7 Y_{1t} + \beta_8 Y_{2t} + u_{4t}$$

\**Journal of Marketing Research*, November 1968, pp. 362–368.

†“A Simultaneous Equation Regression Study of Advertising and Sales of Cigarettes,” *Journal of Marketing Research*, vol. 6, August 1969, pp. 291–300.

where  $Y_1$  = logarithm of sales of filter cigarettes (number of cigarettes) divided by population over age 20

$Y_2$  = logarithm of sales of nonfilter cigarettes (number of cigarettes) divided by population over age 20

$Y_3$  = logarithm of advertising dollars for filter cigarettes divided by population over age 20 divided by advertising price index

$Y_4$  = logarithm of advertising dollars for nonfilter cigarettes divided by population over age 20 divided by advertising price index

$X_1$  = logarithm of disposable personal income divided by population over age 20 divided by consumer price index

$X_2$  = logarithm of price per package of nonfilter cigarettes divided by consumer price index

- a. In the preceding model the  $Y$ 's are endogenous and the  $X$ 's are exogenous. Why does the author assume  $X_2$  to be exogenous?
- b. If  $X_2$  is treated as an endogenous variable, how would you modify the preceding model?
- 18.8.** G. Menges developed the following econometric model for the West German economy\*:

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 I_t + u_{1t}$$

$$I_t = \beta_3 + \beta_4 Y_t + \beta_5 Q_t + u_{2t}$$

$$C_t = \beta_6 + \beta_7 Y_t + \beta_8 C_{t-1} + \beta_9 P_t + u_{3t}$$

$$Q_t = \beta_{10} + \beta_{11} Q_{t-1} + \beta_{12} R_t + u_{4t}$$

where  $Y$  = national income

$I$  = net capital formation

$C$  = personal consumption

$Q$  = profits

$P$  = cost of living index

$R$  = industrial productivity

$t$  = time

$u$  = stochastic disturbances

- a. Which of the variables would you regard as endogenous and which as exogenous?
- b. Is there any equation in the system that can be estimated by the single-equation least-squares method?
- c. What is the reason behind including the variable  $P$  in the consumption function?

\*G. Menges, "Ein Ökonometrisches Modell der Bundesrepublik Deutschland (Vier Strukturgleichungen)," I.F.O. Studien, vol. 5, 1959, pp. 1–22.

**18.9.** L. E. Gallaway and P. E. Smith developed a simple model for the United States economy, which is as follows\*:

$$Y_t = C_t + I_t + G_t$$

$$C_t = \beta_1 + \beta_2 YD_{t-1} + \beta_3 M_t + u_{1t}$$

$$I_t = \beta_4 + \beta_5(Y_{t-1} - Y_{t-2}) + \beta_6 Z_{t-1} + u_{2t}$$

$$G_t = \beta_7 + \beta_8 G_{t-1} + u_{3t}$$

where

$Y$  = gross national product

$C$  = personal consumption expenditure

$I$  = gross private domestic investment

$G$  = government expenditure plus net foreign investment

$YD$  = disposable, or after-tax, income

$M$  = money supply at the beginning of the quarter

$Z$  = property income before taxes

$t$  = time

$u_1, u_2,$  and  $u_3$  = stochastic disturbances

All variables are measured in the first difference form.

From the quarterly data from 1948–1957, the authors applied the least-squares method to each equation individually and obtained the following results:

$$\hat{C}_t = 0.09 + 0.43YD_{t-1} + 0.23M_t \quad R^2 = 0.23$$

$$\hat{I}_t = 0.08 + 0.43(Y_{t-1} - Y_{t-2}) + 0.48Z_t \quad R^2 = 0.40$$

$$\hat{G}_t = 0.13 + 0.67G_{t-1} \quad R^2 = 0.42$$

- a. How would you justify the use of the single-equation least-squares method in this case?
- b. Why are the  $R^2$  values rather low?

## Problems

- 18.10.** Table 18.2 gives you data on  $Y$  (gross domestic product),  $I$  (gross private domestic investment), and  $C$  (personal consumption expenditure) for the United States for the period 1970–1999. All data are in 1996 billions of dollars. Assume that  $C$  is linearly related to  $Y$  as in the simple Keynesian model of income determination of Example 18.2. Obtain OLS estimates of the parameters of the consumption function. Save the results for another look at the same data using the methods developed in Chapter 20.

\*"A Quarterly Econometric Model of the United States," *Journal of American Statistical Association*, vol. 56, 1961, pp. 379–383.

**TABLE 18.2** PERSONAL CONSUMPTION EXPENDITURE, GROSS PRIVATE DOMESTIC INVESTMENT, AND GDP, UNITED STATES, 1970–1999 (Billions of 1996 Dollars)

Observation	<i>C</i>	<i>I</i>	<i>Y</i>	Observation	<i>C</i>	<i>I</i>	<i>Y</i>
1970	2317.5	436.2	3578.0	1985	3820.9	863.4	5717.1
1971	2405.2	485.8	3697.7	1986	3981.2	857.7	5912.4
1972	2550.5	543.0	3998.4	1987	4113.4	879.3	6113.3
1973	2675.9	606.5	4123.4	1988	4279.5	902.8	6368.4
1974	2653.7	561.7	4099.0	1989	4393.7	936.5	6591.9
1975	2710.9	462.2	4084.4	1990	4474.5	907.3	6707.9
1976	2868.9	555.5	4311.7	1991	4466.6	829.5	6676.4
1977	2992.1	639.4	4511.8	1992	4594.5	899.8	6880.0
1978	3124.7	713.0	4760.6	1993	4748.9	977.9	7062.6
1979	3203.2	735.4	4912.1	1994	4928.1	1107.0	7347.7
1980	3193.0	655.3	4900.9	1995	5075.6	1140.6	7543.8
1981	3236.0	715.6	5021.0	1996	5237.5	1242.7	7813.2
1982	3275.5	615.2	4913.3	1997	5423.9	1393.3	8159.5
1983	3454.3	673.7	5132.3	1998	5678.7	1566.8	8515.7
1984	3640.6	871.5	5505.2	1999	5978.8	1669.7	8875.8

Notes: *C* = personal consumption expenditure.

*I* = gross private domestic investment.

*Y* = gross domestic product.

Source: *Economic Report of the President*, 2001, Table B-2, p. 276.

**18.11.** Using the data given in exercise 18.10, regress gross domestic investment *I* on GDP and save the results for further examination in a later chapter.

**18.12.** Consider the macroeconomics identity

$$C + I = Y \quad (= \text{GDP})$$

As before, assume that

$$C_t = \beta_0 + \beta_1 Y_t + u_t$$

and, following the **accelerator model** of macroeconomics, let

$$I_t = \alpha_0 + \alpha_1(Y_t - Y_{t-1}) + v_t$$

where *u* and *v* are error terms. From the data given in exercise 18.10, estimate the accelerator model and save the results for further study.

# 19

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## THE IDENTIFICATION PROBLEM

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In this chapter we consider the nature and significance of the identification problem. The crux of the identification problem is as follows: Recall the demand-and-supply model introduced in Section 18.2. Suppose that we have time series data on  $Q$  and  $P$  only and no additional information (such as income of the consumer, price prevailing in the previous period, and weather condition). The identification problem then consists in seeking an answer to this question: Given only the data on  $P$  and  $Q$ , how do we know whether we are estimating the demand function or the supply function? Alternatively, if we *think* we are fitting a demand function, how do we guarantee that it is, in fact, the demand function that we are estimating and not something else?

A moment's reflection will reveal that an answer to the preceding question is necessary before one proceeds to estimate the parameters of our demand function. In this chapter we shall show how the identification problem is resolved. We first introduce a few notations and definitions and then illustrate the identification problem with several examples. This is followed by the rules that may be used to find out whether an equation in a simultaneous-equation model is identified, that is, whether it is the relationship that we are actually estimating, be it the demand or supply function or something else.

### 19.1 NOTATIONS AND DEFINITIONS

To facilitate our discussion, we introduce the following notations and definitions.

The general  $M$  equations model in  $M$  endogenous, or jointly dependent, variables may be written as Eq. (19.1.1):

$$\begin{aligned}
 Y_{1t} = & \beta_{12}Y_{2t} + \beta_{13}Y_{3t} + \cdots + \beta_{1M}Y_{Mt} \\
 & + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + \cdots + \gamma_{1K}X_{Kt} + u_{1t} \\
 Y_{2t} = & \beta_{21}Y_{1t} + \beta_{23}Y_{3t} + \cdots + \beta_{2M}Y_{Mt} \\
 & + \gamma_{21}X_{1t} + \gamma_{22}X_{2t} + \cdots + \gamma_{2K}X_{Kt} + u_{2t} \\
 Y_{3t} = & \beta_{31}Y_{1t} + \beta_{32}Y_{2t} + \cdots + \beta_{3M}Y_{Mt} \\
 & + \gamma_{31}X_{1t} + \gamma_{32}X_{2t} + \cdots + \gamma_{3K}X_{Kt} + u_{3t} \\
 & \dots\dots\dots \\
 Y_{Mt} = & \beta_{M1}Y_{1t} + \beta_{M2}Y_{2t} + \cdots + \beta_{M,M-1}Y_{M-1,t} \\
 & + \gamma_{M1}X_{1t} + \gamma_{M2}X_{2t} + \cdots + \gamma_{MK}X_{Kt} + u_{Mt}
 \end{aligned}
 \tag{19.1.1}$$

where  $Y_1, Y_2, \dots, Y_M = M$  endogenous, or jointly dependent, variables  
 $X_1, X_2, \dots, X_K = K$  predetermined variables (one of these  $X$  variables  
may take a value of unity to allow for the intercept  
term in each equation)

- $u_1, u_2, \dots, u_M = M$  stochastic disturbances
- $t = 1, 2, \dots, T =$  total number of observations
- $\beta$ 's = coefficients of the endogenous variables
- $\gamma$ 's = coefficients of the predetermined variables

In passing, note that not each and every variable need appear in each equation. As a matter of fact, we see in Section 19.2 that this must not be the case if an equation is to be identified.

As Eq. (19.1.1) shows, the variables entering a simultaneous-equation model are of two types: **endogenous**, that is, those (whose values are) determined within the model; and **predetermined**, that is, those (whose values are) determined outside the model. The endogenous variables are regarded as stochastic, whereas the predetermined variables are treated as nonstochastic.

The predetermined variables are divided into two categories: **exogenous**, current as well as lagged, and **lagged endogenous**. Thus,  $X_{1t}$  is a current (present-time) exogenous variable, whereas  $X_{1(t-1)}$  is a lagged exogenous variable, with a lag of one time period.  $Y_{(t-1)}$  is a lagged endogenous variable with a lag of one time period, but since the value of  $Y_{1(t-1)}$  is known at the current time  $t$ , it is regarded as nonstochastic, hence, a predetermined variable.<sup>1</sup> In short, current exogenous, lagged exogenous, and lagged endogenous

<sup>1</sup>It is assumed implicitly here that the stochastic disturbances, the  $u$ 's, are serially uncorrelated. If this is not the case,  $Y_{t-1}$  will be correlated with the current period disturbance term  $u_t$ . Hence, we cannot treat it as predetermined.

variables are deemed predetermined; their values are not determined by the model in the current time period.

It is up to the model builder to specify which variables are endogenous and which are predetermined. Although (noneconomic) variables, such as temperature and rainfall, are clearly exogenous or predetermined, the model builder must exercise great care in classifying economic variables as endogenous or predetermined: He or she must defend the classification on a priori or theoretical grounds. However, later in the chapter we provide a statistical test of exogeneity.

The equations appearing in (19.1.1) are known as the **structural**, or **behavioral**, equations because they may portray the structure (of an economic model) of an economy or the behavior of an economic agent (e.g., consumer or producer). The  $\beta$ 's and  $\gamma$ 's are known as the **structural parameters** or **coefficients**.

From the structural equations one can solve for the  $M$  endogenous variables and derive the **reduced-form equations** and the associated **reduced-form coefficients**. **A reduced-form equation is one that expresses an endogenous variable solely in terms of the predetermined variables and the stochastic disturbances.** To illustrate, consider the Keynesian model of income determination encountered in Chapter 18:

$$\text{Consumption function: } C_t = \beta_0 + \beta_1 Y_t + u_t \quad 0 < \beta_1 < 1 \quad (18.2.3)$$

$$\text{Income identity: } Y_t = C_t + I_t \quad (18.2.4)$$

In this model  $C$  (consumption) and  $Y$  (income) are the endogenous variables and  $I$  (investment expenditure) is treated as an exogenous variable. Both these equations are structural equations, (18.2.4) being an identity. As usual, the MPC  $\beta_1$  is assumed to lie between 0 and 1.

If (18.2.3) is substituted into (18.2.4), we obtain, after simple algebraic manipulation,

$$Y_t = \Pi_0 + \Pi_1 I_t + w_t \quad (19.1.2)$$

where

$$\begin{aligned} \Pi_0 &= \frac{\beta_0}{1 - \beta_1} \\ \Pi_1 &= \frac{1}{1 - \beta_1} \\ w_t &= \frac{u_t}{1 - \beta_1} \end{aligned} \quad (19.1.3)$$

Equation (19.1.2) is a **reduced-form equation**; it expresses the endogenous variable  $Y$  solely as a function of the exogenous (or predetermined) variable  $I$  and the stochastic disturbance term  $u$ .  $\Pi_0$  and  $\Pi_1$  are the associated **reduced-form coefficients**. Notice that these reduced-form coefficients are nonlinear combinations of the structural coefficient(s).

Substituting the value of  $Y$  from (19.1.2) into  $C$  of (18.2.3), we obtain another reduced-form equation:

$$C_t = \Pi_2 + \Pi_3 I_t + w_t \quad (19.1.4)$$

where

$$\begin{aligned} \Pi_2 &= \frac{\beta_0}{1 - \beta_1} & \Pi_3 &= \frac{\beta_1}{1 - \beta_1} \\ w_t &= \frac{u_t}{1 - \beta_1} \end{aligned} \quad (19.1.5)$$

The reduced-form coefficients, such as  $\Pi_1$  and  $\Pi_3$ , are also known as **impact**, or **short-run, multipliers**, because they measure the immediate impact on the endogenous variable of a unit change in the value of the exogenous variable.<sup>2</sup> If in the preceding Keynesian model the investment expenditure is increased by, say, \$1 and if the MPC is assumed to be 0.8, then from (19.1.3) we obtain  $\Pi_1 = 5$ . This result means that increasing the investment by \$1 will immediately (i.e., in the current time period) lead to an increase in income of \$5, that is, a fivefold increase. Similarly, under the assumed conditions, (19.1.5) shows that  $\Pi_3 = 4$ , meaning that \$1 increase in investment expenditure will lead immediately to \$4 increase in consumption expenditure.

In the context of econometric models, equations such as (18.2.4) or  $Q_t^d = Q_t^s$  (quantity demanded equal to quantity supplied) are known as the *equilibrium conditions*. Identity (18.2.4) states that aggregate income  $Y$  must be equal to aggregate consumption (i.e., consumption expenditure plus investment expenditure). When equilibrium is achieved, the endogenous variables assume their equilibrium values.<sup>3</sup>

Notice an interesting feature of the reduced-form equations. Since only the predetermined variables and stochastic disturbances appear on the right sides of these equations, and since the predetermined variables are assumed to be uncorrelated with the disturbance terms, the OLS method can be applied to estimate the coefficients of the reduced-form equations (the  $\Pi$ 's). From the estimated reduced-form coefficients one may estimate the structural coefficients (the  $\beta$ 's), as shown later. This procedure is known as **indirect least squares** (ILS), and the estimated structural coefficients are called ILS estimates.

We shall study the ILS method in greater detail in Chapter 20. In the meantime, note that since the reduced-form coefficients can be estimated by the

<sup>2</sup>In econometric models the exogenous variables play a crucial role. Very often, such variables are under the direct control of the government. Examples are the rate of personal and corporate taxes, subsidies, unemployment compensation, etc.

<sup>3</sup>For details, see Jan Kmenta, *Elements of Econometrics*, 2d ed., Macmillan, New York, 1986, pp. 723–731.



OLS method, and since these coefficients are combinations of the structural coefficients, the possibility exists that the structural coefficients can be “retrieved” from the reduced-form coefficients, and it is in the estimation of the structural parameters that we may be ultimately interested. How does one retrieve the structural coefficients from the reduced-form coefficients? The answer is given in Section 19.2, an answer that brings out the crux of the identification problem.

## 19.2 THE IDENTIFICATION PROBLEM

By the **identification problem** we mean whether numerical estimates of the parameters of a structural equation can be obtained from the estimated reduced-form coefficients. If this can be done, we say that the particular equation is *identified*. If this cannot be done, then we say that the equation under consideration is *unidentified*, or *underidentified*.

An identified equation may be either *exactly* (or fully or just) *identified* or *overidentified*. It is said to be exactly identified if unique numerical values of the structural parameters can be obtained. It is said to be overidentified if more than one numerical value can be obtained for some of the parameters of the structural equations. The circumstances under which each of these cases occurs will be shown in the following discussion.

The identification problem arises because different sets of structural coefficients may be compatible with the same set of data. To put the matter differently, a given reduced-form equation may be compatible with different structural equations or different hypotheses (models), and it may be difficult to tell which particular hypothesis (model) we are investigating. In the remainder of this section we consider several examples to show the nature of the identification problem.

### Underidentification

Consider once again the demand-and-supply model (18.2.1) and (18.2.2), together with the market-clearing, or equilibrium, condition that demand is equal to supply. By the equilibrium condition, we obtain

$$\alpha_0 + \alpha_1 P_t + u_{1t} = \beta_0 + \beta_1 P_t + u_{2t} \quad (19.2.1)$$

Solving (19.2.1), we obtain the equilibrium price

$$P_t = \Pi_0 + v_t \quad (19.2.2)$$

where

$$\Pi_0 = \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \quad (19.2.3)$$

$$v_t = \frac{u_{2t} - u_{1t}}{\alpha_1 - \beta_1} \quad (19.2.4)$$

Substituting  $P_t$  from (19.2.2) into (18.2.1) or (18.2.2), we obtain the following equilibrium quantity:

$$Q_t = \Pi_1 + w_t \quad (19.2.5)$$

where

$$\Pi_1 = \frac{\alpha_1 \beta_0 - \alpha_0 \beta_1}{\alpha_1 - \beta_1} \quad (19.2.6)$$

$$w_t = \frac{\alpha_1 u_{2t} - \beta_1 u_{1t}}{\alpha_1 - \beta_1} \quad (19.2.7)$$

Incidentally, note that the error terms  $v_t$  and  $w_t$  are linear combinations of the original error terms  $u_1$  and  $u_2$ .

Equations (19.2.2) and (19.2.5) are reduced-form equations. Now our demand-and-supply model contains four structural coefficients  $\alpha_0$ ,  $\alpha_1$ ,  $\beta_0$ , and  $\beta_1$ , but there is no unique way of estimating them. Why? The answer lies in the two reduced-form coefficients given in (19.2.3) and (19.2.6). These reduced-form coefficients contain all four structural parameters, but there is no way in which the four structural unknowns can be estimated from only two reduced-form coefficients. Recall from high school algebra that to estimate four unknowns we must have four (independent) equations, and, in general, to estimate  $k$  unknowns we must have  $k$  (independent) equations. Incidentally, if we run the reduced-form regression (19.2.2) and (19.2.5), we will see that there are no explanatory variables, only the *constants*, and these *constants* will simply give the mean values of  $P$  and  $Q$  (why?).

What all this means is that, given time series data on  $P$  (price) and  $Q$  (quantity) and no other information, there is no way the researcher can guarantee whether he or she is estimating the demand function or the supply function. That is, a given  $P_t$  and  $Q_t$  represent simply the point of intersection of the appropriate demand-and-supply curves because of the equilibrium condition that demand is equal to supply. To see this clearly, consider the scattergram shown in Figure 19.1.

Figure 19.1a gives a few scatterpoints relating  $Q$  to  $P$ . Each scatterpoint represents the intersection of a demand and a supply curve, as shown in Figure 19.1b. Now consider a single point, such as that shown in Figure 19.1c. There is no way we can be sure which demand-and-supply curve of a whole family of curves shown in that panel generated that point. Clearly, some additional information about the nature of the demand-and-supply curves is needed. For example, if the demand curve shifts over time because of change in income, tastes, etc., but the supply curve remains relatively stable, as in Figure 19.1d, the scatterpoints trace out a supply curve. In this situation, we say that the supply curve is identified. By the same token, if the supply curve shifts over time because of changes in weather conditions (in the case of agricultural commodities) or other extraneous factors but the demand curve remains relatively stable, as in Figure 19.1e the scatterpoints trace out a demand curve. In this case, we say that the demand curve is identified.

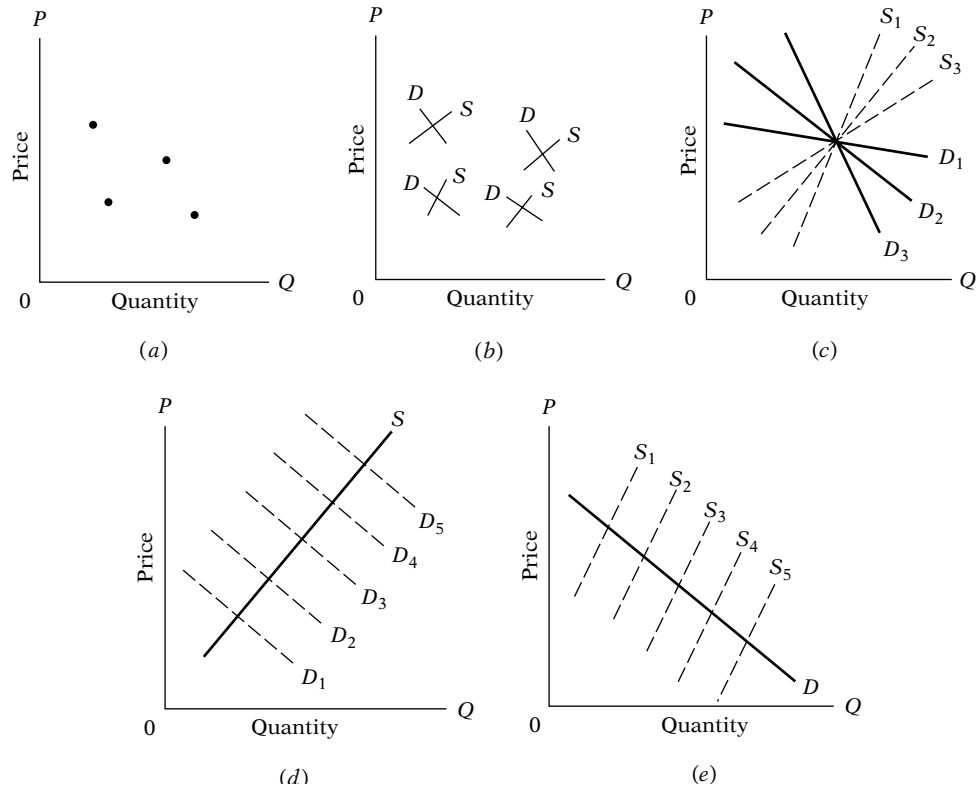


FIGURE 19.1 Hypothetical supply-and-demand functions and the identification problem.

There is an alternative and perhaps more illuminating way of looking at the identification problem. Suppose we multiply (18.2.1) by  $\lambda$  ( $0 \leq \lambda \leq 1$ ) and (18.2.2) by  $1 - \lambda$  to obtain the following equations (*note*: we drop the superscripts on  $Q$ ):

$$\lambda Q_t = \lambda \alpha_0 + \lambda \alpha_1 P_t + \lambda u_{1t} \quad (19.2.8)$$

$$(1 - \lambda) Q_t = (1 - \lambda) \beta_0 + (1 - \lambda) \beta_1 P_t + (1 - \lambda) u_{2t} \quad (19.2.9)$$

Adding these two equations gives the following *linear combination* of the original demand-and-supply equations:

$$Q_t = \gamma_0 + \gamma_1 P_t + w_t \quad (19.2.10)$$

where

$$\begin{aligned} \gamma_0 &= \lambda \alpha_0 + (1 - \lambda) \beta_0 \\ \gamma_1 &= \lambda \alpha_1 + (1 - \lambda) \beta_1 \\ w_t &= \lambda u_{1t} + (1 - \lambda) u_{2t} \end{aligned} \quad (19.2.11)$$

The “bogus,” or “mongrel,” equation (19.2.10) is *observationally indistinguishable* from either (18.2.1) or (18.2.2) because they involve the regression of  $Q$  and  $P$ . Therefore, if we have time series data on  $P$  and  $Q$  only, any of (18.2.1), (18.2.2), or (19.2.10) may be compatible with the same data. In other words, the same data may be compatible with the “hypothesis” (18.2.1), (18.2.2), or (19.2.10), and there is no way we can tell which one of these hypotheses we are testing.

For an equation to be identified, that is, for its parameters to be estimated, it must be shown that the given set of data will not produce a structural equation that looks similar in appearance to the one in which we are interested. If we set out to estimate the demand function, we must show that the given data are not consistent with the supply function or some mongrel equation.

### Just, or Exact, Identification

The reason we could not identify the preceding demand function or the supply function was that the same variables  $P$  and  $Q$  are present in both functions and there is no additional information, such as that indicated in Figure 19.1*d* or *e*. But suppose we consider the following demand-and-supply model:

$$\text{Demand function: } Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} \quad \alpha_1 < 0, \alpha_2 > 0 \quad (19.2.12)$$

$$\text{Supply function: } Q_t = \beta_0 + \beta_1 P_t + u_{2t} \quad \beta_1 > 0 \quad (19.2.13)$$

where  $I$  = income of the consumer, an exogenous variable, and all other variables are as defined previously.

Notice that the only difference between the preceding model and our original demand-and-supply model is that there is an additional variable in the demand function, namely, income. From economic theory of demand we know that income is usually an important determinant of demand for most goods and services. Therefore, its inclusion in the demand function will give us some additional information about consumer behavior. For most commodities income is expected to have a positive effect on consumption ( $\alpha_2 > 0$ ).

Using the market-clearing mechanism, quantity demanded = quantity supplied, we have

$$\alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} = \beta_0 + \beta_1 P_t + u_{2t} \quad (19.2.14)$$

Solving Eq. (19.2.14) provides the following equilibrium value of  $P_t$ :

$$P_t = \Pi_0 + \Pi_1 I_t + v_t \quad (19.2.15)$$

where the reduced-form coefficients are

$$\Pi_0 = \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \quad (19.2.16)$$

$$\Pi_1 = -\frac{\alpha_2}{\alpha_1 - \beta_1}$$

and

$$v_t = \frac{u_{2t} - u_{1t}}{\alpha_1 - \beta_1}$$

Substituting the equilibrium value of  $P_t$  into the preceding demand or supply function, we obtain the following equilibrium quantity:

$$Q_t = \Pi_2 + \Pi_3 I_t + w_t \quad (19.2.17)$$

where

$$\Pi_2 = \frac{\alpha_1 \beta_0 - \alpha_0 \beta_1}{\alpha_1 - \beta_1} \quad (19.2.18)$$

$$\Pi_3 = -\frac{\alpha_2 \beta_1}{\alpha_1 - \beta_1}$$

and

$$w_t = \frac{\alpha_1 u_{2t} - \beta_1 u_{1t}}{\alpha_1 - \beta_1}$$

Since (19.2.15) and (19.2.17) are both reduced-form equations, the OLS method can be applied to estimate their parameters. Now the demand-and-supply model (19.2.12) and (19.2.13) contains five structural coefficients— $\alpha_0, \alpha_1, \alpha_2, \beta_1$ , and  $\beta_2$ . But there are only four equations to estimate them, namely, the four reduced-form coefficients  $\Pi_0, \Pi_1, \Pi_2$ , and  $\Pi_3$  given in (19.2.16) and (19.2.18). Hence, unique solution of all the structural coefficients is not possible. But it can be readily shown that the parameters of the supply function can be identified (estimated) because

$$\begin{aligned} \beta_0 &= \Pi_2 - \beta_1 \Pi_0 \\ \beta_1 &= \frac{\Pi_3}{\Pi_1} \end{aligned} \quad (19.2.19)$$

But there is no unique way of estimating the parameters of the demand function; therefore, it remains underidentified. Incidentally, note that the structural coefficient  $\beta_1$  is a nonlinear function of the reduced-form coefficients, which poses some problems when it comes to estimating the standard error of the estimated  $\beta_1$ , as we shall see in Chapter 20.

To verify that the demand function (19.2.12) cannot be identified (estimated), let us multiply it by  $\lambda$  ( $0 \leq \lambda \leq 1$ ) and (19.2.13) by  $1 - \lambda$  and add them up to obtain the following “mongrel” equation:

$$Q_t = \gamma_0 + \gamma_1 P_t + \gamma_2 I_t + w_t \quad (19.2.20)$$

where

$$\begin{aligned} \gamma_0 &= \lambda\alpha_0 + (1 - \lambda)\beta_0 \\ \gamma_1 &= \lambda\alpha_1 + (1 - \lambda)\beta_1 \\ \gamma_2 &= \lambda\alpha_2 \end{aligned} \quad (19.2.21)$$

and

$$w_t = \lambda u_{1t} + (1 - \lambda)u_{2t}$$

Equation (19.2.20) is observationally indistinguishable from the demand function (19.2.12) although it is distinguishable from the supply function (19.2.13), which does not contain the variable  $I$  as an explanatory variable. Hence, the demand function remains unidentified.

**Notice an interesting fact: It is the presence of an additional variable in the demand function that enables us to identify the supply function!** Why? The inclusion of the income variable in the demand equation provides us some additional information about the variability of the function, as indicated in Figure 19.1*d*. The figure shows how the intersection of the stable supply curve with the shifting demand curve (on account of changes in income) enables us to trace (identify) the supply curve. As will be shown shortly, very often the identifiability of an equation depends on whether it excludes one or more variables that are included in other equations in the model.

But suppose we consider the following demand-and-supply model:

$$\text{Demand function: } Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} \quad \alpha_1 < 0, \alpha_2 > 0 \quad (19.2.12)$$

$$\text{Supply function: } Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \quad \beta_1 > 0, \beta_2 > 0 \quad (19.2.22)$$

where the demand function remains as before but the supply function includes an additional explanatory variable, price lagged one period. The supply function postulates that the quantity of a commodity supplied depends on its current and previous period's price, a model often used to explain the supply of many agricultural commodities. Note that  $P_{t-1}$  is a predetermined variable because its value is known at time  $t$ .

By the market-clearing mechanism we have

$$\alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \quad (19.2.23)$$

Solving this equation, we obtain the following equilibrium price:

$$P_t = \Pi_0 + \Pi_1 I_t + \Pi_2 P_{t-1} + v_t \quad (19.2.24)$$

where

$$\begin{aligned} \Pi_0 &= \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \\ \Pi_1 &= -\frac{\alpha_2}{\alpha_1 - \beta_1} \\ \Pi_2 &= \frac{\beta_2}{\alpha_1 - \beta_1} \\ v_t &= \frac{u_{2t} - u_{1t}}{\alpha_1 - \beta_1} \end{aligned} \quad (19.2.25)$$

Substituting the equilibrium price into the demand or supply equation, we obtain the corresponding equilibrium quantity:

$$Q_t = \Pi_3 + \Pi_4 I_t + \Pi_5 P_{t-1} + w_t \quad (19.2.26)$$

where the reduced-form coefficients are

$$\begin{aligned} \Pi_3 &= \frac{\alpha_1 \beta_0 - \alpha_0 \beta_1}{\alpha_1 - \beta_1} \\ \Pi_4 &= -\frac{\alpha_2 \beta_1}{\alpha_1 - \beta_1} \\ \Pi_5 &= \frac{\alpha_1 \beta_2}{\alpha_1 - \beta_1} \end{aligned} \quad (19.2.27)$$

and

$$w_t = \frac{\alpha_1 u_{2t} - \beta_1 u_{1t}}{\alpha_1 - \beta_1}$$

The demand-and-supply model given in Eqs. (19.2.12) and (19.2.22) contain six structural coefficients— $\alpha_0$ ,  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$ —and there are six reduced-form coefficients— $\Pi_0$ ,  $\Pi_1$ ,  $\Pi_2$ ,  $\Pi_3$ ,  $\Pi_4$ , and  $\Pi_5$ —to estimate them. Thus, we have six equations in six unknowns, and normally we should be able to obtain unique estimates. Therefore, the parameters of both the demand-and-supply equations can be identified, and the system as a whole can be identified. (In exercise 19.2 the reader is asked to express the six structural coefficients in terms of the six reduced-form coefficients given previously to show that unique estimation of the model is possible.)

To check that the preceding demand-and-supply functions are identified, we can also resort to the device of multiplying the demand equation (19.2.12)

by  $\lambda$  ( $0 \leq \lambda \leq 1$ ) and the supply equation (19.2.22) by  $1 - \lambda$  and add them to obtain a mongrel equation. This mongrel equation will contain both the pre-determined variables  $I_t$  and  $P_{t-1}$ ; hence, it will be observationally different from the demand as well as the supply equation because the former does not contain  $P_{t-1}$  and the latter does not contain  $I_t$ .

### Overidentification

For certain goods and services, income as well as wealth of the consumer is an important determinant of demand. Therefore, let us modify the demand function (19.2.12) as follows, keeping the supply function as before:

$$\text{Demand function: } Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{1t} \quad (19.2.28)$$

$$\text{Supply function: } Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \quad (19.2.22)$$

where in addition to the variables already defined,  $R$  represents wealth; for most goods and services, wealth, like income, is expected to have a positive effect on consumption.

Equating demand to supply, we obtain the following equilibrium price and quantity:

$$P_t = \Pi_0 + \Pi_1 I_t + \Pi_2 R_t + \Pi_3 P_{t-1} + v_t \quad (19.2.29)$$

$$Q_t = \Pi_4 + \Pi_5 I_t + \Pi_6 R_t + \Pi_7 P_{t-1} + w_t \quad (19.2.30)$$

where

$$\begin{aligned} \Pi_0 &= \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} & \Pi_1 &= -\frac{\alpha_2}{\alpha_1 - \beta_1} \\ \Pi_2 &= -\frac{\alpha_3}{\alpha_1 - \beta_1} & \Pi_3 &= \frac{\beta_2}{\alpha_1 - \beta_1} \\ \Pi_4 &= \frac{\alpha_1 \beta_0 - \alpha_0 \beta_1}{\alpha_1 - \beta_1} & \Pi_5 &= -\frac{\alpha_2 \beta_1}{\alpha_1 - \beta_1} \\ \Pi_6 &= -\frac{\alpha_3 \beta_1}{\alpha_1 - \beta_1} & \Pi_7 &= \frac{\alpha_1 \beta_2}{\alpha_1 - \beta_1} \\ w_t &= \frac{\alpha_1 u_{2t} - \beta_1 u_{1t}}{\alpha_1 - \beta_1} & v_t &= \frac{u_{2t} - u_{1t}}{\alpha_1 - \beta_1} \end{aligned} \quad (19.2.31)$$

The preceding demand-and-supply model contains seven structural coefficients, but there are eight equations to estimate them—the eight reduced-form coefficients given in (19.2.31); that is, the number of equations is greater than the number of unknowns. As a result, unique estimation of all



the parameters of our model is not possible, which can be shown easily. From the preceding reduced-form coefficients, we can obtain

$$\beta_1 = \frac{\Pi_6}{\Pi_2} \quad (19.2.32)$$

or

$$\beta_1 = \frac{\Pi_5}{\Pi_1} \quad (19.2.33)$$

that is, there are two estimates of the price coefficient in the supply function, and there is no guarantee that these two values or solutions will be identical.<sup>4</sup> Moreover, since  $\beta_1$  appears in the denominators of all the reduced-form coefficients, the ambiguity in the estimation of  $\beta_1$  will be transmitted to other estimates too.

Why was the supply function identified in the system (19.2.12) and (19.2.22) but not in the system (19.2.28) and (19.2.22), although in both cases the supply function remains the same? The answer is that we have “too much,” or an **oversufficiency of information**, to identify the supply curve. This situation is the opposite of the case of underidentification, where there is too little information. The oversufficiency of the information results from the fact that in the model (19.2.12) and (19.2.22) the exclusion of the income variable from the supply function was enough to identify it, but in the model (19.2.28) and (19.2.22) the supply function excludes not only the income variable but also the wealth variable. In other words, in the latter model we put “too many” restrictions on the supply function by requiring it to exclude more variables than necessary to identify it. However, this situation does not imply that overidentification is necessarily bad because we shall see in Chapter 20 how we can handle the problem of too much information, or too many restrictions.

We have now exhausted all the cases. As the preceding discussion shows, an equation in a simultaneous-equation model may be underidentified or identified (either over- or just). The model as a whole is identified if each equation in it is identified. To secure identification, we resort to the reduced-form equations. But in Section 19.3, we consider an alternative and perhaps less time-consuming method of determining whether or not an equation in a simultaneous-equation model is identified.

### 19.3 RULES FOR IDENTIFICATION

As the examples in Section 19.2 show, in principle it is possible to resort to the reduced-form equations to determine the identification of an equation

<sup>4</sup>Notice the difference between under- and overidentification. In the former case, it is impossible to obtain estimates of the structural parameters, whereas in the latter case, there may be several estimates of one or more structural coefficients.

in a system of simultaneous equations. But these examples also show how time-consuming and laborious the process can be. Fortunately, it is not essential to use this procedure. The so-called **order and rank conditions of identification** lighten the task by providing a systematic routine.

To understand the order and rank conditions, we introduce the following notations:

$M$  = number of endogenous variables in the model

$m$  = number of endogenous variables in a given equation

$K$  = number of predetermined variables in the model including the intercept

$k$  = number of predetermined variables in a given equation

### The Order Condition of Identifiability<sup>5</sup>

A necessary (but not sufficient) condition of identification, known as the **order condition**, may be stated in two different but equivalent ways as follows (the necessary as well as sufficient condition of identification will be presented shortly):

#### DEFINITION 19.1

In a model of  $M$  simultaneous equations in order for an equation to be identified, it must exclude *at least*  $M - 1$  variables (endogenous as well as predetermined) appearing in the model. If it excludes exactly  $M - 1$  variables, the equation is just identified. If it excludes more than  $M - 1$  variables, it is overidentified.

#### DEFINITION 19.2

In a model of  $M$  simultaneous equations, in order for an equation to be identified, the number of predetermined variables excluded from the equation must not be less than the number of endogenous variables included in that equation less 1, that is,

$$K - k \geq m - 1 \quad (19.3.1)$$

If  $K - k = m - 1$ , the equation is just identified, but if  $K - k > m - 1$ , it is overidentified.

In exercise 19.1 the reader is asked to prove that the preceding two definitions of identification are equivalent.

To illustrate the order condition, let us revert to our previous examples.

<sup>5</sup>The term **order** refers to the order of a matrix, that is, the number of rows and columns present in a matrix. See **App. B**.

**EXAMPLE 19.1**

$$\text{Demand function: } Q_t = \alpha_0 + \alpha_1 P_t + u_{1t} \quad (18.2.1)$$

$$\text{Supply function: } Q_t = \beta_0 + \beta_1 P_t + u_{2t} \quad (18.2.2)$$

This model has two endogenous variables  $P$  and  $Q$  and no predetermined variables. To be identified, each of these equations must exclude at least  $M - 1 = 1$  variable. Since this is not the case, neither equation is identified.

**EXAMPLE 19.2**

$$\text{Demand function: } Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} \quad (19.2.12)$$

$$\text{Supply function: } Q_t = \beta_0 + \beta_1 P_t + u_{2t} \quad (19.2.13)$$

In this model  $Q$  and  $P$  are endogenous and  $I$  is exogenous. Applying the order condition given in (19.3.1), we see that the demand function is unidentified. On the other hand, the supply function is just identified because it excludes exactly  $M - 1 = 1$  variable  $I_t$ .

**EXAMPLE 19.3**

$$\text{Demand function: } Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} \quad (19.2.12)$$

$$\text{Supply function: } Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \quad (19.2.22)$$

Given that  $P_t$  and  $Q_t$  are endogenous and  $I_t$  and  $P_{t-1}$  are predetermined, Eq. (19.2.12) excludes exactly one variable  $P_{t-1}$  and Eq. (19.2.22) also excludes exactly one variable  $I_t$ . Hence each equation is identified by the order condition. Therefore, the model as a whole is identified.

**EXAMPLE 19.4**

$$\text{Demand function: } Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{1t} \quad (19.2.28)$$

$$\text{Supply function: } Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \quad (19.2.22)$$

In this model  $P_t$  and  $Q_t$  are endogenous and  $I_t$ ,  $R_t$ , and  $P_{t-1}$  are predetermined. The demand function excludes exactly one variable  $P_{t-1}$ , and hence by the order condition it is exactly identified. But the supply function excludes two variables  $I_t$  and  $R_t$ , and hence it is overidentified. As noted before, in this case there are two ways of estimating  $\beta_1$ , the coefficient of the price variable.

Notice a slight complication here. By the order condition the demand function is identified. But if we try to estimate the parameters of this equation from the reduced-form coefficients given in (19.2.31), the estimates will not be unique because  $\beta_1$ , which enters into the computations, takes two values and we shall have to decide which of these values is appropriate. But this complication can be obviated because it is shown in Chapter 20 that in cases of overidentification the method of indirect least squares is not appropriate and should be discarded in favor of other methods. One such method is **two-stage least squares**, which we shall discuss fully in Chapter 20.

As the previous examples show, identification of an equation in a model of simultaneous equations is possible if that equation excludes one or more variables that are present elsewhere in the model. This situation is known as the *exclusion (of variables) criterion*, or *zero restrictions criterion* (the coefficients of variables not appearing in an equation are assumed to have zero values). This criterion is by far the most commonly used method of securing or determining identification of an equation. But notice that the zero restrictions criterion is based on a priori or theoretical expectations that certain variables do not appear in a given equation. It is up to the researcher to spell out clearly why he or she does expect certain variables to appear in some equations and not in others.

### The Rank Condition of Identifiability<sup>6</sup>

The order condition discussed previously is *a necessary but not sufficient* condition for identification; that is, even if it is satisfied, it may happen that an equation is not identified. Thus, in Example 19.2, the supply equation was identified by the order condition because it excluded the income variable  $I_t$ , which appeared in the demand function. But identification is accomplished only if  $\alpha_2$ , the coefficient of  $I_t$  in the demand function, is not zero, that is, if the income variable not only probably but actually does enter the demand function.

More generally, even if the order condition  $K - k \geq m - 1$  is satisfied by an equation, it may be unidentified because the predetermined variables excluded from this equation but present in the model may not all be independent so that there may not be one-to-one correspondence between the structural coefficients (the  $\beta$ 's) and the reduced-form coefficients (the  $\Pi$ 's). That is, we may not be able to estimate the structural parameters from the reduced-form coefficients, as we shall show shortly. Therefore, we need both a necessary and sufficient condition for identification. This is provided by the *rank condition* of identification, which may be stated as follows:

#### RANK CONDITION OF IDENTIFICATION

In a model containing  $M$  equations in  $M$  endogenous variables, an equation is identified if and only if *at least* one nonzero determinant of order  $(M - 1)(M - 1)$  can be constructed from the coefficients of the variables (both endogenous and predetermined) excluded from that particular equation but included in the other equations of the model.

<sup>6</sup>The term **rank** refers to the rank of a matrix and is given by the largest-order square matrix (contained in the given matrix) whose determinant is nonzero. Alternatively, the rank of a matrix is the largest number of linearly independent rows or columns of that matrix. See **App. B**.

As an illustration of the rank condition of identification, consider the following hypothetical system of simultaneous equations in which the  $Y$  variables are endogenous and the  $X$  variables are predetermined.<sup>7</sup>

$$Y_{1t} - \beta_{10} - \beta_{12}Y_{2t} - \beta_{13}Y_{3t} - \gamma_{11}X_{1t} = u_{1t} \quad (19.3.2)$$

$$Y_{2t} - \beta_{20} - \beta_{23}Y_{3t} - \gamma_{21}X_{1t} - \gamma_{22}X_{2t} = u_{2t} \quad (19.3.3)$$

$$Y_{3t} - \beta_{30} - \beta_{31}Y_{1t} - \gamma_{31}X_{1t} - \gamma_{32}X_{2t} = u_{3t} \quad (19.3.4)$$

$$Y_{4t} - \beta_{40} - \beta_{41}Y_{1t} - \beta_{42}Y_{2t} - \gamma_{43}X_{3t} = u_{4t} \quad (19.3.5)$$

To facilitate identification, let us write the preceding system in Table 19.1, which is self-explanatory.

Let us first apply the order condition of identification, as shown in Table 19.2. By the order condition each equation is identified. Let us recheck with the rank condition. Consider the first equation, which excludes variables  $Y_4$ ,  $X_2$ , and  $X_3$  (this is represented by zeros in the first row of Table 19.1). For this equation to be identified, we must obtain at least one nonzero determinant of order  $3 \times 3$  from the coefficients of the variables excluded from this equation but included in other equations. To obtain the determinant we first obtain the relevant matrix of coefficients of variables  $Y_4$ ,  $X_2$ , and  $X_3$  included in the other equations. In the present case there is

**TABLE 19.1**

Equation no.	Coefficients of the variables							
	1	$Y_1$	$Y_2$	$Y_3$	$Y_4$	$X_1$	$X_2$	$X_3$
(19.3.2)	$-\beta_{10}$	1	$-\beta_{12}$	$-\beta_{13}$	0	$-\gamma_{11}$	0	0
(19.3.3)	$-\beta_{20}$	0	1	$-\beta_{23}$	0	$-\gamma_{21}$	$-\gamma_{22}$	0
(19.3.4)	$-\beta_{30}$	$-\beta_{31}$	0	1	0	$-\gamma_{31}$	$-\gamma_{32}$	0
(19.3.5)	$-\beta_{40}$	$-\beta_{41}$	$-\beta_{42}$	0	1	0	0	$-\gamma_{43}$

**TABLE 19.2**

Equation no.	No. of predetermined variables excluded, ( $K - k$ )	No. of endogenous variables included less one, ( $m - 1$ )	Identified?
(19.3.2)	2	2	Exactly
(19.3.3)	1	1	Exactly
(19.3.4)	1	1	Exactly
(19.3.5)	2	2	Exactly

<sup>7</sup>The simultaneous-equation system presented in (19.1.1) may be shown in the following alternative form, which may be convenient for matrix manipulations.

only one such matrix, call it  $\mathbf{A}$ , defined as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & -\gamma_{22} & 0 \\ 0 & -\gamma_{32} & 0 \\ 1 & 0 & -\gamma_{43} \end{bmatrix} \quad (19.3.6)$$

It can be seen that the determinant of this matrix is zero:

$$\det \mathbf{A} = \begin{vmatrix} 0 & -\gamma_{22} & 0 \\ 0 & -\gamma_{32} & 0 \\ 1 & 0 & -\gamma_{43} \end{vmatrix} \quad (19.3.7)$$

Since the determinant is zero, the rank of the matrix (19.3.6), denoted by  $\rho(\mathbf{A})$ , is less than 3. Therefore, Eq. (19.3.2) does not satisfy the rank condition and hence is not identified.

As noted, the rank condition is both a necessary and sufficient condition for identification. Therefore, although the order condition shows that Eq. (19.3.2) is identified, the rank condition shows that it is not. Apparently, the columns or rows of the matrix  $\mathbf{A}$  given in (19.3.6) are not (linearly) independent, meaning that there is some relationship between the variables  $Y_4$ ,  $X_2$ , and  $X_3$ . As a result, we may not have enough information to estimate the parameters of equation (19.3.2); the reduced-form equations for the preceding model will show that it is not possible to obtain the structural coefficients of that equation from the reduced-form coefficients. The reader should verify that by the rank condition Eqs. (19.3.3) and (19.3.4) are also unidentified but Eq. (19.3.5) is identified.

As the preceding discussion shows, *the rank condition tells us whether the equation under consideration is identified or not, whereas the order condition tells us if it is exactly identified or overidentified.*

To apply the rank condition one may proceed as follows:

1. Write down the system in a tabular form, as shown in Table 19.1.
2. Strike out the coefficients of the row in which the equation under consideration appears.
3. Also strike out the columns corresponding to those coefficients in 2 which are nonzero.
4. The entries left in the table will then give only the coefficients of the variables included in the system but not in the equation under consideration. From these entries form all possible matrices, like  $\mathbf{A}$ , of order  $M - 1$  and obtain the corresponding determinants. If at least one nonvanishing or nonzero determinant can be found, the equation in question is (just or over) identified. The rank of the matrix, say,  $\mathbf{A}$ , in this case is exactly equal to  $M - 1$ . If all the possible  $(M - 1)(M - 1)$  determinants are zero, the rank of the matrix  $\mathbf{A}$  is less than  $M - 1$  and the equation under investigation is not identified.

Our discussion of the order and rank conditions of identification leads to the following general principles of identifiability of a structural equation in a system of  $M$  simultaneous equations:

1. If  $K - k > m - 1$  and the rank of the  $\mathbf{A}$  matrix is  $M - 1$ , the equation is overidentified.
2. If  $K - k = m - 1$  and the rank of the matrix  $\mathbf{A}$  is  $M - 1$ , the equation is exactly identified.
3. If  $K - k \geq m - 1$  and the rank of the matrix  $\mathbf{A}$  is less than  $M - 1$ , the equation is underidentified.
4. If  $K - k < m - 1$ , the structural equation is unidentified. The rank of the  $\mathbf{A}$  matrix in this case is bound to be less than  $M - 1$ . (Why?)

Henceforth, when we talk about identification we mean exact identification, or overidentification. There is no point in considering unidentified, or underidentified, equations because no matter how extensive the data, the structural parameters cannot be estimated. However, as shown in Chapter 20, parameters of overidentified as well as just identified equations can be estimated.

Which condition should one use in practice: Order or rank? For large simultaneous-equation models, applying the rank condition is a formidable task. Therefore, as Harvey notes,

Fortunately, the order condition is usually sufficient to ensure identifiability, and although it is important to be aware of the rank condition, a failure to verify it will rarely result in disaster.<sup>8</sup>

#### \*19.4 A TEST OF SIMULTANEITY<sup>9</sup>

If there is no simultaneous equation, or **simultaneity problem**, the OLS estimators produce consistent and efficient estimators. On the other hand, if there is simultaneity, OLS estimators are not even consistent. In the presence of simultaneity, as we will show in Chapter 20, the methods of **two-stage least squares (2SLS)** and **instrumental variables** will give estimators that are consistent and efficient. Oddly, if we apply these alternative methods when there is in fact no simultaneity, these methods yield estimators that are consistent but not efficient (i.e., with smaller variance). All this discussion suggests that we should check for the simultaneity problem before we discard OLS in favor of the alternatives.

As we showed earlier, the simultaneity problem arises because some of the regressors are endogenous and are therefore likely to be correlated with

<sup>8</sup>Andrew Harvey, *The Econometric Analysis of Time Series*, 2d ed., The MIT Press, Cambridge, Mass., 1990, p. 328.

<sup>9</sup>Optional.

<sup>9</sup>The following discussion draws from Robert S. Pindyck and Daniel L. Rubinfeld, *Econometric Models and Economic Forecasts*, 3d ed., McGraw-Hill, New York, 1991, pp. 303–305.

the disturbance, or error, term. Therefore, *a test of simultaneity is essentially a test of whether (an endogenous) regressor is correlated with the error term*. If it is, the simultaneity problem exists, in which case alternatives to OLS must be found; if it is not, we can use OLS. To find out which is the case in a concrete situation, we can use Hausman's specification error test.

### Hausman Specification Test

A version of the Hausman specification error test that can be used for testing the simultaneity problem can be explained as follows<sup>10</sup>:

To fix ideas, consider the following two-equation model:

$$\text{Demand function: } Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{1t} \quad (19.4.1)$$

$$\text{Supply function: } Q_t = \beta_0 + \beta_1 P_t + u_{2t} \quad (19.4.2)$$

where  $P$  = price  
 $Q$  = quantity  
 $I$  = income  
 $R$  = wealth  
 $u$ 's = error terms

Assume that  $I$  and  $R$  are exogenous. Of course,  $P$  and  $Q$  are endogenous.

Now consider the supply function (19.4.2). If there is no simultaneity problem (i.e.,  $P$  and  $Q$  are mutually independent),  $P_t$  and  $u_{2t}$  should be uncorrelated (why?). On the other hand, if there is simultaneity,  $P_t$  and  $u_{2t}$  will be correlated. To find out which is the case, the Hausman test proceeds as follows:

First, from (19.4.1) and (19.4.2) we obtain the following reduced-form equations:

$$P_t = \Pi_0 + \Pi_1 I_t + \Pi_2 R_t + v_t \quad (19.4.3)$$

$$Q_t = \Pi_3 + \Pi_4 I_t + \Pi_3 R_t + w_t \quad (19.4.4)$$

where  $v$  and  $w$  are the reduced-form error terms. Estimating (19.4.3) by OLS we obtain

$$\hat{P}_t = \hat{\Pi}_0 + \hat{\Pi}_1 I_t + \hat{\Pi}_2 R_t \quad (19.4.5)$$

Therefore,

$$P_t = \hat{P}_t + \hat{v}_t \quad (19.4.6)$$

<sup>10</sup>J. A. Hausman, "Specification Tests in Econometrics," *Econometrica*, vol. 46, November 1976, pp. 1251–1271. See also A. Nakamura and M. Nakamura, "On the Relationship among Several Specification Error Tests Presented by Durbin, Wu, and Hausman," *Econometrica*, vol. 49, November 1981, pp. 1583–1588.



where  $\hat{P}_t$  are estimated  $P_t$  and  $\hat{v}_t$  are the estimated residuals. Substituting (19.4.6) into (19.4.2), we get

$$Q_t = \beta_0 + \beta_1 \hat{P}_t + \beta_1 \hat{v}_t + u_{2t} \quad (19.4.7)$$

*Note:* The coefficients of  $P_t$  and  $v_t$  are the same.

Now, under the null hypothesis that there is no simultaneity, the correlation between  $\hat{v}_t$  and  $u_{2t}$  should be zero, asymptotically. Thus, if we run the regression (19.4.7) and find that the coefficient of  $v_t$  in (19.4.7) is statistically zero, we can conclude that there is no simultaneity problem. Of course, this conclusion will be reversed if we find this coefficient to be statistically significant.

Essentially, then, the Hausman test involves the following steps:

**Step 1.** Regress  $P_t$  on  $I_t$  and  $R_t$  to obtain  $\hat{v}_t$ .

**Step 2.** Regress  $Q_t$  on  $\hat{P}_t$  and  $\hat{v}_t$  and perform a  $t$  test on the coefficient of  $\hat{v}_t$ . If it is significant, do not reject the hypothesis of simultaneity; otherwise, reject it.<sup>11</sup> For efficient estimation, however, Pindyck and Rubinfeld suggest regressing  $Q_t$  on  $P_t$  and  $\hat{v}_t$ .<sup>12</sup>

#### EXAMPLE 19.5

##### PINDYCK–RUBINFELD MODEL OF PUBLIC SPENDING<sup>13</sup>

To study the behavior of U.S. state and local government expenditure, the authors developed the following simultaneous equation model:

$$\text{EXP} = \beta_1 + \beta_2 \text{AID} + \beta_3 \text{INC} + \beta_4 \text{POP} + u_i \quad (19.4.8)$$

$$\text{AID} = \delta_1 + \delta_2 \text{EXP} + \delta_3 \text{PS} + v_i \quad (19.4.9)$$

where EXP = state and local government public expenditures  
AID = level of federal grants-in-aid  
INC = income of states  
POP = state population  
PS = population of primary and secondary school children  
 $u$  and  $v$  = error terms

In this model, INC, POP, and PS are regarded exogenous.

Because of the possibility of simultaneity between EXP and AID, the authors first regress AID on INC, POP, and PS (i.e., the reduced-form regression). Let the error term in this regression be  $w_i$ . From this regression the calculated residual is  $\hat{w}_i$ . The authors then regress

(Continued)

<sup>11</sup>If more than one endogenous regressor is involved, we will have to use the  $F$  test.

<sup>12</sup>Pindyck and Rubinfeld, op. cit., p. 304. *Note:* The regressor is  $P_t$  and not  $\hat{P}_t$ .

<sup>13</sup>Pindyck and Rubinfeld, op. cit., pp. 176–177. Notations slightly altered.

**EXAMPLE 19.5** (Continued)

EXP on AID, INC, POP, and  $\hat{w}_i$ , to obtain the following results:

$$\begin{aligned} \widehat{\text{EXP}} &= -89.41 + 4.50\text{AID} + 0.00013\text{INC} - 0.518\text{POP} - 1.39\hat{w}_i \\ t &= (-1.04) \quad (5.89) \quad (3.06) \quad (-4.63) \quad (-1.73) \quad \text{(19.4.10)}^{14} \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad R^2 = 0.99 \end{aligned}$$

At the 5 percent level of significance, the coefficient of  $\hat{w}_i$  is not statistically significant, and therefore, at this level, there is no simultaneity problem. However, at the 10 percent level of significance, it is statistically significant, raising the possibility that the simultaneity problem is present.

Incidentally, the OLS estimation of (19.4.8) is as follows:

$$\begin{aligned} \widehat{\text{EXP}} &= -46.81 + 3.24\text{AID} + 0.00019\text{INC} - 0.597\text{POP} \\ t &= (-0.56) \quad (13.64) \quad (8.12) \quad (-5.71) \quad \text{(19.4.11)} \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad R^2 = 0.993 \end{aligned}$$

Notice an interesting feature of the results given in (19.4.10) and (19.4.11): When simultaneity is explicitly taken into account, the AID variable is less significant although numerically it is greater in magnitude.

**\*19.5 TESTS FOR EXOGENEITY**

We noted earlier that it is the researcher's responsibility to specify which variables are endogenous and which are exogenous. This will depend on the problem at hand and the a priori information the researcher has. But is it possible to develop a statistical test of exogeneity, in the manner of Granger's causality test?

The Hausman test discussed in Section 19.4 can be utilized to answer this question. Suppose we have a three-equation model in three endogenous variables,  $Y_1$ ,  $Y_2$ , and  $Y_3$ , and suppose there are three exogenous variables,  $X_1$ ,  $X_2$ , and  $X_3$ . Further, suppose that the first equation of the model is

$$Y_{1i} = \beta_0 + \beta_2 Y_{2i} + \beta_3 Y_{3i} + \alpha_1 X_{1i} + u_{1i} \quad \text{(19.5.1)}$$

If  $Y_2$  and  $Y_3$  are truly endogenous, we cannot estimate (19.5.1) by OLS (why?). But how do we find that out? We can proceed as follows. We obtain the reduced-form equations for  $Y_2$  and  $Y_3$  (Note: the reduced-form equations will have only predetermined variables on the right-hand side). From these reduced-form equations, we obtain  $\hat{Y}_{2i}$  and  $\hat{Y}_{3i}$ , the predicted values of  $Y_{2i}$  and  $Y_{3i}$ , respectively. Then in the spirit of the Hausman test discussed

<sup>14</sup>As in footnote 12, the authors use AID rather than  $\widehat{\text{AID}}$  as the regressor.  
\*Optional.

earlier, we can estimate the following equation by OLS:

$$Y_{1i} = \beta_0 + \beta_2 Y_{2i} + \beta_3 Y_{3i} + \alpha_1 X_{1i} + \lambda_2 \hat{Y}_{2i} + \lambda_3 \hat{Y}_{3i} + u_{1i} \quad (19.5.2)$$

Using the  $F$  test, we test the hypothesis that  $\lambda_2 = \lambda_3 = 0$ . If this hypothesis is rejected,  $Y_2$  and  $Y_3$  can be deemed endogenous, but if it is not rejected, they can be treated as exogenous. For a concrete example, see exercise 19.16.

## 19.6 SUMMARY AND CONCLUSIONS

1. The problem of identification precedes the problem of estimation.
2. The identification problem asks whether one can obtain unique numerical estimates of the structural coefficients from the estimated reduced-form coefficients.
3. If this can be done, an equation in a system of simultaneous equations is identified. If this cannot be done, that equation is un- or under-identified.
4. An identified equation can be just identified or overidentified. In the former case, unique values of structural coefficients can be obtained; in the latter, there may be more than one value for one or more structural parameters.
5. The identification problem arises because the same set of data may be compatible with different sets of structural coefficients, that is, different models. Thus, in the regression of price on quantity only, it is difficult to tell whether one is estimating the supply function or the demand function, because price and quantity enter both equations.
6. To assess the identifiability of a structural equation, one may apply the technique of **reduced-form equations**, which expresses an endogenous variable solely as a function of predetermined variables.
7. However, this time-consuming procedure can be avoided by resorting to either the **order condition** or the **rank condition** of identification. Although the order condition is easy to apply, it provides only a necessary condition for identification. On the other hand, the rank condition is both a necessary and sufficient condition for identification. If the rank condition is satisfied, the order condition is satisfied, too, although the converse is not true. In practice, though, the order condition is generally adequate to ensure identifiability.
8. In the presence of simultaneity, OLS is generally not applicable, as was shown in Chapter 18. But if one wants to use it nonetheless, it is imperative to test for simultaneity explicitly. The **Hausman specification test** can be used for this purpose.
9. Although in practice deciding whether a variable is endogenous or exogenous is a matter of judgment, one can use the Hausman specification test to determine whether a variable or group of variables is endogenous or exogenous.

**10.** Although they are in the same family, the concepts of causality and exogeneity are different and one may not necessarily imply the other. In practice it is better to keep those concepts separate (see Section 17.14).

## EXERCISES

- 19.1.** Show that the two definitions of the order condition of identification are equivalent.
- 19.2.** Deduce the structural coefficients from the reduced-form coefficients given in (19.2.25) and (19.2.27).
- 19.3.** Obtain the reduced form of the following models and determine in each case whether the structural equations are unidentified, just identified, or overidentified:
- Chap. 18, Example 18.2.
  - Chap. 18, Example 18.3.
  - Chap. 18, Example 18.6.
- 19.4.** Check the identifiability of the models of exercise 19.3 by applying both the order and rank conditions of identification.
- 19.5.** In the model (19.2.22) and (19.2.28) of the text it was shown that the supply equation was overidentified. What restrictions, if any, on the structural parameters will make this equation just identified? Justify the restrictions you impose.
- 19.6.** From the model

$$Y_{1t} = \beta_{10} + \beta_{12}Y_{2t} + \gamma_{11}X_{1t} + u_{1t}$$

$$Y_{2t} = \beta_{20} + \beta_{21}Y_{1t} + \gamma_{22}X_{2t} + u_{2t}$$

the following reduced-form equations are obtained:

$$Y_{1t} = \Pi_{10} + \Pi_{11}X_{1t} + \Pi_{12}X_{2t} + w_t$$

$$Y_{2t} = \Pi_{20} + \Pi_{21}X_{1t} + \Pi_{22}X_{2t} + v_t$$

- Are the structural equations identified?
  - What happens to identification if it is known a priori that  $\gamma_{11} = 0$ ?
- 19.7.** Refer to exercise 19.6. The estimated reduced-form equations are as follows:

$$Y_{1t} = 4 + 3X_{1t} + 8X_{2t}$$

$$Y_{2t} = 2 + 6X_{1t} + 10X_{2t}$$

- Obtain the values of the structural parameters.
  - How would you test the null hypothesis that  $\gamma_{11} = 0$ ?
- 19.8.** The model

$$Y_{1t} = \beta_{10} + \beta_{12}Y_{2t} + \gamma_{11}X_{1t} + u_{1t}$$

$$Y_{2t} = \beta_{20} + \beta_{21}Y_{1t} + u_{2t}$$

produces the following reduced-form equations:

$$Y_{1t} = 4 + 8X_{1t}$$

$$Y_{2t} = 2 + 12X_{1t}$$

- a. Which structural coefficients, if any, can be estimated from the reduced-form coefficients? Demonstrate your contention.
  - b. How does the answer to (a) change if it is known a priori that (1)  $\beta_{12} = 0$  and (2)  $\beta_{10} = 0$ ?
- 19.9.** Determine whether the structural equations of the model given in exercise 18.8 are identified.
- 19.10.** Refer to exercise 18.7 and find out which structural equations can be identified.
- 19.11.** Table 19.3 is a model in five equations with five endogenous variables  $Y$  and four exogenous variables  $X$ :

TABLE 19.3

Equation no.	Coefficients of the variables								
	$Y_1$	$Y_2$	$Y_3$	$Y_4$	$Y_5$	$X_1$	$X_2$	$X_3$	$X_4$
1	1	$\beta_{12}$	0	$\beta_{14}$	0	$\gamma_{11}$	0	0	$\gamma_{14}$
2	0	1	$\beta_{23}$	$\beta_{24}$	0	0	$\gamma_{22}$	$\gamma_{23}$	0
3	$\beta_{31}$	0	1	$\beta_{34}$	$\beta_{35}$	0	0	$\gamma_{33}$	$\gamma_{34}$
4	0	$\beta_{42}$	0	1	0	$\gamma_{41}$	0	$\gamma_{43}$	0
5	$\beta_{51}$	0	0	$\beta_{54}$	1	0	$\gamma_{52}$	$\gamma_{53}$	0

Determine the identifiability of each equation with the aid of the order and rank conditions of identifications.

- 19.12.** Consider the following extended Keynesian model of income determination:

Consumption function:  $C_t = \beta_1 + \beta_2 Y_t - \beta_3 T_t + u_{1t}$

Investment function:  $I_t = \alpha_0 + \alpha_1 Y_{t-1} + u_{2t}$

Taxation function:  $T_t = \gamma_0 + \gamma_1 Y_t + u_{3t}$

Income identity:  $Y_t = C_t + I_t + G_t$

where  $C$  = consumption expenditure  
 $Y$  = income  
 $I$  = investment  
 $T$  = taxes  
 $G$  = government expenditure  
 $u$ 's = the disturbance terms

In the model the endogenous variables are  $C$ ,  $I$ ,  $T$ , and  $Y$  and the predetermined variables are  $G$  and  $Y_{t-1}$ .

By applying the order condition, check the identifiability of each of the equations in the system and of the system as a whole. What would

happen if  $r_t$ , the interest rate, assumed to be exogenous, were to appear on the right-hand side of the investment function?

- 19.13.** Refer to the data given in Table 18.1 of Chapter 18. Using these data, estimate the reduced-form regressions (19.1.2) and (19.1.4). Can you estimate  $\beta_0$  and  $\beta_1$ ? Show your calculations. Is the model identified? Why or why not?
- 19.14.** Suppose we propose yet another definition of the order condition of identifiability:

$$K \geq m + k - 1$$

which states that the number of predetermined variables in the system can be no less than the number of unknown coefficients in the equation to be identified. Show that this definition is equivalent to the two other definitions of the order condition given in the text.

- 19.15.** A simplified version of Suits' model of the watermelon market is as follows\*:

$$\text{Demand equation: } P_t = \alpha_0 + \alpha_1(Q_t/N_t) + \alpha_2(Y_t/N_t) + \alpha_3 F_t + u_{1t}$$

$$\text{Crop supply function: } Q_t = \beta_0 + \beta_1(P_t/W_t) + \beta_2 P_{t-1} + \beta_3 C_{t-1} + \beta_4 T_{t-1} + u_{2t}$$

where  $P$  = price  
 $(Q/N)$  = per capita quantity demanded  
 $(Y/N)$  = per capita income  
 $F_t$  = freight costs  
 $(P/W)$  = price relative to the farm wage rate  
 $C$  = price of cotton  
 $T$  = price of other vegetables  
 $N$  = population

$P$  and  $Q$  are the endogenous variables.

- a.** Obtain the reduced form.  
**b.** Determine whether the demand, the supply, or both functions are identified.
- 19.16.** Consider the following demand-and-supply model for money:

$$\text{Money demand: } M_t^d = \beta_0 + \beta_1 Y_t + \beta_2 R_t + \beta_3 P_t + u_{1t}$$

$$\text{Money supply: } M_t^s = \alpha_0 + \alpha_1 Y_t + u_{2t}$$

where  $M$  = money  
 $Y$  = income  
 $R$  = rate of interest  
 $P$  = price  
 $u$ 's = error terms

\*D. B. Suits, "An Econometric Model of the Watermelon Market," *Journal of Farm Economics*, vol. 37, 1955, pp. 237–251.

**TABLE 19.4** MONEY, GDP, INTEREST RATE, AND CONSUMER PRICE INDEX, UNITED STATES, 1970–1999

Observation	$M_2$	GDP	TBRATE	CPI
1970	626.4000	3578.000	6.458000	38.80000
1971	710.1000	3697.700	4.348000	40.50000
1972	802.1000	3998.400	4.071000	41.80000
1973	855.2000	4123.400	7.041000	44.40000
1974	901.9000	4099.000	7.886000	49.30000
1975	1015.900	4084.400	5.838000	53.80000
1976	1151.700	4311.700	4.989000	56.90000
1977	1269.900	4511.800	5.265000	60.60000
1978	1365.500	4760.600	7.221000	65.20000
1979	1473.100	4912.100	10.04100	72.60000
1980	1599.100	4900.900	11.50600	82.40000
1981	1754.600	5021.000	14.02900	90.90000
1982	1909.500	4913.300	10.68600	96.50000
1983	2126.000	5132.300	8.630000	99.60000
1984	2309.700	5505.200	9.580000	103.9000
1985	2495.400	5717.100	7.480000	107.6000
1986	2732.100	5912.400	5.980000	109.6000
1987	2831.100	6113.300	5.820000	113.6000
1988	2994.300	6368.400	6.690000	118.3000
1989	3158.400	6591.900	8.100000	124.0000
1990	3277.600	6707.900	7.510000	130.7000
1991	3376.800	6676.400	5.420000	136.2000
1992	3430.700	6880.000	3.450000	140.3000
1993	3484.400	7062.600	3.020000	144.5000
1994	3499.000	7347.700	4.290000	148.2000
1995	3641.900	7543.800	5.510000	142.4000
1996	3813.300	7813.200	5.020000	156.9000
1997	4028.900	8159.500	5.070000	160.5000
1998	4380.600	8515.700	4.810000	163.0000
1999	4643.700	8875.800	4.660000	166.6000

Notes:  $M_2 = M_2$  money supply (billions of dollars, seasonally adjusted).

GDP = gross domestic product (billions of dollars, seasonally adjusted).

TBRATE = 3-month treasury bill rate, %.

CPI = Consumer Price Index (1982–1984 = 100).

Source: *Economic Report of the President, 2001*, Tables B-2, B-60, B-73, B-69.

Assume that  $R$  and  $P$  are exogenous and  $M$  and  $Y$  are endogenous. Table 19.4 gives data on  $M$  ( $M_2$  definition),  $Y$  (GDP),  $R$  (3-month Treasury bill rate) and  $P$  (Consumer Price Index), for the United States for 1970–1999.

- Is the demand function identified?
- Is the supply function identified?
- Obtain the expressions for the reduced-form equations for  $M$  and  $Y$ .
- Apply the test of simultaneity to the supply function.
- How would we find out if  $Y$  in the money supply function is in fact endogenous?

# 20

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## SIMULTANEOUS-EQUATION METHODS

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Having discussed the nature of the simultaneous-equation models in the previous two chapters, in this chapter we turn to the problem of estimation of the parameters of such models. At the outset it may be noted that the estimation problem is rather complex because there are a variety of estimation techniques with varying statistical properties. In view of the introductory nature of this text, we shall consider only a few of these techniques. Our discussion will be simple and often heuristic, the finer points being left to the references.

### 20.1 APPROACHES TO ESTIMATION

If we consider the general  $M$  equations model in  $M$  endogenous variables given in (19.1.1), we may adopt two approaches to estimate the structural equations, namely, single-equation methods, also known as **limited information methods**, and system methods, also known as **full information methods**. In the single-equation methods to be considered shortly, we estimate each equation in the system (of simultaneous equations) individually, taking into account any restrictions placed on that equation (such as exclusion of some variables) without worrying about the restrictions on the other equations in the system,<sup>1</sup> hence the name *limited information methods*. In

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<sup>1</sup>For the purpose of identification, however, information provided by other equations will have to be taken into account. But as noted in Chap. 19, estimation is possible only in the case of (fully or over-) identified equations. In this chapter we assume that the identification problem is solved using the techniques of Chap. 19.



the system methods, on the other hand, we estimate all the equations in the model simultaneously, taking due account of all restrictions on such equations by the omission or absence of some variables (recall that for identification such restrictions are essential), hence the name *full information methods*.

As an example, consider the following four-equations model:

$$\begin{aligned}
 Y_{1t} &= \beta_{10} + \beta_{12}Y_{2t} + \beta_{13}Y_{3t} + \beta_{14}Y_{4t} + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + \gamma_{13}X_{3t} + u_{1t} \\
 Y_{2t} &= \beta_{20} + \beta_{21}Y_{1t} + \beta_{23}Y_{3t} + \beta_{24}Y_{4t} + \gamma_{21}X_{1t} + \gamma_{22}X_{2t} + \gamma_{23}X_{3t} + u_{2t} \\
 Y_{3t} &= \beta_{30} + \beta_{31}Y_{1t} + \beta_{32}Y_{2t} + \beta_{34}Y_{4t} + \gamma_{31}X_{1t} + \gamma_{32}X_{2t} + \gamma_{33}X_{3t} + u_{3t} \\
 Y_{4t} &= \beta_{40} + \beta_{41}Y_{1t} + \beta_{42}Y_{2t} + \beta_{43}Y_{3t} + \gamma_{41}X_{1t} + \gamma_{42}X_{2t} + \gamma_{43}X_{3t} + u_{4t}
 \end{aligned}
 \tag{20.1.1}$$

where the  $Y$ 's are the endogenous variables and the  $X$ 's are the exogenous variables. If we are interested in estimating, say, the third equation, the single-equation methods will consider this equation only, noting that variables  $Y_2$  and  $X_3$  are excluded from it. In the systems methods, on the other hand, we try to estimate all four equations simultaneously, taking into account all the restrictions imposed on the various equations of the system.

To preserve the spirit of simultaneous-equation models, ideally one should use the systems method, such as the **full information maximum likelihood (FIML) method**.<sup>2</sup> In practice, however, such methods are not commonly used for a variety of reasons. First, the computational burden is enormous. For example, the comparatively small (20 equations) 1955 Klein–Goldberger model of the U.S. economy had 151 nonzero coefficients, of which the authors estimated only 51 coefficients using the time series data. The Brookings-Social Science Research Council (SSRC) econometric model of the U.S. economy published in 1965 initially had 150 equations.<sup>3</sup> Although such elaborate models may furnish finer details of the various sectors of the economy, the computations are a stupendous task even in these days of high-speed computers, not to mention the cost involved. Second, the systems methods, such as FIML, lead to solutions that are highly nonlinear in the parameters and are therefore often difficult to determine. Third, if there is a specification error (say, a wrong functional form or exclusion of relevant variables) in one or more equations of the system, that error is transmitted to the rest of the system. As a result, the systems methods become very sensitive to specification errors.

<sup>2</sup>For a simple discussion of this method, see Carl F. Christ, *Econometric Models and Methods*, John Wiley & Sons, New York, 1966, pp. 395–401.

<sup>3</sup>James S. Duesenberry, Gary Fromm, Lawrence R. Klein, and Edwin Kuh, eds., *A Quarterly Model of the United States Economy*, Rand McNally, Chicago, 1965.

In practice, therefore, single-equation methods are often used. As Klein puts it,

Single equation methods, in the context of a simultaneous system, may be less sensitive to specification error in the sense that those parts of the system that are correctly specified may not be affected appreciably by errors in specification in another part.<sup>4</sup>

In the rest of the chapter we shall deal with single-equation methods only. Specifically, we shall discuss the following single-equation methods:

1. Ordinary least squares (OLS)
2. Indirect least squares (ILS)
3. Two-stage least squares (2SLS)

## 20.2 RECURSIVE MODELS AND ORDINARY LEAST SQUARES

We saw in Chapter 18 that because of the interdependence between the stochastic disturbance term and the endogenous explanatory variable(s), the OLS method is inappropriate for the estimation of an equation in a system of simultaneous equations. If applied erroneously, then, as we saw in Section 18.3, the estimators are not only biased (in small samples) but also inconsistent; that is, the bias does not disappear no matter how large the sample size. There is, however, one situation where OLS can be applied appropriately even in the context of simultaneous equations. This is the case of the **recursive**, **triangular**, or **causal** models. To see the nature of these models, consider the following three-equation system:

$$\begin{aligned} Y_{1t} &= \beta_{10} && + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + u_{1t} \\ Y_{2t} &= \beta_{20} + \beta_{21}Y_{1t} && + \gamma_{21}X_{1t} + \gamma_{22}X_{2t} + u_{2t} \\ Y_{3t} &= \beta_{30} + \beta_{31}Y_{1t} + \beta_{32}Y_{2t} && + \gamma_{31}X_{1t} + \gamma_{32}X_{2t} + u_{3t} \end{aligned} \quad (20.2.1)$$

where, as usual, the  $Y$ 's and the  $X$ 's are, respectively, the endogenous and exogenous variables. The disturbances are such that

$$\text{cov}(u_{1t}, u_{2t}) = \text{cov}(u_{1t}, u_{3t}) = \text{cov}(u_{2t}, u_{3t}) = 0$$

that is, the same-period disturbances in different equations are uncorrelated (technically, this is the assumption of **zero contemporaneous correlation**).

Now consider the first equation of (20.2.1). Since it contains only the exogenous variables on the right-hand side and since by assumption they are uncorrelated with the disturbance term  $u_{1t}$ , this equation satisfies the critical assumption of the classical OLS, namely, uncorrelatedness between the

<sup>4</sup>Lawrence R. Klein, *A Textbook of Econometrics*, 2d ed., Prentice Hall, Englewood Cliffs, N.J., 1974, p. 150.

explanatory variables and the stochastic disturbances. Hence, OLS can be applied straightforwardly to this equation. Next consider the second equation of (20.2.1), which contains the endogenous variable  $Y_1$  as an explanatory variable along with the nonstochastic  $X$ 's. Now OLS can also be applied to this equation, provided  $Y_{1t}$  and  $u_{2t}$  are uncorrelated. Is this so? The answer is yes because  $u_1$ , which affects  $Y_1$ , is by assumption uncorrelated with  $u_2$ . Therefore, for all practical purposes,  $Y_1$  is a predetermined variable insofar as  $Y_2$  is concerned. Hence, one can proceed with OLS estimation of this equation. Carrying this argument a step further, we can also apply OLS to the third equation in (19.2.1) because both  $Y_1$  and  $Y_2$  are uncorrelated with  $u_3$ .

Thus, in the recursive system OLS can be applied to each equation separately. Actually, we do not have a simultaneous-equation problem in this situation. From the structure of such systems, it is clear that there is no interdependence among the endogenous variables. Thus,  $Y_1$  affects  $Y_2$ , but  $Y_2$  does not affect  $Y_1$ . Similarly,  $Y_1$  and  $Y_2$  influence  $Y_3$  without, in turn, being influenced by  $Y_3$ . In other words, each equation exhibits a unilateral causal dependence, hence the name causal models.<sup>5</sup> Schematically, we have Figure 20.1.

As an example of a recursive system, one may postulate the following model of wage and price determination:

$$\begin{aligned} \text{Price equation:} \quad \dot{P}_t &= \beta_{10} + \beta_{11}\dot{W}_{t-1} + \beta_{12}\dot{R}_t + \beta_{13}\dot{M}_t + \beta_{14}\dot{L}_t + u_{1t} \\ \text{Wage equation:} \quad \dot{W}_t &= \beta_{20} + \beta_{21}UN_t + \beta_{32}\dot{P}_t + u_{2t} \end{aligned} \quad (20.2.2)$$

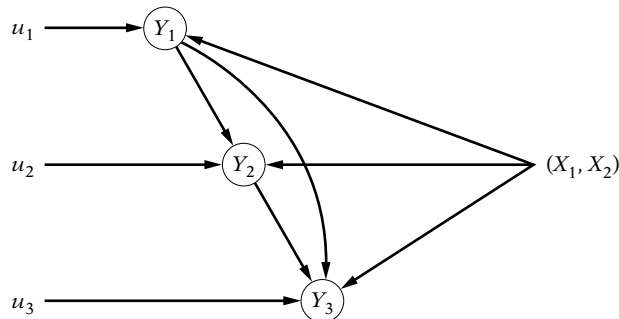


FIGURE 20.1 Recursive model.

