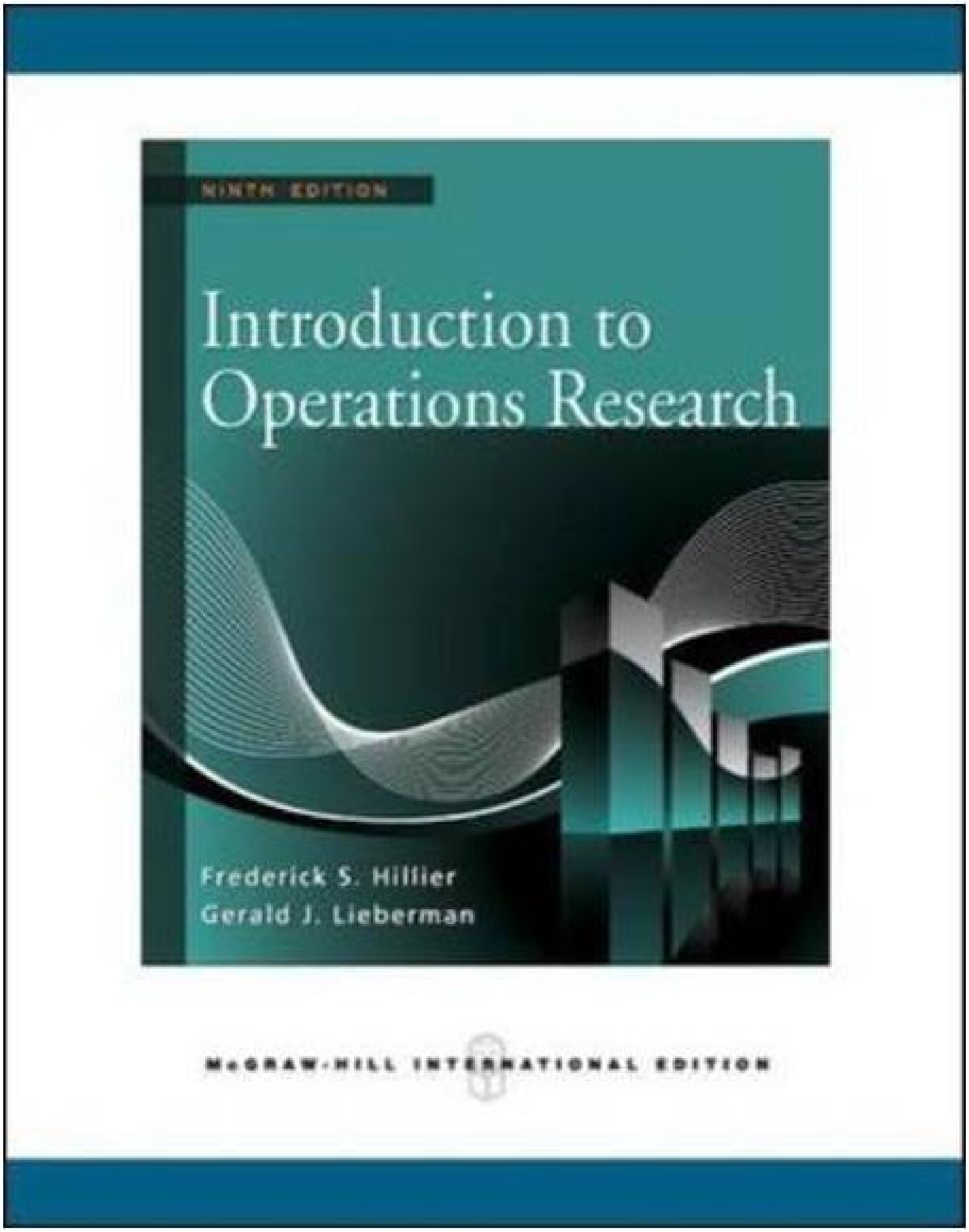
Solutions Manual

Introduction to Operations Research

9th Edition

Frederick S. Hillier



SOLUTIONS MANUAL

For

INTRODUCTION TO OPERATIONS RESEARCH

Ninth Edition

FREDERICK S. HILLIER
Stanford University

GERALD J. LIEBERMAN
Late of Stanford University

Prepared by
PELIN G. CANBOLAT

TABLE OF CONTENTS

SOLUTIONS TO END-OF-CHAPTER PROBLEMS AND CASES

CHAPTER 1: INTRODUCTION

1.3-1.

Answers will vary.

1.3-2.

Answers will vary.

1.3-3.

By using operations research (OR), FedEx managed to survive crises that could drive it out of business. The new planning system provided more flexibility in choosing the destinations that it serves, the routes and the schedules. Improved schedules yielded into faster and more reliable service. OR applied to this complex system with a lot of interdependencies resulted in an efficient use of the assets. With the new system, FedEx maintained a high load factor while being able to service in a reliable, flexible and profitable manner. The model also enabled the company to foresee future risks and to take measures against undesirable outcomes. The systematic approach has been effective in convincing investors and employees about the benefits of the changes. Consequently, "today FedEx is one of the nation's largest integrated, multi-conveyance freight carriers" [p. 32].

CHAPTER 2: OVERVIEW OF THE OPERATIONS RESEARCH MODELING APPROACH

2.1-1.

(a) The rise of electronic brokerage firms in the late 90s was a threat against full-service financial service firms like Merrill Lynch. Electronic trading offered very low costs, which were hard to compete with for full-service firms. With banks, discount brokers and electronic trading firms involved, the competition was fierce. Merrill Lynch needed an urgent response to these changes in order to survive.

(b) "The group's mission is to aid strategic decision making in complex business situations through quantitative modeling and analysis" [p.8].

(c) The data obtained for each client consisted of "data for six categories of revenue, four categories of account type, nine asset allocation categories, along with data on number of trades, mutual fund exchanges and redemptions, sales of zero coupon bonds, and purchases of new issues" [p. 10].

(d) As a result of this study, two main pricing options, viz., an asset-based pricing option and a direct online pricing option were offered to the clients. The first targeted the clients who want advice from a financial advisor. The clients who would choose this option would be charged at a fixed rate of the value of their assets and would not pay for each trade. The latter pricing option was for the clients who want to invest online and who do not want advice. These self-directed investors would be charged for every trade.

(e) "The benefits were significant and fell into four areas: seizing the marketplace initiative, finding the pricing sweet spot, improving financial performance, and adopting the approach in other strategic initiatives" [p.15].

2.1-2.

(a) This study arose from GM's efforts to survive the competition of the late 80s. Various factors, including the rise of foreign imports, the increase in customer expectations and the pricing constraints, forced GM to close plants and to incur large financial losses. While trying to copy Japanese production methods directly, GM was suffering from "missing production targets, working unscheduled overtime, experiencing high scrap costs, and executing throughput-improvement initiatives with disappointing results" [p. 7]. The real problems were not understood and the company was continuously losing money while the managers kept disagreeing about solutions.

(b) The goal of this study was "to improve the throughput performance of existing and new manufacturing systems through coordinated efforts in three areas: modeling and algorithms, data collection, and throughput-improvement processes" [p. 7].

(c) The data collection was automated by using programmable logic controllers (PLCs). The software kept track of the production events including "machine faults and blocking and starving events" [p. 13] and recorded their duration. The summary of this data was then transferred to a centralized database, which converted this to workstation-performance characteristics and used in validating the models, determining the bottleneck processes and enhancing throughput.

(d) The improved production throughput resulted in more than $2.1 billion in documented savings and increased revenue.

2.1-3.

(a) San Francisco Police Department has a total police force of 1900, with 850 officers on patrol. The total budget of SFPD in 1986 was $176 million with patrol coverage cost of $79 million. This brings out the importance of the problem.