<sup>5</sup>The alternative name *triangular* stems from the fact that if we form the matrix of the coefficients of the endogenous variables given in (20.2.1), we obtain the following triangular matrix:

$$\begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{array} \begin{bmatrix} Y_1 & Y_2 & Y_3 \\ 1 & 0 & 0 \\ \beta_{21} & 1 & 0 \\ \beta_{31} & \beta_{32} & 1 \end{bmatrix}$$

Note that the entries above the main diagonal are zeros (why?).

where  $\dot{P}$  = rate of change of price per unit of output  
 $\dot{W}$  = rate of change of wages per employee  
 $\dot{R}$  = rate of change of price of capital  
 $\dot{M}$  = rate of change of import prices  
 $\dot{L}$  = rate of change of labor productivity  
UN = unemployment rate, %<sup>6</sup>

The price equation postulates that the rate of change of price in the current period is a function of the rates of change in the prices of capital and of raw material, the rate of change in labor productivity, and the rate of change in wages in the previous period. The wage equation shows that the rate of change in wages in the current period is determined by the current period rate of change in price and the unemployment rate. It is clear that the causal chain runs from  $\dot{W}_{t-1} \rightarrow \dot{P}_t \rightarrow \dot{W}_t$ , and hence OLS may be applied to estimate the parameters of the two equations individually.

Although recursive models have proved to be useful, most simultaneous-equation models do not exhibit such a unilateral cause-and-effect relationship. Therefore, OLS, in general, is inappropriate to estimate a single equation in the context of a simultaneous-equation model.<sup>7</sup>

There are some who argue that, although OLS is generally inapplicable to simultaneous-equation models, one can use it, if only as a standard or norm of comparison. That is, one can estimate a structural equation by OLS, with the resulting properties of biasedness, inconsistency, etc. Then the same equation may be estimated by other methods especially designed to handle the simultaneity problem and the results of the two methods compared, at least qualitatively. In many applications the results of the inappropriately applied OLS may not differ very much from those obtained by more sophisticated methods, as we shall see later. In principle, one should not have much objection to the production of the results based on OLS so long as estimates based on alternative methods devised for simultaneous-equation models are also given. In fact, this approach might give us some idea about how badly OLS does in situations when it is applied inappropriately.<sup>8</sup>

<sup>6</sup>Note: The dotted symbol means "time derivative." For example,  $\dot{P} = dP/dt$ . For discrete time series,  $dP/dt$  is sometimes approximated by  $\Delta P/\Delta t$ , where the symbol  $\Delta$  is the first difference operator, which was originally introduced in Chap. 12.

<sup>7</sup>It is important to keep in mind that we are assuming that the disturbances across equations are contemporaneously uncorrelated. If this is not the case, we may have to resort to the Zellner SURE (seemingly unrelated regressions) estimation technique to estimate the parameters of the recursive system. See A. Zellner, "An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregation Bias," *Journal of the American Statistical Association*, vol. 57, 1962, pp. 348–368.

<sup>8</sup>It may also be noted that in small samples the alternative estimators, like the OLS estimators, are also biased. But the OLS estimator has the "virtue" that it has minimum variance among these alternative estimators. But this is true of small samples only.

### 20.3 ESTIMATION OF A JUST IDENTIFIED EQUATION: THE METHOD OF INDIRECT LEAST SQUARES (ILS)

For a just or exactly identified structural equation, the method of obtaining the estimates of the structural coefficients from the OLS estimates of the reduced-form coefficients is known as the **method of indirect least squares (ILS)**, and the estimates thus obtained are known as the **indirect least-squares estimates**. ILS involves the following three steps:

**Step 1.** We first obtain the reduced-form equations. As noted in Chapter 19, these reduced-form equations are obtained from the structural equations in such a manner that the dependent variable in each equation is the only endogenous variable and is a function solely of the predetermined (exogenous or lagged endogenous) variables and the stochastic error term(s).

**Step 2.** We apply OLS to the reduced-form equations individually. This operation is permissible since the explanatory variables in these equations are predetermined and hence uncorrelated with the stochastic disturbances. The estimates thus obtained are consistent.<sup>9</sup>

**Step 3.** We obtain estimates of the original structural coefficients from the estimated reduced-form coefficients obtained in Step 2. As noted in Chapter 19, if an equation is exactly identified, there is a one-to-one correspondence between the structural and reduced-form coefficients; that is, one can derive unique estimates of the former from the latter.

As this three-step procedure indicates, the name ILS derives from the fact that structural coefficients (the object of primary enquiry in most cases) are obtained indirectly from the OLS estimates of the reduced-form coefficients.

#### An Illustrative Example

Consider the demand-and-supply model introduced in Section 19.2, which for convenience is given below with a slight change in notation:

$$\text{Demand function: } Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 X_t + u_{1t} \quad (20.3.1)$$

$$\text{Supply function: } Q_t = \beta_0 + \beta_1 P_t + u_{2t} \quad (20.3.2)$$

where  $Q$  = quantity

$P$  = price

$X$  = income or expenditure

Assume that  $X$  is exogenous. As noted previously, the supply function is exactly identified whereas the demand function is not identified.

<sup>9</sup>In addition to being consistent, the estimates “may be best unbiased and/or asymptotically efficient, depending respectively upon whether (i) the  $z$ 's [=  $X$ 's] are exogenous and not merely predetermined [i.e., do not contain lagged values of endogenous variables] and/or (ii) the distribution of the disturbances is normal.” (W. C. Hood and Tjalling C. Koopmans, *Studies in Econometric Method*, John Wiley & Sons, New York, 1953, p. 133.)

The reduced-form equations corresponding to the preceding structural equations are

$$P_t = \Pi_0 + \Pi_1 X_t + w_t \quad (20.3.3)$$

$$Q_t = \Pi_2 + \Pi_3 X_t + v_t \quad (20.3.4)$$

where the  $\Pi$ 's are the reduced-form coefficients and are (nonlinear) combinations of the structural coefficients, as shown in Eqs. (19.2.16) and (19.2.18), and where  $w$  and  $v$  are linear combinations of the structural disturbances  $u_1$  and  $u_2$ .

Notice that each reduced-form equation contains only one endogenous variable, which is the dependent variable and which is a function solely of the exogenous variable  $X$  (income) and the stochastic disturbances. Hence, the parameters of the preceding reduced-form equations may be estimated by OLS. These estimates are

$$\hat{\Pi}_1 = \frac{\sum p_t x_t}{\sum x_t^2} \quad (20.3.5)$$

$$\hat{\Pi}_0 = \bar{P} - \hat{\Pi}_1 \bar{X} \quad (20.3.6)$$

$$\hat{\Pi}_3 = \frac{\sum q_t x_t}{\sum x_t^2} \quad (20.3.7)$$

$$\hat{\Pi}_2 = \bar{Q} - \hat{\Pi}_3 \bar{X} \quad (20.3.8)$$

where the lowercase letters, as usual, denote deviations from sample means and where  $\bar{Q}$  and  $\bar{P}$  are the sample mean values of  $Q$  and  $P$ . As noted previously, the  $\hat{\Pi}_i$ 's are consistent estimators and under appropriate assumptions are also minimum variance unbiased or asymptotically efficient (see footnote 9).

Since our primary objective is to determine the structural coefficients, let us see if we can estimate them from the reduced-form coefficients. Now as shown in Section 19.2, the supply function is exactly identified. Therefore, its parameters can be estimated uniquely from the reduced-form coefficients as follows:

$$\beta_0 = \Pi_2 - \beta_1 \Pi_0 \quad \text{and} \quad \beta_1 = \frac{\Pi_3}{\Pi_1}$$

Hence, the estimates of these parameters can be obtained from the estimates of the reduced-form coefficients as

$$\hat{\beta}_0 = \hat{\Pi}_2 - \hat{\beta}_1 \hat{\Pi}_0 \quad (20.3.9)$$

$$\hat{\beta}_1 = \frac{\hat{\Pi}_3}{\hat{\Pi}_1} \quad (20.3.10)$$

which are the ILS estimators. Note that the parameters of the demand function cannot be thus estimated (however, see exercise 20.13).

**TABLE 20.1** CROP PRODUCTION, CROP PRICES, AND PER CAPITA PERSONAL CONSUMPTION EXPENDITURES, 1982 DOLLARS; UNITED STATES, 1970–1991

Year	Index of crop production (1977 = 100), $Q$	Index of crop prices received by farmers (1977 = 100), $P$	Real per capita personal consumption expenditure, $X$
1970	77	52	3,152
1971	86	56	3,372
1972	87	60	3,658
1973	92	91	4,002
1974	84	117	4,337
1975	93	105	4,745
1976	92	102	5,241
1977	100	100	5,772
1978	102	105	6,384
1979	113	116	7,035
1980	101	125	7,677
1981	117	134	8,375
1982	117	121	8,868
1983	88	128	9,634
1984	111	138	10,408
1985	118	120	11,184
1986	109	107	11,843
1987	108	106	12,568
1988	92	126	13,448
1989	107	134	14,241
1990	114	127	14,996
1991	111	130	15,384

*Source: Economic Report of the President, 1993. Data on  $Q$  (Table B-94), on  $P$  (Table B-96), on  $X$  (Table B-5).*

To give some numerical results, we obtained the data shown in Table 20.1. First we estimate the reduced-form equations, regressing separately price and quantity on per capital real consumption expenditure. The results are as follows:

$$\begin{aligned} \hat{P}_t &= 72.3091 + 0.0043X_t \\ \text{se} &= (9.2002) \quad (0.0009) & \mathbf{(20.3.11)} \\ t &= (7.8595) \quad (4.4104) & R^2 = 0.4930 \end{aligned}$$

$$\begin{aligned} \hat{Q}_t &= 84.0702 + 0.0020X_t \\ \text{se} &= (4.8960) \quad (0.0005) & \mathbf{(20.3.12)} \\ t &= (17.1711) \quad (3.7839) & R^2 = 0.4172 \end{aligned}$$

Using (20.3.9) and (20.3.10), we obtain these ILS estimates:

$$\hat{\beta}_0 = 51.0562 \quad \mathbf{(20.3.13)}$$

$$\hat{\beta}_1 = 0.4566 \quad \mathbf{(20.3.14)}$$

Therefore, the estimated ILS regression is<sup>10</sup>

$$\hat{Q}_t = 51.0562 + 0.4566P_t \quad (20.3.15)$$

For comparison, we give the results of the (inappropriately applied) OLS regression of  $Q$  on  $P$ :

$$\begin{aligned} \hat{Q}_t &= 65.1719 + 0.3272P_t \\ \text{se} &= (9.3294) \quad (0.0835) \\ t &= (6.9856) \quad (3.9203) \quad R^2 = 0.4345 \end{aligned} \quad (20.3.16)$$

These results show how OLS can distort the “true” picture when it is applied in inappropriate situations.

### Properties of ILS Estimators

We have seen that the estimators of the reduced-form coefficients are consistent and under appropriate assumptions also best unbiased or asymptotically efficient (see footnote 9). Do these properties carry over to the ILS estimators? It can be shown that the ILS estimators inherit all the asymptotic properties of the reduced-form estimators, such as consistency and asymptotic efficiency. But (the small sample) properties such as unbiasedness do not generally hold true. It is shown in Appendix 20A, Section 20A.1, that the ILS estimators  $\hat{\beta}_0$  and  $\hat{\beta}_1$  of the supply function given previously are biased but the bias disappears as the sample size increases indefinitely (that is, the estimators are consistent).<sup>11</sup>

### 20.4 ESTIMATION OF AN OVERIDENTIFIED EQUATION: THE METHOD OF TWO-STAGE LEAST SQUARES (2SLS)

Consider the following model:

$$\text{Income function:} \quad Y_{1t} = \beta_{10} + \beta_{11}Y_{2t} + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + u_{1t} \quad (20.4.1)$$

$$\text{Money supply function:} \quad Y_{2t} = \beta_{20} + \beta_{21}Y_{1t} + u_{2t} \quad (20.4.2)$$

<sup>10</sup>We have not presented the standard errors of the estimated structural coefficients because, as noted previously, these coefficients are generally nonlinear functions of the reduced-form coefficients and there is no simple method of estimating their standard errors from the standard errors of the reduced-form coefficients. For large-sample size, however, standard errors of the structural coefficients can be obtained approximately. For details, see Jan Kmenta, *Elements of Econometrics*, Macmillan, New York, 1971, p. 444.

<sup>11</sup>Intuitively this can be seen as follows:  $E(\hat{\beta}_1) = \beta_1$  if  $E(\hat{\Pi}_3/\hat{\Pi}_1) = (\Pi_3/\Pi_1)$ . Now even if  $E(\hat{\Pi}_3) = \Pi_3$  and  $E(\hat{\Pi}_1) = \Pi_1$ , it can be shown that  $E(\hat{\Pi}_3/\hat{\Pi}_1) \neq E(\hat{\Pi}_3)/E(\hat{\Pi}_1)$ ; that is, the expectation of the ratio of two variables is not equal to the ratio of the expectations of the two variables. However, as shown in App. 20A.1,  $\text{plim}(\hat{\Pi}_3/\hat{\Pi}_1) = \text{plim}(\hat{\Pi}_3)/\text{plim}(\hat{\Pi}_1) = \Pi_3/\Pi_1$  since  $\hat{\Pi}_3$  and  $\hat{\Pi}_1$  are consistent estimators.



where  $Y_1$  = income  
 $Y_2$  = stock of money  
 $X_1$  = investment expenditure  
 $X_2$  = government expenditure on goods and services

The variables  $X_1$  and  $X_2$  are exogenous.

The income equation, a hybrid of quantity-theory–Keynesian approaches to income determination, states that income is determined by money supply, investment expenditure, and government expenditure. The *money supply function* postulates that the stock of money is determined (by the Federal Reserve System) on the basis of the level of income. Obviously, we have a simultaneous-equation problem, which can be checked by the simultaneity test discussed in Chapter 19.

Applying the order condition of identification, we can see that the income equation is underidentified whereas the money supply equation is overidentified. There is not much that can be done about the income equation short of changing the model specification. The overidentified money supply function may not be estimated by ILS because there are two estimates of  $\beta_{21}$  (the reader should verify this via the reduced-form coefficients).

As a matter of practice, one may apply OLS to the money supply equation, but the estimates thus obtained will be inconsistent in view of the likely correlation between the stochastic explanatory variable  $Y_1$  and the stochastic disturbance term  $u_2$ . Suppose, however, we find a “proxy” for the stochastic explanatory variable  $Y_1$  such that, although “resembling”  $Y_1$  (in the sense that it is highly correlated with  $Y_1$ ), it is uncorrelated with  $u_2$ . Such a proxy is also known as an **instrumental variable** (see Chapter 17). If one can find such a proxy, OLS can be used straightforwardly to estimate the money supply function. But how does one obtain such an instrumental variable? One answer is provided by the **two-stage least squares** (2SLS), developed independently by Henri Theil<sup>12</sup> and Robert Basmann.<sup>13</sup> As the name indicates, the method involves two successive applications of OLS. The process is as follows:

**Stage 1.** To get rid of the likely correlation between  $Y_1$  and  $u_2$ , regress first  $Y_1$  on *all* the predetermined variables in the *whole system*, not just that equation. In the present case, this means regressing  $Y_1$  on  $X_1$  and  $X_2$  as follows:

$$Y_{1t} = \hat{\Pi}_0 + \hat{\Pi}_1 X_{1t} + \hat{\Pi}_2 X_{2t} + \hat{u}_t \quad (20.4.3)$$

where  $\hat{u}_t$  are the usual OLS residuals. From Eq. (20.4.3) we obtain

$$\hat{Y}_{1t} = \hat{\Pi}_0 + \hat{\Pi}_1 X_{1t} + \hat{\Pi}_2 X_{2t} \quad (20.4.4)$$

<sup>12</sup>Henri Theil, “Repeated Least-Squares Applied to Complete Equation Systems,” The Hague: The Central Planning Bureau, The Netherlands, 1953 (mimeographed).

<sup>13</sup>Robert L. Basmann, “A Generalized Classical Method of Linear Estimation of Coefficients in a Structural Equation,” *Econometrica*, vol. 25, 1957, pp. 77–83.

where  $\hat{Y}_{1t}$  is an estimate of the mean value of  $Y$  conditional upon the fixed  $X$ 's. Note that (20.4.3) is nothing but a reduced-form regression because only the exogenous or predetermined variables appear on the right-hand side.

Equation (20.4.3) can now be expressed as

$$Y_{1t} = \hat{Y}_{1t} + \hat{u}_t \quad (20.4.5)$$

which shows that the stochastic  $Y_1$  consists of two parts:  $\hat{Y}_{1t}$ , which is a linear combination of the nonstochastic  $X$ 's, and a random component  $\hat{u}_t$ . Following the OLS theory,  $\hat{Y}_{1t}$  and  $\hat{u}_t$  are uncorrelated. (Why?)

**Stage 2.** The overidentified money supply equation can now be written as

$$\begin{aligned} Y_{2t} &= \beta_{20} + \beta_{21}(\hat{Y}_{1t} + \hat{u}_t) + u_{2t} \\ &= \beta_{20} + \beta_{21}\hat{Y}_{1t} + (u_{2t} + \beta_{21}\hat{u}_t) \\ &= \beta_{20} + \beta_{21}\hat{Y}_{1t} + u_t^* \end{aligned} \quad (20.4.6)$$

where  $u_t^* = u_{2t} + \beta_{21}\hat{u}_t$ .

Comparing (20.4.6) with (20.4.2), we see that they are very similar in appearance, the only difference being that  $Y_1$  is replaced by  $\hat{Y}_1$ . What is the advantage of (20.4.6)? It can be shown that although  $Y_1$  in the original money supply equation is correlated or likely to be correlated with the disturbance term  $u_2$  (hence rendering OLS inappropriate),  $\hat{Y}_{1t}$  in (20.4.6) is uncorrelated with  $u_t^*$  asymptotically, that is, in the large sample (or more accurately, as the sample size increases indefinitely). As a result, OLS can be applied to (20.4.6), which will give consistent estimates of the parameters of the money supply function.<sup>14</sup>

As this two-stage procedure indicates, the basic idea behind 2SLS is to “purify” the stochastic explanatory variable  $Y_1$  of the influence of the stochastic disturbance  $u_2$ . This goal is accomplished by performing the reduced-form regression of  $Y_1$  on all the predetermined variables in the system (Stage 1), obtaining the estimates  $\hat{Y}_{1t}$  and replacing  $Y_{1t}$  in the original equation by the estimated  $\hat{Y}_{1t}$ , and then applying OLS to the equation thus transformed (Stage 2). The estimators thus obtained are consistent; that is, they converge to their true values as the sample size increases indefinitely.

<sup>14</sup>But note that in small samples  $\hat{Y}_{1t}$  is likely to be correlated with  $u_t^*$ . The reason is as follows: From (20.4.4) we see that  $\hat{Y}_{1t}$  is a weighted linear combination of the predetermined  $X$ 's, with  $\hat{\Pi}$ 's as the weights. Now even if the predetermined variables are truly nonstochastic, the  $\hat{\Pi}$ 's, being estimators, are stochastic. Therefore,  $\hat{Y}_{1t}$  is stochastic too. Now from our discussion of the reduced-form equations and indirect least-squares estimation, it is clear that the reduced-coefficients, the  $\hat{\Pi}$ 's, are functions of the stochastic disturbances, such as  $u_2$ . And since  $\hat{Y}_{1t}$  depends on the  $\hat{\Pi}$ 's, it is likely to be correlated with  $u_2$ , which is a component of  $u_t^*$ . As a result,  $\hat{Y}_{1t}$  is expected to be correlated with  $u_t^*$ . But as noted previously, this correlation disappears as the sample size tends to infinity. The upshot of all this is that in small samples the 2SLS procedure may lead to biased estimation.

To illustrate 2SLS further, let us modify the income–money supply model as follows:

$$Y_{1t} = \beta_{10} + \beta_{12}Y_{2t} + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + u_{1t} \quad (20.4.7)$$

$$Y_{2t} = \beta_{20} + \beta_{21}Y_{1t} + \gamma_{23}X_{3t} + \gamma_{24}X_{4t} + u_{2t} \quad (20.4.8)$$

where, in addition to the variables already defined,  $X_3$  = income in the previous time period and  $X_4$  = money supply in the previous period. Both  $X_3$  and  $X_4$  are predetermined.

It can be readily verified that both Eqs. (20.4.7) and (20.4.8) are overidentified. To apply 2SLS, we proceed as follows: In Stage 1 we regress the endogenous variables on *all* the predetermined variables in the system. Thus,

$$Y_{1t} = \hat{\Pi}_{10} + \hat{\Pi}_{11}X_{1t} + \hat{\Pi}_{12}X_{2t} + \hat{\Pi}_{13}X_{3t} + \hat{\Pi}_{14}X_{4t} + \hat{u}_{1t} \quad (20.4.9)$$

$$Y_{2t} = \hat{\Pi}_{20} + \hat{\Pi}_{21}X_{1t} + \hat{\Pi}_{22}X_{2t} + \hat{\Pi}_{23}X_{3t} + \hat{\Pi}_{24}X_{4t} + \hat{u}_{2t} \quad (20.4.10)$$

In Stage 2 we replace  $Y_1$  and  $Y_2$  in the original (structural) equations by their estimated values from the preceding two regressions and then run the OLS regressions as follows:

$$Y_{1t} = \beta_{10} + \beta_{12}\hat{Y}_{2t} + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + u_{1t}^* \quad (20.4.11)$$

$$Y_{2t} = \beta_{20} + \beta_{21}\hat{Y}_{1t} + \gamma_{23}X_{3t} + \gamma_{24}X_{4t} + u_{2t}^* \quad (20.4.12)$$

where  $u_{1t}^* = u_{1t} + \beta_{12}\hat{u}_{2t}$  and  $u_{2t}^* = u_{2t} + \beta_{21}\hat{u}_{1t}$ . The estimates thus obtained will be consistent.

Note the following features of 2SLS.

1. It can be applied to an individual equation in the system without directly taking into account any other equation(s) in the system. Hence, for solving econometric models involving a large number of equations, 2SLS offers an economical method. For this reason the method has been used extensively in practice.

2. Unlike ILS, which provides multiple estimates of parameters in the overidentified equations, 2SLS provides only one estimate per parameter.

3. It is easy to apply because all one needs to know is the total number of exogenous or predetermined variables in the system without knowing any other variables in the system.

4. Although specially designed to handle overidentified equations, the method can also be applied to exactly identified equations. But then ILS and 2SLS will give identical estimates. (Why?)

5. If the  $R^2$  values in the reduced-form regressions (that is, Stage 1 regressions) are very high, say, in excess of 0.8, the classical OLS estimates

and 2SLS estimates will be very close. But this result should not be surprising because if the  $R^2$  value in the first stage is very high, it means that the estimated values of the endogenous variables are very close to their actual values, and hence the latter are less likely to be correlated with the stochastic disturbances in the original structural equations. (Why?)<sup>15</sup> If, however, the  $R^2$  values in the first-stage regressions are very low, the 2SLS estimates will be practically meaningless because we shall be replacing the original  $Y$ 's in the second-stage regression by the estimated  $\hat{Y}$ 's from the first-stage regressions, which will essentially represent the disturbances in the first-stage regressions. In other words, in this case, the  $\hat{Y}$ 's will be very poor proxies for the original  $Y$ 's.

6. Notice that in reporting the ILS regression in (20.3.15) we did not state the standard errors of the estimated coefficients (for reasons explained in footnote 10). But we can do this for the 2SLS estimates because the structural coefficients are directly estimated from the second-stage (OLS) regressions. There is, however, a caution to be exercised. The estimated standard errors in the second-stage regressions need to be modified because, as can be seen from Eq. (20.4.6), the error term  $u_t^*$  is, in fact, the original error term  $u_{2t}$  plus  $\beta_{21}\hat{u}_t$ . Hence, the variance of  $u_t^*$  is not exactly equal to the variance of the original  $u_{2t}$ . However, the modification required can be easily effected by the formula given in Appendix 20A, Section 20A.2.

7. In using the 2SLS, bear in mind the following remarks of Henri Theil:

The statistical justification of the 2SLS is of the large-sample type. When there are no lagged endogenous variables, . . . the 2SLS coefficient estimators are consistent if the exogenous variables are constant in repeated samples and if the disturbance[s] [appearing in the various behavioral or structural equations] . . . are independently and identically distributed with zero means and finite variances. . . . If these two conditions are satisfied, the sampling distribution of 2SLS coefficient estimators becomes approximately normal for large samples. . . .

When the equation system contains lagged endogenous variables, the consistency and large-sample normality of the 2SLS coefficient estimators require an additional condition, . . . that as the sample increases the mean square of the values taken by each lagged endogenous variable converges in probability to a positive limit. . . .

If the [disturbances appearing in the various structural equations are] *not* independently distributed, lagged endogenous variables are not independent of the current operation of the equation system . . . , which means these variables are not really predetermined. If these variables are nevertheless treated as predetermined in the 2SLS procedure, the resulting estimators are not consistent.<sup>16</sup>

<sup>15</sup>In the extreme case if  $R^2 = 1$  in the first-stage regression, the endogenous explanatory variable in the original (overidentified) equation will be practically nonstochastic (why?).

<sup>16</sup>Henri Theil, *Introduction to Econometrics*, Prentice Hall, Englewood Cliffs, N.J., 1978, pp. 341–342.

## 20.5 2SLS: A NUMERICAL EXAMPLE

To illustrate the 2SLS method, consider the income–money supply model given previously in Eqs. (20.4.1) and (20.4.2). As shown, the money supply equation is overidentified. To estimate the parameters of this equation, we resort to the two-stage least-squares method. The data required for analysis are given in Table 20.2; this table also gives some data that are required to answer some of the questions given in the exercises.

**Stage 1 Regression.** We first regress the stochastic explanatory variable income  $Y_1$ , represented by GDP, on the predetermined variables private

**TABLE 20.2** GDP, M2, GPDI, FEDEXP, TB6, USA, 1970–1999

YEAR	GDP ( $Y_1$ )	M2 ( $Y_2$ )	GPDI ( $X_1$ )	FEDEXP ( $X_2$ )	TB6 ( $X_3$ )
1970	3578.000	626.4000	436.2000	198.6000	6.562000
1971	3697.700	710.1000	485.8000	216.6000	4.511000
1972	3998.400	802.1000	543.0000	240.0000	4.466000
1973	4123.400	855.2000	606.5000	259.7000	7.178000
1974	4099.000	901.9000	561.7000	291.2000	7.926000
1975	4084.400	1015.900	462.2000	345.4000	6.122000
1976	4311.700	1151.700	555.5000	371.9000	5.266000
1977	4511.800	1269.900	639.4000	405.0000	5.510000
1978	4760.600	1365.500	713.0000	444.2000	7.572000
1979	4912.100	1473.100	735.4000	489.6000	10.01700
1980	4900.900	1599.100	655.3000	576.6000	11.37400
1981	5021.000	1754.600	715.6000	659.3000	13.77600
1982	4913.300	1909.500	615.2000	732.1000	11.08400
1983	5132.300	2126.000	673.7000	797.8000	8.750000
1984	5505.200	2309.700	871.5000	856.1000	9.800000
1985	5717.100	2495.400	863.4000	924.6000	7.660000
1986	5912.400	2732.100	857.7000	978.5000	6.030000
1987	6113.300	2831.100	879.3000	1018.400	6.050000
1988	6368.400	2994.300	902.8000	1066.200	6.920000
1989	6591.900	3158.400	936.5000	1140.300	8.040000
1990	6707.900	3277.600	907.3000	1228.700	7.470000
1991	6676.400	3376.800	829.5000	1287.600	5.490000
1992	6880.000	3430.700	899.8000	1418.900	3.570000
1993	7062.600	3484.400	977.9000	1471.500	3.140000
1994	7347.700	3499.000	1107.000	1506.000	4.660000
1995	7543.800	3641.900	1140.600	1575.700	5.590000
1996	7813.200	3813.300	1242.700	1635.900	5.090000
1997	8159.500	4028.900	1393.300	1678.800	5.180000
1998	8515.700	4380.600	1566.800	1705.000	4.850000
1999	8875.800	4643.700	1669.700	1750.200	4.760000

Notes:  $Y_1$  = GDP = gross domestic product (\$, billions, seasonally adjusted)  
 $Y_2$  = M2 = M2 money supply (\$, billions, seasonally adjusted)  
 $X_1$  = GPDI = gross private domestic investment (\$, billions, seasonally adjusted)  
 $X_2$  = FEDEXP = Federal government expenditure (\$, billions, seasonally adjusted)  
 $X_3$  = TB6 = 6-month treasury bill rate (%).

Source: *Economic Report of the President*, 2001. Tables B-2, B-69, B-73, B-84.

investment  $X_1$  and government expenditure  $X_2$ , obtaining the following results:

$$\begin{aligned}\hat{Y}_{1t} &= 2587.351 + 1.6707X_{1t} + 1.9693X_{2t} \\ \text{se} &= (72.0011) \quad (0.1646) \quad (0.0983) \\ t &= (35.9349) \quad (10.1489) \quad (20.0200) \quad R^2 = 0.9947\end{aligned}\tag{20.5.1}$$

**Stage 2 Regression.** We now estimate the money supply function (20.4.2), replacing the endogenous variable  $Y_1$  by  $Y_1$  estimated from (20.5.1) ( $= \hat{Y}_1$ ). The results are as follows:

$$\begin{aligned}\hat{Y}_{2t} &= -2198.297 + 0.7916\hat{Y}_{1t} \\ \text{se} &= (139.0986) \quad (0.0232) \\ t &= (-15.8038) \quad (34.0502) \quad R^2 = 0.9764\end{aligned}\tag{20.5.2}$$

As we pointed out previously, the estimated standard errors given in (20.5.2) need to be corrected in the manner suggested in Appendix 20.A, Section 20A.2. Effecting this correction (most econometric packages can do it now), we obtain the following results:

$$\begin{aligned}\hat{Y}_{2t} &= -2198.297 + 0.7915\hat{Y}_{1t} \\ \text{se} &= (126.9598) \quad (0.0212) \\ t &= (-17.3149) \quad (37.3057) \quad R^2 = 0.9803\end{aligned}\tag{20.5.3}$$

As noted in Appendix 20A, Section 20A.2, the standard errors given in (20.5.3) do not differ much from those given in (20.5.2) because the  $R^2$  in Stage 1 regression is very high.

**OLS Regression.** For comparison, we give the regression of money stock on income as shown in (20.4.2) without “purging” the stochastic  $Y_{1t}$  of the influence of the stochastic disturbance term.

$$\begin{aligned}\hat{Y}_{2t} &= -2195.468 + 0.7911Y_{1t} \\ \text{se} &= (126.6460) \quad (0.0211) \\ t &= (-17.3354) \quad (37.3812) \quad R^2 = 0.9803\end{aligned}\tag{20.5.4}$$

Comparing the “inappropriate” OLS results with the Stage 2 regression, we see that the two regressions are virtually the same. Does this mean that

the 2SLS procedure is worthless? Not at all. That in the present situation the two results are practically identical should not be surprising because, as noted previously, the  $R^2$  value in the first stage is very high, thus making the estimated  $\hat{Y}_{1t}$  virtually identical with the actual  $Y_{1t}$ . Therefore, in this case the OLS and second-stage regressions will be more or less similar. But there is no guarantee that this will happen in every application. An implication, then, is that in overidentified equations one should not accept the classical OLS procedure without checking the second-stage regression(s).

**Simultaneity between GDP and Money Supply.** Let us find out if GDP ( $Y_1$ ) and money supply ( $Y_2$ ) are mutually dependent. For this purpose we use the Hausman test of simultaneity discussed in Chapter 19.

First we regress GDP on  $X_1$  (investment expenditure) and  $X_2$  (government expenditure), the exogenous variables in the system (i.e., we estimate the reduced-form regression.) From this regression we obtain the estimated GDP and the residuals  $\hat{v}_t$ , as suggested in Eq. (19.4.7). Then we regress money supply on estimated GDP and  $v_t$  to obtain the following results:

$$\begin{aligned} \hat{Y}_{2t} &= -2198.297 + 0.7915\hat{Y}_{1t} + 0.6984\hat{v}_t \\ \text{se} &= (129.0548) \quad (0.0215) \quad (0.2970) \quad \mathbf{(20.5.5)} \\ t &= (-17.0338) \quad (36.70016) \quad (2.3511) \end{aligned}$$

Since the  $t$  value of  $\hat{v}_t$  is statistically significant (the  $p$  value is 0.0263), we cannot reject the hypothesis of simultaneity between money supply and GDP, which should not be surprising. (*Note:* Strictly speaking, this conclusion is valid only in large samples; technically, it is only valid as the sample size increases indefinitely.)

**Hypothesis Testing.** Suppose we want to test the hypothesis that income has no effect on money demand. Can we test this hypothesis with the usual  $t$  test from the estimated regression (20.5.2)? Yes, provided the sample is large and provided we correct the standard errors as shown in (20.5.3), we can use the  $t$  test to test the significance of an individual coefficient and the  $F$  test to test joint significance of two or more coefficients, using formula (8.5.7).<sup>17</sup>

What happens if the error term in a structural equation is autocorrelated and or correlated with the error term in another structural equation in the system? A full answer to this question will take us beyond the scope of

<sup>17</sup>But take this precaution: The restricted and unrestricted RSS in the numerator must be calculated using predicted  $Y$  (as in Stage 2 of 2SLS) and the RSS in the denominator is calculated using actual rather than predicted values of the regressors. For an accessible discussion of this point, see T. Dudley Wallace and J. Lew Silver, *Econometrics: An Introduction*, Addison-Wesley, Reading, Mass., 1988, Sec. 8.5.



the book and is better left for the references (see the reference given in footnote 7). Nevertheless, estimation techniques (such as Zellner's SURE technique) do exist to handle these complications.

## 20.6 ILLUSTRATIVE EXAMPLES

In this section we consider some applications of the simultaneous-equation methods.

### EXAMPLE 20.1

#### ADVERTISING, CONCENTRATION, AND PRICE MARGINS

To study the interrelationships among advertising, concentration (as measured by the concentration ratio), and price-cost margins, Allyn D. Strickland and Lenord W. Weiss formulated the following three-equation model.<sup>18</sup>

*Advertising intensity function:*

$$Ad/S = a_0 + a_1M + a_2(CD/S) + a_3C + a_4C^2 + a_5Gr + a_6Dur \quad (20.6.1)$$

*Concentration function:*

$$C = b_0 + b_1(Ad/S) + b_2(MES/S) \quad (20.6.2)$$

*Price-cost margin function:*

$$M = c_0 + c_1(K/S) + c_2Gr + c_3C + c_4GD + c_5(Ad/S) + c_6(MES/S) \quad (20.6.3)$$

where Ad = advertising expense  
S = value of shipments  
C = four-firm concentration ratio  
CD = consumer demand  
MES = minimum efficient scale  
M = price/cost margin  
Gr = annual rate of growth of industrial production  
Dur = dummy variable for durable goods industry  
K = capital stock  
GD = measure of geographic dispersion of output

By the order conditions for identifiability, Eq. (20.6.2) is overidentified, whereas (20.6.1) and (20.6.3) are exactly identified.

The data for the analysis came largely from the 1963 Census of Manufacturers and covered 408 of the 417 four-digit manufacturing industries. The three equations were first estimated by OLS, yielding the results shown in Table 20.3. To correct for the simultaneous-equation bias, the authors reestimated the model using 2SLS. The ensuing results are given in Table 20.4. We leave it to the reader to compare the two results.

(Continued)

<sup>18</sup>See their "Advertising, Concentration, and Price-Cost Margins," *Journal of Political Economy*, vol. 84, no. 5, 1976, pp. 1109–1121.



**EXAMPLE 20.1** (Continued)

**TABLE 20.3** OLS ESTIMATES OF THREE EQUATIONS (*t* Ratios in Parentheses)

	Dependent variable		
	Ad/S Eq. (20.6.1)	<i>C</i> Eq. (20.6.2)	<i>M</i> Eq. (20.6.3)
Constant	-0.0314 (-7.45)	0.2638 (25.93)	0.1682 (17.15)
<i>C</i>	0.0554 (3.56)	—	0.0629 (2.89)
<i>C</i> <sup>2</sup>	-0.0568 (-3.38)	—	—
<i>M</i>	0.1123 (9.84)	—	—
CD/ <i>S</i>	0.0257 (8.94)	—	—
Gr	0.0387 (1.64)	—	0.2255 (2.61)
Dur	-0.0021 (-1.11)	—	—
Ad/ <i>S</i>	—	1.1613 (3.3)	1.6536 (11.00)
MES/ <i>S</i>	—	4.1852 (18.99)	0.0686 (0.54)
<i>K</i> / <i>S</i>	—	—	0.1123 (8.03)
GD	—	—	-0.0003 (-2.90)
<i>R</i> <sup>2</sup>	0.374	0.485	0.402
df	401	405	401

**TABLE 20.4**

TWO-STAGE LEAST-SQUARES ESTIMATES OF THREE EQUATIONS  
(*t* Ratios in Parentheses)

	Dependent variable		
	Ad/S Eq. (20.6.1)	<i>C</i> Eq. (20.6.2)	<i>M</i> Eq. (20.6.3)
Constant	-0.0245 (-3.86)	0.2591 (21.30)	0.1736 (14.66)
<i>C</i>	0.0737 (2.84)	—	0.0377 (0.93)
<i>C</i> <sup>2</sup>	-0.0643 (-2.64)	—	—
<i>M</i>	0.0544 (2.01)	—	—
CD/ <i>S</i>	0.0269 (8.96)	—	—
Gr	0.0539 (2.09)	—	0.2336 (2.61)
Dur	-0.0018 (-0.93)	—	—
Ad/ <i>S</i>	—	1.5347 (2.42)	1.6256 (5.52)
MES/ <i>S</i>	—	4.169 (18.84)	0.1720 (0.92)
<i>K</i> / <i>S</i>	—	—	0.1165 (7.30)
GD	—	—	-0.0003 (-2.79)

**EXAMPLE 20.2**

KLEIN'S MODEL I

In Example 18.6 we discussed briefly the pioneering model of Klein. Initially, the model was estimated for the period 1920–1941. The underlying data are given in Table 20.5; and OLS, reduced-form, and 2SLS estimates are given in Table 20.6. We leave it to the reader to interpret these results.

(Continued)

**EXAMPLE 20.2** (Continued)

**TABLE 20.5** UNDERLYING DATA FOR KLEIN'S MODEL I

Year	C*	P	W	I	K <sub>-1</sub>	X	W'	G	T
1920	39.8	12.7	28.8	2.7	180.1	44.9	2.2	2.4	3.4
1921	41.9	12.4	25.5	-0.2	182.8	45.6	2.7	3.9	7.7
1922	45.0	16.9	29.3	1.9	182.6	50.1	2.9	3.2	3.9
1923	49.2	18.4	34.1	5.2	184.5	57.2	2.9	2.8	4.7
1924	50.6	19.4	33.9	3.0	189.7	57.1	3.1	3.5	3.8
1925	52.6	20.1	35.4	5.1	192.7	61.0	3.2	3.3	5.5
1926	55.1	19.6	37.4	5.6	197.8	64.0	3.3	3.3	7.0
1927	56.2	19.8	37.9	4.2	203.4	64.4	3.6	4.0	6.7
1928	57.3	21.1	39.2	3.0	207.6	64.5	3.7	4.2	4.2
1929	57.8	21.7	41.3	5.1	210.6	67.0	4.0	4.1	4.0
1930	55.0	15.6	37.9	1.0	215.7	61.2	4.2	5.2	7.7
1931	50.9	11.4	34.5	-3.4	216.7	53.4	4.8	5.9	7.5
1932	45.6	7.0	29.0	-6.2	213.3	44.3	5.3	4.9	8.3
1933	46.5	11.2	28.5	-5.1	207.1	45.1	5.6	3.7	5.4
1934	48.7	12.3	30.6	-3.0	202.0	49.7	6.0	4.0	6.8
1935	51.3	14.0	33.2	-1.3	199.0	54.4	6.1	4.4	7.2
1936	57.7	17.6	36.8	2.1	197.7	62.7	7.4	2.9	8.3
1937	58.7	17.3	41.0	2.0	199.8	65.0	6.7	4.3	6.7
1938	57.5	15.3	38.2	-1.9	201.8	60.9	7.7	5.3	7.4
1939	61.6	19.0	41.6	1.3	199.9	69.5	7.8	6.6	8.9
1940	65.0	21.1	45.0	3.3	201.2	75.7	8.0	7.4	9.6
1941	69.7	23.5	53.3	4.9	204.5	88.4	8.5	13.8	11.6

\*Interpretation of column heads is listed in Example 18.6.  
Source: These data from G. S. Maddala, *Econometrics*, McGraw-Hill, New York, 1977, p. 238.

**TABLE 20.6\*** OLS, REDUCED FORM AND 2SLS ESTIMATES OF KLEIN'S MODEL I

**OLS:**

$$\begin{aligned} \hat{C} &= 16.237 + 0.193P + 0.796(W + W') + 0.089P_{-1} & \bar{R}^2 &= 0.978 & DW &= 1.367 \\ & (1.203) (0.091) (0.040) (0.090) \\ \hat{I} &= 10.125 + 0.479P + 0.333P_{-1} - 0.112K_{-1} & \bar{R}^2 &= 0.919 & DW &= 1.810 \\ & (5.465) (0.097) (0.100) (0.026) \\ \hat{W} &= 0.064 + 0.439X + 0.146X_{-1} + 0.130t & \bar{R}^2 &= 0.985 & DW &= 1.958 \\ & (1.151) (0.032) (0.037) (0.031) \end{aligned}$$

**Reduced-form:**

$$\begin{aligned} \hat{P} &= 46.383 + 0.813P_{-1} - 0.213K_{-1} + 0.015X_{-1} + 0.297t - 0.926T + 0.443G \\ & (10.870) (0.444) (0.067) (0.252) (0.154) (0.385) (0.373) \\ & \bar{R}^2 = 0.753 \quad DW = 1.854 \\ \widehat{W + W'} &= 40.278 + 0.823P_{-1} - 0.144K_{-1} + 0.115X_{-1} + 0.881t - 0.567T + 0.859G \\ & (8.787) (0.359) (0.054) (0.204) (0.124) (0.311) (0.302) \\ & \bar{R}^2 = 0.949 \quad DW = 2.395 \\ \hat{X} &= 78.281 + 1.724P_{-1} - 0.319K_{-1} + 0.094X_{-1} + 0.878t - 0.565T + 1.317G \\ & (18.860) (0.771) (0.110) (0.438) (0.267) (0.669) (0.648) \\ & \bar{R}^2 = 0.882 \quad DW = 2.049 \end{aligned}$$

**2SLS:**

$$\begin{aligned} \hat{C} &= 16.543 + 0.019P + 0.810(W + W') + 0.214P_{-1} & \bar{R}^2 &= 0.9726 \\ & (1.464) (0.130) (0.044) (0.118) \\ \hat{I} &= 20.284 + 0.149P + 0.616P_{-1} - 0.157K_{-1} & \bar{R}^2 &= 0.8643 \\ & (8.361) (0.191) (0.180) (0.040) \\ \hat{W} &= 0.065 + 0.438X + 0.146X_{-1} + 0.130t & \bar{R}^2 &= 0.9852 \\ & (1.894) (0.065) (0.070) (0.053) \end{aligned}$$

\*Interpretation of variables is listed in Example 18.6 (standard errors in parentheses).  
Source: G. S. Maddala, *Econometrics*, McGraw-Hill, New York, 1977, p. 242.

**EXAMPLE 20.3**

## THE CAPITAL ASSET PRICING MODEL EXPRESSED AS A RECURSIVE SYSTEM

In a rather unusual application of recursive simultaneous-equation modeling, Cheng F. Lee and W. P. Lloyd<sup>19</sup> estimated the following model for the oil industry:

$$R_{1t} = \alpha_1 + \gamma_1 M_t + u_{1t}$$

$$R_{2t} = \alpha_2 + \beta_{21} R_{1t} + \gamma_2 M_t + u_{2t}$$

$$R_{3t} = \alpha_3 + \beta_{31} R_{1t} + \beta_{32} R_{2t} + \gamma_3 M_t + u_{3t}$$

$$R_{4t} = \alpha_4 + \beta_{41} R_{1t} + \beta_{42} R_{2t} + \beta_{43} R_{3t} + \gamma_4 M_t + u_{4t}$$

$$R_{5t} = \alpha_5 + \beta_{51} R_{1t} + \beta_{52} R_{2t} + \beta_{53} R_{3t} + \beta_{54} R_{4t} + \gamma_5 M_t + u_{5t}$$

$$R_{6t} = \alpha_6 + \beta_{61} R_{1t} + \beta_{62} R_{2t} + \beta_{63} R_{3t} + \beta_{64} R_{4t} + \beta_{65} R_{5t} + \gamma_6 M_t + u_{6t}$$

$$R_{7t} = \alpha_7 + \beta_{71} R_{1t} + \beta_{72} R_{2t} + \beta_{73} R_{3t} + \beta_{74} R_{4t} + \beta_{75} R_{5t} + \beta_{76} R_{6t} + \gamma_7 M_t + u_{7t}$$

where  $R_1$  = rate of return on security 1 (= Imperial Oil)

$R_2$  = rate of return on security 2 (= Sun Oil)

⋮

$R_7$  = rate of return on security 7 (= Standard of Indiana)

$M_t$  = rate of return on the market index

$u_{it}$  = disturbances ( $i = 1, 2, \dots, 7$ )

Before we present the results, the obvious question is: How do we choose which is security 1, which is security 2, and so on? Lee and Lloyd answer this question purely empirically. They regress the rate of return on security  $i$  on the rates of return of the remaining six securities and observe the resulting  $R^2$ . Thus, there will be seven such regressions. Then they order the estimated  $R^2$  values, from the lowest to the highest. The security having the lowest  $R^2$  is designated as security 1 and the one having the highest  $R^2$  is designated as 7. The idea behind this is intuitively simple. If the  $R^2$  of the rate of return of, say, Imperial Oil, is lowest with respect to the other six securities, it would suggest that this security is affected least by the movements in the returns of the other securities. Therefore, the causal ordering, if any, runs from this security to the others and there is no feedback from the other securities.

Although one may object to such a purely empirical approach to causal ordering, let us present their empirical results nonetheless, which are given in Table 20.7.

In exercise 5.5 we introduced the *characteristic line* of modern investment theory, which is simply the regression of the rate of return on security  $i$  on the market rate of return. The slope coefficient, known as the *beta coefficient*, is a measure of the volatility of the security's return. What the Lee-Lloyd regression results suggest is that there are significant intra-industry relationships between security returns, apart from the common market influence represented by the market portfolio. Thus, Standard of Indiana's return depends not only on the market rate of return but also on the rates of return on Shell Oil, Phillips Petroleum, and

(Continued)

<sup>19</sup>"The Capital Asset Pricing Model Expressed as a Recursive System: An Empirical Investigation," *Journal of Financial and Quantitative Analysis*, June 1976, pp. 237-249.

**EXAMPLE 20.3** (Continued)

**TABLE 20.7** RECURSIVE SYSTEM ESTIMATES FOR THE OIL INDUSTRY

	Linear form dependent variables						
	Standard of Indiana	Shell Oil	Phillips Petroleum	Union Oil	Standard of Ohio	Sun Oil	Imperial Oil
Standard of Indiana							
Shell Oil	0.2100* (2.859)						
Phillips Petroleum	0.2293* (2.176)	0.0791 (1.065)					
Union Oil	0.1754* (2.472)	0.2171* (3.177)	0.2225* (2.337)				
Standard of Ohio	-0.0794 (-1.294)	0.0147 (0.235)	0.4248* (5.501)	0.1468* (1.735)			
Sun Oil	0.1249 (1.343)	0.1710* (1.843)	0.0472 (0.355)	0.1339 (0.908)	0.0499 (0.271)		
Imperial Oil	-0.1077 (-1.412)	0.0526 (0.6804)	0.0354 (0.319)	0.1580 (1.290)	-0.2541* (-1.691)	0.0828 (0.971)	
Constant	0.0868 (0.681)	-0.0384 (1.296)	-0.0127 (-0.068)	-0.2034 (0.986)	0.3009 (1.204)	0.2013 (1.399)	0.3710* (2.161)
Market index	0.3681* (2.165)	0.4997* (3.039)	0.2884 (1.232)	0.7609* (3.069)	0.9089* (3.094)	0.7161* (4.783)	0.6432* (3.774)
R <sup>2</sup>	0.5020	0.4658	0.4106	0.2532	0.0985	0.2404	0.1247
Durbin- Watson	2.1083	2.4714	2.2306	2.3468	2.2181	2.3109	1.9592

\*Denotes significance at 0.10 level or better for two-tailed test.  
 Note: The *t* values appear in parentheses beneath the coefficients.  
 Source: Cheng F. Lee and W. P. Lloyd, op. cit., Table 3b.

Union Oil. To put the matter differently, the movement in the rate of return on Standard of Indiana can be better explained if in addition to the market rate of return we also consider the rates of return experienced by Shell Oil, Phillips Petroleum, and Union Oil.

**EXAMPLE 20.4**

REVISED FORM OF ST. LOUIS MODEL<sup>20</sup>

The well-known, and often controversial, St. Louis model originally developed in the late 1960s has been revised from time to time. One such revision is given in Table 20.8, and the empirical results based on this revised model are given in Table 20.9. (Note: A dot over a variable means the growth rate of that variable.) The model basically consists of Eqs. (1), (2), (4), and (5) in Table 20.8, the other equations representing the definitions. Equation (1) was

(Continued)

<sup>20</sup>Review, Federal Reserve Bank of St. Louis, May 1982, p. 14.

**EXAMPLE 20.4** (Continued)

**TABLE 20.8** THE ST. LOUIS MODEL

$$\begin{aligned}
 (1) \quad \dot{Y}_t &= C1 + \sum_{i=0}^4 CM_i(\dot{M}_{t-i}) + \sum_{i=0}^4 CE(\dot{E}_{t-i}) + \varepsilon 1_t \\
 (2) \quad \dot{P}_t &= C2 + \sum_{i=1}^4 CPE_i(\dot{PE}_{t-i}) + \sum_{i=0}^5 CD_i(\dot{X}_{t-i} - \dot{XF}_{t-i}^*) \\
 &\quad + CPA(\dot{PA}_t) + CDUM1(DUM1) + CDUM2(DUM2) + \varepsilon 2_t \\
 (3) \quad \dot{PA}_t &= \sum_{i=1}^{21} CPRL_i(\dot{P}_{t-i}) \\
 (4) \quad RL_t &= C3 + \sum_{i=0}^{20} CPRL_i(\dot{P}_{t-i}) + \varepsilon 3_t \\
 (5) \quad U_t - UF_t &= CG(GAP_t) + CG1(GAP_{t-1}) + \varepsilon 4_t \\
 (6) \quad Y_t &= (P_t/100)(X_t) \\
 (7) \quad \dot{Y}_t &= [(Y_t/Y_{t-1})^4 - 1]100 \\
 (8) \quad \dot{X}_t &= [(X_t/X_{t-1})^4 - 1]100 \\
 (9) \quad \dot{P}_t &= [(P_t/P_{t-1})^4 - 1]100 \\
 (10) \quad GAP_t &= [(XF_t/X_t)/XF_t]100 \\
 (11) \quad \dot{XF}_t^* &= [(XF_t/X_{t-1})^4 - 1]100
 \end{aligned}$$

$Y$  = nominal GNP  
 $M$  = money stock (M1)  
 $E$  = high employment expenditures  
 $P$  = GNP deflator (1972 = 100)  
 $PE$  = relative price of energy  
 $X$  = output in 1972 dollars  
 $XF$  = potential output (Rasche/Tatom)  
 $RL$  = corporate bond rate  
 $U$  = unemployment rate  
 $UF$  = unemployment rate at full employment  
 $DUM1$  = control dummy (1971–III to 1973–I = 1; 0 elsewhere)  
 $DUM2$  = postcontrol dummy (1973–II to 1975–I = 1; 0 elsewhere)  
*Source:* Federal Reserve Bank of St. Louis, *Review*, May 1982, p. 14.

estimated by OLS. Equations (1), (2), and (4) were estimated using the Almon distributed-lag method with (endpoint) constraints on the coefficients. Where relevant, the equations were corrected for first-order ( $\rho_1$ ) and/or second-order ( $\rho_2$ ) serial correlation.

Examining the results, we observe that it is the rate of growth in the money supply that primarily determines the rate of growth of (nominal) GNP and not the rate of growth in high-employment expenditures. The sum of the  $M$  coefficients is 1.06, suggesting that a 1 percent (sustained) increase in the money supply on the average leads to about 1.06 percent increase in the nominal GNP. On the other hand, the sum of the  $E$  coefficients, about 0.05, suggests that a change in high-employment government expenditure has little impact on the rate of growth of nominal GNP. It is left to the reader to interpret the results of the other regressions reported in Table 20.9.

(Continued)

**EXAMPLE 20.4** (Continued)

**TABLE 20.9**

IN-SAMPLE ESTIMATION: 1960-I TO 1980-IV (Absolute Value of *t* Statistic in Parentheses)

(1)	$\widehat{Y}_t = 2.44 + 0.40\dot{M}_t + 0.39\dot{M}_{t-1} + 0.22\dot{M}_{t-2} + 0.06\dot{M}_{t-3} - 0.01\dot{M}_{t-4}$ $+ 0.06\dot{E}_t + 0.02\dot{E}_{t-1} - 0.02\dot{E}_{t-2} - 0.02\dot{E}_{t-3} + 0.01\dot{E}_{t-4}$	<p>(2.15) (3.38) (5.06) (2.18) (0.82) (0.11)</p> <p>(1.46) (0.63) (0.57) (0.52) (0.34)</p>	<p><math>R^2 = 0.39</math>    <math>se = 3.50</math>    <math>DW = 2.02</math></p>
(2)	$\widehat{P}_t = 0.96 + 0.01\dot{P}E_{t-1} + 0.04\dot{P}E_{t-2} - 0.01\dot{P}E_{t-3} + 0.02\dot{P}E_{t-4}$ $- 0.00(\dot{X}_t - \dot{X}F_t^*) + 0.01(\dot{X}_{t-1} - \dot{X}F_{t-1}^*) + 0.02(\dot{X}_{t-2} - \dot{X}F_{t-2}^*)$ $+ 0.02(\dot{X}_{t-3} - \dot{X}F_{t-3}^*) + 0.02(\dot{X}_{t-4} - \dot{X}F_{t-4}^*) + 0.01(\dot{X}_{t-5} - \dot{X}F_{t-5}^*)$ $+ 1.03(\dot{P}A_t) - 0.61(DUM1_t) + 1.65(DUM2_t)$	<p>(2.53) (0.75) (1.96) (0.73) (1.38)</p> <p>(0.18) (1.43) (4.63)</p> <p>(3.00) (2.42) (2.16)</p> <p>(10.49) (1.02) (2.71)</p>	<p><math>R^2 = 0.80</math>    <math>se = 1.28</math>    <math>DW = 1.97</math>    <math>\hat{\rho} = 0.12</math></p>
(4)	$\widehat{RL}_t = 2.97 + 0.96 \sum_{i=0}^{20} \dot{P}_{t-i}$	<p>(3.12) (5.22)</p>	<p><math>R^2 = 0.32</math>    <math>se = 0.33</math>    <math>DW = 1.76</math>    <math>\hat{\rho} = 0.94</math></p>
(5)	$\widehat{U}_t - \widehat{UF}_t = 0.28(GAP_t) + 0.14(GAP_{t-1})$	<p>(11.89) (6.31)</p>	<p><math>R^2 = 0.63</math>    <math>se = 0.17</math>    <math>DW = 1.95</math>    <math>\hat{\rho}_1 = 1.43</math>    <math>\hat{\rho}_2 = 0.52</math></p>

Source: Federal Reserve Bank of St. Louis, *Review*, May 1982, p. 14.

**20.7 SUMMARY AND CONCLUSIONS**

1. Assuming that an equation in a simultaneous-equation model is identified (either exactly or over-), we have several methods to estimate it.
2. These methods fall into two broad categories: *Single-equation methods* and *systems methods*.
3. For reasons of economy, specification errors, etc. the single-equation methods are by far the most popular. A unique feature of these methods is that one can estimate a single-equation in a multiequation model without worrying too much about other equations in the system. (*Note:* For identification purposes, however, the other equations in the system count.)
4. Three commonly used single-equation methods are **OLS**, **ILS**, and **2SLS**.
5. Although OLS is, in general, inappropriate in the context of simultaneous-equation models, it can be applied to the so-called **recursive models** where there is a definite but unidirectional cause-and-effect relationship among the endogenous variables.

6. The method of ILS is suited for just or exactly identified equations. In this method OLS is applied to the reduced-form equation, and it is from the reduced-form coefficients that one estimates the original structural coefficients.

7. The method of 2SLS is especially designed for overidentified equations, although it can also be applied to exactly identified equations. But then the results of 2SLS and ILS are identical. The basic idea behind 2SLS is to replace the (stochastic) endogenous explanatory variable by a linear combination of the predetermined variables in the model and use this combination as the explanatory variable in lieu of the original endogenous variable. The 2SLS method thus resembles the **instrumental variable method** of estimation in that the linear combination of the predetermined variables serves as an instrument, or proxy, for the endogenous regressor.

8. A noteworthy feature of both ILS and 2SLS is that the estimates obtained are consistent, that is, as the sample size increases indefinitely, the estimates converge to their true population values. The estimates may not satisfy small-sample properties, such as unbiasedness and minimum variance. Therefore, the results obtained by applying these methods to small samples and the inferences drawn from them should be interpreted with due caution.

## EXERCISES

### Questions

- 20.1. State whether each of the following statements is true or false:
- The method of OLS is not applicable to estimate a structural equation in a simultaneous-equation model.
  - In case an equation is not identified, 2SLS is not applicable.
  - The problem of simultaneity does not arise in a recursive simultaneous-equation model.
  - The problems of simultaneity and exogeneity mean the same thing.
  - The 2SLS and other methods of estimating structural equations have desirable statistical properties only in large samples.
  - There is no such thing as an  $R^2$  for the simultaneous-equation model as a whole.
  - \*g. The 2SLS and other methods of estimating structural equations are not applicable if the equation errors are autocorrelated and/or are correlated across equations.
  - h. If an equation is exactly identified, ILS and 2SLS give identical results.
- 20.2. Why is it unnecessary to apply the two-stage least-squares method to exactly identified equations?

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\*Optional.

**20.3.** Consider the following modified Keynesian model of income determination:

$$C_t = \beta_{10} + \beta_{11}Y_t + u_{1t}$$

$$I_t = \beta_{20} + \beta_{21}Y_t + \beta_{22}Y_{t-1} + u_{2t}$$

$$Y_t = C_t + I_t + G_t$$

where  $C$  = consumption expenditure  
 $I$  = investment expenditure  
 $Y$  = income  
 $G$  = government expenditure  
 $G_t$  and  $Y_{t-1}$  are assumed predetermined

- Obtain the reduced-form equations and determine which of the preceding equations are identified (either just or over).
- Which method will you use to estimate the parameters of the over-identified equation and of the exactly identified equation? Justify your answer.

**20.4.** Consider the following results\*:

$$\text{OLS: } \widehat{W}_t = 0.276 + 0.258\dot{P}_t + 0.046\dot{P}_{t-1} + 4.959V_t \quad R^2 = 0.924$$

$$\text{OLS: } \widehat{P}_t = 2.693 + 0.232\dot{W}_t - 0.544\dot{X}_t + 0.247\dot{M}_t + 0.064\dot{M}_{t-1} \quad R^2 = 0.982$$

$$\text{2SLS: } \widehat{W}_t = 0.272 + 0.257\dot{P}_t + 0.046\dot{P}_{t-1} + 4.966V_t \quad R^2 = 0.920$$

$$\text{2SLS: } \widehat{P}_t = 2.686 + 0.233\dot{W}_t - 0.544\dot{X}_t + 0.246\dot{M}_t + 0.046\dot{M}_{t-1} \quad R^2 = 0.981$$

where  $\dot{W}_t$ ,  $\dot{P}_t$ ,  $\dot{M}_t$ , and  $\dot{X}_t$  are percentage changes in earnings, prices, import prices, and labor productivity (all percentage changes are over the previous year), respectively, and where  $V_t$  represents unfilled job vacancies (percentage of total number of employees).

“Since the OLS and 2SLS results are practically identical, 2SLS is meaningless.” Comment.

<sup>†</sup>**20.5.** Assume that production is characterized by the Cobb–Douglas production function

$$Q_i = AK_i^\alpha L_i^\beta$$

where  $Q$  = output  
 $K$  = capital input  
 $L$  = labor input  
 $A$ ,  $\alpha$ , and  $\beta$  = parameters  
 $i$  =  $i$ th firm

\*Source: *Prices and Earnings in 1951–1969: An Econometric Assessment*, Department of Employment, United Kingdom, Her Majesty’s Stationery Office, London, 1971, p. 30.

<sup>†</sup>Optional.



Given the price of final output  $P$ , the price of labor  $W$ , and the price of capital  $R$ , and assuming profit maximization, we obtain the following empirical model of production:

*Production function:*

$$\ln Q_i = \ln A + \alpha \ln K_i + \beta \ln L_i + \ln u_{1i} \quad (1)$$

*Marginal product of labor function:*

$$\ln Q_i = -\ln \beta + \ln L_i + \ln \frac{W}{P} + \ln u_{2i} \quad (2)$$

*Marginal product of capital function:*

$$\ln Q_i = -\ln \alpha + \ln K_i + \ln \frac{R}{P} + \ln u_{3i} \quad (3)$$

where  $u_1$ ,  $u_2$ , and  $u_3$  are stochastic disturbances.

In the preceding model there are three equations in three endogenous variables  $Q$ ,  $L$ , and  $K$ .  $P$ ,  $R$ , and  $W$  are exogenous.

- What problems do you encounter in estimating the model if  $\alpha + \beta = 1$ , that is, when there are constant returns to scale?
- Even if  $\alpha + \beta \neq 1$ , can you estimate the equations? Answer by considering the identifiability of the system.
- If the system is not identified, what can be done to make it identifiable?

*Note:* Equations (2) and (3) are obtained by differentiating  $Q$  with respect to labor and capital, respectively, setting them equal to  $W/P$  and  $R/P$ , transforming the resulting expressions into logarithms, and adding (the logarithm of) the disturbance terms.

- 20.6.** Consider the following demand-and-supply model for money:

$$\text{Demand for money: } M_t^d = \beta_0 + \beta_1 Y_t + \beta_2 R_t + \beta_3 P_t + u_{1t}$$

$$\text{Supply of money: } M_t^s = \alpha_0 + \alpha_1 Y_t + u_{2t}$$

where  $M$  = money  
 $Y$  = income  
 $R$  = rate of interest  
 $P$  = price

Assume that  $R$  and  $P$  are predetermined.

- Is the demand function identified?
  - Is the supply function identified?
  - Which method would you use to estimate the parameters of the identified equation(s)? Why?
  - Suppose we modify the supply function by adding the explanatory variables  $Y_{t-1}$  and  $M_{t-1}$ . What happens to the identification problem? Would you still use the method you used in **c**? Why or why not?
- 20.7.** Refer to exercise 18.10. For the two-equation system there obtain the reduced-form equations and estimate their parameters. Estimate the

indirect least-squares regression of consumption on income and compare your results with the OLS regression.

### Problems

**20.8.** Consider the following model:

$$R_t = \beta_0 + \beta_1 M_t + \beta_2 Y_t + u_{1t}$$

$$Y_t = \alpha_0 + \alpha_1 R_t + u_{2t}$$

where  $M_t$  (money supply) is exogenous,  $R_t$  is the interest rate, and  $Y_t$  is GDP.

- How would you justify the model?
- Are the equations identified?
- Using the data given in Table 20.2, estimate the parameters of the identified equations. Justify the method(s) you use.

**20.9.** Suppose we change the model in exercise 20.8 as follows:

$$R_t = \beta_0 + \beta_1 M_t + \beta_2 Y_t + \beta_3 Y_{t-1} + u_{1t}$$

$$Y_t = \alpha_0 + \alpha_1 R_t + u_{2t}$$

- Find out if the system is identified.
- Using the data given in Table 20.2, estimate the parameters of the identified equation(s).

**20.10.** Consider the following model:

$$R_t = \beta_0 + \beta_1 M_t + \beta_2 Y_t + u_{1t}$$

$$Y_t = \alpha_0 + \alpha_1 R_t + \alpha_2 I_t + u_{2t}$$

where the variables are as defined in exercise 20.8. Treating  $I$  (domestic investment) and  $M$  exogenously, determine the identification of the system. Using the data of Table 20.2, estimate the parameters of the identified equation(s).

**20.11.** Suppose we change the model of exercise 20.10 as follows:

$$R_t = \beta_0 + \beta_1 M_t + \beta_2 Y_t + u_{1t}$$

$$Y_t = \alpha_0 + \alpha_1 R_t + \alpha_2 I_t + u_{2t}$$

$$I_t = \gamma_0 + \gamma_1 R_t + u_{3t}$$

Assume that  $M$  is determined exogenously.

- Find out which of the equations are identified.
  - Estimate the parameters of the identified equation(s) using the data given in Table 20.2. Justify your method(s).
- 20.12.** Verify the standard errors reported in (20.5.3).  
**20.13.** Return to the demand-supply model given in Eqs. (20.3.1) and (20.3.2). Suppose the supply function is altered as follows:

$$Q_t = \beta_0 + \beta_1 P_{t-1} + u_{2t}$$

where  $P_{t-1}$  is the price prevailing in the previous period.

- If  $X$  (expenditure) and  $P_{t-1}$  are predetermined, is there a simultaneity problem?

- b. If there is, are the demand and supply functions each identified? If they are, obtain their reduced-form equations and estimate them from the data given in Table 20.1.
- c. From the reduced-form coefficients, can you derive the structural coefficients? Show the necessary computations.
- 20.14. Class Exercise:** Consider the following simple macroeconomic model for the U.S. economy, say, for the period 1960–1999.\*

*Private consumption function:*

$$C_t = \alpha_0 + \alpha_1 Y_t + \alpha_2 C_{t-1} + u_{1t} \quad \alpha_1 > 0, 0 < \alpha_2 < 1$$

*Private gross investment function:*

$$I_t = \beta_0 + \beta_1 Y_t + \beta_2 R_t + \beta_3 I_{t-1} + u_{2t} \quad \beta_1 > 0, \beta_2 < 0, 0 < \beta_3 < 1$$

*A money demand function:*

$$R_t = \lambda_0 + \lambda_1 Y_t + \lambda_2 M_{t-1} + \lambda_3 P_t + \lambda_4 R_{t-1} + u_{3t} \\ \lambda_1 > 0, \lambda_2 < 0, \lambda_3 > 0, 0 < \lambda_4 < 1$$

*Income identity:*

$$Y_t = C_t + I_t + G_t$$

where  $C$  = real private consumption;  $I$  = real gross private investment,  $G$  = real government expenditure,  $Y$  = real GDP,  $M$  = M2 money supply at current prices,  $R$  = long-term interest rate (%), and  $P$  = Consumer Price Index. The endogenous variables are  $C$ ,  $I$ ,  $R$ , and  $Y$ . The predetermined variables are:  $C_{t-1}$ ,  $I_{t-1}$ ,  $M_{t-1}$ ,  $P_t$ ,  $R_{t-1}$ , and  $G_t$  plus the intercept term. The  $u$ 's are the error terms.

- a. Using the order condition of identification, determine which of the four equations are identified, either exact or over.
- b. Which method(s) do you use to estimate the identified equations?
- c. Obtain suitable data from government and/or private sources, estimate the model, and comment on your results.

## APPENDIX 20A

### 20A.1 BIAS IN THE INDIRECT LEAST-SQUARES ESTIMATORS

To show that the ILS estimators, although consistent, are biased, we use the demand-and-supply model given in Eqs. (20.3.1) and (20.3.2). From (20.3.10) we obtain

$$\hat{\beta}_1 = \frac{\hat{\Pi}_3}{\hat{\Pi}_1}$$

\*Adapted from H. R. Seddighi, K. A. Lawler, and A. V. Katos, *Econometrics: A Practical Approach*, Routledge, New York, 2000, p. 204.

Now

$$\hat{\Pi}_3 = \frac{\sum q_t x_t}{\sum x_t^2} \quad \text{from (20.3.7)}$$

and

$$\hat{\Pi}_1 = \frac{\sum p_t x_t}{\sum x_t^2} \quad \text{from (20.3.5)}$$

Therefore, on substitution, we obtain

$$\hat{\beta}_1 = \frac{\sum q_t x_t}{\sum p_t x_t} \quad (1)$$

Using (20.3.3) and (20.3.4), we obtain

$$p_t = \Pi_1 x_t + (w_t - \bar{w}) \quad (2)$$

$$q_t = \Pi_3 x_t + (v_t - \bar{v}) \quad (3)$$

where  $\bar{w}$  and  $\bar{v}$  are the mean values of  $w_t$  and  $v_t$ , respectively.

Substituting (2) and (3) into (1), we obtain

$$\begin{aligned} \hat{\beta}_1 &= \frac{\Pi_3 \sum x_t^2 + \sum (v_t - \bar{v}) x_t}{\Pi_1 \sum x_t^2 + \sum (w_t - \bar{w}) x_t} \\ &= \frac{\Pi_3 + \sum (v_t - \bar{v}) x_t / \sum x_t^2}{\Pi_1 + \sum (w_t - \bar{w}) x_t / \sum x_t^2} \end{aligned} \quad (4)$$

Since the expectation operator  $E$  is a linear operator, we cannot take the expectation of (4), although it is clear that  $\hat{\beta}_1 \neq (\Pi_3/\Pi_1)$  generally. (Why?)

But as the sample size tends to infinity, we can obtain

$$\text{plim}(\hat{\beta}_1) = \frac{\text{plim} \Pi_3 + \text{plim} \sum (v_t - \bar{v}) x_t / \sum x_t^2}{\text{plim} \Pi_1 + \text{plim} \sum (w_t - \bar{w}) x_t / \sum x_t^2} \quad (5)$$

where use is made of the properties of plim, namely, that

$$\text{plim}(A + B) = \text{plim} A + \text{plim} B \quad \text{and} \quad \text{plim} \left( \frac{A}{B} \right) = \frac{\text{plim} A}{\text{plim} B}$$

Now as the sample size is increased indefinitely, the second term in both the denominator and the numerator of (5) tends to zero (why?), yielding

$$\text{plim}(\hat{\beta}_1) = \frac{\Pi_3}{\Pi_1} \quad (6)$$

showing that, although biased,  $\hat{\beta}_1$  is a consistent estimator of  $\beta_1$ .

## 20A.2 ESTIMATION OF STANDARD ERRORS OF 2SLS ESTIMATORS

The purpose of this appendix is to show that the standard errors of the estimates obtained from the second-stage regression of the 2SLS procedure, using the formula applicable in OLS estimation, are not the “proper” estimates of the “true” standard errors. To see this, we use the income–money supply model given in (20.4.1) and (20.4.2). We estimate the parameters of the overidentified money supply function from the second-stage regression as

$$Y_{2t} = \beta_{20} + \beta_{21}\hat{Y}_{1t} + u_t^* \quad (20.4.6)$$

where

$$u_t^* = u_{2t} + \beta_{21}\hat{u}_{1t} \quad (7)$$

Now when we run regression (20.4.6), the standard error of, say,  $\hat{\beta}_{21}$  is obtained from the following expression:

$$\text{var}(\hat{\beta}_{21}) = \frac{\hat{\sigma}_{u^*}^2}{\sum \hat{y}_{1t}^2} \quad (8)$$

where

$$\hat{\sigma}_{u^*}^2 = \frac{\sum (\hat{u}_t^*)^2}{n-2} = \frac{\sum (Y_{2t} - \hat{\beta}_{20} - \hat{\beta}_{21}\hat{Y}_{1t})^2}{n-2} \quad (9)$$

But  $\sigma_{u^*}^2$  is not the same thing as  $\hat{\sigma}_{u_2}^2$ , where the latter is an unbiased estimate of the true variance of  $u_2$ . This difference can be readily verified from (7). To obtain the true (as defined previously)  $\hat{\sigma}_{u_2}^2$ , we proceed as follows:

$$\hat{u}_{2t} = Y_{2t} - \hat{\beta}_{20} - \hat{\beta}_{21}Y_{1t}$$

where  $\hat{\beta}_{20}$  and  $\hat{\beta}_{21}$  are the estimates from the second-stage regression. Hence,

$$\hat{\sigma}_{u_2}^2 = \frac{\sum (Y_{2t} - \hat{\beta}_{20} - \hat{\beta}_{21}Y_{1t})^2}{n-2} \quad (10)$$

Note the difference between (9) and (10): In (10) we use actual  $Y_1$  rather than the estimated  $\hat{Y}_1$  from the first-stage regression.

Having estimated (10), the easiest way to correct the standard errors of coefficients estimated in the second-stage regression is to multiply each one of them by  $\hat{\sigma}_{u_2}/\hat{\sigma}_{u^*}$ . Note that if  $Y_{1t}$  and  $\hat{Y}_{1t}$  are very close, that is, the  $R^2$  in the first-stage regression is very high, the correction factor  $\hat{\sigma}_{u_2}/\hat{\sigma}_{u^*}$  will be close to 1, in which case the estimated standard errors in the second-stage regression may be taken as the true estimates. But in other situations, we shall have to use the preceding correction factor.

# 21

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## TIME SERIES ECONOMETRICS: SOME BASIC CONCEPTS

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We noted in Chapter 1 that one of the important types of data used in empirical analysis is **time series** data. In this and the following chapter we take a closer look at such data not only because of the frequency with which they are used in practice but also because they pose several challenges to econometricians and practitioners.

*First*, empirical work based on time series data assumes that the underlying time series is **stationary**. Although we have discussed the concept of stationarity intuitively in Chapter 1, we discuss it more fully in this chapter. More specifically, we will try to find out what stationarity means and why one should worry about it.

*Second*, in Chapter 12, on autocorrelation, we discussed several causes of autocorrelation. Sometimes autocorrelation results because the underlying time series is nonstationary.

*Third*, in regressing a time series variable on another time series variable(s), one often obtains a very high  $R^2$  (in excess of 0.9) even though there is no meaningful relationship between the two variables. Sometimes we expect no relationship between two variables, yet a regression of one on the other variable often shows a significant relationship. This situation exemplifies the problem of **spurious**, or **nonsense, regression**, whose nature will be explored shortly. It is therefore very important to find out if the relationship between economic variables is spurious or nonsensical. We will see in this chapter how spurious regressions can arise if time series are not stationary.

*Fourth*, some financial time series, such as stock prices, exhibit what is known as the **random walk phenomenon**. This means the best prediction

of the price of a stock, say IBM, tomorrow is equal to its price today plus a purely random shock (or error term). If this were in fact the case, forecasting asset prices would be a futile exercise.

*Fifth*, regression models involving time series data are often used for forecasting. In view of the preceding discussion, we would like to know if such forecasting is valid if the underlying time series are not stationary.

*Finally*, causality tests of Granger and Sims that we discussed in Chapter 17 assume that the time series involved in analysis are stationary. Therefore, tests of stationarity should precede tests of causality.

At the outset a disclaimer is in order. The topic of time series analysis is so vast and evolving and some of the mathematics underlying the various techniques of time series analysis is so involved that the best we hope to achieve in an introductory text like this is to give the reader a glimpse of some of the fundamental concepts of time series analysis. For those who want to pursue this topic further, we provide references.<sup>1</sup>

## 21.1 A LOOK AT SELECTED U.S. ECONOMIC TIME SERIES

To set the stage, and to give the reader a feel for the somewhat esoteric concepts of time series analysis to be developed in this chapter, it might be useful to consider several U.S. economic time series of general interest. The time series we consider are: (1) GDP (gross domestic product), (2) PDI (personal disposable income), (3) PCE (personal consumption expenditure), (4) profits (corporate profits after tax), and (5) dividends (net corporate dividend); all data are in billions of 1987 dollars and are for the quarterly periods of 1970–1991, for a total of 88 quarterly observations. The raw data are given in Table 21.1.

<sup>1</sup>At the introductory level, these references may be helpful: Gary Koop, *Analysis of Economic Data*, John Wiley & Sons, New York, 2000; Jeff B. Cromwell, Walter C. Labys, and Michel Terraza, *Univariate Tests for Time Series Models*, Sage Publications, California, Ansbury Park, 1994; Jeff B. Cromwell, Michael H. Hannan, Walter C. Labys, and Michel Terraza, *Multivariate Tests for Time Series Models*, Sage Publications, California, Ansbury Park, 1994; H. R. Seddighi, K. A. Lawler, and A. V. Katos, *Econometrics: A Practical Approach*, Routledge, New York, 2000. At the intermediate level, see Walter Enders, *Applied Econometric Time Series*, John Wiley & Sons, New York, 1995; Kerry Patterson, *An Introduction to Applied Econometrics: A Time Series Approach*, St. Martin's Press, New York, 2000; T. C. Mills, *The Econometric Modelling of Financial Time Series*, 2d ed., Cambridge University Press, New York, 1999; Marno Verbeek, *A Guide to Modern Econometrics*, John Wiley & Sons, New York, 2000; Wojciech W. Charemza and Derek F. Deadman, *New Directions in Econometric Practice: General to Specific Modelling and Vector Autoregression*, 2d ed., Edward Elgar Publisher, New York, 1997. At the advanced level, see Hamilton, J. D., *Time Series Analysis*, Princeton University Press, Princeton, N.J., 1994, and G. S. Maddala and In-Moo Kim, *Unit Roots, Cointegration, and Structural Change*, Cambridge University Press, 1998. At the applied level, see B. Bhaskara Rao, ed., *Cointegration for the Applied Economist*, St. Martin's Press, New York, 1994, and Chandan Mukherjee, Howard White, and Marc Wuyts, *Econometrics and Data Analysis for Developing Countries*, Routledge, New York, 1998.