Like most police departments, SFPD was also operated with manually designed schedules. It was impossible to know if the manual schedules were optimal in serving residents' needs. It was difficult to evaluate alternative policies for scheduling and deploying officers. There was also the problem of poor response time and low productivity, pressure of increasing demands for service with decreasing budgets. The scheduling system was facing the problem of providing the highest possible correlation between the number of officers needed and the number actually on duty during each hour. All these problems led the Task Force to search for a new system and thus undertake this study.

(b) After reviewing the manual system, the Task Force decided to search for a new system. The criteria it specified included the following six directives :

-- the system must use the CAD (computer aided dispatching) system, which provides a large and rich data base on resident calls for service. The CAD system was used to dispatch patrol officers to call for service and to maintain operating statistics such as call types, waiting times, travel time and total time consumed in servicing calls. The directive was to use this data on calls for service and consumed times to establish work load by day of week and hour of day

-- it must generate optimal and realistic integer schedules that meet management policy guidelines using a micro-computer

-- it must allow easy adjustment of optimal schedules to accommodate human considerations without sacrificing productivity

-- it must create schedules in less than 30 minutes and make changes in less than 60 seconds

-- it must be able to perform both tactical scheduling and strategic policy testing in one integrated system

-- the user interface must be flexible and easy, allowing the users (captains) to decide the sequence of functions to be executed instead of forcing them to follow a restrictive sequence.

2.1-4.

(a) Taking all the statistics of AIDS cases into account it was inferred that just one-third of all cases nation-wide involved some aspect of Injection Drug Use(IDU). But in contrast to this national picture, over 60% of 500 cases reported in New Haven , Connecticut was traced to drug use. Though it was realized previously , by 1987 it was clear that the dominant mode of HIV transmission in New Haven was the practice of needle sharing for drug injection.

This was the background of the study and in 1987 a street outreach program was implemented which included a survey of drug addicts with partial intent to determine why IDUs continued to share needles given the threat of HIV infection and AIDS. It was claimed by the survey respondents that IDUs shared needles since they were scared and feared arrest for possessing a syringe without prescription which was forbidden by law in Connecticut. Respondents also pointed out difficulties involved in entering drug treatment program. The officials recognized that logical intervention was needle exchange whereby IDUs exchanged their used needles for clean ones. This would remove infectious drug injection equipment from circulation and also ease access to clean needles. Further, contacts made as a result of needle exchange might lead some active IDUs to consider counseling or enter drug treatment. After a lot of lobbying finally the bill for the first legal needle exchange program became effective on July 1, 1990.

(b) The design for the needle exchange program was achieved over the summer of 1980. The relevant committee decided that IDUs would be treated with respect and so no identification information was asked of program clients. The program began operating on November 13, 1990.

The needle exchange operate on an outreach basis. A van donated by Yale university visits neighborhoods with high concentration of IDUs. Outreach staff members try to educate the clients over there by different means like distributing literature documenting risks of HIV infection, dispensing condoms , clean packets, etc.

The primary goal of needle exchange is to reduce incidence of new HIV infection among IDUs. While studies showed consistent self-reported reductions in risky behavior among IDUs participating in needle exchange programs the studies were not convincing. So the mechanics of needle exchange require that the behavior of needles must change. What was required was to reduce the time needles spend circulating in the population. As needles circulate for shorter period of time, needles share fewer people which lower the number of infected needles in the pool of circulating needles which in effect lowers chances of an IDU becoming infected being injected with a previously infected needle. To use this theory required invention of new data collection system which is as follows.

A syringe tracking and testing is a system developed to interview the needles returned to the program. All clients participating in the needle exchange are given unique code names and every needle distributed receives a code. Everytime a client exchanges needles, an outreach worker records the date and location of exchange. He also records the code name of the client receiving the needles alongside the codes of the needles. The client then places the returned needles in a canister to which the worker puts a label with the date and location of exchange and code name of client.

All returned needles are brought to a laboratory at Yale University where a technician collates the information on the canister labels with the tracking numbers on the returned needles. For non-program or street needles returned to needle exchange , the location, date, and client code are recorded. A sample of the returned needles are tested for HIV.

(c) The initial results from this system were both shocking and decisive. At the start of the program, the IDUs presented the needles in their possession for clear needles. These street needles are representative of risk faced by an IDU prior to operation of needle exchange which showed a prevalence level of 67.5 percent which tested HIV positive. As of middle of March 1991, 50.3 percent of the program needles tested positive. Since March 1991, additional program needles have been tested of which 40.5 percent tested positive. This gave further support to the protection offered by needle exchange program.

Though these results are encouraging, they do not link operations of needle exchange to changes in the rate of new HIV infections . To achieve this required a development of a mathematical model describing HIV transmission among IDUs via needle sharing. The syringe tracking and testing system in concert with limited observation obtained from surveying program clients provided data required to estimate parameters for this model. Though the model developed was conservative, the results were interesting. It estimated that in absence of behavioral changes on part of IDUs in the program, rate of new HIV infections among needle exchange clients would drop. It estimates a 33 percent reduction in new HIV infections.

(d) To understand the impact of this study requires both a local and national perspective. In the local aspect, it is possible to construct a conservative estimate of the actual number of infections averted. As many clients who joined the needle exchange apparently dropped out, the conservative impact of the program can be estimated by multiplying the cumulative number of person years spent in the program over all clients by the incidence reduction of 2 HIV infections per 100 client years. This assumes all those who apparently dropped out of the program are truly recidivists, an assumption that may be patently false. Calculations have shown that between $1 million and $2 million dollar in public health care expenditure have been avoided over the first two years of the program.

This only hints at the true impact of this work. Needle exchange has been returned to the menu of legitimate AIDS intervention in major American cities in large part due to the evaluation of New Haven. In some calculations made as to how much public health care costs could be avoided only the annual reduction in HIV incidence among needle exchange program clients is considered, as opposed to changes in lifetime probability of acquiring HIV infection. While decrease in lifetime risk will be less than decrease in annual incidence the effect of placing clients in drug treatment via needle exchange has been ignored. If this point is considered impact of needle exchange on probability of their acquiring HIV could be substantial.

2.2-1.

The financial benefits that resulted from this study include savings of $40 million in 2001 and of $5 million in 2002. The savings for any major disruption have been between $1 and $5 million. The new system enabled Continental Airlines to operate in an efficient and cost-effective manner in case of disruptions. The time to recover and the costs associated with disruptions are reduced. What-if analysis allowed the company to evaluate various scenarios in short periods of time. Since the complete reliable data can be generated quickly, the company reacts to facts rather than forecasts. These improvements in handling irregularities resulted in better and more reliable service and hence happier customers.

2.2-2.

(a) Swift & Company operates in an industry that involves highly skilled labor, many production pathways and perishable products. To generate profit, the company needs to make an efficient use of every single animal procured. Before this study, Swift was not able to meet the shipping deadlines and as a result of this, it was forced to offer discounts. The consequences of this practice included highly reduced profits, inaccurate forecasts and very low reliability. The company had to find a way to come up with the best product mix and to survive in this business defined by volatility and velocity.

(b) The purpose of the scheduling models is "to fix the production schedule for the next shift and to create a projection of short order" [p. 74]. They generate shift-level and daily schedule for 28 days. The capable-to-promise (CTP) models "determine whether a plant can ship a requested order-line-item quantity on the requested date and time given the availability of cattle and constraints on the plants' capacity during the 90-day model horizon" [p. 75]. The starting inventory, committed orders, and production schedule generated by the CTP models are inputs to the available-to-promise (ATP) models. Every 15 minutes, the ATP models determine the unsold production of each shift and alert the salespeople to undesirable inventory levels.

(c) The company now uses 45 optimization models.

(d) As a result of this study, the key performance measure, namely the weekly percent-sold position has increased by 22%. The company can now allocate resources to the production of required products rather than wasting them. The inventory resulting from this approach is much lower than what it used to be before. Since the resources are used effectively to satisfy the demand, the production is sold out. The company does not need to offer discounts as often as before. The customers order earlier to make sure that they can get what they want by the time they want. This in turn allows Swift to operate even more efficiently. The temporary storage costs are reduced by 90%. The customers are now more satisfied with Swift. With this study, Swift gained a considerable competitive advantage. The monetary benefits in the first years was $12.74 million, including the increase in the profit from optimizing the product mix, the decrease in the cost of lost sales, in the frequency of discount offers and in the number of lost customers. The main nonfinancial benefits are the increased reliability and a good reputation in the business.

2.2-3.

(a) The Dutch Government has been facing problems regarding its water management the past it was too much water but now it is the scarcity of fresh water and pollution due to increased industrialization and a growing population with high standard of living. Some features of the Dutch landscape exaggerate the problem.

Netherlands , one of the densely populated countries of the world and the seventh largest wealthiest nation derives a huge amount of wealth from crops grown in irrigated land. Since agriculture is the largest user of fresh water in Netherlands water shortages can cause large economic losses. The Rhine river is Netherlands major source of surface water for agriculture, irrigation and other purposes. Along with other rivers and canals it is a major artery for the inland shipping fleet of Western Europe. Low water levels in rivers and canals can cause shipping delays and economic losses too because only partially laden ships can navigate the inland waterways. Besides this mines and industries along the Rhine discharge into the water different type of pollutants which also contribute to the rivers increasing level of salinity which in turn damage crops and threaten environment and personal health. Power plants on the banks can degrade quality of water by discharging excess heat into streams. which may in turn endanger the ecological balance in the neighborhood. Besides salinity, the most important water quality problem is eutrophication heavy growth of algae in relatively stagnant water of storage reservoirs and lakes which cause the water to smell and taste foul.

Though in mid 1970s supply met the demand of fresh surface water except in dry years, it was predicted not to be true for late 1980s. But ground water sources were already facing scarcity. Rapid increases in ground water extraction recently have resulted in drop of its level in many areas. This in turn can cause agricultural and environmental damage in areas where water level was higher. Facing such water management problem the Netherlands Government agency responsible for water control and public works, Rijkswaterstaat commissioned an analysis on which to base a new national water management policy. which resulted in PAWN , the Policy Analysis for the Water Management of the Netherlands in April, 1977.

(b) The purpose of the five mathematical models are as follows:

The Water Distribution Model is the heart of the analytic method. The infrastructure of the surface water system consists of rivers and canals that transport water, lakes and reservoirs that store water, weirs, locks and lock bypasses , sluices and pumping stations that are used to control the transport of water throughout the country. The model simulates the major components of this system in detail and contains aggregated representations of the other components. The model provides information on the water management system, including flows, level of water, extractions, discharges, depth of shipping and concentration of pollutants. It provides information on different costs, including investment and operating for technical and managerial tactics and irrigation, as well as shortage and salinity losses for agriculture, low water shipping loss and shipping delay losses. This information is provided for each 10-day period and is given in a summary of totals on averages for the entire year.

The Industry Response Simulation Model : When water becomes more expensive or less available, firms respond by modifying their production process to consume less water; usually with an accompanying increase in costs, part or all which may be passed on to their customers. To find out these responses and their costs, PAWN developed and used this model. The model simulates the behavior of industrial firms in response to a change in ground water extractions or an increase in the price of drinking water. In determining behavior, the model assumes that each firm will choose the least costly alternative available.

Although the model was developed to investigate the effect of ground water charges on industry it is also used to examine effect of imposing quotas that restrict ground water extractions.

Electric Power Reallocation and Cost Model : The Water Distribution Model provides an excess temperature table that shows rise in temperature at one node resulting from a reference heat discharge by a power plant at another node. To obtain the excess temperatures created by heat of power plant discharges other than the reference, this model scales the correct entries in the table by the ratio of the new to reference discharge.

The model calculates the optimal generating schedule for two basic conditions: one in which the thermal standards are relaxed and the other in which it is imposed. The difference is the cost attributable to the thermal standards, the thermal penalty cost. The model repeats this process for each 10-day period in the year and calculates the total thermal penalty annually as well as some other statistics.

The Nutrient Model : Eutrophication, a heavy growth of algae called an algae bloom occurs in the still water of lakes and reservoirs. This model estimates the amounts of nutrients , phosphates, nitrogen and silicon available to algae, given the nutrient flows entering the lake. The model calculates the composition of a column of water 1 square meter in area and as deep as the lake under investigation , in contact with the air and with the bottom sedimentations. The important nutrient processes include the inflow and outflow of nutrient bearing water, the flux of nutrients from the bottom, and the flux of nutrient to and from algae.

The Algae Bloom Model : PAWN used this model to analyze the effect on algae blooms of circumstances, including introducing control tactics. It predicts weekly size and species composition of the algae bloom, given amounts of nutrients and solar energy available to the algae.

(c) PAWN compares policies in terms of their impacts. In choosing impact measures, primary criterion was that they are sufficient to span quite a number of objectives. It includes both national and general water management as well as specific objectives mentioned by different interest groups. The objectives also had to reflect both equity and efficiency.

Impacts on water management system include investment and operating cost of technical and managerial tactics, as well as flood risk in the Ijssal lake.

Direct impact on users include change in profit, expenditure, revenue for each user group like agriculture, shipping, electric power generation, industries, drinking water supply companies.

Environmental impacts include violation of water quality standards, damage to nature areas caused by construction of new facilities and total amount of ground water extracted.

Impact on entire nation include net monetary benefit to nation after deducting transfer payments, total economic effects-- both Government revenues and charges-- in production , employment and imports-- that occurs in both industries directly involved in construction of major new facilities and interrelated industries and effects on public health.

PAWN also pays attention to distributional effects that show uneven distribution of monetary benefit and costs among producers, consumers, Government and uneven distribution of other impacts among different groups and locations.

(d) The several tangible benefits are:

-- the building of Brielse Mier pipeline which will yield $38 million investment savings and $15 million annual net benefit in decreased salinity damage to agriculture.

-- rejection of plan to build the second dike to separate Markermeer from an adjacent saline lake-- saving more than $95 million in investment costs, 0.2% of Dutch domestic product.

-- implementation of new flushing policy for Markermeer is expected to yield net benefits between $1.2 million and $5.4 million per year.

-- adoption of a more stringent thermal standard of an increase of 3 degree Celsius for canals by Dutch since PAWN showed it was practical and not costly. This led to a decrease in locally harmful ecological effects of power plant heat discharges.

The intangible benefits are:

-- drastic changes in Dutch approach to eutrophication, their most serious water quality problem

-- implementation of all recommendations by PAWN would lead to an expected profit between \$53 million and \$128 million per year

-- to deal with ground water extraction problem, priority had to be given to industry and drinking water companies. If practical methods to replenish the ground water cannot be devised, regulatory measures will show growth ot ground water sprinkling.

-- comprehensive methodology developed by PAWN has been adopted by the Government, other departments, laboratories and used in several major studies.

-- PAWN provides method to educate decision makers and train analysts in analyses of complex natural resource and environmental questions.

-- the general approach and some of the techniques have potentially wide applicability.

2.2-4.

(a) The author's example of a model in natural sciences is Newton's Law of Universal Gravitation. Tough he says it is one of the most important models in Physics this does not account for all details. For example, it is only approximate if the particles are objects with non-spherical shapes and model ignores relativity.

The model in OR identified by him is the Economic Order Quantity (EOQ) model. Like Newton's Law in Physics, this model too is simple and highlights important features of the real world. It identifies some critical relationships and also shows that a single model can be used for all types of orders. This model too ignores details of the real world which might be considered important. But just like Newton's law, EOQ model is one of the most important one in MS/OR.

(b) The MS/OR profession is often compared to natural sciences. Basic precepts in natural sciences can be used to guide research in MS/OR. The author believes a greater understanding of these precepts can provide needed focus form the profession and help resolve some recent debates.