**TABLE 21.1** MACROECONOMICS DATA, UNITED STATES, 1970-I TO 1991-IV

Quarter	GDP	PDI	PCE	Profits	Dividend	Quarter	GDP	PDI	PCE	Profits	Dividend
1970-I	2872.8	1990.6	1800.5	44.7	24.5	1981-I	3860.5	2783.7	2475.5	159.5	64.0
1970-II	2860.3	2020.1	1807.5	44.4	23.9	1981-II	3844.4	2776.7	2476.1	143.7	68.4
1970-III	2896.6	2045.3	1824.7	44.9	23.3	1981-III	3864.5	2814.1	2487.4	147.6	71.9
1970-IV	2873.7	2045.2	1821.2	42.1	23.1	1981-IV	3803.1	2808.8	2468.6	140.3	72.4
1971-I	2942.9	2073.9	1849.9	48.8	23.8	1982-I	3756.1	2795.0	2484.0	114.4	70.0
1971-II	2947.4	2098.0	1863.5	50.7	23.7	1982-II	3771.1	2824.8	2488.9	114.0	68.4
1971-III	2966.0	2106.6	1876.9	54.2	23.8	1982-III	3754.4	2829.0	2502.5	114.6	69.2
1971-IV	2980.8	2121.1	1904.6	55.7	23.7	1982-IV	3759.6	2832.6	2539.3	109.9	72.5
1972-I	3037.3	2129.7	1929.3	59.4	25.0	1983-I	3783.5	2843.6	2556.5	113.6	77.0
1972-II	3089.7	2149.1	1963.3	60.1	25.5	1983-II	3886.5	2867.0	2604.0	133.0	80.5
1972-III	3125.8	2193.9	1989.1	62.8	26.1	1983-III	3944.4	2903.0	2639.0	145.7	83.1
1972-IV	3175.5	2272.0	2032.1	68.3	26.5	1983-IV	4012.1	2960.6	2678.2	141.6	84.2
1973-I	3253.3	2300.7	2063.9	79.1	27.0	1984-I	4089.5	3033.2	2703.8	155.1	83.3
1973-II	3267.6	2315.2	2062.0	81.2	27.8	1984-II	4144.0	3065.9	2741.1	152.6	82.2
1973-III	3264.3	2337.9	2073.7	81.3	28.3	1984-III	4166.4	3102.7	2754.6	141.8	81.7
1973-IV	3289.1	2382.7	2067.4	85.0	29.4	1984-IV	4194.2	3118.5	2784.8	136.3	83.4
1974-I	3259.4	2334.7	2050.8	89.0	29.8	1985-I	4221.8	3123.6	2824.9	125.2	87.2
1974-II	3267.6	2304.5	2059.0	91.2	30.4	1985-II	4254.8	3189.6	2849.7	124.8	90.8
1974-III	3239.1	2315.0	2065.5	97.1	30.9	1985-III	4309.0	3156.5	2893.3	129.8	94.1
1974-IV	3226.4	2313.7	2039.9	86.8	30.5	1985-IV	4333.5	3178.7	2895.3	134.2	97.4
1975-I	3154.0	2282.5	2051.8	75.8	30.0	1986-I	4390.5	3227.5	2922.4	109.2	105.1
1975-II	3190.4	2390.3	2086.9	81.0	29.7	1986-II	4387.7	3281.4	2947.9	106.0	110.7
1975-III	3249.9	2354.4	2114.4	97.8	30.1	1986-III	4412.6	3272.6	2993.7	111.0	112.3
1975-IV	3292.5	2389.4	2137.0	103.4	30.6	1986-IV	4427.1	3266.2	3012.5	119.2	111.0
1976-I	3356.7	2424.5	2179.3	108.4	32.6	1987-I	4460.0	3295.2	3011.5	140.2	108.0
1976-II	3369.2	2434.9	2194.7	109.2	35.0	1987-II	4515.3	3241.7	3046.8	157.9	105.5
1976-III	3381.0	2444.7	2213.0	110.0	36.6	1987-III	4559.3	3285.7	3075.8	169.1	105.1
1976-IV	3416.3	2459.5	2242.0	110.3	38.3	1987-IV	4625.5	3335.8	3074.6	176.0	106.3
1977-I	3466.4	2463.0	2271.3	121.5	39.2	1988-I	4655.3	3380.1	3128.2	195.5	109.6
1977-II	3525.0	2490.3	2280.8	129.7	40.0	1988-II	4704.8	3386.3	3147.8	207.2	113.3
1977-III	3574.4	2541.0	2302.6	135.1	41.4	1988-III	4734.5	3407.5	3170.6	213.4	117.5
1977-IV	3567.2	2556.2	2331.6	134.8	42.4	1988-IV	4779.7	3443.1	3202.9	226.0	121.0
1978-I	3591.8	2587.3	2347.1	137.5	43.5	1989-I	4809.8	3473.9	3200.9	221.3	124.6
1978-II	3707.0	2631.9	2394.0	154.0	44.5	1989-II	4832.4	3450.9	3208.6	206.2	127.1
1978-III	3735.6	2653.2	2404.5	158.0	46.6	1989-III	4845.6	3466.9	3241.1	195.7	129.1
1978-IV	3779.6	2680.9	2421.6	167.8	48.9	1989-IV	4859.7	3493.0	3241.6	203.0	130.7
1979-I	3780.8	2699.2	2437.9	168.2	50.5	1990-I	4880.8	3531.4	3258.8	199.1	132.3
1979-II	3784.3	2697.6	2435.4	174.1	51.8	1990-II	4900.3	3545.3	3258.6	193.7	132.5
1979-III	3807.5	2715.3	2454.7	178.1	52.7	1990-III	4903.3	3547.0	3281.2	196.3	133.8
1979-IV	3814.6	2728.1	2465.4	173.4	54.5	1990-IV	4855.1	3529.5	3251.8	199.0	136.2
1980-I	3830.8	2742.9	2464.6	174.3	57.6	1991-I	4824.0	3514.8	3241.1	189.7	137.8
1980-II	3732.6	2692.0	2414.2	144.5	58.7	1991-II	4840.7	3537.4	3252.4	182.7	136.7
1980-III	3733.5	2722.5	2440.3	151.0	59.3	1991-III	4862.7	3539.9	3271.2	189.6	138.1
1980-IV	3808.5	2777.0	2469.2	154.6	60.5	1991-IV	4868.0	3547.5	3271.1	190.3	138.5

Notes: GDP (Gross Domestic Product), billions of 1987 dollars, p. A-96. PDI (Personal disposable income), billions of 1987 dollars, p. A-112. PCE (Personal consumption expenditure), billions of 1987 dollars, p. A-96. Profits (corporate profits after tax), billions of dollars, p. A-110. Dividends (net corporate dividend payments), billions of dollars, p. A-110.

Source: U.S. Department of Commerce, Bureau of Economic Analysis, *Business Statistics, 1963-1991*, June 1992.



Figure 21.1 is a plot of the data for GDP, PDI, and PCE, and Figure 21.2 presents the other two time series. A visual plot of the data is usually the first step in the analysis of any time series. The first impression that we get from these graphs is that all the time series shown in Figures 21.1 and 21.2 seem to be “trending” upward, albeit with fluctuations. Suppose we wanted to speculate on the shape of these curves over the quarterly period, say, from

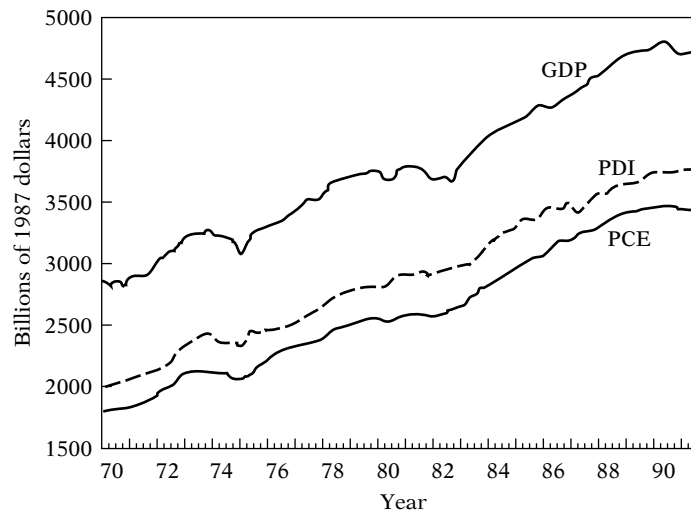


FIGURE 21.1 GDP, PDI, and PCE, United States, 1970–1991 (quarterly).

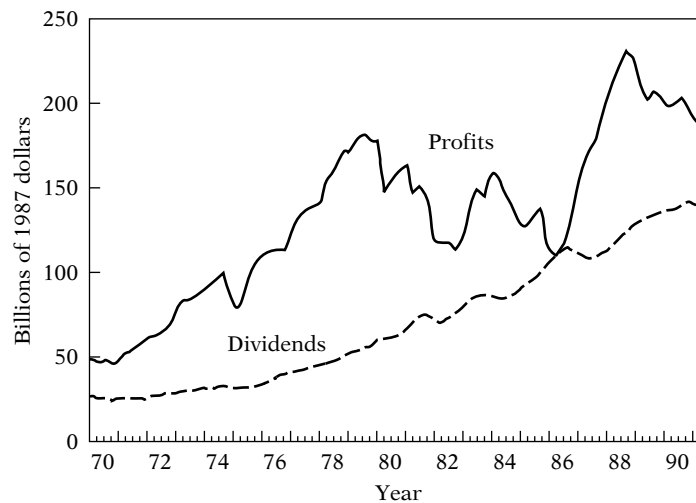


FIGURE 21.2 Profits and dividends, United States, 1970–1991 (quarterly).

1992–I to 1996–IV.<sup>2</sup> Can we simply mentally extend the curves shown in the above figures? Perhaps we can if we know the statistical, or stochastic, mechanism, or the **data generating process (DGP)**, that generated these curves? But what is that mechanism? To answer this and related questions, we need to study some “new” vocabulary that has been developed by time series analysts, to which we now turn.

## 21.2 KEY CONCEPTS<sup>3</sup>

What is this vocabulary? It consists of concepts such as these:

1. Stochastic processes
2. Stationarity processes
3. Purely random processes
4. Nonstationary processes
5. Integrated variables
6. Random walk models
7. Cointegration
8. Deterministic and stochastic trends
9. Unit root tests

In what follows we will discuss each of these concepts. Our discussion will often be heuristic. Wherever possible and helpful, we will provide appropriate examples.

## 21.3 STOCHASTIC PROCESSES

*A random or stochastic process is a collection of random variables ordered in time.*<sup>4</sup> If we let  $Y$  denote a random variable, and if it is continuous, we denote it as  $Y(t)$ , but if it is discrete, we denoted it as  $Y_t$ . An example of the former is an electrocardiogram, and an example of the latter is GDP, PDI, etc. Since most economic data are collected at discrete points in time, for our purpose we will use the notation  $Y_t$  rather than  $Y(t)$ . If we let  $Y$  represent GDP, for our data we have  $Y_1, Y_2, Y_3, \dots, Y_{86}, Y_{87}, Y_{88}$ , where the subscript 1 denotes the first observation (i.e., GDP for the first quarter of 1970) and the subscript 88 denotes the last observation (i.e., GDP for the fourth quarter of 1991). *Keep in mind that each of these  $Y$ 's is a random variable.*

In what sense can we regard GDP as a stochastic process? Consider for instance the GDP of \$2872.8 billion for 1970–I. In theory, the GDP figure for

<sup>2</sup>Of course, we have the actual data for this period now and could compare it with the data that is “predicted” on the basis of the earlier period.

<sup>3</sup>The following discussion is based on Maddala et al., op. cit., and Charemza et al., op. cit.

<sup>4</sup>The term “stochastic” comes from the Greek word “stokhos,” which means a target or bull’s-eye. If you have ever thrown darts on a dart board with the aim of hitting the bull’s-eye, how often did you hit the bull’s-eye? Out of a hundred darts you may be lucky to hit the bull’s-eye only a few times; at other times the darts will be spread randomly around the bull’s-eye.

the first quarter of 1970 could have been any number, depending on the economic and political climate then prevailing. The figure of 2872.8 is a particular **realization** of all such possibilities.<sup>5</sup> Therefore, we can say that GDP is a stochastic process and the actual values we observed for the period 1970–I to 1991–IV are a particular realization of that process (i.e., sample). The distinction between the stochastic process and its realization is akin to the distinction between population and sample in cross-sectional data. Just as we use sample data to draw inferences about a population, in time series we use the realization to draw inferences about the underlying stochastic process.

### Stationary Stochastic Processes

A type of stochastic process that has received a great deal of attention and scrutiny by time series analysts is the so-called **stationary stochastic process**. Broadly speaking, *a stochastic process is said to be stationary if its mean and variance are constant over time and the value of the covariance between the two time periods depends only on the distance or gap or lag between the two time periods and not the actual time at which the covariance is computed*. In the time series literature, such a stochastic process is known as a **weakly stationary**, or **covariance stationary**, or **second-order stationary**, or **wide sense, stochastic process**. For the purpose of this chapter, and in most practical situations, this type of stationarity often suffices.<sup>6</sup>

To explain weak stationarity, let  $Y_t$  be a stochastic time series with these properties:

$$\text{Mean:} \quad E(Y_t) = \mu \quad (21.3.1)$$

$$\text{Variance:} \quad \text{var}(Y_t) = E(Y_t - \mu)^2 = \sigma^2 \quad (21.3.2)$$

$$\text{Covariance:} \quad \gamma_k = E[(Y_t - \mu)(Y_{t+k} - \mu)] \quad (21.3.3)$$

where  $\gamma_k$ , the covariance (or autocovariance) at lag  $k$ , is the covariance between the values of  $Y_t$  and  $Y_{t+k}$ , that is, between two  $Y$  values  $k$  periods apart. If  $k = 0$ , we obtain  $\gamma_0$ , which is simply the variance of  $Y (= \sigma^2)$ ; if  $k = 1$ ,  $\gamma_1$  is the covariance between two adjacent values of  $Y$ , the type of covariance we encountered in Chapter 12 (recall the Markov first-order autoregressive scheme).

Suppose we shift the origin of  $Y$  from  $Y_t$  to  $Y_{t+m}$  (say, from the first quarter of 1970 to the first quarter of 1975 for our GDP data). Now if  $Y_t$  is to be stationary, the mean, variance, and autocovariances of  $Y_{t+m}$  must be the

<sup>5</sup>You can think of the value of \$2872.8 billion as the mean value of all possible values of GDP for the first quarter of 1970.

<sup>6</sup>A time series is strictly stationary if *all* the moments of its probability distribution and not just the first two (i.e., mean and variance) are invariant over time. If, however, the stationary process is normal, the weakly stationary stochastic process is also strictly stationary, for the normal stochastic process is fully specified by its two moments, the mean and the variance.

same as those of  $Y_t$ . In short, if a time series is stationary, its mean, variance, and autocovariance (at various lags) remain the same no matter at what point we measure them; that is, they are time invariant. Such a time series will tend to return to its mean (called **mean reversion**) and fluctuations around this mean (measured by its variance) will have a broadly constant amplitude.<sup>7</sup>

If a time series is not stationary in the sense just defined, it is called a **nonstationary time series** (keep in mind we are talking only about weak stationarity). In other words, a nonstationary time series will have a *time-varying mean or a time-varying variance or both*.

Why are stationary time series so important? Because if a time series is nonstationary, we can study its behavior only for the time period under consideration. Each set of time series data will therefore be for a particular episode. As a consequence, it is not possible to generalize it to other time periods. Therefore, for the purpose of forecasting, such (nonstationary) time series may be of little practical value.

How do we know that a particular time series is stationary? In particular, are the time series shown in Figures 21.1 and 21.2 stationary? We will take this important topic up in Sections 21.8 and 21.9, where we will consider several tests of stationarity. But if we depend on common sense, it would seem that the time series depicted in Figures 21.1 and 21.2 are nonstationary, at least in the mean values. But more on this later.

Before we move on, we mention a special type of stochastic process (or time series), namely, a **purely random**, or **white noise**, process. We call a stochastic process purely random if it has zero mean, constant variance  $\sigma^2$ , and is serially uncorrelated.<sup>8</sup> You may recall that the error term  $u_t$ , entering the classical normal linear regression model that we discussed in **Part I** of this book was assumed to be a white noise process, which we denoted as  $u_t \sim \text{IIDN}(0, \sigma^2)$ ; that is,  $u_t$  is independently and identically distributed as a normal distribution with zero mean and constant variance.

### Nonstationary Stochastic Processes

Although our interest is in stationary time series, one often encounters nonstationary time series, the classic example being the **random walk model** (RWM).<sup>9</sup> It is often said that asset prices, such as stock prices or exchange rates, follow a random walk; that is, they are nonstationary. We distinguish two types of random walks: (1) random walk without drift (i.e., no constant or intercept term) and (2) random walk with drift (i.e., a constant term is present).

<sup>7</sup>This point has been made by Keith Cuthbertson, Stephen G. Hall, and Mark P. Taylor, *Applied Econometric Techniques*, The University of Michigan Press, 1995, p. 130.

<sup>8</sup>If it is also independent, such a process is called **strictly white noise**.

<sup>9</sup>The term random walk is often compared with a drunkard's walk. Leaving a bar, the drunkard moves a random distance  $u_t$  at time  $t$ , and, continuing to walk indefinitely, will eventually drift farther and farther away from the bar. The same is said about stock prices. Today's stock price is equal to yesterday's stock price plus a random shock.

**Random Walk without Drift.** Suppose  $u_t$  is a white noise error term with mean 0 and variance  $\sigma^2$ . Then the series  $Y_t$  is said to be a random walk if

$$Y_t = Y_{t-1} + u_t \quad (21.3.4)$$

In the random walk model, as (21.3.4) shows, the value of  $Y$  at time  $t$  is equal to its value at time  $(t - 1)$  plus a random shock; thus it is an AR(1) model in the language of Chapters 12 and 17. We can think of (21.3.4) as a regression of  $Y$  at time  $t$  on its value lagged one period. Believers in the **efficient capital market hypothesis** argue that stock prices are essentially random and therefore there is no scope for profitable speculation in the stock market: If one could predict tomorrow's price on the basis of today's price, we would all be millionaires.

Now from (21.3.4) we can write

$$Y_1 = Y_0 + u_1$$

$$Y_2 = Y_1 + u_2 = Y_0 + u_1 + u_2$$

$$Y_3 = Y_2 + u_3 = Y_0 + u_1 + u_2 + u_3$$

In general, if the process started at some time 0 with a value of  $Y_0$ , we have

$$Y_t = Y_0 + \sum u_t \quad (21.3.5)$$

Therefore,

$$E(Y_t) = E\left(Y_0 + \sum u_t\right) = Y_0 \quad (\text{why?}) \quad (21.3.6)$$

In like fashion, it can be shown that

$$\text{var}(Y_t) = t\sigma^2 \quad (21.3.7)$$

As the preceding expression shows, the mean of  $Y$  is equal to its initial, or starting, value, which is constant, but as  $t$  increases, its variance increases indefinitely, thus violating a condition of stationarity. In short, the RWM without drift is a nonstationary stochastic process. In practice  $Y_0$  is often set at zero, in which case  $E(Y_t) = 0$ .

An interesting feature of RWM is the *persistence of random shocks* (i.e., random errors), which is clear from (21.3.5):  $Y_t$  is the sum of initial  $Y_0$  plus the sum of random shocks. As a result, the impact of a particular shock does not die away. For example, if  $u_2 = 2$  rather than  $u_2 = 0$ , then all  $Y_t$ 's from  $Y_2$  onward will be 2 units higher and the effect of this shock never dies out. That is why random walk is said to have an *infinite memory*. As Kerry Patterson notes, random walk remembers the shock forever<sup>10</sup>; that is, it has infinite memory.

<sup>10</sup>Kerry Patterson, op cit., Chap. 6.

Interestingly, if you write (21.3.4) as

$$(Y_t - Y_{t-1}) = \Delta Y_t = u_t \quad (21.3.8)$$

where  $\Delta$  is the first difference operator that we discussed in Chapter 12. It is easy to show that, while  $Y_t$  is nonstationary, its first difference is stationary. In other words, the first differences of a random walk time series are stationary. But we will have more to say about this later.

**Random Walk with Drift.** Let us modify (21.3.4) as follows:

$$Y_t = \delta + Y_{t-1} + u_t \quad (21.3.9)$$

where  $\delta$  is known as the **drift parameter**. The name drift comes from the fact that if we write the preceding equation as

$$Y_t - Y_{t-1} = \Delta Y_t = \delta + u_t \quad (21.3.10)$$

it shows that  $Y_t$  drifts upward or downward, depending on  $\delta$  being positive or negative. Note that model (21.3.9) is also an AR(1) model.

Following the procedure discussed for random walk without drift, it can be shown that for the random walk with drift model (21.3.9),

$$E(Y_t) = Y_0 + t \cdot \delta \quad (21.3.11)$$

$$\text{var}(Y_t) = t\sigma^2 \quad (21.3.12)$$

As you can see, for RWM with drift the mean as well as the variance increases over time, again violating the conditions of (weak) stationarity. In short, RWM, with or without drift, is a nonstationary stochastic process.

To give a glimpse of the random walk with and without drift, we conducted two simulations as follows:

$$Y_t = Y_0 + u_t \quad (21.3.13)$$

where  $u_t$  are white noise error terms such that each  $u_t \sim N(0, 1)$ ; that is, each  $u_t$  follows the standard normal distribution. From a random number generator, we obtained 500 values of  $u$  and generated  $Y_t$  as shown in (21.3.13). We assumed  $Y_0 = 0$ . Thus, (21.3.13) is an RWM without drift.

Now consider

$$Y_t = \delta + Y_0 + u_t \quad (21.3.14)$$

which is RWM with drift. We assumed  $u_t$  and  $Y_0$  as in (21.3.13) and assumed that  $\delta = 2$ .

The graphs of models (21.3.13) and (21.3.14), respectively, are in Figures 21.3 and 21.4. The reader can compare these two diagrams in light of our discussion of the RWM with and without drift.

The random walk model is an example of what is known in the literature as a **unit root process**. Since this term has gained tremendous currency in the time series literature, we next explain what a unit root process is.

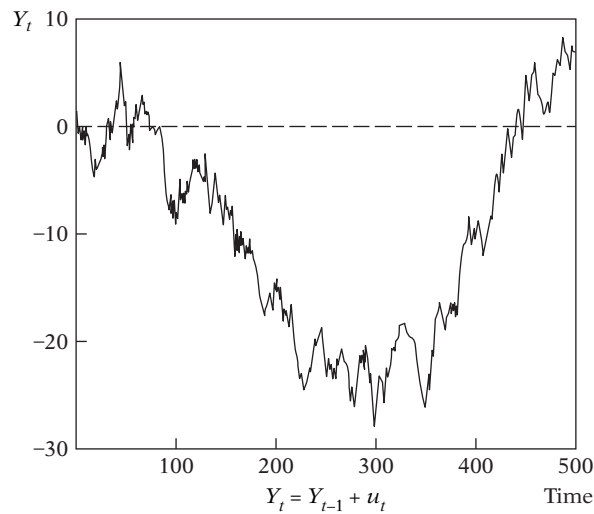


FIGURE 21.3 A random walk without drift.

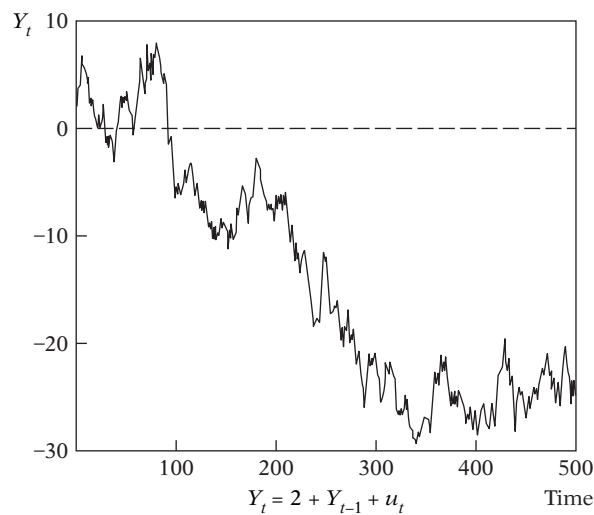


FIGURE 21.4 A random walk with drift.

## 21.4 UNIT ROOT STOCHASTIC PROCESS

Let us write the RWM (21.3.4) as:

$$Y_t = \rho Y_{t-1} + u_t \quad -1 \leq \rho \leq 1 \quad (21.4.1)$$

This model resembles the Markov first-order autoregressive model that we discussed in the chapter on autocorrelation. If  $\rho = 1$ , (21.4.1) becomes a RWM (without drift). If  $\rho$  is in fact 1, we face what is known as the **unit root problem**, that is, a situation of nonstationarity; we already know that in this case the variance of  $Y_t$  is not stationary. The name unit root is due to the fact that  $\rho = 1$ .<sup>11</sup> Thus the terms *nonstationarity*, *random walk*, and *unit root* can be treated as synonymous.

If, however,  $|\rho| \leq 1$ , that is if the absolute value of  $\rho$  is less than one, then it can be shown that the time series  $Y_t$  is stationary in the sense we have defined it.<sup>12</sup>

In practice, then, it is important to find out if a time series possesses a unit root.<sup>13</sup> In Section 21.9 we will discuss several tests of unit root, that is, several tests of stationarity. In that section we will also determine whether the time series depicted in Figures 21.1 and 21.2 are stationary. Perhaps the reader might suspect that they are not. But we shall see.

## 21.5 TREND STATIONARY (TS) AND DIFFERENCE STATIONARY (DS) STOCHASTIC PROCESSES

The distinction between stationary and nonstationary stochastic processes (or time series) has a crucial bearing on whether the trend (the slow long-run evolution of the time series under consideration) observed in the constructed time series in Figures 21.3 and 21.4 or in the actual economic time series of Figures 21.1 and 21.2 is **deterministic** or **stochastic**. Broadly speaking, if the trend in a time series is completely predictable and not variable, we call it a deterministic trend, whereas if it is not predictable, we call it a stochastic trend. To make the definition more formal, consider the following model of the time series  $Y_t$ .

$$Y_t = \beta_1 + \beta_2 t + \beta_3 Y_{t-1} + u_t \quad (21.5.1)$$

<sup>11</sup>A technical point: If  $\rho = 1$ , we can write (21.4.1) as  $Y_t - Y_{t-1} = u_t$ . Now using the **lag operator**  $L$  so that  $LY_t = Y_{t-1}$ ,  $L^2 Y_t = Y_{t-2}$ , and so on, we can write (21.4.1) as  $(1 - L)Y_t = u_t$ . The term unit root refers to the root of the polynomial in the lag operator. If you set  $(1 - L) = 0$ , we obtain,  $L = 1$ , hence the name unit root.

<sup>12</sup>If in (21.4.1) it is assumed that the initial value of  $Y$  ( $= Y_0$ ) is zero,  $|\rho| \leq 1$ , and  $u_t$  is white noise and distributed normally with zero mean and unit variance, then it follows that  $E(Y_t) = 0$  and  $\text{var}(Y_t) = 1/(1 - \rho^2)$ . Since both these are constants, by the definition of weak stationarity,  $Y_t$  is stationary. On the other hand, as we saw before, if  $\rho = 1$ ,  $Y_t$  is a random walk or nonstationary.

<sup>13</sup>A time series may contain more than one unit root. But we will discuss this situation later in the chapter.



where  $u_t$  is a white noise error term and where  $t$  is time measured chronologically. Now we have the following possibilities:

**Pure random walk:** If in (21.5.1)  $\beta_1 = 0$ ,  $\beta_2 = 0$ ,  $\beta_3 = 1$ , we get

$$Y_t = Y_{t-1} + u_t \quad (21.5.2)$$

which is nothing but a RWM without drift and is therefore nonstationary. But note that, if we write (21.5.2) as

$$\Delta Y_t = (Y_t - Y_{t-1}) = u_t \quad (21.3.8)$$

it becomes stationary, as noted before. Hence, a RWM without drift is a **difference stationary process (DSP)**.

**Random walk with drift:** If in (21.5.1)  $\beta_1 \neq 0$ ,  $\beta_2 = 0$ ,  $\beta_3 = 1$ , we get

$$Y_t = \beta_1 + Y_{t-1} + u_t \quad (21.5.3)$$

which is a random walk with drift and is therefore nonstationary. If we write it as

$$(Y_t - Y_{t-1}) = \Delta Y_t = \beta_1 + u_t \quad (21.5.3a)$$

this means  $Y_t$  will exhibit a positive ( $\beta_1 > 0$ ) or negative ( $\beta_1 < 0$ ) trend (see Figure 21.4). Such a trend is called a **stochastic trend**. Equation (21.5.3a) is a DSP process because the nonstationarity in  $Y_t$  can be eliminated by taking first differences of the time series.

**Deterministic trend:** If in (21.5.1),  $\beta_1 \neq 0$ ,  $\beta_2 \neq 0$ ,  $\beta_3 = 0$ , we obtain

$$Y_t = \beta_1 + \beta_2 t + u_t \quad (21.5.4)$$

which is called a **trend stationary process (TSP)**. Although the mean of  $Y_t$  is  $\beta_1 + \beta_2 t$ , which is not constant, its variance ( $= \sigma^2$ ) is. Once the values of  $\beta_1$  and  $\beta_2$  are known, the mean can be forecast perfectly. Therefore, if we subtract the mean of  $Y_t$  from  $Y_t$ , the resulting series will be stationary, hence the name **trend stationary**. This procedure of removing the (deterministic) trend is called **detrending**.

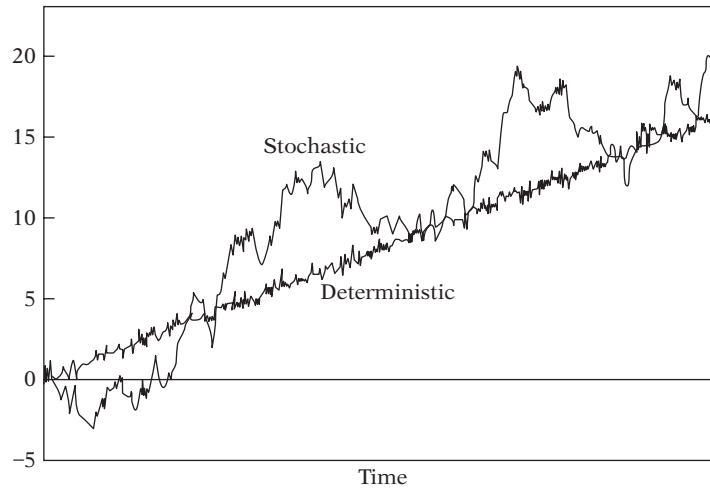
**Random walk with drift and deterministic trend:** If in (21.5.1),  $\beta_1 \neq 0$ ,  $\beta_2 \neq 0$ ,  $\beta_3 = 1$ , we obtain:

$$Y_t = \beta_1 + \beta_2 t + Y_{t-1} + u_t \quad (21.5.5)$$

we have a random walk with drift and a deterministic trend, which can be seen if we write this equation as

$$\Delta Y_t = \beta_1 + \beta_2 t + u_t \quad (21.5.5a)$$

which means that  $Y_t$  is nonstationary.



**FIGURE 21.5** Deterministic versus stochastic trend.

Source: Charemza et al., op. cit., p. 91.

**Deterministic trend with stationary AR(1) component:** If in (21.5.1)  $\beta_1 \neq 0$ ,  $\beta_2 \neq 0$ ,  $\beta_3 < 1$ , then we get

$$Y_t = \beta_1 + \beta_2 t + \beta_3 Y_{t-1} + u_t \quad (21.5.6)$$

which is stationary around the deterministic trend.

To see the difference between stochastic and deterministic trends, consider Figure 21.5.<sup>14</sup> The series named stochastic in this figure is generated by an RWM:  $Y_t = 0.5 + Y_{t-1} + u_t$ , where 500 values of  $u_t$  were generated from a standard normal distribution and where the initial value of  $Y$  was set at 1. The series named deterministic is generated as follows:  $Y_t = 0.5t + u_t$ , where  $u_t$  were generated as above and where  $t$  is time measured chronologically.

As you can see from Figure 21.5, in the case of the deterministic trend, the deviations from the trend line (which represents nonstationary mean) are purely random and they die out quickly; they do not contribute to the long-run development of the time series, which is determined by the trend component  $0.5t$ . In the case of the stochastic trend, on the other hand, the random component  $u_t$  affects the long-run course of the series  $Y_t$ .

## 21.6 INTEGRATED STOCHASTIC PROCESSES

The random walk model is but a specific case of a more general class of stochastic processes known as **integrated processes**. Recall that the RWM without drift is nonstationary, but its first difference, as shown in (21.3.8), is

<sup>14</sup>The following discussion is based on Wojciech W. Charemza et al., op. cit., pp. 89–91.

stationary. Therefore, we call the RWM without drift **integrated of order 1**, denoted as  $I(1)$ . Similarly, if a time series has to be differenced twice (i.e., take the first difference of the first differences) to make it stationary, we call such a time series **integrated of order 2**.<sup>15</sup> In general, if a (nonstationary) time series has to be differenced  $d$  times to make it stationary, that time series is said to be **integrated of order  $d$** . A time series  $Y_t$  integrated of order  $d$  is denoted as  $Y_t \sim I(d)$ . If a time series  $Y_t$  is stationary to begin with (i.e., it does not require any differencing), it is said to be integrated of order zero, denoted by  $Y_t \sim I(0)$ . Thus, we will use the terms “stationary time series” and “time series integrated of order zero” to mean the same thing.

Most economic time series are generally  $I(1)$ ; that is, they generally become stationary only after taking their first differences. Are the time series shown in Figures 21.1 and 21.2  $I(1)$  or of higher order? We will examine them in Sections 21.8 and 21.9.

### Properties of Integrated Series

The following properties of integrated time series may be noted: Let  $X_t$ ,  $Y_t$ , and  $Z_t$  be three time series.

**1.** If  $X_t \sim I(0)$  and  $Y_t \sim I(1)$ , then  $Z_t = (X_t + Y_t) = I(1)$ ; that is, a linear combination or sum of stationary and nonstationary time series is nonstationary.

**2.** If  $X_t \sim I(d)$ , then  $Z_t = (a + bX_t) = I(d)$ , where  $a$  and  $b$  are constants. That is, a linear combination of an  $I(d)$  series is also  $I(d)$ . Thus, if  $X_t \sim I(0)$ , then  $Z_t = (a + bX_t) \sim I(0)$ .

**3.** If  $X_t \sim I(d_1)$  and  $Y_t \sim I(d_2)$ , then  $Z_t = (aX_t + bY_t) \sim I(d_2)$ , where  $d_1 < d_2$ .

**4.** If  $X_t \sim I(d)$  and  $Y_t \sim I(d)$ , then  $Z_t = (aX_t + bY_t) \sim I(d^*)$ ;  $d^*$  is generally equal to  $d$ , but in some cases  $d^* < d$  (see the topic of cointegration in Section 21.11).

As you can see from the preceding statements, one has to pay careful attention in combining two or more time series that are integrated of different order.

To see why this is important, consider the two-variable regression model discussed in Chapter 3, namely,  $Y_t = \beta_1 + \beta_2 X_t + u_t$ . Under the classical OLS assumptions, we know that

$$\hat{\beta}_2 = \frac{\sum x_t y_t}{\sum x_t^2} \quad (21.6.1)$$

where the small letters, as usual, indicate deviation from mean values. Suppose  $Y_t$  is  $I(0)$ , but  $X_t$  is  $I(1)$ ; that is, the former is stationary and the latter is

<sup>15</sup>For example if  $Y_t$  is  $I(2)$ , then  $\Delta\Delta Y_t = \Delta(Y_t - Y_{t-1}) = \Delta Y_t - \Delta Y_{t-1} = Y_t - 2Y_{t-1} + Y_{t-2}$  will become stationary. But note that  $\Delta\Delta Y_t = \Delta^2 Y_t \neq Y_t - Y_{t-2}$ .

not. Since  $X_t$  is nonstationary, its variance will increase indefinitely, thus dominating the numerator term in (21.6.1) with the result that  $\hat{\beta}_2$  will converge to zero asymptotically (i.e., in large samples) and it will not even have an asymptotic distribution.<sup>16</sup>

## 21.7 THE PHENOMENON OF SPURIOUS REGRESSION

To see why stationary time series are so important, consider the following two random walk models:

$$Y_t = Y_{t-1} + u_t \quad (21.7.1)$$

$$X_t = X_{t-1} + v_t \quad (21.7.2)$$

where we generated 500 observations of  $u_t$  from  $u_t \sim N(0, 1)$  and 500 observations of  $v_t$  from  $v_t \sim N(0, 1)$  and assumed that the initial values of both  $Y$  and  $X$  were zero. We also assumed that  $u_t$  and  $v_t$  are serially uncorrelated as well as mutually uncorrelated. As you know by now, both these time series are nonstationary; that is, they are  $I(1)$  or exhibit stochastic trends.

Suppose we regress  $Y_t$  on  $X_t$ . Since  $Y_t$  and  $X_t$  are uncorrelated  $I(1)$  processes, the  $R^2$  from the regression of  $Y$  on  $X$  should tend to zero; that is, there should not be any relationship between the two variables. But wait till you see the regression results:

Variable	Coefficient	Std. error	t statistic
C	-13.2556	0.6203	-21.36856
X	0.3376	0.0443	7.61223
$R^2 = 0.1044$		$d = 0.0121$	

As you can see, the coefficient of  $X$  is highly statistically significant, and, although the  $R^2$  value is low, it is statistically significantly different from zero. From these results, you may be tempted to conclude that there is a significant statistical relationship between  $Y$  and  $X$ , whereas a priori there should be none. This is in a nutshell the **phenomenon of spurious or nonsense regression**, first discovered by Yule.<sup>17</sup> Yule showed that (spurious) correlation could persist in nonstationary time series even if the sample is very large. That there is something wrong in the preceding regression is suggested by the extremely low Durbin-Watson  $d$  value, which suggests very

<sup>16</sup>This point is due to Maddala et al., op. cit., p. 26.

<sup>17</sup>Yule, G. U., "Why Do We Sometimes Get Nonsense Correlations Between Time Series? A Study in Sampling and the Nature of Time Series," *Journal of the Royal Statistical Society*, vol. 89, 1926, pp. 1-64. For extensive Monte Carlo simulations on spurious regression see C. W. J. Granger and P. Newbold, "Spurious Regressions in Econometrics," *Journal of Econometrics*, vol. 2, 1974, pp. 111-120.

strong first-order autocorrelation. According to Granger and Newbold, *an  $R^2 > d$  is a good rule of thumb to suspect that the estimated regression is spurious*, as in the example above.

That the regression results presented above are meaningless can be easily seen from regressing the first differences of  $Y_t (= \Delta Y_t)$  on the first differences of  $X_t (= \Delta X_t)$ ; remember that although  $Y_t$  and  $X_t$  are nonstationary, their first differences are stationary. In such a regression you will find that  $R^2$  is practically zero, as it should be, and the Durbin–Watson  $d$  is about 2. In Exercise 21.24 you are asked to run this regression and verify the statement just made.

Although dramatic, this example is a strong reminder that one should be extremely wary of conducting regression analysis based on time series that exhibit stochastic trends. And one should therefore be extremely cautious in reading too much in the regression results based on  $I(1)$  variables. For an example, see exercise 21.26. To some extent, this is true of time series subject to deterministic trends, an example of which is given in exercise 21.25.

## 21.8 TESTS OF STATIONARITY

By now the reader probably has a good idea about the nature of stationary stochastic processes and their importance. In practice we face two important questions: (1) How do we find out if a given time series is stationary? (2) If we find that a given time series is not stationary, is there a way that it can be made stationary? We take up the first question in this section and discuss the second question in Section 21.10.

Before we proceed, keep in mind that we are primarily concerned with weak, or covariance, stationarity.

Although there are several tests of stationarity, we discuss only those that are prominently discussed in the literature. In this section we discuss two tests: (1) graphical analysis and (2) the correlogram test. Because of the importance attached to it in the recent past, we discuss the *unit root test* in the next section. We illustrate these tests with appropriate examples.

### 1. Graphical Analysis

As noted earlier, before one pursues formal tests, it is always advisable to plot the time series under study, as we have done in Figures 21.1 and 21.2 for the data given in Table 21.1. Such a plot gives an initial clue about the likely nature of the time series. Take, for instance, the GDP time series shown in Figure 21.1. You will see that over the period of study GDP has been increasing, that is, showing an upward trend, suggesting perhaps that the mean of the GDP has been changing. This perhaps suggests that the GDP series is not stationary. This is also more or less true of the other U.S. economic time series shown in Figure 21.2. Such an intuitive feel is the starting point of more formal tests of stationarity.

## 2. Autocorrelation Function (ACF) and Correlogram

One simple test of stationarity is based on the so-called **autocorrelation function (ACF)**. The ACF at lag  $k$ , denoted by  $\rho_k$ , is defined as

$$\begin{aligned}\rho_k &= \frac{\gamma_k}{\gamma_0} \\ &= \frac{\text{covariance at lag } k}{\text{variance}}\end{aligned}\quad (21.8.1)$$

where covariance at lag  $k$  and variance are as defined before. Note that if  $k = 0$ ,  $\rho_0 = 1$  (why?)

Since both covariance and variance are measured in the same units of measurement,  $\rho_k$  is a *unitless*, or *pure, number*. It lies between  $-1$  and  $+1$ , as any correlation coefficient does. If we plot  $\rho_k$  against  $k$ , the graph we obtain is known as the **population correlogram**.

Since in practice we only have a realization (i.e., sample) of a stochastic process, we can only compute the **sample autocorrelation function (SAFC)**,  $\hat{\rho}_k$ . To compute this, we must first compute the **sample covariance** at lag  $k$ ,  $\hat{\gamma}_k$ , and the **sample variance**,  $\hat{\gamma}_0$ , which are defined as<sup>18</sup>

$$\hat{\gamma}_k = \frac{\sum(Y_t - \bar{Y})(Y_{t+k} - \bar{Y})}{n} \quad (21.8.2)$$

$$\hat{\gamma}_0 = \frac{\sum(Y_t - \bar{Y})^2}{n} \quad (21.8.3)$$

where  $n$  is the sample size and  $\bar{Y}$  is the sample mean.

Therefore, the sample autocorrelation function at lag  $k$  is

$$\hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\gamma}_0} \quad (21.8.4)$$

which is simply the ratio of sample covariance (at lag  $k$ ) to sample variance. A plot of  $\hat{\rho}_k$  against  $k$  is known as the **sample correlogram**.

How does a sample correlogram enable us to find out if a particular time series is stationary? For this purpose, let us first present the sample correlograms of a purely white noise random process and of a random walk process. Return to the driftless RWM (21.3.13). There we generated a sample of 500 error terms, the  $u$ 's, from the standard normal distribution. The correlogram of these 500 purely random error terms is as shown in Figure 21.6; we have shown this correlogram up to 30 lags. We will comment shortly on how one chooses the lag length.

For the time being, just look at the column labeled AC, which is the sample autocorrelation function, and the first diagram on the left, labeled

<sup>18</sup>Strictly speaking, we should divide the sample covariance at lag  $k$  by  $(n - k)$  and the sample variance by  $(n - 1)$  rather than by  $n$  (why?) where  $n$  is the sample size.

Sample: 2 500  
Included observations: 499

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob	
		1	-0.022	-0.022	0.2335	0.629
		2	-0.019	-0.020	0.4247	0.809
		3	-0.009	-0.010	0.4640	0.927
		4	-0.031	-0.031	0.9372	0.919
		5	-0.070	-0.072	3.4186	0.636
		6	-0.008	-0.013	3.4493	0.751
		7	0.048	0.045	4.6411	0.704
		8	-0.069	-0.070	7.0385	0.532
		9	0.022	0.017	7.2956	0.606
		10	-0.004	-0.011	7.3059	0.696
		11	0.024	0.025	7.6102	0.748
		12	0.024	0.027	7.8993	0.793
		13	0.026	0.021	8.2502	0.827
		14	-0.047	-0.046	9.3726	0.806
		15	-0.037	-0.030	10.074	0.815
		16	-0.026	-0.031	10.429	0.843
		17	-0.029	-0.024	10.865	0.863
		18	-0.043	-0.050	11.807	0.857
		19	0.038	0.028	12.575	0.860
		20	0.099	0.093	17.739	0.605
		21	0.001	0.007	17.739	0.665
		22	0.065	0.060	19.923	0.588
		23	0.053	0.055	21.404	0.556
		24	-0.017	-0.004	21.553	0.606
		25	-0.024	-0.005	21.850	0.644
		26	-0.008	-0.008	21.885	0.695
		27	-0.036	-0.027	22.587	0.707
		28	0.053	0.072	24.068	0.678
		29	-0.004	-0.011	24.077	0.725
		30	-0.026	-0.025	24.445	0.752

**FIGURE 21.6** Correlogram of white noise error term  $u$ . AC = autocorrelation, PAC = partial autocorrelation (see Chapter 22), Q-Stat =  $Q$  statistic, Prob = probability.

autocorrelation. The solid vertical line in this diagram represents the zero axis; observations above the line are positive values and those below the line are negative values. As is very clear from this diagram, for a purely white noise process the autocorrelations at various lags hover around zero. *This is the picture of a correlogram of a stationary time series.* Thus, if the correlogram of an actual (economic) time series resembles the correlogram of a white noise time series, we can say that time series is probably stationary.

Now look at the correlogram of a random walk series, as generated, say, by (21.3.13). The picture is as shown in Figure 21.7. The most striking feature of this correlogram is that the autocorrelation coefficients at various lags are very high even up to a lag of 33 quarters. As a matter of fact, if we consider lags of up to 60 quarters, the autocorrelation coefficients are quite high; the

Sample: 2 500  
Included observations: 499

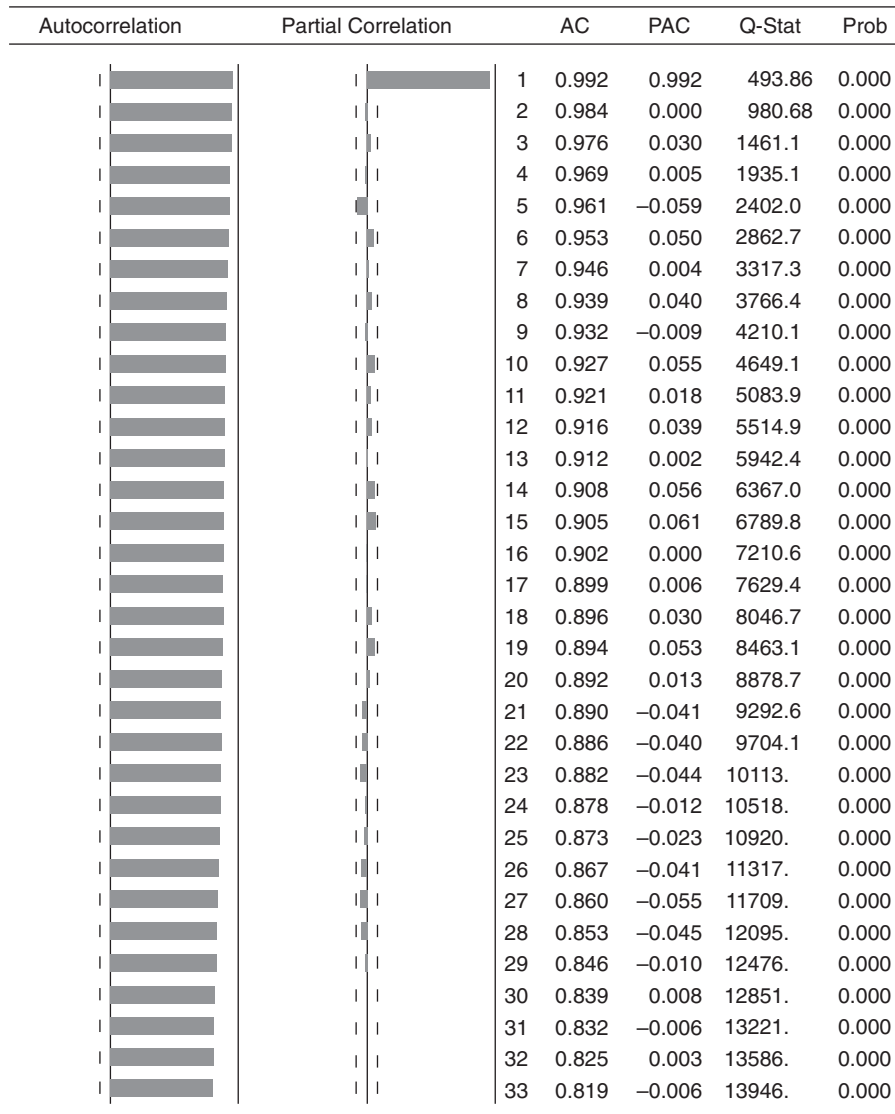


FIGURE 21.7 Correlogram of a random walk time series. See Figure 21.6 for definitions.



coefficient is about 0.7 at lag 60. Figure 21.7 is the typical correlogram of a nonstationary time series: The autocorrelation coefficient starts at a very high value and declines very slowly toward zero as the lag lengthens.

Now let us take a concrete economic example. Let us examine the correlogram of the GDP time series given in Table 21.1. The correlogram up to 25 lags is shown in Figure 21.8. The GDP correlogram up to 25 lags also shows a pattern similar to the correlogram of the random walk model in Figure 21.7. The autocorrelation coefficient starts at a very high value at lag 1 (0.969) and declines very slowly. Thus it seems that the GDP time series is nonstationary. If you plot the correlograms of the other U.S. economic time series shown in Figures 21.1 and 21.2, you will also see a similar pattern, leading to the conclusion that all these time series are nonstationary; they may be nonstationary in mean or variance or both.

Sample: 1970–1 1991–4  
Included observations: 88

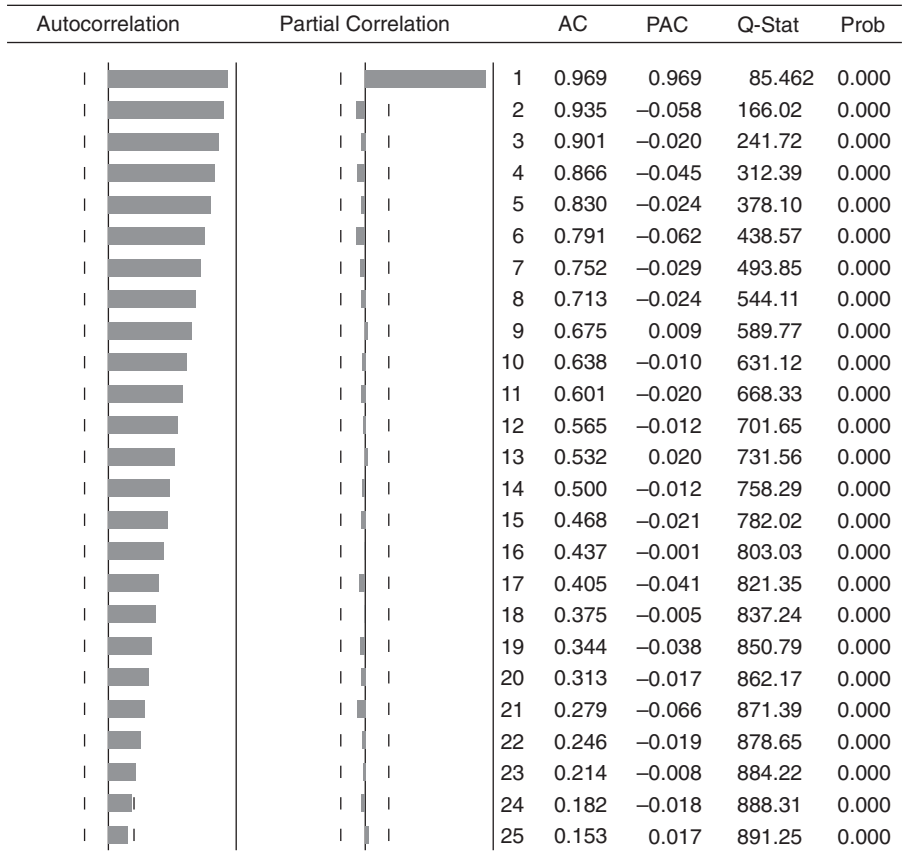


FIGURE 21.8 Correlogram of U.S. GDP, 1970–I to 1991–IV. See Figure 21.6 for definitions.

Two practical questions may be posed here. First, how do we choose the lag length to compute the ACF? Second, how do you decide whether a correlation coefficient at a certain lag is statistically significant? The answer follows.

**The Choice of Lag Length.** This is basically an empirical question. A rule of thumb is to compute ACF up to one-third to one-quarter the length of the time series. Since for our economic data we have 88 quarterly observations, by this rule lags of 22 to 29 quarters will do. The best practical advice is to start with sufficiently large lags and then reduce them by some statistical criterion, such as the *Akaike* or *Schwarz information criterion* that we discussed in Chapter 13. Alternatively, one can use the following statistical tests.

### Statistical Significance of Autocorrelation Coefficients

Consider, for instance, the correlogram of the GDP time series given in Figure 21.8. How do we decide whether the correlation coefficient of 0.638 at lag 10 (quarters) is statistically significant? The statistical significance of any  $\hat{\rho}_k$  can be judged by its standard error. Bartlett has shown that if a time series is purely random, that is, it exhibits white noise (see Figure 21.6), the sample autocorrelation coefficients  $\hat{\rho}_k$  are *approximately*<sup>19</sup>

$$\hat{\rho}_k \sim N(0, 1/n) \quad (21.8.5)$$

that is, in large samples the sample autocorrelation coefficients are normally distributed with zero mean and variance equal to one over the sample size. Since we have 88 observations, the variance is  $1/88 = 0.01136$  and the standard error is  $\sqrt{0.01136} = 0.1066$ . Then following the properties of the standard normal distribution, the 95% confidence interval for any (population)  $\rho_k$  is:

$$\hat{\rho}_k \pm 1.96(0.1066) \quad (21.8.6)$$

In other words,

$$\text{Prob}(\hat{\rho}_k - 0.2089 \leq \rho_k \leq \hat{\rho}_k + 0.2089) = 0.95 \quad (21.8.7)$$

If the preceding interval includes the value of zero, we do not reject the hypothesis that the true  $\rho_k$  is zero, but if this interval does not include 0, we reject the hypothesis that the true  $\rho_k$  is zero. Applying this to the estimated value of  $\hat{\rho}_{10} = 0.638$ , the reader can verify that the 95% confidence interval for true  $\rho_{10}$  is  $(0.638 \pm 0.2089)$  or  $(0.4291, 0.8469)$ .<sup>20</sup> Obviously, this inter-

<sup>19</sup>M. S. Bartlett, "On the Theoretical Specification of Sampling Properties of Autocorrelated Time Series," *Journal of the Royal Statistical Society, Series B*, vol. 27, 1946, pp. 27-41.

val does not include the value of zero, suggesting that we are 95% confident that the true  $\rho_{10}$  is significantly different from zero.<sup>21</sup> As you can check, even at lag 20 the estimated  $\rho_{20}$  is statistically significant at the 5% level.

Instead of testing the statistical significance of any individual autocorrelation coefficient, we can test the *joint hypothesis* that all the  $\rho_k$  up to certain lags are simultaneously equal to zero. This can be done by using the **Q statistic** developed by Box and Pierce, which is defined as<sup>22</sup>

$$Q = n \sum_{k=1}^m \hat{\rho}_k^2 \quad (21.8.8)$$

where  $n$  = sample size and  $m$  = lag length. The  $Q$  statistic is often used as a test of whether a time series is white noise. In large samples, it is *approximately* distributed as the chi-square distribution with  $m$  df. In an application, if the computed  $Q$  exceeds the critical  $Q$  value from the chi-square distribution at the chosen level of significance, one can reject the null hypothesis that all the (true)  $\rho_k$  are zero; at least some of them must be nonzero.

A variant of the Box–Pierce  $Q$  statistic is the **Ljung–Box (LB) statistic**, which is defined as<sup>23</sup>

$$LB = n(n+2) \sum_{k=1}^m \left( \frac{\hat{\rho}_k^2}{n-k} \right) \sim \chi^2 m \quad (21.8.9)$$

Although in large samples both  $Q$  and LB statistics follow the chi-square distribution with  $m$  df, the LB statistic has been found to have better (more powerful, in the statistical sense) small-sample properties than the  $Q$  statistic.<sup>24</sup>

Returning to the GDP example given in Figure 21.8, the value of the LB statistic up to lag 25 is about 891.25. The probability of obtaining such an LB value under the null hypothesis that the sum of 25 squared estimated autocorrelation coefficients is zero is practically zero, as the last column of that figures shows. Therefore, the conclusion is that the GDP time series is nonstationary, therefore reinforcing our hunch from Figure 21.1 that the GDP series may be nonstationary. In exercise 21.16 you are asked to confirm that the other four U.S. economic time series are also nonstationary.

<sup>20</sup>Our sample size of 88 observations, although not very large, is reasonably large to use the normal approximation.

<sup>21</sup>Alternatively, if you divide the estimated value of any  $\rho_k$  by the standard error of  $(\sqrt{1/n})$ , for sufficiently large  $n$ , you will obtain the standard  $Z$  value, whose probability can be easily obtained from the standard normal table. Thus for the estimated  $\rho_{10} = 0.638$ , the  $Z$  value is  $0.638/0.1066 = 5.98$  (approx.). If the true  $\rho_{10}$  were in fact zero, the probability of obtaining a  $Z$  value of as much as 5.98 or greater is very small, thus rejecting the hypothesis that the true  $\rho_{10}$  is zero.

<sup>22</sup>G. E. P. Box and D. A. Pierce, "Distribution of Residual Autocorrelations in Autoregressive Integrated Moving Average Time Series Models," *Journal of the American Statistical Association*, vol. 65, 1970, pp. 1509–1526.

<sup>23</sup>G. M. Ljung and G. P. E. Box, "On a Measure of Lack of Fit in Time Series Models," *Biometrika*, vol. 66, 1978, pp. 66–72.

<sup>24</sup>The  $Q$  and LB statistics may not be appropriate in every case. For a critique, see Maddala et al., op. cit., p. 19.

## 21.9 THE UNIT ROOT TEST

A test of stationarity (or nonstationarity) that has become widely popular over the past several years is the **unit root test**. We will first explain it, then illustrate it and then consider some limitations of this test.

The starting point is the unit root (stochastic) process that we discussed in Section 21.4. We start with

$$Y_t = \rho Y_{t-1} + u_t \quad -1 \leq \rho \leq 1 \quad (21.4.1)$$

where  $u_t$  is a white noise error term.

We know that if  $\rho = 1$ , that is, in the case of the unit root, (21.4.1) becomes a random walk model without drift, which we know is a nonstationary stochastic process. Therefore, why not simply regress  $Y_t$  on its (one-period) lagged value  $Y_{t-1}$  and find out if the estimated  $\rho$  is statistically equal to 1? If it is, then  $Y_t$  is nonstationary. This is the general idea behind the unit root test of stationarity.

For theoretical reasons, we manipulate (21.4.1) as follows: Subtract  $Y_{t-1}$  from both sides of (21.4.1) to obtain:

$$\begin{aligned} Y_t - Y_{t-1} &= \rho Y_{t-1} - Y_{t-1} + u_t \\ &= (\rho - 1)Y_{t-1} + u_t \end{aligned} \quad (21.9.1)$$

which can be alternatively written as:

$$\Delta Y_t = \delta Y_{t-1} + u_t \quad (21.9.2)$$

where  $\delta = (\rho - 1)$  and  $\Delta$ , as usual, is the first-difference operator.

In practice, therefore, instead of estimating (21.4.1), we estimate (21.9.2) and test the (null) hypothesis that  $\delta = 0$ . If  $\delta = 0$ , then  $\rho = 1$ , that is we have a unit root, meaning the time series under consideration is nonstationary.

Before we proceed to estimate (21.9.2), it may be noted that if  $\delta = 0$ , (21.9.2) will become

$$\Delta Y_t = (Y_t - Y_{t-1}) = u_t \quad (21.9.3)$$

Since  $u_t$  is a white noise error term, it is stationary, which means that the first differences of a random walk time series are stationary, a point we have already made before.

Now let us turn to the estimation of (21.9.2). This is simple enough; all we have to do is to take the first differences of  $Y_t$  and regress them on  $Y_{t-1}$  and see if the estimated slope coefficient in this regression ( $= \hat{\delta}$ ) is zero or not. If it is zero, we conclude that  $Y_t$  is nonstationary. But if it is negative, we conclude that  $Y_t$  is stationary.<sup>25</sup> The only question is which test we use to

<sup>25</sup>Since  $\delta = (\rho - 1)$ , for stationarity  $\rho$  must be less than one. For this to happen  $\delta$  must be negative.

find out if the estimated coefficient of  $Y_{t-1}$  in (21.9.2) is zero or not. You might be tempted to say, why not use the usual  $t$  test? Unfortunately, under the null hypothesis that  $\delta = 0$  (i.e.,  $\rho = 1$ ), the  $t$  value of the estimated coefficient of  $Y_{t-1}$  does not follow the  $t$  distribution even in large samples; that is, it does not have an asymptotic normal distribution.

What is the alternative? Dickey and Fuller have shown that under the null hypothesis that  $\delta = 0$ , the estimated  $t$  value of the coefficient of  $Y_{t-1}$  in (21.9.2) follows the  $\tau$  (**tau**) **statistic**.<sup>26</sup> These authors have computed the critical values of the *tau statistic* on the basis of Monte Carlo simulations. A sample of these critical values is given in **Appendix D**, Table D.7. The table is limited, but MacKinnon has prepared more extensive tables, which are now incorporated in several econometric packages.<sup>27</sup> In the literature the **tau statistic or test** is known as the **Dickey–Fuller (DF) test**, in honor of its discoverers. Interestingly, if the hypothesis that  $\delta = 0$  is rejected (i.e., the time series is stationary), we can use the usual (Student’s)  $t$  test.

The actual procedure of implementing the DF test involves several decisions. In discussing the nature of the unit root process in Sections 21.4 and 21.5, we noted that a random walk process may have no drift, or it may have drift or it may have both deterministic and stochastic trends. To allow for the various possibilities, the DF test is estimated in three different forms, that is, under three different null hypotheses.

$$Y_t \text{ is a random walk:} \quad \Delta Y_t = \delta Y_{t-1} + u_t \quad (21.9.2)$$

$$Y_t \text{ is a random walk with drift:} \quad \Delta Y_t = \beta_1 + \delta Y_{t-1} + u_t \quad (21.9.4)$$

$$Y_t \text{ is a random walk with drift} \\ \text{around a stochastic trend:} \quad \Delta Y_t = \beta_1 + \beta_2 t + \delta Y_{t-1} + u_t \quad (21.9.5)$$

where  $t$  is the time or trend variable. In each case, the *null hypothesis* is that  $\delta = 0$ ; that is, there is a unit root—the time series is nonstationary. The alternative hypothesis is that  $\delta$  is less than zero; that is, the time series is stationary.<sup>28</sup> If the null hypothesis is rejected, it means that  $Y_t$  is a stationary time series with zero mean in the case of (21.9.2), that  $Y_t$  is stationary with a nonzero mean [ $= \beta_1/(1 - \rho)$ ] in the case of (21.9.4), and that  $Y_t$  is stationary around a deterministic trend in (21.9.5).

<sup>26</sup>D. A. Dickey and W. A. Fuller, “Distribution of the Estimators for Autoregressive Time Series with a Unit Root,” *Journal of the American Statistical Association*, vol. 74, 1979, pp. 427–431. See also W. A. Fuller, *Introduction to Statistical Time Series*, John Wiley & Sons, New York, 1976.

<sup>27</sup>J. G. MacKinnon, “Critical Values of Cointegration Tests,” in R. E. Engle and C. W. J. Granger, eds., *Long-Run Economic Relationships: Readings in Cointegration*, Chap. 13, Oxford University Press, New York, 1991.

<sup>28</sup>We rule out the possibility that  $\delta > 0$ , because in that case  $\rho > 1$ , in which case the underlying time series will be explosive.

*It is extremely important to note that the critical values of the tau test to test the hypothesis that  $\delta = 0$ , are different for each of the preceding three specifications of the DF test, which can be seen clearly from **Appendix D**, Table D.7. Moreover, if, say, specification (21.9.4) is correct, but we estimate (21.9.2), we will be committing a specification error, whose consequences we already know from Chapter 13. The same is true if we estimate (21.9.4) rather than the true (21.9.5). Of course, there is no way of knowing which specification is correct to begin with. Some trial and error is inevitable, data mining notwithstanding.*

The actual estimation procedure is as follows: Estimate (21.9.2), or (21.9.3), or (21.9.4) by OLS; divide the estimated coefficient of  $Y_{t-1}$  in each case by its standard error to compute the ( $\tau$ ) tau statistic; and refer to the DF tables (or any statistical package). If the computed absolute value of the tau statistic ( $|\tau|$ ) exceeds the DF or MacKinnon critical tau values, we reject the hypothesis that  $\delta = 0$ , in which case the time series is stationary. On the other hand, if the computed  $|\tau|$  does not exceed the critical tau value, we do not reject the null hypothesis, in which case the time series is nonstationary. Make sure that you use the appropriate critical  $\tau$  values.

Let us return to the U.S. GDP time series. For this series, the results of the three regressions (21.9.2), (21.9.4), and (21.9.5) are as follows: The dependent variable in each case is  $\Delta Y_t = \Delta \text{GDP}_t$

$$\widehat{\Delta \text{GDP}}_t = 0.00576 \text{GDP}_{t-1} \quad (21.9.6)$$

$$t = (5.7980) \quad R^2 = -0.0152 \quad d = 1.34$$

$$\widehat{\Delta \text{GDP}}_t = 28.2054 - 0.00136 \text{GDP}_{t-1} \quad (21.9.7)$$

$$t = (1.1576) \quad (-0.2191) \quad R^2 = 0.00056 \quad d = 1.35$$

$$\widehat{\Delta \text{GDP}}_t = 190.3857 + 1.4776t - 0.0603 \text{GDP}_{t-1} \quad (21.9.8)$$

$$t = (1.8389) \quad (1.6109) \quad (-1.6252)$$

$$R^2 = 0.0305 \quad d = 1.31$$

Our primary interest here is in the  $t$  ( $= \tau$ ) value of the  $\text{GDP}_{t-1}$  coefficient. The critical 1, 5, and 10 percent  $\tau$  values for model (21.9.6) are  $-2.5897$ ,  $-1.9439$ , and  $-1.6177$ , respectively, and are  $-3.5064$ ,  $-2.8947$ , and  $-2.5842$  for model (21.9.7) and  $-4.0661$ ,  $-3.4614$ , and  $-3.1567$  for model (21.3.8). As noted before, these critical values are different for the three models.

Before we examine the results, we have to decide which of the three models may be appropriate. We should rule out model (21.9.6) because the coefficient of  $\text{GDP}_{t-1}$ , which is equal to  $\delta$  is positive. But since  $\delta = (\rho - 1)$ , a positive  $\delta$  would imply that  $\rho > 1$ . Although a theoretical possibility, we rule this

case out because in this case the GDP time series would be explosive.<sup>29</sup> That leaves us with models (21.9.7) and (21.9.8). In both cases the estimated  $\delta$  coefficient is negative, implying that the estimated  $\rho$  is less than 1. For these two models, the estimated  $\rho$  values are 0.9986 and 0.9397, respectively. The only question now is if these values are statistically significantly below 1 for us to declare that the GDP time series is stationary.

For model (21.9.7) the estimated  $\tau$  value is  $-0.2191$ , which in absolute value is below even the 10 percent critical value of  $-2.5842$ . Since, in absolute terms, the former is smaller than the latter, our conclusion is that the GDP time series is not stationary.<sup>30</sup>

The story is the same for model (21.9.8). The computed  $\tau$  value of  $-1.6252$  is less than even the 10 percent critical  $\tau$  value of  $-3.1567$  in absolute terms.

Therefore, on the basis of graphical analysis, the correlogram, and the Dickey-Fuller test, the conclusion is that for the quarterly periods of 1970 to 1991, the U.S. GDP time series was nonstationary; i.e., it contained a unit root.

### The Augmented Dickey-Fuller (ADF) Test

In conducting the DF test as in (21.9.2), (21.9.4), or (21.9.5), it was assumed that the error term  $u_t$  was uncorrelated. But in case the  $u_t$  are correlated, Dickey and Fuller have developed a test, known as the **augmented Dickey-Fuller (ADF) test**. This test is conducted by “augmenting” the preceding three equations by adding the lagged values of the dependent variable  $\Delta Y_t$ . To be specific, suppose we use (21.9.5). The ADF test here consists of estimating the following regression:

$$\Delta Y_t = \beta_1 + \beta_2 t + \delta Y_{t-1} + \sum_{i=1}^m \alpha_i \Delta Y_{t-i} + \varepsilon_t \quad (21.9.9)$$

where  $\varepsilon_t$  is a pure white noise error term and where  $\Delta Y_{t-1} = (Y_{t-1} - Y_{t-2})$ ,  $\Delta Y_{t-2} = (Y_{t-2} - Y_{t-3})$ , etc. The number of lagged difference terms to include is often determined empirically, the idea being to include enough terms so that the error term in (21.9.9) is serially uncorrelated. In ADF we still test whether  $\delta = 0$  and the ADF test follows the same asymptotic distribution as the DF statistic, so the same critical values can be used.

<sup>29</sup>More technically, since (21.9.2) is a first-order difference equation, the so-called stability condition requires that  $|\rho| < 1$ .

<sup>30</sup>Another way of stating this is that the computed  $\tau$  value should be more negative than the critical  $\tau$  value, which is not the case here. Hence the conclusion stays. Since in general  $\delta$  is expected to be negative, the estimated  $\tau$  statistic will have a negative sign. Therefore, a large negative  $\tau$  value is generally an indication of stationarity.



To give a glimpse of this procedure, we estimated (21.9.9) for the GDP series using one lagged difference of GDP; the results were as follows<sup>31</sup>:

$$\begin{aligned} \widehat{\Delta \text{GDP}}_t &= 234.9729 + 1.8921t - 0.0786\text{GDP}_{t-1} + 0.3557\Delta \text{GDP}_{t-1} \\ t &= (2.3833) \quad (2.1522) \quad (-2.2152) \quad (3.4647) \\ R^2 &= 0.1526 \quad d = 2.0858 \end{aligned} \tag{21.9.10}$$

The  $t$  ( $= \tau$ ) value of the  $\text{GDP}_{t-1}$  coefficient ( $= \delta$ ) is  $-2.2152$ , but this value in absolute terms is much less than even the 10 percent critical  $\tau$  value of  $-3.1570$ , again suggesting that even after taking care of possible autocorrelation in the error term, the GDP series is nonstationary.

#### Testing the Significance of More Than One Coefficient: The $F$ Test

Suppose we estimate model (21.9.5) and test the hypothesis that  $\beta_1 = \beta_2 = 0$ , that is, the model is RWM without drift and trend. To test this joint hypothesis, we can use the *restricted F* test discussed in Chapter 8. That is, we estimate (21.9.5) (the unrestricted regression) and estimate (21.9.5), dropping the intercept and trend. Then we use the restricted  $F$  test as shown in Eq. (8.7.9), except that we cannot use the conventional  $F$  table to get the critical  $F$  values. As they did with the  $\tau$  statistic, Dickey and Fuller have developed critical  $F$  values for this situation, a sample of which is given in **Appendix D**, Table D.7. An example is presented in exercise 21.27.

#### The Phillips–Perron (PP) Unit Root Tests<sup>32</sup>

An important assumption of the DF test is that the error terms  $u_t$  are independently and identically distributed. The ADF test adjusts the DF test to take care of possible serial correlation in the error terms by adding the lagged difference terms of the regressand. Phillips and Perron use *nonparametric statistical methods* to take care of the serial correlation in the error terms without adding lagged difference terms. Since the asymptotic distribution of the PP test is the same as the ADF test statistic, we will not pursue this topic here.

#### A Critique of the Unit Root Tests<sup>33</sup>

We have discussed several unit root tests and there are several more. The question is: Why are there so many unit root tests? The answer lies in the

<sup>31</sup>Higher-order lagged differences were considered but they were insignificant.

<sup>32</sup>P. C. B. Phillips and P. Perron, "Testing for a Unit Root in Time Series Regression," *Biometrika*, vol. 75, 1988, pp. 335–346. The PP test is now included in several software packages.

<sup>33</sup>For detailed discussion, see Terrence C. Mills, *op. cit.*, pp. 87–88.



**size** and **power** of these tests. By size of a test we mean the level of significance (i.e., the probability of committing a Type I error) and by power of a test we mean the probability of rejecting the null hypothesis when it is false. The power of a test is calculated by subtracting the probability of a Type II error from 1; Type II error is the probability of accepting a false null hypothesis. The maximum power is 1. Most unit root tests are based on the null hypothesis that the time series under consideration has a unit root; that is, it is nonstationary. The alternative hypothesis is that the time series is stationary.

**Size of Test.** You will recall from Chapter 13 the distinction we made between the nominal and the true levels of significance. The DF test is sensitive to the way it is conducted. Remember that we discussed three varieties of the DF test: (1) a pure random walk, (2) a random walk with drift, and (3) a random walk with drift and trend. If, for example, the true model is (1) but we estimate (2), and conclude that, say, on the 5 percent level that the time series is stationary, this conclusion may be wrong because the true level of significance in this case is much larger than 5 percent.<sup>34</sup> The size distortion could also result from excluding moving average (MA) components from the model (on moving average, see Chapter 22).

**Power of Test.** Most tests of the DF type have low power; that is, they tend to accept the null of unit root more frequently than is warranted. That is, these tests may find a unit root even when none exists. There are several reasons for this. *First*, the power depends on the (time) *span* of the data more than mere size of the sample. For a given sample size  $n$ , the power is greater when the span is large. Thus, unit root test(s) based on 30 observations over a span of 30 years may have more power than that based on, say, 100 observations over a span of 100 days. *Second*, if  $\rho \approx 1$  but not exactly 1, the unit root test may declare such a time series nonstationary. *Third*, these types of tests assume a single unit root; that is, they assume that the given time series is  $I(1)$ . But if a time series is integrated of order higher than 1, say,  $I(2)$ , there will be more than one unit root. In the latter case one may use the **Dickey-Pantula test**.<sup>35</sup> *Fourth*, if there are structural breaks in a time series (see the chapter on dummy variables) due to, say, the OPEC oil embargoes, the unit root tests may not catch them.

In applying the unit root tests one should therefore keep in mind the limitations of the tests. Of course, there have been modifications of these tests by Perron and Ng, Elliot, Rothenberg and Stock, Fuller, and Leybourne.<sup>36</sup> Because of this, Maddala and Kim advocate that the traditional DF, ADF,

<sup>34</sup>For a Monte Carlo experiment about this, see Charemza et al., op. cit., p. 114.

<sup>35</sup>D. A. Dickey and S. Pantula, "Determining the Order of Differencing in Autoregressive Processes," *Journal of Business and Economic Statistics*, vol. 5, 1987, pp. 455-461.

<sup>36</sup>A discussion of these tests can be found in Maddala et al., op. cit., Chap. 4.

and PP tests should be discarded. As econometric software packages incorporate the new tests, that may very well happen. But it should be added that as yet there is no uniformly powerful test of the unit root hypothesis.

### 21.10 TRANSFORMING NONSTATIONARY TIME SERIES

Now that we know the problems associated with nonstationary time series, the practical question is what to do. To avoid the spurious regression problem that may arise from regressing a nonstationary time series on one or more nonstationary time series, we have to transform nonstationary time series to make them stationary. The transformation method depends on whether the time series are difference stationary (DSP) or trend stationary (TSP). We consider each of these methods in turn.

#### Difference-Stationary Processes

If a time series has a unit root, the first differences of such time series are stationary.<sup>37</sup> Therefore, the solution here is to take the first differences of the time series.

Returning to our U.S. GDP time series, we have already seen that it has a unit root. Let us now see what happens if we take the first differences of the GDP series.

Let  $\Delta\text{GDP}_t = (\text{GDP}_t - \text{GDP}_{t-1})$ . For convenience, let  $D_t = \Delta\text{GDP}_t$ . Now consider the following regression:

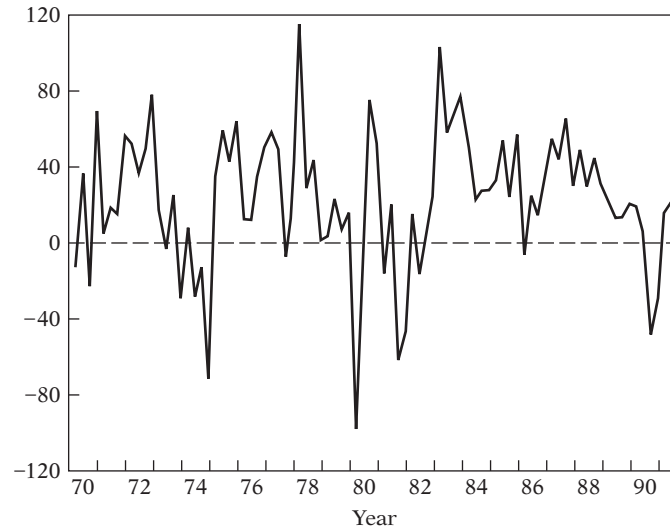
$$\begin{aligned} \widehat{\Delta D}_t &= 16.0049 - 0.06827D_{t-1} \\ t &= (3.6402) \quad (-6.6303) && (21.10.1) \\ R^2 &= 0.3435 \quad d = 2.0344 \end{aligned}$$

The 1 percent critical DF  $\tau$  value is  $-3.5073$ . Since the computed  $\tau (= t)$  is more negative than the critical value, we conclude that the first-differenced GDP is stationary; that is, it is  $I(0)$ . It is as shown in Figure 21.9. If you compare Figure 21.9 with Figure 21.1, you will see the obvious difference between the two.

#### Trend-Stationary Process

As we have seen in Figure 21.5, a TSP is stationary around the trend line. Hence, the simplest way to make such a time series stationary is to regress it on time and the residuals from this regression will then be stationary.

<sup>37</sup>If a time series is  $I(2)$ , it will contain two unit roots, in which case we will have to difference it twice. If it is  $I(d)$ , it has to be differenced  $d$  times, where  $d$  is any integer.



**FIGURE 21.9** First differences of U.S. GDP, 1970–1991 (quarterly).

In other words, run the following regression:

$$Y_t = \beta + \beta_2 t + u_t \quad (21.10.2)$$

where  $Y_t$  is the time series under study and where  $t$  is the trend variable measured chronologically.

Now

$$\hat{u}_t = (Y_t - \hat{\beta}_1 - \hat{\beta}_2 t) \quad (21.10.3)$$

will be stationary.  $\hat{u}_t$  is known as a (linearly) **detrended time series**.

It is important to note that the trend may be nonlinear. For example, it could be

$$Y_t = \beta_1 + \beta_2 t + \beta_3 t^2 + u_t \quad (21.10.4)$$

which is a quadratic trend series. If that is the case, the residuals from (21.10.4) will now be (quadratically) detrended time series.

It should be pointed out that if a time series is DSP but we treat it as TSP, this is called **underdifferencing**. On the other hand, if a time series is TSP but we treat it as DSP, this is called **overdifferencing**. The consequences of these types of specification errors can be serious, depending on how one handles the serial correlation properties of the resulting error terms.<sup>38</sup>

In passing it may be noted that most macroeconomic time series are DSP rather than TSP.

<sup>38</sup>For a detailed discussion of this, see Maddala et al., op. cit., Sec. 2.7.

### 21.11 COINTEGRATION: REGRESSION OF A UNIT ROOT TIME SERIES ON ANOTHER UNIT ROOT TIME SERIES

We have warned that the regression of a nonstationary time series on another nonstationary time series may produce a spurious regression. Let us suppose that we consider the PCE and PDI time series given in Table 21.1. Subjecting these time series individually to unit root analysis, you will find that they both are  $I(1)$ ; that is, they contain a unit root. Suppose, then, that we regress PCE on PDI as follows:

$$\text{PCE}_t = \beta_1 + \beta_2 \text{PDI}_t + u_t \quad (21.11.1)$$

Let us write this as:

$$u_t = \text{PCE}_t - \beta_1 - \beta_2 \text{PDI}_t \quad (21.11.2)$$

Suppose we now subject  $u_t$  to unit root analysis and find that it is stationary; that is, it is  $I(0)$ . This is an interesting situation, for although  $\text{PCE}_t$  and  $\text{PDI}_t$  are individually  $I(1)$ , that is, they have stochastic trends, their linear combination (21.11.2) is  $I(0)$ . So to speak, the linear combination cancels out the stochastic trends in the two series. If you take consumption and income as two  $I(1)$  variables, savings defined as (income – consumption) could be  $I(0)$ . As a result, a regression of consumption on income as in (21.11.1) would be meaningful (i.e., not spurious). In this case we say that the two variables are **cointegrated**. Economically speaking, two variables will be cointegrated if they have a long-term, or equilibrium, relationship between them. Economic theory is often expressed in equilibrium terms, such as Fisher’s quantity theory of money or the theory of purchasing parity (PPP), just to name a few.

In short, provided we check that the residuals from regressions like (21.11.1) are  $I(0)$  or stationary, the traditional regression methodology (including the  $t$  and  $F$  tests) that we have considered extensively is applicable to data involving (nonstationary) time series. The valuable contribution of the concepts of unit root, cointegration, etc. is to force us to find out if the regression residuals are stationary. As Granger notes, “A test for cointegration can be thought of as a pre-test to avoid ‘spurious regression’ situations.”<sup>39</sup>

In the language of cointegration theory, a regression such as (21.11.1) is known as a **cointegrating regression** and the slope parameter  $\beta_2$  is known as the **cointegrating parameter**. The concept of cointegration can be extended to a regression model containing  $k$  regressors. In this case we will have  $k$  cointegrating parameters.

#### Testing for Cointegration

A number of methods for testing cointegration have been proposed in the literature. We consider here two comparatively simple methods: (1) the DF

<sup>39</sup>C. W. J. Granger, “Developments in the Study of Co-Integrated Economic Variables,” *Oxford Bulletin of Economics and Statistics*, vol. 48, 1986, p. 226.

or ADF unit root test on the residuals estimated from the cointegrating regression and (2) the cointegrating regression Durbin–Watson (CRDW) test.<sup>40</sup>

**Engle–Granger (EG) or Augmented Engle–Granger (AEG) Test.** We already know how to apply the DF or ADF unit root tests. All we have to do is estimate a regression like (21.11.1), obtain the residuals, and use the DF or ADF tests.<sup>41</sup> There is one precaution to exercise, however. Since the estimated  $u_t$  are based on the *estimated* cointegrating parameter  $\beta_2$ , the DF and ADF critical significance values are not quite appropriate. Engle and Granger have calculated these values, which can be found in the references.<sup>42</sup> Therefore, the DF and ADF tests in the present context are known as **Engle–Granger (EG)** and **augmented Engle–Granger (AEG)** tests. However, several software packages now present these critical values along with other outputs.