To be useful, an MS/OR model must possess some qualities as models in natural science. The most important of them are :

-- understandibility

-- verifiability

-- reproducibility

The extent to which a model can be understood depends on tools available for evaluation. But models have inherent values which can be interpreted on inspection. In 1960s many MS/OR professionals tried to model the behavior of automobile traffic. The most successful models examined traffic from macro point of view and found similarities between traffic flow and fluid flow. Less successful models examined behavior of individual drivers with complicated queuing expressions. The latter models though more accurate have less value. They are too difficult to interpret.

A model should be verifiable and based on observable phenomena. It must capture the essence of a problem faced by MS/OR practitioners. It must include important parameters, decision variables, aims and objectives and their relationships.

As a criterion for publication, the phenomena underlying the model must be reproducible. To have a broader appeal the model must be sufficiently general.

2.3-1.

(a) Towards the end of 90s, Philips Electronics faced challenges in coordinating its supply chains. Decentralized short-term planning was no longer very reliable. The spread of the information to various branches of the global supply chains was taking a lot of time and the information was distorted while it was being transferred. To deal with the uncertainty, the companies had to keep high inventory levels.

(b) The ultimate purpose of this study was "to improve competitiveness by improving customer service, increasing sales and margins, and reducing obsolescence and inventories" [p. 38]. To achieve this, the project team aimed at designing a collaborative-planning (CP) process that would improve trust and collaboration between partners and accelerate decision making.

(c) "The algorithm can generate feasible plans within seconds. In fact, the calculation of the plan is hardly noticeable to the people participating in the weekly CP meeting. The speed of the algorithm also allows planners to compute multiple plans during the meeting, creating an interactive planning environment. The software environment also provides strong problem-solving support, used extensively during the CP meetings. One such capability is called backward pegging. It exploits the one-to-one relationship between the storage of an end item in some future period and a constraining stock on hand or scheduled receipt of one or more upstream items. Thus, the backward-pegging mechanism makes the actual material bottlenecks in the network visible" [p. 41-42].

(d) The four steps of the collaborative-planning process are gathering data, deciding, escalating and deploying.

(e) This study allowed the companies to solve complex problems quickly, to exploit profitable opportunities and to enhance trust within the supply chain. The information is now conveyed to other parties in a shorter time and more accurately. As a result of this, the companies can have accurate information about the availability of material at different stages. This results in the reduction of inventory and obsolescence as well as the ability to respond promptly to the changes in market conditions. The benefit from decreasing inventory and obsolescence is around $5 million per year in total. Nonfinancial benefits include enhanced flexibility and reliability throughout the chain.

2.3-2.

(a) The role of evaluating a model is to extract information from it. It entails two, often simultaneous activities -- identifying alternatives and calculating objectives.

The most known technique for identifying alternatives is optimization. The process yields a single solution which maximizes or minimizes a single objective function. The most prevalent technique used for identifying multiple alternatives is sensitivity analysis. The process can show how the optimum changes when model parameters change or can provide near-optimal alternative solutions.

The author views that optimization should not be the sole goal, not just because models are abstractions of real world but because does not provide adequate information for making decisions. Its objective is to find only one solution. But the decision maker probably would prefer information on several alternatives. Though sensitivity analysis increases effectiveness of optimization , it is deficient. It only yields alternative solution near optimum. The decision maker rather needs unique solutions which offer distinct alternatives.

So the author opines that research should be devoted to identify multiple alternatives. One may begin in the solution process itself. Each solution is a feasible alternative, which the decision maker may choose over the optimum. New algorithms may be designed to identify distinct alternatives.

The second step of evaluation should involve calculating quantifiable objective for each alternative.

Thus summarizing, the author views that although optimization has dominated research in MS/OR it is but one technique for addressing one part of MS/OR process. It is deficient since does not provide adequate information for making important decisions/ Complex decisions rather require information on many alternatives and also an understanding of basic trade-offs and principles. Optimization alone cannot provide this information.

(b) The key to MS/OR is not only possessing knowledge. Though different practitioners take different approaches -- three key steps being

-- modeling
-- evaluating
-- deciding , which are all complementary.

In MS/OR systematized knowledge is reflected in better decisions. The key to good decisions is knowledge and judgment. Modeling and evaluation form a systematized way for acquiring knowledge; judgment is acquired through experience.

The problems which do not require judgment are the ones which can be formulated with well-defined objective functions and solved automatically with algorithms which are pretty efficient; an example being the shortest path algorithm. On the other hand, there are problems which are easy to formulate but difficult to solve, example a carpet store owner would not argue with the objective of the cutting stock problem but may not be happy with solutions provided by available software. He would benefit from models that offer help in cutting the carpet. Combining knowledge from modeling with judgment of store owner would give best result.

Generally, important questions facing management are not well-defined as shortest path or cutting stock problem. Neither there are related well-defined problems which can be optimized, example the facilities layout problem.

Thus the roles are all complementary. Most depend on both judgment of decision maker and knowledge gained from modeling and evaluating.

2.4-1

The credibility of analyses and therefore the probability that policies based upon them will be implemented depends on the perceived validity of the models.

The process of model validation though is a burden helps to learn lessons which may not lead to just improvements in the model but also to changes in the scientific theory and public policy. This happened in PAWN with the Nutrient model and eutrophication. When PAWN was started, the Dutch eutrophication control strategy was to decrease phosphate discharges into surface water from point sources mostly sewerage treatment plants.

To find out how effective this strategy is the Algae Bloom model was applied to some major Dutch lakes. It was revealed that in most cases this required enormous percentage decrease in phosphate concentrations.

Next question was what was to be done to achieve a particular percentage decrease in phosphate concentration. The Dutch strategy was based on the fact that large amount of phosphates and other nutrients accumulated in bottom of the lakes was bound permanently to the bottom and hence unavailable to support algae blooms. This was contradicted both in the Nutrient Model calibration process and validation process.

Studies taken convinced that nutrients particularly phosphate can be liberated from bottom sediments both in normal steady mode and explosive mode. This conclusion was widely accepted in the scientific community.

But the conclusion implied that use of a phosphate reduction program as the only way to limit algae bloom would have hardly any immediate success. But analysis with the Algae Bloom model suggests other tactics which could be effective and combination of tactics should be tailored to individual lakes.

2.4-2 The author feels that observation and experimentation are not emphasized in the MS/OR literature or in the training of its workers as much as experience would lead one to believe. As examples he has given some experiences with the US AirForce in early '50s which strengthens his belief.

He opines that observing actual operations as part of analysis process provides a required base for understanding what is going on in a problem situation. They can help to point out difficulties being encountered, suggest hypothesis and theories that may account for problems and offer evidence regarding the validity of the models built as part of problem solving process.

If a problem is in regard to a non-existing system or an operating system fulfills an important function that must continue, so that controlled experiments with are not possible-- one can build a theory about relevant phenomena and analyze the theory but numerical results obtained in this way clearly can be viewed with suspicion. Alternatively if a similar system exists, one can extrapolate from results with it to make estimates about the prospective system. Infact, administrative emergencies or an executive desire to try something new may cause the behavior of a system already in existence to change. The analyst may then be able to collect data useful for analyzing how the system would operate under changed circumstances or for identifying problems that might crop up under different operating regimes.

From his personal experiences he gives evidence to give substance to these remarks of his.

If data was used from one system to predict performance of another he believes that the parameter values form observing another similar system can be useful, and incorporating such estimates in a crude study can be better than not doing a study at all. Parameter values from one context to another cannot be expected to support detailed findings, but even crude findings are enough to provide indispensable information on which to base policy.

He has also analyzed the results of a continent wide Air Defense exercise. He says here that analysis must be carefully planned, and planning must begin early. Early work serves to put attention on the structure of the work and issues to be faced as well as other responsibilities.

Thus, in nutshell, the author views that skills involved in observation and experimentation are enumerous and should be part of the tool kit of many MS/OR analysts. He views that discriminating observation and carefully planned experimentation and analysis are central to MS/OR.