Let us illustrate these tests. We first regressed PCE on PDI and obtained the following regression:

$$\begin{aligned}\widehat{\text{PCE}}_t &= -171.4412 + 0.9672\text{PDI}_t \\ t &= (-7.4808) \quad (119.8712) \qquad (21.11.3) \\ R^2 &= 0.9940 \quad d = 0.5316\end{aligned}$$

Since PCE and PDI are individually nonstationary, there is the possibility that this regression is spurious. But when we performed a unit root test on the residuals obtained from (21.11.3), we obtained the following results:

$$\begin{aligned}\widehat{\Delta\hat{u}}_t &= -0.2753\hat{u}_{t-1} \\ t &= (-3.7791) \qquad (21.11.4) \\ R^2 &= 0.1422 \quad d = 2.2775\end{aligned}$$

The Engle–Granger 1 percent critical  $\tau$  value is  $-2.5899$ . Since the computed  $\tau (= t)$  value is much more negative than this, our conclusion is that the residuals from the regression of PCE on PDI are  $I(0)$ ; that is, they are

<sup>40</sup>There is this difference between tests for unit roots and tests for cointegration. As David A. Dickey, Dennis W. Jansen, and Daniel I. Thornton observe, “Tests for unit roots are performed on univariate [i.e., single] time series. In contrast, cointegration deals with the relationship among a group of variables, where (unconditionally) each has a unit root.” See their article, “A Primer on Cointegration with an Application to Money and Income,” *Economic Review*, Federal Reserve Bank of St. Louis, March–April 1991, p. 59. As the name suggests, this article is an excellent introduction to cointegration testing.

<sup>41</sup>If PCE and PDI are not cointegrated, any linear combination of them will be nonstationary and, therefore, the  $u_t$  will also be nonstationary.

<sup>42</sup>R. F. Engle and C. W. Granger, “Co-integration and Error Correction: Representation, Estimation and Testing,” *Econometrica*, vol. 55, 1987, pp. 251–276.

stationary. Hence, (21.11.3) is a cointegrating regression and this regression is not spurious, even though individually the two variables are nonstationary. One can call (21.11.3) the **static** or **long run** consumption function and interpret its parameters as long run parameters. Thus, 0.9672 represents the long-run, or equilibrium, marginal propensity to consumer (MPC).

**Cointegrating Regression Durbin–Watson (CRDW) Test.** An alternative, and quicker, method of finding out whether PCE and PDI are cointegrated is the CRDW test, whose critical values were first provided by Sargan and Bhargava.<sup>43</sup> In CRDW we use the Durbin–Watson  $d$  obtained from the cointegrating regression, such as  $d = 0.5316$  given in (21.11.3). But now the null hypothesis is that  $d = 0$  rather than the standard  $d = 2$ . This is because in Chapter 12 we observed that  $d \approx 2(1 - \hat{\rho})$ , so if there is to be a unit root, the estimated  $\rho$  will be about 1, which implies that  $d$  will be about zero.

On the basis of 10,000 simulations formed from 100 observations each, the 1, 5, and 10 percent critical values to test the hypothesis that the true  $d = 0$  are 0.511, 0.386, and 0.322, respectively. Thus, if the computed  $d$  value is smaller than, say, 0.511, we reject the null hypothesis of cointegration at the 1 percent level. In our example, the value of 0.5316 is above this critical value, suggesting that PCE and PDI are cointegrated, thus reinforcing the finding on the basis of the EG test.<sup>44</sup>

*To sum up*, our conclusion, based on both the EG and CRDW tests, is that PCE and PDI are cointegrated.<sup>45</sup> Although they individually exhibit random walks, there seems to be a stable long-run relationship between them; they will not wander away from each other, which is evident from Figure 21.1.

### Cointegration and Error Correction Mechanism (ECM)

We just showed that PCE and PDI are cointegrated; that is, there is a long-term, or equilibrium, relationship between the two. Of course, in the short run there may be disequilibrium. Therefore, one can treat the error term in (21.11.2) as the “equilibrium error.” And we can use this error term to tie the short-run behavior of PCE to its long-run value. The **error correction mechanism (ECM)** first used by Sargan<sup>46</sup> and later popularized by Engle

<sup>43</sup>J. D. Sargan and A. S. Bhargava, “Testing Residuals from Least-Squares Regression for being Generated by the Gaussian Random Walk,” *Econometrica*, vol. 51, 1983, pp. 153–174.

<sup>44</sup>There is considerable debate about the superiority of CRDW over DF, which can be found in the references. The debate revolves around the power of the two statistics, that is, the probability of not committing a Type II error. Engle and Granger, for example, prefer the ADF to the CRDW test.

<sup>45</sup>The EG and CRDW tests are now supplemented (supplanted?) by more powerful tests developed by Johansen. But the discussion of the **Johansen method** is beyond the scope of this book because the mathematics involved is quite complex, although several software packages now use the Johansen method.

<sup>46</sup>J. D. Sargan, “Wages and Prices in the United Kingdom: A Study in Econometric Methodology,” in K. F. Wallis and D. F. Hendry, eds., *Quantitative Economics and Econometric Analysis*, Basil Blackwell, Oxford, U.K., 1984.

and Granger corrects for disequilibrium. An important theorem, known as the **Granger representation theorem**, states that if two variables  $Y$  and  $X$  are cointegrated, then the relationship between the two can be expressed as ECM. To see what this means, let us revert to our PCE–PDI example. Now consider the following model:

$$\Delta \text{PCE}_t = \alpha_0 + \alpha_1 \Delta \text{PDI}_t + \alpha_2 u_{t-1} + \varepsilon_t \quad (21.11.5)$$

where  $\Delta$  as usual denotes the first difference operator,  $\varepsilon_t$  is a random error term, and  $u_{t-1} = (\text{PCE}_{t-1} - \beta_1 - \beta_2 \text{PDI}_{t-1})$ , that is, the one-period lagged value of the error from the cointegrating regression (21.11.1).

ECM equation (21.11.5) states that  $\Delta \text{PCE}$  depends on  $\Delta \text{PDI}$  and also on the equilibrium error term.<sup>47</sup> If the latter is nonzero, then the model is out of equilibrium. Suppose  $\Delta \text{PDI}$  is zero and  $u_{t-1}$  is positive. This means  $\text{PCE}_{t-1}$  is too high to be in equilibrium, that is,  $\text{PCE}_{t-1}$  is above its equilibrium value of  $(\alpha_0 + \alpha_1 \text{PDI}_{t-1})$ . Since  $\alpha_2$  is expected to be negative, the term  $\alpha_2 u_{t-1}$  is negative and, therefore,  $\Delta \text{PCE}_t$  will be negative to restore the equilibrium. That is, if  $\text{PCE}_t$  is above its equilibrium value, it will start falling in the next period to correct the equilibrium error; hence the name ECM. By the same token, if  $u_{t-1}$  is negative (i.e., PCE is below its equilibrium value),  $\alpha_2 u_{t-1}$  will be positive, which will cause  $\Delta \text{PCE}_t$  to be positive, leading  $\text{PCE}_t$  to rise in period  $t$ . Thus, the absolute value of  $\alpha_2$  decides how quickly the equilibrium is restored. In practice, we estimate  $u_{t-1}$  by  $\hat{u}_{t-1} = (\text{PCE}_t - \hat{\beta}_1 - \hat{\beta}_2 \text{PDI}_t)$ .

Returning to our illustrative example, the empirical counterpart of (21.11.5) is:

$$\begin{aligned} \widehat{\Delta \text{PCE}}_t &= 11.6918 + 0.2906 \Delta \text{PDI}_t - 0.0867 \hat{u}_{t-1} \\ t &= (5.3249) \quad (4.1717) \quad (-1.6003) \quad (21.11.6) \\ R^2 &= 0.1717 \quad d = 1.9233 \end{aligned}$$

Statistically, the equilibrium error term is zero, suggesting that PCE adjusts to changes in PDI in the same time period. As (21.11.6) shows, short-run changes in PDI have a positive impact on short-run changes in personal consumption. One can interpret 0.2906 as the short-run marginal propensity to consume (MPC); the long-run MPC is given by the estimated (static) equilibrium relation (21.11.3) as 0.9672.

Before we conclude this section, the caution sounded by S. G. Hall is worth remembering:

While the concept of cointegration is clearly an important theoretical underpinning of the error correction model there are still a number of problems surrounding its

<sup>47</sup>The following discussion is based on Gary Koop, op. cit., pp. 159–160 and Kerry Peterson, op. cit., Sec. 8.5.



practical application; the critical values and small sample performance of many of these tests are unknown for a wide range of models; informed inspection of the correlogram may still be an important tool.<sup>48</sup>

## 21.12 SOME ECONOMIC APPLICATIONS

We conclude this chapter by considering some concrete examples.

### EXAMPLE 21.1

M1 MONTHLY MONEY SUPPLY IN THE UNITED STATES, JANUARY 1951 TO SEPTEMBER 30, 1999

Figure 21.10 shows the M1 money supply for the United States from January 1951 to September 30, 1999. From our knowledge of stationarity, it seems that the M1 money supply time series is nonstationary, which can be confirmed by unit root analysis. (Note: to save space, we have not given the actual data, which can be obtained from the Federal Reserve Board

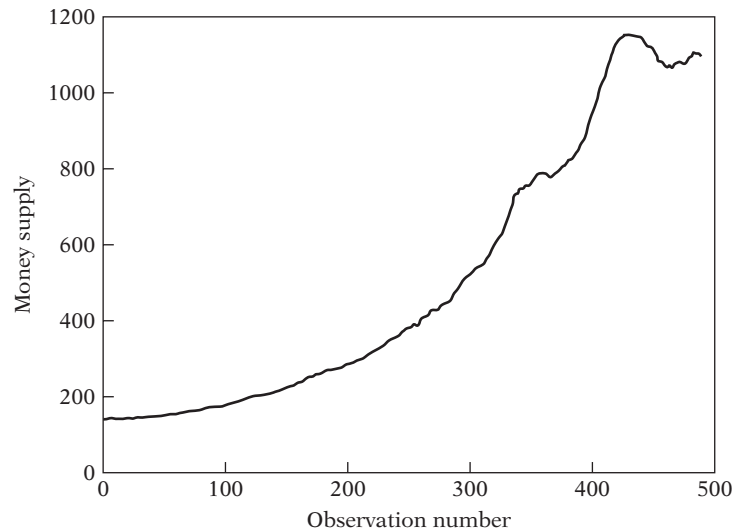


FIGURE 21.10 U.S. money supply over 1951:01 to 1999:09.

(Continued)

<sup>48</sup>S. G. Hall, "An Application of the Granger and Engle Two-Step Estimation Procedure to the United Kingdom Aggregate Wage Data," *Oxford Bulletin of Economics and Statistics*, vol. 48, no. 3, August 1986, p. 238. See also John Y. Campbell and Pierre Perron, "Pitfalls and Opportunities: What Macroeconomists Should Know about Unit Roots," NBER (National Bureau of Economic Research) *Macroeconomics Annual 1991*, pp. 141–219.



**EXAMPLE 21.1** (Continued)

or the Federal Reserve Bank of St. Louis.)

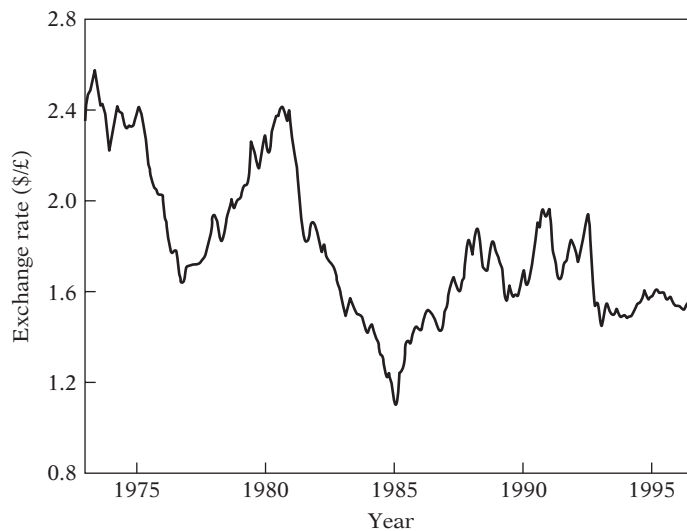
$$\begin{aligned} \Delta \hat{M}_t &= 0.2618 + 0.0159t - 0.0044M_{t-1} \\ t &= (0.7919) \quad (4.4227) \quad (-3.0046) && \text{(21.12.1)} \\ &&& R^2 = 0.0670 \quad d = 0.7172 \end{aligned}$$

The 1, 5, and 10 percent critical  $\tau$  values are  $-3.9811$ ,  $-3.4210$ , and  $-3.1329$ . Since the  $t$  value of  $-3.0046$  is less negative than any of these critical values, the conclusion is that the  $M1$  time series is nonstationary; that is, it contains a unit root or it is  $I(1)$ . Even when several lagged values of  $\Delta M_t$  (à la ADF) were introduced, the conclusion did not change. On the other hand, the first differences of the  $M1$  money supply were found to be stationary (check this out).

**EXAMPLE 21.2**

THE U.S./U.K. EXCHANGE RATE: JANUARY 1, 1973, TO OCTOBER 10, 1996

Figure 21.11 gives the graph of the (\$/£) exchange rate from January 1973 to October 1996, for a total of 286 observations. By now you should be able to spot this time series as nonstationary. Carrying out the unit root tests, we obtained the following  $\tau$  statistics:  $-1.2749$  (no intercept, no trend),  $-1.7710$  (intercept), and  $-1.6269$  (intercept and trend). Each of these statistics, in absolute value, was less than its critical  $\tau$  value from the appropriate DF tables, thus confirming the graphical impression that the U.S./U.K. exchange rate time series is nonstationary.



**FIGURE 21.11** U.S./U.K. exchange rate: January 1973 to October 1996.

**EXAMPLE 21.3**

U.S. CONSUMER PRICE INDEX (CPI), JANUARY 1947 TO JANUARY 2000

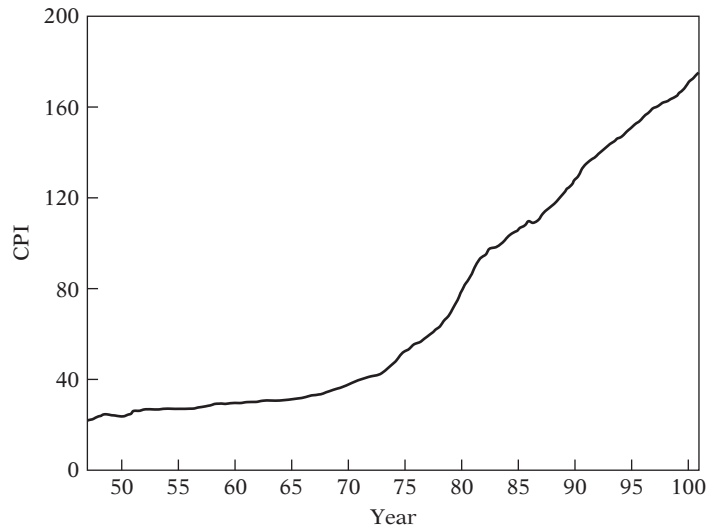
Figure 21.12 shows the U.S. CPI from January 1947 to January 2000 for a total of 649 observations. The CPI series, like the M1 series considered previously, shows a sustained upward trend. The unit root exercise gave the following results:

$$\widehat{\Delta CPI}_t = -0.0094 + 0.00051t - 0.00066CPI_{t-1} + 0.5473\Delta CPI_{t-1}$$

$$t = (-0.6538) \quad (4.3431) \quad (-1.5472) \quad (16.4448) \quad \mathbf{(21.12.2)}$$

$$R^2 = 0.5177 \quad d = 2.1410$$

The  $t (= \tau)$  value of  $CPI_{t-1}$  is  $-1.5472$ . The 10 percent critical value is  $-3.1317$ . Since, in absolute terms, the computed  $\tau$  is less than the critical  $\tau$ , the conclusion is that CPI is not a stationary time series. We can characterize it as having a stochastic trend (why?). However, if you take the first differences of the CPI series, you will find them to be stationary. Hence CPI is a difference-stationary (DS) time series.



**FIGURE 21.12** U.S. CPI, January 1947 to January 2000.

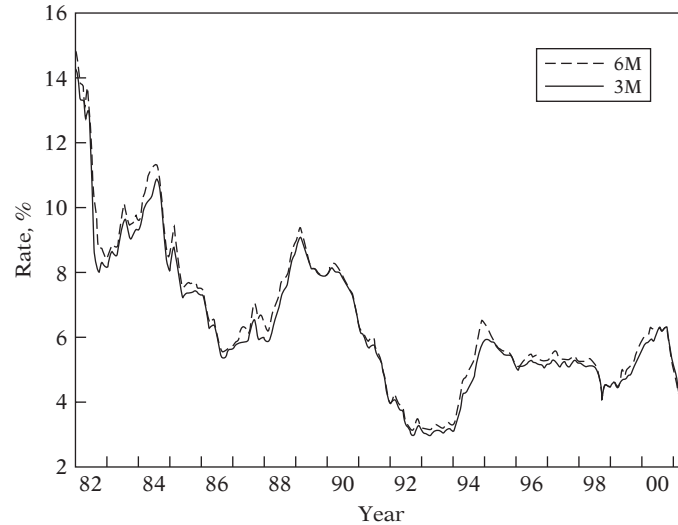
**EXAMPLE 21.4**

ARE 3-MONTH AND 6-MONTH TREASURY BILL RATES COINTEGRATED?

Figure 21.13 plots (constant maturity) 3-month and 6-month U.S. Treasury bill (T bill) rates from January 1982 to June 2001, for a total of 234 observations. Does the graph show that the two rates are cointegrated; that is, is there an equilibrium relationship between the two? From financial theory, we would expect that to be the case, otherwise arbitrageurs will exploit any discrepancy between the short and the long rates. First of all, let us see if the two time series are stationary.

(Continued)

**EXAMPLE 20.1** (Continued)



**FIGURE 21.13**  
Three- and six-month Treasury bill rates (constant maturity).

On the basis of the pure random walk model (i.e., no intercept, no trend), both the rates were stationary. Including intercept, trend, and one lagged difference, the results suggested that the two rates might be trend stationary; the trend coefficient in both cases was negative and significant at about the 7 percent level. So, depending on which results we accept, the two rates are either stationary or trend stationary.

Regressing the 6-month T bill rate (TB6) on the 3 month T-bill rate, we obtained the following regression.

$$\widehat{TB6}_t = -0.0456 + 1.0466TB3_t$$

$$t = (-1.1207) \quad (171.6239) \quad R^2 = 0.9921 \quad d = 0.4055 \quad (21.12.3)$$

Applying the unit root test to the residuals from the preceding regression, we found that the residuals were stationary, suggesting that the 3- and 6-month T bill rates were cointegrated. Using this knowledge, we obtained the following error correction model (ECM):

$$\Delta \widehat{TB6}_t = -0.0067 + 0.9360 \Delta TB3_t - 0.2030 \hat{u}_{t-1}$$

$$t = (-0.8662) \quad (41.9592) \quad (-5.3837) \quad (21.12.4)$$

$$R^2 = 0.8852 \quad d = 1.5604$$

where  $\hat{u}_{t-1}$  is the lagged value of the error correction term from the preceding period. As these results show, 0.20 of the discrepancy in the two rates in the previous month is eliminated this month.<sup>49</sup> Besides, short-run changes in the 3-month T bill rate are quickly reflected in the 6-month T bill rate, as the slope coefficient between the two is 0.9360. This should not be a surprising finding in view of the efficiency of the U.S. money markets.

<sup>49</sup>Since both T bill rates are in percent form, this would suggest that if the 6-month TB rate was higher than the 3-month TB rate more than expected a priori in the last month, this month it will be reduced by 0.20 percentage points to restore the long-run relationship between the two interest rates. For the underlying theory about the relationship between short- and long-run interest rates, see any money and banking textbook and read up on the term structure of interest rates.

### 21.13 SUMMARY AND CONCLUSIONS

1. Regression analysis based on time series data implicitly assumes that the underlying time series are stationary. The classical  $t$  tests,  $F$  tests, etc. are based on this assumption.

2. In practice most economic time series are nonstationary.

3. A stochastic process is said to be **weakly stationary** if its mean, variance, and autocovariances are constant over time (i.e., they are time-invariant).

4. At the informal level, weak stationarity can be tested by the **correlogram** of a time series, which is a graph of autocorrelation at various lags. For stationary time series, the correlogram tapers off quickly, whereas for nonstationary time series it dies off gradually. For a purely random series, the autocorrelations at all lags 1 and greater are zero.

5. At the formal level, stationarity can be checked by finding out if the time series contains a unit root. The **Dickey-Fuller (DF)** and **augmented Dickey-Fuller (ADF)** tests can be used for this purpose.

6. An economic time series can be **trend stationary (TS)** or **difference stationary (DS)**. A TS time series has a deterministic trend, whereas a DS time series has a variable, or stochastic, trend. The common practice of including the time or trend variable in a regression model to detrend the data is justifiable only for TS time series. The DF and ADF tests can be applied to determine whether a time series is TS or DS.

7. Regression of one time series variable on one or more time series variables often can give nonsensical or spurious results. This phenomenon is known as **spurious regression**. One way to guard against it is to find out if the time series are cointegrated.

8. **Cointegration** means that despite being individually nonstationary, a linear combination of two or more time series can be stationary. The EG, AEG, and CRDW tests can be used to find out if two or more time series are cointegrated.

9. Cointegration of two (or more) time series suggests that there is a long-run, or equilibrium, relationship between them.

10. The **error correction mechanism (ECM)** developed by Engle and Granger is a means of reconciling the short-run behavior of an economic variable with its long-run behavior.

11. The field of time series econometrics is evolving. The established results and tests are in some cases tentative and a lot more work remains. An important question that needs an answer is why some economic time series are stationary and some are nonstationary.

### EXERCISES

#### Questions

21.1. What is meant by weak stationarity?

21.2. What is meant by an integrated time series?

- 21.3. What is the meaning of a unit root?
- 21.4. If a time series is  $I(3)$ , how many times would you have to difference it to make it stationary?
- 21.5. What are Dickey–Fuller (DF) and augmented DF tests?
- 21.6. What are Engle–Granger (EG) and augmented EG tests?
- 21.7. What is the meaning of cointegration?
- 21.8. What is the difference, if any, between tests of unit roots and tests of cointegration?
- 21.9. What is spurious regression?
- 21.10. What is the connection between cointegration and spurious regression?
- 21.11. What is the difference between a deterministic trend and a stochastic trend?
- 21.12. What is meant by a trend-stationary process (TSP) and a difference-stationary process (DSP)?
- 21.13. What is a random walk (model)?
- 21.14. “For a random walk stochastic process, the variance is infinite.” Do you agree? Why?
- 21.15. What is the error correction mechanism (ECM)? What is its relation with cointegration?

### Problems

- 21.16. Using the data given in Table 21.1, obtain sample correlograms up to 25 lags for the time series PCE, PDI, Profits, and Dividends. What general pattern do you see? Intuitively, which one(s) of these time series seem to be stationary?
- 21.17. For each of the time series of exercise 21.16, use the DF test to find out if these series contain a unit root. If a unit root exists, how would you characterize such a time series?
- 21.18. Continue with exercise 21.17. How would you decide if the ADF test is more appropriate than the DF test?
- 21.19. Consider the dividends and profits time series given in Table 21.1. Since dividends depend on profits, consider the following simple model:

$$\text{Dividends}_t = \beta_1 + \beta_2 \text{Profits}_t + u_t$$

- a. Would you expect this regression to suffer from the spurious regression phenomenon? Why?
- b. Are Dividends and Profits time series cointegrated? How do you test for this explicitly? If, after testing, you find that they are cointegrated, would your answer in **a** change?
- c. Employ the error correction mechanism (ECM) to study the short- and long-run behavior of dividends in relation to profits.
- d. If you examine the Dividends and Profits series individually, do they exhibit stochastic or deterministic trends? What tests do you use?
- \*e. Assume Dividends and Profits are cointegrated. Then, instead of regressing dividends on profits, you regress profits on dividends. Is such a regression valid?

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\*Optional.

- 21.20.** Take the first differences of the time series given in Table 21.1 and plot them. Also obtain a correlogram of each time series up to 25 lags. What strikes you about these correlograms?
- 21.21.** Instead of regressing dividends on profits in level form, suppose you regress the first difference of dividends on the first difference of profits. Would you include the intercept in this regression? Why or why not? Show the calculations.
- 21.22.** Continue with the previous exercise. How would you test the first-difference regression for stationarity? In the present example, what would you expect a priori and why? Show all the calculations.
- 21.23.** From the U.K. private sector housing starts ( $X$ ) for the period 1948 to 1984, Terence Mills obtained the following regression results\*:

$$\begin{aligned}\widehat{\Delta X}_t &= 31.03 - 0.188X_{t-1} \\ \text{se} &= (12.50) \quad (0.080) \\ (t = )\tau & \quad (-2.35)\end{aligned}$$

*Note:* The 5 percent critical  $\tau$  value is  $-2.95$  and the 10 percent critical  $\tau$  value is  $-2.60$ .

- a.** On the basis of these results, is the housing starts time series stationary or nonstationary? Alternatively, is there a unit root in this time series? How do you know?
- b.** If you were to use the usual  $t$  test, is the observed  $t$  value statistically significant? On this basis, would you have concluded that this time series is stationary?
- c.** Now consider the following regression results:

$$\begin{aligned}\widehat{\Delta^2 X}_t &= 4.76 - 1.39\Delta X_{t-1} + 0.313\Delta^2 X_{t-1} \\ \text{se} &= (5.06) \quad (0.236) \quad (0.163) \\ (t = )\tau & \quad (-5.89)\end{aligned}$$

where  $\Delta^2$  is the second difference operator, that is, the first difference of the first difference. The estimated  $\tau$  value is now statistically significant. What can you say now about the stationarity of the time series in question?

*Note:* The purpose of the preceding regression is to find out if there is a **second unit root** in the time series.

- 21.24.** Generate two random walk series as indicated in (21.7.1) and (21.7.2) and regress one on the other. Repeat this exercise but now use their first differences and verify that in this regression the  $R^2$  value is about zero and the Durbin-Watson  $d$  is close to 2.

\*Terence C. Mills, op. cit., p. 127. Notation slightly altered.

- 21.25.** To show that two variables, each with deterministic trend, can lead to spurious regression, Charemza et al. obtained the following regression based on 30 observations\*:

$$\hat{Y}_t = 5.92 + 0.030X_t$$

$$t = (9.9) \quad (21.2)$$

$$R^2 = 0.92 \quad d = 0.06$$

where  $Y_1 = 1, Y_2 = 2, \dots, Y_n = n$  and  $X_1 = 1, X_2 = 4, \dots, X_n = n^2$ .

- What kind of trend does  $Y$  exhibit? and  $X$ ?
  - Plot the two variables and plot the regression line. What general conclusion do you draw from this plot?
- 21.26.** From the data for the period 1971–I to 1988–IV for Canada, the following regression results were obtained:

$$1. \quad \widehat{\ln M1}_t = -10.2571 + 1.5975 \ln \text{GDP}_t$$

$$t = (-12.9422) \quad (25.8865)$$

$$R^2 = 0.9463 \quad d = 0.3254$$

$$2. \quad \widehat{\Delta \ln M1}_t = 0.0095 + 0.5833 \Delta \ln \text{GDP}_t$$

$$t = (2.4957) \quad (1.8958)$$

$$R^2 = 0.0885 \quad d = 1.7399$$

$$3. \quad \Delta \hat{u}_t = -0.1958 \hat{u}_{t-1}$$

$$(t = \tau) (-2.2521)$$

$$R^2 = 0.1118 \quad d = 1.4767$$

where  $M1 = M1$  money supply,  $\text{GDP} =$  gross domestic product, both measured in billions of Canadian dollars,  $\ln$  is natural log, and  $\hat{u}_t$  represent the estimated residuals from regression 1.

- Interpret regressions 1 and 2.
- Do you suspect that regression 1 is spurious? Why?
- Is regression 2 spurious? How do you know?
- From the results of regression 3, would you change your conclusion in **b**? And why?
- Now consider the following regression:

$$\widehat{\Delta \ln M1}_t = 0.0084 + 0.7340 \Delta \ln \text{GDP}_t - 0.0811 \hat{u}_{t-1}$$

$$t = (2.0496) \quad (2.0636) \quad (-0.8537)$$

$$R^2 = 0.1066 \quad d = 1.6697$$

What does this regression tell you? Does this help you decide if regression 1 is spurious or not?

\*Charemza et al., op. cit., p. 93.

**21.27.** The following regressions are based on the CPI data for the United States for the period 1960–1999, for a total of 40 annual observations:

1. 
$$\widehat{\Delta\text{CPI}}_t = 0.0372\text{CPI}_{t-1}$$
$$t = (9.6427)$$
$$R^2 = 0.0304 \quad d = 0.5259 \quad \text{RSS} = 203.6222$$
2. 
$$\widehat{\Delta\text{CPI}}_t = 1.8052 + 0.0208\text{CPI}_{t-1}$$
$$t = (2.5000) \quad (2.7583)$$
$$R^2 = 0.1705 \quad d = 0.6030 \quad \text{RSS} = 174.1966$$
3. 
$$\widehat{\Delta\text{CPI}}_t = 1.8790 + 0.5706t - 0.1158\text{CPI}_{t-1}$$
$$t = (3.1460) \quad (4.2576) \quad (-3.5443)$$
$$R^2 = 0.4483 \quad d = 0.7969 \quad \text{RSS} = 115.8579$$

where RSS = residual sum of squares.

- a. Examining the preceding regressions, what can you say about stationarity of the CPI time series?
- b. How would you choose among the three models?
- c. Equation (1) is Eq. (3) minus the intercept and trend. Which test would you use to decide if the implied restrictions of model 1 are valid? (*Hint:* Use the Dickey–Fuller  $t$  and  $F$  tests. Use the approximate values given in **Appendix D**, Table D.7.)



# 22

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## TIME SERIES ECONOMETRICS: FORECASTING

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We noted in the **Introduction** that forecasting is an important part of econometric analysis, for some people probably the most important. How do we forecast economic variables, such as GDP, inflation, exchange rates, stock prices, unemployment rates, and myriad other economic variables? In this chapter we discuss two methods of forecasting that have become quite popular: (1) **autoregressive integrated moving average (ARIMA)**, popularly known as the **Box-Jenkins** methodology,<sup>1</sup> and (2) **vector autoregression (VAR)**.

In this chapter we also discuss the special problems involved in forecasting prices of financial assets, such as stock prices and exchange rates. These asset prices are characterized by the phenomenon known as **volatility clustering**, that is, periods in which they exhibit wide swings for an extended time period followed by a period of comparative tranquility. One only has to look at the Dow Jones Index in the recent past. The so-called **autoregressive conditional heteroscedasticity (ARCH)** or **generalized autoregressive conditional heteroscedasticity (GARCH)** models can capture such volatility clustering.

The topic of economic forecasting is vast, and specialized books have been written on this subject. Our objective in this chapter is to give the reader just a glimpse of this subject. The interested reader may consult the references for further study. Fortunately, most modern econometric packages have user-friendly introductions to several techniques discussed in this chapter.

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<sup>1</sup>G. P. E. Box and G. M. Jenkins, *Time Series Analysis: Forecasting and Control*, revised ed., Holden Day, San Francisco, 1978.

The linkage between this chapter and the previous chapter is that the forecasting methods discussed below assume that the underlying time series are stationary or they can be made stationary with appropriate transformations. As we progress through this chapter, you will see the use of the several concepts that we introduced in the last chapter.

## 22.1 APPROACHES TO ECONOMIC FORECASTING

Broadly speaking, there are five approaches to economic forecasting based on time series data: (1) exponential smoothing methods, (2) single-equation regression models, (3) simultaneous-equation regression models, (4) autoregressive integrated moving average models (ARIMA), and (5) vector autoregression.

### Exponential Smoothing Methods<sup>2</sup>

These are essentially methods of fitting a suitable curve to historical data of a given time series. There are a variety of these methods, such as *single exponential smoothing*, *Holt's linear method*, and *Holt-Winters' method* and their variations. Although still used in several areas of business and economic forecasting, these are now supplemented (supplanted?) by the other four methods mentioned previously. We will not discuss them in this chapter, for that would take us far afield.

### Single-Equation Regression Models

The bulk of this book has been devoted to single-equation regression models. As an example of a single-equation model, consider the demand function for automobiles. On the basis of economic theory, we postulate that the demand for automobiles is a function of automobile prices, advertising expenditure, income of the consumer, interest rate (as a measure of the cost of borrowing), and other relevant variables (e.g., family size, travel distance to work). From time series data, we estimate an appropriate model of auto demand (either a linear, log-linear or nonlinear), which can be used for forecasting demand for autos in the future. Of course, as noted in Chapter 5, forecasting errors increase rapidly if we go too far out in the future.

### Simultaneous-Equation Regression Models<sup>3</sup>

In Chapters 18, 19, and 20 we considered simultaneous-equation models. In their heyday during the 1960s and 1970s, elaborate models of the U.S.

<sup>2</sup>For a comparatively simple exposition of these methods, see Spyros Makridakis, Steven C. Wheelwright, and Rob J. Hyndman, *Forecasting Methods and Applications*, 3d ed., John Wiley & Sons, New York, 1998.

<sup>3</sup>For a textbook treatment of the use of simultaneous-equation models in forecasting, see Robert S. Pindyck and Daniel L. Rubinfeld, *Econometric Models & Economic Forecasts*, 4th ed., McGraw-Hill, New York, 1998, Part III.

economy based on simultaneous equations dominated economic forecasting. But since then the glamor of such forecasting models has subsided because of their poor forecasting performance, especially since the 1973 and 1979 oil price shocks (due to OPEC oil embargoes) and also because of the so-called **Lucas critique**.<sup>4</sup> The thrust of this critique, as you may recall, is that the parameters estimated from an econometric model are dependent on the policy prevailing at the time the model was estimated and will change if there is a policy change. In short, the estimated parameters are not invariant in the presence of policy changes.

For example, in October 1979 the Fed changed its monetary policy dramatically. Instead of targeting interest rates, it announced it would henceforth monitor the rate of growth of the money supply. With such a pronounced change, an econometric model estimated from past data will have little forecasting value in the new regime. These days the Fed's emphasis has changed from controlling the money supply to controlling the short-term interest rate (the federal funds rate).

### ARIMA Models

The publication by Box and Jenkins of *Time Series Analysis: Forecasting and Control* (op. cit.) ushered in a new generation of forecasting tools. Popularly known as the Box–Jenkins (BJ) methodology, but technically known as the ARIMA methodology, the emphasis of these methods is not on constructing single-equation or simultaneous-equation models but on analyzing the probabilistic, or stochastic, properties of economic time series on their own under the philosophy *let the data speak for themselves*. Unlike the regression models, in which  $Y_t$  is explained by  $k$  regressor  $X_1, X_2, X_3, \dots, X_k$ , the BJ-type time series models allow  $Y_t$  to be explained by past, or lagged, values of  $Y$  itself and stochastic error terms. For this reason, ARIMA models are sometimes called *atheoretic* models because they are not derived from any economic theory—and economic theories are often the basis of simultaneous-equation models.

In passing, note that our emphasis in this chapter is on *univariate* ARIMA models, that is, ARIMA models pertaining to a single time series. But the analysis can be extended to multivariate ARIMA models.

### VAR Models

**VAR** methodology superficially resembles simultaneous-equation modeling in that we consider several endogenous variables together. But each endogenous variable is explained by its lagged, or past, values and the lagged values of all other endogenous variables in the model; usually, there are no exogenous variables in the model.

<sup>4</sup>Robert E. Lucas, "Econometric Policy Evaluation: A Critique," in Carnegie–Rochester Conference Series, *The Phillips Curve*, North-Holland, Amsterdam, 1976, pp. 19–46. This article, among others, earned Lucas a Nobel Prize in economics.

In the rest of this chapter we discuss the fundamentals of Box–Jenkins and VAR approaches to economic forecasting. Our discussion is elementary and heuristic. The reader wishing to pursue this subject further is advised to consult the references.<sup>5</sup>

## 22.2 AR, MA, AND ARIMA MODELING OF TIME SERIES DATA

To introduce several ideas, some old and some new, let us work with the GDP time series data for the United States given in Table 21.1. A plot of this time series is already given in Figures 21.1 (undifferenced GDP) and 21.9 (first-differenced GDP); recall that GDP in level form is nonstationary but in the (first) differenced form it is stationary.

If a time series is stationary, we can model it in a variety of ways.

### An Autoregressive (AR) Process

Let  $Y_t$  represent GDP at time  $t$ . If we model  $Y_t$  as

$$(Y_t - \delta) = \alpha_1(Y_{t-1} - \delta) + u_t \quad (22.2.1)$$

where  $\delta$  is the mean of  $Y$  and where  $u_t$  is an uncorrelated random error term with zero mean and constant variance  $\sigma^2$  (i.e., it is *white noise*), then we say that  $Y_t$  follows a **first-order autoregressive**, or **AR(1)**, stochastic process, which we have already encountered in Chapter 12. Here the value of  $Y$  at time  $t$  depends on its value in the previous time period and a random term; the  $Y$  values are expressed as deviations from their mean value. In other words, this model says that the forecast value of  $Y$  at time  $t$  is simply some proportion ( $= \alpha_1$ ) of its value at time  $(t - 1)$  plus a random shock or disturbance at time  $t$ ; again the  $Y$  values are expressed around their mean values.

But if we consider this model,

$$(Y_t - \delta) = \alpha_1(Y_{t-1} - \delta) + \alpha_3(Y_{t-2} - \delta) + u_t \quad (22.2.2)$$

then we say that  $Y_t$  follows a **second-order autoregressive**, or **AR(2)**, process. That is, the value of  $Y$  at time  $t$  depends on its value in the previous two time periods, the  $Y$  values being expressed around their mean value  $\delta$ .

In general, we can have

$$(Y_t - \delta) = \alpha_1(Y_{t-1} - \delta) + \alpha_2(Y_{t-2} - \delta) + \cdots + \alpha_p(Y_{t-p} - \delta) + u_t \quad (22.2.3)$$

in which case  $Y_t$  is a **pth-order autoregressive**, or **AR(p)**, process.

<sup>5</sup>See Pindyck and Rubinfeld, op. cit., Part 3; Alan Pankratz, *Forecasting with Dynamic Regression Models*, John Wiley & Sons, New York, 1991 (this is an applied book); and Andrew Harvey, *The Econometric Analysis of Time Series*, The MIT Press, 2d ed., Cambridge, Mass., 1990 (this is a rather advanced book). A thorough but accessible discussion can also be found in Terence C. Mills, *Time Series Techniques for Economists*, Cambridge University Press, New York, 1990.

Notice that in all the preceding models only the current and previous  $Y$  values are involved; there are no other regressors. In this sense, we say that the “data speak for themselves.” They are a kind of *reduced form model* that we encountered in our discussion of the simultaneous-equation models.

### A Moving Average (MA) Process

The AR process just discussed is not the only mechanism that may have generated  $Y$ . Suppose we model  $Y$  as follows:

$$Y_t = \mu + \beta_0 u_t + \beta_1 u_{t-1} \quad (22.2.4)$$

where  $\mu$  is a constant and  $u$ , as before, is the white noise stochastic error term. Here  $Y$  at time  $t$  is equal to a constant plus a moving average of the current and past error terms. Thus, in the present case, we say that  $Y$  follows a **first-order moving average**, or an **MA(1)**, process.

But if  $Y$  follows the expression

$$Y_t = \mu + \beta_0 u_t + \beta_1 u_{t-1} + \beta_2 u_{t-2} \quad (22.2.5)$$

then it is an **MA(2)** process. More generally,

$$Y_t = \mu + \beta_0 u_t + \beta_1 u_{t-1} + \beta_2 u_{t-2} + \cdots + \beta_q u_{t-q} \quad (22.2.6)$$

is an **MA( $q$ )** process. In short, a moving average process is simply a linear combination of white noise error terms.

### An Autoregressive and Moving Average (ARMA) Process

Of course, it is quite likely that  $Y$  has characteristics of both AR and MA and is therefore *ARMA*. Thus,  $Y_t$  follows an **ARMA(1, 1)** process if it can be written as

$$Y_t = \theta + \alpha_1 Y_{t-1} + \beta_0 u_t + \beta_1 u_{t-1} \quad (22.2.7)$$

because there is one autoregressive and one moving average term. In (22.2.7)  $\theta$  represents a constant term.

In general, in an **ARMA( $p, q$ )** process, there will be  $p$  autoregressive and  $q$  moving average terms.

### An Autoregressive Integrated Moving Average (ARIMA) Process

The time series models we have already discussed are based on the assumption that the time series involved are (weakly) stationary in the sense defined in Chapter 21. Briefly, the mean and variance for a weakly stationary time series are constant and its covariance is time-invariant. But we know that many economic time series are nonstationary, that is, they are *integrated*; for example, the economic time series in Table 21.1 are integrated.

But we also saw in Chapter 21 that if a time series is integrated of order 1 [i.e., it is  $I(1)$ ], its first differences are  $I(0)$ , that is, stationary. Similarly, if a

time series is  $I(2)$ , its second difference is  $I(0)$ . In general, if a time series is  $I(d)$ , after differencing it  $d$  times we obtain an  $I(0)$  series.

Therefore, if we have to difference a time series  $d$  times to make it stationary and then apply the ARMA( $p, q$ ) model to it, we say that the original time series is **ARIMA( $p, d, q$ )**, that is, it is an **autoregressive integrated moving average** time series, where  $p$  denotes the number of autoregressive terms,  $d$  the number of times the series has to be differenced before it becomes stationary, and  $q$  the number of moving average terms. Thus, an ARIMA(2, 1, 2) time series has to be differenced once ( $d = 1$ ) before it becomes stationary and the (first-differenced) stationary time series can be modeled as an ARMA(2, 2) process, that is, it has two AR and two MA terms. Of course, if  $d = 0$  (i.e., a series is stationary to begin with), ARIMA( $p, d = 0, q$ ) = ARMA( $p, q$ ). Note that an ARIMA( $p, 0, 0$ ) process means a purely AR( $p$ ) stationary process; an ARIMA(0, 0,  $q$ ) means a purely MA( $q$ ) stationary process. Given the values of  $p, d$ , and  $q$ , one can tell what process is being modeled.

The important point to note is that to use the Box–Jenkins methodology, we must have either a stationary time series or a time series that is stationary after one or more differencings. The reason for assuming stationarity can be explained as follows:

The objective of B–J [Box–Jenkins] is to identify and estimate a statistical model which can be interpreted as having generated the sample data. If this estimated model is then to be used for forecasting we must assume that the features of this model are constant through time, and particularly over future time periods. Thus the simple reason for requiring stationary data is that any model which is inferred from these data can itself be interpreted as stationary or stable, therefore providing valid basis for forecasting.<sup>6</sup>

## 22.3 THE BOX–JENKINS (BJ) METHODOLOGY

The million-dollar question obviously is: Looking at a time series, such as the U.S. GDP series in Figure 21.1, how does one know whether it follows a purely AR process (and if so, what is the value of  $p$ ) or a purely MA process (and if so, what is the value of  $q$ ) or an ARMA process (and if so, what are the values of  $p$  and  $q$ ) or an ARIMA process, in which case we must know the values of  $p, d$ , and  $q$ . The BJ methodology comes in handy in answering the preceding question. The method consists of four steps:

**Step 1. Identification.** That is, find out the appropriate values of  $p, d$ , and  $q$ . We will show shortly how the **correlogram** and **partial correlogram** aid in this task.

**Step 2. Estimation.** Having identified the appropriate  $p$  and  $q$  values, the next stage is to estimate the parameters of the autoregressive and moving average terms included in the model. Sometimes this calculation can be done by simple least squares but sometimes we will have to resort to

<sup>6</sup>Michael Pokorny, *An Introduction to Econometrics*, Basil Blackwell, New York, 1987, p. 343.

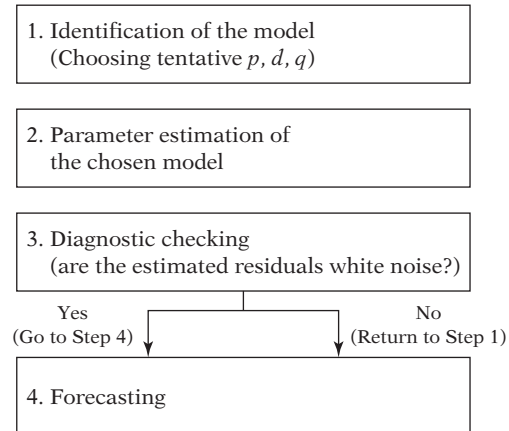


FIGURE 22.1 The Box–Jenkins methodology.

nonlinear (in parameter) estimation methods. Since this task is now routinely handled by several statistical packages, we do not have to worry about the actual mathematics of estimation; the enterprising student may consult the references on that.

**Step 3. Diagnostic checking.** Having chosen a particular ARIMA model, and having estimated its parameters, we next see whether the chosen model fits the data reasonably well, for it is possible that another ARIMA model might do the job as well. This is why Box–Jenkins ARIMA modeling is more an art than a science; considerable skill is required to choose the right ARIMA model. One simple test of the chosen model is to see if the residuals estimated from this model are white noise; if they are, we can accept the particular fit; if not, we must start over. **Thus, the BJ methodology is an iterative process** (see Figure 22.1).

**Step 4. Forecasting.** One of the reasons for the popularity of the ARIMA modeling is its success in forecasting. In many cases, the forecasts obtained by this method are more reliable than those obtained from the traditional econometric modeling, particularly for short-term forecasts. Of course, each case must be checked.

With this general discussion, let us look at these four steps in some detail. Throughout, we will use the GDP data given in Table 21.1 to illustrate the various points.

## 22.4 IDENTIFICATION

The chief tools in identification are the **autocorrelation function (ACF)**, the **partial autocorrelation function (PACF)**, and the resulting **correlograms**, which are simply the plots of ACFs and PACFs against the lag length.

In the previous chapter we defined the (population) ACF ( $\rho_k$ ) and the sample ACF ( $\hat{\rho}_k$ ). The concept of partial autocorrelation is analogous to the



concept of partial regression coefficient. In the  $k$ -variable multiple regression model, the  $k$ th regression coefficient  $\beta_k$  measures the rate of change in the mean value of the regressand for a unit change in the  $k$ th regressor  $X_k$ , holding the influence of all other regressors constant.

In similar fashion the **partial autocorrelation**  $\rho_{kk}$  measures correlation between (time series) observations that are  $k$  time periods apart after controlling for correlations at intermediate lags (i.e., lag less than  $k$ ). In other words, partial autocorrelation is the correlation between  $Y_t$  and  $Y_{t-k}$  after removing the effect of the intermediate  $Y$ 's.<sup>7</sup> In Section 7.11 we already introduced the concept of partial correlation in the regression context and showed its relation to simple correlations. Such partial correlations are now routinely computed by most statistical packages.

In Figure 22.2 we show the correlogram and partial correlogram of the GDP series. From this figure, two facts stand out: First, the ACF declines

Lag	Sample ACF ( $\hat{\rho}_k$ )	Sample PACF ( $\hat{\rho}_{kk}$ )
1	0.969	0.969
2	0.935	-0.058
3	0.901	-0.020
4	0.866	-0.045
5	0.830	-0.024
6	0.791	-0.062
7	0.752	-0.029
8	0.713	-0.024
9	0.675	-0.009
10	0.638	-0.010
11	0.601	-0.020
12	0.565	-0.012
13	0.532	-0.020
14	0.500	-0.012
15	0.468	-0.021
16	0.437	-0.001
17	0.405	-0.041
18	0.375	-0.005
19	0.344	-0.038
20	0.313	-0.017
21	0.279	-0.066
22	0.246	-0.019
23	0.214	-0.008
24	0.182	-0.018
25	0.153	0.017

95% confidence interval
95% confidence interval

FIGURE 22.2 Correlogram and partial correlogram, GDP, United States, 1970–I to 1991–IV.

<sup>7</sup>In time series data a large proportion of correlation between  $Y_t$  and  $Y_{t-k}$  may be due to the correlations they have with the intervening lags  $Y_{t-1}, Y_{t-2}, \dots, Y_{t-k+1}$ . The partial correlation  $\rho_{kk}$  removes the influence of these intervening variables.





approximate 95% confidence limits for  $\rho_k$  are  $-0.2089$  and  $+0.2089$ . (*Note:* As discussed in Chapter 21, these confidence limits are asymptotic and so can be considered approximate.) But at all other lags, they are not statistically different from zero. This is also true of the partial autocorrelations,  $\hat{\rho}_{kk}$ .

Now how do the correlograms given in Figure 22.3 enable us to find the ARMA pattern of the GDP time series? (*Note:* We will consider only the first-differenced GDP series because it is stationary.) One way of accomplishing this is to consider the ACF and PACF and the associated correlograms of a selected number of ARMA processes, such as AR(1), AR(2), MA(1), MA(2), ARMA(1, 1), ARIMA(2, 2), and so on. Since each of these stochastic processes exhibits typical patterns of ACF and PACF, if the time series under study fits one of these patterns we can identify the time series with that process. Of course, we will have to apply diagnostic tests to find out if the chosen ARMA model is reasonably accurate.

To study the properties of the various standard ARIMA processes would consume a lot of space. What we plan to do is to give general guidelines (see Table 22.1); the references can give the details of the various stochastic processes.

Notice that the ACFs and PACFs of AR( $p$ ) and MA( $q$ ) processes have opposite patterns; in the AR( $p$ ) case the AC declines geometrically or exponentially but the PACF cuts off after a certain number of lags, whereas the opposite happens to an MA( $q$ ) process.

Geometrically, these patterns are shown in Figure 22.4.

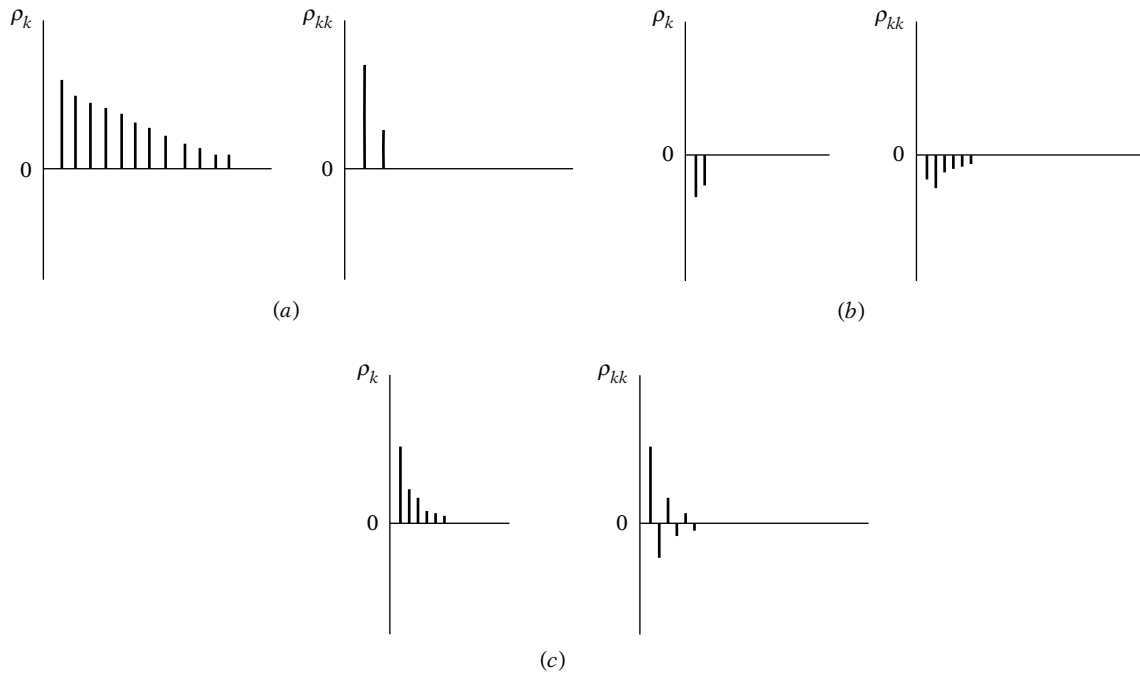
**A Warning.** Since in practice we do not observe the theoretical ACFs and PACFs and rely on their sample counterparts, the estimated ACFs and PACFs will not match exactly their theoretical counterparts. What we are looking for is the resemblance between theoretical and sample ACFs and PACFs so that they can point us in the right direction in constructing ARIMA models. And that is why ARIMA modeling requires a great deal of skill, which of course comes from practice.

**ARIMA Identification of U.S. GDP.** Returning to the correlogram and partial correlogram of the stationary (after first-differencing) U.S. GDP for 1970–I to 1991–IV given in Figure 22.3, what do we see?

TABLE 22.1 THEORETICAL PATTERNS OF ACF AND PACF

Type of model	Typical pattern of ACF	Typical pattern of PACF
AR( $p$ )	Decays exponentially or with damped sine wave pattern or both	Significant spikes through lags $p$
MA( $q$ )	Significant spikes through lags $q$	Declines exponentially
ARMA( $p, q$ )	Exponential decay	Exponential decay

*Note:* The terms exponential and geometric decay mean the same things (recall our discussion of the Koyck distributed lag).



**FIGURE 22.4** ACF and PACF of selected stochastic processes: (a) AR(2):  $\alpha_1 = 0.5, \alpha_2 = 0.3$ ; (b) MA(2):  $\beta_1 = 0.5, \beta_2 = 0.3$ ; (c) ARMA(1, 1):  $\alpha_1 = 0.5, \beta_1 = 0.5$ .

Remembering that the ACF and PACF shown there are sample quantities, we do not have a nice pattern as suggested in Table 22.1. The autocorrelations decline up to lag 4, then except at lags 8 and 12, the rest of them are statistically not different from zero (the solid lines shown in this figure give the approximate 95% confidence limits). The partial autocorrelations with spikes at lag 1, 8, and 12 seem statistically significant but the rest are not; if the partial correlation coefficient were significant only at lag 1, we could have identified this as an AR(1) model. Let us therefore assume that the process that generated the (first-differenced) GDP is at the most an AR(12) process. Of course, we do not have to include all the AR terms up to 12, for from the partial correlogram we know that only the AR terms at lag 1, 8, and 12 are significant.

## 22.5 ESTIMATION OF THE ARIMA MODEL

Let  $Y_t^*$  denote the first differences of U.S. GDP. Then our tentatively identified AR model is

$$Y_t^* = \delta + \alpha_1 Y_{t-1}^* + \alpha_8 Y_{t-8}^* + \alpha_{12} Y_{t-12}^* \quad (22.5.1)$$

Using Eviews, we obtained the following estimates:

$$\begin{aligned} \widehat{Y}_t^* &= 23.0894 + 0.3428Y_{t-1}^* - 0.2994Y_{t-8}^* - 0.2644Y_{t-12}^* \\ \text{se} &= (2.9774) \quad (0.0987) \quad (0.1016) \quad (0.0986) \\ t &= (7.7547) \quad (3.4695) \quad (-2.9475) \quad (-2.6817) \end{aligned} \quad (22.5.2)$$

$$R^2 = 0.2931 \quad d = 1.7663$$

We leave it as an exercise for the reader to estimate a model that contains only  $Y_{t-1}^*$  and a model that contains both  $Y_{t-1}^*$  and  $Y_{t-8}^*$  terms and to compare the results with those given in (22.5.2).

### 22.6 DIAGNOSTIC CHECKING

How do we know that the model in (22.5.2) is a reasonable fit to the data? One simple diagnostic is to obtain residuals from (22.5.2) and obtain the ACF and PACF of these residuals, say, up to lag 25. The estimated AC and PACF are shown in Figure 22.5. As this figure shows, none of the autocorrelations

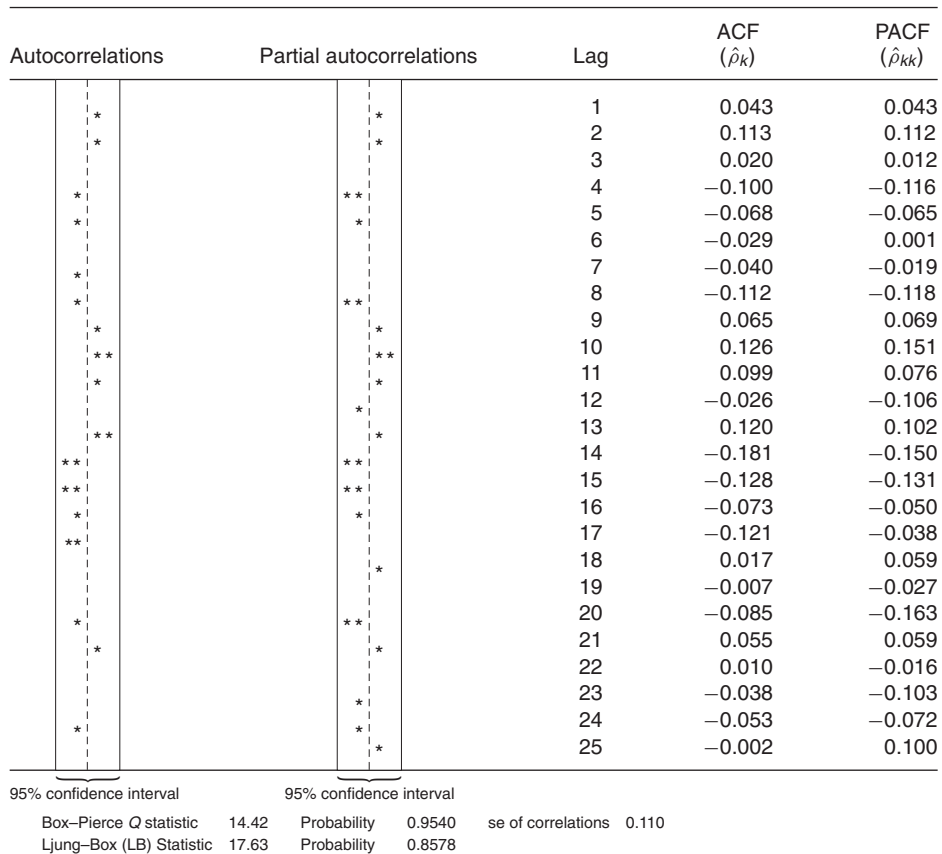


FIGURE 22.5 Correlograms of the residuals obtained from ARIMA model (22.5.2).

and partial autocorrelations is individually statistically significant. Nor is the sum of the 25 squared autocorrelations, as shown by the Box–Pierce  $Q$  and Ljung–Box LB statistics (see Chapter 21), statistically significant. In other words, the correlograms of both autocorrelation and partial autocorrelation give the impression that the residuals estimated from (22.5.2) are purely random. Hence, there may not be any need to look for another ARIMA model.

## 22.7 FORECASTING

Remember that the GDP data are for the period 1970–I to 1991–IV. Suppose, on the basis of model (22.5.2), we want to forecast GDP for the first four quarters of 1992. But in (22.5.2) the dependent variable is *change* in the GDP over the previous quarter. Therefore, if we use (22.5.2), what we can obtain are the forecasts of GDP changes between the first quarter of 1992 and the fourth quarter of 1991, second quarter of 1992 over the first quarter of 1992, etc.

To obtain the forecast of GDP level rather than its changes, we can “undo” the first-difference transformation that we had used to obtain the changes. (More technically, we *integrate* the first-differenced series.) Thus, to obtain the forecast value of GDP (not  $\Delta$ GDP) for 1992–I, we rewrite model (22.5.1) as

$$\begin{aligned} Y_{1992-I} - Y_{1991-IV} &= \delta + \alpha_1[Y_{1991-IV} - Y_{1991-III}] \\ &\quad + \alpha_8[Y_{1989-IV} - Y_{1989-III}] \\ &\quad + \alpha_{12}[Y_{1988-IV} - Y_{1988-III}] + u_{1992-I} \end{aligned} \quad (22.7.1)$$

That is,

$$\begin{aligned} Y_{1992-I} &= \delta + (1 + \alpha_1)Y_{1991-IV} - \alpha_1 Y_{1991-III} + \alpha_8 Y_{1989-IV} - \alpha_8 Y_{1989-III} \\ &\quad + \alpha_{12} Y_{1988-IV} - \alpha_{12} Y_{1988-III} + u_{1992-I} \end{aligned} \quad (22.7.2)$$

The values of  $\delta$ ,  $\alpha_1$ ,  $\alpha_8$ , and  $\alpha_{12}$  are already known from the estimated regression (22.5.2). The value of  $u_{1992-I}$  is assumed to be zero (why?). Therefore, we can easily obtain the forecast value of  $Y_{1992-I}$ . The numerical estimate of this forecast value is<sup>9</sup>

$$\begin{aligned} \hat{Y}_{1992-I} &= 23.0894 + (1 + 0.3428)Y_{1991-IV} - 0.3428Y_{1991-III} \\ &\quad + (-0.2994)Y_{1989-IV} - (-0.2994)Y_{1989-III} \\ &\quad + (-0.2644)Y_{1988-IV} - (-0.2644)Y_{1988-III} \\ &= 23.0894 + 1.3428(4868) - 0.3428(4862.7) \\ &\quad - 0.2994(4859.7) + 0.2994(4845.6) - 0.2644(4779.7) \\ &\quad + 0.2644(4734.5) \\ &= 4876.7 \quad (\text{approx.}) \end{aligned}$$

<sup>9</sup>Although standard computer packages do this computation routinely, we show the detailed calculations to illustrate the mechanics involved.

Thus the forecast value of GDP for 1992–I is about \$4877 billion (1987 dollars). Incidentally, the actual value of real GDP for 1992–I was \$4873.7 billion; the forecast error was an overestimate of \$3 billion.

Note that if you were to use (22.5.2) to compute the forecast change of GDP from 1991–IV to 1992–I, you would obtain this figure as  $-\$4.25$  billion.

## 22.8 FURTHER ASPECTS OF THE BJ METHODOLOGY

In the preceding paragraphs we have provided but a sketchy introduction to the BJ modeling. There are many aspects of this methodology that we have not considered for lack of space, for example, **seasonality**. Many time series exhibit seasonal behavior. Examples are sales by department stores in conjunction with major holidays, seasonal consumption of ice cream, travels during public holidays, etc. If, for example, we had data on department stores sales by quarters, the sales figures will show spikes in the fourth quarter. In such situations, one can remove the seasonal influence by taking fourth-quarter differences of the sales figures and then decide what kind of ARIMA model to fit.

We have analyzed only a single time series at a time. But nothing prevents the BJ methodology from being extended to the simultaneous study of two or more time series. A foray into that topic would take us far afield. The interested reader may want to consult the references.<sup>10</sup> In the following section, however, we discuss this topic in the context of what is known as vector autoregression.

## 22.9 VECTOR AUTOREGRESSION (VAR)

In Chapters 18 to 20 we considered simultaneous, or structural, equation models. In such models some variables are treated as endogenous and some as exogenous or predetermined (exogenous plus lagged endogenous). Before we estimate such models, we have to make sure that the equations in the system are identified (either exactly or over-). This identification is often achieved by assuming that some of the predetermined variables are present only in some equations. This decision is often subjective and has been severely criticized by Christopher Sims.<sup>11</sup>

According to Sims, if there is true simultaneity among a set of variables, they should all be treated on an equal footing; there should not be any a priori distinction between endogenous and exogenous variables. It is in this spirit that Sims developed his **VAR** model.

The seeds of this model were already sown in the Granger causality test discussed in Chapter 17. In Eqs. (17.14.1) and (17.14.2), which explain current GDP in terms of lagged money supply and lagged GDP and current money supply in terms of lagged money supply and lagged GDP, we are

<sup>10</sup>For an accessible treatment of this subject, see Terence C. Mills, op. cit., Part III.

<sup>11</sup>C. A. Sims, "Macroeconomics and Reality," *Econometrica*, vol. 48, 1980, pp. 1–48.

essentially treating GDP and money supply as a pair of endogenous variables. There are no exogenous variables in this system.

Similarly, in Example 17.13 we examined the nature of causality between money and interest rate in Canada. In the money equation, only the lagged values of money and interest rate appear, and in the interest rate equation only the lagged values of interest rate and money appear.

Both these examples are illustrations of **vector autoregressive models**; the term autoregressive is due to the appearance of the lagged value of the dependent variable on the right-hand side and the term vector is due to the fact that we are dealing with a vector of two (or more) variables.

### Estimation or VAR

Returning to the Canadian money–interest rate example, we saw that when we introduced six lags of each variable as regressors, we could not reject the hypothesis that there was bilateral causality between money ( $M_1$ ) and interest rate,  $R$  (90-day corporate interest rate). That is,  $M_1$  affects  $R$  and  $R$  affects  $M_1$ . These kinds of situations are ideally suited for the application of VAR.

To explain how a VAR is estimated, we will continue with the preceding example. For now we assume that each equation contains  $k$  lag values of  $M$  (as measured by  $M_1$ ) and  $R$ . In this case, one can estimate each of the following equations by OLS.<sup>12</sup>

$$M_{1t} = \alpha + \sum_{j=1}^k \beta_j M_{t-j} + \sum_{j=1}^k \gamma_j R_{t-j} + u_{1t} \quad (22.9.1)$$

$$R_t = \alpha' + \sum_{j=1}^k \theta_j M_{t-j} + \sum_{j=1}^k \gamma_j R_{t-j} + u_{2t} \quad (22.9.2)$$

where the  $u$ 's are the stochastic error terms, called **impulses** or **innovations** or **shocks** in the language of VAR.

Before we estimate (22.9.1) and (22.9.2) we have to decide on the maximum lag length,  $k$ . This is an empirical question. We have 40 observations in all. Including too many lagged terms will consume degrees of freedom, not to mention introducing the possibility of multicollinearity. Including too few lags will lead to specification errors. One way of deciding this question is to use a criterion like the Akaike or Schwarz and choose that model that gives the lowest values of these criteria. There is no question that some trial and error is inevitable.

To illustrate the mechanics, we initially used four lags ( $k = 4$ ) of each variable and using Eviews 4 we obtained the estimates of the parameters of the

<sup>12</sup>One can use the **SURE** (seemingly unrelated regression) technique to estimate the two equations together. However, since each regression contains the same number of lagged endogenous variables, the OLS estimation of each equation separately produces identical (and efficient) estimates.

**TABLE 22.2** VECTOR AUTOREGRESSION ESTIMATES BASED ON 4 LAGS

Sample (adjusted): 1980–I to 1987–IV  
 Included observations: 32 after adjusting endpoints  
 Standard errors in ( ) and *t* statistics in [ ]

	$M_1$	$R$
$M_1$ (–1)	1.076737 (0.20174) [5.33733]	0.001282 (0.00067) [1.90083]
$M_1$ (–2)	0.173433 (0.31444) [0.55157]	–0.002140 (0.00105) [–2.03584]
$M_1$ (–3)	–0.366465 (0.34687) [–1.05648]	0.002176 (0.00116) [1.87699]
$M_1$ (–4)	0.077602 (0.20789) [0.37329]	–0.001479 (0.00069) [–2.12855]
$R$ (–1)	–275.0293 (57.2174) [–4.80675]	1.139310 (0.19127) [5.95670]
$R$ (–2)	227.1750 (95.3947) [2.38142]	–0.309053 (0.31888) [–0.96917]
$R$ (–3)	8.511851 (96.9176) [0.08783]	0.052361 (0.32397) [0.16162]
$R$ (–4)	–50.19926 (64.7554) [–0.77521]	0.001076 (0.21646) [0.00497]
$C$	2413.827 (1622.65) [1.48759]	4.919000 (5.42416) [0.90687]
$R^2$	0.988154	0.852890
Adj. $R^2$	0.984034	0.801721
Sum square residuals	4820241.	53.86233
SE equation	457.7944	1.530307
$F$ statistic	239.8315	16.66815
Log likelihood	–236.1676	–53.73716
Akaike A/C	15.32298	3.921073
Schwarz SC	15.73521	4.333311
Mean dependent	28514.53	11.67292
SD dependent	3623.058	3.436688
Determinant residual covariance	490782.3	
Log likelihood (df adjusted)	–300.4722	
Akaike information criterion	19.90451	
Schwarz criterion	20.72899	

preceding two equations, which are given in Table 22.2. Note that although our sample runs from 1979–1 to 1988–4, we used the sample for the period 1979–1 to 1987–4 and saved the last four observations to check the forecasting accuracy of the fitted VAR.

Since the preceding equations are OLS regressions, the output of the regression given in Table 22.2 is to be interpreted in the usual fashion. Of course, with several lags of the same variables, each estimated coefficient will not be statistically significant, possibly because of multicollinearity. But collectively, they may be significant on the basis of the standard  $F$  test.

Let us examine the results presented in Table 22.2. First consider the  $M_1$  regression. Individually, only  $M_1$  at lag 1 and  $R$  at lags 1 and 2 are statistically significant. But the  $F$  value is so high that we cannot reject the hypothesis that collectively all the lagged terms are statistically significant. Turning to the interest rate regression, we see that all the four lagged money



terms are individually statistically significant (at 10 percent or better level), whereas only the 1-period lagged interest rate variable is significant.

For comparative purposes, we present in Table 22.3 the VAR results based on only 2 lags of each endogenous variable. Here you will see that in the money regression the 1-period-lagged money variable and both lagged interest rate terms are individually statistically significant. In the interest rate regression, both lagged money terms (at about 5 percent level) and one lagged interest term are individually significant.

If we have to make a choice between the model given in Table 22.2 and that given in Table 22.3, which would we choose? The Akaike and Schwarz information values for the model in Table 22.2 are, respectively, 15.32 and 15.73, whereas the corresponding values for Table 22.3 are 15.10 and 15.33. Since the lower the values of Akaike and Schwarz statistics, the better the model, on that basis it seems the more parsimonious model given in Table 22.3 is preferable. We also considered 6 lags of each of the endogenous variables and found that the values of Akaike and Schwarz statistics were 15.37 and 15.98, respectively. Again, the choice seems to be the model with two lagged terms of each endogenous variable, that is, the model in Table 22.3.

**TABLE 22.3** VECTOR AUTOREGRESSION ESTIMATES BASED ON 2 LAGS

Sample (adjusted): 1979–III to 1987–IV  
Included observations: 34 after adjusting endpoints  
Standard errors in ( ) and *t* statistics in [ ]

	$M_1$	$R$
$M_1(-1)$	1.037537 (0.16048) [6.46509]	0.001091 (0.00059) [1.85825]
$M_1(-2)$	-0.044661 (0.15591) [-0.28646]	-0.001255 (0.00057) [-2.19871]
$R(-1)$	-234.8850 (45.5224) [-5.15977]	1.069081 (0.16660) [6.41708]
$R(-2)$	160.1560 (48.5283) [3.30026]	-0.223364 (0.17760) [-1.25768]
$C$	1451.977 (1185.59) [1.22468]	5.796434 (4.33894) [1.33591]
$R^2$	0.988198	0.806660
Adj. $R^2$	0.986571	0.779993
Sum square residuals	5373510.	71.97054
SE equation	430.4573	1.575355
$F$ statistic	607.0720	30.24878
Log likelihood	-251.7446	-60.99215
Akaike A/C	15.10263	3.881891
Schwarz SC	15.32709	4.106356
Mean dependent	28216.26	11.75049
SD dependent	3714.506	3.358613
Determinant residual covariance	458485.4	
Log likelihood (df adjusted)	-318.0944	
Akaike information criterion	19.29967	
Schwarz criterion	19.74860	

### Forecasting with VAR

Suppose we choose the model given in Table 22.3. We can use it for the purpose of forecasting the values of  $M_1$  and  $R$ . Remember that our data covers the period 1979–I to 1988–IV, but we have not used the values for 1988 in estimating the VAR models. Now suppose we want to forecast the value of  $M_1$  for 1988–I, that is, the first quarter of 1988. The forecast value for 1988–I can be obtained as follows:

$$\begin{aligned}\hat{M}_{1988-I} = & 1451.977 + 1.0375M_{1987-IV} - 0.0446M_{1987-III} \\ & - 234.8850R_{1987-IV} + 160.1560R_{1987-III}\end{aligned}$$

where the coefficient values are obtained from Table 22.3. Now using the appropriate values of  $M$  and  $R$  from Table 17.3, the forecast value of money for the first quarter of 1988 can be seen to be 36,996 (millions of Canadian dollars). The actual value of  $M$  for 1988–I was 36,480, which means that our model overpredicted the actual value by about 516 (millions of dollars), which is about 1.4 percent of the actual  $M$  for 1988–I. Of course, these estimates will change, depending on how many lagged values we consider in the VAR model. It is left as an exercise for the reader to forecast the value of  $R$  for the first quarter of 1988 and compare it with its actual value for that quarter.

### VAR and Causality

You may recall that we discussed the topic of causality in Chapter 17. There we considered the Granger's and Sims's tests of causality. Is there any connection between VAR and causality? In Chapter 17 (Section 14) we saw that up to 2, 4, and 6 lags there was bilateral causality between  $M_1$  and  $R$ , but at lag 8 there was no causality between the two variables. Thus, the results are mixed. Now you may recall from Chapter 21 the Granger representation theorem. One of the implications of this theorem is that if two variables, say,  $X_t$  and  $Y_t$  are cointegrated and each is individually  $I(1)$ , that is, integrated of order 1 (i.e., each is individually nonstationary), then either  $X_t$  must Granger-cause  $Y_t$  or  $Y_t$  must Granger-cause  $X_t$ .

In our illustrative example this means if  $M_1$  and  $R$  are individually  $I(1)$ , but are cointegrated, then either  $M_1$  must Granger-cause  $R$  or  $R$  must Granger-cause  $M_1$ . This means we must first find out if the two variables are  $I(1)$  individually and then find out if they are cointegrated. If this is not the case, then the whole question of causality may become moot. In exercise 22.22, the reader is asked to find out if the two variables are nonstationary but are cointegrated. If you do the exercise, you will find that there is some *weak* evidence of cointegration between  $M_1$  and  $R$ , which is why the causality tests discussed in Section 17.14 were equivocal.

### Some Problems with VAR Modeling

The advocates of VAR emphasize these virtues of the method: (1) The method is simple; one does not have to worry about determining which variables are endogenous and which ones exogenous. All variables in VAR are endogenous.<sup>13</sup> (2) Estimation is simple; that is, the usual OLS method can be applied to each equation separately. (3) The forecasts obtained by this method are in many cases better than those obtained from the more complex simultaneous-equation models.<sup>14</sup>

But the critics of VAR modeling point out the following problems:

1. Unlike simultaneous-equation models, a VAR model is *a-theoretic* because it uses less prior information. Recall that in simultaneous-equation models exclusion or inclusion of certain variables plays a crucial role in the identification of the model.

2. Because of its emphasis on forecasting, VAR models are less suited for policy analysis.

3. The biggest practical challenge in VAR modeling is to choose the appropriate lag length. Suppose you have a three-variable VAR model and you decide to include eight lags of each variable in each equation. You will have 24 lagged parameters in each equation plus the constant term, for a total of 25 parameters. Unless the sample size is large, estimating that many parameters will consume a lot of degrees of freedom with all the problems associated with that.<sup>15</sup>

4. Strictly speaking, in an  $m$ -variable VAR model, all the  $m$  variables should be (jointly) stationary. If that is not the case, we will have to transform the data appropriately (e.g., by first-differencing). As Harvey notes, the results from the transformed data may be unsatisfactory. He further notes that “The usual approach adopted by VAR *aficionados* is therefore to work in levels, even if some of these series are nonstationary. In this case, it is important to recognize the effect of unit roots on the distribution of estimators.”<sup>16</sup> Worse yet, if the model contains a mix of  $I(0)$  and  $I(1)$  variables, that is, a mix of stationary and nonstationary variables, transforming the data will not be easy.

5. Since the individual coefficients in the estimated VAR models are often difficult to interpret, the practitioners of this technique often estimate the so-called **impulse response function (IRF)**. The IRF traces out the response of the dependent variable in the VAR system to shocks in the error

<sup>13</sup>Sometimes purely exogenous variables are included to allow for trend and seasonal factors.

<sup>14</sup>See, for example, T. Kinal and J. B. Ratner, “Regional Forecasting Models with Vector Autoregression: The Case of New York State,” Discussion Paper #155, Department of Economics, State University of New York at Albany, 1982.

<sup>15</sup>If we have an  $m$ -equation VAR model with  $p$  lagged values of the  $m$  variables, in all we have to estimate  $(m + pm^2)$  parameters.

<sup>16</sup>Andrew Harvey, *The Econometric Analysis of Time Series*, The MIT Press, 2d ed., Cambridge, Mass., 1990, p. 83.

terms, such as  $u_1$  and  $u_2$  in Eqs. (22.9.1) and (22.9.2). Suppose  $u_1$  in the  $M_1$  equation increases by a value of one standard deviation. Such a shock or change will change  $M_1$  in the current as well as future periods. But since  $M_1$  appears in the  $R$  regression, the change in  $u_1$  will also have an impact on  $R$ . Similarly, a change of one standard deviation in  $u_2$  of the  $R$  equation will have an impact on  $M_1$ . The IRF traces out the impact of such shocks for several periods in the future. Although the utility of such IRF analysis has been questioned by researchers, it is the centerpiece of VAR analysis.<sup>17</sup>

For a comparison of the performance of VAR with other forecasting techniques, the reader may consult the references.<sup>18</sup>

### An Application of VAR: A VAR Model of the Texas Economy

To test the conventional wisdom, “As the oil patch goes, so goes the Texas economy,” Thomas Fomby and Joseph Hirschberg developed a three-variable VAR model of the Texas economy for the period 1974–I to 1988–I.<sup>19</sup> The three variables considered were (1) percentage change in real price of oil, (2) percentage change in Texas nonagricultural employment, and (3) percentage change in nonagricultural employment in the rest of the United States. The authors introduced the constant term and two lagged values of each variable in each equation. Therefore, the number of parameters estimated in each equation was seven. The results of the OLS estimation of the VAR model are given in Table 22.4. The  $F$  tests given in this table are to test the hypothesis that collectively the various lagged coefficients are zero. Thus, the  $F$  test for the  $x$  variable (percentage change in real price of oil) shows that both the lagged terms of  $x$  are statistically different from zero; the probability of obtaining an  $F$  value of 12.5536 under the null hypothesis that they are both simultaneously equal to zero is very low, about 0.00004. On the other hand, collectively, the two lagged  $y$  values (percentage change in Texas nonagricultural employment) are not significantly different from zero to explain  $x$ ; the  $F$  value is only 1.36. All other  $F$  statistics are to be interpreted similarly.

On the basis of these and other results presented in their paper, Fomby and Hirschberg conclude that the conventional wisdom about the Texas economy is not quite accurate, for after the initial instability resulting from OPEC oil shocks, the Texas economy is now less dependent on fluctuations in the price of oil.

<sup>17</sup>D. E. Runkle, “Vector Autoregression and Reality,” *Journal of Business and Economic Statistics*, vol. 5, 1987, pp. 437–454.

<sup>18</sup>S. McNeese, “Forecasting Accuracy of Alternative Techniques: A Comparison of U.S. Macroeconomic Forecasts,” *Journal of Business and Economic Statistics*, vol. 4, 1986, pp. 5–15; E. Mahmoud, “Accuracy in Forecasting: A Survey,” *Journal of Forecasting*, vol. 3, 1984, pp. 139–159.

<sup>19</sup>Thomas B. Fomby and Joseph G. Hirschberg, “Texas in Transition: Dependence on Oil and the National Economy,” *Economic Review*, Federal Reserve Bank of Dallas, January 1989, pp. 11–28.

**TABLE 22.4** ESTIMATION RESULTS FOR SECOND-ORDER\* TEXAS VAR SYSTEM: 1974-I TO 1988-I

Dependent variable: x (percentage change in real price of oil)				
Variable	Lag	Coefficient	Standard error	Significance level
x	1	0.7054	0.1409	0.8305E-5
x	2	-0.3351	0.1500	0.3027E-1
y	1	-1.3525	2.7013	0.6189
y	2	3.4371	2.4344	0.1645
z	1	3.4566	2.8048	0.2239
z	2	-4.8703	2.7500	0.8304E-1
Constant	0	-0.9983E-2	0.1696E-1	0.5589
$\bar{R}^2 = 0.2982$ ; $Q(21) = 8.2618$ ( $P = 0.9939$ )				
Tests for joint significance, dependent variable = x				
Variable			F-statistic	Significance level
x			12.5536	0.4283E-4
y			1.3646	0.2654
z			1.5693	0.2188
Dependent variable: y (percentage change in Texas nonagricultural employment)				
Variable	Lag	Coefficient	Standard error	Significance level
x	1	0.2228E-1	0.8759E-2	0.1430E-1
x	2	-0.1883E-2	0.9322E-2	0.8407
y	1	0.6462	0.1678	0.3554E-3
y	2	0.4234E-1	0.1512	0.7807
z	1	0.2655	0.1742	0.1342
z	2	-0.1715	0.1708	0.3205
Constant	0	-0.1602E-2	0.1053E-1	0.1351
$\bar{R}^2 = 0.6316$ ; $Q(21) = 21.5900$ ( $P = 0.4234$ )				
Tests for joint significance, dependent variable = y				
Variable			F-statistic	Significance level
x			3.6283	0.3424E-4
y			19.1440	0.8287E-6
z			1.1684	0.3197
Dependent variable: z (percentage change in nonagricultural employment in rest of United States)				
Variable	Lag	Coefficient	Standard error	Significance level
x	1	-0.8330E-2	0.6849E-2	0.2299
x	2	0.3635E-2	0.7289E-2	0.6202
y	1	0.3849	0.1312	0.5170E-2
y	2	-0.4805	0.1182	0.1828E-2
z	1	0.7226	0.1362	0.3004E-5
z	2	-0.1366E-1	0.1336	0.9190
Constant	0	-0.2387E-2	0.8241E-3	0.5701E-2
$\bar{R}^2 = 0.6503$ ; $Q(21) = 15.6182$ ( $P = 0.7907$ )				
Tests for joint significance, dependent variable = z				
Variable			F-statistic	Significance level
x			0.7396	0.4827
y			8.2714	0.8360E-3
z			27.9609	0.1000E-7

\*Two-lagged terms of each variable.

Source: *Economic Review*, Federal Reserve Bank of Dallas, January 1989, p. 21.

## 22.10 MEASURING VOLATILITY IN FINANCIAL TIME SERIES: THE ARCH AND GARCH MODELS

As noted in the introduction to this chapter, financial time series, such as stock prices, exchange rates, inflation rates, etc. often exhibit the phenomenon of **volatility clustering**, that is, periods in which their prices show wide swings for an extended time period followed by periods in which there is relative calm. As Philip Franses notes:

Since such [financial time series] data reflect the result of trading among buyers and sellers at, for example, stock markets, various sources of news and other exogenous economic events may have an impact on the time series pattern of asset prices. Given that news can lead to various interpretations, and also given that specific economic events like an oil crisis can last for some time, we often observe that large positive and large negative observations in financial time series tend to appear in clusters.<sup>20</sup>

Knowledge of volatility is of crucial importance in many areas. For example, considerable macroeconomic work has been done in studying the variability of inflation over time. For some decision makers, inflation in itself may not be bad, but its variability is bad because it makes financial planning difficult.

The same is true of importers, exporters, and traders in foreign exchange markets, for variability in the exchange rates means huge losses or profits. Investors in the stock market are obviously interested in the volatility of stock prices, for high volatility could mean huge losses or gains and hence greater uncertainty. In volatile markets it is difficult for companies to raise capital in the capital markets.

How do we model financial time series that may experience such volatility? For example, how do we model time series of stock prices, exchange rates, inflation, etc.? A characteristic of most of these financial time series is that in their *level form* they are random walks; that is, they are nonstationary. On the other hand, in the first difference form, they are generally stationary, as we saw in the case of GDP series in the previous chapter even though GDP is not strictly a financial time series.

Therefore, instead of modeling the levels of financial time series, why not model their first differences? But these first differences often exhibit wide swings, or **volatility**, suggesting that the variance of financial time series varies over time. How can we model such “varying variance”? This is where the so-called **autoregressive conditional heteroscedasticity (ARCH)** model originally developed by Engle comes in handy.<sup>21</sup>

As the name suggests, heteroscedasticity, or unequal variance, may have an autoregressive structure in that heteroscedasticity observed over differ-

<sup>20</sup>Philip Hans Franses, *Time Series Models for Business and Economic Forecasting*, Cambridge University Press, New York, 1998, p. 155.

<sup>21</sup>R. Engle, “Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation,” *Econometrica*, vol. 50, no. 1, 1982, pp. 987–1007. See also A. Bera and M. Higgins, “ARCH Models: Properties, Estimation and Testing,” *Journal of Economic Surveys*, vol. 7, 1993, pp. 305–366.

ent periods may be autocorrelated. To see what all this means, let us consider a concrete example.

#### U.S./U.K. EXCHANGE RATE: AN EXAMPLE

Figure 22.6 gives *logs* of the monthly U.S./U.K. exchange rate (dollars per pound) for the period 1973 to 1995, for a total of 276 monthly observations. As you can see from this figure, there are considerable ups and downs in the exchange rate over the sample period. To see this more vividly, in Figure 22.7 we plot the changes in the logs of the exchange rate; note that changes in the log of a variable denote relative changes, which, if multiplied by 100, give percentage changes. As you can observe, the relative changes in the U.S./U.K. exchange rate show periods of wide swings for some time period and periods of rather moderate swings in other time periods, thus exemplifying the phenomenon of volatility clustering.

Now the practical question is: How do we statistically measure volatility? Let us illustrate this with our exchange rate example.

Let  $Y_t$  = U.S./U.K. exchange rate

$Y_t^*$  = log of  $Y_t$

$dY_t^* = Y_t^* - Y_{t-1}^*$  = relative change in the exchange rate

$d\bar{Y}_t^*$  = mean of  $dY_t^*$

$X_t = dY_t^* - d\bar{Y}_t^*$

Thus,  $X_t$  is the mean-adjusted relative change in the exchange rate. Now we can use  $X_t^2$  as a measure of volatility. Being a squared quantity, its value will be high in periods when there are big changes in the prices of financial assets and its value will be comparatively small when there are modest changes in the prices of financial assets.<sup>22</sup>

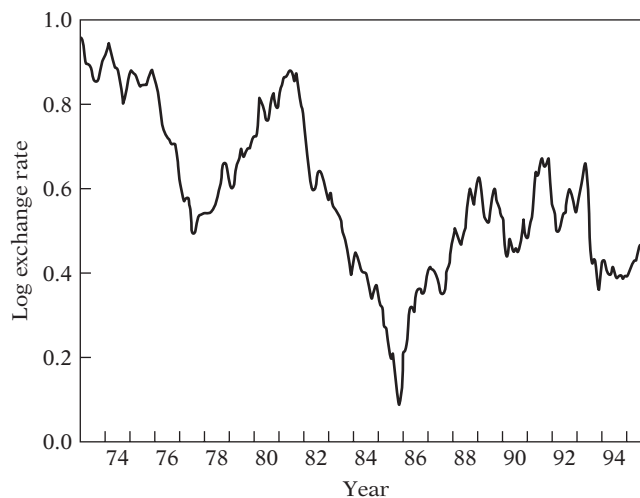


FIGURE 22.6 Log of U.S./U.K. exchange rate, 1973–1995 (monthly)

(Continued)

<sup>22</sup>You might wonder why we do not use the variance of  $X_t = \sum X_t^2/n$  as a measure of volatility. This is because we want to take into account changing volatility of asset prices over time. If we use the variance of  $X_t$ , it will only be a single value for a given data set.



U.S./U.K. EXCHANGE RATE: AN EXAMPLE (Continued)

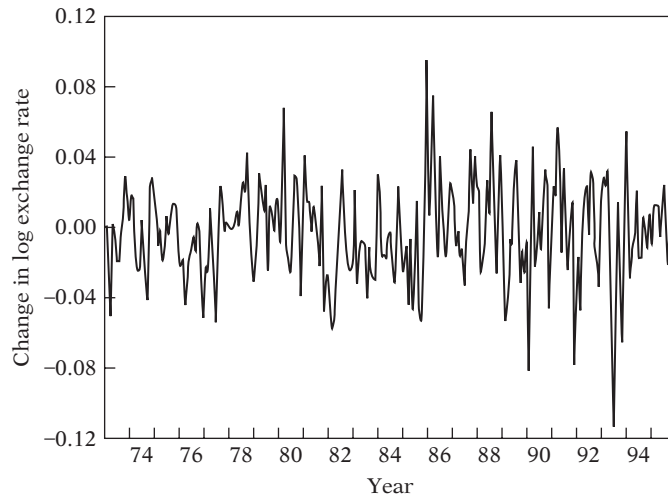


FIGURE 22.7 Change in the log of U.S./U.K. exchange rate.

Accepting  $X_t^2$  as a measure of volatility, how do we know if it changes over time? Suppose we consider the following AR(1), or ARIMA (1, 0, 0), model:

$$X_t^2 = \beta_0 + \beta_1 X_{t-1}^2 + u_t \quad (22.10.1)$$

This model postulates that volatility in the current period is related to its value in the previous period plus a white noise error term. If  $\beta_2$  is positive, it suggests that if volatility was high in the previous period, it will continue to be high in the current period, indicating volatility clustering. If  $\beta_1$  is zero, then there is no volatility clustering. The statistical significance of the estimated  $\beta_2$  can be judged by the usual  $t$  test.

There is nothing to prevent us from considering an AR( $p$ ) model of volatility such that

$$X_t^2 = \beta_0 + \beta_1 X_{t-1}^2 + \beta_2 X_{t-2}^2 + \dots + \beta_p X_{t-p}^2 + u_t \quad (22.10.2)$$

This model suggests that volatility in the current period is related to volatility in the past  $p$  periods, the value of  $p$  being an empirical question. This empirical question can be resolved by one or more of the model selection criteria that we discussed in Chapter 13 (e.g., the Akaike information measure). We can test the significance of any individual  $\beta$  coefficient by the  $t$  test and the collective significance of two or more coefficients by the usual  $F$  test.

Model (22.10.1) is an example of an **ARCH(1)** model and (22.10.2) is called an **ARCH( $p$ )** model, where  $p$  represents the number of autoregressive terms in the model.

Before proceeding further, let us illustrate the ARCH model with the U.S./U.K. exchange rate data. The results of the ARCH(1) model were as follows.

$$X_t^2 = 0.0006 + 0.1694 X_{t-1}^2 \quad (22.10.3)$$

$t = (6.7831) \quad (2.8355) \quad R^2 = 0.0287 \quad d = 1.9972$

where  $X_t^2$  is as defined before.

(Continued)



## U.S./U.K. EXCHANGE RATE: AN EXAMPLE (Continued)

Since the coefficient of the lagged term is highly significant ( $p$  value of about 0.005), it seems volatility clustering is present in the present instance. We tried higher-order ARCH models, but only the AR(1) model turned out to be significant.

How would we test for the ARCH effect in a regression model in general that is based on time series data? To be more specific, let us consider the  $k$ -variable linear regression model:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \cdots + \beta_k X_{kt} + u_t \quad (22.10.4)$$

and assume that *conditional* on the information available at time  $(t - 1)$ , the disturbance term is distributed as

$$u_t \sim N\left[0, \left(\alpha_0 + \alpha_1 u_{t-1}^2\right)\right] \quad (22.10.5)$$

that is,  $u_t$  is normally distributed with zero mean and

$$\text{var}(u_t) = \left(\alpha_0 + \alpha_1 u_{t-1}^2\right) \quad (22.10.6)$$

that is, the variance of  $u_t$  follows an ARCH(1) process.

The normality of  $u_t$  is not new to us. What is new is that the variance of  $u$  at time  $t$  is dependent on the squared disturbance at time  $(t - 1)$ , thus giving the appearance of serial correlation.<sup>23</sup> Of course, the error variance may depend not only on one lagged term of the squared error term but also on several lagged squared terms as follows:

$$\text{var}(u_t) = \sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_p u_{t-p}^2 \quad (22.10.7)$$

If there is no autocorrelation in the error variance, we have

$$H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_p = 0 \quad (22.10.8)$$

in which case  $\text{var}(u_t) = \alpha_0$ , and we do not have the ARCH effect.

Since we do not directly observe  $\sigma_t^2$ , Engle has shown that running the following regression can easily test the preceding null hypothesis:

$$\hat{u}_t^2 = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{u}_{t-1}^2 + \hat{\alpha}_2 \hat{u}_{t-2}^2 + \cdots + \hat{\alpha}_p \hat{u}_{t-p}^2 \quad (22.10.9)$$

where  $\hat{u}_t$ , as usual, denote the OLS variance obtained from the original regression model (22.10.4).

One can test the null hypothesis  $H_0$  by the usual  $F$  test, or alternatively, by computing  $nR^2$ , where  $R^2$  is the coefficient of determination from the auxiliary regression (22.10.9). It can be shown that

$$nR_{\text{asy}}^2 \sim \chi_p^2 \quad (22.10.10)$$

that is, in large samples  $nR^2$  follows the chi-square distribution with  $df$  equal to the number of autoregressive terms in the auxiliary regression.

Before we proceed to illustrate, make sure that you do not confuse autocorrelation of the error term as discussed in Chapter 12 and the ARCH model. In the ARCH model it is the (conditional) variance of  $u_t$  that depends on the (squared) previous error terms, thus giving the impression of autocorrelation.

<sup>23</sup>A technical note: Remember that for our classical linear model the variance of  $u_t$  was assumed to be  $\sigma^2$ , which in the present context becomes unconditional variance. If  $\alpha_1 < 1$ , the stability condition, we can write  $\sigma^2 = \alpha_0 + \alpha_1 \sigma^2$ ; that is,  $\sigma^2 = \alpha_0 / (1 - \alpha_1)$ . This shows that the unconditional variance of  $u$  does not depend on  $t$ , but does depend on the ARCH parameter  $\alpha_1$ .

## NEW YORK STOCK EXCHANGE PRICE CHANGES

As a further illustration of the ARCH effect, Figure 22.8 presents monthly percentage change in the NYSE (New York Stock Exchange) Index for the period 1952–1995.<sup>24</sup> It is evident from this graph that the percent price changes in the NYSE Index exhibit considerable volatility. Notice especially the wide swing around the 1987 crash in stock prices.

To capture the volatility in the stock return seen in the figure, let us consider a very simple model:

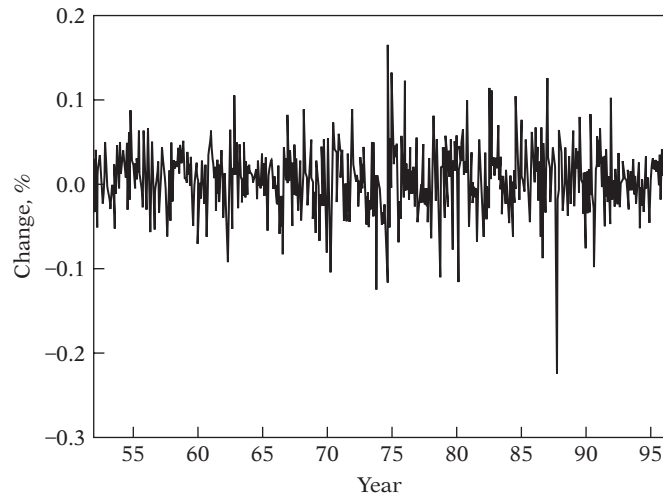
$$Y_t = \beta_1 + u_t \quad (22.10.11)$$

where  $Y_t$  = percent change in the NYSE stock index and  $u_t$  = random error term.

Notice that besides the intercept, there is no other explanatory variable in the model. From the data, we obtained the following OLS regression:

$$\begin{aligned} \hat{Y}_t &= 0.00686 \\ t &= (3.8835) \\ d &= 1.9215 \end{aligned} \quad (22.10.12)$$

What does this intercept denote? It is simply the average percent rate of return on the NYSE index, or the mean value of  $Y_t$  (can you verify this?). Thus over the sample period the average monthly return on the NYSE index was about 0.0069 percent.



**FIGURE 22.8**  
Monthly percent change in the NYSE Price Index, 1952–1995.

(Continued)

<sup>24</sup>This graph and the regression results presented below are based on the data collected by Gary Koop, *Analysis of Economic Data*, John Wiley & Sons, New York, 2000 (data from the data disk). The monthly percentage change in the stock price index can be regarded as a rate of return on the index.

## NEW YORK STOCK EXCHANGE PRICE CHANGES (Continued)

Now we obtain the residuals from the preceding regression and estimate the ARCH(1) model, which gave the following results:

$$\hat{u}_t^2 = 0.00145 + 0.1167\hat{u}_{t-1}^2$$

$$t = (8.8929) \quad (2.6934) \quad (22.10.13)$$

$$R^2 = 0.0136 \quad d = 2.0121$$

where  $\hat{u}_t$  is the estimated residual from regression (22.10.12).

Since the lagged squared disturbance term is statistically significant ( $p$  value of about 0.007), it seems the error variances are correlated; that is, there is an ARCH effect. We tried higher-order ARCH models but only ARCH(1) was statistically significant.

### What To Do if ARCH Is Present

Recall that we have discussed several methods of correcting for heteroscedasticity, which basically involved applying OLS to transformed data. Remember that OLS applied to transformed data is generalized least squares (GLS). If the ARCH effect is found, we will have to use GLS. We will not pursue the technical details, for they are beyond the scope of this book.<sup>25</sup> Fortunately, software packages such as Eviews, Shazam, Microfit, and Pc-Give have now user-friendly routines to estimate such models.

### A Word on the Durbin–Watson $d$ and the ARCH Effect

We have reminded the reader several times that a significant  $d$  statistic may not always mean that there is significant autocorrelation in the data at hand. Very often a significant  $d$  value is an indication of the model specification errors that we discussed in Chapter 13. Now we have an additional specification error, that due to the ARCH effect. Therefore, in a time series regression, if a significant  $d$  value is obtained, we should test for the ARCH effect before accepting the  $d$  statistic at its face value. An example is given in exercise 22.23.

### A Note on the GARCH Model

Since its “discovery” in 1982, ARCH modeling has become a growth industry, with all kinds of variations on the original model. One that has become

<sup>25</sup>Consult Russell Davidson and James G. MacKinnon, *Estimation and Inference in Econometrics*, Oxford University Press, New York, 1993, Sec. 16.4 and William H. Greene, *Econometric Analysis*, 4th ed., Prentice Hall, Englewood Cliffs, N.J., 2000, Sec. 18.5.

popular is the **generalized autoregressive conditional heteroscedasticity (GARCH)** model, originally proposed by Bollerslev.<sup>26</sup> The simplest GARCH model is the GARCH(1, 1) model, which can be written as:

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 \sigma_{t-1}^2 \quad (22.10.14)$$

which says that the conditional variance of  $u$  at time  $t$  depends not only on the squared error term in the previous time period [as in ARCH(1)] but also on its conditional variance in the previous time period. This model can be generalized to a GARCH( $p, q$ ) model in which there are  $p$  lagged terms of the squared error term and  $q$  terms of the lagged conditional variances.

We will not pursue the technical details of these models, as they are involved, except to point out that a GARCH(1, 1) model is equivalent to an ARCH(2) model and a GARCH( $p, q$ ) model is equivalent to an ARCH( $p + q$ ) model.<sup>27</sup>

For our U.S./U.K. exchange rate and NYSE stock return examples, we have already stated that an ARCH(2) model was not significant, suggesting that perhaps a GARCH(1, 1) model is not appropriate in these cases.

## 22.11 CONCLUDING EXAMPLES

We conclude this chapter by considering a few additional examples that illustrate some of the points we have made in this chapter.

### THE RELATIONSHIP BETWEEN THE HELP-WANTED INDEX (HWI) AND THE UNEMPLOYMENT RATE (UN) FROM JANUARY 1969 TO JANUARY 2000

To study causality between HWI and UN, two indicators of labor market conditions in the United States, Marc A. Giammatteo considered the following regression model<sup>28</sup>:

$$HWI_t = \alpha_0 + \sum_{i=1}^{25} \alpha_i UN_{t-i} + \sum_{j=1}^{25} \beta_j HWI_{t-j} \quad (22.11.1)$$

$$UN_t = \alpha_0 + \sum_{i=1}^{25} \lambda_i UN_{t-i} + \sum_{j=1}^{25} \delta_j HWI_{t-j} \quad (22.11.2)$$

To save space we will not present the actual regression results, but the main conclusion that emerges from this study is that there is bilateral causality between the two labor market indicators and this conclusion did not change when the lag length was varied. The data on HWI and UN are given in the data disk.

<sup>26</sup>T. Bollerslev, "Generalized Autoregressive Conditional Heteroscedasticity," *Journal of Econometrics*, vol. 31, 1986, pp. 307–326.

<sup>27</sup>For details, see Davidson and MacKinnon, op. cit., pp. 558–560.

<sup>28</sup>Marc A. Giammatteo (West Point, Class of 2000), "The Relationship between the Help Wanted Index and the Unemployment Rate," unpublished term paper. (Notations altered to conform to our notation.)

**ARIMA MODELING OF THE YEN/DOLLAR EXCHANGE RATE:  
JANUARY 1971 TO DECEMBER 1998<sup>29</sup>**

The yen/dollar exchange rate (¥/\$) is a key exchange rate. From the logarithms of the monthly ¥/\$, it was found that in the level form this exchange rate showed the typical pattern of a nonstationary time series. But examining the first differences, it was found that they were stationary; the graph here pretty much resembles Figure 22.8.

Unit root analysis confirmed that the first differences of the logs of ¥/\$ were stationary. After examining the correlogram of the log first differences, we estimated the following ARIMA(1, 0, 2) model:

$$\hat{Y}_t = -0.0034 + 0.9678\hat{Y}_{t-1} + -0.5866u_{t-1} - 0.4057u_{t-2}$$

$$t = (-4.3638) \quad (67.5439) \quad (-11.4361) \quad (-7.9532) \quad \mathbf{(22.11.3)}$$

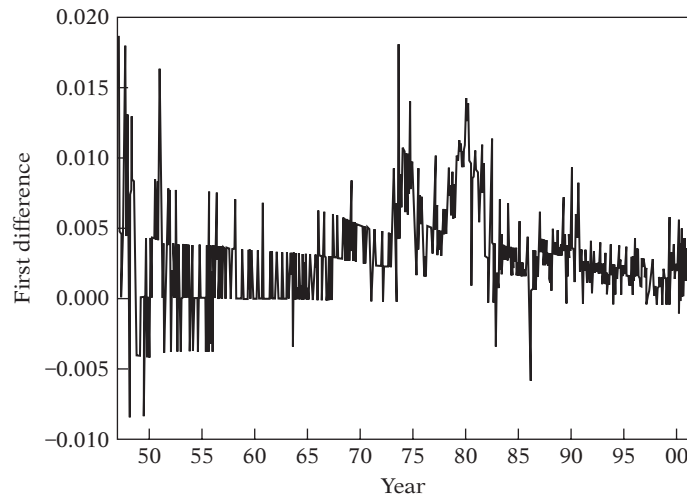
$$R^2 = 0.1454 \quad d = 1.9803$$

where  $Y_t$  = first differences of the logs of ¥/\$ and  $u$  is a white noise error term.

To save space, we have provided the data underlying the preceding analysis in the data disk. Using these data, the reader is urged to try other models and compare their forecasting performances.

**ARCH MODEL OF THE U.S. INFLATION RATE: JANUARY 1947 TO JANUARY 2001**

To see if the ARCH effect is present in the U.S. inflation rate as measured by the CPI, we obtained CPI data from January 1947 to January 2001. The plot of the logarithms of the CPI showed that the time series was nonstationary. But the plot of the first differences of the logs of the CPI, as shown in Figure 22.9, show considerable volatility even though the first differences are stationary.



**FIGURE 22.9** First differences of the logs of CPI.

(Continued)

<sup>29</sup>I am thankful to Gregory M. Ogborn and Marc C. Ogborn (West Point, Class of 2001) for collecting and analyzing the data.

## ARCH MODEL OF THE U.S. INFLATION RATE: . . . (Continued)

Following the procedure outlined in regressions (22.10.12) and (22.10.13), we first regressed the logged first differences of CPI on a constant and obtained residuals from this equation. Squaring these residuals, we obtained the following ARCH(3) model:

$$\widehat{u}_t^2 = 0.000052 + 0.3399\widehat{u}_{t-1}^2 + 0.1338\widehat{u}_{t-2}^2 + 0.0920\widehat{u}_{t-3}^2$$

$$t = (5.1893) \quad (8.7270) \quad (3.5620) \quad (2.5387) \quad (22.11.4)$$

$$R^2 = 0.2153 \quad d = 2.0334$$

As you can see, there is quite a bit of persistence in the volatility, as volatility in the current month depends on volatility in the preceding 3 months. The reader is advised to obtain CPI data from government sources and try to see if another model does a better job, preferably a GARCH model.

**22.12 SUMMARY AND CONCLUSIONS**

**1.** Box–Jenkins and VAR approaches to economic forecasting are alternatives to traditional single- and simultaneous-equation models.

**2.** To forecast the values of a time series, the basic Box–Jenkins strategy is as follows:

**a.** First examine the series for stationarity. This step can be done by computing the autocorrelation function (ACF) and the partial autocorrelation function (PACF) or by a formal unit root analysis. The correlograms associated with ACF and PACF are often good visual diagnostic tools.

**b.** If the time series is not stationary, difference it one or more times to achieve stationarity.

**c.** The ACF and PACF of the stationary time series are then computed to find out if the series is purely autoregressive or purely of the moving average type or a mixture of the two. From broad guidelines given in Table 22.1 one can then determine the values of  $p$  and  $q$  in the ARMA process to be fitted. At this stage the chosen ARMA( $p, q$ ) model is tentative.

**d.** The tentative model is then estimated.

**e.** The residuals from this tentative model are examined to find out if they are white noise. If they are, the tentative model is probably a good approximation to the underlying stochastic process. If they are not, the process is started all over again. Therefore, the Box–Jenkins method is iterative.

**f.** The model finally selected can be used for forecasting.

**3.** The VAR approach to forecasting considers several time series at a time. The distinguishing features of VAR are as follows:

**a.** It is a truly simultaneous system in that all variables are regarded as endogenous.

**b.** In VAR modeling the value of a variable is expressed as a linear function of the past, or lagged, values of that variable and all other variables included in the model.

**c.** If each equation contains the same number of lagged variables in the system, it can be estimated by OLS without resorting to any systems method, such as two-stage least squares (2SLS) or seemingly unrelated regressions (SURE).

**d.** This simplicity of VAR modeling may be its drawback. In view of the limited number of observations that are generally available in most economic analyses, introduction of several lags of each variable can consume a lot of degrees of freedom.<sup>30</sup>

**e.** If there are several lags in each equation, it is not always easy to interpret each coefficient, especially if the signs of the coefficients alternate. For this reason one examines the impulse response function (IRF) in VAR modeling to find out how the dependent variable responds to a shock administered to one or more equations in the system.

**f.** There is considerable debate and controversy about the superiority of the various forecasting methods. Single-equation, simultaneous-equation, Box–Jenkins, and VAR methods of forecasting have their admirers as well as detractors. All one can say is that there is no single method that will suit all situations. If that were the case, there would be no need for discussing the various alternatives. One thing is sure: The Box–Jenkins and VAR methodologies have now become an integral part of econometrics.

**4.** We also considered in this chapter a special class of models, ARCH and GARCH, which are especially useful in analyzing financial time series, such as stock prices, inflation rates, and exchange rates. A distinguishing feature of these models is that the error variance may be correlated over time because of the phenomenon of volatility clustering. In this connection we also pointed out that in many cases a significant Durbin–Watson  $d$  may in fact be due to the ARCH or GARCH effect.

## EXERCISES

### Questions

- 22.1.** What are the major methods of economic forecasting?
- 22.2.** What are the major differences between simultaneous-equation and Box–Jenkins approaches to economic forecasting?
- 22.3.** Outline the major steps involved in the application of the Box–Jenkins approach to forecasting.
- 22.4.** What happens if Box–Jenkins techniques are applied to time series that are nonstationary?

<sup>30</sup>Followers of Bayesian statistics believe that this problem can be minimized. See R. Litterman, “A Statistical Approach to Economic Forecasting,” *Journal of Business and Economic Statistics*, vol. 4, 1986, pp. 1–4.

- 22.5. What are the differences between Box–Jenkins and VAR approaches to economic forecasting?
- 22.6. In what sense is VAR atheoretic?
- 22.7. “If the primary object is forecasting, VAR will do the job.” Critically evaluate this statement.
- 22.8. Since the number of lags to be introduced in a VAR model can be a subjective question, how does one decide how many lags to introduce in a concrete application?
- 22.9. Comment on this statement: “Box–Jenkins and VAR are prime examples of measurement without theory.”
- 22.10. What is the connection, if any, between Granger causality tests and VAR modeling?

### Problems

- 22.11. Consider the data on PDI (personal disposable income) given in Table 21.1. Suppose you want to fit a suitable ARIMA model to these data. Outline the steps involved in carrying out this task.
- 22.12. Repeat exercise 22.11 for the PCE (personal consumption expenditure) data given in Table 21.1.
- 22.13. Repeat exercise 22.11 for the profits data given in Table 21.1.
- 22.14. Repeat exercise 22.11 for the dividends data given in Table 21.1.
- 22.15. In Section 13.9 you were introduced to the Schwarz criterion to determine lag length. How would you use this criterion to determine the appropriate lag length in a VAR model?
- 22.16. Using the data on PCE and PDI given in Table 21.1, develop a bivariate VAR model for the period 1970–I to 1990–IV. Use this model to forecast the values of these variables for the four quarters of 1991 and compare the forecast values with the actual values given in Table 21.1.
- 22.17. Repeat exercise 22.16, using the data on dividends and profits.
- \*22.18. Refer to any statistical package and estimate the impulse response function for a period of up to 8 lags for the VAR model that you developed in exercise 22.16.
- \*22.19. Repeat exercise 22.18 for the VAR model that you developed in exercise 22.17.
- 22.20. Refer to the VAR regression results given in Table 22.4. From the various  $F$  tests reported in the three regressions given there, what can you say about the nature of causality in the three variables?
- 22.21. Continuing with exercise 20.20, can you guess why the authors chose to express the three variables in the model in percentage change form rather than the levels of these variables? (*Hint*: Stationarity.)
- 22.22. Using the Canadian data given in Table 17.3, find out if  $M_1$  and  $R$  are stationary random variables? If not, are they cointegrated? Show the necessary calculations.
- 22.23. Continue with the data given in Table 17.3. Now consider the following simple model of money demand in Canada:

$$\ln M_{1t} = \beta_1 + \beta_2 \ln \text{GDP}_t + \beta_3 \ln R_t + u_t$$

---

\*Optional.



- a. How would you interpret the parameters of this model?  
b. Obtain the residuals from this model and find out if there is any ARCH effect.
- 22.24.** Refer to the ARCH(3) model given in (22.11.4). Using the same data we estimated the following ARCH(1) model:

$$\hat{u}_t^2 = 0.00000078 + 0.3737\hat{u}_{t-1}^2$$

$$t = (7.5843) \quad (10.2351)$$

$$R^2 = 0.1397 \quad d = 1.9896$$

How would you choose between the two models? Show the necessary calculations.

---

## A REVIEW OF SOME STATISTICAL CONCEPTS

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This appendix provides a very sketchy introduction to some of the statistical concepts encountered in the text. The discussion is nonrigorous, and no proofs are given because several excellent books on statistics do that job very well. Some of these books are listed at the end of this appendix.

### A.1 SUMMATION AND PRODUCT OPERATORS

The Greek capital letter  $\sum$  (sigma) is used to indicate summation. Thus,

$$\sum_{i=1}^n x_i = x_1 + x_2 + \cdots + x_n$$

Some of the important properties of the summation operator  $\sum$  are

1.  $\sum_{i=k}^n k = nk$ , where  $k$  is constant. Thus,  $\sum_{i=1}^4 3 = 4 \cdot 3 = 12$ .
2.  $\sum_{i=1}^n kx_i = k \sum_{i=1}^n x_i$ , where  $k$  is a constant.
3.  $\sum_{i=1}^n (a + bx_i) = na + b \sum_{i=1}^n x_i$ , where  $a$  and  $b$  are constants and where use is made of properties 1 and 2 above.
4.  $\sum_{i=1}^n (x_i + y_i) = \sum_{i=1}^n x_i + \sum_{i=1}^n y_i$ .

The summation operator can also be extended to multiple sums. Thus,  $\sum \sum$ , the double summation operator, is defined as

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m x_{ij} &= \sum_{i=1}^n (x_{i1} + x_{i2} + \cdots + x_{im}) \\ &= (x_{11} + x_{21} + \cdots + x_{n1}) + (x_{12} + x_{22} + \cdots + x_{n2}) \\ &\quad + \cdots + (x_{1m} + x_{2m} + \cdots + x_{nm}) \end{aligned}$$

Some of the properties of  $\sum \sum$  are

1.  $\sum_{i=1}^n \sum_{j=1}^m x_{ij} = \sum_{j=1}^m \sum_{i=1}^n x_{ij}$ ; that is, the order in which the double summation is performed is interchangeable.
2.  $\sum_{i=1}^n \sum_{j=1}^m x_i y_j = \sum_{i=1}^n x_i \sum_{j=1}^m y_j$ .
3.  $\sum_{i=1}^n \sum_{j=1}^m (x_{ij} + y_{ij}) = \sum_{i=1}^n \sum_{j=1}^m x_{ij} + \sum_{i=1}^n \sum_{j=1}^m y_{ij}$ .
4.  $[\sum_{i=1}^n x_i]^2 = \sum_{i=1}^n x_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n x_i x_j = \sum_{i=1}^n x_i^2 + 2 \sum_{i < j} x_i x_j$ .

The product operator  $\prod$  is defined as

$$\prod_{i=1}^n x_i = x_1 \cdot x_2 \cdots x_n$$

Thus,

$$\prod_{i=1}^3 x_i = x_1 \cdot x_2 \cdot x_3$$

## A.2 SAMPLE SPACE, SAMPLE POINTS, AND EVENTS

The set of all possible outcomes of a random, or chance, experiment is called the **population**, or **sample space**, and each member of this sample space is called a **sample point**. Thus, in the experiment of tossing two coins, the sample space consists of these four possible outcomes:  $HH$ ,  $HT$ ,  $TH$ , and  $TT$ , where  $HH$  means a head on the first toss and also a head on the second toss,  $HT$  means a head on the first toss and a tail on the second toss, and so on. Each of the preceding occurrences constitutes a sample point.

An **event** is a subset of the sample space. Thus, if we let  $A$  denote the occurrence of one head and one tail, then, of the preceding possible outcomes, only two belong to  $A$ , namely  $HT$  and  $TH$ . In this case  $A$  constitutes an event. Similarly, the occurrence of two heads in a toss of two coins is an event. Events are said to be **mutually exclusive** if the occurrence of one event precludes the occurrence of another event. If in the preceding example  $HH$  occurs, the occurrence of the event  $HT$  at the same time is not possible. Events are said to be (collectively) **exhaustive** if they exhaust all the possible outcomes of an experiment. Thus, in the example, the events (a) two heads, (b) two tails, and (c) one tail, one head exhaust all the outcomes; hence they are (collectively) exhaustive events.

## A.3 PROBABILITY AND RANDOM VARIABLES

### Probability

Let  $A$  be an event in a sample space. By  $P(A)$ , the probability of the event  $A$ , we mean the proportion of times the event  $A$  will occur in repeated trials of an experiment. Alternatively, in a total of  $n$  possible equally likely out-

comes of an experiment, if  $m$  of them are favorable to the occurrence of the event  $A$ , we define the ratio  $m/n$  as the **relative frequency** of  $A$ . For large values of  $n$ , this relative frequency will provide a very good approximation of the probability of  $A$ .

**Properties of Probability.**  $P(A)$  is a real-valued function<sup>1</sup> and has these properties:

1.  $0 \leq P(A) \leq 1$  for every  $A$ .
2. If  $A, B, C, \dots$  constitute an exhaustive set of events, then  $P(A + B + C + \dots) = 1$ , where  $A + B + C$  means  $A$  or  $B$  or  $C$ , and so forth.
3. If  $A, B, C, \dots$  are mutually exclusive events, then

$$P(A + B + C + \dots) = P(A) + P(B) + P(C) + \dots$$

#### EXAMPLE 1

Consider the experiment of throwing a die numbered 1 through 6. The sample space consists of the outcomes 1, 2, 3, 4, 5, and 6. These six events therefore exhaust the entire sample space. The probability of any one of these numbers showing up is  $1/6$  since there are six equally likely outcomes and any one of them has an equal chance of showing up. Since 1, 2, 3, 4, 5, and 6 form an exhaustive set of events,  $P(1 + 2 + 3 + 4 + 5 + 6) = 1$  where 1, 2, 3, ... means the probability of number 1 or number 2 or number 3, etc. And since 1, 2, ..., 6 are mutually exclusive events in that two numbers cannot occur simultaneously,  $P(1 + 2 + 3 + 4 + 5 + 6) = P(1) + P(2) + \dots + P(6) = 1$ .

## Random Variables

A variable whose value is determined by the outcome of a chance experiment is called a **random variable** (rv). Random variables are usually denoted by the capital letters  $X, Y, Z$ , and so on, and the values taken by them are denoted by small letters  $x, y, z$ , and so on.

A random variable may be either **discrete** or **continuous**. A discrete rv takes on only a finite (or countably infinite) number of values.<sup>2</sup> For example, in throwing two dice, each numbered 1 to 6, if we define the random variable  $X$  as the sum of the numbers showing on the dice, then  $X$  will take one of these values: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12. Hence it is a discrete random variable. A continuous rv, on the other hand, is one that can take on any value in some interval of values. Thus, the height of an individual is a

<sup>1</sup>A function whose domain and range are subsets of real numbers is commonly referred to as a real-valued function. For details, see Alpha C. Chiang, *Fundamental Methods of Mathematical Economics*, 3d ed., McGraw-Hill, 1984, Chap. 2.

<sup>2</sup>For a simple discussion of the notion of countably infinite sets, see R. G. D. Allen, *Basic Mathematics*, Macmillan, London, 1964, p. 104.

continuous variable—in the range, say, 60 to 65 inches it can take any value, depending on the precision of measurement.

#### A.4 PROBABILITY DENSITY FUNCTION (PDF)

##### Probability Density Function of a Discrete Random Variable

Let  $X$  be a discrete rv taking distinct values  $x_1, x_2, \dots, x_n, \dots$ . Then the function

$$\begin{aligned} f(x) &= P(X = x_i) && \text{for } i = 1, 2, \dots, n, \dots \\ &= 0 && \text{for } x \neq x_i \end{aligned}$$

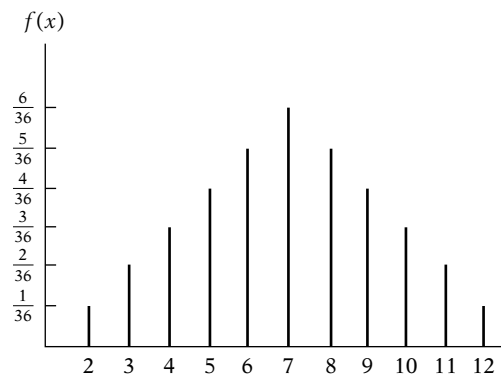
is called the **discrete probability density function** (PDF) of  $X$ , where  $P(X = x_i)$  means the probability that the discrete rv  $X$  takes the value of  $x_i$ .

##### EXAMPLE 2

In a throw of two dice, the random variable  $X$ , the sum of the numbers shown on two dice, can take one of the 11 values shown. The PDF of this variable can be shown to be as follows (see also Figure A.1):

$$\begin{aligned} x = & 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \\ f(x) = & \left(\frac{1}{36}\right)\left(\frac{2}{36}\right)\left(\frac{3}{36}\right)\left(\frac{4}{36}\right)\left(\frac{5}{36}\right)\left(\frac{6}{36}\right)\left(\frac{5}{36}\right)\left(\frac{4}{36}\right)\left(\frac{3}{36}\right)\left(\frac{2}{36}\right)\left(\frac{1}{36}\right) \end{aligned}$$

These probabilities can be easily verified. In all there are 36 possible outcomes, of which one is favorable to number 2, two are favorable to number 3 (since the sum 3 can occur either as 1 on the first die and 2 on the second die or 2 on the first die and 1 on the second die), and so on.



**FIGURE A.1**  
Density function of the discrete random variable of Example 2.

### Probability Density Function of a Continuous Random Variable

Let  $X$  be a continuous rv. Then  $f(x)$  is said to be the PDF of  $X$  if the following conditions are satisfied:

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_a^b f(x) dx = P(a \leq x \leq b)$$

where  $f(x) dx$  is known as the *probability element* (the probability associated with a small interval of a continuous variable) and where  $P(a \leq x \leq b)$  means the probability that  $X$  lies in the interval  $a$  to  $b$ . Geometrically, we have Figure A.2.

For a continuous rv, in contrast with a discrete rv, the probability that  $X$  takes a specific value is zero<sup>3</sup>; probability for such a variable is measurable only over a given range or interval, such as  $(a, b)$  shown in Figure A.2.

#### EXAMPLE 3

Consider the following density function:

$$f(x) = \frac{1}{9}x^2 \quad 0 \leq x \leq 3$$

It can be readily verified that  $f(x) \geq 0$  for all  $x$  in the range 0 to 3 and that  $\int_0^3 \frac{1}{9}x^2 dx = 1$ . (Note: The integral is  $(\frac{1}{27}x^3 \Big|_0^3) = 1$ .) If we want to evaluate the above PDF between, say, 0 and 1, we obtain  $\int_0^1 \frac{1}{9}x^2 dx = (\frac{1}{27}x^3 \Big|_0^1) = \frac{1}{27}$ ; that is, the probability that  $x$  lies between 0 and 1 is  $1/27$ .

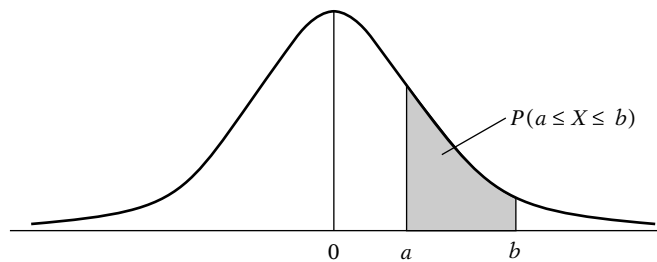


FIGURE A.2 Density function of a continuous random variable.

<sup>3</sup>Note:  $\int_a^a f(x) dx = 0$ .

**Joint Probability Density Functions**

**Discrete Joint PDF.** Let  $X$  and  $Y$  be two discrete random variables. Then the function

$$\begin{aligned} f(x, y) &= P(X = x \text{ and } Y = y) \\ &= 0 \quad \text{when } X \neq x \text{ and } Y \neq y \end{aligned}$$

is known as the **discrete joint probability density function** and gives the (joint) probability that  $X$  takes the value of  $x$  and  $Y$  takes the value of  $y$ .

**EXAMPLE 4**

The following table gives the joint PDF of the discrete variables  $X$  and  $Y$ .

		X			
		-2	0	2	3
Y	3	0.27	0.08	0.16	0
	6	0	0.04	0.10	0.35

This table tells us that the probability that  $X$  takes the value of  $-2$  while  $Y$  simultaneously takes the value of  $3$  is  $0.27$  and that the probability that  $X$  takes the value of  $3$  while  $Y$  takes the value of  $6$  is  $0.35$ , and so on.

**Marginal Probability Density Function**

In relation to  $f(x, y)$ ,  $f(x)$  and  $f(y)$  are called **individual**, or **marginal**, probability density functions. These marginal PDFs are derived as follows:

$$f(x) = \sum_y f(x, y) \quad \text{marginal PDF of } X$$

$$f(y) = \sum_x f(x, y) \quad \text{marginal PDF of } Y$$

where, for example,  $\sum_y$  means the sum over all values of  $Y$  and  $\sum_x$  means the sum over all values of  $X$ .

**EXAMPLE 5**

Consider the data given in Example 4. The marginal PDF of  $X$  is obtained as follows:

$$f(x = -2) = \sum_y f(x, y) = 0.27 + 0 = 0.27$$

$$f(x = 0) = \sum_y f(x, y) = 0.08 + 0.04 = 0.12$$

$$f(x = 2) = \sum_y f(x, y) = 0.16 + 0.10 = 0.26$$

$$f(x = 3) = \sum_y f(x, y) = 0 + 0.35 = 0.35$$

(Continued)

**EXAMPLE 5** (Continued)

Likewise, the marginal PDF of  $Y$  is obtained as

$$f(y = 3) = \sum_x f(x, y) = 0.27 + 0.08 + 0.16 + 0 = 0.51$$

$$f(y = 6) = \sum_x f(x, y) = 0 + 0.04 + 0.10 + 0.35 = 0.49$$

As this example shows, to obtain the marginal PDF of  $X$  we add the column numbers, and to obtain the marginal PDF of  $Y$  we add the row numbers. Notice that  $\sum_x f(x)$  over all values of  $X$  is 1, as is  $\sum_y f(y)$  over all values of  $Y$  (why?).

**Conditional PDF.** As noted in Chapter 2, in regression analysis we are often interested in studying the behavior of one variable conditional upon the values of another variable(s). This can be done by considering the conditional PDF. The function

$$f(x | y) = P(X = x | Y = y)$$

is known as the **conditional PDF** of  $X$ ; it gives the probability that  $X$  takes on the value of  $x$  given that  $Y$  has assumed the value  $y$ . Similarly,

$$f(y | x) = P(Y = y | X = x)$$

which gives the *conditional PDF of  $Y$* .

The conditional PDFs may be obtained as follows:

$$f(x | y) = \frac{f(x, y)}{f(y)} \quad \text{conditional PDF of } X$$

$$f(y | x) = \frac{f(x, y)}{f(x)} \quad \text{conditional PDF of } Y$$

As the preceding expressions show, the conditional PDF of one variable can be expressed as the ratio of the joint PDF to the marginal PDF of another (conditioning) variable.

**EXAMPLE 6**

Continuing with Examples 4 and 5, let us compute the following conditional probabilities:

$$f(X = -2 | Y = 3) = \frac{f(X = -2, Y = 3)}{f(Y = 3)} = 0.27/0.51 = 0.53$$

Notice that the unconditional probability  $f(X = -2)$  is 0.27, but if  $Y$  has assumed the value of 3, the probability that  $X$  takes the value of  $-2$  is 0.53.

$$f(X = 2 | Y = 6) = \frac{f(X = 2, Y = 6)}{f(Y = 6)} = 0.10/0.49 = 0.20$$

Again note that the unconditional probability that  $X$  takes the value of 2 is 0.26, which is different from 0.20, which is its value if  $Y$  assumes the value of 6.



### Statistical Independence

Two random variables  $X$  and  $Y$  are statistically independent if and only if

$$f(x, y) = f(x)f(y)$$

that is, if the joint PDF can be expressed as the product of the marginal PDFs.

#### EXAMPLE 7

A bag contains three balls numbered 1, 2, and 3. Two balls are drawn at random, with replacement, from the bag (i.e., the first ball drawn is replaced before the second is drawn). Let  $X$  denote the number of the first ball drawn and  $Y$  the number of the second ball drawn. The following table gives the joint PDF of  $X$  and  $Y$ .

		X		
		1	2	3
Y	1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
	2	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
	3	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

Now  $f(X = 1, Y = 1) = \frac{1}{9}$ ,  $f(X = 1) = \frac{1}{3}$  (obtained by summing the first column), and  $f(Y = 1) = \frac{1}{3}$  (obtained by summing the first row). Since  $f(X, Y) = f(X)f(Y)$  in this example we can say that the two variables are statistically independent. It can be easily checked that for any other combination of  $X$  and  $Y$  values given in the above table the joint PDF factors into individual PDFs.

It can be shown that the  $X$  and  $Y$  variables given in Example 4 are not statistically independent since the product of the two marginal PDFs is not equal to the joint PDF. (Note:  $f(X, Y) = f(X)f(Y)$  must be true for all combinations of  $X$  and  $Y$  if the two variables are to be statistically independent.)

**Continuous Joint PDF.** The PDF  $f(x, y)$  of two continuous variables  $X$  and  $Y$  is such that

$$f(x, y) \geq 0$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

$$\int_c^d \int_a^b f(x, y) dx dy = P(a \leq x \leq b, c \leq y \leq d)$$

**EXAMPLE 8**

Consider the following PDF.

$$f(x, y) = 2 - x - y \quad 0 \leq x \leq 1; 0 \leq y \leq 1$$

It is obvious that  $f(x, y) \geq 0$ . Moreover<sup>4</sup>

$$\int_0^1 \int_0^1 (2 - x - y) dx dy = 1$$

The marginal PDF of  $X$  and  $Y$  can be obtained as

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \text{marginal PDF of } X$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad \text{marginal PDF of } Y$$

**EXAMPLE 9**

The two marginal PDFs of the joint PDF given in Example 8 are as follows:

$$f(x) = \int_0^1 f(x, y) dy = \int_0^1 (2 - x - y) dy$$

$$\left( 2y - xy - \frac{y^2}{2} \right) \Big|_0^1 = \frac{3}{2} - x \quad 0 \leq x \leq 1$$

$$f(y) = \int_0^1 (2 - x - y) dx$$

$$\left( 2x - xy - \frac{x^2}{2} \right) \Big|_0^1 = \frac{3}{2} - y \quad 0 \leq y \leq 1$$

To see if the two variables of Example 8 are statistically independent, we need to find out if  $f(x, y) = f(x)f(y)$ . Since  $(2 - x - y) \neq (\frac{3}{2} - x)(\frac{3}{2} - y)$ , we can say that the two variables are not statistically independent.

<sup>4</sup>

$$\int_0^1 \left[ \int_0^1 (2 - x - y) dx \right] dy = \int_0^1 \left[ \left( 2x - \frac{x^2}{2} - xy \right) \Big|_0^1 \right] dy$$

$$= \int_0^1 \left( \frac{3}{2} - y \right) dy$$

$$= \left( \frac{3}{2}y - \frac{y^2}{2} \right) \Big|_0^1 = 1$$

*Note:* The expression  $(\frac{3}{2}y - y^2/2) \Big|_0^1$  means the expression in the parentheses is to be evaluated at the upper limit value of 1 and the lower limit value of 0; the latter value is subtracted from the former to obtain the value of the integral. Thus, in the preceding example the limits are  $(\frac{3}{2} - \frac{1}{2})$  at  $y = 1$  and 0 at  $y = 0$ , giving the value of the integral as 1.

**A.5 CHARACTERISTICS OF PROBABILITY DISTRIBUTIONS**

A probability distribution can often be summarized in terms of a few of its characteristics, known as the **moments** of the distribution. Two of the most widely used moments are the **mean**, or **expected value**, and the **variance**.

**Expected Value**

The expected value of a discrete rv  $X$ , denoted by  $E(X)$ , is defined as follows:

$$E(X) = \sum_x xf(x)$$

where  $\sum_x$  means the sum over all values of  $X$  and where  $f(x)$  is the (discrete) PDF of  $X$ .

**EXAMPLE 10**

Consider the probability distribution of the sum of two numbers in a throw of two dice given in Example 2. (See Figure A.1.) Multiplying the various  $X$  values given there by their probabilities and summing over all the observations, we obtain:

$$\begin{aligned} E(X) &= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{26}\right) + 4\left(\frac{3}{36}\right) + \cdots + 12\left(\frac{1}{36}\right) \\ &= 7 \end{aligned}$$

which is the average value of the sum of numbers observed in a throw of two dice.

**EXAMPLE 11**

Estimate  $E(X)$  and  $E(Y)$  for the data given in Example 4. We have seen that

$x$	-2	0	2	3
$f(x)$	0.27	0.12	0.26	0.35

Therefore,

$$\begin{aligned} E(X) &= \sum_x xf(x) \\ &= (-2)(0.27) + (0)(0.12) + (2)(0.26) + (3)(0.35) \\ &= 1.03 \end{aligned}$$

Similarly,

$y$	3	6
$f(y)$	0.51	0.49

$$\begin{aligned} E(Y) &= \sum_y yf(y) \\ &= (3)(0.51) + (6)(0.49) \\ &= 4.47 \end{aligned}$$

(Continued)

**EXAMPLE 11** (Continued)

The expected value of a continuous rv is defined as

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

The only difference between this case and the expected value of a discrete rv is that we replace the summation symbol by the integral symbol.

**EXAMPLE 12**

Let us find out the expected value of the continuous PDF given in Example 3.

$$\begin{aligned} E(X) &= \int_0^3 x \left( \frac{x^2}{9} \right) dx \\ &= \frac{1}{9} \left[ \left( \frac{x^4}{4} \right) \right]_0^3 \\ &= \frac{9}{4} \\ &= 2.25 \end{aligned}$$

**Properties of Expected Values**

1. The expected value of a constant is the constant itself. Thus, if  $b$  is a constant,  $E(b) = b$ .
2. If  $a$  and  $b$  are constants,

$$E(aX + b) = aE(X) + b$$

This can be generalized. If  $X_1, X_2, \dots, X_N$  are  $N$  random variables and  $a_1, a_2, \dots, a_N$  and  $b$  are constants, then

$$E(a_1X_1 + a_2X_2 + \dots + a_NX_N + b) = a_1E(X_1) + a_2E(X_2) + \dots + a_NE(X_N) + b$$

3. If  $X$  and  $Y$  are independent random variables, then

$$E(XY) = E(X)E(Y)$$

That is, the expectation of the product  $XY$  is the product of the (individual) expectations of  $X$  and  $Y$ .

4. If  $X$  is a random variable with PDF  $f(x)$  and if  $g(X)$  is any function of  $X$ , then

$$\begin{aligned} E[g(X)] &= \sum_x g(X)f(x) && \text{if } X \text{ is discrete} \\ &= \int_{-\infty}^{\infty} g(X)f(x)dx && \text{if } X \text{ is continuous} \end{aligned}$$

Thus, if  $g(X) = X^2$ ,

$$E(X^2) = \sum_x x^2 f(X) \quad \text{if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} x^2 f(X) dx \quad \text{if } X \text{ is continuous}$$

**EXAMPLE 13**

Consider the following PDF:

$x$	-2	1	2
$f(x)$	$\frac{5}{8}$	$\frac{1}{8}$	$\frac{2}{8}$

Then

$$E(X) = -2\left(\frac{5}{8}\right) + 1\left(\frac{1}{8}\right) + 2\left(\frac{2}{8}\right)$$

$$= -\frac{5}{8}$$

and

$$E(X^2) = 4\left(\frac{5}{8}\right) + 1\left(\frac{1}{8}\right) + 4\left(\frac{2}{8}\right)$$

$$= \frac{29}{8}$$

**Variance**

Let  $X$  be a random variable and let  $E(X) = \mu$ . The distribution, or spread, of the  $X$  values around the expected value can be measured by the variance, which is defined as

$$\text{var}(X) = \sigma_X^2 = E(X - \mu)^2$$

The positive square root of  $\sigma_X^2$ ,  $\sigma_X$ , is defined as the **standard deviation** of  $X$ . The variance or standard deviation gives an indication of how closely or widely the individual  $X$  values are spread around their mean value.

The variance defined previously is computed as follows:

$$\text{var}(X) = \sum_x (X - \mu)^2 f(x) \quad \text{if } X \text{ is a discrete rv}$$

$$= \int_{-\infty}^{\infty} (X - \mu)^2 f(x) dx \quad \text{if } X \text{ is a continuous rv}$$

For computational convenience, the variance formula given above can also be expressed as

$$\text{var}(X) = \sigma_X^2 = E(X - \mu)^2$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - [E(X)]^2$$

Applying this formula, it can be seen that the variance of the random variable given in Example 13 is  $\frac{29}{8} - \left(-\frac{5}{8}\right)^2 = \frac{207}{64} = 3.23$ .

**EXAMPLE 14**

Let us find the variance of the random variable given in Example 3.

$$\text{var}(X) = E(X^2) - [E(X)]^2$$

Now

$$\begin{aligned} E(X^2) &= \int_0^3 x^2 \left(\frac{x^2}{9}\right) dx \\ &= \int_0^3 \frac{x^4}{9} dx \\ &= \frac{1}{9} \left[ \frac{x^5}{5} \right]_0^3 \\ &= 243/45 \\ &= 27/5 \end{aligned}$$

Since  $E(X) = \frac{9}{4}$  (see Example 12), we finally have

$$\begin{aligned} \text{var}(X) &= 243/45 - \left(\frac{9}{4}\right)^2 \\ &= 243/720 = 0.34 \end{aligned}$$

**Properties of Variance**

1.  $E(X - \mu)^2 = E(X^2) - \mu^2$ , as noted before.
2. The variance of a constant is zero.
3. If  $a$  and  $b$  are constants, then

$$\text{var}(aX + b) = a^2 \text{var}(X)$$

4. If  $X$  and  $Y$  are *independent* random variables, then

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$$

$$\text{var}(X - Y) = \text{var}(X) + \text{var}(Y)$$

This can be generalized to more than two independent variables.

5. If  $X$  and  $Y$  are *independent* rv's and  $a$  and  $b$  are constants, then

$$\text{var}(aX + bY) = a^2 \text{var}(X) + b^2 \text{var}(Y)$$

**Covariance**

Let  $X$  and  $Y$  be two rv's with means  $\mu_x$  and  $\mu_y$ , respectively. Then the **covariance** between the two variables is defined as

$$\text{cov}(X, Y) = E\{(X - \mu_x)(Y - \mu_y)\} = E(XY) - \mu_x \mu_y$$

It can be readily seen that the variance of a variable is the covariance of that variable with itself.

The covariance is computed as follows:

$$\begin{aligned}\text{cov}(X, Y) &= \sum_y \sum_x (X - \mu_x)(Y - \mu_y)f(x, y) \\ &= \sum_y \sum_x XYf(x, y) - \mu_x\mu_y\end{aligned}$$

if  $X$  and  $Y$  are discrete random variables, and

$$\begin{aligned}\text{cov}(X, Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (X - \mu_x)(Y - \mu_y)f(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} XYf(x, y) dx dy - \mu_x\mu_y\end{aligned}$$

if  $X$  and  $Y$  are continuous random variables.

### Properties of Covariance

1. If  $X$  and  $Y$  are independent, their covariance is zero, for

$$\begin{aligned}\text{cov}(X, Y) &= E(XY) - \mu_x\mu_y \\ &= \mu_x\mu_y - \mu_x\mu_y && \text{since } E(XY) = E(X)E(Y) = \mu_x\mu_y \\ &= 0 && \text{when } X \text{ and } Y \text{ are independent}\end{aligned}$$

- 2.

$$\text{cov}(a + bX, c + dY) = bd \text{cov}(X, Y)$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are constants.

#### EXAMPLE 15

Let us find out the covariance between discrete random variables  $X$  and  $Y$  whose joint PDF is as shown in Example 4. From Example 11 we already know that  $\mu_x = E(X) = 1.03$  and  $\mu_y = E(Y) = 4.47$ .

$$\begin{aligned}E(XY) &= \sum_y \sum_x XYf(x, y) \\ &= (-2)(3)(0.27) + (0)(3)(0.08) + (2)(3)(0.16) + (3)(3)(0) \\ &\quad + (-2)(6)(0) + (0)(6)(0.04) + (2)(6)(0.10) + (3)(6)(0.35) \\ &= 6.84\end{aligned}$$

Therefore,

$$\begin{aligned}\text{cov}(X, Y) &= E(XY) - \mu_x\mu_y \\ &= 6.84 - (1.03)(4.47) \\ &= 2.24\end{aligned}$$

**Correlation Coefficient**

The (population) correlation coefficient  $\rho$  (rho) is defined as

$$\rho = \frac{\text{cov}(X, Y)}{\sqrt{\{\text{var}(X) \text{var}(Y)\}}} = \frac{\text{cov}(X, Y)}{\sigma_x \sigma_y}$$

Thus defined,  $\rho$  is a measure of *linear* association between two variables and lies between  $-1$  and  $+1$ ,  $-1$  indicating perfect negative association and  $+1$  indicating perfect positive association.

From the preceding formula, it can be seen that

$$\text{cov}(X, Y) = \rho \sigma_x \sigma_y$$

**EXAMPLE 16**

Estimate the coefficient of correlation for the data of Example 4.

From the PDFs given in Example 11 it can be easily shown that  $\sigma_x = 2.05$  and  $\sigma_y = 1.50$ . We have already shown that  $\text{cov}(X, Y) = 2.24$ . Therefore, applying the preceding formula we estimate  $\rho$  as  $2.24/(2.05)(1.50) = 0.73$ .

**Variations of Correlated Variables.** Let  $X$  and  $Y$  be two rv's. Then

$$\begin{aligned} \text{var}(X + Y) &= \text{var}(X) + \text{var}(Y) + 2 \text{cov}(X, Y) \\ &= \text{var}(X) + \text{var}(Y) + 2\rho\sigma_x\sigma_y \\ \text{var}(X - Y) &= \text{var}(X) + \text{var}(Y) - 2 \text{cov}(X, Y) \\ &= \text{var}(X) + \text{var}(Y) - 2\rho\sigma_x\sigma_y \end{aligned}$$

If, however,  $X$  and  $Y$  are independent,  $\text{cov}(X, Y)$  is zero, in which case the  $\text{var}(X + Y)$  and  $\text{var}(X - Y)$  are both equal to  $\text{var}(X) + \text{var}(Y)$ , as noted previously.

The preceding results can be generalized as follows. Let  $\sum_{i=1}^n X_i = X_1 + X_2 + \cdots + X_n$ , then the variance of the linear combination  $\sum X_i$  is

$$\begin{aligned} \text{var}\left(\sum_{i=1}^n x_i\right) &= \sum_{i=1}^n \text{var} X_i + 2 \sum_{i < j} \text{cov}(X_i, X_j) \\ &= \sum_{i=1}^n \text{var} X_i + 2 \sum_{i < j} \rho_{ij} \sigma_i \sigma_j \end{aligned}$$

where  $\rho_{ij}$  is the correlation coefficient between  $X_i$  and  $X_j$  and where  $\sigma_i$  and  $\sigma_j$  are the standard deviations of  $X_i$  and  $X_j$ .



Thus,

$$\begin{aligned}\text{var}(X_1 + X_2 + X_3) &= \text{var } X_1 + \text{var } X_2 + \text{var } X_3 + 2 \text{cov}(X_1, X_2) \\ &\quad + 2 \text{cov}(X_1, X_3) + 2 \text{cov}(X_2, X_3) \\ &= \text{var } X_1 + \text{var } X_2 + \text{var } X_3 + 2\rho_{12}\sigma_1\sigma_2 \\ &\quad + 2\rho_{13}\sigma_1\sigma_3 + 2\rho_{23}\sigma_2\sigma_3\end{aligned}$$

where  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  are, respectively, the standard deviations of  $X_1$ ,  $X_2$ , and  $X_3$  and where  $\rho_{12}$  is the correlation coefficient between  $X_1$  and  $X_2$ ,  $\rho_{13}$  that between  $X_1$  and  $X_3$ , and  $\rho_{23}$  that between  $X_2$  and  $X_3$ .

### Conditional Expectation and Conditional Variance

Let  $f(x, y)$  be the joint PDF of random variables  $X$  and  $Y$ . The conditional expectation of  $X$ , given  $Y = y$ , is defined as

$$\begin{aligned}E(X|Y = y) &= \sum_x xf(x|Y = y) && \text{if } X \text{ is discrete} \\ &= \int_{-\infty}^{\infty} xf(x|Y = y)dx && \text{if } X \text{ is continuous}\end{aligned}$$

where  $E(X|Y = y)$  means the conditional expectation of  $X$  given  $Y = y$  and where  $f(x|Y = y)$  is the conditional PDF of  $X$ . The conditional expectation of  $Y$ ,  $E(Y|X = x)$ , is defined similarly.

**Conditional Expectation.** Note that  $E(X|Y)$  is a random variable because it is a function of the conditioning variable  $Y$ . However,  $E(X|Y = y)$ , where  $y$  is a specific value of  $Y$ , is a constant.

**Conditional Variance.** The conditional variance of  $X$  given  $Y = y$  is defined as

$$\begin{aligned}\text{var}(X|Y = y) &= E\{[X - E(X|Y = y)]^2 | Y = y\} \\ &= \sum_x [X - E(X|Y = y)]^2 f(x|Y = y) && \text{if } X \text{ is discrete} \\ &= \int_{-\infty}^{\infty} [X - E(X|Y = y)]^2 f(x|Y = y) dx && \text{if } X \text{ is continuous}\end{aligned}$$

**EXAMPLE 17**

Compute  $E(Y | X = 2)$  and  $\text{var}(Y | X = 2)$  for the data given in Example 4.

$$\begin{aligned} E(Y | X = 2) &= \sum_y yf(Y = y | X = 2) \\ &= 3f(Y = 3 | X = 2) + 6f(Y = 6 | X = 2) \\ &= 3(0.16/0.26) + 6(0.10/0.26) \\ &= 4.15 \end{aligned}$$

*Note:*  $f(Y = 3 | X = 2) = f(Y = 3, X = 2) / f(X = 2) = 0.16/0.26$ , and  $f(Y = 6 | X = 2) = f(Y = 6, X = 2) / f(X = 2) = 0.10/0.26$ , so

$$\begin{aligned} \text{var}(Y | X = 2) &= \sum_y [Y - E(Y | X = 2)]^2 f(Y | X = 2) \\ &= (3 - 4.15)^2(0.16/0.26) + (6 - 4.15)^2(0.10/0.26) \\ &= 2.13 \end{aligned}$$

**Properties of Conditional Expectation and Conditional Variance**

1. If  $f(X)$  is a function of  $X$ , then  $E(f(X) | X) = f(X)$ , that is, the function of  $X$  behaves as a constant in computation of its expectation conditional on  $X$ . Thus,  $[E(X^3 | X)] = E(X^3)$ ; this is because, if  $X$  is known,  $X^3$  is also known.

2. If  $f(X)$  and  $g(X)$  are functions of  $X$ , then

$$E[f(X)Y + g(X) | X] = f(X)E(Y | X) + g(X)$$

For example,  $E[XY + cX^2 | X] = XE(Y | X) + cX^2$ , where  $c$  is a constant.

3. If  $X$  and  $Y$  are independent,  $E(Y | X) = E(Y)$ . That is, if  $X$  and  $Y$  are independent random variables, then the conditional expectation of  $Y$ , given  $X$ , is the same as the unconditional expectation of  $Y$ .

4. **The law of iterated expectations.** It is interesting to note the following relation between the unconditional expectation of a random variable  $Y$ ,  $E(Y)$ , and its conditional expectation based on another random variable  $X$ ,  $E(Y | X)$ :

$$E(Y) = E_X[E(Y | X)]$$

This is known as the law of iterated expectations, which in the present context states that the marginal, or unconditional, expectation of  $Y$  is equal to the expectation of its conditional expectation, the symbol  $E_X$  denoting that the expectation is taken over the values of  $X$ . Put simply, this law states that if we first obtain  $E(Y | X)$  as a function of  $X$  and take its expected value over the distribution of  $X$  values, you wind up with  $E(Y)$ , the unconditional expectation of  $Y$ . The reader can verify this using the data given in Example 4.

5. If  $X$  and  $Y$  are independent, then  $\text{var}(Y|X) = \text{var}(Y)$ .
6.  $\text{var}(Y) = E[\text{var}(Y|X)] + \text{var}[E(Y|X)]$ ; that is, the (unconditional) variance of  $Y$  is equal to expectation of the conditional variance of  $Y$  plus the variance of the conditional expectation of  $Y$ .

### Higher Moments of Probability Distributions

Although mean, variance, and covariance are the most frequently used summary measures of univariate and multivariate PDFs, we occasionally need to consider higher moments of the PDFs, such as the third and the fourth moments. The third and fourth moments of a univariate PDF  $f(x)$  around its mean value ( $\mu$ ) are defined as

$$\text{Third moment: } E(X - \mu)^3$$

$$\text{Fourth moment: } E(X - \mu)^4$$

In general, the  $r$ th moment about the mean is defined as

$$r\text{th moment: } E(X - \mu)^r$$

The third and fourth moments of a distribution are often used in studying the “shape” of a probability distribution, in particular, its **skewness**,  $S$  (i.e., lack of symmetry) and **kurtosis**,  $K$  (i.e., tallness or flatness), as shown in Figure A.3.

One measure of skewness is defined as

$$\begin{aligned} S &= \frac{E(X - \mu)^3}{\sigma^3} \\ &= \frac{\text{third moment about the mean}}{\text{cube of the standard deviation}} \end{aligned}$$

A commonly used measure of kurtosis is given by

$$\begin{aligned} K &= \frac{E(X - \mu)^4}{[E(X - \mu)^2]^2} \\ &= \frac{\text{fourth moment about mean}}{\text{square of the second moment}} \end{aligned}$$

PDFs with values of  $K$  less than 3 are called **platykurtic** (fat or short-tailed), and those with values greater than 3 are called **leptokurtic** (slim or long-tailed). See Figure A.3. A PDF with a kurtosis value of 3 is known as **mesokurtic**, of which the normal distribution is the prime example. (See the discussion of the normal distribution in Section A.6.)

We will show shortly how the measures of skewness and kurtosis can be combined to determine whether a random variable follows a normal

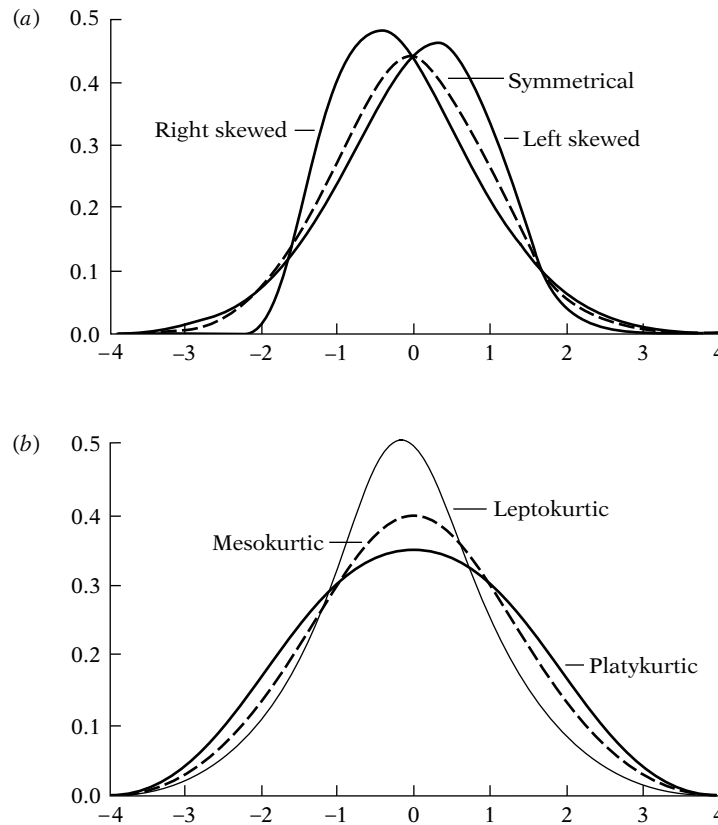


FIGURE A.3 (a) Skewness; (b) kurtosis.

distribution. Recall that our hypothesis-testing procedure, as in the  $t$  and  $F$  tests, is based on the assumption (at least in small or finite samples) that the underlying distribution of the variable (or sample statistic) is normal. It is therefore very important to find out in concrete applications whether this assumption is fulfilled.

## A.6 SOME IMPORTANT THEORETICAL PROBABILITY DISTRIBUTIONS

In the text extensive use is made of the following probability distributions.

### Normal Distribution

The best known of all the theoretical probability distributions is the normal distribution, whose bell-shaped picture is familiar to anyone with a modicum of statistical knowledge.

A (continuous) random variable  $X$  is said to be normally distributed if its PDF has the following form:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2}\right) \quad -\infty < x < \infty$$

where  $\mu$  and  $\sigma^2$ , known as the *parameters of the distribution*, are, respectively, the mean and the variance of the distribution. The properties of this distribution are as follows:

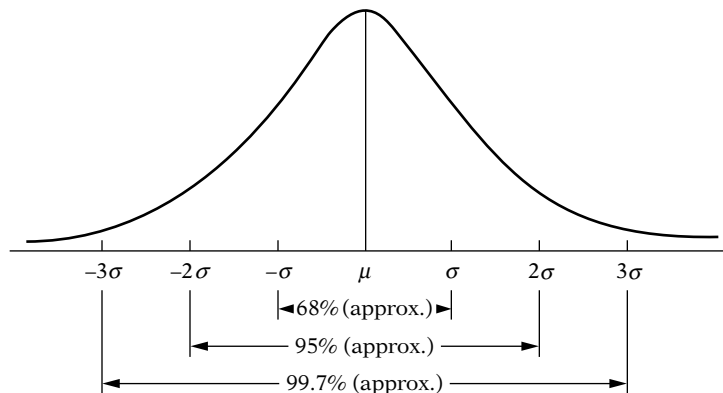
1. It is symmetrical around its mean value.
2. Approximately 68 percent of the area under the normal curve lies between the values of  $\mu \pm \sigma$ , about 95 percent of the area lies between  $\mu \pm 2\sigma$ , and about 99.7 percent of the area lies between  $\mu \pm 3\sigma$ , as shown in Figure A.4.

3. The normal distribution depends on the two parameters  $\mu$  and  $\sigma^2$ , so once these are specified, one can find the probability that  $X$  will lie within a certain interval by using the PDF of the normal distribution. But this task can be lightened considerably by referring to Table D.1 of **Appendix D**. To use this table, we convert the given normally distributed variable  $X$  with mean  $\mu$  and  $\sigma^2$  into a **standardized normal variable**  $Z$  by the following transformation:

$$Z = \frac{x - \mu}{\sigma}$$

An important property of any standardized variable is that its mean value is zero and its variance is unity. Thus  $Z$  has zero mean and unit variance. Substituting  $z$  into the normal PDF given previously, we obtain

$$f(Z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} Z^2\right)$$



**FIGURE A.4** Areas under the normal curve.

which is the PDF of the standardized normal variable. The probabilities given in **Appendix D**, Table D.1, are based on this standardized normal variable.

By convention, we denote a normally distributed variable as

$$X \sim N(\mu, \sigma^2)$$

where  $\sim$  means “distributed as,”  $N$  stands for the normal distribution, and the quantities in the parentheses are the two parameters of the normal distribution, namely, the mean and the variance. Following this convention,

$$X \sim N(0, 1)$$

means  $X$  is a normally distributed variable with zero mean and unit variance. In other words, it is a standardized normal variable  $Z$ .

#### EXAMPLE 18

Assume that  $X \sim N(8, 4)$ . What is the probability that  $X$  will assume a value between  $X_1 = 4$  and  $X_2 = 12$ ? To compute the required probability, we compute the  $Z$  values as

$$Z_1 = \frac{X_1 - \mu}{\sigma} = \frac{4 - 8}{2} = -2$$

$$Z_2 = \frac{X_2 - \mu}{\sigma} = \frac{12 - 8}{2} = +2$$

Now from Table D.1 we observe that  $\Pr(0 \leq Z \leq 2) = 0.4772$ . Then, by symmetry, we have  $\Pr(-2 \leq Z \leq 0) = 0.4772$ . Therefore, the required probability is  $0.4772 + 0.4772 = 0.9544$ . (See Figure A.4.)

#### EXAMPLE 19

What is the probability that in the preceding example  $X$  exceeds 12?

The probability that  $X$  exceeds 12 is the same as that  $Z$  exceeds 2. From Table D.1 it is obvious that this probability is  $(0.5 - 0.4772)$  or 0.0228.

**4.** Let  $X_1 \sim N(\mu_1, \sigma_1^2)$  and  $X_2 \sim N(\mu_2, \sigma_2^2)$  and assume that they are independent. Now consider the linear combination

$$Y = aX_1 + bX_2$$

where  $a$  and  $b$  are constants. Then it can be shown that

$$Y \sim N[(a\mu_1 + b\mu_2), (a^2\sigma_1^2 + b^2\sigma_2^2)]$$

This result, which states that a *linear combination of normally distributed variables is itself normally distributed*, can be easily generalized to a linear combination of more than two normally distributed variables.

**5. Central limit theorem.** Let  $X_1, X_2, \dots, X_n$  denote  $n$  independent random variables, all of which have the same PDF with mean  $= \mu$  and variance  $= \sigma^2$ . Let  $\bar{X} = \sum X_i/n$  (i.e., the sample mean). Then as  $n$  increases indefinitely (i.e.,  $n \rightarrow \infty$ ),

$$\bar{X} \underset{n \rightarrow \infty}{\sim} N\left(\mu, \frac{\sigma^2}{n}\right)$$

That is,  $\bar{X}$  approaches the normal distribution with mean  $\mu$  and variance  $\sigma^2/n$ . Notice that this result holds true regardless of the form of the PDF. As a result, it follows that

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0, 1)$$

That is,  $Z$  is a standardized normal variable.

**6.** The third and fourth moments of the normal distribution around the mean value are as follows:

$$\text{Third moment: } E(X - \mu)^3 = 0$$

$$\text{Fourth moment: } E(X - \mu)^4 = 3\sigma^4$$

*Note:* All odd-powered moments about the mean value of a normally distributed variable are zero.

**7.** As a result, and following the measures of skewness and kurtosis discussed earlier, for a normal PDF skewness  $= 0$  and kurtosis  $= 3$ ; that is, a normal distribution is symmetric and mesokurtic. Therefore, a simple test of normality is to find out whether the computed values of skewness and kurtosis depart from the norms of 0 and 3. This is in fact the logic underlying the **Jarque–Bera (JB) test of normality** discussed in the text:

$$JB = n \left[ \frac{S^2}{6} + \frac{(K - 3)^2}{24} \right] \quad (5.12.1)$$

where  $S$  stands for skewness and  $K$  for kurtosis. Under the null hypothesis of normality,  $JB$  is distributed as a **chi-square** statistic with 2 df.

**8.** The mean and the variance of a normally distributed random variable are independent in that one is not a function of the other.

### The $\chi^2$ (Chi-Square) Distribution

Let  $Z_1, Z_2, \dots, Z_k$  be *independent* standardized normal variables (i.e., normal variables with zero mean and unit variance). Then the quantity

$$Z = \sum_{i=1}^k Z_i^2$$

is said to possess the  $\chi^2$  distribution with  $k$  degrees of freedom (df), where the term df means the number of independent quantities in the previous sum. A chi-square-distributed variable is denoted by  $\chi_k^2$ , where the subscript  $k$  indicates the df. Geometrically, the chi-square distribution appears in Figure A.5.

Properties of the  $\chi^2$  distribution are as follows:

1. As Figure A.5 shows, the  $\chi^2$  distribution is a skewed distribution, the degree of the skewness depending on the df. For comparatively few df, the distribution is highly skewed to the right; but as the number of df increases, the distribution becomes increasingly symmetrical. As a matter of fact, for df in excess of 100, the variable

$$\frac{\sqrt{2\chi^2} - \sqrt{2k-1}}{\sqrt{2k-1}}$$

can be treated as a standardized normal variable, where  $k$  is the df.

2. The mean of the chi-square distribution is  $k$ , and its variance is  $2k$ , where  $k$  is the df.

3. If  $Z_1$  and  $Z_2$  are two independent chi-square variables with  $k_1$  and  $k_2$  df, then the sum  $Z_1 + Z_2$  is also a chi-square variable with  $\text{df} = k_1 + k_2$ .

#### EXAMPLE 20

What is the probability of obtaining a  $\chi^2$  value of 40 or greater, given the df of 20?

As Table D.4 shows, the probability of obtaining a  $\chi^2$  value of 39.9968 or greater (20 df) is 0.005. Therefore, the probability of obtaining a  $\chi^2$  value of 40 or greater is less than 0.005, a rather small probability.

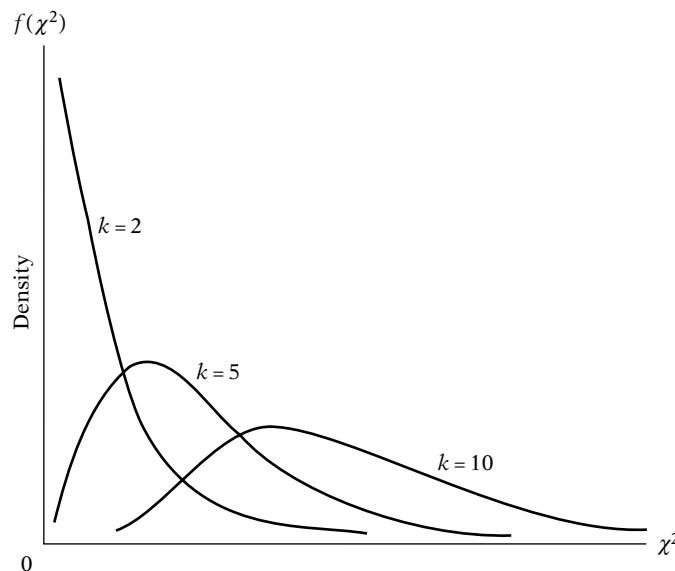


FIGURE A.5 Density function of the  $\chi^2$  variable.