Observing actual operations and collection of data allow us to discern problems, develop hypotheses and validate models needing skill.

Similarly, accurate and complete data are required to estimate validity. Program evaluation brings together many of the issues of observation and experimentation.

Thus issues of scientific and professional craft related to observation and experimentation should occur important places in experience, literature and training of MS/OR workers.

2.4-3

(a) The author views that analysts do not believe that a model can be completely validated. He further opines that policy models can at best be invalidated. Thus the objective of validation or invalidation attempts is to increase the degree of confidence that the events obtained from the model will take place under conditions assumed. After trying all invalidation procedures, one will have a good understanding of strengths and weaknesses of the model and will be able to meet criticisms of omissions. Knowing the limitations of the model will enable one to express proper confidence on its results.

(b) Model Validity deals with correspondence of the model to the real world and related to pointing out all stated and implied assumptions, identification and inclusion of all decision variables and hypothesized relations among variables. Different assumptions are made and the analyst compares each assumption and hypothesis to the internal and external problem environments viewed by the decision maker and comments on the extent of divergence.

Data validity deals with raw and structured data, where structured data is manipulated raw data. Raw data validity is concerned with measurement problems and determining if the data is accurate, impartial and representative. Structured data validity needs review of each step of the manipulation and is a part of model verification.

Logical/mathematical validity deals with translating the model form into a numerical, computer process that produces solutions. There is no standard method to determine this. Approaches include comparing model outcomes with expected or historical results and a close scrutiny of the model form and its numerical representation on a flow chart.

Predictive validity is analyzing errors between actual and predicted outcomes for a model's components and relationships. Here one looks for errors and their magnitudes, why they exist and if how they can be corrected.

Operational validity attempts to assess the importance of errors found under technical validity. It must find out if the use of the model is appropriate for the observed and expected errors. It also deals with the fact whether the model can produce unacceptable answers for proper ranges of parameter values.

Dynamic validity is concerned with determining how the model will be maintained during its life cycle so it will continue to be an accepted representation of the real system. The two areas of interest thus are update and review.

(c) Sensitivity analysis plays an important role in testing the operational validity of a model. In this, values of model parameters are varied over some range of interest to determine if and how the recommended solution changes. If the solution is sensitive to certain parameter changes, the decision maker may want the model analysts to explore further or justify in detail values of these parameters. Sensitivity analysis also involves the relationship between small changes in parameter values and magnitude of related changes in outputs.

(d) Validating a model tests the agreement between behavior of the model and the real world system being modeled. Models of a non-existing system are the difficult to validate. Three concepts apply here : face validity or expert opinion, variable -parameter validity and sensitivity analysis and hypothesis validity. Though these concepts are applicable to all models, models of real systems can be subjected to further tests. Validity is measured by how well the real-system compares with model-generated data.The model is replicatively valid if it matches data acquired from the real system. It is predictively valid when it matches data before getting the data from the real system. A model is structurally valid if it reproduces the observed real system behavior as well as reflects the way in which real system works to produce this.

The author views that there is no validation methodology appropriate for all models. He says that a decision-aiding model can never be completely validated as there are never real data about the alternatives not implemented. Thus, analysts must be careful in devising, implementing, interpreting and reporting validation tests for their models.

(e) Basic validation steps have been cited in page 616 of the article.

2.5-1

(a) In late 1970s oil companies began to experience downward pressure on profitability due to rapid and continuing changes in external environment. Partially in response to these pressures Texaco's Computer Information Systems department developed an improved on-line interactive gasoline blending system called OMEGA. It was first installed in 1983 and is now used in all seven Texaco US refineries and in two foreign plants.

(b) Simple interactive user interface makes OMEGA easy to use. All input data can be entered by hand. OMEGA can interface with refinery data acquisition system. The user can access stock qualities, stock availabilities, blend specification and requirements, starting values and limits, optimization options, automatic stock selection, automatic blend specification and several other options.

Several features aid the user in performing planning functions. By choosing appropriate options user can obtain optimization options. User also has other options.

Each refinery uses different set of features depending on its availability of blending stocks. These vary depending on the configuration of the refinery and particular crudes being refined. Availability and easy use of OMEGA features has provided engineers and blenders with powerful and easy tool.

(c) OMEGA is constantly being updated and extended. It had to be modified to take into account EPA's regulation for a lead phase down for regular-leaded gasoline so that now OMEGA could be more accurate for these lower lead levels.

OMEGA is continuously modified to reflect changes in refinery operations. Differences in refineries required changes to the system.

When Texaco began installing OMEGA in their foreign refineries, additional changes had to be made to handle different requirements of different countries.

Improvements to OMEGA are needed to enable it to answer the new and unanticipated what-if questions often asked by refinery engineers.

(d) Each refinery uses OMEGA in varying degrees and for various purposes depending on their needs, complexity and configuration. Below the typical usage of the system is pointed out.

On a monthly basis, refineries use OMEGA to develop a gasoline blending plan for the month. The refinery planning model's projected blending stock volumes are input to OMEGA. The blending planner calculates 3 to 8 blends in a single OMEGA run. The refinery planning model's blend compositions are input into OMEGA as initial values. Once a reasonable blend is developed, the marketing department is contacted to discuss resulting grade splits. After marketing department does their job a finalized blending plan is developed for the month. The scheduler determines when each of the grades will be blended. All these work are done by using OMEGA.

(e) OMEGA contributes to overall profitability. To measure actual benefit, a method tried was comparing blend composition that blenders used with and without OMEGA. Here OMEGA achieved as much as 30 percent increase in profit. Average increase in profit is approximately 5 percent of gross gasoline revenue. If OMEGA is used to calculate blending recipes fewer blends fail to meet their quality specification. OMEGA's more reliable gasoline grade-split estimates provide significant aid to those developing marketing strategies and refinery production targets. OMEGA is used for what-if case studies performed for example for economic analysis of refinery improvement projects and analysis of how proposed Government regulations would affect Texaco. OMEGA's features have enabled Texaco with capacity to do things not possible with previous blending system, for example, to deal with mix stocks, consider new grades of gasoline, more control on inventory, etc. OMEGA's features make it easy and quick to explore new avenues of profitability for a refinery.

2.5-2

(a) Yellow Freight System, Inc. was founded in 1926 as a regional motor carrier serving the Mid-West. Today it is one of the largest motor carriers in the country. From a mixed operations in 1970s, Yellow now predominantly serves the less-than-truckload (LTL) portion of the freight market. The '80s were a difficult decade for the motor carrier industry. Deregulation made the way for tremendous opportunity for growth but also presented management with new and difficult challenges to manage these larger operations more efficiently than before. After 1980, motor carriers were forced to compete on price, which led to a lot of pressure to cut costs. The result was decrease in transportation rates. Between 1980 to 1990, transportation rates translated to a drop in real terms of 29%. In addition to real rate decreases, the shipping community in response to intense international competition, started to increase their expectation in service. For many shippers, Yellow Freight is a full partner in their total quality management programs. Another important component of the logistics system is timely delivery of freight. Service reliability is also critical. This heightened emphasis on service was a problem for some long-standing operating practices used by national LTL carriers. The effect of these pressures can be seen in the tremendous attrition the industry suffered. Out of top 20 revenue producing LTL carriers in 1979, only 6 are there today. In this period, Yellow Freight grew from 248 to 630 terminals. This growth has had the effect of creating an extremely large and complex operation. The large network also needs a greater degree of coordination.

In 1986, Yellow initiated a project to improve its ability to manage a complex system. Yellow was interested in using modern network method to simulate and optimize a large network. The project had a main goal-- improved service and service reliability through better management control of the network. This goal was supplemented by broader management objectives. There was also an expectation that improved planning would lead to higher productivity level and lower costs. Consequently, a project team was formed.