### Student's $t$ Distribution

If  $Z_1$  is a standardized normal variable [that is,  $Z_1 \sim N(0, 1)$ ] and another variable  $Z_2$  follows the chi-square distribution with  $k$  df and is distributed independently of  $Z_1$ , then the variable defined as

$$\begin{aligned}t &= \frac{Z_1}{\sqrt{(Z_2/k)}} \\ &= \frac{Z_1\sqrt{k}}{\sqrt{Z_2}}\end{aligned}$$

follows Student's  $t$  distribution with  $k$  df. A  $t$ -distributed variable is often designated as  $t_k$ , where the subscript  $k$  denotes the df. Geometrically, the  $t$  distribution is shown in Figure A.6.

Properties of the Student's  $t$  distribution are as follows:

1. As Figure A.6 shows, the  $t$  distribution, like the normal distribution, is symmetrical, but it is flatter than the normal distribution. But as the df increase, the  $t$  distribution approximates the normal distribution.

2. The mean of the  $t$  distribution is zero, and its variance is  $k/(k - 2)$ .

The  $t$  distribution is tabulated in Table D.2.

#### EXAMPLE 21

Given  $df = 13$ , what is the probability of obtaining a  $t$  value (a) of about 3 or greater, (b) of about  $-3$  or smaller, and (c) of  $|t|$  of about 3 or greater, where  $|t|$  means the absolute value (i.e., disregarding the sign) of  $t$ ?

From Table D.2, the answers are (a) about 0.005, (b) about 0.005 because of the symmetry of the  $t$  distribution, and (c) about  $0.01 = 2(0.005)$ .

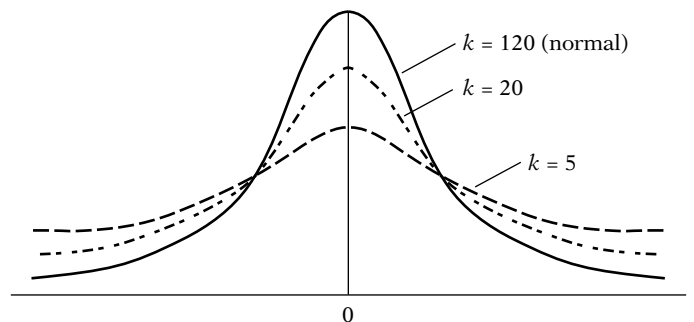


FIGURE A.6 Student's  $t$  distribution for selected degrees of freedom.

### The $F$ Distribution

If  $Z_1$  and  $Z_2$  are independently distributed chi-square variables with  $k_1$  and  $k_2$  df, respectively, the variable

$$F = \frac{Z_1/k_1}{Z_2/k_2}$$

follows (Fisher's)  $F$  distribution with  $k_1$  and  $k_2$  df. An  $F$ -distributed variable is denoted by  $F_{k_1, k_2}$  where the subscripts indicate the df associated with the two  $Z$  variables,  $k_1$  being called the *numerator df* and  $k_2$  the *denominator df*. Geometrically, the  $F$  distribution is shown in Figure A.7.

The  $F$  distribution has the following properties:

1. Like the chi-square distribution, the  $F$  distribution is skewed to the right. But it can be shown that as  $k_1$  and  $k_2$  become large, the  $F$  distribution approaches the normal distribution.

2. The mean value of an  $F$ -distributed variable is  $k_2/(k_2 - 2)$ , which is defined for  $k_2 > 2$ , and its variance is

$$\frac{2k_2^2(k_1 + k_2 - 2)}{k_1(k_2 - 2)^2(k_2 - 4)}$$

which is defined for  $k_2 > 4$ .

3. The square of a  $t$ -distributed random variable with  $k$  df has an  $F$  distribution with 1 and  $k$  df. Symbolically,

$$t_k^2 = F_{1, k}$$

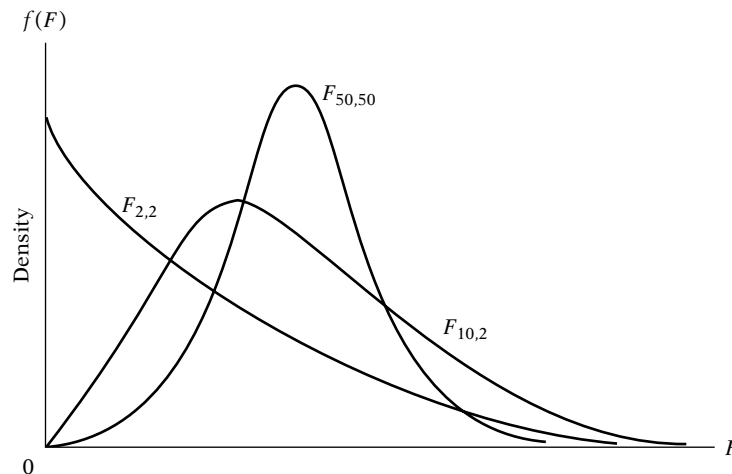


FIGURE A.7  $F$  distribution for various degrees of freedom.

**EXAMPLE 22**

Given  $k_1 = 10$  and  $k_2 = 8$ , what is the probability of obtaining an  $F$  value (a) of 3.4 or greater and (b) of 5.8 or greater?

As Table D.3 shows, these probabilities are (a) approximately 0.05 and (b) approximately 0.01.

4. If the denominator df,  $k_2$ , is fairly large, the following relationship holds between the  $F$  and the chi-square distributions:

$$k_1 F \sim \chi_{k_1}^2$$

That is, for fairly large denominator df, the numerator df times the  $F$  value is approximately the same as a chi-square value with numerator df.

**EXAMPLE 23**

Let  $k_1 = 20$  and  $k_2 = 120$ . The 5 percent critical  $F$  value for these df is 1.48. Therefore,  $k_1 F = (20)(1.48) = 29.6$ . From the chi-square distribution for 20 df, the 5 percent critical chi-square value is about 31.41.

In passing, note that since for large df the  $t$ , chi-square, and  $F$  distributions approach the normal distribution, these three distributions are known as the *distributions related to the normal distribution*.

**The Bernoulli Binomial Distribution**

A random variable  $X$  is said to follow a distribution named after Bernoulli (a Swiss mathematician) if its probability density (or mass) function (PDF) is:

$$P(X = 0) = 1 - p$$

$$P(X = 1) = p$$

where  $p$ ,  $0 \leq p \leq 1$ , is the probability that some event is a “success,” such as the probability of obtaining a head in a toss of a coin. For such a variable,

$$E(X) = [1 \times p(X = 1) + 0 \times p(X = 0)] = p$$

$$\text{var}(X) = pq$$

where  $q = (1 - p)$ , that is, the probability of a “failure.”

**Binomial Distribution**

The binomial distribution is the generalization of the Bernoulli distribution. Let  $n$  denote the number of independent trials, each of which results in a

“success” with probability  $p$  and a “failure” with a probability  $q = (1 - p)$ . If  $X$  represents the number of successes in the  $n$  trials, then  $X$  is said to follow the binomial distribution whose PDF is:

$$f(X) = \binom{n}{x} p^x (1 - p)^{n-x}$$

where  $x$  represents the number of successes in  $n$  trials and where

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

where  $n!$ , read as  $n$  factorial, means  $n(n-1)(n-2) \cdots 1$ .

The binomial is a two-parameter distribution,  $n$  and  $p$ . For this distribution,

$$E(X) = np$$

$$\text{var}(X) = np(1 - p) = npq$$

For example, if you toss a coin 100 times and want to find out the probability of obtaining 60 heads, you put  $p = 0.5$ ,  $n = 100$  and  $x = 60$  in the above formula. Computer routines exist to evaluate such probabilities.

You can see how the binomial distribution is a generalization of the Bernoulli distribution.

### The Poisson Distribution

A random  $X$  variable is said to have the Poisson distribution if its PDF is:

$$f(X) = \frac{e^{-\lambda} \lambda^x}{x!} \quad \text{for } x = 0, 1, 2, \dots, \lambda > 0$$

The Poisson distribution depends on a single parameter,  $\lambda$ . A distinguishing feature of the Poisson distribution is that its variance is equal to its expected value, which is  $\lambda$ . That is,

$$E(X) = \text{var}(X) = \lambda$$

The Poisson model, as we saw in the chapter on nonlinear regression models, is used to model rare or infrequent phenomena, such as the number of phone calls received in a span of, say, 5 minutes, or the number of speeding tickets received in a span of an hour, or the number of patents received by a firm, say, in a year.

## A.7 STATISTICAL INFERENCE: ESTIMATION

In Section A.6 we considered several theoretical probability distributions. Very often we know or are willing to assume that a random variable  $X$  follows a particular probability distribution but do not know the value(s) of the parameter(s) of the distribution. For example, if  $X$  follows the normal

distribution, we may want to know the value of its two parameters, namely, the mean and the variance. To estimate the unknowns, the usual procedure is to assume that we have a **random sample** of size  $n$  from the known probability distribution and use the sample data to estimate the unknown parameters.<sup>5</sup> This is known as the **problem of estimation**. In this section, we take a closer look at this problem. The problem of estimation can be broken down into two categories: point estimation and interval estimation.

### Point Estimation

To fix the ideas, let  $X$  be a random variable with PDF  $f(x; \theta)$ , where  $\theta$  is the parameter of the distribution (for simplicity of discussion only, we are assuming that there is only one unknown parameter; our discussion can be readily generalized). Assume that we know the functional form—that is, we know the theoretical PDF, such as the  $t$  distribution—but do not know the value of  $\theta$ . Therefore, we draw a random sample of size  $n$  from this known PDF and then develop a function of the sample values such that

$$\hat{\theta} = f(x_1, x_2, \dots, x_n)$$

provides us an estimate of the true  $\theta$ .  $\hat{\theta}$  is known as a **statistic**, or an **estimator**, and a particular numerical value taken by the estimator is known as an **estimate**. Note that  $\hat{\theta}$  can be treated as a random variable because it is a function of the sample data.  $\hat{\theta}$  provides us with a rule, or formula, that tells us how we may estimate the true  $\theta$ . Thus, if we let

$$\hat{\theta} = \frac{1}{n}(x_1 + x_2 + \dots + x_n) = \bar{X}$$

where  $\bar{X}$  is the sample mean, then  $\bar{X}$  is an estimator of the true mean value, say,  $\mu$ . If in a specific case  $\bar{X} = 50$ , this provides an *estimate of*  $\mu$ . The estimator  $\hat{\theta}$  obtained previously is known as a **point estimator** because it provides only a single (point) estimate of  $\theta$ .

### Interval Estimation

Instead of obtaining only a single estimate of  $\theta$ , suppose we obtain two estimates of  $\theta$  by constructing two estimators  $\hat{\theta}_1(x_1, x_2, \dots, x_n)$  and  $\hat{\theta}_2(x_1, x_2, \dots, x_n)$ , and say with some confidence (i.e., probability) that the interval between  $\hat{\theta}_1$  and  $\hat{\theta}_2$  includes the true  $\theta$ . Thus, in interval estimation, in contrast with point estimation, we provide a range of possible values within which the true  $\theta$  may lie.

<sup>5</sup>Let  $X_1, X_2, \dots, X_n$  be  $n$  random variables with joint PDF  $f(x_1, x_2, \dots, x_n)$ . If we can write

$$f(x_1, x_2, \dots, x_n) = f(x_1)f(x_2)\cdots f(x_n)$$

where  $f(x)$  is the common PDF of each  $X$ , then  $x_1, x_2, \dots, x_n$  are said to constitute a random sample of size  $n$  from a population with PDF  $f(x_i)$ .

The key concept underlying interval estimation is the notion of the **sampling, or probability distribution, of an estimator**. For example, it can be shown that if a variable  $X$  is normally distributed, then the sample mean  $\bar{X}$  is also normally distributed with mean  $= \mu$  (the true mean) and variance  $= \sigma^2/n$ , where  $n$  is the sample size. In other words, the sampling, or probability, distribution of the estimator  $\bar{X}$  is  $\bar{X} \sim N(\mu, \sigma^2/n)$ . As a result, if we construct the interval

$$\bar{X} \pm 2 \frac{\sigma}{\sqrt{n}}$$

and say that the probability is approximately 0.95, or 95 percent, that intervals like it will include the true  $\mu$ , we are in fact constructing an interval estimator for  $\mu$ . Note that the interval given previously is random since it is based on  $\bar{X}$ , which will vary from sample to sample.

More generally, in interval estimation we construct two estimators  $\hat{\theta}_1$  and  $\hat{\theta}_2$ , both functions of the sample  $X$  values, such that

$$\Pr(\hat{\theta}_1 \leq \theta \leq \hat{\theta}_2) = 1 - \alpha \quad 0 < \alpha < 1$$

That is, we can state that the probability is  $1 - \alpha$  that the interval from  $\hat{\theta}_1$  to  $\hat{\theta}_2$  contains the true  $\theta$ . This interval is known as a **confidence interval** of size  $1 - \alpha$  for  $\theta$ ,  $1 - \alpha$  being known as the **confidence coefficient**. If  $\alpha = 0.05$ , then  $1 - \alpha = 0.95$ , meaning that if we construct a confidence interval with a confidence coefficient of 0.95, then in repeated such constructions resulting from repeated sampling we shall be right in 95 out of 100 cases if we maintain that the interval contains the true  $\theta$ . When the confidence coefficient is 0.95, we often say that we have a 95% confidence interval. In general, if the confidence coefficient is  $1 - \alpha$ , we say that we have a  $100(1 - \alpha)\%$  confidence interval. Note that  $\alpha$  is known as the **level of significance**, or the probability of committing a Type I error. This topic is discussed in Section A.8.

#### EXAMPLE 24

Suppose that the distribution of height of men in a population is normally distributed with mean  $= \mu$  inches and  $\sigma = 2.5$  inches. A sample of 100 men drawn randomly from this population had an average height of 67 inches. Establish a 95% confidence interval for the mean height ( $= \mu$ ) in the population as a whole.

As noted,  $\bar{X} \sim N(\mu, \sigma^2/n)$ , which in this case becomes  $\bar{X} \sim N(\mu, 2.5^2/100)$ . From Table D.1 one can see that

$$\bar{X} - 1.96 \left( \frac{\sigma}{\sqrt{n}} \right) \leq \mu \leq \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}$$

covers 95% of the area under the normal curve. Therefore, the preceding interval provides a 95% confidence

interval for  $\mu$ . Plugging in the given values of  $\bar{X}$ ,  $\sigma$ , and  $n$ , we obtain the 95% confidence interval as

$$66.51 \leq \mu \leq 67.49$$

In repeated such measurements, intervals thus established will include the true  $\mu$  with 95 percent confidence. A technical point may be noted here. Although we can say that the probability that the random interval  $[\bar{X} \pm 1.96(\sigma/\sqrt{n})]$  includes  $\mu$  is 95 percent, we *cannot* say that the probability is 95 percent that the particular interval (66.51, 67.49) includes  $\mu$ . Once this interval is fixed, the probability that it will include  $\mu$  is either 0 or 1. What we can say is that if we construct 100 such intervals, 95 out of the 100 intervals will include the true  $\mu$ ; we cannot guarantee that one particular interval will necessarily include  $\mu$ .

## Methods of Estimation

Broadly speaking, there are three methods of parameter estimation: (1) least squares (LS), (2) maximum likelihood, and (3) method of moments (MOM) and its extension, the generalized method of moments (GMM). We have devoted considerable time to illustrate the LS method. In Chapter 4 we introduced the ML method in the regression context. But the method is of much broader application.

The key idea behind the ML is the **likelihood function**. To illustrate this, suppose the random variable  $X$  has PDF  $f(X, \theta)$  which depends on a single parameter  $\theta$ . We know the PDF (e.g., Bernoulli or binomial) but do not know the parameter value. Suppose we obtain a random sample of  $nX$  values. The joint PDF of these  $n$  values is:

$$g(x_1, x_2, \dots, x_n; \theta)$$

Because it is a random sample, we can write the preceding joint PDF as a product of the individual PDF as

$$g(x_1, x_2, \dots, x_n; \theta) = f(x_1; \theta)f(x_2; \theta) \cdots f(x_n; \theta)$$

The joint PDF has a dual interpretation. If  $\theta$  is known, we interpret it as joint probability of observing the given sample values. On the other hand, we can treat it as a function of  $\theta$  for given values of  $x_1, x_2, \dots, x_n$ . On the latter interpretation, we call the joint PDF the **likelihood function (LF)** and write it as

$$L(\theta; x_1, x_2, \dots, x_n) = f(x_1; \theta)f(x_2; \theta) \cdots f(x_n; \theta)$$

Observe the role reversal of  $\theta$  in the joint probability density function and the likelihood function.

The ML estimator of  $\theta$  is that value of  $\theta$  that maximizes the (sample) likelihood function,  $L$ . For mathematical convenience, we often take the log of the likelihood, called the **log-likelihood function (log L)**. Following the calculus rules of maximization, we differentiate the log-likelihood function with respect to the unknown and equate the resulting derivative to zero. The resulting value of the estimator is called the **maximum-likelihood estimator**. One can apply the second-order condition of maximization to assure that the value we have obtained is in fact the maximum value.

In case there is more than one unknown parameter, we differentiate the log-likelihood function with respect to each unknown, set the resulting expressions to zero, and solve them simultaneously to obtain the values of the unknown parameters. We have already shown this for the multiple regression model (see the appendix to Chapter 4).

**EXAMPLE 25**

Assume that the random variable  $X$  follows the Poisson distribution with the mean value of  $\lambda$ . Suppose  $x_1, x_2, \dots, x_n$  are independent Poisson random variables each with mean  $\lambda$ . Suppose we want to find out the ML estimator of  $\lambda$ . The likelihood function here is:

$$L(x_1, x_2, \dots, x_n; \lambda) = \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \frac{e^{-\lambda} \lambda^{x_2}}{x_2!} \dots \frac{e^{-\lambda} \lambda^{x_n}}{x_n!}$$

$$= \frac{e^{-n\lambda} \lambda^{\sum x_i}}{x_1! x_2! \dots x_n!}$$

This is a rather unwieldy-expression, but if we take its log, it becomes

$$\log(x_1, x_2, \dots, x_n; \lambda) = -n\lambda + \sum x_i \log \lambda - \log c$$

where  $\log c = \prod x_i!$ . Differentiating the preceding expression with respect to  $\lambda$ , we obtain  $(-n + (\sum x_i)/\lambda)$ . By setting this last expression to zero, we obtain  $\lambda_{ml} = (\sum x_i)/n = \bar{X}$ , which is the ML estimator of the unknown  $\lambda$ .

**The Method of Moments.** We have given a glimpse of MOM in exercise 3.4 in the so-called **analogy principle** in which the sample moments try to duplicate the properties of their population counterparts. The GMM, which is a generalization of MOM, is now becoming more popular, but not at the introductory level. Hence we will not pursue it here.

The desirable statistical properties fall into two categories: small-sample, or finite-sample, properties and large-sample, or asymptotic, properties. Underlying both these sets of properties is the notion that an estimator has a sampling, or probability, distribution.

**Small-Sample Properties**

**Unbiasedness.** An estimator  $\hat{\theta}$  is said to be an unbiased estimator of  $\theta$  if the expected value of  $\hat{\theta}$  is equal to the true  $\theta$ ; that is,

$$E(\hat{\theta}) = \theta$$

or

$$E(\hat{\theta}) - \theta = 0$$

If this equality does not hold, then the estimator is said to be biased, and the bias is calculated as

$$\text{bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

Of course, if  $E(\hat{\theta}) = \theta$ —that is,  $\hat{\theta}$  is an unbiased estimator—the bias is zero.

Geometrically, the situation is as depicted in Figure A.8. In passing, note that unbiasedness is a property of repeated sampling, not of any given sample: keeping the sample size fixed, we draw several samples, each time obtaining an estimate of the unknown parameter. The average value of these



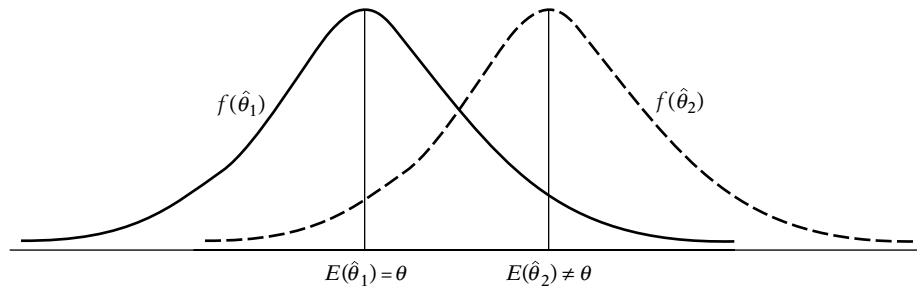


FIGURE A.8 Biased and unbiased estimators.

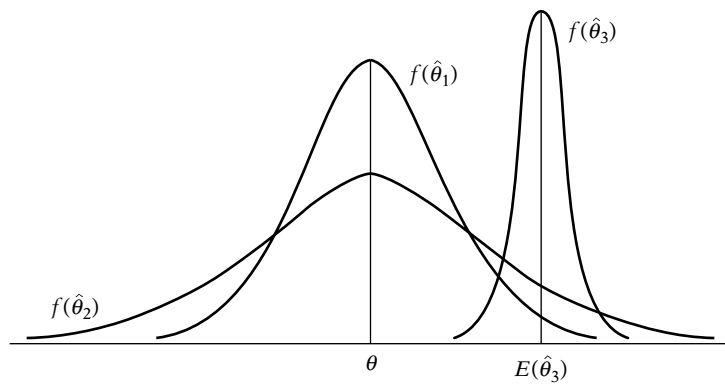


FIGURE A.9 Distribution of three estimators of  $\theta$ .

estimates is expected to be equal to the true value if the estimator is to be unbiased.

**Minimum Variance.**  $\hat{\theta}_1$  is said to be a minimum-variance estimator of  $\theta$  if the variance of  $\hat{\theta}_1$  is smaller than or at most equal to the variance of  $\hat{\theta}_2$ , which is any other estimator of  $\theta$ . Geometrically, we have Figure A.9, which shows three estimators of  $\theta$ , namely  $\hat{\theta}_1$ ,  $\hat{\theta}_2$ , and  $\hat{\theta}_3$ , and their probability distributions. As shown, the variance of  $\hat{\theta}_3$  is smaller than that of either  $\hat{\theta}_1$  or  $\hat{\theta}_2$ . Hence, assuming only the three possible estimators, in this case  $\hat{\theta}_3$  is a minimum-variance estimator. But note that  $\hat{\theta}_3$  is a biased estimator (why?).

**Best Unbiased, or Efficient, Estimator.** If  $\hat{\theta}_1$  and  $\hat{\theta}_2$  are two unbiased estimators of  $\theta$ , and the variance of  $\hat{\theta}_1$  is smaller than or at most equal to the variance of  $\hat{\theta}_2$ , then  $\hat{\theta}_1$  is a **minimum-variance unbiased, or best unbiased, or efficient**, estimator. Thus, in Figure A.9, of the two unbiased estimators  $\hat{\theta}_1$  and  $\hat{\theta}_2$ ,  $\hat{\theta}_1$  is best unbiased, or efficient.

**Linearity.** An estimator  $\hat{\theta}$  is said to be a linear estimator of  $\theta$  if it is a linear function of the sample observations. Thus, the sample mean defined as

$$\bar{X} = \frac{1}{n} \sum X_i = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

is a linear estimator because it is a linear function of the  $X$  values.

**Best Linear Unbiased Estimator (BLUE).** If  $\hat{\theta}$  is linear, is unbiased, and has minimum variance in the class of all linear unbiased estimators of  $\theta$ , then it is called a *best linear unbiased estimator*, or **BLUE** for short.

**Minimum Mean-Square-Error (MSE) Estimator.** The MSE of an estimator  $\hat{\theta}$  is defined as

$$\text{MSE}(\hat{\theta}) = E(\hat{\theta} - \theta)^2$$

This is in contrast with the variance of  $\hat{\theta}$ , which is defined as

$$\text{var}(\hat{\theta}) = E[\hat{\theta} - E(\hat{\theta})]^2$$

The difference between the two is that  $\text{var}(\hat{\theta})$  measures the dispersion of the distribution of  $\hat{\theta}$  around its mean or expected value, whereas  $\text{MSE}(\hat{\theta})$  measures dispersion around the true value of the parameter. The relationship between the two is as follows:

$$\begin{aligned} \text{MSE}(\hat{\theta}) &= E(\hat{\theta} - \theta)^2 \\ &= E[\hat{\theta} - E(\hat{\theta}) + E(\hat{\theta}) - \theta]^2 \\ &= E[\hat{\theta} - E(\hat{\theta})]^2 + E[E(\hat{\theta}) - \theta]^2 + 2E[\hat{\theta} - E(\hat{\theta})][E(\hat{\theta}) - \theta] \\ &= E[\hat{\theta} - E(\hat{\theta})]^2 + E[E(\hat{\theta}) - \theta]^2 \quad \text{since the last term is zero}^6 \\ &= \text{var}(\hat{\theta}) + \text{bias}(\hat{\theta})^2 \\ &= \text{variance of } \hat{\theta} \text{ plus square bias} \end{aligned}$$

Of course, if the bias is zero,  $\text{MSE}(\hat{\theta}) = \text{var}(\hat{\theta})$ .

The minimum MSE criterion consists in choosing an estimator whose MSE is the least in a competing set of estimators. But notice that even if such an estimator is found, there is a tradeoff involved—to obtain minimum variance you may have to accept some bias. Geometrically, the situation is as shown in Figure A.10. In this figure,  $\hat{\theta}_2$  is slightly biased, but its variance is smaller than that of the unbiased estimator  $\hat{\theta}_1$ . In practice, however, the minimum MSE criterion is used when the best unbiased criterion is incapable of producing estimators with smaller variances.

<sup>6</sup>The last term can be written as  $2\{[E(\hat{\theta})]^2 - [E(\hat{\theta})]^2 - \theta E(\hat{\theta}) + \theta E(\hat{\theta})\} = 0$ . Also note that  $E[E(\hat{\theta}) - \theta]^2 = [E(\hat{\theta}) - \theta]^2$ , since the expected value of a constant is simply the constant itself.

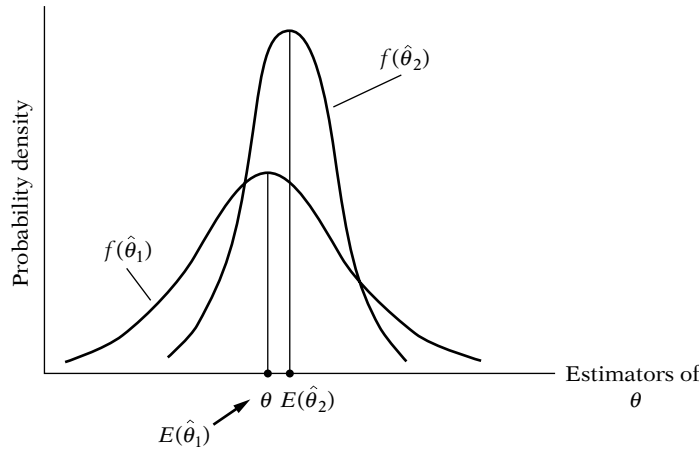


FIGURE A.10 Tradeoff between bias and variance.

### Large-Sample Properties

It often happens that an estimator does not satisfy one or more of the desirable statistical properties in small samples. But as the sample size increases indefinitely, the estimator possesses several desirable statistical properties. These properties are known as the **large-sample**, or **asymptotic, properties**.

**Asymptotic Unbiasedness.** An estimator  $\hat{\theta}$  is said to be an asymptotically unbiased estimator of  $\theta$  if

$$\lim_{n \rightarrow \infty} E(\hat{\theta}_n) = \theta$$

where  $\hat{\theta}_n$  means that the estimator is based on a sample size of  $n$  and where  $\lim$  means limit and  $n \rightarrow \infty$  means that  $n$  increases indefinitely. In words,  $\hat{\theta}$  is an asymptotically unbiased estimator of  $\theta$  if its expected, or mean, value approaches the true value as the sample size gets larger and larger. As an example, consider the following measure of the sample variance of a random variable  $X$ :

$$S^2 = \frac{\sum (X_i - \bar{X})^2}{n}$$

It can be shown that

$$E(S^2) = \sigma^2 \left( 1 - \frac{1}{n} \right)$$

where  $\sigma^2$  is the true variance. It is obvious that in a small sample  $S^2$  is biased, but as  $n$  increases indefinitely,  $E(S^2)$  approaches true  $\sigma^2$ ; hence it is asymptotically unbiased.

**Consistency.**  $\hat{\theta}$  is said to be a consistent estimator if it approaches the true value  $\theta$  as the sample size gets larger and larger. Figure A.11 illustrates this property.

In this figure we have the distribution of  $\hat{\theta}$  based on sample sizes of 25, 50, 80, and 100. As the figure shows,  $\hat{\theta}$  based on  $n = 25$  is biased since its sampling distribution is not centered on the true  $\theta$ . But as  $n$  increases, the distribution of  $\hat{\theta}$  not only tends to be more closely centered on  $\theta$  (i.e.,  $\hat{\theta}$  becomes less biased) but its variance also becomes smaller. If in the limit (i.e., when  $n$  increases indefinitely) the distribution of  $\hat{\theta}$  collapses to the single point  $\theta$ , that is, if the distribution of  $\hat{\theta}$  has zero spread, or variance, we say that  $\hat{\theta}$  is a **consistent estimator** of  $\theta$ .

More formally, an estimator  $\hat{\theta}$  is said to be a consistent estimator of  $\theta$  if the probability that the absolute value of the difference between  $\hat{\theta}$  and  $\theta$  is less than  $\delta$  (an arbitrarily small positive quantity) approaches unity. Symbolically,

$$\lim_{n \rightarrow \infty} P\{|\hat{\theta} - \theta| < \delta\} = 1 \quad \delta > 0$$

where  $P$  stands for probability. This is often expressed as

$$\text{plim}_{n \rightarrow \infty} \hat{\theta} = \theta$$

where plim means probability limit.

Note that the properties of unbiasedness and consistency are conceptually very much different. The property of unbiasedness can hold for any sample size, whereas consistency is strictly a large-sample property.

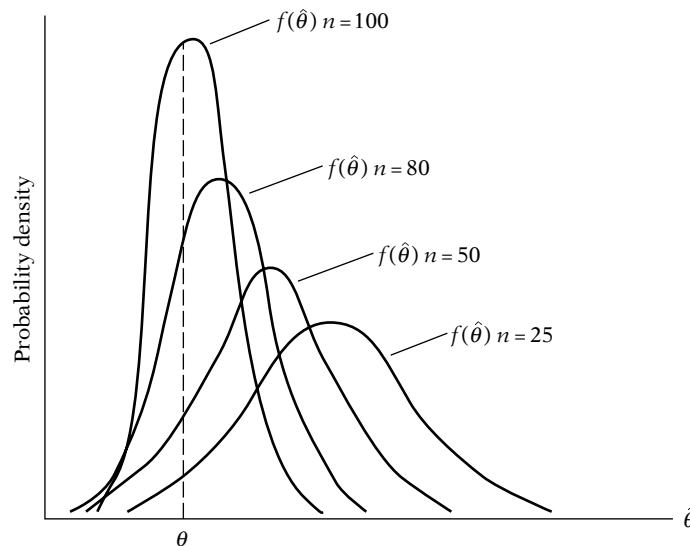


FIGURE A.11 The distribution of  $\hat{\theta}$  as sample size increases.

A *sufficient condition* for consistency is that the bias and variance both tend to zero as the sample size increases indefinitely.<sup>7</sup> Alternatively, a sufficient condition for consistency is that the  $MSE(\hat{\theta})$  tends to zero as  $n$  increases indefinitely. (For  $MSE(\hat{\theta})$ , see the discussion presented previously.)

**EXAMPLE 26**

Let  $X_1, X_2, \dots, X_n$  be a random sample from a distribution with mean  $\mu$  and variance  $\sigma^2$ . Show that the sample mean  $\bar{X}$  is a consistent estimator of  $\mu$ .

From elementary statistics it is known that  $E(\bar{X}) = \mu$  and  $\text{var}(\bar{X}) = \sigma^2/n$ . Since  $E(\bar{X}) = \mu$  regardless of the sample size, it is unbiased. Moreover, as  $n$  increases indefinitely,  $\text{var}(\bar{X})$  tends toward zero. Hence,  $\bar{X}$  is a consistent estimator of  $\mu$ .

The following rules about probability limits are noteworthy.

**1. Invariance (Slutsky property).** If  $\hat{\theta}$  is a consistent estimator of  $\theta$  and if  $h(\hat{\theta})$  is any continuous function of  $\hat{\theta}$ , then

$$\text{plim}_{n \rightarrow \infty} h(\hat{\theta}) = h(\theta)$$

What this means is that if  $\hat{\theta}$  is a consistent estimator of  $\theta$ , then  $1/\hat{\theta}$  is also a consistent estimator of  $1/\theta$  and that  $\log(\hat{\theta})$  is also a consistent estimator of  $\log(\theta)$ . Note that this property does not hold true of the expectation operator  $E$ ; that is, if  $\hat{\theta}$  is an unbiased estimator of  $\theta$  [that is,  $E(\hat{\theta}) = \theta$ ], it is *not true* that  $1/\hat{\theta}$  is an unbiased estimator of  $1/\theta$ ; that is,  $E(1/\hat{\theta}) \neq 1/E(\hat{\theta}) \neq 1/\theta$ .

**2.** If  $b$  is a constant, then

$$\text{plim}_{n \rightarrow \infty} b = b$$

That is, the probability limit of a constant is the same constant.

**3.** If  $\hat{\theta}_1$  and  $\hat{\theta}_2$  are consistent estimators, then

$$\text{plim}(\hat{\theta}_1 + \hat{\theta}_2) = \text{plim} \hat{\theta}_1 + \text{plim} \hat{\theta}_2$$

$$\text{plim}(\hat{\theta}_1 \hat{\theta}_2) = \text{plim} \hat{\theta}_1 \text{plim} \hat{\theta}_2$$

$$\text{plim}\left(\frac{\hat{\theta}_1}{\hat{\theta}_2}\right) = \frac{\text{plim} \hat{\theta}_1}{\text{plim} \hat{\theta}_2}$$

The last two properties, in general, do not hold true of the expectation operator  $E$ . Thus,  $E(\hat{\theta}_1/\hat{\theta}_2) \neq E(\hat{\theta}_1)/E(\hat{\theta}_2)$ . Similarly,  $E(\hat{\theta}_1 \hat{\theta}_2) \neq E(\hat{\theta}_1)E(\hat{\theta}_2)$ . If, however,  $\hat{\theta}_1$  and  $\hat{\theta}_2$  are independently distributed,  $E(\hat{\theta}_1 \hat{\theta}_2) = E(\hat{\theta}_1)E(\hat{\theta}_2)$ , as noted previously.

<sup>7</sup>More technically,  $\lim_{n \rightarrow \infty} E(\hat{\theta}_n) = \theta$  and  $\lim_{n \rightarrow \infty} \text{var}(\hat{\theta}_n) = 0$ .

**Asymptotic Efficiency.** Let  $\hat{\theta}$  be an estimator of  $\theta$ . The variance of the asymptotic distribution of  $\hat{\theta}$  is called the **asymptotic variance** of  $\hat{\theta}$ . If  $\hat{\theta}$  is consistent and its asymptotic variance is smaller than the asymptotic variance of all other consistent estimators of  $\theta$ ,  $\hat{\theta}$  is called **asymptotically efficient**.

**Asymptotic Normality.** An estimator  $\hat{\theta}$  is said to be asymptotically normally distributed if its sampling distribution tends to approach the normal distribution as the sample size  $n$  increases indefinitely. For example, statistical theory shows that if  $X_1, X_2, \dots, X_n$  are independent normally distributed variables with the same mean  $\mu$  and the same variance  $\sigma^2$ , the sample mean  $\bar{X}$  is also normally distributed with mean  $\mu$  and variance  $\sigma^2/n$  in small as well as large samples. But if the  $X_i$  are independent with mean  $\mu$  and variance  $\sigma^2$  but are not necessarily from the normal distribution, then the sample mean  $\bar{X}$  is asymptotically normally distributed with mean  $\mu$  and variance  $\sigma^2/n$ ; that is, as the sample size  $n$  increases indefinitely, the sample mean tends to be normally distributed with mean  $\mu$  and variance  $\sigma^2/n$ . That is in fact the central limit theorem discussed previously.

## A.8 STATISTICAL INFERENCE: HYPOTHESIS TESTING

Estimation and hypothesis testing constitute the twin branches of classical statistical inference. Having examined the problem of estimation, we briefly look at the problem of testing statistical hypotheses.

The problem of hypothesis testing may be stated as follows. Assume that we have an rv  $X$  with a known PDF  $f(x; \theta)$ , where  $\theta$  is the parameter of the distribution. Having obtained a random sample of size  $n$ , we obtain the point estimator  $\hat{\theta}$ . Since the true  $\theta$  is rarely known, we raise the question: Is the estimator  $\hat{\theta}$  “compatible” with some hypothesized value of  $\theta$ , say,  $\theta = \theta^*$ , where  $\theta^*$  is a specific numerical value of  $\theta$ ? In other words, could our sample have come from the PDF  $f(x; \theta) = \theta^*$ ? In the language of hypothesis testing  $\theta = \theta^*$  is called the **null** (or maintained) **hypothesis** and is generally denoted by  $H_0$ . The null hypothesis is tested against an **alternative hypothesis**, denoted by  $H_1$ , which, for example, may state that  $\theta \neq \theta^*$ . (Note: In some textbooks,  $H_0$  and  $H_1$  are designated by  $H_1$  and  $H_2$ , respectively.)

The null hypothesis and the alternative hypothesis can be **simple** or **composite**. A hypothesis is called *simple* if it specifies the value(s) of the parameter(s) of the distribution; otherwise it is called a *composite* hypothesis. Thus, if  $X \sim N(\mu, \sigma^2)$  and we state that

$$H_0: \mu = 15 \quad \text{and} \quad \sigma = 2$$

it is a simple hypothesis, whereas

$$H_0: \mu = 15 \quad \text{and} \quad \sigma > 2$$

is a composite hypothesis because here the value of  $\sigma$  is not specified.

To test the null hypothesis (i.e., to test its validity), we use the sample information to obtain what is known as the **test statistic**. Very often this test statistic turns out to be the point estimator of the unknown parameter. Then we try to find out the *sampling, or probability, distribution* of the test statistic and use the **confidence interval or test of significance** approach to test the null hypothesis. The mechanics are illustrated below.

To fix the ideas, let us revert to Example 23, which was concerned with the height ( $X$ ) of men in a population. We are told that

$$X_i \sim N(\mu, \sigma^2) = N(\mu, 2.5^2)$$

$$\bar{X} = 67 \quad n = 100$$

Let us assume that

$$H_0: \mu = \mu^* = 69$$

$$H_1: \mu \neq 69$$

The question is: Could the sample with  $\bar{X} = 67$ , the test statistic, have come from the population with the mean value of 69? Intuitively, we may not reject the null hypothesis if  $\bar{X}$  is “sufficiently close” to  $\mu^*$ ; otherwise we may reject it in favor of the alternative hypothesis. But how do we decide that  $\bar{X}$  is “sufficiently close” to  $\mu^*$ ? We can adopt two approaches, (1) confidence interval and (2) test of significance, both leading to identical conclusions in any specific application.

### The Confidence Interval Approach

Since  $X_i \sim N(\mu, \sigma^2)$ , we know that the test statistic  $\bar{X}$  is distributed as

$$\bar{X} \sim N(\mu, \sigma^2/n)$$

Since we know the probability distribution of  $\bar{X}$ , why not establish, say, a  $100(1 - \alpha)$  confidence interval for  $\mu$  based on  $\bar{X}$  and see whether this confidence interval includes  $\mu = \mu^*$ ? If it does, we may not reject the null hypothesis; if it does not, we may reject the null hypothesis. Thus, if  $\alpha = 0.05$ , we will have a 95% confidence interval and if this confidence interval includes  $\mu^*$ , we may not reject the null hypothesis—95 out of 100 intervals thus established are likely to include  $\mu^*$ .

The actual mechanics are as follows: since  $\bar{X} \sim N(\mu, \sigma^2/n)$ , it follows that

$$Z_i = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

that is, a standard normal variable. Then from the normal distribution table we know that

$$\Pr(-1.96 \leq Z_i \leq 1.96) = 0.95$$

That is,

$$\Pr\left(-1.96 \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq 1.96\right) = 0.95$$

which, on rearrangement, gives

$$\Pr\left[\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}\right] = 0.95$$

This is a 95% confidence interval for  $\mu$ . Once this interval has been established, the test of the null hypothesis is simple. All that we have to do is to see whether  $\mu = \mu^*$  lies in this interval. If it does, we may not reject the null hypothesis; if it does not, we may reject it.

Turning to our example, we have already established a 95% confidence interval for  $\mu$ , which is

$$66.51 \leq \mu \leq 67.49$$

This interval obviously does not include  $\mu = 69$ . Therefore, we can reject the null hypothesis that the true  $\mu$  is 69 with a 95% confidence coefficient. Geometrically, the situation is as depicted in Figure A.12.

In the language of hypothesis testing, the confidence interval that we have established is called the **acceptance region** and the area(s) outside the acceptance region is (are) called the **critical region(s)**, or **region(s) of rejection** of the null hypothesis. The lower and upper limits of the acceptance region (which demarcate it from the rejection regions) are called the **critical values**. In this language of hypothesis testing, if the hypothesized value falls inside the acceptance region, one may not reject the null hypothesis; otherwise one may reject it.

It is important to note that in deciding to reject or not reject  $H_0$ , we are likely to commit two types of errors: (1) we may reject  $H_0$  when it is, in fact,

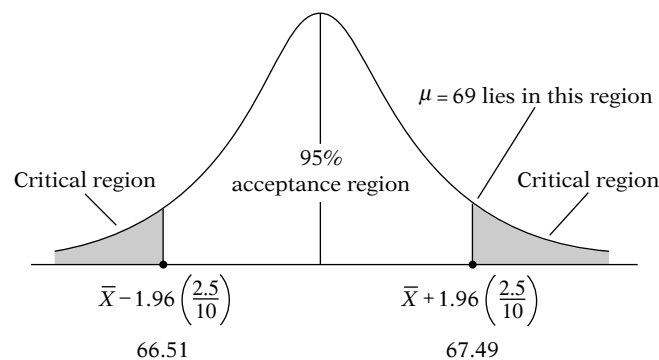


FIGURE A.12 95% confidence interval for  $\mu$ .



true; this is called a **type I error** (thus, in the preceding example  $\bar{X} = 67$  could have come from the population with a mean value of 69), or (2) we may not reject  $H_0$  when it is, in fact, false; this is called a **type II error**. Therefore, a hypothesis test does not establish the value of true  $\mu$ . It merely provides a means of deciding whether we may act as if  $\mu = \mu^*$ .

**Type I and Type II Errors.** Schematically, we have

Decision	State of nature	
	$H_0$ is true	$H_0$ is false
Reject	Type I error	No error
Do not reject	No error	Type II error

Ideally, we would like to minimize both type I and type II errors. But unfortunately, for any given sample size, it is not possible to minimize both the errors simultaneously. The classical approach to this problem, embodied in the work of Neyman and Pearson, is to assume that a type I error is likely to be more serious in practice than a type II error. Therefore, one should try to keep the probability of committing a type I error at a fairly low level, such as 0.01 or 0.05, and then try to minimize the probability of having a type II error as much as possible.

In the literature, the probability of a type I error is designated as  $\alpha$  and is called the **level of significance**, and the probability of a type II error is designated as  $\beta$ . The probability of *not* committing a type II error is called the **power of the test**. *Put differently, the power of a test is its ability to reject a false null hypothesis.* The classical approach to hypothesis testing is to fix  $\alpha$  at levels such as 0.01 (or 1 percent) or 0.05 (5 percent) and then try to maximize the power of the test; that is to minimize  $\beta$ .

It is important that the reader understands the concept of the power of a test, which is best explained with an example.<sup>8</sup>

Let  $X \sim N(\mu, 100)$ ; that is,  $X$  is normally distributed with mean  $\mu$  and variance 100. Assume that  $\alpha = 0.05$ . Suppose we have a sample of 25 observations, which gives a sample mean value of  $\bar{X}$ . Suppose further we entertain the hypothesis  $H_0: \mu = 50$ . Since  $X$  is normally distributed, we know that the sample mean is also normally distributed as:  $\bar{X} \sim N(\mu, 100/25)$ . Hence under the stated null hypothesis that  $\mu = 50$ , the 95% confidence interval for  $\bar{X}$  is  $(\mu \pm 1.96(\sqrt{100/25}) = \mu \pm 3.92$ , that is, (46.08 to 53.92). Therefore, the critical region consists of all values of  $\bar{X}$  less than 46.08 or greater than 53.92. That is, we will reject the null hypothesis that the true mean is 50 if a sample mean value is found below 46.08 or greater than 53.92.

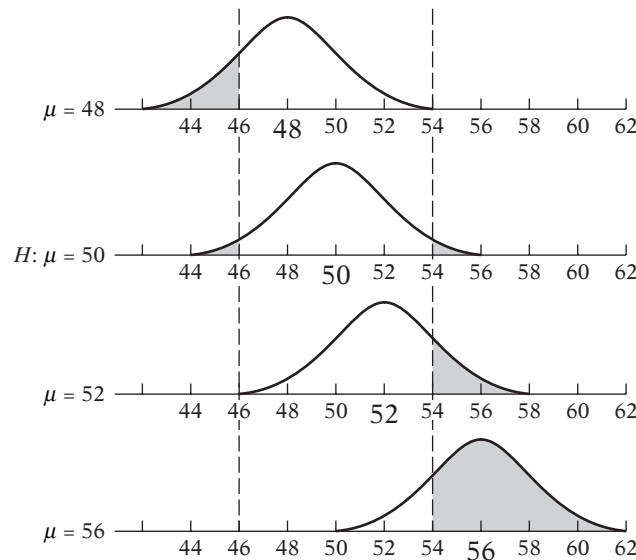
<sup>8</sup>The following discussion and the figures are based on Helen M. Walker and Joseph Lev, *Statistical Inference*, Holt, Rinehart and Winston, New York, 1953, pp. 161–162.

But what is the probability that  $\bar{X}$  will lie in the preceding critical region(s) if the true  $\mu$  has a value different from 50? Suppose there are three alternative hypotheses:  $\mu = 48$ ,  $\mu = 52$ , and  $\mu = 56$ . If any of these alternatives is true, it will be the actual mean of the distribution of  $\bar{X}$ . The standard error is unchanged for the three alternatives since  $\sigma^2$  is still assumed to be 100.

The shaded areas in Figure A.13 show the probabilities that  $\bar{X}$  will fall in the critical region if each of the alternative hypotheses is true. As you can check, these probabilities are 0.17 (for  $\mu = 48$ ), 0.05 (for  $\mu = 50$ ), 0.17 (for  $\mu = 52$ ) and 0.85 (for  $\mu = 56$ ). As you can see from this figure, whenever the true value of  $\mu$  differs substantially from the hypothesis under consideration (which here is  $\mu = 50$ ), the probability of rejecting the hypothesis is high but when the true value is not very different from the value given under the null hypothesis, the probability of rejection is small. Intuitively, this should make sense if the null and alternative hypotheses are very closely bunched.

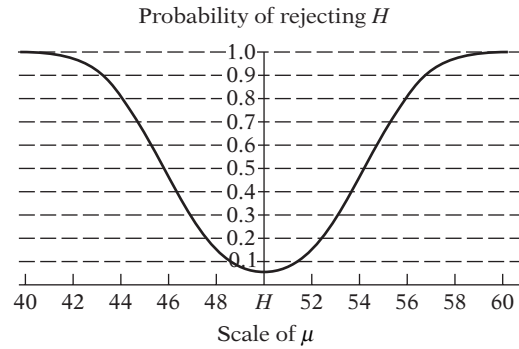
This can be seen further if you consider Figure A.14, which is called the **power function graph**, and the curve shown there is called the **power curve**.

The reader will by now realize that the confidence coefficient  $(1 - \alpha)$  discussed earlier is simply one minus the probability of committing a type



**FIGURE A.13** Distribution of  $X$  when  $N = 25$ ,  $\sigma = 10$ , and  $\mu = 48, 50, 52$ , or  $56$ . Under  $H: \mu = 50$ , the critical region with  $\alpha = 0.05$  is  $\bar{X} < 46.1$  and  $\bar{X} > 53.9$ . The shaded area indicates the probability that  $\bar{X}$  will fall into the critical region. This probability is:

0.17 if $\mu = 48$	0.17 if $\mu = 52$
0.05 if $\mu = 50$	0.85 if $\mu = 56$



**FIGURE A.14** Power function of test of hypothesis  $\mu = 50$  when  $N = 25$ ,  $\sigma = 10$ , and  $\alpha = 0.05$ .

I error. Thus a 95 percent confidence coefficient means that we are prepared to accept at the most a 5 percent probability of committing a type I error—we do not want to reject the true hypothesis by more than 5 out of 100 times.

**The  $p$  Value, or Exact Level of Significance.** Instead of preselecting  $\alpha$  at arbitrary levels, such as 1, 5, or 10 percent, one can obtain the  **$p$  (probability) value, or exact level of significance** of a test statistic. The  $p$  value is defined as *the lowest significance level at which a null hypothesis can be rejected*.

Suppose that in an application involving 20 df we obtain a  $t$  value of 3.552. Now the  $p$  value, or the exact probability, of obtaining a  $t$  value of 3.552 or greater can be seen from Table D.2 as 0.001 (one-tailed) or 0.002 (two-tailed). We can say that the observed  $t$  value of 3.552 is statistically significant at the 0.001 or 0.002 level, depending on whether we are using a one-tail or two-tail test.

Several statistical packages now routinely print out the  $p$  value of the estimated test statistics. Therefore, the reader is advised to give the  $p$  value wherever possible.

### The Test of Significance Approach

Recall that

$$Z_i = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

In any given application,  $\bar{X}$  and  $n$  are known (or can be estimated), but the true  $\mu$  and  $\sigma$  are not known. But if  $\sigma$  is specified and we assume (under  $H_0$ ) that  $\mu = \mu^*$ , a specific numerical value, then  $Z_i$  can be directly computed and we can easily look at the normal distribution table to find the probability of obtaining the computed  $Z$  value. If this probability is small, say, less than 5 percent or 1 percent, we can reject the null hypothesis—if the

hypothesis were true, the chances of obtaining the particular  $Z$  value should be very high. This is the general idea behind the test of significance approach to hypothesis testing. The key idea here is the test statistic (here the  $Z$  statistic) and its probability distribution under the assumed value  $\mu = \mu^*$ . Appropriately, in the present case, the test is known as the **Z test**, since we use the  $Z$  (standardized normal) value.

Returning to our example, if  $\mu = \mu^* = 69$ , the  $Z$  statistic becomes

$$\begin{aligned} Z &= \frac{\bar{X} - \mu^*}{\sigma/\sqrt{n}} \\ &= \frac{67 - 69}{2.5/\sqrt{100}} \\ &= -2/0.25 = -8 \end{aligned}$$

If we look at the normal distribution table D.1, we see that the probability of obtaining such a  $Z$  value is extremely small. (*Note:* The probability of a  $Z$  value exceeding 3 or  $-3$  is about 0.001. Therefore, the probability of  $Z$  exceeding 8 is even smaller.) Therefore, we can reject the null hypothesis that  $\mu = 69$ ; given this value, our chance of obtaining  $\bar{X}$  of 67 is extremely small. We therefore doubt that our sample came from the population with a mean value of 69. Diagrammatically, the situation is depicted in Figure A.15.

In the language of test of significance, when we say that a test (statistic) is significant, we generally mean that we can reject the null hypothesis. And the test statistic is regarded as significant if the probability of our obtaining it is equal to or less than  $\alpha$ , the probability of committing a type I error. Thus if  $\alpha = 0.05$ , we know that the probability of obtaining a  $Z$  value of  $-1.96$  or  $1.96$  is 5 percent (or 2.5 percent in each tail of the standardized normal distribution). In our illustrative example  $Z$  was  $-8$ . Hence the probability of obtaining such a  $Z$  value is much smaller than 2.5 percent, well below our pre-specified probability of committing a type I error. That is why the computed value of  $Z = -8$  is statistically significant; that is, we reject the null

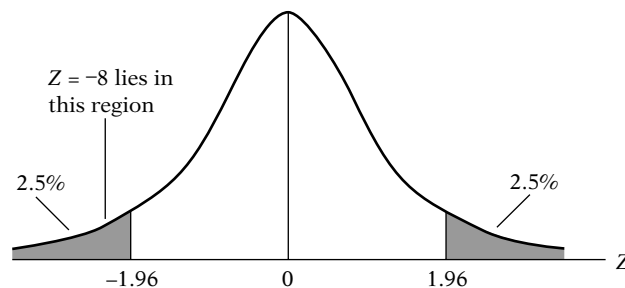


FIGURE A.15 The distribution of the  $Z$  statistic.

hypothesis that the true  $\mu^*$  is 69. Of course, we reached the same conclusion using the confidence interval approach to hypothesis testing.

We now summarize the steps involved in testing a statistical hypothesis:

**Step 1.** State the null hypothesis  $H_0$  and the alternative hypothesis  $H_1$  (e.g.,  $H_0: \mu = 69$  and  $H_1: \mu \neq 69$ ).

**Step 2.** Select the test statistic (e.g.,  $\bar{X}$ ).

**Step 3.** Determine the probability distribution of the test statistic (e.g.,  $\bar{X} \sim N(\mu, \sigma^2/n)$ ).

**Step 4.** Choose the level of significance (i.e., the probability of committing a type I error)  $\alpha$ .

**Step 5.** Using the probability distribution of the test statistic, establish a  $100(1 - \alpha)\%$  confidence interval. If the value of the parameter under the null hypothesis (e.g.,  $\mu = \mu^* = 69$ ) lies in this confidence region, the region of acceptance, do not reject the null hypothesis. But if it falls outside this interval (i.e., it falls into the region of rejection), you may reject the null hypothesis. Keep in mind that in not rejecting or rejecting a null hypothesis you are taking a chance of being wrong  $\alpha$  percent of the time.

## REFERENCES

For the details of the material covered in this appendix, the reader may consult the following references:

Hoel, Paul G.: *Introduction to Mathematical Statistics*, 4th ed., John Wiley & Sons, New York, 1974. This book provides a fairly simple introduction to various aspects of mathematical statistics.

Freund, John E., and Ronald E. Walpole: *Mathematical Statistics*, 3d ed., Prentice Hall, Englewood Cliffs, N.J., 1980. Another introductory textbook in mathematical statistics.

Mood, Alexander M., Franklin A. Graybill, and Duane C. Boes: *Introduction to the Theory of Statistics*, 3d ed., McGraw-Hill, New York, 1974. This is a comprehensive introduction to the theory of statistics but is somewhat more difficult than the preceding two textbooks.

Newbold, Paul: *Statistics for Business and Economics*, Prentice Hall, Englewood Cliffs, N.J., 1984. A comprehensive nonmathematical introduction to statistics with lots of worked-out problems.



**Column Vector**

A matrix consisting of  $M$  rows and only one column is called a **column vector**. Letting the boldface lowercase letters denote vectors, an example of a column vector is

$$\mathbf{x}_{4 \times 1} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 9 \end{bmatrix}$$

**Row Vector**

A matrix consisting of only one row and  $N$  columns is called a **row vector**.

$$\mathbf{x}_{1 \times 4} = [1 \quad 2 \quad 5 \quad -4] \quad \mathbf{y}_{1 \times 5} = [0 \quad 5 \quad -9 \quad 6 \quad 10]$$

**Transposition**

The transpose of an  $M \times N$  matrix  $\mathbf{A}$ , denoted by  $\mathbf{A}'$  (read as  $\mathbf{A}$  prime or  $\mathbf{A}$  transpose) is an  $N \times M$  matrix obtained by interchanging the rows and columns of  $\mathbf{A}$ ; that is, the  $i$ th row of  $\mathbf{A}$  becomes the  $i$ th column of  $\mathbf{A}'$ . For example,

$$\mathbf{A}_{3 \times 2} = \begin{bmatrix} 4 & 5 \\ 3 & 1 \\ 5 & 0 \end{bmatrix} \quad \mathbf{A}'_{2 \times 3} = \begin{bmatrix} 4 & 3 & 5 \\ 5 & 1 & 0 \end{bmatrix}$$

Since a vector is a special type of matrix, the transpose of a row vector is a column vector and the transpose of a column vector is a row vector. Thus

$$\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{x}' = [4 \quad 5 \quad 6]$$

We shall follow the convention of indicating the row vectors by primes.

**Submatrix**

Given any  $M \times N$  matrix  $\mathbf{A}$ , if all but  $r$  rows and  $s$  columns of  $\mathbf{A}$  are deleted, the resulting matrix of order  $r \times s$  is called a **submatrix** of  $\mathbf{A}$ . Thus, if

$$\mathbf{A}_{3 \times 3} = \begin{bmatrix} 3 & 5 & 7 \\ 8 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

and we delete the third row and the third column of  $\mathbf{A}$ , we obtain

$$\mathbf{B}_{2 \times 2} = \begin{bmatrix} 3 & 5 \\ 8 & 2 \end{bmatrix}$$

which is a submatrix of  $\mathbf{A}$  whose order is  $2 \times 2$ .

## B.2 TYPES OF MATRICES

### Square Matrix

A matrix that has the same number of rows as columns is called a **square matrix**.

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 5 & 8 \\ 7 & 3 & 1 \\ 4 & 5 & 0 \end{bmatrix}$$

### Diagonal Matrix

A square matrix with at least one nonzero element on the main diagonal (running from the upper-left-hand corner to the lower-right-hand corner) and zeros elsewhere is called a **diagonal matrix**.

$$\mathbf{A}_{2 \times 2} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \mathbf{B}_{3 \times 3} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

### Scalar Matrix

A diagonal matrix whose diagonal elements are all equal is called a **scalar matrix**. An example is the variance-covariance matrix of the population disturbance of the classical linear regression model given in equation (C.2.3), namely,

$$\text{var-cov}(\mathbf{u}) = \begin{bmatrix} \sigma^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2 \end{bmatrix}$$

### Identity, or Unit, Matrix

A diagonal matrix whose diagonal elements are all 1 is called an **identity**, or **unit, matrix** and is denoted by  $\mathbf{I}$ . It is a special kind of scalar matrix.

$$\mathbf{I}_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{I}_{4 \times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

### Symmetric Matrix

A square matrix whose elements above the main diagonal are mirror images of the elements below the main diagonal is called a **symmetric matrix**. Alternatively, a symmetric matrix is such that its transpose is equal to itself; that is,  $\mathbf{A} = \mathbf{A}'$ . That is, the element  $a_{ij}$  of  $\mathbf{A}$  is equal to the element  $a_{ji}$  of  $\mathbf{A}'$ . An example is the variance-covariance matrix given in equation (C.2.2). Another example is the correlation matrix given in (C.5.1).



### Null Matrix

A matrix whose elements are all zero is called a **null matrix** and is denoted by  $\mathbf{0}$ .

### Null Vector

A row or column vector whose elements are all zero is called a **null vector** and is also denoted by  $\mathbf{0}$ .

### Equal Matrices

Two matrices  $\mathbf{A}$  and  $\mathbf{B}$  are said to be equal if they are of the same order and their corresponding elements are equal; that is,  $a_{ij} = b_{ij}$  for all  $i$  and  $j$ . For example, the matrices

$$\mathbf{A}_{3 \times 3} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & 2 \\ 5 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_{3 \times 3} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & 2 \\ 5 & 1 & 3 \end{bmatrix}$$

are equal; that is  $\mathbf{A} = \mathbf{B}$ .

## B.3 MATRIX OPERATIONS

### Matrix Addition

Let  $\mathbf{A} = [a_{ij}]$  and  $\mathbf{B} = [b_{ij}]$ . If  $\mathbf{A}$  and  $\mathbf{B}$  are of the same order, we define matrix addition as

$$\mathbf{A} + \mathbf{B} = \mathbf{C}$$

where  $\mathbf{C}$  is of the same order as  $\mathbf{A}$  and  $\mathbf{B}$  and is obtained as  $c_{ij} = a_{ij} + b_{ij}$  for all  $i$  and  $j$ ; that is,  $\mathbf{C}$  is obtained by adding the corresponding elements of  $\mathbf{A}$  and  $\mathbf{B}$ . If such addition can be effected,  $\mathbf{A}$  and  $\mathbf{B}$  are said to be *conformable* for addition. For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & -1 & 3 \\ -2 & 0 & 1 & 5 \end{bmatrix}$$

and  $\mathbf{C} = \mathbf{A} + \mathbf{B}$ , then

$$\mathbf{C} = \begin{bmatrix} 3 & 3 & 3 & 8 \\ 4 & 7 & 9 & 14 \end{bmatrix}$$

### Matrix Subtraction

Matrix subtraction follows the same principle as matrix addition except that  $\mathbf{C} = \mathbf{A} - \mathbf{B}$ ; that is, we subtract the elements of  $\mathbf{B}$  from the corresponding elements of  $\mathbf{A}$  to obtain  $\mathbf{C}$ , provided  $\mathbf{A}$  and  $\mathbf{B}$  are of the same order.

### Scalar Multiplication

To multiply a matrix  $\mathbf{A}$  by a scalar  $\lambda$  (a real number), we multiply each element of the matrix by  $\lambda$ :

$$\lambda\mathbf{A} = [\lambda a_{ij}]$$

For example, if  $\lambda = 2$  and

$$\mathbf{A} = \begin{bmatrix} -3 & 5 \\ 8 & 7 \end{bmatrix}$$

then

$$\lambda\mathbf{A} = \begin{bmatrix} -6 & 10 \\ 16 & 14 \end{bmatrix}$$

### Matrix Multiplication

Let  $\mathbf{A}$  be  $M \times N$  and  $\mathbf{B}$  be  $N \times P$ . Then the product  $\mathbf{AB}$  (in that order) is defined to be a new matrix  $\mathbf{C}$  of order  $M \times P$  such that

$$c_{ij} = \sum_{k=1}^N a_{ik}b_{kj} \quad \begin{array}{l} i = 1, 2, \dots, M \\ j = 1, 2, \dots, P \end{array}$$

That is, the element in the  $i$ th row and the  $j$ th column of  $\mathbf{C}$  is obtained by multiplying the elements of the  $i$ th row of  $\mathbf{A}$  by the corresponding elements of the  $j$ th column of  $\mathbf{B}$  and summing over all terms; this is known as the *row by column* rule of multiplication. Thus, to obtain  $c_{11}$ , the element in the first row and the first column of  $\mathbf{C}$ , we multiply the elements in the first row of  $\mathbf{A}$  by the corresponding elements in the first column of  $\mathbf{B}$  and sum over all terms. Similarly, to obtain  $c_{12}$ , we multiply the elements in the first row of  $\mathbf{A}$  by the corresponding elements in the second column of  $\mathbf{B}$  and sum over all terms, and so on.

Note that for multiplication to exist, matrices  $\mathbf{A}$  and  $\mathbf{B}$  must be conformable with respect to multiplication; that is, the number of columns in  $\mathbf{A}$  must be equal to the number of rows in  $\mathbf{B}$ . If, for example,

$$\mathbf{A}_{2 \times 3} = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_{3 \times 2} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 6 & 2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB} = \mathbf{C}_{2 \times 2} &= \begin{bmatrix} (3 \times 2) + (4 \times 3) + (7 \times 6) & (3 \times 1) + (4 \times 5) + (7 \times 2) \\ (5 \times 2) + (6 \times 3) + (1 \times 6) & (5 \times 1) + (6 \times 5) + (1 \times 2) \end{bmatrix} \\ &= \begin{bmatrix} 60 & 37 \\ 34 & 37 \end{bmatrix} \end{aligned}$$

But if

$$\mathbf{A}_{2 \times 3} = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_{2 \times 2} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

the product  $\mathbf{AB}$  is not defined since  $\mathbf{A}$  and  $\mathbf{B}$  are not conformable with respect to multiplication.

### Properties of Matrix Multiplication

1. Matrix multiplication is not necessarily *commutative*; that is, in general,  $\mathbf{AB} \neq \mathbf{BA}$ . Therefore, the order in which the matrices are multiplied is very important.  $\mathbf{AB}$  means that  $\mathbf{A}$  is *postmultiplied* by  $\mathbf{B}$  or  $\mathbf{B}$  is *premultiplied* by  $\mathbf{A}$ .

2. Even if  $\mathbf{AB}$  and  $\mathbf{BA}$  exist, the resulting matrices may not be of the same order. Thus, if  $\mathbf{A}$  is  $M \times N$  and  $\mathbf{B}$  is  $N \times M$ ,  $\mathbf{AB}$  is  $M \times M$  whereas  $\mathbf{BA}$  is  $N \times N$ , hence of different order.

3. Even if  $\mathbf{A}$  and  $\mathbf{B}$  are both square matrices, so that  $\mathbf{AB}$  and  $\mathbf{BA}$  are both defined, the resulting matrices will not be necessarily equal. For example, if

$$\mathbf{A} = \begin{bmatrix} 4 & 7 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 6 & 8 \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} 46 & 76 \\ 15 & 31 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 19 & 17 \\ 48 & 58 \end{bmatrix}$$

and  $\mathbf{AB} \neq \mathbf{BA}$ . An example of  $\mathbf{AB} = \mathbf{BA}$  is when both  $\mathbf{A}$  and  $\mathbf{B}$  are identity matrices.

4. A row vector postmultiplied by a column vector is a scalar. Thus, consider the ordinary least-squares residuals  $\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n$ . Letting  $\mathbf{u}$  be a column vector and  $\mathbf{u}'$  be a row vector, we have

$$\begin{aligned} \hat{\mathbf{u}}' \hat{\mathbf{u}} &= [\hat{u}_1 \quad \hat{u}_2 \quad \hat{u}_3 \quad \cdots \quad \hat{u}_n] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \vdots \\ \hat{u}_n \end{bmatrix} \\ &= \hat{u}_1^2 + \hat{u}_2^2 + \hat{u}_3^2 + \cdots + \hat{u}_n^2 \\ &= \sum \hat{u}_i^2 \quad \text{a scalar [see Eq. (C.3.5)]} \end{aligned}$$

5. A column vector postmultiplied by a row vector is a matrix. As an example, consider the population disturbances of the classical linear regression model, namely,  $u_1, u_2, \dots, u_n$ . Letting  $\mathbf{u}$  be a column vector and  $\mathbf{u}'$  a row vector, we obtain

$$\begin{aligned} \mathbf{u} \mathbf{u}' &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} [u_1 \quad u_2 \quad u_3 \quad \cdots \quad u_n] \\ &= \begin{bmatrix} u_1^2 & u_1 u_2 & u_1 u_3 & \cdots & u_1 u_n \\ u_2 u_1 & u_2^2 & u_2 u_3 & \cdots & u_2 u_n \\ \dots & \dots & \dots & \dots & \dots \\ u_n u_1 & u_n u_2 & u_n u_3 & \cdots & u_n^2 \end{bmatrix} \end{aligned}$$

which is a matrix of order  $n \times n$ . Note that the preceding matrix is symmetrical.

6. A matrix postmultiplied by a column vector is a column vector.
7. A row vector postmultiplied by a matrix is a row vector.
8. Matrix multiplication is *associative*; that is,  $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ , where  $\mathbf{A}$  is  $M \times N$ ,  $\mathbf{B}$  is  $N \times P$ , and  $\mathbf{C}$  is  $P \times K$ .
9. Matrix multiplication is distributive with respect to addition; that is,  $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$  and  $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ .

### Matrix Transposition

We have already defined the process of matrix transposition as interchanging the rows and the columns of a matrix (or a vector). We now state some of the properties of transposition.

1. The transpose of a transposed matrix is the original matrix itself. Thus,  $(\mathbf{A}')' = \mathbf{A}$ .
2. If  $\mathbf{A}$  and  $\mathbf{B}$  are conformable for addition, then  $\mathbf{C} = \mathbf{A} + \mathbf{B}$  and  $\mathbf{C}' = (\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$ . That is, the transpose of the sum of two matrices is the sum of their transposes.
3. If  $\mathbf{AB}$  is defined, then  $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$ . That is, the transpose of the product of two matrices is the product of their transposes in the reverse order. This can be generalized:  $(\mathbf{ABCD})' = \mathbf{D}'\mathbf{C}'\mathbf{B}'\mathbf{A}'$ .
4. The transpose of an identity matrix  $\mathbf{I}$  is the identity matrix itself; that is  $\mathbf{I}' = \mathbf{I}$ .
5. The transpose of a scalar is the scalar itself. Thus, if  $\lambda$  is a scalar,  $\lambda' = \lambda$ .
6. The transpose of  $(\lambda\mathbf{A})'$  is  $\lambda\mathbf{A}'$  where  $\lambda$  is a scalar. [Note:  $(\lambda\mathbf{A})' = \mathbf{A}'\lambda' = \mathbf{A}'\lambda = \lambda\mathbf{A}'$ .]
7. If  $\mathbf{A}$  is a square matrix such that  $\mathbf{A} = \mathbf{A}'$ , then  $\mathbf{A}$  is a symmetric matrix. (Cf. the definition of symmetric matrix given previously.)

### Matrix Inversion

An inverse of a square matrix  $\mathbf{A}$ , denoted by  $\mathbf{A}^{-1}$  (read  $\mathbf{A}$  inverse), if it exists, is a unique square matrix such that

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where  $\mathbf{I}$  is an identity matrix whose order is the same as that of  $\mathbf{A}$ . For example

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \quad \mathbf{A}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{6}{8} & -\frac{1}{4} \end{bmatrix} \quad \mathbf{AA}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

We shall see how  $\mathbf{A}^{-1}$  is computed after we study the topic of determinants. In the meantime, note these properties of the inverse.

1.  $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ ; that is, the inverse of the product of two matrices is the product of their inverses in the reverse order.
2.  $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$ ; that is, the transpose of  $\mathbf{A}$  inverse is the inverse of  $\mathbf{A}$  transpose.

**B.4 DETERMINANTS**

To every square matrix,  $\mathbf{A}$ , there corresponds a number (scalar) known as the determinant of the matrix, which is denoted by  $\det \mathbf{A}$  or by the symbol  $|\mathbf{A}|$ , where  $|\cdot|$  means “the determinant of.” Note that a matrix per se has no numerical value, but the determinant of a matrix is a number.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -7 \\ 2 & 5 & 0 \\ 3 & 8 & 6 \end{bmatrix} \quad |\mathbf{A}| = \begin{vmatrix} 1 & 3 & -7 \\ 2 & 5 & 0 \\ 3 & 8 & 6 \end{vmatrix}$$

The  $|\mathbf{A}|$  in this example is called a determinant of order 3 because it is associated with a matrix of order  $3 \times 3$ .

**Evaluation of a Determinant**

The process of finding the value of a determinant is known as the *evaluation*, *expansion*, or *reduction* of the determinant. This is done by manipulating the entries of the matrix in a well-defined manner.

**Evaluation of a  $2 \times 2$  Determinant.** If

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

its determinant is evaluated as follows:

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

which is obtained by cross-multiplying the elements on the main diagonal and subtracting from it the cross-multiplication of the elements on the other diagonal of matrix  $\mathbf{A}$ , as indicated by the arrows.

**Evaluation of a  $3 \times 3$  Determinant.** If

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then

$$|\mathbf{A}| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

A careful examination of the evaluation of a  $3 \times 3$  determinant shows:

1. Each term in the expansion of the determinant contains one and only one element from each row and each column.
2. The number of elements in each term is the same as the number of rows (or columns) in the matrix. Thus, a  $2 \times 2$  determinant has two elements

in each term of its expansion, a  $3 \times 3$  determinant has three elements in each term of its expansion, and so on.

3. The terms in the expansion alternate in sign from  $+$  to  $-$ .

4. A  $2 \times 2$  determinant has two terms in its expansion, and a  $3 \times 3$  determinant has six terms in its expansion. The general rule is: The determinant of order  $N \times N$  has  $N! = N(N-1)(N-2) \cdots 3 \cdot 2 \cdot 1$  terms in its expansion, where  $N!$  is read “ $N$  factorial.” Following this rule, a determinant of order  $5 \times 5$  will have  $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$  terms in its expansion.<sup>1</sup>

### Properties of Determinants

1. A matrix whose determinantal value is zero is called a **singular matrix**, whereas a matrix with a nonzero determinant is called a **nonsingular matrix**. The inverse of a matrix as defined before does not exist for a singular matrix.

2. If all the elements of any row of  $\mathbf{A}$  are zero, its determinant is zero. Thus,

$$|\mathbf{A}| = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = 0$$

3.  $|\mathbf{A}'| = |\mathbf{A}|$ ; that is, the determinants of  $\mathbf{A}$  and  $\mathbf{A}$  transpose are the same.

4. Interchanging any two rows or any two columns of a matrix  $\mathbf{A}$  changes the sign of  $|\mathbf{A}|$ .

#### EXAMPLE

If

$$\mathbf{A} = \begin{bmatrix} 6 & 9 \\ -1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$$

where  $\mathbf{B}$  is obtained by interchanging the rows of  $\mathbf{A}$ , then

$$\begin{aligned} |\mathbf{A}| &= 24 - (-9) & \text{and} & & |\mathbf{B}| &= -9 - (24) \\ &= 33 & & & &= -33 \end{aligned}$$

5. If every element of a row or a column of  $\mathbf{A}$  is multiplied by a scalar  $\lambda$ , then  $|\mathbf{A}|$  is multiplied by  $\lambda$ .

<sup>1</sup>To evaluate the determinant of an  $N \times N$  matrix,  $\mathbf{A}$ , see the references.

**EXAMPLE**

If

$$\lambda = 5 \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 5 & -8 \\ 2 & 4 \end{bmatrix}$$

and we multiply the first row of  $\mathbf{A}$  by 5 to obtain

$$\mathbf{B} = \begin{bmatrix} 25 & -40 \\ 2 & 4 \end{bmatrix}$$

it can be seen that  $|\mathbf{A}| = 36$  and  $|\mathbf{B}| = 180$ , which is  $5|\mathbf{A}|$ .

6. If two rows or columns of a matrix are identical, its determinant is zero.

7. If one row or a column of a matrix is a multiple of another row or column of that matrix, its determinant is zero. Thus, if

$$\mathbf{A} = \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}$$

where the first row of  $\mathbf{A}$  is twice its second row,  $|\mathbf{A}| = 0$ . More generally, if any row (column) of a matrix is a linear combination of other rows (columns), its determinant is zero.

8.  $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ ; that is, the determinant of the product of two matrices is the product of their (individual) determinants.

**Rank of a Matrix**

The rank of a matrix is the order of the largest square submatrix whose determinant is not zero.

**EXAMPLE**

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 6 \\ 0 & 4 & 5 \\ 3 & 2 & 1 \end{bmatrix}$$

It can be seen that  $|\mathbf{A}| = 0$ . In other words,  $\mathbf{A}$  is a singular matrix. Hence although its order is  $3 \times 3$ , its rank is less than 3. Actually, it is 2, because we can find a  $2 \times 2$  submatrix whose determinant is not zero. For example, if we delete the first row and the first column of  $\mathbf{A}$ , we obtain

$$\mathbf{B} = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$$

whose determinant is  $-6$ , which is nonzero. Hence the rank of  $\mathbf{A}$  is 2. As noted previously, the inverse of a singular matrix does not exist. Therefore, for an  $N \times N$  matrix  $\mathbf{A}$ , its rank must be  $N$  for its inverse to exist; if it is less than  $N$ ,  $\mathbf{A}$  is singular.

**Minor**

If the  $i$ th row and  $j$ th column of an  $N \times N$  matrix  $\mathbf{A}$  are deleted, the determinant of the resulting submatrix is called the **minor** of the element  $a_{ij}$  (the element at the intersection of the  $i$ th row and the  $j$ th column) and is denoted by  $|\mathbf{M}_{ij}|$ .

**EXAMPLE**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The minor of  $a_{11}$  is

$$|\mathbf{M}_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

Similarly, the minor of  $a_{21}$  is

$$|\mathbf{M}_{21}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{13}a_{32}$$

The minors of other elements of  $\mathbf{A}$  can be found similarly.

**Cofactor**

The cofactor of the element  $a_{ij}$  of an  $N \times N$  matrix  $\mathbf{A}$ , denoted by  $c_{ij}$ , is defined as

$$c_{ij} = (-1)^{i+j} |\mathbf{M}_{ij}|$$

In other words, a cofactor is a *signed* minor, the sign being positive if  $i + j$  is even and being negative if  $i + j$  is odd. Thus, the cofactor of the element  $a_{11}$  of the  $3 \times 3$  matrix  $\mathbf{A}$  given previously is  $a_{22}a_{33} - a_{23}a_{32}$ , whereas the cofactor of the element  $a_{21}$  is  $-(a_{12}a_{33} - a_{13}a_{32})$  since the sum of the subscripts 2 and 1 is 3, which is an odd number.

**Cofactor Matrix.** Replacing the elements  $a_{ij}$  of a matrix  $\mathbf{A}$  by their cofactors, we obtain a matrix known as the **cofactor matrix** of  $\mathbf{A}$ , denoted by  $(\text{cof } \mathbf{A})$ .

**Adjoint Matrix.** The adjoint matrix, written as  $(\text{adj } \mathbf{A})$ , is the transpose of the cofactor matrix; that is,  $(\text{adj } \mathbf{A}) = (\text{cof } \mathbf{A})'$ .

**B.5 FINDING THE INVERSE OF A SQUARE MATRIX**

If  $\mathbf{A}$  is square and nonsingular (that is,  $|\mathbf{A}| \neq 0$ ), its inverse  $\mathbf{A}^{-1}$  can be found as follows:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} (\text{adj } \mathbf{A})$$



The steps involved in the computation are as follows:

1. Find the determinant of  $\mathbf{A}$ . If it is nonzero, proceed to step 2.
2. Replace each element  $a_{ij}$  of  $\mathbf{A}$  by its cofactor to obtain the cofactor matrix.
3. Transpose the cofactor matrix to obtain the adjoint matrix.
4. Divide each element of the adjoint matrix by  $|\mathbf{A}|$ .

#### EXAMPLE

Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 4 \\ 2 & 1 & 3 \end{bmatrix}$$

**Step 1.** We first find the determinant of the matrix. Applying the rules of expanding a  $3 \times 3$  determinant given previously, we obtain  $|\mathbf{A}| = -24$ .

**Step 2.** We now obtain the cofactor matrix, say,  $\mathbf{C}$ :

$$\mathbf{C} = \begin{bmatrix} \begin{vmatrix} 7 & 4 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 7 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 17 & -7 & -9 \\ -3 & -3 & 3 \\ -13 & 11 & -3 \end{bmatrix}$$

**Step 3.** Transposing the preceding cofactor matrix, we obtain the following adjoint matrix:

$$(\text{adj } \mathbf{A}) = \begin{bmatrix} 17 & -3 & -13 \\ -7 & -3 & 11 \\ -9 & 3 & -3 \end{bmatrix}$$

**Step 4.** We now divide the elements of  $(\text{adj } \mathbf{A})$  by the determinantal value of  $-24$  to obtain

$$\mathbf{A}^{-1} = -\frac{1}{24} \begin{bmatrix} 17 & -3 & -13 \\ -7 & -3 & 11 \\ -9 & 3 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{17}{24} & \frac{3}{24} & \frac{13}{24} \\ \frac{7}{24} & \frac{3}{24} & -\frac{11}{24} \\ \frac{9}{24} & -\frac{3}{24} & \frac{3}{24} \end{bmatrix}$$

It can be readily verified that

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is an identity matrix. The reader should verify that for the illustrative example given in Appendix C the inverse of the  $\mathbf{X}'\mathbf{X}$  matrix is as shown in Eq. (C.10.5).

## B.6 MATRIX DIFFERENTIATION

To follow the material in Appendix CA, Section CA.2, we need some rules regarding matrix differentiation.

### RULE 1

If  $\mathbf{a}' = [a_1 \ a_2 \ \dots \ a_n]$  is a row vector of numbers, and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is a column vector of the variables  $x_1, x_2, \dots, x_n$ , then

$$\frac{\partial(\mathbf{a}'\mathbf{x})}{\partial\mathbf{x}} = \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

### RULE 2

Consider the matrix  $\mathbf{x}'\mathbf{A}\mathbf{x}$  such that

$$\mathbf{x}'\mathbf{A}\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n] \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$\frac{\partial(\mathbf{x}'\mathbf{A}\mathbf{x})}{\partial\mathbf{x}} = 2\mathbf{A}\mathbf{x}$$

which is a column vector of  $n$  elements, or

$$\frac{\partial(\mathbf{x}'\mathbf{A}\mathbf{x})}{\partial\mathbf{x}} = 2\mathbf{x}'\mathbf{A}$$

which is a row vector of  $n$  elements.

## REFERENCES

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APPENDIX **C**

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**THE MATRIX APPROACH  
TO LINEAR REGRESSION  
MODEL**

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This appendix presents the classical linear regression model involving  $k$  variables ( $Y$  and  $X_2, X_3, \dots, X_k$ ) in matrix algebra notation. Conceptually, the  $k$ -variable model is a logical extension of the two- and three-variable models considered thus far in this text. Therefore, this appendix presents very few new concepts save for the matrix notation.<sup>1</sup>

A great advantage of matrix algebra over scalar algebra (elementary algebra dealing with scalars or real numbers) is that it provides a compact method of handling regression models involving any number of variables; once the  $k$ -variable model is formulated and solved in matrix notation, the solution applies to one, two, three, or any number of variables.

**C.1 THE  $k$ -VARIABLE LINEAR REGRESSION MODEL**

If we generalize the two- and three-variable linear regression models, the  $k$ -variable population regression model (PRF) involving the dependent variable  $Y$  and  $k - 1$  explanatory variables  $X_2, X_3, \dots, X_k$  may be written as

$$\text{PRF: } Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \dots + \beta_k X_{ki} + u_i \quad i = 1, 2, 3, \dots, n \quad (\text{C.1.1})$$

where  $\beta_1$  = the intercept,  $\beta_2$  to  $\beta_k$  = partial slope coefficients,  $u$  = stochastic disturbance term, and  $i$  =  $i$ th observation,  $n$  being the size of the population. The PRF (C.1.1) is to be interpreted in the usual manner: It gives the mean or expected value of  $Y$  conditional upon the fixed (in repeated sampling) values of  $X_2, X_3, \dots, X_k$ , that is,  $E(Y | X_{2i}, X_{3i}, \dots, X_{ki})$ .

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<sup>1</sup>Readers not familiar with matrix algebra should review **App. B** before proceeding any further. **Appendix B** provides the essentials of matrix algebra needed to follow this appendix.

Equation (C.1.1) is a shorthand expression for the following set of  $n$  simultaneous equations:

$$\begin{aligned} Y_1 &= \beta_1 + \beta_2 X_{21} + \beta_3 X_{31} + \cdots + \beta_k X_{k1} + u_1 \\ Y_2 &= \beta_1 + \beta_2 X_{22} + \beta_3 X_{32} + \cdots + \beta_k X_{k2} + u_2 \\ &\dots\dots\dots \\ Y_n &= \beta_1 + \beta_2 X_{2n} + \beta_3 X_{3n} + \cdots + \beta_k X_{kn} + u_n \end{aligned} \tag{C.1.2}$$

Let us write the system of equations (C.1.2) in an alternative but more illuminating way as follows<sup>2</sup>:

$$\begin{aligned} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} &= \begin{bmatrix} 1 & X_{21} & X_{31} & \cdots & X_{k1} \\ 1 & X_{22} & X_{32} & \cdots & X_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{2n} & X_{3n} & \cdots & X_{kn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \\ \mathbf{y} &= \mathbf{X} \boldsymbol{\beta} + \mathbf{u} \\ n \times 1 & \quad n \times k \quad k \times 1 \quad n \times 1 \end{aligned} \tag{C.1.3}$$

where  $\mathbf{y} = n \times 1$  column vector of observations on the dependent variable  $Y$   
 $\mathbf{X} = n \times k$  matrix giving  $n$  observations on  $k - 1$  variables  $X_2$  to  $X_k$ ,  
the first column of 1's representing the intercept term (this matrix is also known as the **data matrix**)  
 $\boldsymbol{\beta} = k \times 1$  column vector of the unknown parameters  $\beta_1, \beta_2, \dots, \beta_k$   
 $\mathbf{u} = n \times 1$  column vector of  $n$  disturbances  $u_i$

Using the rules of matrix multiplication and addition, the reader should verify that systems (C.1.2) and (C.1.3) are equivalent.

System (C.1.3) is known as the *matrix representation of the general ( $k$ -variable) linear regression model*. It can be written more compactly as

$$\begin{matrix} \mathbf{y} & = & \mathbf{X} & \boldsymbol{\beta} & + & \mathbf{u} \\ n \times 1 & & n \times k & k \times 1 & & n \times 1 \end{matrix} \tag{C.1.4}$$

Where there is no confusion about the dimensions or orders of the matrix  $\mathbf{X}$  and the vectors  $\mathbf{y}$ ,  $\boldsymbol{\beta}$ , and  $\mathbf{u}$ , Eq. (C.1.4) may be written simply as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \tag{C.1.5}$$

As an illustration of the matrix representation, consider the two-variable consumption-income model considered in Chapter 3, namely,  $Y_i = \beta_1 + \beta_2 X_i + u_i$ , where  $Y$  is consumption expenditure and  $X$  is income. Using the

<sup>2</sup>Following the notation introduced in **App. B**, we shall represent vectors by lowercase boldfaced letters and matrices by uppercase boldfaced letters.

data given in Table 3.2, we may write the matrix formulation as

$$\begin{bmatrix} 70 \\ 65 \\ 90 \\ 95 \\ 110 \\ 115 \\ 120 \\ 140 \\ 155 \\ 150 \end{bmatrix} = \begin{bmatrix} 1 & 80 \\ 1 & 100 \\ 1 & 120 \\ 1 & 140 \\ 1 & 160 \\ 1 & 180 \\ 1 & 200 \\ 1 & 220 \\ 1 & 240 \\ 1 & 260 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{bmatrix} \quad (\text{C.1.6})$$

$$\begin{matrix} \mathbf{y} & = & \mathbf{X} & \boldsymbol{\beta} & + & \mathbf{u} \\ 10 \times 1 & & 10 \times 2 & 2 \times 1 & & 10 \times 1 \end{matrix}$$

As in the two- and three-variable cases, our objective is to estimate the parameters of the multiple regression (C.1.1) and to draw inferences about them from the data at hand. In matrix notation this amounts to estimating  $\boldsymbol{\beta}$  and drawing inferences about this  $\boldsymbol{\beta}$ . For the purpose of estimation, we may use the method of ordinary least squares (OLS) or the method of maximum likelihood (ML). But as noted before, these two methods yield identical estimates of the regression coefficients.<sup>3</sup> Therefore, we shall confine our attention to the method of OLS.

## C.2 ASSUMPTIONS OF THE CLASSICAL LINEAR REGRESSION MODEL IN MATRIX NOTATION

The assumptions underlying the classical linear regression model are given in Table C.1; they are presented both in scalar notation and in matrix notation. Assumption 1 given in (C.2.1) means that the expected value of the disturbance vector  $\mathbf{u}$ , that is, of each of its elements, is zero. More explicitly,  $E(\mathbf{u}) = \mathbf{0}$  means

$$E \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} E(u_1) \\ E(u_2) \\ \vdots \\ E(u_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (\text{C.2.1})$$

<sup>3</sup>The proof that this is so in the  $k$ -variable case can be found in the footnote reference given in Chap. 4.

**TABLE C.1** ASSUMPTIONS OF THE CLASSICAL LINEAR REGRESSION MODEL

Scalar notation	Matrix notation
1. $E(u_i) = 0$ , for each $i$ (3.2.1)	1. $E(\mathbf{u}) = \mathbf{0}$ where $\mathbf{u}$ and $\mathbf{0}$ are $n \times 1$ column vectors, $\mathbf{0}$ being a null vector
2. $E(u_i u_j) = 0$ $i \neq j$ (3.2.5) $= \sigma^2$ $i = j$ (3.2.2)	2. $E(\mathbf{u}\mathbf{u}') = \sigma^2 \mathbf{I}$ where $\mathbf{I}$ is an $n \times n$ identity matrix
3. $X_2, X_3, \dots, X_k$ are nonstochastic or fixed	3. The $n \times k$ matrix $\mathbf{X}$ is nonstochastic, that is, it consists of a set of fixed numbers
4. There is no exact linear relationship among the $X$ variables, that is, no multicollinearity (7.1.7)	4. The rank of $\mathbf{X}$ is $\rho(\mathbf{X}) = k$ , where $k$ is the number of columns in $\mathbf{X}$ and $k$ is less than the number of observations, $n$
5. For hypothesis testing, $u_i \sim N(0, \sigma^2)$ (4.2.4)	5. The $\mathbf{u}$ vector has a multivariate normal distribution, i.e., $\mathbf{u} \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$

Assumption 2 [Eq. (C.2.2)] is a compact way of expressing the two assumptions given in (3.2.5) and (3.2.2) by the scalar notation. To see this, we can write

$$E(\mathbf{u}\mathbf{u}') = E \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} [u_1 \quad u_2 \quad \cdots \quad u_n]$$

where  $\mathbf{u}'$  is the transpose of the column vector  $\mathbf{u}$ , or a row vector. Performing the multiplication, we obtain

$$E(\mathbf{u}\mathbf{u}') = E \begin{bmatrix} u_1^2 & u_1 u_2 & \cdots & u_1 u_n \\ u_2 u_1 & u_2^2 & \cdots & u_2 u_n \\ \dots & \dots & \dots & \dots \\ u_n u_1 & u_n u_2 & \cdots & u_n^2 \end{bmatrix}$$

Applying the expectations operator  $E$  to each element of the preceding matrix, we obtain

$$E(\mathbf{u}\mathbf{u}') = \begin{bmatrix} E(u_1^2) & E(u_1 u_2) & \cdots & E(u_1 u_n) \\ E(u_2 u_1) & E(u_2^2) & \cdots & E(u_2 u_n) \\ \dots & \dots & \dots & \dots \\ E(u_n u_1) & E(u_n u_2) & \cdots & E(u_n^2) \end{bmatrix} \quad (\text{C.2.2})$$

Because of the assumptions of homoscedasticity and no serial correlation, matrix (C.2.2) reduces to

$$\begin{aligned}
 E(\mathbf{uu}') &= \begin{bmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \sigma^2 \end{bmatrix} \\
 &= \sigma^2 \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (\text{C.2.3}) \\
 &= \sigma^2 \mathbf{I}
 \end{aligned}$$

where  $\mathbf{I}$  is an  $n \times n$  identity matrix.

Matrix (C.2.2) [and its representation given in (C.2.3)] is called the **variance-covariance matrix** of the disturbances  $u_i$ ; the elements on the main diagonal of this matrix (running from the upper left corner to the lower right corner) give the variances, and the elements off the main diagonal give the covariances.<sup>4</sup> Note that the variance-covariance matrix is **symmetric**: The elements above and below the main diagonal are reflections of one another.

Assumption 3 states that the  $n \times k$  matrix  $\mathbf{X}$  is nonstochastic; that is, it consists of fixed numbers. As noted previously, our regression analysis is conditional regression analysis, conditional upon the fixed values of the  $X$  variables.

Assumption 4 states that the  $\mathbf{X}$  matrix has full column rank equal to  $k$ , the number of columns in the matrix. This means that the columns of the  $X$  matrix are linearly independent; that is, there is no **exact linear relationship** among the  $X$  variables. In other words there is no multicollinearity. In scalar notation this is equivalent to saying that there exists no set of numbers  $\lambda_1, \lambda_2, \dots, \lambda_k$  not all zero such that [cf. (7.1.8)]

$$\lambda_1 X_{1i} + \lambda_2 X_{2i} + \cdots + \lambda_k X_{ki} = 0 \quad (\text{C.2.4})$$

where  $X_{1i} = 1$  for all  $i$  (to allow for the column of 1's in the  $\mathbf{X}$  matrix). In matrix notation, (C.2.4) can be represented as

$$\boldsymbol{\lambda}' \mathbf{x} = 0 \quad (\text{C.2.5})$$

where  $\boldsymbol{\lambda}'$  is a  $1 \times k$  row vector and  $\mathbf{x}$  is a  $k \times 1$  column vector.

<sup>4</sup>By definition, the variance of  $u_i = E[u_i - E(u_i)]^2$  and the covariance between  $u_i$  and  $u_j = E[u_i - E(u_i)][u_j - E(u_j)]$ . But because of the assumption  $E(u_i) = 0$  for each  $i$ , we have the variance-covariance matrix (C.2.3).

If an exact linear relationship such as (C.2.4) exists, the variables are said to be collinear. If, on the other hand, (C.2.4) holds true only if  $\lambda_1 = \lambda_2 = \lambda_3 = \dots = 0$ , then the  $X$  variables are said to be linearly independent. An intuitive reason for the *no multicollinearity* assumption was given in Chapter 7, and we explored this assumption further in Chapter 10.

### C.3 OLS ESTIMATION

To obtain the OLS estimate of  $\beta$ , let us first write the  $k$ -variable sample regression (SRF):

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \dots + \hat{\beta}_k X_{ki} + \hat{u}_i \quad (\text{C.3.1})$$

which can be written more compactly in matrix notation as

$$\mathbf{y} = \mathbf{X}\hat{\boldsymbol{\beta}} + \hat{\mathbf{u}} \quad (\text{C.3.2})$$

and in matrix form as

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_{21} & X_{31} & \cdots & X_{k1} \\ 1 & X_{22} & X_{32} & \cdots & X_{k2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & X_{2n} & X_{3n} & \cdots & X_{kn} \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} + \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} \quad (\text{C.3.3})$$

$$\begin{matrix} \mathbf{y} & = & \mathbf{X} & \hat{\boldsymbol{\beta}} & + & \hat{\mathbf{u}} \\ n \times 1 & & n \times k & k \times 1 & & n \times 1 \end{matrix}$$

where  $\hat{\boldsymbol{\beta}}$  is a  $k$ -element column vector of the OLS estimators of the regression coefficients and where  $\hat{\mathbf{u}}$  is an  $n \times 1$  column vector of  $n$  residuals.

As in the two- and three-variable models, in the  $k$ -variable case the OLS estimators are obtained by minimizing

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \dots - \hat{\beta}_k X_{ki})^2 \quad (\text{C.3.4})$$

where  $\sum \hat{u}_i^2$  is the residual sum of squares (RSS). In matrix notation, this amounts to minimizing  $\hat{\mathbf{u}}'\hat{\mathbf{u}}$  since

$$\hat{\mathbf{u}}'\hat{\mathbf{u}} = [\hat{u}_1 \quad \hat{u}_2 \quad \cdots \quad \hat{u}_n] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} = \hat{u}_1^2 + \hat{u}_2^2 + \dots + \hat{u}_n^2 = \sum \hat{u}_i^2 \quad (\text{C.3.5})$$

Now from (C.3.2) we obtain

$$\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}} \quad (\text{C.3.6})$$





Note these features of the  $(\mathbf{X}'\mathbf{X})$  matrix: (1) It gives the raw sums of squares and cross products of the  $X$  variables, one of which is the intercept term taking the value of 1 for each observation. The elements on the main diagonal give the raw sums of squares, and those off the main diagonal give the raw sums of cross products (by *raw* we mean in original units of measurement). (2) It is symmetrical since the cross product between  $X_{2i}$  and  $X_{3i}$  is the same as that between  $X_{3i}$  and  $X_{2i}$ . (3) It is of order  $(k \times k)$ , that is,  $k$  rows and  $k$  columns.

In (C.3.10) the known quantities are  $(\mathbf{X}'\mathbf{X})$  and  $(\mathbf{X}'\mathbf{y})$  (the cross product between the  $X$  variables and  $y$ ) and the unknown is  $\hat{\beta}$ . Now using matrix algebra, if the inverse of  $(\mathbf{X}'\mathbf{X})$  exists, say,  $(\mathbf{X}'\mathbf{X})^{-1}$ , then premultiplying both sides of (C.3.10) by this inverse, we obtain

$$(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

But since  $(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X}) = \mathbf{I}$ , an identity matrix of order  $k \times k$ , we get

$$\mathbf{I}\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

or

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \mathbf{y} \quad (\text{C.3.11})$$

$k \times 1 \quad k \times k \quad (k \times n) \quad (n \times 1)$

**Equation (C.3.11) is a fundamental result of the OLS theory in matrix notation.** It shows how the  $\hat{\beta}$  vector can be estimated from the given data. Although (C.3.11) was obtained from (C.3.9), it can be obtained directly from (C.3.7) by differentiating  $\hat{\mathbf{u}}'\hat{\mathbf{u}}$  with respect to  $\hat{\beta}$ . The proof is given in Appendix CA, Section CA.2.

### An Illustration

As an illustration of the matrix methods developed so far, let us rework the consumption–income example of Chapter 3, whose data are reproduced in (C.1.6). For the two-variable case we have

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

$$(\mathbf{X}'\mathbf{X}) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ X_1 & X_2 & X_3 & \cdots & X_n \end{bmatrix} \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ 1 & X_3 \\ \cdots & \cdots \\ 1 & X_N \end{bmatrix} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}$$

and

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ X_1 & X_2 & X_3 & \cdots & X_n \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum X_i Y_i \end{bmatrix}$$

Using the data given in (C.1.6), we obtain

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 10 & 1700 \\ 1700 & 322000 \end{bmatrix}$$

and

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1110 \\ 205500 \end{bmatrix}$$

Using the rules of matrix inversion given in **Appendix B**, we can see that the inverse of the preceding ( $\mathbf{X}'\mathbf{X}$ ) matrix is

$$\mathbf{X}'\mathbf{X}^{-1} = \begin{bmatrix} 0.97576 & -0.005152 \\ -0.005152 & 0.0000303 \end{bmatrix}$$

Therefore,

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 0.97576 & -0.005152 \\ -0.005152 & 0.0000303 \end{bmatrix} \begin{bmatrix} 1110 \\ 205500 \end{bmatrix} \\ &= \begin{bmatrix} 24.4545 \\ 0.5079 \end{bmatrix} \end{aligned}$$

Previously we obtained  $\hat{\beta}_1 = 24.4545$  and  $\hat{\beta}_2 = 0.5091$  using the computer. The difference between the two estimates is due to the rounding errors. In passing, note that in working on a desk calculator it is essential to obtain results to several significant digits to minimize the rounding errors.

### Variance–Covariance Matrix of $\hat{\boldsymbol{\beta}}$

Matrix methods enable us to develop formulas not only for the variance of  $\hat{\beta}_i$ , any given element of  $\hat{\boldsymbol{\beta}}$ , but also for the covariance between any two elements of  $\hat{\boldsymbol{\beta}}$ , say,  $\hat{\beta}_i$  and  $\hat{\beta}_j$ . We need these variances and covariances for the purpose of statistical inference.

By definition, the variance–covariance matrix of  $\hat{\boldsymbol{\beta}}$  is [cf. (C.2.2)]

$$\text{var-cov}(\hat{\boldsymbol{\beta}}) = E\{[\hat{\boldsymbol{\beta}} - E(\hat{\boldsymbol{\beta}})][\hat{\boldsymbol{\beta}} - E(\hat{\boldsymbol{\beta}})]'\}$$

which can be written explicitly as

$$\text{var-cov}(\hat{\boldsymbol{\beta}}) = \begin{bmatrix} \text{var}(\hat{\beta}_1) & \text{cov}(\hat{\beta}_1, \hat{\beta}_2) & \cdots & \text{cov}(\hat{\beta}_1, \hat{\beta}_k) \\ \text{cov}(\hat{\beta}_2, \hat{\beta}_1) & \text{var}(\hat{\beta}_2) & \cdots & \text{cov}(\hat{\beta}_2, \hat{\beta}_k) \\ \dots & \dots & \dots & \dots \\ \text{cov}(\hat{\beta}_k, \hat{\beta}_1) & \text{cov}(\hat{\beta}_k, \hat{\beta}_2) & \cdots & \text{var}(\hat{\beta}_k) \end{bmatrix} \quad (\text{C.3.12})$$

It is shown in Appendix CA, Section CA.3, that the preceding variance-covariance matrix can be obtained from the following formula:

$$\text{var-cov}(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1} \quad (\text{C.3.13})$$

where  $\sigma^2$  is the homoscedastic variance of  $u_i$  and  $(\mathbf{X}'\mathbf{X})^{-1}$  is the inverse matrix appearing in Eq. (C.3.11), which gives the OLS estimator  $\hat{\boldsymbol{\beta}}$ .

In the two- and three-variable linear regression models an unbiased estimator of  $\sigma^2$  was given by  $\hat{\sigma}^2 = \sum \hat{u}_i^2 / (n - 2)$  and  $\hat{\sigma}^2 = \sum \hat{u}_i^2 / (n - 3)$ , respectively. In the  $k$ -variable case, the corresponding formula is

$$\begin{aligned} \hat{\sigma}^2 &= \frac{\sum \hat{u}_i^2}{n - k} \\ &= \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n - k} \end{aligned} \quad (\text{C.3.14})$$

where there are now  $n - k$  df. (Why?)

Although in principle  $\hat{\mathbf{u}}'\hat{\mathbf{u}}$  can be computed from the estimated residuals, in practice it can be obtained directly as follows. Recalling that  $\sum \hat{u}_i^2 (= \text{RSS}) = \text{TSS} - \text{ESS}$ , in the two-variable case we may write

$$\sum \hat{u}_i^2 = \sum y_i^2 - \hat{\beta}_2^2 \sum x_i^2 \quad (\text{3.3.6})$$

and in the three-variable case

$$\sum \hat{u}_i^2 = \sum y_i^2 - \hat{\beta}_2 \sum y_i x_{2i} - \hat{\beta}_3 \sum y_i x_{3i} \quad (\text{7.4.19})$$

By extending this principle, it can be seen that for the  $k$ -variable model

$$\sum \hat{u}_i^2 = \sum y_i^2 - \hat{\beta}_2 \sum y_i x_{2i} - \cdots - \hat{\beta}_k \sum y_i x_{ki} \quad (\text{C.3.15})$$

In matrix notation,

$$\text{TSS: } \sum y_i^2 = \mathbf{y}'\mathbf{y} - n\bar{Y}^2 \quad (\text{C.3.16})$$

$$\text{ESS: } \hat{\beta}_2 \sum y_i x_{2i} + \cdots + \hat{\beta}_k \sum y_i x_{ki} = \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2 \quad (\text{C.3.17})$$

where the term  $n\bar{Y}^2$  is known as the correction for mean.<sup>6</sup> Therefore,

$$\hat{\mathbf{u}}'\hat{\mathbf{u}} = \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} \quad (\text{C.3.18})$$

Once  $\hat{\mathbf{u}}'\hat{\mathbf{u}}$  is obtained,  $\hat{\sigma}^2$  can be easily computed from (C.3.14), which, in turn, will enable us to estimate the variance-covariance matrix (C.3.13).

For our illustrative example,

$$\begin{aligned} \hat{\mathbf{u}}'\hat{\mathbf{u}} &= 132100 - [24.4545 \quad 0.5091] \begin{bmatrix} 1110 \\ 205500 \end{bmatrix} \\ &= 337.373 \end{aligned}$$

Hence,  $\hat{\sigma}^2 = (337.273/8) = 42.1591$ , which is approximately the value obtained previously in Chapter 3.

### Properties of OLS Vector $\hat{\boldsymbol{\beta}}$

In the two- and three-variable cases we know that the OLS estimators are linear and unbiased, and in the class of all linear unbiased estimators they have minimum variance (the Gauss-Markov property). In short, the OLS estimators are best linear unbiased estimators (BLUE). This property extends to the entire  $\hat{\boldsymbol{\beta}}$  vector; that is,  $\hat{\boldsymbol{\beta}}$  is linear (each of its elements is a linear function of  $Y$ , the dependent variable).  $E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$ , that is, the expected value of each element of  $\hat{\boldsymbol{\beta}}$  is equal to the corresponding element of the true  $\boldsymbol{\beta}$ , and in the class of all linear unbiased estimators of  $\boldsymbol{\beta}$ , the OLS estimator  $\hat{\boldsymbol{\beta}}$  has minimum variance.

The proof is given in Appendix CA, Section CA.4. As stated in the introduction, the  $k$ -variable case is in most cases a straight extension of the two- and three-variable cases.

### C.4 THE COEFFICIENT OF DETERMINATION $R^2$ IN MATRIX NOTATION

The coefficient of determination  $R^2$  has been defined as

$$R^2 = \frac{\text{ESS}}{\text{TSS}}$$

In the two-variable case,

$$R^2 = \frac{\hat{\beta}_2^2 \sum x_i^2}{\sum y_i^2} \quad (\text{3.5.6})$$

and in the three-variable case

$$R^2 = \frac{\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i}}{\sum y_i^2} \quad (\text{7.5.5})$$

<sup>6</sup>Note:  $\sum y_i^2 = \sum (Y_i - \bar{Y})^2 = \sum Y_i^2 - n\bar{Y}^2 = \mathbf{y}'\mathbf{y} - n\bar{Y}^2$ . Therefore, without the correction term,  $\mathbf{y}'\mathbf{y}$  will give simply the raw sum of squares, not the sum of squared deviations.

Generalizing we obtain for the  $k$ -variable case

$$R^2 = \frac{\hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i} + \dots + \hat{\beta}_k \sum y_i x_{ki}}{\sum y_i^2} \quad (\text{C.4.1})$$

By using (C.3.16) and (C.3.17), Eq. (C.4.1) can be written as

$$R^2 = \frac{\hat{\beta}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2}{\mathbf{y}'\mathbf{y} - n\bar{Y}^2} \quad (\text{C.4.2})$$

which gives the matrix representation of  $R^2$ .

For our illustrative example,

$$\begin{aligned} \hat{\beta}'\mathbf{X}'\mathbf{y} &= [24.3571 \quad 0.5079] \begin{bmatrix} 1,110 \\ 205,500 \end{bmatrix} \\ &= 131,409.831 \\ \mathbf{y}'\mathbf{y} &= 132,100 \end{aligned}$$

and

$$n\bar{Y}^2 = 123,210$$

Plugging these values into (C.4.2), we see that  $R^2 = 0.9224$ , which is about the same as obtained before, save for the rounding errors.

### C.5 THE CORRELATION MATRIX

In the previous chapters we came across the zero-order, or simple, correlation coefficients  $r_{12}, r_{13}, r_{23}$ , and the partial, or first-order, correlations  $r_{12.3}, r_{13.2}, r_{23.1}$ , and their interrelationships. In the  $k$ -variable case, we shall have in all  $k(k-1)/2$  zero-order correlation coefficients. (Why?) These  $k(k-1)/2$  correlations can be put into a matrix, called the **correlation matrix  $\mathbf{R}$**  as follows:

$$\begin{aligned} \mathbf{R} &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & \dots & r_{1k} \\ r_{21} & r_{22} & r_{23} & \dots & r_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ r_{k1} & r_{k2} & r_{k3} & \dots & r_{kk} \end{bmatrix} \\ &= \begin{bmatrix} 1 & r_{12} & r_{13} & \dots & r_{1k} \\ r_{21} & 1 & r_{23} & \dots & r_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ r_{k1} & r_{k2} & r_{k3} & \dots & 1 \end{bmatrix} \end{aligned} \quad (\text{C.5.1})$$

where the subscript 1, as before, denotes the dependent variable  $Y$  ( $r_{12}$  means correlation coefficient between  $Y$  and  $X_2$ , and so on) and where use is made of the fact the coefficient of correlation of a variable with respect to itself is always 1 ( $r_{11} = r_{22} = \dots = r_{kk} = 1$ ).

From the correlation matrix  $\mathbf{R}$  one can obtain correlation coefficients of first order (see Chapter 7) and of higher order such as  $r_{12,34\dots k}$ . (See exercise C.4.) Many computer programs routinely compute the  $\mathbf{R}$  matrix. We have used the correlation matrix in Chapter 10.

### C.6 HYPOTHESIS TESTING ABOUT INDIVIDUAL REGRESSION COEFFICIENTS IN MATRIX NOTATION

For reasons spelled out in the previous chapters, if our objective is inference as well as estimation, we shall have to assume that the disturbances  $u_i$  follow some probability distribution. Also for reasons given previously, in regression analysis we usually assume that each  $u_i$  follows the normal distribution with zero mean and constant variance  $\sigma^2$ . In matrix notation, we have

$$\mathbf{u} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}) \quad (\text{C.6.1})$$

where  $\mathbf{u}$  and  $\mathbf{0}$  are  $n \times 1$  column vectors and  $\mathbf{I}$  is an  $n \times n$  identity matrix,  $\mathbf{0}$  being the **null vector**.

Given the normality assumption, we know that in two- and three-variable linear regression models (1) the OLS estimators  $\hat{\beta}_i$  and the ML estimators  $\tilde{\beta}_i$  are identical, but the ML estimator  $\tilde{\sigma}^2$  is biased, although this bias can be removed by using the unbiased OLS estimator  $\hat{\sigma}^2$ ; and (2) the OLS estimators  $\hat{\beta}_i$  are also normally distributed. Generalizing, in the  $k$ -variable case we can show that

$$\hat{\boldsymbol{\beta}} \sim N[\boldsymbol{\beta}, \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}] \quad (\text{C.6.2})$$

that is, each element of  $\hat{\boldsymbol{\beta}}$  is normally distributed with mean equal to the corresponding element of true  $\boldsymbol{\beta}$  and the variance given by  $\sigma^2$  times the appropriate diagonal element of the inverse matrix  $(\mathbf{X}'\mathbf{X})^{-1}$ .

Since in practice  $\sigma^2$  is unknown, it is estimated by  $\hat{\sigma}^2$ . Then by the usual shift to the  $t$  distribution, it follows that each element of  $\hat{\boldsymbol{\beta}}$  follows the  $t$  distribution with  $n - k$  df. Symbolically,

$$t = \frac{\hat{\beta}_i - \beta_i}{\text{se}(\hat{\beta}_i)} \quad (\text{C.6.3})$$

with  $n - k$  df, where  $\hat{\beta}_i$  is any element of  $\hat{\boldsymbol{\beta}}$ .

The  $t$  distribution can therefore be used to test hypotheses about the true  $\beta_i$  as well as to establish confidence intervals about it. The actual mechanics have already been illustrated in Chapters 5 and 8. For a fully worked example, see Section C.10.

**C.7 TESTING THE OVERALL SIGNIFICANCE OF REGRESSION:  
ANALYSIS OF VARIANCE IN MATRIX NOTATION**

In Chapter 8 we developed the ANOVA technique (1) to test the overall significance of the estimated regression, that is, to test the null hypothesis that the true (partial) slope coefficients are simultaneously equal to zero, and (2) to assess the incremental contribution of an explanatory variable. The ANOVA technique can be easily extended to the  $k$ -variable case. Recall that the ANOVA technique consists of decomposing the TSS into two components: the ESS and the RSS. The matrix expressions for these three sums of squares are already given in (C.3.16), (C.3.17), and (C.3.18), respectively. The degrees of freedom associated with these sums of squares are  $n - 1$ ,  $k - 1$ , and  $n - k$ , respectively. (Why?) Then, following Chapter 8, Table 8.1, we can set up Table C.2.

Assuming that the disturbances  $u_i$  are normally distributed and the null hypothesis is  $\beta_2 = \beta_3 = \dots = \beta_k = 0$ , and following Chapter 8, one can show that

$$F = \frac{(\hat{\beta}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2)/(k - 1)}{(\mathbf{y}'\mathbf{y} - \hat{\beta}'\mathbf{X}'\mathbf{y})/(n - k)} \tag{C.7.1}$$

follows the  $F$  distribution with  $k - 1$  and  $n - k$  df.

In Chapter 8 we saw that, under the assumptions stated previously, there is a close relationship between  $F$  and  $R^2$ , namely,

$$F = \frac{R^2/(k - 1)}{(1 - R^2)/(n - k)} \tag{8.5.11}$$

Therefore, the ANOVA Table C.2 can be expressed as Table C.3. One advantage of Table C.3 over Table C.2 is that the entire analysis can be done in terms of  $R^2$ ; one need not consider the term  $(\mathbf{y}'\mathbf{y} - n\bar{Y}^2)$ , for it drops out in the  $F$  ratio.

**TABLE C.2** MATRIX FORMULATION OF THE ANOVA TABLE FOR  $k$ -VARIABLE LINEAR REGRESSION MODEL

Source of variation	SS	df	MSS
Due to regression (that is, due to $X_2, X_3, \dots, X_k$ )	$\hat{\beta}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2$	$k - 1$	$\frac{\hat{\beta}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2}{k - 1}$
Due to residuals	$\mathbf{y}'\mathbf{y} - \hat{\beta}'\mathbf{X}'\mathbf{y}$	$n - k$	$\frac{\mathbf{y}'\mathbf{y} - \hat{\beta}'\mathbf{X}'\mathbf{y}}{n - k}$
Total	$\mathbf{y}'\mathbf{y} - n\bar{Y}^2$	$n - 1$	



**TABLE C.3**  $k$ -VARIABLE ANOVA TABLE IN MATRIX FORM IN TERMS OF  $R^2$ 

Source of variation	SS	df	MSS
Due to regression (that is, due to $X_2, X_3, \dots, X_k$ )	$R^2(\mathbf{y}'\mathbf{y} - n\bar{Y}^2)$	$k - 1$	$\frac{R^2(\mathbf{y}'\mathbf{y} - n\bar{Y}^2)}{k - 1}$
Due to residuals	$(1 - R^2)(\mathbf{y}'\mathbf{y} - n\bar{Y}^2)$	$n - k$	$\frac{(1 - R^2)(\mathbf{y}'\mathbf{y} - n\bar{Y}^2)}{n - k}$
Total	$\mathbf{y}'\mathbf{y} - n\bar{Y}^2$	$n - 1$	

### C.8 TESTING LINEAR RESTRICTIONS: GENERAL $F$ TESTING USING MATRIX NOTATION

In Section 8.7 we introduced the general  $F$  test to test the validity of linear restrictions imposed on one or more parameters of the  $k$ -variable linear regression model. The appropriate test was given in (8.7.9) [or its equivalent (8.7.10)]. The matrix counterpart of (8.7.9) can be easily derived.

Let

$\hat{\mathbf{u}}_R$  = the residual vector from the restricted least-squares regression

$\hat{\mathbf{u}}_{UR}$  = the residual vector from the unrestricted least-squares regression

Then

$\hat{\mathbf{u}}_R' \hat{\mathbf{u}}_R = \sum \hat{u}_R^2 = \text{RSS from the restricted regression}$

$\hat{\mathbf{u}}_{UR}' \hat{\mathbf{u}}_{UR} = \sum \hat{u}_{UR}^2 = \text{RSS from the unrestricted regression}$

$m$  = number of linear restrictions

$k$  = number of parameters (including the intercept) in the unrestricted regression

$n$  = number of observations

The matrix counterpart of (8.7.9) is then

$$F = \frac{(\hat{\mathbf{u}}_R' \hat{\mathbf{u}}_R - \hat{\mathbf{u}}_{UR}' \hat{\mathbf{u}}_{UR})/m}{(\hat{\mathbf{u}}_{UR}' \hat{\mathbf{u}}_{UR})/(n - k)} \quad (\text{C.8.1})$$

which follows the  $F$  distribution with  $(m, n - k)$  df. As usual, if the computed  $F$  value from (C.8.1) exceeds the critical  $F$  value, we can reject the restricted regression; otherwise, we do not reject it.

### C.9 PREDICTION USING MULTIPLE REGRESSION: MATRIX FORMULATION

In Section 8.9 we discussed, using scalar notation, how the estimated multiple regression can be used for predicting (1) the mean and (2) individual values of  $Y$ , given the values of the  $X$  regressors. In this section we show how to express these predictions in matrix form. We also present the formulas to estimate the variances and standard errors of the predicted values; in Chapter 8 we noted that these formulas are better handled in matrix

notation, for the scalar or algebraic expressions of these formulas become rather unwieldy.

### Mean Prediction

Let

$$\mathbf{X}_0 = \begin{bmatrix} 1 \\ X_{02} \\ X_{03} \\ \vdots \\ X_{0k} \end{bmatrix} \quad (\text{C.9.1})$$

be the vector of values of the  $X$  variables for which we wish to predict  $\hat{Y}_0$ , the mean prediction of  $Y$ .

Now the estimated multiple regression, in scalar form, is

$$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \cdots + \hat{\beta}_k X_{ki} + u_i \quad (\text{C.9.2})$$

which in matrix notation can be written compactly as

$$\hat{Y}_i = \mathbf{x}'_i \hat{\boldsymbol{\beta}} \quad (\text{C.9.3})$$

where  $\mathbf{x}'_i = [1 \ X_{2i} \ X_{3i} \ \cdots \ X_{ki}]$  and

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix}$$

Equation (C.9.2) or (C.9.3) is of course the mean prediction of  $Y_i$  corresponding to given  $\mathbf{x}'_i$ .

If  $\mathbf{x}'_i$  is as given in (C.9.1), (C.9.3) becomes

$$(\hat{Y}_i | \mathbf{x}'_0) = \mathbf{x}'_0 \hat{\boldsymbol{\beta}} \quad (\text{C.9.4})$$

where, of course, the values of  $\mathbf{x}_0$  are specified. Note that (C.9.4) gives an unbiased prediction of  $E(Y_i | \mathbf{x}'_0)$ , since  $E(\mathbf{x}'_0 \hat{\boldsymbol{\beta}}) = \mathbf{x}'_0 \hat{\boldsymbol{\beta}}$ . (Why?)

### Variance of Mean Prediction

The formula to estimate the variance of  $(\hat{Y}_0 | \mathbf{x}'_0)$  is as follows<sup>7</sup>:

$$\text{var}(\hat{Y}_0 | \mathbf{x}'_0) = \sigma^2 \mathbf{x}'_0 (\mathbf{X}'\mathbf{X})^{-1} \mathbf{x}_0 \quad (\text{C.9.5})$$

where  $\sigma^2$  is the variance of  $u_i$ ,  $\mathbf{x}'_0$  are the given values of the  $X$  variables for which we wish to predict, and  $(\mathbf{X}'\mathbf{X})$  is the matrix given in (C.3.9). In practice, we replace  $\sigma^2$  by its unbiased estimator  $\hat{\sigma}^2$ .

We will illustrate mean prediction and its variance in the next section.

<sup>7</sup>For derivation, see J. Johnston, *Econometrics Methods*, McGraw-Hill, 3d ed., New York, 1984, pp. 195–196.

**Individual Prediction**

As pointed out in Chapters 5 and 8, the individual prediction of  $Y (= Y_0)$  is also given by (C.9.3) or more specifically by (C.9.4). The difference between mean and individual predictions lies in their variances.

**Variance of Individual Prediction**

The formula for the variance of an individual prediction is as follows<sup>8</sup>:

$$\text{var}(Y_0 | \mathbf{x}_0) = \sigma^2[1 + \mathbf{x}'_0(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_0] \quad (\text{C.9.6})$$

where  $\text{var}(Y_0 | \mathbf{x}_0)$  stands for  $E[Y_0 - \hat{Y}_0 | X]^2$ . In practice we replace  $\sigma^2$  by its unbiased estimator  $\hat{\sigma}^2$ . We illustrate this formula in the next section.

**C.10 SUMMARY OF THE MATRIX APPROACH:  
AN ILLUSTRATIVE EXAMPLE**

Consider the data given in Table C.4. These data pertain to per capita personal consumption expenditure (PPCE) and per capita personal disposable income (PPDI) and time or the trend variable. By including the trend variable in the model, we are trying to find out the relationship of PPCE to PPDI net of the trend variable (which may represent a host of other factors, such as technology, change in tastes, etc.)

For empirical purposes, therefore, the regression model is

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \hat{u}_i \quad (\text{C.10.1})$$

where  $Y$  = per capita consumption expenditure,  $X_2$  = per capita disposable income, and  $X_3$  = time. The data required to run the regression (C.10.1) are given in Table C.4.

**TABLE C.4** PER CAPITA PERSONAL CONSUMPTION EXPENDITURE (PPCE) AND PER CAPITA PERSONAL DISPOSABLE INCOME (PPDI) IN THE UNITED STATES, 1956–1970, IN 1958 DOLLARS

PPCE, $Y$	PPDI, $X_2$	Time, $X_3$	PPCE, $Y$	PPDI, $X_2$	Time, $X_3$
1673	1839	1 (= 1956)	1948	2126	9
1688	1844	2	2048	2239	10
1666	1831	3	2128	2336	11
1735	1881	4	2165	2404	12
1749	1883	5	2257	2487	13
1756	1910	6	2316	2535	14
1815	1969	7	2324	2595	15 (= 1970)
1867	2016	8			

Source: *Economic Report of the President*, January 1972, Table B-16.

<sup>8</sup>Ibid.

In matrix notation, our problem may be shown as follows:

$$\begin{bmatrix} 1673 \\ 1688 \\ 1666 \\ 1735 \\ 1749 \\ 1756 \\ 1815 \\ 1867 \\ 1948 \\ 2048 \\ 2128 \\ 2165 \\ 2257 \\ 2316 \\ 2324 \end{bmatrix} = \begin{bmatrix} 1 & 1839 & 1 \\ 1 & 1844 & 2 \\ 1 & 1831 & 3 \\ 1 & 1881 & 4 \\ 1 & 1883 & 5 \\ 1 & 1910 & 6 \\ 1 & 1969 & 7 \\ 1 & 2016 & 8 \\ 1 & 2126 & 9 \\ 1 & 2239 & 10 \\ 1 & 2336 & 11 \\ 1 & 2404 & 12 \\ 1 & 2487 & 13 \\ 1 & 2535 & 14 \\ 1 & 2595 & 15 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} + \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \\ \hat{u}_6 \\ \hat{u}_7 \\ \hat{u}_8 \\ \hat{u}_9 \\ \hat{u}_{10} \\ \hat{u}_{11} \\ \hat{u}_{12} \\ \hat{u}_{13} \\ \hat{u}_{14} \\ \hat{u}_{15} \end{bmatrix} \tag{C.10.2}$$

$$\begin{matrix} \mathbf{y} & = & \mathbf{X} & \hat{\boldsymbol{\beta}} & + & \hat{\mathbf{u}} \\ 15 \times 1 & & 15 \times 3 & 3 \times 1 & & 15 \times 1 \end{matrix}$$

From the preceding data we obtain the following quantities:

$$\begin{aligned} \bar{Y} &= 1942.333 & \bar{X}_2 &= 2126.333 & \bar{X}_3 &= 8.0 \\ \sum (Y_i - \bar{Y})^2 &= 830,121.333 \\ \sum (X_{2i} - \bar{X}_2)^2 &= 1,103,111.333 & \sum (X_{3i} - \bar{X}_3)^2 &= 280.0 \end{aligned}$$

$$\begin{aligned} \mathbf{X}'\mathbf{X} &= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ X_{21} & X_{22} & X_{23} & \cdots & X_{2n} \\ X_{31} & X_{32} & X_{33} & \cdots & X_{3n} \end{bmatrix} \begin{bmatrix} 1 & X_{21} & X_{31} \\ 1 & X_{22} & X_{32} \\ 1 & X_{23} & X_{33} \\ \vdots & \vdots & \vdots \\ 1 & X_{2n} & X_{3n} \end{bmatrix} \\ &= \begin{bmatrix} n & \sum X_{2i} & \sum X_{3i} \\ \sum X_{2i} & \sum X_{2i}^2 & \sum X_{2i} X_{3i} \\ \sum X_{3i} & \sum X_{2i} X_{3i} & \sum X_{3i}^2 \end{bmatrix} \\ &= \begin{bmatrix} 15 & 31,895 & 120 \\ 31,895 & 68,922.513 & 272,144 \\ 120 & 272,144 & 1240 \end{bmatrix} \tag{C.10.3} \end{aligned}$$

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} 29,135 \\ 62,905,821 \\ 247,934 \end{bmatrix} \tag{C.10.4}$$

Using the rules of matrix inversion given in **Appendix B**, one can see that

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 37.232491 & -0.0225082 & 1.336707 \\ -0.0225082 & 0.0000137 & -0.0008319 \\ 1.336707 & -0.0008319 & 0.054034 \end{bmatrix} \quad (\text{C.10.5})$$

Therefore,

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \begin{bmatrix} 300.28625 \\ 0.74198 \\ 8.04356 \end{bmatrix} \quad (\text{C.10.6})$$

The residual sum of squares can now be computed as

$$\begin{aligned} \sum \hat{u}_i^2 &= \hat{\mathbf{u}}'\hat{\mathbf{u}} \\ &= \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} \\ &= 57,420,003 - [300.28625 \quad 0.74198 \quad 8.04356] \begin{bmatrix} 29,135 \\ 62,905,821 \\ 247,934 \end{bmatrix} \\ &= 1976.85574 \end{aligned} \quad (\text{C.10.7})$$

whence we obtain

$$\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{12} = 164.73797 \quad (\text{C.10.8})$$

The variance–covariance matrix for  $\hat{\boldsymbol{\beta}}$  can therefore be shown as

$$\text{var-cov}(\hat{\boldsymbol{\beta}}) = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 6133.650 & -3.70794 & 220.20634 \\ -3.70794 & 0.00226 & -0.13705 \\ 220.20634 & -0.13705 & 8.90155 \end{bmatrix} \quad (\text{C.10.9})$$

The diagonal elements of this matrix give the variances of  $\hat{\beta}_1$ ,  $\hat{\beta}_2$ , and  $\hat{\beta}_3$ , respectively, and their positive square roots give the corresponding standard errors.

From the previous data, it can be readily verified that

$$\text{ESS: } \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2 = 828,144.47786 \quad (\text{C.10.10})$$

$$\text{TSS: } \mathbf{y}'\mathbf{y} - n\bar{Y}^2 = 830,121.333 \quad (\text{C.10.11})$$

Therefore,

$$\begin{aligned} R^2 &= \frac{\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2}{\mathbf{y}'\mathbf{y} - n\bar{Y}^2} \\ &= \frac{828,144.47786}{830,121.333} \\ &= 0.99761 \end{aligned} \quad (\text{C.10.12})$$

Applying (7.8.4) the **adjusted coefficient of determination** can be seen to be

$$\bar{R}^2 = 0.99722 \quad (\text{C.10.13})$$

Collecting our results thus far, we have

$$\begin{aligned} \hat{Y}_i &= 300.28625 + 0.74198X_{2i} + 8.04356X_{3i} \\ &\quad (78.31763) \quad (0.04753) \quad (2.98354) \\ t &= (3.83421) \quad (15.60956) \quad (2.69598) \\ R^2 &= 0.99761 \quad \bar{R}^2 = 0.99722 \quad \text{df} = 12 \end{aligned} \quad (\text{C.10.14})$$

The interpretation of (C.10.14) is this: If both  $X_2$  and  $X_3$  are fixed at zero value, the average value of per capita personal consumption expenditure is estimated at about \$300. As usual, this mechanical interpretation of the intercept should be taken with a grain of salt. The partial regression coefficient of 0.74198 means that, holding all other variables constant, an increase in per capita income of, say, a dollar is accompanied by an increase in the mean per capita personal consumption expenditure of about 74 cents. In short, the marginal propensity to consume is estimated to be about 0.74, or 74 percent. Similarly, holding all other variables constant, the mean per capita personal consumption expenditure increased at the rate of about \$8 per year during the period of the study, 1956–1970. The  $R^2$  value of 0.9976 shows that the two explanatory variables accounted for over 99 percent of the variation in per capita consumption expenditure in the United States over the period 1956–1970. Although  $\bar{R}^2$  dips slightly, it is still very high.

Turning to the statistical significance of the estimated coefficients, we see from (C.10.14) that each of the estimated coefficients is *individually* statistically significant at, say, the 5 percent level of significance: The ratios of the estimated coefficients to their standard errors (that is,  $t$  ratios) are 3.83421, 15.61077, and 2.69598, respectively. Using a two-tail  $t$  test at the 5 percent level of significance, we see that the critical  $t$  value for 12 df is 2.179. Each of the computed  $t$  values exceeds this critical value. Hence, individually we may reject the null hypothesis that the true population value of the relevant coefficient is zero.

As noted previously, we cannot apply the usual  $t$  test to test the hypothesis that  $\beta_2 = \beta_3 = 0$  simultaneously because the  $t$ -test procedure assumes that an independent sample is drawn every time the  $t$  test is applied. If the same sample is used to test hypotheses about  $\beta_2$  and  $\beta_3$  simultaneously, it is likely that the estimators  $\hat{\beta}_2$  and  $\hat{\beta}_3$  are correlated, thus violating the assumption underlying the  $t$ -test procedure.<sup>9</sup> As a matter of fact, a look at the variance-covariance matrix of  $\hat{\beta}$  given in (C.10.9) shows that the estimators  $\hat{\beta}_2$  and  $\hat{\beta}_3$  are negatively correlated (the covariance between the two is  $-0.13705$ ). Hence we cannot use the  $t$  test to test the null hypothesis that  $\beta_2 = \beta_3 = 0$ .

<sup>9</sup>See Sec. 8.5 for details.

But recall that a null hypothesis like  $\beta_2 = \beta_3 = 0$ , simultaneously, can be tested by the analysis of variance technique and the attendant  $F$  test, which were introduced in Chapter 8. For our problem, the analysis of variance table is Table C.5. Under the usual assumptions, we obtain

$$F = \frac{414,072.3893}{164.73797} = 2513.52 \quad (\text{C.10.15})$$

which is distributed as the  $F$  distribution with 2 and 12 df. The computed  $F$  value is obviously highly significant; we can reject the null hypothesis that  $\beta_2 = \beta_3 = 0$ , that is, that per capita personal consumption expenditure is not linearly related to per capita disposable income and trend.

In Section C.9 we discussed the mechanics of forecasting, mean as well as individual. Assume that for 1971 the PPDI figure is \$2610 and we wish to forecast the PPCE corresponding to this figure. Then, the mean as well as individual forecast of PPCE for 1971 is the same and is given as

$$\begin{aligned} (\text{PPCE}_{1971} | \text{PPDI}_{1971}, X_3 = 16) &= \mathbf{x}'_{1971} \hat{\boldsymbol{\beta}} \\ &= [1 \quad 2610 \quad 16] \begin{bmatrix} 300.28625 \\ 0.74198 \\ 8.04356 \end{bmatrix} \\ &= 2365.55 \end{aligned} \quad (\text{C.10.16})$$

where use is made of (C.9.3).

The variances of  $\hat{Y}_{1971}$  and  $Y_{1971}$ , as we know from Section C.9, are different and are as follows:

$$\begin{aligned} \text{var}(\hat{Y}_{1971} | \mathbf{x}'_{1971}) &= \hat{\sigma}^2 [\mathbf{x}'_{1971} (\mathbf{X}'\mathbf{X})^{-1} \mathbf{x}_{1971}] \\ &= 164.73797 [1 \quad 2610 \quad 16] (\mathbf{X}'\mathbf{X})^{-1} \begin{bmatrix} 1 \\ 2610 \\ 16 \end{bmatrix} \end{aligned} \quad (\text{C.10.17})$$

where  $(\mathbf{X}'\mathbf{X})^{-1}$  is as shown in (C.10.5). Substituting this into (C.10.17), the reader should verify that

$$\text{var}(\hat{Y}_{1971} | \mathbf{x}'_{1971}) = 48.6426 \quad (\text{C.10.18})$$

**TABLE C.5** THE ANOVA TABLE FOR THE DATA OF TABLE C.4

Source of variation	SS	df	MSS
Due to $X_2, X_3$	828,144.47786	2	414,072.3893
Due to residuals	1,976.85574	12	164.73797
Total	830,121.33360	14	

and therefore

$$\text{se}(\hat{Y}_{1971} | \mathbf{x}'_{1971}) = 6.9744$$

We leave it to the reader to verify, using (C.9.6), that

$$\text{var}(Y_{1971} | \mathbf{x}'_{1971}) = 213.3806 \quad (\text{C.10.19})$$

and

$$\text{se}(Y_{1971} | \mathbf{x}'_{1971}) = 14.6076$$

*Note:*  $\text{var}(Y_{1971} | \mathbf{x}'_{1971}) = E[Y_{1971} - \hat{Y}_{1971} | \mathbf{x}'_{1971}]^2$ .

In Section C.5 we introduced the correlation matrix  $\mathbf{R}$ . For our data, the correlation matrix is as follows:

$$R = \begin{array}{c} Y \\ X_2 \\ X_3 \end{array} \begin{array}{ccc} Y & X_2 & X_3 \\ \left[ \begin{array}{ccc} 1 & 0.9980 & 0.9743 \\ 0.9980 & 1 & 0.9664 \\ 0.9743 & 0.9664 & 1 \end{array} \right] \end{array} \quad (\text{C.10.20})$$

Note that in (C.10.20) we have bordered the correlation matrix by the variables of the model so that we can readily identify which variables are involved in the computation of the correlation coefficient. Thus, the coefficient 0.9980 in the first row of matrix (C.10.20) tells us that it is the correlation coefficient between  $Y$  and  $X_2$  (that is,  $r_{12}$ ). From the zero-order correlations given in the correlation matrix (C.10.20) one can easily derive the first-order correlation coefficients. (See exercise C.7.)

### C.11 GENERALIZED LEAST SQUARES (GLS)

On several occasions we have mentioned that OLS is a special case of GLS. To see this, return to Eq. (C.2.2). To take into account heteroscedastic variances [the elements on the main diagonal of (C.2.2)] and autocorrelations in the error terms [the elements off the main diagonal of (C.2.2)], assume that

$$E(\mathbf{u}\mathbf{u}') = \sigma^2\mathbf{V} \quad (\text{C.11.1})$$

where  $\mathbf{V}$  is a known  $n \times n$  matrix.

Therefore, if our model is:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

where  $E(\mathbf{u}) = \mathbf{0}$  and  $\text{var-cov}(\mathbf{u}) = \sigma^2\mathbf{V}$ . In case  $\sigma^2$  is unknown, which is typically the case,  $\mathbf{V}$  then represents the assumed structure of variances and covariances among the random errors  $u_t$ .



Under the stated condition of the variance–covariance of the error terms, it can be shown that

$$\beta^{\text{gls}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y} \quad (\text{C.11.2})$$

$\beta^{\text{gls}}$  is known as the **generalized least-squares (GLS) estimator** of  $\beta$ .

It can also be shown that

$$\text{var-cov}(\beta^{\text{gls}}) = \sigma^2(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1} \quad (\text{C.11.3})$$

It can be proved that  $\beta^{\text{gls}}$  is the best linear unbiased estimator of  $\beta$ .

If it is assumed that the variance of each error term is the same constant  $\sigma^2$  and the error terms are mutually uncorrelated, then the  $\mathbf{V}$  matrix reduces to the identity matrix, as shown in (C.2.3). If the error terms are mutually uncorrelated but they have different (i.e., heteroscedastic) variances, then the  $\mathbf{V}$  matrix will be diagonal with the unequal variances along the main diagonal. Of course, if there is heteroscedasticity as well as autocorrelation, then the  $\mathbf{V}$  matrix will have entries on the main diagonal as well as on the off diagonal.

The real problem in practice is that we do not know  $\sigma^2$  as well as the true variances and covariances (i.e., the structure of the  $\mathbf{V}$  matrix). As a solution, we can use the method of **estimated (or feasible) generalized least squares (EGLS)**. Here we first estimate our model by OLS disregarding the problems of heteroscedasticity and/or autocorrelation. We obtain the residuals from this model and form the (estimated) variance–covariance matrix of the error term by replacing the entries in the expression just before (C.2.2) by the estimated  $u$ , namely,  $\hat{u}$ . It can be shown that EGLS estimators are consistent estimators of GLS. Symbolically,

$$\beta^{\text{egls}} = (\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{y} \quad (\text{C.11.4})$$

$$\text{var-cov}(\beta^{\text{egls}}) = \sigma^2(\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{X})^{-1} \quad (\text{C.11.5})$$

where  $\hat{\mathbf{V}}$  is an estimate of  $\mathbf{V}$ .

## C.12 SUMMARY AND CONCLUSIONS

The primary purpose of this appendix was to introduce the matrix approach to classical linear regression model. Although very few new concepts of regression analysis were introduced, the matrix notation provides a compact method of dealing with linear regression models involving any number of variables.

In concluding this appendix, note that if the  $Y$  and  $X$  variables are measured in the deviation form, that is, as deviations from their sample means, there are a few changes in the formulas presented previously. These changes

**TABLE C.6** *k*-VARIABLE REGRESSION MODEL IN ORIGINAL UNITS AND IN THE DEVIATION FORM\*

Original units		Deviation form
$\mathbf{y} = \mathbf{X}\hat{\boldsymbol{\beta}} + \hat{\mathbf{u}}$	(C.3.2)	$\mathbf{y} = \mathbf{X}\hat{\boldsymbol{\beta}} + \hat{\mathbf{u}}$ The column of 1's in the $\mathbf{X}$ matrix drops out. (Why?)
$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$	(C.3.11)	Same
$\text{var-cov}(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$	(C.3.13)	Same
$\hat{\mathbf{u}}'\hat{\mathbf{u}} = \mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y}$	(C.3.18)	Same
$\sum y_i^2 = \mathbf{y}'\mathbf{y} - n\bar{Y}^2$	(C.3.16)	$\sum y_i^2 = \mathbf{y}'\mathbf{y}$ <span style="float:right">(C.12.1)</span>
$\text{ESS} = \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2$	(C.3.17)	$\text{ESS} = \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y}$ <span style="float:right">(C.12.2)</span>
$R^2 = \frac{\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} - n\bar{Y}^2}{\mathbf{y}'\mathbf{y} - n\bar{Y}^2}$	(C.4.2)	$R^2 = \frac{\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y}}{\mathbf{y}'\mathbf{y}}$ <span style="float:right">(C.12.3)</span>

\*Note that although in both cases the symbols for the matrices and vectors are the same, in the deviation form the elements of the matrices and vectors are assumed to be deviations rather than the raw data. Note also that in the deviation form  $\hat{\boldsymbol{\beta}}$  is of order  $k - 1$  and the var-cov ( $\hat{\boldsymbol{\beta}}$ ) is of order  $(k - 1)(k - 1)$ .

are listed in Table C.6.<sup>10</sup> As this table shows, in the deviation form the correction for mean  $n\bar{Y}^2$  drops out from the TSS and ESS. (Why?) This loss results in a change for the formula for  $R^2$ . Otherwise, most of the formulas developed in the original units of measurement hold true for the deviation form.

### EXERCISES

**C.1.** For the illustrative example discussed in Section C.10 the  $\mathbf{X}'\mathbf{X}$  and  $\mathbf{X}'\mathbf{y}$  using the data in the deviation form are as follows:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 1,103,111.333 & 16,984 \\ 16,984 & 280 \end{bmatrix}$$

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} 955,099.333 \\ 14,854.000 \end{bmatrix}$$

- Estimate  $\beta_2$  and  $\beta_3$ .
- How would you estimate  $\beta_1$ ?
- Obtain the variance of  $\hat{\beta}_2$  and  $\hat{\beta}_3$  and their covariances.
- Obtain  $R^2$  and  $\bar{R}^2$ .
- Comparing your results with those given in Section C.10, what do you find are the advantages of the deviation form?

**C.2.** Refer to exercise 22.23. Using the data given therein, set up the appropriate  $(\mathbf{X}'\mathbf{X})$  matrix and the  $\mathbf{X}'\mathbf{y}$  vector and estimate the parameter vector  $\boldsymbol{\beta}$  and its variance-covariance matrix. Also obtain  $R^2$ . How would you test

<sup>10</sup>In these days of high-speed computers there may not be need for the deviation form. But it simplifies formulas and therefore calculations if one is working with a desk calculator and dealing with large numbers.

the hypothesis that the elasticities of M1 with respect to GDP and interest rate  $R$  are numerically the same?

**C.3. Testing the equality of two regression coefficients.** Suppose that you are given the following regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$$

and you want to test the hypothesis that  $\beta_2 = \beta_3$ . If we assume that the  $u_i$  are normally distributed, it can be shown that

$$t = \frac{\hat{\beta}_2 - \hat{\beta}_3}{\sqrt{\text{var}(\hat{\beta}_2) + \text{var}(\hat{\beta}_3) - 2 \text{cov}(\hat{\beta}_2, \hat{\beta}_3)}}$$

follows the  $t$  distribution with  $n - 3$  df (see Section 8.6). (In general, for the  $k$ -variable case the df are  $n - k$ .) Therefore, the preceding  $t$  test can be used to test the null hypothesis  $\beta_2 = \beta_3$ .

Apply the preceding  $t$  test to test the hypothesis that the true values of  $\beta_2$  and  $\beta_3$  in the regression (C.10.14) are identical.

*Hint:* Use the var-cov matrix of  $\beta$  given in (C.10.9).

**C.4. Expressing higher-order correlations in terms of lower-order correlations.** Correlation coefficients of order  $p$  can be expressed in terms of correlation coefficients of order  $p - 1$  by the following **reduction formula**:

$$r_{12.345\dots p} = \frac{r_{12.345\dots(p-1)} - [r_{1p.345\dots(p-1)}r_{2p.345\dots(p-1)}]}{\sqrt{[1 - r_{1p.345\dots(p-1)}^2]}\sqrt{[1 - r_{2p.345\dots(p-1)}^2]}}$$

Thus,

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{1 - r_{13}^2}\sqrt{1 - r_{23}^2}}$$

as found in Chapter 7.

You are given the following correlation matrix:

$$\mathbf{R} = \begin{matrix} & \begin{matrix} Y & X_2 & X_3 & X_4 & X_5 \end{matrix} \\ \begin{matrix} Y \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{matrix} & \begin{bmatrix} 1 & 0.44 & -0.34 & -0.31 & -0.14 \\ & 1 & 0.25 & -0.19 & -0.35 \\ & & 1 & 0.44 & 0.33 \\ & & & 1 & 0.85 \\ & & & & 1 \end{bmatrix} \end{matrix}$$

Find the following:

a.  $r_{12.345}$

b.  $r_{12.34}$

c.  $r_{12.3}$

d.  $r_{13.245}$

e.  $r_{13.24}$

f.  $r_{13.2}$

**C.5. Expressing higher-order regression coefficients in terms of lower-order regression coefficients.** A regression coefficient of order  $p$  can

be expressed in terms of a regression coefficient of order  $p - 1$  by the following reduction formula:

$$\hat{\beta}_{12.345\dots p} = \frac{\hat{\beta}_{12.345\dots(p-1)} - [\hat{\beta}_{1p.345\dots(p-1)}\hat{\beta}_{p2.345\dots(p-1)}]}{1 - \hat{\beta}_{2p.345\dots(p-1)}\hat{\beta}_{p2.345\dots(p-1)}}$$

Thus,

$$\hat{\beta}_{12.3} = \frac{\hat{\beta}_{12} - \hat{\beta}_{13}\hat{\beta}_{32}}{1 - \hat{\beta}_{23}\hat{\beta}_{32}}$$

where  $\beta_{12.3}$  is the slope coefficient in the regression of  $y$  on  $X_2$  holding  $X_3$  constant. Similarly,  $\beta_{12.34}$  is the slope coefficient in the regression of  $Y$  on  $X_2$  holding  $X_3$  and  $X_4$  constant, and so on.

Using the preceding formula, find expressions for the following regression coefficients in terms of lower-order regression coefficients:  $\hat{\beta}_{12.3456}$ ,  $\hat{\beta}_{12.345}$ , and  $\hat{\beta}_{12.34}$ .

**C.6.** Establish the following identity:

$$\hat{\beta}_{12.3}\hat{\beta}_{23.1}\hat{\beta}_{31.2} = r_{12.3}r_{23.1}r_{31.2}$$

**C.7.** For the correlation matrix  $\mathbf{R}$  given in (C.10.20) find all the first-order partial correlation coefficients.

**C.8.** In studying the variation in crime rates in certain large cities in the United States, Ogburn obtained the following data\*:

$\bar{Y} = 19.9$	$S_1 = 7.9$	$Y$	$\left[ \begin{array}{ccccc} 1 & 0.44 & -0.34 & -0.31 & -0.14 \\ & 1 & 0.25 & -0.19 & -0.35 \\ & & 1 & 0.44 & 0.33 \\ & & & 1 & 0.85 \\ & & & & 1 \end{array} \right]$
$\bar{X}_2 = 49.2$	$S_2 = 1.3$	$X_2$	
$\bar{X}_3 = 10.2$	$S_3 = 4.6$	$X_3$	
$\bar{X}_4 = 481.4$	$S_4 = 74.4$	$X_4$	
$\bar{X}_5 = 41.6$	$S_5 = 10.8$	$X_5$	

where  $Y$  = crime rate, number of known offenses per thousand of population

$X_2$  = percentage of male inhabitants

$X_3$  = percentage of total inhabitants who are foreign-born males

$X_4$  = number of children under 5 years of age per thousand married women between ages 15 and 44 years

$X_5$  = church membership, number of church members 13 years of age and over per 100 of total population 13 years of age and over;  $S_1$  to  $S_5$  are the sample standard deviations of variables  $Y$  through  $X_5$  and  $\mathbf{R}$  is the correlation matrix

**a.** Treating  $Y$  as the dependent variable, obtain the regression of  $Y$  on the four  $X$  variables and interpret the estimated regression.

\*W. F. Ogburn, "Factors in the Variation of Crime among Cities," *Journal of American Statistical Association*, vol. 30, 1935, p. 12.

- b. Obtain  $r_{12,3}$ ,  $r_{14,35}$ , and  $r_{15,34}$ .
- c. Obtain  $R^2$  and test the hypothesis that all partial slope coefficients are simultaneously equal to zero.
- C.9.** The following table gives data on output and total cost of production of a commodity in the short run. (See Example 7.4.)

Output	Total cost, \$
1	193
2	226
3	240
4	244
5	257
6	260
7	274
8	297
9	350
10	420

To test whether the preceding data suggest the U-shaped average and marginal cost curves typically encountered in the short run, one can use the following model:

$$Y_i = \beta_1 + \beta_2 X_i + \beta_3 X_i^2 + \beta_4 X_i^3 + u_i$$

where  $Y$  = total cost and  $X$  = output. The additional explanatory variables  $X_i^2$  and  $X_i^3$  are derived from  $X$ .

- a. Express the data in the deviation form and obtain  $(\mathbf{X}'\mathbf{X})$ ,  $(\mathbf{X}'\mathbf{y})$ , and  $(\mathbf{X}'\mathbf{X})^{-1}$ .
- b. Estimate  $\beta_2$ ,  $\beta_3$ , and  $\beta_4$ .
- c. Estimate the var-cov matrix of  $\hat{\beta}$ .
- d. Estimate  $\beta_1$ . Interpret  $\hat{\beta}_1$  in the context of the problem.
- e. Obtain  $R^2$  and  $\bar{R}^2$ .
- f. A priori, what are the signs of  $\beta_2$ ,  $\beta_3$ , and  $\beta_4$ ? Why?
- g. From the total cost function given previously obtain expressions for the marginal and average cost functions.
- h. Fit the average and marginal cost functions to the data and comment on the fit.
- i. If  $\beta_3 = \beta_4 = 0$ , what is the nature of the marginal cost function? How would you test the hypothesis that  $\beta_3 = \beta_4 = 0$ ?
- j. How would you derive the total variable cost and average variable cost functions from the given data?
- C.10.** In order to study the labor force participation of urban poor families (families earning less than \$3943 in 1969), the data in Table C.7 were obtained from the 1970 Census of Population.
- a. Using the regression model  $Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + u_i$ , obtain the estimates of the regression coefficients and interpret your results.
- b. A priori, what are the expected signs of the regression coefficients in the preceding model and why?

**TABLE C.7** LABOR FORCE PARTICIPATION EXPERIENCE OF THE URBAN POOR: CENSUS TRACTS, NEW YORK CITY, 1970

Tract no.	% in labor force, $Y^*$	Mean family income, $X_2^\dagger$	Mean family size, $X_3$	Unemployment rate, $X_4^\ddagger$
137	64.3	1,998	2.95	4.4
139	45.4	1,114	3.40	3.4
141	26.6	1,942	3.72	1.1
142	87.5	1,998	4.43	3.1
143	71.3	2,026	3.82	7.7
145	82.4	1,853	3.90	5.0
147	26.3	1,666	3.32	6.2
149	61.6	1,434	3.80	5.4
151	52.9	1,513	3.49	12.2
153	64.7	2,008	3.85	4.8
155	64.9	1,704	4.69	2.9
157	70.5	1,525	3.89	4.8
159	87.2	1,842	3.53	3.9
161	81.2	1,735	4.96	7.2
163	67.9	1,639	3.68	3.6

\* $Y$  = family heads under 65 years old

$^\dagger X_2$  = dollars

$^\ddagger X_4$  = percent of civilian labor force unemployed

Source: Census Tracts: New York, Bureau of the Census, U.S. Department of Commerce, 1970.

- c. How would you test the hypothesis that the overall unemployment rate has no effect on the labor force participation of the urban poor in the census tracts given in the accompanying table?
- d. Should any variables be dropped from the preceding model? Why?
- e. What other variables would you consider for inclusion in the model?
- C.11.** In an application of the Cobb–Douglas production function the following results were obtained:

$$\widehat{\ln Y}_i = 2.3542 + 0.9576 \ln X_{2i} + 0.8242 \ln X_{3i}$$

$$(0.3022) \quad (0.3571)$$

$$R^2 = 0.8432 \quad df = 12$$

where  $Y$  = output,  $X_2$  = labor input, and  $X_3$  = capital input, and where the figures in parentheses are the estimated standard errors.

- a. As noted in Chapter 7, the coefficients of the labor and capital inputs in the preceding equation give the elasticities of output with respect to labor and capital. Test the hypothesis that these elasticities are *individually* equal to unity.
- b. Test the hypothesis that the labor and capital elasticities are equal, assuming (i) the covariance between the estimated labor and capital coefficients is zero, and (ii) it is  $-0.0972$ .
- c. How would you test the overall significance of the preceding regression equation?

\***C.12.** Express the likelihood function for the  $k$ -variable regression model in matrix notation and show that  $\hat{\boldsymbol{\beta}}$ , the vector of maximum likelihood estimators, is identical to  $\hat{\boldsymbol{\beta}}$ , the vector of OLS estimators of the  $k$ -variable regression model.

**C.13. Regression using standardized variables.** Consider the following sample regression functions (SRFs):

$$Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_{2i} + \hat{\beta}_3 X_{3i} + \hat{u}_i \quad (1)$$

$$Y_i^* = b_1 + b_2 X_{2i}^* + b_3 X_{3i}^* + \hat{u}_i^* \quad (2)$$

where

$$Y_i^* = \frac{Y_i - \bar{Y}}{s_Y}$$

$$X_{2i}^* = \frac{X_{2i} - \bar{X}_2}{s_2}$$

$$X_{3i}^* = \frac{X_{3i} - \bar{X}_3}{s_3}$$

where the  $s$ 's denote the sample standard deviations. As noted in Chapter 6, Section 6.3, the starred variables above are known as the *standardized variables*. These variables have zero means and unit (= 1) standard deviations. Expressing all the variables in the deviation form, show the following for model (2):

**a.**  $\mathbf{X}'\mathbf{X} = \begin{bmatrix} 1 & r_{23} \\ r_{23} & 1 \end{bmatrix} n$

**b.**  $\mathbf{X}'\mathbf{y} = \begin{bmatrix} r_{12} \\ r_{13} \end{bmatrix} n$

**c.**  $\mathbf{X}'\mathbf{X}^{-1} = \frac{1}{n(1-r_{23}^2)} \begin{bmatrix} 1 & -r_{23} \\ -r_{23} & 1 \end{bmatrix}$

**d.**  $\hat{\boldsymbol{\beta}} = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \frac{1}{1-r_{23}^2} \begin{bmatrix} r_{12} - r_{23}r_{13} \\ r_{13} - r_{23}r_{12} \end{bmatrix}$

**e.**  $b_1 = 0$

Also establish the relationship between the  $b$ 's and the  $\hat{\beta}$ 's.

(Note that in the preceding relations  $n$  denotes the sample size;  $r_{12}$ ,  $r_{13}$ , and  $r_{23}$  denote the correlations between  $Y$  and  $X_2$ , between  $Y$  and  $X_3$ , and between  $X_2$  and  $X_3$ , respectively.)

**C.14.** Verify the Eqs. (C.10.18) and (C.10.19).

\***C.15. Constrained least-squares.** Assume

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \quad (1)$$

which we want to estimate subject to a set of equality restrictions or constraints:

$$\mathbf{R}\boldsymbol{\beta} = \mathbf{r} \quad (2)$$

---

\*Optional

where  $\mathbf{R}$  is a *known* matrix of order  $q \times k$  ( $q \leq k$ ) and  $\mathbf{r}$  is a *known* vector of  $q$  elements. To illustrate, suppose our model is

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \beta_5 X_{5i} + u_i \quad (3)$$

and suppose we want to estimate this model subject to these restrictions:

$$\begin{aligned} \beta_2 - \beta_3 &= 0 \\ \beta_4 + \beta_5 &= 1 \end{aligned} \quad (4)$$

We can use some of the techniques discussed in Chapter 8 to incorporate these restrictions (e.g.,  $\beta_2 = \beta_3$  and  $\beta_4 = 1 - \beta_5$ , thus removing  $\beta_2$  and  $\beta_4$  from the model) and test for the validity of these restrictions by the  $F$  test discussed there. But a more direct way of estimating (3) incorporating the restrictions (4) directly in the estimating procedure is first to express the restrictions in the form of Eq. (2), which in the present case becomes

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{r} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (5)$$

Letting  $\hat{\boldsymbol{\beta}}^*$  denote the restricted least-squares or constrained least-squares estimator, one can show that  $\hat{\boldsymbol{\beta}}^*$  can be estimated by the following formula\*:

$$\hat{\boldsymbol{\beta}}^* = \hat{\boldsymbol{\beta}} + (\mathbf{X}'\mathbf{X})^{-1} \mathbf{R}'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1} \mathbf{R}]^{-1} (\mathbf{r} - \mathbf{R}\hat{\boldsymbol{\beta}}) \quad (6)$$

where  $\hat{\boldsymbol{\beta}}$  is the usual (unconstrained) estimator estimated from the usual formula  $(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y}$ .

- What is the  $\beta$  vector in (3)?
- Given this  $\beta$  vector, verify that the  $\mathbf{R}$  matrix and  $\mathbf{r}$  vector given in (5) do in fact incorporate the restrictions in (4).
- Write down the  $\mathbf{R}$  and  $\mathbf{r}$  in the following cases:
  - $\beta_2 = \beta_3 = \beta_4 = 2$
  - $\beta_2 = \beta_3$  and  $\beta_4 = \beta_5$
  - $\beta_2 - 3\beta_3 = 5\beta_4$
  - $\beta_2 + 3\beta_3 = 0$
- When will  $\hat{\boldsymbol{\beta}}^* = \hat{\boldsymbol{\beta}}$ ?

## APPENDIX CA

### CA.1 DERIVATION OF $k$ NORMAL OR SIMULTANEOUS EQUATIONS

Differentiating

$$\sum \hat{u}_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \cdots - \hat{\beta}_k X_{ki})^2$$

\*See J. Johnston, *op. cit.*, p. 205.



partially with respect to  $\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_k$ , we obtain

$$\begin{aligned} \frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_1} &= 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \dots - \hat{\beta}_k X_{ki})(-1) \\ \frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_2} &= 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{2i} - \dots - \hat{\beta}_k X_{ki})(-X_{2i}) \\ &\dots \dots \dots \\ \frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_k} &= 2 \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_{ki} - \dots - \hat{\beta}_k X_{ki})(-X_{ki}) \end{aligned}$$

Setting the preceding partial derivatives equal to zero and rearranging the terms, we obtain the  $k$  normal equations given in (C.3.8).

**CA.2 MATRIX DERIVATION OF NORMAL EQUATIONS**

From (C.3.7) we obtain

$$\hat{\mathbf{u}}' \hat{\mathbf{u}} = \mathbf{y}' \mathbf{y} - 2 \hat{\boldsymbol{\beta}}' \mathbf{X}' \mathbf{y} + \hat{\boldsymbol{\beta}}' \mathbf{X}' \mathbf{X} \hat{\boldsymbol{\beta}}$$

Using rules of matrix differentiation given in **Appendix B**, we obtain

$$\frac{\partial(\hat{\mathbf{u}}' \hat{\mathbf{u}})}{\partial \hat{\boldsymbol{\beta}}} = -2 \mathbf{X}' \mathbf{y} + 2 \mathbf{X}' \mathbf{X} \hat{\boldsymbol{\beta}}$$

Setting the preceding equation to zero gives

$$(\mathbf{X}' \mathbf{X}) \hat{\boldsymbol{\beta}} = \mathbf{X}' \mathbf{y}$$

whence  $\hat{\boldsymbol{\beta}} = (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{y}$ , provided the inverse exists.

**CA.3 VARIANCE-COVARIANCE MATRIX OF  $\hat{\boldsymbol{\beta}}$**

From (C.3.11) we obtain

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{y}$$

Substituting  $\mathbf{y} = \mathbf{X} \boldsymbol{\beta} + \mathbf{u}$  into the preceding expression gives

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' (\mathbf{X} \boldsymbol{\beta} + \mathbf{u}) \\ &= (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{X} \boldsymbol{\beta} + (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{u} \\ &= \boldsymbol{\beta} + (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{u} \end{aligned} \tag{1}$$

Therefore,

$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{u} \tag{2}$$

By definition

$$\begin{aligned}\text{var-cov}(\hat{\beta}) &= E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] \\ &= E\{[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}][(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}']\} \\ &= E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\mathbf{u}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]\end{aligned}\quad (3)$$

where in the last step use is made of the fact that  $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$ .

Noting that the  $X$ 's are nonstochastic, on taking expectation of (3) we obtain

$$\begin{aligned}\text{var-cov}(\hat{\beta}) &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E(\mathbf{u}\mathbf{u}')\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\end{aligned}$$

which is the result given in (C.3.13). Note that in deriving the preceding result use is made of the assumption that  $E(\mathbf{u}\mathbf{u}') = \sigma^2\mathbf{I}$ .

#### CA.4 BLUE PROPERTY OF OLS ESTIMATORS

From (C.3.11) we have

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \quad (1)$$

Since  $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$  is a matrix of fixed numbers,  $\hat{\beta}$  is a linear function of  $Y$ . Hence, by definition it is a linear estimator.

Recall that the PRF is

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{u} \quad (2)$$

Substituting this into (1), we obtain

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + \mathbf{u}) \quad (3)$$

$$= \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \quad (4)$$

since  $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X} = \mathbf{I}$ .

Taking expectation of (4) gives

$$\begin{aligned}E(\hat{\beta}) &= E(\beta) + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E(\mathbf{u}) \\ &= \beta\end{aligned}\quad (5)$$

since  $E(\beta) = \beta$  (why?) and  $E(\mathbf{u}) = \mathbf{0}$  by assumption, which shows that  $\hat{\beta}$  is an unbiased estimator of  $\beta$ .

Let  $\hat{\beta}^*$  be any other linear estimator of  $\beta$ , which can be written as

$$\beta^* = [(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' + \mathbf{C}]\mathbf{y} \quad (6)$$

where  $\mathbf{C}$  is a matrix of constants.