(b) The development effort at Yellow started with an existing model as a base and then were modified. The result of this effort was SYSNET. SYSNET is more than 80,000 lines of FORTRAN code for performing sophisticated optimizations using modern network tools. They developed an innovative, interactive optimization technology that puts human beings in the loop, placing sophisticated, up-to-date optimization methods in their hands. These methods were required in the development of a system that would handle the entire network without resorting to heuristic methods to decrease the size of the problem. As a result , user is able to analyze impacts of changes in the whole network in a simple but interactive fashion. Projects can be completed earlier new with greater precision. Decisions on shipment consolidations are now optimized taking into account the system effect of each decision.

Yellow uses SYSNET for two sets of applications:

-- main use is tactical load planning, which involves monthly planning and revision of set of instructions that govern handling and consolidation of shipments through the network.

-- the second set of applications involve longer range planning of the network itself. These problems cover the location and sizing of new facilities, and long range decisions that govern the flow of freight between terminals.

At Yellow SYSNET is more than just a piece of code. It embodies an entire planning methodology adopted by all levels of the company. From strategic planning studies communicated to high-level management to network routing instructions sent right to the field, SYSNET has become a comprehensive planning process that has allowed management to maintain control of a large complex operation. In addition, Yellow uses SYSNET as the central tool in the design and evaluation of projects of over $10 million in annual savings.

(c) The interactive aspects of the code proved important in two respects:

-- the user was needed to guide the search for changes in the network. For example, user may know that freight levels are in the rise in the Midwest or a particular breakbulk is facing problems with capacity. In other cases, user may know that current solution is a local minimum and a major change in the network is needed to achieve an overall improvement. A human being can easily point out these spatial patterns and test for promising configurations.

-- the second use proved critical to the adoption of the system and was the user's capability to accept and reject suggestions made by the computer. SYSNET displays suggested changes and allows the user to evaluate each one in terms of difficult to quantify factors. Also local factors, such as work rules or special operating practices that are not incorporated into the model can be accounted for by a knowledgeable user.

(d) For strategic planning, the outputs from SYSNET are a set of reports used to prepare management summaries on different options. SYSNET is also used on a operational basis to perform load planning. In this role, SYSNET is used to maintain a file that determines the actual routing of shipments through the service network. This file, which contains the load planning, is accessed directly by systems that are used by every terminal manager in the field. SYSNET'S control of load planning and its capability to communicate these instructions to the field is the most important accomplishment of the project.

(e) SYSNET's effect can be seen in four areas :

-- quality of planning practices and management culture

-- cost savings resulting directly from improvement in load planning

-- in analyzing projects
-- improved service to customers from more reliable transportation

Qualitative changes includes the following :

-- management had more control over network operations. SYSNET now allows managers to have direct control. The new load pattern closely controls the loading of directs and management can quickly change the load pattern in response to changing needs.

-- it could set realistic performance standards. SYSNET allowed Yellow to set direct loading standards based on anticipated freight levels, creating more realistic performance expectations.

-- planners can better understand the total system now. Yellow can now evaluate new projects and ideas based on their impact on the entire system

-- SYSNET allows managers to analyze projects formally before making decisions

-- with SYSNET managers can analyze new options quickly in response to changing situations

-- Analysts can now try new ideas on computers which ultimately leads to new ideas in the field

-- because of SYSNET, Yellow is more open now to use of new information technologies

-- the new system has reduced claims. SYSNET has had a substantial impact on management culture at Yellow

Performance improvement due to better load planning include :

A study was undertaken to estimate savings that could be attributed to SYSNET. Total cost savings for the system were estimated at over $7.3 million annually. Savings in breakbulk handling costs also increased.

Besides this, reducing shipments handled in the long run may bring down investments in fixed facilities.

SYSNET brought down the cost of routing trailers in part by identifying directs with lower transportation costs-- savings due to better routing of trailers were estimated to be $1 million annually.

Ongoing projects include :

Operations planning uses SYSNET to scrutinize various projects with a wide range from relocating breakbulks to realigning satellites with breakbulks. Using SYSNET, operational planning now completes over 200 projects per year, mostly on an informal, exploratory basis. SYSNET'S speed in evaluating different ideas is critical to this process.

In 1990, Yellow used SYSNET to identify over $10 million in annual savings from different projects. SYSNET improved the speed with which such analyses could be completed and expanded the scope of each project thus allowing Yellow to study system impacts with more precision than before. SYSNET thus has played a main role in identification, design and evaluation of these projects.

Improved service includes :

Savings from SYSNET are substantial compared to the cost of its development and implementation. Following the implementation of SYSNET management can be better focused on improved service.

Yellow continues to use SYSNET for a number of planning projects and to continuously monitor and improve the load planning system which is now used directly within linehaul operations group responsible for day-to-day management of flows through the system. In addition, Yellow is using SYSNET as a foundation to expand the use of optimization methods for the other aspects of its operations.

SYSNET is now very popular within the company for its capability to carry out accurate, comprehensive network planning projects.

2.6-1

(a) Implementing this major change in operations needed involvement and support of all levels of the company. Process started with acceptance of system with operation planning department. Operation planning was responsible for guiding the project and managing with close cooperation from the information services department and all aspects of the implementation. The systems acceptance was ma lot due to use of interactive optimization which gave users the support needed to optimize such a large network while simultaneously keeping them in close control of the entire process. Users could also analyze suggested changes to the network based on changes in flows and costs, which could be compared against actual field totals.

The next step was to validate the cost model. They were able to compare both total system costs and different subcategories against actual cost summaries for these categories. The individual cost categories within SYSNET consistently match corporate statistics within a few percent and total costs often match with 1 or 2 percent.

The validation of the cost model, both in totality and individual components, played a vital role in gaining upper management's acceptance. The interactive reports and features that convinced operations planning also played a strong role in winning support of top management. They ran sessions for upper management to demonstrate how SYSNET made suggestions and generated supporting reports to back-up the numbers. They also demonstrated how standard operating practices could be detrimental and why coordinating the entire network was important. By taking all these efforts, they gained the needed confidence of upper management required to support a field implementation.

(b) With the support of upper management, they were able to develop an implementation strategy. The controlled direct program changed operating philosophy so drastically that a single corporate-wide transition was viewed as not safe. In implementing SYSNET, Yellow made a systematic change in the way it loaded directs. SYSNET encourages a greater proportion of directs to be loaded onto breakbulks. It was not possible to change this operation methods so easily over the whole network. It was also difficult to do it in a piecemeal fashion. To deal with this problem, they developed a phased implementation strategy that started with smallest breakbulks in the system and went up to larger ones. Careful planning made sure that no breakbulk would be over capacity during the intermediate stages of the process. The entire implementation was so planned as to ensure that no breakbulk would find itself over capacity during the transition period.

(c) To communicate the new concept to terminal managers in the field involved three steps :

-- designing new support tools so that SYSNET routing instructions were easy to follow

-- training terminal managers and dock personnel to use these new system and most important

-- convincing terminal managers that the new approach was a good idea.

They developed two new support tools to assist field operations :

-- first was a set of reports that managers or dock supervisors could access from their local computer terminals which would give them immediate access to SYSNET load pattern.

-- second, was a revised shipment movement bill. This provides a very high level of control over the routing of individual shipments.

The Operations Planning department handled training by organizing series of visits to all 25 breakbulks. During each visit, the staff members explained the principles behind the controlled direct program, new reports and use of new routing directions. Follow-up was done by phone calls.

The most important task was to convince terminal managers of the logic behind the new operations strategy. Terminal managers needed to understand that they had to follow the load planning since it was designed to coordinate different parts of the system. They used examples to illustrate the effect their decisions could have on other terminals. Generally, people in the field accepted the principle that their decisions should be coordinated with those in the rest of the system

(d) Following the implementation of SYSNET, they developed a target that represented the expected number of directs that they should be loading based on the SYSNET plan. Yellow then measured terminal manager's performance based on how close they were to this target. After some period, it deemed compliance with the plan so good that it now measures terminal managers performance on other activities and Yellow continues to monitor compliance with the load plan informally. It then contacts managers that appear to be not in compliance to determine the reasons. In short, SYSNET has changed load planning from a decentralized process that depended on local management incentives to a centralized process that relies on monitoring and enforcement.