Substituting for  $\mathbf{y}$  from (2) into (6), we get

$$\begin{aligned} \hat{\beta}^* &= [(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' + \mathbf{C}](\mathbf{X}\beta + \mathbf{u}) \\ &= \beta + \mathbf{C}\mathbf{X}\beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} + \mathbf{C}\mathbf{u} \end{aligned} \quad (7)$$

Now if  $\hat{\beta}^*$  is to be an unbiased estimator of  $\beta$ , we must have

$$\mathbf{C}\mathbf{X} = \mathbf{0} \quad (\text{Why?}) \quad (8)$$

Using (8), (7) can be written as

$$\hat{\beta}^* - \beta = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} + \mathbf{C}\mathbf{u} \quad (9)$$

By definition, the var-cov ( $\hat{\beta}^*$ ) is

$$E(\hat{\beta}^* - \beta)(\hat{\beta}^* - \beta)' = E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} + \mathbf{C}\mathbf{u}][(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} + \mathbf{C}\mathbf{u}]' \quad (10)$$

Making use of the properties of matrix inversion and transposition and after algebraic simplification, we obtain

$$\begin{aligned} \text{var-cov}(\hat{\beta}^*) &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1} + \sigma^2\mathbf{C}\mathbf{C}' \\ &= \text{var-cov}(\hat{\beta}) + \sigma^2\mathbf{C}\mathbf{C}' \end{aligned} \quad (11)$$

which shows that the variance–covariance matrix of the alternative unbiased linear estimator  $\hat{\beta}^*$  is equal to the variance–covariance matrix of the OLS estimator  $\hat{\beta}$  plus  $\sigma^2$  times  $\mathbf{C}\mathbf{C}'$ , which is a positive semidefinite\* matrix. Hence the variances of a given element of  $\hat{\beta}^*$  must necessarily be equal to or greater than the corresponding element of  $\hat{\beta}$ , which shows that  $\hat{\beta}$  is BLUE. Of course, if  $\mathbf{C}$  is a null matrix, i.e.,  $\mathbf{C} = \mathbf{0}$ , then  $\hat{\beta}^* = \hat{\beta}$ , which is another way of saying that if we have found a BLUE estimator, it must be the least-squares estimator  $\hat{\beta}$ .

\*See references in **App. B**.

APPENDIX **D**

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## STATISTICAL TABLES

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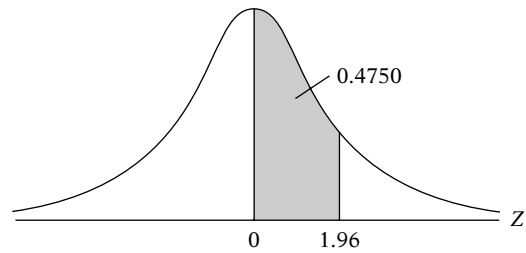
Table D.1	Areas under the standardized normal distribution
Table D.2	Percentage points of the $t$ distribution
Table D.3	Upper percentage points of the $F$ distribution
Table D.4	Upper percentage points of the $\chi^2$ distribution
Table D.5	Durbin–Watson $d$ statistic: Significance points of $d_L$ and $d_U$ at 0.05 and 0.01 levels of significance
Table D.6	Critical values of runs in the runs test
Table D.7	1% and 5% critical Dickey–Fuller $t (= \tau)$ and $F$ values for unit root tests

**TABLE D.1** AREAS UNDER THE STANDARDIZED NORMAL DISTRIBUTION

**Example**

$$\Pr(0 \leq Z \leq 1.96) = 0.4750$$

$$\Pr(Z \geq 1.96) = 0.5 - 0.4750 = 0.025$$



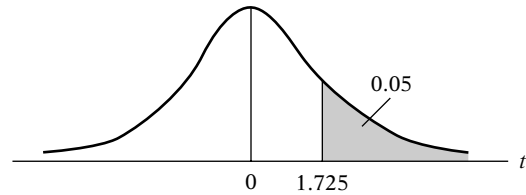
Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4454	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

*Note:* This table gives the area in the right-hand tail of the distribution (i.e.,  $Z \geq 0$ ). But since the normal distribution is symmetrical about  $Z = 0$ , the area in the left-hand tail is the same as the area in the corresponding right-hand tail. For example,  $P(-1.96 \leq Z \leq 0) = 0.4750$ . Therefore,  $P(-1.96 \leq Z \leq 1.96) = 2(0.4750) = 0.95$ .

**TABLE D.2** PERCENTAGE POINTS OF THE *t* DISTRIBUTION

**Example**

$\Pr(t > 2.086) = 0.025$   
 $\Pr(t > 1.725) = 0.05$  for  $df = 20$   
 $\Pr(|t| > 1.725) = 0.10$



df	Pr	0.25	0.10	0.05	0.025	0.01	0.005	0.001
		0.50	0.20	0.10	0.05	0.02	0.010	0.002
1		1.000	3.078	6.314	12.706	31.821	63.657	318.31
2		0.816	1.886	2.920	4.303	6.965	9.925	22.327
3		0.765	1.638	2.353	3.182	4.541	5.841	10.214
4		0.741	1.533	2.132	2.776	3.747	4.604	7.173
5		0.727	1.476	2.015	2.571	3.365	4.032	5.893
6		0.718	1.440	1.943	2.447	3.143	3.707	5.208
7		0.711	1.415	1.895	2.365	2.998	3.499	4.785
8		0.706	1.397	1.860	2.306	2.896	3.355	4.501
9		0.703	1.383	1.833	2.262	2.821	3.250	4.297
10		0.700	1.372	1.812	2.228	2.764	3.169	4.144
11		0.697	1.363	1.796	2.201	2.718	3.106	4.025
12		0.695	1.356	1.782	2.179	2.681	3.055	3.930
13		0.694	1.350	1.771	2.160	2.650	3.012	3.852
14		0.692	1.345	1.761	2.145	2.624	2.977	3.787
15		0.691	1.341	1.753	2.131	2.602	2.947	3.733
16		0.690	1.337	1.746	2.120	2.583	2.921	3.686
17		0.689	1.333	1.740	2.110	2.567	2.898	3.646
18		0.688	1.330	1.734	2.101	2.552	2.878	3.610
19		0.688	1.328	1.729	2.093	2.539	2.861	3.579
20		0.687	1.325	1.725	2.086	2.528	2.845	3.552
21		0.686	1.323	1.721	2.080	2.518	2.831	3.527
22		0.686	1.321	1.717	2.074	2.508	2.819	3.505
23		0.685	1.319	1.714	2.069	2.500	2.807	3.485
24		0.685	1.318	1.711	2.064	2.492	2.797	3.467
25		0.684	1.316	1.708	2.060	2.485	2.787	3.450
26		0.684	1.315	1.706	2.056	2.479	2.779	3.435
27		0.684	1.314	1.703	2.052	2.473	2.771	3.421
28		0.683	1.313	1.701	2.048	2.467	2.763	3.408
29		0.683	1.311	1.699	2.045	2.462	2.756	3.396
30		0.683	1.310	1.697	2.042	2.457	2.750	3.385
40		0.681	1.303	1.684	2.021	2.423	2.704	3.307
60		0.679	1.296	1.671	2.000	2.390	2.660	3.232
120		0.677	1.289	1.658	1.980	2.358	2.617	3.160
∞		0.674	1.282	1.645	1.960	2.326	2.576	3.090

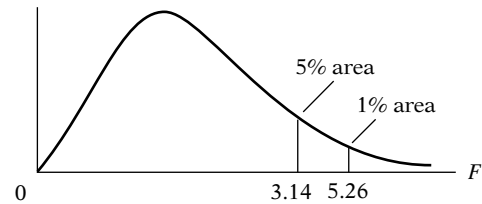
Note: The smaller probability shown at the head of each column is the area in one tail; the larger probability is the area in both tails.

Source: From E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians*, vol. 1, 3d ed., table 12, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika*.

**TABLE D.3** UPPER PERCENTAGE POINTS OF THE *F* DISTRIBUTION

**Example**

Pr( $F > 1.59$ ) = 0.25  
 Pr( $F > 2.42$ ) = 0.10 for df  $N_1 = 10$   
 Pr( $F > 3.14$ ) = 0.05 and  $N_2 = 9$   
 Pr( $F > 5.26$ ) = 0.01



df for denominator $N_2$	df for numerator $N_1$												
	Pr	1	2	3	4	5	6	7	8	9	10	11	12
1	.25	5.83	7.50	8.20	8.58	8.82	8.98	9.10	9.19	9.26	9.32	9.36	9.41
	.10	39.9	49.5	53.6	55.8	57.2	58.2	58.9	59.4	59.9	60.2	60.5	60.7
	.05	161	200	216	225	230	234	237	239	241	242	243	244
2	.25	2.57	3.00	3.15	3.23	3.28	3.31	3.34	3.35	3.37	3.38	3.39	3.39
	.10	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.40	9.41
	.05	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4
3	.25	2.02	2.28	2.36	2.39	2.41	2.42	2.43	2.44	2.44	2.44	2.45	2.45
	.10	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.22
	.05	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74
4	.25	1.81	2.00	2.05	2.06	2.07	2.08	2.08	2.08	2.08	2.08	2.08	2.08
	.10	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.91	3.90
	.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91
5	.25	1.69	1.85	1.88	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89
	.10	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.28	3.27
	.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.71	4.68
6	.25	1.62	1.76	1.78	1.79	1.79	1.78	1.78	1.78	1.77	1.77	1.77	1.77
	.10	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.92	2.90
	.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00
7	.25	1.57	1.70	1.72	1.72	1.71	1.71	1.70	1.70	1.69	1.69	1.69	1.68
	.10	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.68	2.67
	.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57
8	.25	1.54	1.66	1.67	1.66	1.66	1.65	1.64	1.64	1.63	1.63	1.63	1.62
	.10	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.52	2.50
	.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28
9	.25	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58
	.10	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.40	2.38
	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07
10	.25	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58
	.10	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.40	2.38
	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07
11	.25	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58
	.10	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.40	2.38
	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07
12	.25	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58
	.10	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.40	2.38
	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07

Source: From E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians*, vol. 1, 3d ed., table 18, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika*.

df for numerator $N_1$													df for denom- inator $N_2$
15	20	24	30	40	50	60	100	120	200	500	$\infty$	Pr	
9.49	9.58	9.63	9.67	9.71	9.74	9.76	9.78	9.80	9.82	9.84	9.85	.25	1
61.2	61.7	62.0	62.3	62.5	62.7	62.8	63.0	63.1	63.2	63.3	63.3	.10	
246	248	249	250	251	252	252	253	253	254	254	254	.05	2
3.41	3.43	3.43	3.44	3.45	3.45	3.46	3.47	3.47	3.48	3.48	3.48	.25	
9.42	9.44	9.45	9.46	9.47	9.47	9.47	9.48	9.48	9.49	9.49	9.49	.10	3
19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	.05	
99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	.01	4
2.46	2.46	2.46	2.47	2.47	2.47	2.47	2.47	2.47	2.47	2.47	2.47	.25	
5.20	5.18	5.18	5.17	5.16	5.15	5.15	5.14	5.14	5.14	5.14	5.13	.10	5
8.70	8.66	8.64	8.62	8.59	8.58	8.57	8.55	8.55	8.54	8.53	8.53	.05	
26.9	26.7	26.6	26.5	26.4	26.4	26.3	26.2	26.2	26.2	26.1	26.1	.01	6
2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	.25	
3.87	3.84	3.83	3.82	3.80	3.80	3.79	3.78	3.78	3.77	3.76	3.76	.10	7
5.86	5.80	5.77	5.75	5.72	5.70	5.69	5.66	5.66	5.65	5.64	5.63	.05	
14.2	14.0	13.9	13.8	13.7	13.7	13.7	13.6	13.6	13.5	13.5	13.5	.01	8
1.89	1.88	1.88	1.88	1.88	1.88	1.87	1.87	1.87	1.87	1.87	1.87	.25	
3.24	3.21	3.19	3.17	3.16	3.15	3.14	3.13	3.12	3.12	3.11	3.10	.10	9
4.62	4.56	4.53	4.50	4.46	4.44	4.43	4.41	4.40	4.39	4.37	4.36	.05	
9.72	9.55	9.47	9.38	9.29	9.24	9.20	9.13	9.11	9.08	9.04	9.02	.01	10
1.76	1.76	1.75	1.75	1.75	1.75	1.74	1.74	1.74	1.74	1.74	1.74	.25	
2.87	2.84	2.82	2.80	2.78	2.77	2.76	2.75	2.74	2.73	2.73	2.72	.10	11
3.94	3.87	3.84	3.81	3.77	3.75	3.74	3.71	3.70	3.69	3.68	3.67	.05	
7.56	7.40	7.31	7.23	7.14	7.09	7.06	6.99	6.97	6.93	6.90	6.88	.01	12
1.68	1.67	1.67	1.66	1.66	1.66	1.65	1.65	1.65	1.65	1.65	1.65	.25	
2.63	2.59	2.58	2.56	2.54	2.52	2.51	2.50	2.49	2.48	2.48	2.47	.10	13
3.51	3.44	3.41	3.38	3.34	3.32	3.30	3.27	3.27	3.25	3.24	3.23	.05	
6.31	6.16	6.07	5.99	5.91	5.86	5.82	5.75	5.74	5.70	5.67	5.65	.01	14
1.62	1.61	1.60	1.60	1.59	1.59	1.59	1.58	1.58	1.58	1.58	1.58	.25	
2.46	2.42	2.40	2.38	2.36	2.35	2.34	2.32	2.32	2.31	2.30	2.29	.10	15
3.22	3.15	3.12	3.08	3.04	3.02	3.01	2.97	2.97	2.95	2.94	2.93	.05	
5.52	5.36	5.28	5.20	5.12	5.07	5.03	4.96	4.95	4.91	4.88	4.86	.01	16
1.57	1.56	1.56	1.55	1.55	1.54	1.54	1.53	1.53	1.53	1.53	1.53	.25	
2.34	2.30	2.28	2.25	2.23	2.22	2.21	2.19	2.18	2.17	2.17	2.16	.10	17
3.01	2.94	2.90	2.86	2.83	2.80	2.79	2.76	2.75	2.73	2.72	2.71	.05	
4.96	4.81	4.73	4.65	4.57	4.52	4.48	4.42	4.40	4.36	4.33	4.31	.01	18

(Continued)



**TABLE D.3** UPPER PERCENTAGE POINTS OF THE *F* DISTRIBUTION (Continued)

df for denom- inator $N_2$	df for numerator $N_1$												
	Pr	1	2	3	4	5	6	7	8	9	10	11	12
10	.25	1.49	1.60	1.60	1.59	1.59	1.58	1.57	1.56	1.56	1.55	1.55	1.54
	.10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.30	2.28
	.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91
	.01	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71
11	.25	1.47	1.58	1.58	1.57	1.56	1.55	1.54	1.53	1.53	1.52	1.52	1.51
	.10	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.23	2.21
	.05	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79
	.01	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40
12	.25	1.46	1.56	1.56	1.55	1.54	1.53	1.52	1.51	1.51	1.50	1.50	1.49
	.10	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.17	2.15
	.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69
	.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16
13	.25	1.45	1.55	1.55	1.53	1.52	1.51	1.50	1.49	1.49	1.48	1.47	1.47
	.10	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.12	2.10
	.05	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60
	.01	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96
14	.25	1.44	1.53	1.53	1.52	1.51	1.50	1.49	1.48	1.47	1.46	1.46	1.45
	.10	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.08	2.05
	.05	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53
	.01	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.86	3.80
15	.25	1.43	1.52	1.52	1.51	1.49	1.48	1.47	1.46	1.46	1.45	1.44	1.44
	.10	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.04	2.02
	.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48
	.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73	3.67
16	.25	1.42	1.51	1.51	1.50	1.48	1.47	1.46	1.45	1.44	1.44	1.44	1.43
	.10	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	2.01	1.99
	.05	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42
	.01	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55
17	.25	1.42	1.51	1.50	1.49	1.47	1.46	1.45	1.44	1.43	1.43	1.42	1.41
	.10	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.98	1.96
	.05	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38
	.01	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52	3.46
18	.25	1.41	1.50	1.49	1.48	1.46	1.45	1.44	1.43	1.42	1.42	1.41	1.40
	.10	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.96	1.93
	.05	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34
	.01	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37
19	.25	1.41	1.49	1.49	1.47	1.46	1.44	1.43	1.42	1.41	1.41	1.40	1.40
	.10	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.94	1.91
	.05	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31
	.01	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36	3.30
20	.25	1.40	1.49	1.48	1.46	1.45	1.44	1.43	1.42	1.41	1.40	1.39	1.39
	.10	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.92	1.89
	.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28
	.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23

df for numerator $N_1$													df for denom- inator $N_2$
15	20	24	30	40	50	60	100	120	200	500	$\infty$	Pr	
1.53	1.52	1.52	1.51	1.51	1.50	1.50	1.49	1.49	1.49	1.48	1.48	.25	10
2.24	2.20	2.18	2.16	2.13	2.12	2.11	2.09	2.08	2.07	2.06	2.06	.10	
2.85	2.77	2.74	2.70	2.66	2.64	2.62	2.59	2.58	2.56	2.55	2.54	.05	
4.56	4.41	4.33	4.25	4.17	4.12	4.08	4.01	4.00	3.96	3.93	3.91	.01	
1.50	1.49	1.49	1.48	1.47	1.47	1.47	1.46	1.46	1.46	1.45	1.45	.25	11
2.17	2.12	2.10	2.08	2.05	2.04	2.03	2.00	2.00	1.99	1.98	1.97	.10	
2.72	2.65	2.61	2.57	2.53	2.51	2.49	2.46	2.45	2.43	2.42	2.40	.05	
4.25	4.10	4.02	3.94	3.86	3.81	3.78	3.71	3.69	3.66	3.62	3.60	.01	
1.48	1.47	1.46	1.45	1.45	1.44	1.44	1.43	1.43	1.43	1.42	1.42	.25	12
2.10	2.06	2.04	2.01	1.99	1.97	1.96	1.94	1.93	1.92	1.91	1.90	.10	
2.62	2.54	2.51	2.47	2.43	2.40	2.38	2.35	2.34	2.32	2.31	2.30	.05	
4.01	3.86	3.78	3.70	3.62	3.57	3.54	3.47	3.45	3.41	3.38	3.36	.01	
1.46	1.45	1.44	1.43	1.42	1.42	1.42	1.41	1.41	1.40	1.40	1.40	.25	13
2.05	2.01	1.98	1.96	1.93	1.92	1.90	1.88	1.88	1.86	1.85	1.85	.10	
2.53	2.46	2.42	2.38	2.34	2.31	2.30	2.26	2.25	2.23	2.22	2.21	.05	
3.82	3.66	3.59	3.51	3.43	3.38	3.34	3.27	3.25	3.22	3.19	3.17	.01	
1.44	1.43	1.42	1.41	1.41	1.40	1.40	1.39	1.39	1.39	1.38	1.38	.25	14
2.01	1.96	1.94	1.91	1.89	1.87	1.86	1.83	1.83	1.82	1.80	1.80	.10	
2.46	2.39	2.35	2.31	2.27	2.24	2.22	2.19	2.18	2.16	2.14	2.13	.05	
3.66	3.51	3.43	3.35	3.27	3.22	3.18	3.11	3.09	3.06	3.03	3.00	.01	
1.43	1.41	1.41	1.40	1.39	1.39	1.38	1.38	1.37	1.37	1.36	1.36	.25	15
1.97	1.92	1.90	1.87	1.85	1.83	1.82	1.79	1.79	1.77	1.76	1.76	.10	
2.40	2.33	2.29	2.25	2.20	2.18	2.16	2.12	2.11	2.10	2.08	2.07	.05	
3.52	3.37	3.29	3.21	3.13	3.08	3.05	2.98	2.96	2.92	2.89	2.87	.01	
1.41	1.40	1.39	1.38	1.37	1.37	1.36	1.36	1.35	1.35	1.34	1.34	.25	16
1.94	1.89	1.87	1.84	1.81	1.79	1.78	1.76	1.75	1.74	1.73	1.72	.10	
2.35	2.28	2.24	2.19	2.15	2.12	2.11	2.07	2.06	2.04	2.02	2.01	.05	
3.41	3.26	3.18	3.10	3.02	2.97	2.93	2.86	2.84	2.81	2.78	2.75	.01	
1.40	1.39	1.38	1.37	1.36	1.35	1.35	1.34	1.34	1.34	1.33	1.33	.25	17
1.91	1.86	1.84	1.81	1.78	1.76	1.75	1.73	1.72	1.71	1.69	1.69	.10	
2.31	2.23	2.19	2.15	2.10	2.08	2.06	2.02	2.01	1.99	1.97	1.96	.05	
3.31	3.16	3.08	3.00	2.92	2.87	2.83	2.76	2.75	2.71	2.68	2.65	.01	
1.39	1.38	1.37	1.36	1.35	1.34	1.34	1.33	1.33	1.32	1.32	1.32	.25	18
1.89	1.84	1.81	1.78	1.75	1.74	1.72	1.70	1.69	1.68	1.67	1.66	.10	
2.27	2.19	2.15	2.11	2.06	2.04	2.02	1.98	1.97	1.95	1.93	1.92	.05	
3.23	3.08	3.00	2.92	2.84	2.78	2.75	2.68	2.66	2.62	2.59	2.57	.01	
1.38	1.37	1.36	1.35	1.34	1.33	1.33	1.32	1.32	1.31	1.31	1.30	.25	19
1.86	1.81	1.79	1.76	1.73	1.71	1.70	1.67	1.67	1.65	1.64	1.63	.10	
2.23	2.16	2.11	2.07	2.03	2.00	1.98	1.94	1.93	1.91	1.89	1.88	.05	
3.15	3.00	2.92	2.84	2.76	2.71	2.67	2.60	2.58	2.55	2.51	2.49	.01	
1.37	1.36	1.35	1.34	1.33	1.33	1.32	1.31	1.31	1.30	1.30	1.29	.25	20
1.84	1.79	1.77	1.74	1.71	1.69	1.68	1.65	1.64	1.63	1.62	1.61	.10	
2.20	2.12	2.08	2.04	1.99	1.97	1.95	1.91	1.90	1.88	1.86	1.84	.05	
3.09	2.94	2.86	2.78	2.69	2.64	2.61	2.54	2.52	2.48	2.44	2.42	.01	

(Continued)

**TABLE D.3** UPPER PERCENTAGE POINTS OF THE  $F$  DISTRIBUTION (Continued)

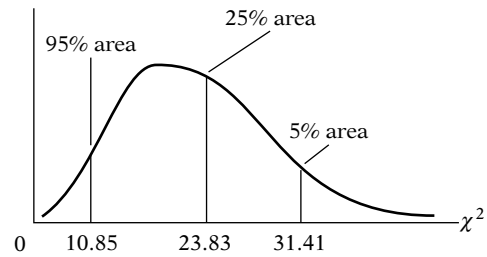
df for denom- inator $N_2$	df for numerator $N_1$												
	Pr	1	2	3	4	5	6	7	8	9	10	11	12
22	.25	1.40	1.48	1.47	1.45	1.44	1.42	1.41	1.40	1.39	1.39	1.38	1.37
	.10	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.88	1.86
	.05	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23
	.01	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12
24	.25	1.39	1.47	1.46	1.44	1.43	1.41	1.40	1.39	1.38	1.38	1.37	1.36
	.10	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.85	1.83
	.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.21	2.18
	.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03
26	.25	1.38	1.46	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.37	1.36	1.35
	.10	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.84	1.81
	.05	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15
	.01	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96
28	.25	1.38	1.46	1.45	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34
	.10	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.81	1.79
	.05	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12
	.01	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90
30	.25	1.38	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.36	1.35	1.35	1.34
	.10	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.79	1.77
	.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09
	.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84
40	.25	1.36	1.44	1.42	1.40	1.39	1.37	1.36	1.35	1.34	1.33	1.32	1.31
	.10	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.73	1.71
	.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00
	.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66
60	.25	1.35	1.42	1.41	1.38	1.37	1.35	1.33	1.32	1.31	1.30	1.29	1.29
	.10	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.68	1.66
	.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92
	.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50
120	.25	1.34	1.40	1.39	1.37	1.35	1.33	1.31	1.30	1.29	1.28	1.27	1.26
	.10	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.62	1.60
	.05	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.87	1.83
	.01	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.40	2.34
200	.25	1.33	1.39	1.38	1.36	1.34	1.32	1.31	1.29	1.28	1.27	1.26	1.25
	.10	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66	1.63	1.60	1.57
	.05	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80
	.01	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.34	2.27
$\infty$	.25	1.32	1.39	1.37	1.35	1.33	1.31	1.29	1.28	1.27	1.25	1.24	1.24
	.10	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.57	1.55
	.05	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75
	.01	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.18

df for numerator $N_1$													df for denom- inator $N_2$
15	20	24	30	40	50	60	100	120	200	500	$\infty$	Pr	
1.36	1.34	1.33	1.32	1.31	1.31	1.30	1.30	1.30	1.29	1.29	1.28	.25	22
1.81	1.76	1.73	1.70	1.67	1.65	1.64	1.61	1.60	1.59	1.58	1.57	.10	
2.15	2.07	2.03	1.98	1.94	1.91	1.89	1.85	1.84	1.82	1.80	1.78	.05	
2.98	2.83	2.75	2.67	2.58	2.53	2.50	2.42	2.40	2.36	2.33	2.31	.01	
1.35	1.33	1.32	1.31	1.30	1.29	1.29	1.28	1.28	1.27	1.27	1.26	.25	24
1.78	1.73	1.70	1.67	1.64	1.62	1.61	1.58	1.57	1.56	1.54	1.53	.10	
2.11	2.03	1.98	1.94	1.89	1.86	1.84	1.80	1.79	1.77	1.75	1.73	.05	
2.89	2.74	2.66	2.58	2.49	2.44	2.40	2.33	2.31	2.27	2.24	2.21	.01	
1.34	1.32	1.31	1.30	1.29	1.28	1.28	1.26	1.26	1.26	1.25	1.25	.25	26
1.76	1.71	1.68	1.65	1.61	1.59	1.58	1.55	1.54	1.53	1.51	1.50	.10	
2.07	1.99	1.95	1.90	1.85	1.82	1.80	1.76	1.75	1.73	1.71	1.69	.05	
2.81	2.66	2.58	2.50	2.42	2.36	2.33	2.25	2.23	2.19	2.16	2.13	.01	
1.33	1.31	1.30	1.29	1.28	1.27	1.27	1.26	1.25	1.25	1.24	1.24	.25	28
1.74	1.69	1.66	1.63	1.59	1.57	1.56	1.53	1.52	1.50	1.49	1.48	.10	
2.04	1.96	1.91	1.87	1.82	1.79	1.77	1.73	1.71	1.69	1.67	1.65	.05	
2.75	2.60	2.52	2.44	2.35	2.30	2.26	2.19	2.17	2.13	2.09	2.06	.01	
1.32	1.30	1.29	1.28	1.27	1.26	1.26	1.25	1.24	1.24	1.23	1.23	.25	30
1.72	1.67	1.64	1.61	1.57	1.55	1.54	1.51	1.50	1.48	1.47	1.46	.10	
2.01	1.93	1.89	1.84	1.79	1.76	1.74	1.70	1.68	1.66	1.64	1.62	.05	
2.70	2.55	2.47	2.39	2.30	2.25	2.21	2.13	2.11	2.07	2.03	2.01	.01	
1.30	1.28	1.26	1.25	1.24	1.23	1.22	1.21	1.21	1.20	1.19	1.19	.25	40
1.66	1.61	1.57	1.54	1.51	1.48	1.47	1.43	1.42	1.41	1.39	1.38	.10	
1.92	1.84	1.79	1.74	1.69	1.66	1.64	1.59	1.58	1.55	1.53	1.51	.05	
2.52	2.37	2.29	2.20	2.11	2.06	2.02	1.94	1.92	1.87	1.83	1.80	.01	
1.27	1.25	1.24	1.22	1.21	1.20	1.19	1.17	1.17	1.16	1.15	1.15	.25	60
1.60	1.54	1.51	1.48	1.44	1.41	1.40	1.36	1.35	1.33	1.31	1.29	.10	
1.84	1.75	1.70	1.65	1.59	1.56	1.53	1.48	1.47	1.44	1.41	1.39	.05	
2.35	2.20	2.12	2.03	1.94	1.88	1.84	1.75	1.73	1.68	1.63	1.60	.01	
1.24	1.22	1.21	1.19	1.18	1.17	1.16	1.14	1.13	1.12	1.11	1.10	.25	120
1.55	1.48	1.45	1.41	1.37	1.34	1.32	1.27	1.26	1.24	1.21	1.19	.10	
1.75	1.66	1.61	1.55	1.50	1.46	1.43	1.37	1.35	1.32	1.28	1.25	.05	
2.19	2.03	1.95	1.86	1.76	1.70	1.66	1.56	1.53	1.48	1.42	1.38	.01	
1.23	1.21	1.20	1.18	1.16	1.14	1.12	1.11	1.10	1.09	1.08	1.06	.25	200
1.52	1.46	1.42	1.38	1.34	1.31	1.28	1.24	1.22	1.20	1.17	1.14	.10	
1.72	1.62	1.57	1.52	1.46	1.41	1.39	1.32	1.29	1.26	1.22	1.19	.05	
2.13	1.97	1.89	1.79	1.69	1.63	1.58	1.48	1.44	1.39	1.33	1.28	.01	
1.22	1.19	1.18	1.16	1.14	1.13	1.12	1.09	1.08	1.07	1.04	1.00	.25	$\infty$
1.49	1.42	1.38	1.34	1.30	1.26	1.24	1.18	1.17	1.13	1.08	1.00	.10	
1.67	1.57	1.52	1.46	1.39	1.35	1.32	1.24	1.22	1.17	1.11	1.00	.05	
2.04	1.88	1.79	1.70	1.59	1.52	1.47	1.36	1.32	1.25	1.15	1.00	.01	

**TABLE D.4** UPPER PERCENTAGE POINTS OF THE  $\chi^2$  DISTRIBUTION

**Example**

$\Pr(\chi^2 > 10.85) = 0.95$   
 $\Pr(\chi^2 > 23.83) = 0.25$  for  $df = 20$   
 $\Pr(\chi^2 > 31.41) = 0.05$



Degrees of freedom \ Pr	.995	.990	.975	.950	.900
1	$392704 \times 10^{-10}$	$157088 \times 10^{-9}$	$982069 \times 10^{-9}$	$393214 \times 10^{-8}$	.0157908
2	.0100251	.0201007	.0506356	.102587	.210720
3	.0717212	.114832	.215795	.351846	.584375
4	.206990	.297110	.484419	.710721	1.063623
5	.411740	.554300	.831211	1.145476	1.61031
6	.675727	.872085	1.237347	1.63539	2.20413
7	.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.94030	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.30380
13	3.56503	4.10691	5.00874	5.89186	7.04150
14	4.07468	4.66043	5.62872	6.57063	7.78953
15	4.60094	5.22935	6.26214	7.26094	8.54675
16	5.14224	5.81221	6.90766	7.96164	9.31223
17	5.69724	6.40776	7.56418	8.67176	10.0852
18	6.26481	7.01491	8.23075	9.39046	10.8649
19	6.84398	7.63273	8.90655	10.1170	11.6509
20	7.43386	8.26040	9.59083	10.8508	12.4426
21	8.03366	8.89720	10.28293	11.5913	13.2396
22	8.64272	9.54249	10.9823	12.3380	14.0415
23	9.26042	10.19567	11.6885	13.0905	14.8479
24	9.88623	10.8564	12.4011	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1603	12.1981	13.8439	15.3791	17.2919
27	11.8076	12.8786	14.5733	16.1513	18.1138
28	12.4613	13.5648	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7083	19.7677
30	13.7867	14.9535	16.7908	18.4926	20.5992
40	20.7065	22.1643	24.4331	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7642	37.6886
60	35.5346	37.4848	40.4817	43.1879	46.4589
70	43.2752	45.4418	48.7576	51.7393	55.3290
80	51.1720	53.5400	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2912
100*	67.3276	70.0648	74.2219	77.9295	82.3581

\*For  $df$  greater than 100 the expression  $\sqrt{2\chi^2} - \sqrt{2k-1} = Z$  follows the standardized normal distribution, where  $k$  represents the degrees of freedom.

.750	.500	.250	.100	.050	.025	.010	.005
.1015308	.454937	1.32330	2.70554	3.84146	5.02389	6.63490	7.87944
.575364	1.38629	2.77259	4.60517	5.99147	7.37776	9.21034	10.5966
1.212534	2.36597	4.10835	6.25139	7.81473	9.34840	11.3449	12.8381
1.92255	3.35670	5.38527	7.77944	9.48773	11.1433	13.2767	14.8602
2.67460	4.35146	6.62568	9.23635	11.0705	12.8325	15.0863	16.7496
3.45460	5.34812	7.84080	10.6446	12.5916	14.4494	16.8119	18.5476
4.25485	6.34581	9.03715	12.0170	14.0671	16.0128	18.4753	20.2777
5.07064	7.34412	10.2188	13.3616	15.5073	17.5346	20.0902	21.9550
5.89883	8.34283	11.3887	14.6837	16.9190	19.0228	21.6660	23.5893
6.73720	9.34182	12.5489	15.9871	18.3070	20.4831	23.2093	25.1882
7.58412	10.3410	13.7007	17.2750	19.6751	21.9200	24.7250	26.7569
8.43842	11.3403	14.8454	18.5494	21.0261	23.3367	26.2170	28.2995
9.29906	12.3398	15.9839	19.8119	22.3621	24.7356	27.6883	29.8194
10.1653	13.3393	17.1170	21.0642	23.6848	26.1190	29.1413	31.3193
11.0365	14.3389	18.2451	22.3072	24.9958	27.4884	30.5779	32.8013
11.9122	15.3385	19.3688	23.5418	26.2962	28.8454	31.9999	34.2672
12.7919	16.3381	20.4887	24.7690	27.5871	30.1910	33.4087	35.7185
13.6753	17.3379	21.6049	25.9894	28.8693	31.5264	34.8053	37.1564
14.5620	18.3376	22.7178	27.2036	30.1435	32.8523	36.1908	38.5822
15.4518	19.3374	23.8277	28.4120	31.4104	34.1696	37.5662	39.9968
16.3444	20.3372	24.9348	29.6151	32.6705	35.4789	38.9321	41.4010
17.2396	21.3370	26.0393	30.8133	33.9244	36.7807	40.2894	42.7956
18.1373	22.3369	27.1413	32.0069	35.1725	38.0757	41.6384	44.1813
19.0372	23.3367	28.2412	33.1963	36.4151	39.3641	42.9798	45.5585
19.9393	24.3366	29.3389	34.3816	37.6525	40.6465	44.3141	46.9278
20.8434	25.3364	30.4345	35.5631	38.8852	41.9232	45.6417	48.2899
21.7494	26.3363	31.5284	36.7412	40.1133	43.1944	46.9630	49.6449
22.6572	27.3363	32.6205	37.9159	41.3372	44.4607	48.2782	50.9933
23.5666	28.3362	33.7109	39.0875	42.5569	45.7222	49.5879	52.3356
24.4776	29.3360	34.7998	40.2560	43.7729	46.9792	50.8922	53.6720
33.6603	39.3354	45.6160	51.8050	55.7585	59.3417	63.6907	66.7659
42.9421	49.3349	56.3336	63.1671	67.5048	71.4202	76.1539	79.4900
52.2938	59.3347	66.9814	74.3970	79.0819	83.2976	88.3794	91.9517
61.6983	69.3344	77.5766	85.5271	90.5312	95.0231	100.425	104.215
71.1445	79.3343	88.1303	96.5782	101.879	106.629	112.329	116.321
80.6247	89.3342	98.6499	107.565	113.145	118.136	124.116	128.299
90.1332	99.3341	109.141	118.498	124.342	129.561	135.807	140.169

Source: Abridged from E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians*, vol. 1, 3d ed., table 8, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika*.

970 APPENDIX D: STATISTICAL TABLES

**TABLE D.5A**  
DURBIN-WATSON  $d$  STATISTIC: SIGNIFICANCE POINTS OF  $d_L$  AND  $d_U$  AT 0.05 LEVEL OF SIGNIFICANCE

$n$	$k' = 1$		$k' = 2$		$k' = 3$		$k' = 4$		$k' = 5$		$k' = 6$		$k' = 7$		$k' = 8$		$k' = 9$		$k' = 10$	
	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$
6	0.610	1.400	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7	0.700	1.356	0.467	1.896	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8	0.763	1.332	0.559	1.777	0.368	2.287	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9	0.824	1.320	0.629	1.699	0.455	2.128	0.296	2.588	—	—	—	—	—	—	—	—	—	—	—	—
10	0.879	1.320	0.697	1.641	0.525	2.016	0.376	2.414	0.243	2.822	—	—	—	—	—	—	—	—	—	—
11	0.927	1.324	0.658	1.604	0.595	1.928	0.444	2.283	0.316	2.645	0.203	3.005	—	—	—	—	—	—	—	—
12	0.971	1.331	0.812	1.579	0.658	1.864	0.512	2.177	0.379	2.506	0.268	2.832	0.171	3.149	—	—	—	—	—	—
13	1.010	1.340	0.861	1.562	0.715	1.816	0.574	2.094	0.445	2.390	0.328	2.692	0.230	2.985	0.147	3.266	—	—	—	—
14	1.045	1.350	0.905	1.551	0.767	1.779	0.632	2.030	0.505	2.296	0.389	2.572	0.286	2.848	0.200	3.111	0.127	3.360	—	—
15	1.077	1.361	0.946	1.543	0.814	1.750	0.685	1.977	0.562	2.220	0.447	2.472	0.343	2.727	0.251	2.979	0.175	3.216	0.111	3.438
16	1.106	1.371	0.982	1.539	0.857	1.728	0.734	1.935	0.615	2.157	0.502	2.388	0.398	2.624	0.304	2.860	0.222	3.090	0.155	3.304
17	1.133	1.381	1.015	1.536	0.897	1.710	0.779	1.900	0.664	2.104	0.554	2.318	0.451	2.537	0.356	2.757	0.272	2.975	0.198	3.184
18	1.158	1.391	1.046	1.535	0.933	1.696	0.820	1.872	0.710	2.060	0.603	2.257	0.502	2.461	0.407	2.667	0.321	2.873	0.244	3.073
19	1.180	1.401	1.074	1.536	0.967	1.685	0.859	1.848	0.752	2.023	0.649	2.206	0.549	2.396	0.456	2.589	0.369	2.783	0.290	2.974
20	1.201	1.411	1.100	1.537	0.998	1.676	0.894	1.828	0.792	1.991	0.692	2.162	0.595	2.339	0.502	2.521	0.416	2.704	0.336	2.885
21	1.221	1.420	1.125	1.538	1.026	1.669	0.927	1.812	0.829	1.964	0.732	2.124	0.637	2.290	0.547	2.460	0.461	2.633	0.380	2.806
22	1.239	1.429	1.147	1.541	1.053	1.664	0.958	1.797	0.863	1.940	0.769	2.090	0.677	2.246	0.588	2.407	0.504	2.571	0.424	2.734
23	1.257	1.437	1.168	1.543	1.078	1.660	0.986	1.785	0.895	1.920	0.804	2.061	0.715	2.208	0.628	2.360	0.545	2.514	0.465	2.670
24	1.273	1.446	1.188	1.546	1.101	1.656	1.013	1.775	0.925	1.902	0.837	2.035	0.751	2.174	0.666	2.318	0.584	2.464	0.506	2.613
25	1.288	1.454	1.206	1.550	1.123	1.654	1.038	1.767	0.953	1.886	0.868	2.012	0.784	2.144	0.702	2.280	0.621	2.419	0.544	2.560
26	1.302	1.461	1.224	1.553	1.143	1.652	1.062	1.759	0.979	1.873	0.897	1.992	0.816	2.117	0.735	2.246	0.657	2.379	0.581	2.513
27	1.316	1.469	1.240	1.556	1.162	1.651	1.084	1.753	1.004	1.861	0.925	1.974	0.845	2.093	0.767	2.216	0.691	2.342	0.616	2.470
28	1.328	1.476	1.255	1.560	1.181	1.650	1.104	1.747	1.028	1.850	0.951	1.958	0.874	2.071	0.798	2.188	0.723	2.309	0.650	2.431
29	1.341	1.483	1.270	1.563	1.198	1.650	1.124	1.743	1.050	1.841	0.975	1.944	0.900	2.052	0.826	2.164	0.753	2.278	0.682	2.396
30	1.352	1.489	1.284	1.567	1.214	1.650	1.143	1.739	1.071	1.833	0.998	1.931	0.926	2.034	0.854	2.141	0.782	2.251	0.712	2.363
31	1.363	1.496	1.297	1.570	1.229	1.650	1.160	1.735	1.090	1.825	1.020	1.920	0.950	2.018	0.879	2.120	0.810	2.226	0.741	2.333
32	1.373	1.502	1.309	1.574	1.244	1.650	1.177	1.732	1.109	1.819	1.041	1.909	0.972	2.004	0.904	2.102	0.836	2.203	0.769	2.306
33	1.383	1.508	1.321	1.577	1.258	1.651	1.193	1.730	1.127	1.813	1.061	1.900	0.994	1.991	0.927	2.085	0.861	2.181	0.795	2.281
34	1.393	1.514	1.333	1.580	1.271	1.652	1.208	1.728	1.144	1.808	1.080	1.891	1.015	1.979	0.950	2.069	0.885	2.162	0.821	2.257
35	1.402	1.519	1.343	1.584	1.283	1.653	1.222	1.726	1.160	1.803	1.097	1.884	1.034	1.967	0.971	2.054	0.908	2.144	0.845	2.236
36	1.411	1.525	1.354	1.587	1.295	1.654	1.236	1.724	1.175	1.799	1.114	1.877	1.053	1.957	0.991	2.041	0.930	2.127	0.868	2.216
37	1.419	1.530	1.364	1.590	1.307	1.655	1.249	1.723	1.190	1.795	1.131	1.870	1.071	1.948	1.011	2.029	0.951	2.112	0.891	2.198
38	1.427	1.535	1.373	1.594	1.318	1.656	1.261	1.722	1.204	1.792	1.146	1.864	1.088	1.939	1.029	2.017	0.970	2.098	0.912	2.180
39	1.435	1.540	1.382	1.597	1.328	1.658	1.273	1.722	1.218	1.789	1.161	1.859	1.104	1.932	1.047	2.007	0.990	2.085	0.932	2.164
40	1.442	1.544	1.391	1.600	1.338	1.659	1.285	1.721	1.230	1.786	1.175	1.854	1.120	1.924	1.064	1.997	1.008	2.072	0.952	2.149
45	1.475	1.566	1.430	1.615	1.383	1.666	1.336	1.720	1.287	1.776	1.238	1.835	1.189	1.895	1.139	1.958	1.089	2.022	1.038	2.088
50	1.503	1.585	1.462	1.628	1.421	1.674	1.378	1.721	1.335	1.771	1.291	1.822	1.246	1.875	1.201	1.930	1.156	1.986	1.110	2.044
55	1.528	1.601	1.490	1.641	1.452	1.681	1.414	1.724	1.374	1.768	1.334	1.814	1.294	1.861	1.253	1.909	1.212	1.959	1.170	2.010
60	1.549	1.616	1.514	1.652	1.480	1.689	1.444	1.727	1.408	1.767	1.372	1.808	1.335	1.850	1.298	1.894	1.260	1.939	1.222	1.984
65	1.567	1.629	1.536	1.662	1.503	1.696	1.471	1.731	1.438	1.767	1.404	1.805	1.370	1.843	1.336	1.882	1.301	1.923	1.266	1.964
70	1.583	1.641	1.554	1.672	1.525	1.703	1.494	1.735	1.464	1.768	1.433	1.802	1.401	1.837	1.369	1.873	1.337	1.910	1.305	1.948
75	1.598	1.652	1.571	1.680	1.543	1.709	1.515	1.739	1.487	1.770	1.458	1.801	1.428	1.834	1.399	1.867	1.369	1.901	1.339	1.935
80	1.611	1.662	1.586	1.688	1.560	1.715	1.534	1.743	1.507	1.772	1.480	1.801	1.453	1.831	1.425	1.861	1.397	1.893	1.369	1.925
85	1.624	1.671	1.600	1.696	1.575	1.721	1.550	1.747	1.525	1.774	1.500	1.801	1.474	1.829	1.448	1.857	1.422	1.886	1.396	1.916
90	1.635	1.679	1.612	1.703	1.589	1.726	1.566	1.751	1.542	1.776	1.518	1.801	1.494	1.827	1.469	1.854	1.445	1.881	1.420	1.909
95	1.645	1.687	1.623	1.709	1.602	1.732	1.579	1.755	1.557	1.778	1.535	1.802	1.512	1.827	1.489	1.852	1.465	1.877	1.442	1.903
100	1.654	1.694	1.634	1.715	1.613	1.736	1.592	1.758	1.571	1.780	1.550	1.803	1.528	1.826	1.506	1.850	1.484	1.874	1.462	1.898
150	1.720	1.746	1.706	1.760	1.693	1.774	1.679	1.788	1.665	1.802	1.651	1.817	1.637	1.832	1.622	1.847	1.608	1.862	1.594	1.877
200	1.758	1.778	1.748	1.789	1.738	1.799	1.728	1.810	1.718	1.820	1.707	1.831	1.697	1.841	1.686	1.852	1.675	1.863	1.665	1.874

n	k' = 11		k' = 12		k' = 13		k' = 14		k' = 15		k' = 16		k' = 17		k' = 18		k' = 19		k' = 20	
	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>
16	0.098	3.503	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
17	0.138	3.378	0.087	3.557	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
18	0.177	3.265	0.123	3.441	0.078	3.603	—	—	—	—	—	—	—	—	—	—	—	—	—	—
19	0.220	3.159	0.160	3.335	0.111	3.496	0.070	3.642	—	—	—	—	—	—	—	—	—	—	—	—
20	0.263	3.063	0.200	3.234	0.145	3.395	0.100	3.542	0.063	3.676	—	—	—	—	—	—	—	—	—	—
21	0.307	2.976	0.240	3.141	0.182	3.300	0.132	3.448	0.091	3.583	0.058	3.705	—	—	—	—	—	—	—	—
22	0.349	2.897	0.281	3.057	0.220	3.211	0.166	3.358	0.120	3.495	0.083	3.619	0.052	3.731	—	—	—	—	—	—
23	0.391	2.826	0.322	2.979	0.259	3.128	0.202	3.272	0.153	3.409	0.110	3.535	0.076	3.650	0.048	3.753	—	—	—	—
24	0.431	2.761	0.362	2.908	0.297	3.053	0.239	3.193	0.186	3.327	0.141	3.454	0.101	3.572	0.070	3.678	0.044	3.773	—	—
25	0.470	2.702	0.400	2.844	0.335	2.983	0.275	3.119	0.221	3.251	0.172	3.376	0.130	3.494	0.094	3.604	0.065	3.702	0.041	3.790
26	0.508	2.649	0.438	2.784	0.373	2.919	0.312	3.051	0.256	3.179	0.205	3.303	0.160	3.420	0.120	3.531	0.087	3.632	0.060	3.724
27	0.544	2.600	0.475	2.730	0.409	2.859	0.348	2.987	0.291	3.112	0.238	3.233	0.191	3.349	0.149	3.460	0.112	3.563	0.081	3.658
28	0.578	2.555	0.510	2.680	0.445	2.805	0.383	2.928	0.325	3.050	0.271	3.168	0.222	3.283	0.178	3.392	0.138	3.495	0.104	3.592
29	0.612	2.515	0.544	2.634	0.479	2.755	0.418	2.874	0.359	2.992	0.305	3.107	0.254	3.219	0.208	3.327	0.166	3.431	0.129	3.528
30	0.643	2.477	0.577	2.592	0.512	2.708	0.451	2.823	0.392	2.937	0.337	3.050	0.286	3.160	0.238	3.266	0.195	3.368	0.156	3.465
31	0.674	2.443	0.608	2.553	0.545	2.665	0.484	2.776	0.425	2.887	0.370	2.996	0.317	3.103	0.269	3.208	0.224	3.309	0.183	3.406
32	0.703	2.411	0.638	2.517	0.576	2.625	0.515	2.733	0.457	2.840	0.401	2.946	0.349	3.050	0.299	3.153	0.253	3.252	0.211	3.348
33	0.731	2.382	0.668	2.484	0.606	2.588	0.546	2.692	0.488	2.796	0.432	2.899	0.379	3.000	0.329	3.100	0.283	3.198	0.239	3.293
34	0.758	2.355	0.695	2.454	0.634	2.554	0.575	2.654	0.518	2.754	0.462	2.854	0.409	2.954	0.359	3.051	0.312	3.147	0.267	3.240
35	0.783	2.330	0.722	2.425	0.662	2.521	0.604	2.619	0.547	2.716	0.492	2.813	0.439	2.910	0.388	3.005	0.340	3.099	0.295	3.190
36	0.808	2.306	0.748	2.398	0.689	2.492	0.631	2.586	0.575	2.680	0.520	2.774	0.467	2.868	0.417	2.961	0.369	3.053	0.323	3.142
37	0.831	2.285	0.772	2.374	0.714	2.464	0.657	2.555	0.602	2.646	0.548	2.738	0.495	2.829	0.445	2.920	0.397	3.009	0.351	3.097
38	0.854	2.265	0.796	2.351	0.739	2.438	0.683	2.526	0.628	2.614	0.575	2.703	0.522	2.792	0.472	2.880	0.424	2.968	0.378	3.054
39	0.875	2.246	0.819	2.329	0.763	2.413	0.707	2.499	0.653	2.585	0.600	2.671	0.549	2.757	0.499	2.843	0.451	2.929	0.404	3.013
40	0.896	2.228	0.840	2.309	0.785	2.391	0.731	2.473	0.678	2.557	0.626	2.641	0.575	2.724	0.525	2.808	0.477	2.892	0.430	2.974
45	0.988	2.156	0.938	2.225	0.887	2.296	0.838	2.367	0.788	2.439	0.740	2.512	0.692	2.586	0.644	2.659	0.598	2.733	0.553	2.807
50	1.064	2.103	1.019	2.163	0.973	2.225	0.927	2.287	0.882	2.350	0.836	2.414	0.792	2.479	0.747	2.544	0.703	2.610	0.660	2.675
55	1.129	2.062	1.087	2.116	1.045	2.170	1.003	2.225	0.961	2.281	0.919	2.338	0.877	2.396	0.836	2.454	0.795	2.512	0.754	2.571
60	1.184	2.031	1.145	2.079	1.106	2.127	1.068	2.177	1.029	2.227	0.990	2.278	0.951	2.330	0.913	2.382	0.874	2.434	0.836	2.487
65	1.231	2.006	1.195	2.049	1.160	2.093	1.124	2.138	1.088	2.183	1.052	2.229	1.016	2.276	0.980	2.323	0.944	2.371	0.908	2.419
70	1.272	1.986	1.239	2.026	1.206	2.066	1.172	2.106	1.139	2.148	1.105	2.189	1.072	2.232	1.038	2.275	1.005	2.318	0.971	2.362
75	1.308	1.970	1.277	2.006	1.247	2.043	1.215	2.080	1.184	2.118	1.153	2.156	1.121	2.195	1.090	2.235	1.058	2.275	1.027	2.315
80	1.340	1.957	1.311	1.991	1.283	2.024	1.253	2.059	1.224	2.093	1.195	2.129	1.165	2.165	1.136	2.201	1.106	2.238	1.076	2.275
85	1.369	1.946	1.342	1.977	1.315	2.009	1.287	2.040	1.260	2.073	1.232	2.105	1.205	2.139	1.177	2.172	1.149	2.206	1.121	2.241
90	1.395	1.937	1.369	1.966	1.344	1.995	1.318	2.025	1.292	2.055	1.266	2.085	1.240	2.116	1.213	2.148	1.187	2.179	1.160	2.211
95	1.418	1.929	1.394	1.956	1.370	1.984	1.345	2.012	1.321	2.040	1.296	2.068	1.271	2.097	1.247	2.126	1.222	2.156	1.197	2.186
100	1.439	1.923	1.416	1.948	1.393	1.974	1.371	2.000	1.347	2.026	1.324	2.053	1.301	2.080	1.277	2.108	1.253	2.135	1.229	2.164
150	1.579	1.892	1.564	1.908	1.550	1.924	1.535	1.940	1.519	1.956	1.504	1.972	1.489	1.989	1.474	2.006	1.458	2.023	1.443	2.040
200	1.654	1.885	1.643	1.896	1.632	1.908	1.621	1.919	1.610	1.931	1.599	1.943	1.588	1.955	1.576	1.967	1.565	1.979	1.554	1.991

Source: This table is an extension of the original Durbin-Watson table and is reproduced from N. E. Savin and K. J. White, "The Durbin-Watson Test for Serial Correlation with Extreme Small Samples or Many Regressors," *Econometrica*, vol. 45, November 1977, pp. 1989-96 and as corrected by R. W. Farebrother, *Econometrica*, vol. 48, September 1980, p. 1554. Reprinted by permission of the Econometric Society.

Note: n = number of observations, k' = number of explanatory variables excluding the constant term.

**EXAMPLE**

If n = 40 and k' = 4, d<sub>L</sub> = 1.285 and d<sub>U</sub> = 1.721. If a computed d value is less than 1.285, there is evidence of positive first-order serial correlation; if it is greater than

1.721, there is no evidence of positive first-order serial correlation; but if d lies between the lower and the upper limit, there is inconclusive evidence regarding the presence or absence of positive first-order serial correlation.



972 APPENDIX D: STATISTICAL TABLES

**TABLE D.5B**  
DURBIN-WATSON  $d$  STATISTIC: SIGNIFICANCE POINTS OF  $d_L$  AND  $d_U$  AT 0.01 LEVEL OF SIGNIFICANCE

$n$	$k' = 1$		$k' = 2$		$k' = 3$		$k' = 4$		$k' = 5$		$k' = 6$		$k' = 7$		$k' = 8$		$k' = 9$		$k' = 10$	
	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$	$d_L$	$d_U$
6	0.390	1.142	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7	0.435	1.036	0.294	1.676	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8	0.497	1.003	0.345	1.489	0.229	2.102	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9	0.554	0.998	0.408	1.389	0.279	1.875	0.183	2.433	—	—	—	—	—	—	—	—	—	—	—	—
10	0.604	1.001	0.466	1.333	0.340	1.733	0.230	2.193	0.150	2.690	—	—	—	—	—	—	—	—	—	—
11	0.653	1.010	0.519	1.297	0.396	1.640	0.286	2.030	0.193	2.453	0.124	2.892	—	—	—	—	—	—	—	—
12	0.697	1.023	0.569	1.274	0.449	1.575	0.339	1.913	0.244	2.280	0.164	2.665	0.105	3.053	—	—	—	—	—	—
13	0.738	1.038	0.616	1.261	0.499	1.526	0.391	1.826	0.294	2.150	0.211	2.490	0.140	2.838	0.090	3.182	—	—	—	—
14	0.776	1.054	0.660	1.254	0.547	1.490	0.441	1.757	0.343	2.049	0.257	2.354	0.183	2.667	0.122	2.981	0.078	3.287	—	—
15	0.811	1.070	0.700	1.252	0.591	1.464	0.488	1.704	0.391	1.967	0.303	2.244	0.226	2.530	0.161	2.817	0.107	3.101	0.068	3.374
16	0.844	1.086	0.737	1.252	0.633	1.446	0.532	1.663	0.437	1.900	0.349	2.153	0.269	2.416	0.200	2.681	0.142	2.944	0.094	3.201
17	0.874	1.102	0.772	1.255	0.672	1.432	0.574	1.630	0.480	1.847	0.393	2.078	0.313	2.319	0.241	2.566	0.179	2.811	0.127	3.053
18	0.902	1.118	0.805	1.259	0.708	1.422	0.613	1.604	0.522	1.803	0.435	2.015	0.355	2.238	0.282	2.467	0.216	2.697	0.160	2.925
19	0.928	1.132	0.835	1.265	0.742	1.415	0.650	1.584	0.561	1.767	0.476	1.963	0.396	2.169	0.322	2.381	0.255	2.597	0.196	2.813
20	0.952	1.147	0.863	1.271	0.773	1.411	0.685	1.567	0.598	1.737	0.515	1.918	0.436	2.110	0.362	2.308	0.294	2.510	0.232	2.714
21	0.975	1.161	0.890	1.277	0.803	1.408	0.718	1.554	0.633	1.712	0.552	1.881	0.474	2.059	0.400	2.244	0.331	2.434	0.268	2.625
22	0.997	1.174	0.914	1.284	0.831	1.407	0.748	1.543	0.667	1.691	0.587	1.849	0.510	2.015	0.437	2.188	0.368	2.367	0.304	2.548
23	1.018	1.187	0.938	1.291	0.858	1.407	0.777	1.534	0.698	1.673	0.620	1.821	0.545	1.977	0.473	2.140	0.404	2.308	0.340	2.479
24	1.037	1.199	0.960	1.298	0.882	1.407	0.805	1.528	0.728	1.658	0.652	1.797	0.578	1.944	0.507	2.097	0.439	2.255	0.375	2.417
25	1.055	1.211	0.981	1.305	0.906	1.409	0.831	1.523	0.756	1.645	0.682	1.776	0.610	1.915	0.540	2.059	0.473	2.209	0.409	2.362
26	1.072	1.222	1.001	1.312	0.928	1.411	0.855	1.518	0.783	1.635	0.711	1.759	0.640	1.889	0.572	2.026	0.505	2.168	0.441	2.313
27	1.089	1.233	1.019	1.319	0.949	1.413	0.878	1.515	0.808	1.626	0.738	1.743	0.669	1.867	0.602	1.997	0.536	2.131	0.473	2.269
28	1.104	1.244	1.037	1.325	0.969	1.415	0.900	1.513	0.832	1.618	0.764	1.729	0.696	1.847	0.630	1.970	0.566	2.098	0.504	2.229
29	1.119	1.254	1.054	1.332	0.988	1.418	0.921	1.512	0.855	1.611	0.788	1.718	0.723	1.830	0.658	1.947	0.595	2.068	0.533	2.193
30	1.133	1.263	1.070	1.339	1.006	1.421	0.941	1.511	0.877	1.606	0.812	1.707	0.748	1.814	0.684	1.925	0.622	2.041	0.562	2.160
31	1.147	1.273	1.085	1.345	1.023	1.425	0.960	1.510	0.897	1.601	0.834	1.698	0.772	1.800	0.710	1.906	0.649	2.017	0.589	2.131
32	1.160	1.282	1.100	1.352	1.040	1.428	0.979	1.510	0.917	1.597	0.856	1.690	0.794	1.788	0.734	1.889	0.674	1.995	0.615	2.104
33	1.172	1.291	1.114	1.358	1.055	1.432	0.996	1.510	0.936	1.594	0.876	1.683	0.816	1.776	0.757	1.874	0.698	1.975	0.641	2.080
34	1.184	1.299	1.128	1.364	1.070	1.435	1.012	1.511	0.954	1.591	0.896	1.677	0.837	1.766	0.779	1.860	0.722	1.957	0.665	2.057
35	1.195	1.307	1.140	1.370	1.085	1.439	1.028	1.512	0.971	1.589	0.914	1.671	0.857	1.757	0.800	1.847	0.744	1.940	0.689	2.037
36	1.206	1.315	1.153	1.376	1.098	1.442	1.043	1.513	0.988	1.588	0.932	1.666	0.877	1.749	0.821	1.836	0.766	1.925	0.711	2.018
37	1.217	1.323	1.165	1.382	1.112	1.446	1.058	1.514	1.004	1.586	0.950	1.662	0.895	1.742	0.841	1.825	0.787	1.911	0.733	2.001
38	1.227	1.330	1.176	1.388	1.124	1.449	1.072	1.515	1.019	1.585	0.966	1.658	0.913	1.735	0.860	1.816	0.807	1.899	0.754	1.985
39	1.237	1.337	1.187	1.393	1.137	1.453	1.085	1.517	1.034	1.584	0.982	1.655	0.930	1.729	0.878	1.807	0.826	1.887	0.774	1.970
40	1.246	1.344	1.198	1.398	1.148	1.457	1.098	1.518	1.048	1.584	0.997	1.652	0.946	1.724	0.895	1.799	0.844	1.876	0.749	1.956
45	1.288	1.376	1.245	1.423	1.201	1.474	1.156	1.528	1.111	1.584	1.065	1.643	1.019	1.704	0.974	1.768	0.927	1.834	0.881	1.902
50	1.324	1.403	1.285	1.446	1.245	1.491	1.205	1.538	1.164	1.587	1.123	1.639	1.081	1.692	1.039	1.748	0.997	1.805	0.955	1.864
55	1.356	1.427	1.320	1.466	1.284	1.506	1.247	1.548	1.209	1.592	1.172	1.638	1.134	1.685	1.095	1.734	1.057	1.785	1.018	1.837
60	1.383	1.449	1.350	1.484	1.317	1.520	1.283	1.558	1.249	1.598	1.214	1.639	1.179	1.682	1.144	1.726	1.108	1.771	1.072	1.817
65	1.407	1.468	1.377	1.500	1.346	1.534	1.315	1.568	1.283	1.604	1.251	1.642	1.218	1.680	1.186	1.720	1.153	1.761	1.120	1.802
70	1.429	1.485	1.400	1.515	1.372	1.546	1.343	1.578	1.313	1.611	1.283	1.645	1.253	1.680	1.223	1.716	1.192	1.754	1.162	1.792
75	1.448	1.501	1.422	1.529	1.395	1.557	1.368	1.587	1.340	1.617	1.313	1.649	1.284	1.682	1.256	1.714	1.227	1.748	1.199	1.783
80	1.466	1.515	1.441	1.541	1.416	1.568	1.390	1.595	1.364	1.624	1.338	1.653	1.312	1.683	1.285	1.714	1.259	1.745	1.232	1.777
85	1.482	1.528	1.458	1.553	1.435	1.578	1.411	1.603	1.386	1.630	1.362	1.657	1.337	1.685	1.312	1.714	1.287	1.743	1.262	1.773
90	1.496	1.540	1.474	1.563	1.452	1.587	1.429	1.611	1.406	1.636	1.383	1.661	1.360	1.687	1.336	1.714	1.312	1.741	1.288	1.769
95	1.510	1.552	1.489	1.573	1.468	1.596	1.446	1.618	1.425	1.642	1.403	1.666	1.381	1.690	1.358	1.715	1.336	1.741	1.313	1.767
100	1.522	1.562	1.503	1.583	1.482	1.604	1.462	1.625	1.441	1.647	1.421	1.670	1.400	1.693	1.378	1.717	1.357	1.741	1.335	1.765
150	1.611	1.637	1.598	1.651	1.584	1.665	1.571	1.679	1.557	1.693	1.543	1.708	1.530	1.722	1.515	1.737	1.501	1.752	1.486	1.767
200	1.664	1.684	1.653	1.693	1.643	1.704	1.633	1.715	1.623	1.725	1.613	1.735	1.603	1.746	1.592	1.757	1.582	1.768	1.571	1.779

APPENDIX D: STATISTICAL TABLES 973

n	k' = 11		k' = 12		k' = 13		k' = 14		k' = 15		k' = 16		k' = 17		k' = 18		k' = 19		k' = 20	
	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>	d <sub>L</sub>	d <sub>U</sub>
16	0.060	3.446	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
17	0.084	3.286	0.053	3.506	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
18	0.113	3.146	0.075	3.358	0.047	3.357	—	—	—	—	—	—	—	—	—	—	—	—	—	—
19	0.145	3.023	0.102	3.227	0.067	3.420	0.043	3.601	—	—	—	—	—	—	—	—	—	—	—	—
20	0.178	2.914	0.131	3.109	0.092	3.297	0.061	3.474	0.038	3.639	—	—	—	—	—	—	—	—	—	—
21	0.212	2.817	0.162	3.004	0.119	3.185	0.084	3.358	0.055	3.521	0.035	3.671	—	—	—	—	—	—	—	—
22	0.246	2.729	0.194	2.909	0.148	3.084	0.109	3.252	0.077	3.412	0.050	3.562	0.032	3.700	—	—	—	—	—	—
23	0.281	2.651	0.227	2.822	0.178	2.991	0.136	3.155	0.100	3.311	0.070	3.459	0.046	3.597	0.029	3.725	—	—	—	—
24	0.315	2.580	0.260	2.744	0.209	2.906	0.165	3.065	0.125	3.218	0.092	3.363	0.065	3.501	0.043	3.629	0.027	3.747	—	—
25	0.348	2.517	0.292	2.674	0.240	2.829	0.194	2.982	0.152	3.131	0.116	3.274	0.085	3.410	0.060	3.538	0.039	3.657	0.025	3.766
26	0.381	2.460	0.324	2.610	0.272	2.758	0.224	2.906	0.180	3.050	0.141	3.191	0.107	3.325	0.079	3.452	0.055	3.572	0.036	3.682
27	0.413	2.409	0.356	2.552	0.303	2.694	0.253	2.836	0.208	2.976	0.167	3.113	0.131	3.245	0.100	3.371	0.073	3.490	0.051	3.602
28	0.444	2.363	0.387	2.499	0.333	2.635	0.283	2.772	0.237	2.907	0.194	3.040	0.156	3.169	0.122	3.294	0.093	3.412	0.068	3.524
29	0.474	2.321	0.417	2.451	0.363	2.582	0.313	2.713	0.266	2.843	0.222	2.972	0.182	3.098	0.146	3.220	0.114	3.338	0.087	3.450
30	0.503	2.283	0.447	2.407	0.393	2.533	0.342	2.659	0.294	2.785	0.249	2.909	0.208	3.032	0.171	3.152	0.137	3.267	0.107	3.379
31	0.531	2.248	0.475	2.367	0.422	2.487	0.371	2.609	0.322	2.730	0.277	2.851	0.234	2.970	0.196	3.087	0.160	3.201	0.128	3.311
32	0.558	2.216	0.503	2.330	0.450	2.446	0.399	2.563	0.350	2.680	0.304	2.797	0.261	2.912	0.221	3.026	0.184	3.137	0.151	3.246
33	0.585	2.187	0.530	2.296	0.477	2.408	0.426	2.520	0.377	2.633	0.331	2.746	0.287	2.858	0.246	2.969	0.209	3.078	0.174	3.184
34	0.610	2.160	0.556	2.266	0.503	2.373	0.452	2.481	0.404	2.590	0.357	2.699	0.313	2.808	0.272	2.915	0.233	3.022	0.197	3.126
35	0.634	2.136	0.581	2.237	0.529	2.340	0.478	2.444	0.430	2.550	0.383	2.655	0.339	2.761	0.297	2.865	0.257	2.969	0.221	3.071
36	0.658	2.113	0.605	2.210	0.554	2.310	0.504	2.410	0.455	2.512	0.409	2.614	0.364	2.717	0.322	2.818	0.282	2.919	0.244	3.019
37	0.680	2.092	0.628	2.186	0.578	2.282	0.528	2.379	0.480	2.477	0.434	2.576	0.389	2.675	0.347	2.774	0.306	2.872	0.268	2.969
38	0.702	2.073	0.651	2.164	0.601	2.256	0.552	2.350	0.504	2.445	0.458	2.540	0.414	2.637	0.371	2.733	0.330	2.828	0.291	2.923
39	0.723	2.055	0.673	2.143	0.623	2.232	0.575	2.323	0.528	2.414	0.482	2.507	0.438	2.600	0.395	2.694	0.354	2.787	0.315	2.879
40	0.744	2.039	0.694	2.123	0.645	2.210	0.597	2.297	0.551	2.386	0.505	2.476	0.461	2.566	0.418	2.657	0.377	2.748	0.338	2.838
45	0.835	1.972	0.790	2.044	0.744	2.118	0.700	2.193	0.655	2.269	0.612	2.346	0.570	2.424	0.528	2.503	0.488	2.582	0.448	2.661
50	0.913	1.925	0.871	1.987	0.829	2.051	0.787	2.116	0.746	2.182	0.705	2.250	0.665	2.318	0.625	2.387	0.586	2.456	0.548	2.526
55	0.979	1.891	0.940	1.945	0.902	2.002	0.863	2.059	0.825	2.117	0.786	2.176	0.748	2.237	0.711	2.298	0.674	2.359	0.637	2.421
60	1.037	1.865	1.001	1.914	0.965	1.964	0.929	2.015	0.893	2.067	0.857	2.120	0.822	2.173	0.786	2.227	0.751	2.283	0.716	2.338
65	1.087	1.845	1.053	1.889	1.020	1.934	0.986	1.980	0.953	2.027	0.919	2.075	0.886	2.123	0.852	2.172	0.819	2.221	0.786	2.272
70	1.131	1.831	1.099	1.870	1.068	1.911	1.037	1.953	1.005	1.995	0.974	2.038	0.943	2.082	0.911	2.127	0.880	2.172	0.849	2.217
75	1.170	1.819	1.141	1.856	1.111	1.893	1.082	1.931	1.052	1.970	1.023	2.009	0.993	2.049	0.964	2.090	0.934	2.131	0.905	2.172
80	1.205	1.810	1.177	1.844	1.150	1.878	1.122	1.913	1.094	1.949	1.066	1.984	1.039	2.022	1.011	2.059	0.983	2.097	0.955	2.135
85	1.236	1.803	1.210	1.834	1.184	1.866	1.158	1.898	1.132	1.931	1.106	1.965	1.080	1.999	1.053	2.033	1.027	2.068	1.000	2.104
90	1.264	1.798	1.240	1.827	1.215	1.856	1.191	1.886	1.166	1.917	1.141	1.948	1.116	1.979	1.091	2.012	1.066	2.044	1.041	2.077
95	1.290	1.793	1.267	1.821	1.244	1.848	1.221	1.876	1.197	1.905	1.174	1.934	1.150	1.963	1.126	1.993	1.102	2.023	1.079	2.054
100	1.314	1.790	1.292	1.816	1.270	1.841	1.248	1.868	1.225	1.895	1.203	1.922	1.181	1.949	1.158	1.977	1.136	2.006	1.113	2.034
150	1.473	1.783	1.458	1.799	1.444	1.814	1.429	1.830	1.414	1.847	1.400	1.863	1.385	1.880	1.370	1.897	1.355	1.913	1.340	1.931
200	1.561	1.791	1.550	1.801	1.539	1.813	1.528	1.824	1.518	1.836	1.507	1.847	1.495	1.860	1.484	1.871	1.474	1.883	1.462	1.896

Note: n = number of observations  
k' = number of explanatory variables excluding the constant term.  
Source: Savin and White, op. cit., by permission of the Econometric Society.

**TABLE D.6A** CRITICAL VALUES OF RUNS IN THE RUNS TEST

$N_1$	$N_2$																			
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
2											2	2	2	2	2	2	2	2	2	
3					2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	
4				2	2	2	3	3	3	3	3	3	3	3	4	4	4	4	4	
5			2	2	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	
6		2	2	3	3	3	3	4	4	4	4	4	5	5	5	5	6	6	6	
7		2	2	3	3	3	4	4	4	5	5	5	5	6	6	6	6	6	6	
8		2	3	3	3	4	4	5	5	5	6	6	6	6	6	7	7	7	7	
9		2	3	3	4	4	5	5	5	6	6	6	7	7	7	7	8	8	8	
10		2	3	3	4	5	5	5	6	6	7	7	7	8	8	8	8	8	8	
11		2	3	4	4	5	5	6	6	7	7	7	8	8	8	9	9	9	9	
12	2	2	3	4	4	5	6	6	7	7	7	8	8	8	9	9	9	10	10	
13	2	2	3	4	5	5	6	6	7	7	8	8	9	9	9	10	10	10	10	
14	2	2	3	4	5	5	6	7	7	8	8	9	9	9	10	10	10	11	11	
15	2	3	3	4	5	6	6	7	7	8	8	9	9	10	10	11	11	11	12	
16	2	3	4	4	5	6	6	7	8	8	9	9	10	10	11	11	11	12	12	
17	2	3	4	4	5	6	7	7	8	9	9	10	10	11	11	11	12	12	13	
18	2	3	4	5	5	6	7	8	8	9	9	10	10	11	11	12	12	13	13	
19	2	3	4	5	6	6	7	8	8	9	10	10	11	11	12	12	13	13	13	
20	2	3	4	5	6	6	7	8	9	9	10	10	11	12	12	13	13	13	14	

Note: Tables D.6A and D.6B give the critical values of runs  $n$  for various values of  $N_1$  (+ symbol) and  $N_2$  (– symbol). For the one-sample runs test, any value of  $n$  that is equal to or smaller than that shown in Table D.6A or equal to or larger than that shown in Table D.6B is significant at the 0.05 level.

Source: Sidney Siegel, *Nonparametric Statistics for the Behavioral Sciences*, McGraw-Hill Book Company, New York, 1956, table F, pp. 252–253. The tables have been adapted by Siegel from the original source: Frieda S. Swed and C. Eisenhart, “Tables for Testing Randomness of Grouping in a Sequence of Alternatives,” *Annals of Mathematical Statistics*, vol. 14, 1943. Used by permission of McGraw-Hill Book Company and *Annals of Mathematical Statistics*.

**TABLE D.6B** CRITICAL VALUES OF RUNS IN THE RUNS TEST

$N_1$	$N_2$																			
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
2																				
3																				
4				9	9															
5			9	10	10	11	11													
6			9	10	11	12	12	13	13	13	13									
7				11	12	13	13	14	14	14	14	15	15	15						
8				11	12	13	14	14	15	15	16	16	16	16	17	17	17	17	17	17
9					13	14	14	15	16	16	16	17	17	18	18	18	18	18	18	18
10					13	14	15	16	16	17	17	18	18	18	19	19	19	20	20	20
11					13	14	15	16	17	17	18	19	19	19	20	20	20	21	21	21
12					13	14	16	16	17	18	19	19	20	20	21	21	21	22	22	22
13						15	16	17	18	19	19	20	20	21	21	22	22	23	23	23
14						15	16	17	18	19	20	20	21	22	22	23	23	23	24	24
15						15	16	18	18	19	20	21	22	22	23	23	24	24	24	25
16							17	18	19	20	21	21	22	23	23	24	25	25	25	25
17							17	18	19	20	21	22	23	23	24	25	25	26	26	26
18							17	18	19	20	21	22	23	24	25	25	26	26	26	27
19							17	18	20	21	22	23	23	24	25	26	26	27	27	27
20							17	18	20	21	22	23	24	25	25	26	27	27	27	28

**EXAMPLE**

In a sequence of 30 observations consisting of 20 + signs ( $= N_1$ ) and 10 - signs ( $= N_2$ ), the critical values of runs at the 0.05 level of significance are 9 and 20, as shown by Tables D.6A and D.6B, respectively. Therefore, if in an application it is found that the number of runs is equal to or less than 9 or equal to or greater than 20, one can reject (at the 0.05 level of significance) the hypothesis that the observed sequence is random.

**TABLE D.7** 1% AND 5% CRITICAL DICKEY-FULLER  $t (= \tau)$  AND  $F$  VALUES FOR UNIT ROOT TESTS

Sample size	$t_{nc}^*$		$t_c^*$		$t_{ct}^*$		$F^\dagger$		$F^\ddagger$	
	1%	5%	1%	5%	1%	5%	1%	5%	1%	5%
25	-2.66	-1.95	-3.75	-3.00	-4.38	-3.60	10.61	7.24	8.21	5.68
50	-2.62	-1.95	-3.58	-2.93	-4.15	-3.50	9.31	6.73	7.02	5.13
100	-2.60	-1.95	-3.51	-2.89	-4.04	-3.45	8.73	6.49	6.50	4.88
250	-2.58	-1.95	-3.46	-2.88	-3.99	-3.43	8.43	6.34	6.22	4.75
500	-2.58	-1.95	-3.44	-2.87	-3.98	-3.42	8.34	6.30	6.15	4.71
$\infty$	-2.58	-1.95	-3.43	-2.86	-3.96	-3.41	8.27	6.25	6.09	4.68

\*Subscripts nc, c, and ct denote, respectively, that there is no constant, a constant, and a constant and trend term in the regression (21.9.5).

†The critical  $F$  values are for the joint hypothesis that the constant and  $\delta$  terms in (21.9.5) are simultaneously equal to zero.

‡The critical  $F$  values are for the joint hypothesis that the constant, trend, and  $\delta$  terms in (21.9.5) are simultaneously equal to zero.

Source: Adapted from W. A. Fuller, *Introduction to Statistical Time Series*, John Wiley & Sons, New York, 1976, p. 373 (for the  $\tau$  test), and D.A. Dickey and W. A. Fuller, "Likelihood Ratio Statistics for Autoregressive Time Series with a Unit Root," *Econometrica*, vol. 49, 1981, p. 1063.

APPENDIX **E**

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## ECONOMIC DATA ON THE WORLD WIDE WEB\*

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*Economic Statistics Briefing Room:* An excellent source of data on output, income, employment, unemployment, earnings, production and business activity, prices and money, credits and security markets, and international statistics.

**<http://www.whitehouse.gov/fsbr/esbr.htm>**

*Federal Reserve System Beige Book:* Gives a summary of current economic conditions by Federal Reserve District. There are 12 Federal Reserve Districts.

**<http://www.bog.frb.fed.us/fomc/bb/current>**

*Government Information Sharing Project:* Provides regional economic information; 1990 population and housing census; 1992 economic census; agriculture census for 1982, 1987, 1992; data on U.S. imports and exports 1991–1995; 1990 Equal Employment Opportunity information.

**<http://govinfo.kerr.orst.edu>**

*National Bureau of Economic Research (NBER) Home Page:* This highly regarded private economic research institute has extensive data on asset prices, labor, productivity, money supply, business cycle indicators, etc. NBER has many links to other Web sites.

**<http://www.nber.org>**

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\*Adapted from *Annual Editions: Microeconomics 98/99*, ed. Don Cole, Dushkin/McGraw-Hill, Connecticut, 1998. It should be noted that this list is by no means exhaustive. The sources listed here are updated continually.

*Panel Study:* Provides data on longitudinal survey of representative sample of U.S. individuals and families. These data are collected annually since 1968.

**<http://www.umich.edu/~psid>**

*Resources for Economists on the Internet:* Very comprehensive source of information and data on many economic activities with links to many Web sites. A very valuable source for academic and nonacademic economists.

**<http://econwpa.wvstl.edu/EconFAQ/EconFaq.html>**

*The Federal Web Locator:* Provides information on almost every sector of the federal government, with international links.

**<http://www.law.vill.edu/Fed-Agency/fedwebloc.html>**

*WebEC:WWW Resources in Economics:* A most comprehensive library of economic facts and figures.

**<http://wuecon.wustl.edu/~adnetec/WebEc/WebEc.html>**

*American Stock Exchange:* Information on some 700 companies listed on the second largest stock market.

**<http://www.amex.com/>**

*Bureau of Economic Analysis (BEA) Home Page:* This agency of the U.S. Department of Commerce, which publishes the *Survey of Current Business*, is an excellent source of data on all kinds of economic activities.

**<http://www.bea.doc.gov/>**

*Business Cycle Indicators:* You will find data on about 256 economic times series.

**<http://www.globalexposure.com/bci.html>**

*CIA Publication:* You will find the *World Fact Book* (annual) and *Handbook of International Statistics*.

**<http://www.odic.gov/cia/publications/pubs.html>**

*Energy Information Administration (DOE):* Economic information and data on each fuel category.

**<http://www.eia.doe.gov/>**

*FRED Database:* Federal Reserve Bank of St. Louis publishes historical economic and social data, which include interest rates, monetary and business indicators, exchange rates, etc.

**<http://www.stls.frb.org/fred/fred.html>**

*International Trade Administration:* Offers many Web links to trade statistics, cross-country programs, etc.

**<http://www.ita.doc.gov/>**

*STAT-USA Databases:* The National Trade Data Bank provides the most comprehensive source of international trade data and export promotion information. There is also extensive data on demographic, political, and socioeconomic conditions for several countries.

**<http://www.stat-usa.gov/BEN/databases.html>**

*Statistical Resources on the Web/Economics:* An excellent source of statistics collated from various federal bureaus, economic indicators, the Federal Reserve Board, data on consumer price, and Web links to other sources.

**<http://www.lib.umich.edu/libhome/Documents.centers/stecon.html>**

*Bureau of Labor Statistics:* The home page data related to various aspects of employment, unemployment, and earnings and provides links to other statistical Web sites.

**<http://stats.bls.gov:80/>**

*U.S. Census Bureau Home Page:* Prime source of social, demographic, and economic data on income, employment, income distribution, and poverty.

**<http://www.census.gov/>**

*General Social Survey:* Annual personal interview survey data on U.S. households that began in 1972. More than 35,000 have responded to some 2500 different questions covering a variety of data.

**<http://www.icpsr.umich.edu/GSS/>**

*Institute for Research on Poverty:* Data collected by nonpartisan and nonprofit university-based research center on a variety of questions relating to poverty and social inequality.

**<http://www.ssc.wisc.edu/irp/>**

*Social Security Administration:* The official Web site of the Social Security Administration with a variety of data.

**<http://www.sa.gov/>**

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*Note:* For a list of the seminal articles on the various topics discussed in this book, please refer to the extensive bibliography given at the end of each chapter in Fomby et al., cited above.

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