2.6-2

(a) The information processing industry has experienced several decades of sustained profitable growth. Recently, competition has intensified leading to quick advances in computer technology. This in turn leads to proliferation of both-end products and services. These trends are especially relevant for after-sales service. Maintaining a service parts logistic system to support products installed in the field is essential to competing in this industry.

Growth in both sales and scope of products offered has dramatically increased the number of spare parts that must be maintained. For IBM, the number of installed machines and the annual usage of spare parts have both increased. This growth has increased the dollar value of service inventories, which are used to maintain the very high levels of service expected by IBM's customers. IBM has developed an extensive multiple-echelon logistic structure to provide ready service for the large population of installed machines, which are distributed through the United States.

IBM developed a large and sophisticated inventory management system to provide customers with prompt and reliable service. A fast changing business environment and pressures to decrease investment in inventory led IBM to look for improvements in its control system.

In response to these new needs, IBM initiated the development of a new planning and control system for management of service parts. The result of this was the creation and implementation of a system called Optimizer.

(b) The complicating factors faced by the OR team are as follows :

- -- there are more than 15 million part-location combinations
- -- there are more than 50000 product-location combinations
- -- frequent updating (weekly) of system control parameters was a requirement in response to changes in the service environment and installed base
- -- success of the system is important to IBM's daily operations and so can have a major impact on its future sales and revenues
- -- employees could be expected to protest against any change since the existing control system was working and sophisticated and overall parts logistics problem was complex.

(c) The system developed in this phase had minimum interface to provide data inputs and multi-echelon algorithm without any improvements. Most of the big changes from the original design was in this phase.

They discovered that the echelon structure was in reality more complex than the one used in the analytic model. Consequently, they had to develop extensions to the demand pass-up methodology and incorporate them into the model.

The test was conducted in early 1986 and led to the finding that the value of the total inventory generated by the new system was smaller than expected. It was discovered that the problem was due to differences in criticality of parts. The algorithm made extensive use of inexpensive, non-functional parts to meet product-service objective. Another problem found out at this stage was the churn (instability) in the recommended stock levels every week. Although stock levels are expected to change periodically in response to changing failure rates and to changes in the installed base, it is desirable to keep the stock levels quasi-static in order to avoid logistic and supply problems. They developed control procedures and changed the model to take care of this problem.

(d) In this phase, they completed all functions required for implementation and developed a measurement system to monitor the field implementation test. After being done with the system coding for this phase, they conducted an extensive user acceptance test. Every program module was tested individually and jointly. Finally, a field implementation test went live on 7 machine types in early 1987. The working of the system fulfilled expectations. Scope of the field test was slowly expanded. Results were monitored on a weekly and then monthly basis by the measurement system.

(e) In this phase, they completed the development and installation of all the functions currently in place in Optimizer. The system was able to provide the specified service performance for all parts and locations. Improvements were made. User acceptance testing and integration of final system went smoothly. The project staging helped to sustain support for the project by demonstrating concrete progress throughout the implementation process. It also helped to eradicate problems in formulation and algorithm and programming bugs early. So very few problems occurred when the system went live in a national basis. The final Optimizer system for national implementation consisted of four major modules :

-- a forecasting system module
-- a data delivery system module
-- a decision system that solves multi-echelon stock control problem
-- the PIMS interface system

(f) The implementation of Optimizer yielded a variety of benefits :

-- a decrease in investment on inventory
-- improved services
-- enhanced flexibility in responding to changing service requirements
-- provision of a planning capability
-- improved understanding of the impact of parts operations
-- increased responsive of the control system
-- increased efficiency of NSD human resources
-- identifying the role of functional parts in providing product service is an example of benefits derived from implementation of Optimizer
-- ability to run Optimizer on a weekly basis has increased responsiveness of entire parts inventory system
-- for machines controlled by Optimizer, inventory analysts no longer have to specify parts stocking lists for each echelon in order to make sure that service objectives are attained. They can now focus on other critical management issues.

Optimizer thus has proved to be an extremely valuable planning and operating control tool.

2.7-1.

Answers will vary.

2.7-2.

Answers will vary.

2.7-3.

Answers will vary.

CHAPTER 3: INTRODUCTION TO LINEAR PROGRAMMING

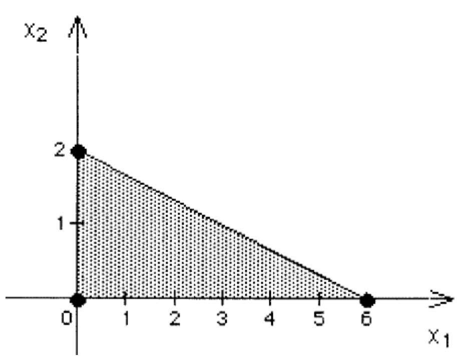
3.1-1.

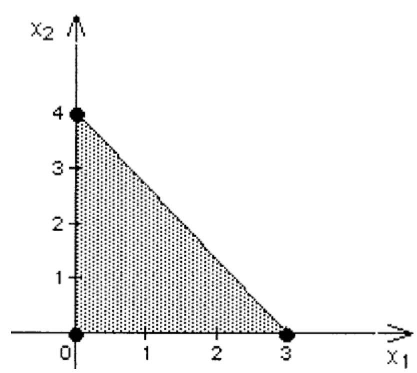
Swift & Company solved a series of LP problems to identify an optimal production schedule. The first in this series is the scheduling model, which generates a shift-level schedule for a 28-day horizon. The objective is to minimize the difference of the total cost and the revenue. The total cost includes the operating costs and the penalties for shortage and capacity violation. The constraints include carcass availability, production, inventory and demand balance equations, and limits on the production and inventory. The second LP problem solved is that of capable-to-promise models. This is basically the same LP as the first one, but excludes coproduct and inventory. The third type of LP problem arises from the available-to-promise models. The objective is to maximize the total available production subject to production and inventory balance equations.

As a result of this study, the key performance measure, namely the weekly percent-sold position has increased by 22%. The company can now allocate resources to the production of required products rather than wasting them. The inventory resulting from this approach is much lower than what it used to be before. Since the resources are used effectively to satisfy the demand, the production is sold out. The company does not need to offer discounts as often as before. The customers order earlier to make sure that they can get what they want by the time they want. This in turn allows Swift to operate even more efficiently. The temporary storage costs are reduced by 90%. The customers are now more satisfied with Swift. With this study, Swift gained a considerable competitive advantage. The monetary benefits in the first years was $12.74 million, including the increase in the profit from optimizing the product mix, the decrease in the cost of lost sales, in the frequency of discount offers and in the number of lost customers. The main nonfinancial benefits are the increased reliability and a good reputation in the business.

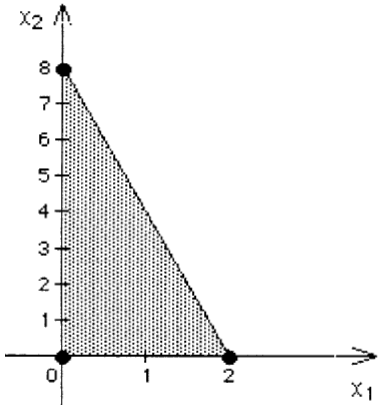
3.1-2.

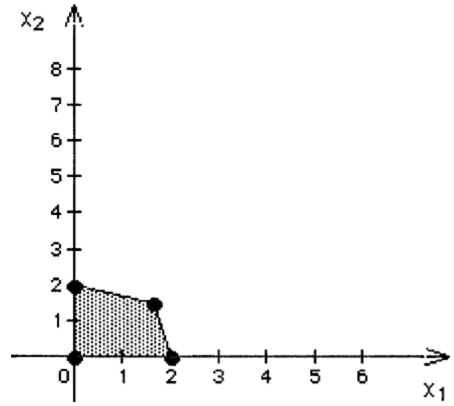
(a) (b)





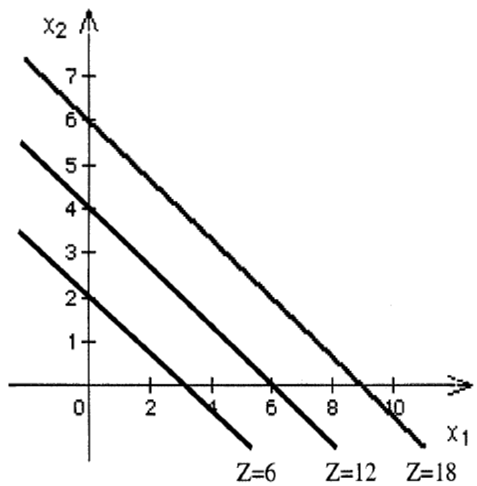
(c) (d)





3.1-3.

(a)



(b)

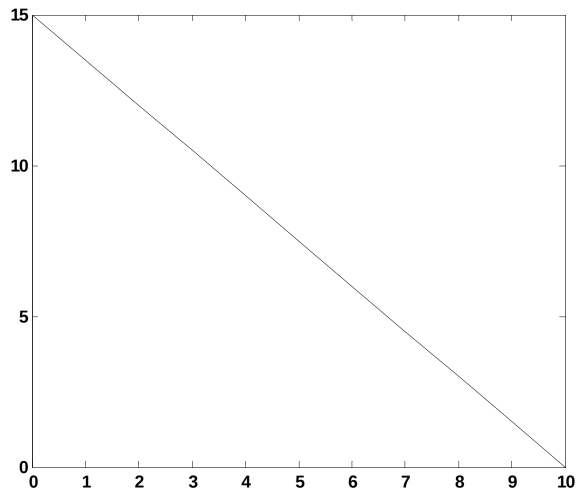
	Slope-Intercept Form	Slope	Intercept
$Z = 6$	$x_2 = -\frac{2}{3}x_1 + 2$	$-\frac{2}{3}$	2
$Z = 12$	$x_2 = -\frac{2}{3}x_1 + 4$	$-\frac{2}{3}$	4
$Z = 18$	$x_2 = -\frac{2}{3}x_1 + 6$	$-\frac{2}{3}$	6

3.1-4.

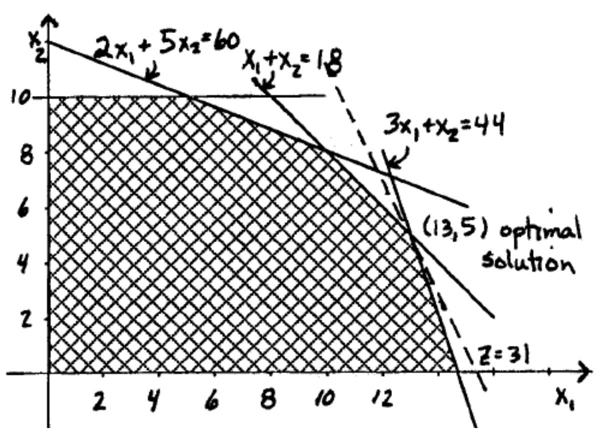
(a) $x_2 = -\frac{3}{2}x_1 + 15$

(b) The slope is $-3/2$, the intercept is 15.

(c)

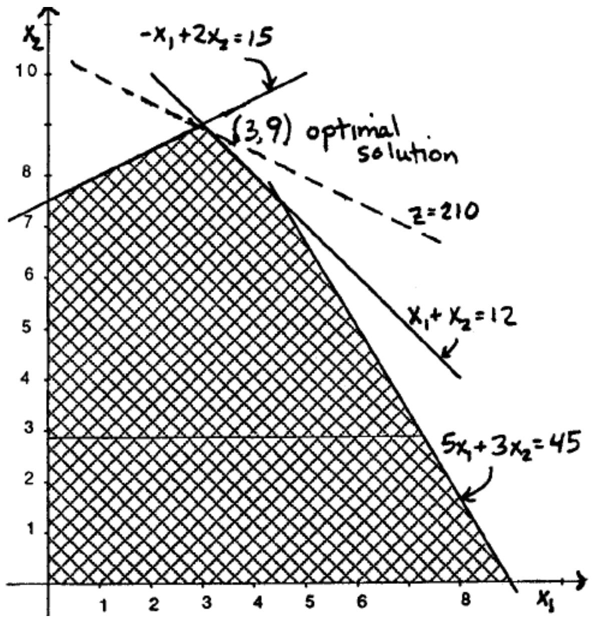


3.1-5.
Optimal Solution: $(x_1^*, x_2^*) = (13, 5)$ and $Z^* = 31$



3.1-6.

Optimal Solution: $(x_1^*, x_2^*) = (3, 9)$ and $Z^* = 210$



3.1-7.

(a) As in the Wyndor Glass Co. problem, we want to find the optimal levels of two activities that compete for limited resources. Let W be the number of wood-framed windows to produce and A be the number of aluminum-framed windows to produce. The data of the problem is summarized in the table below.

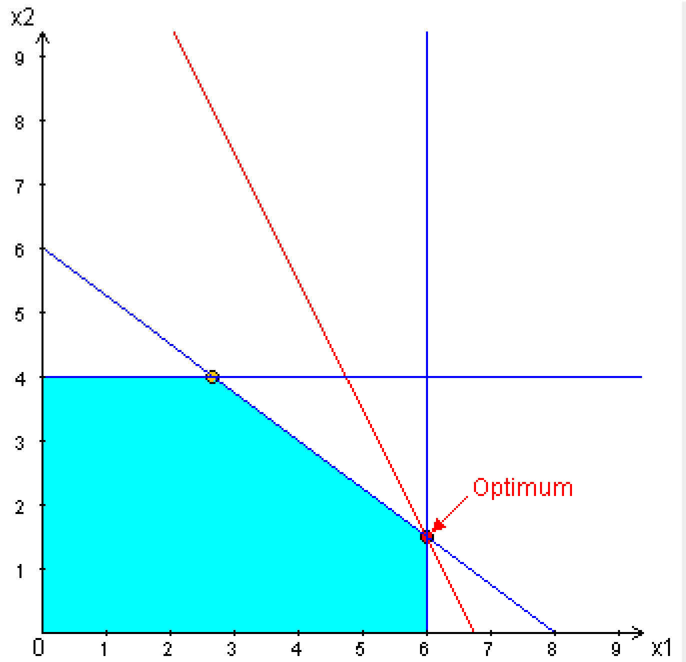
	Resource Usage per Unit of Activity		
Resource	**Wood-framed**	**Aluminum-framed**	**Available Amount**
Glass	6	8	48
Aluminum	0	1	4
Wood	1	0	6
Unit Profit	\$180	\$90	

(b) maximize $P = 180W + 90A$

subject to

$$\begin{aligned} 6W + 8A &\leq 48 \\ W &\leq 6 \\ A &\leq 4 \\ W, A &\geq 0 \end{aligned}$$

(c) Optimal Solution: $(W, A) = (x_1^*, x_2^*) = (6, 1.5)$ and $P^* = 1215$



(d) From Sensitivity Analysis in IOR Tutorial, the allowable range for the profit per wood-framed window is between 67.5 and infinity. As long as all the other parameters are fixed and the profit per wood-framed window is larger than \$67.5, the solution found in (c) stays optimal. Hence, when it is \$120 instead of \$180, it is still optimal to produce 6 wood-framed and 1.5 aluminum-framed windows and this results in a total profit of \$855. However, when it is decreased to \$60, the optimal solution is to make 2.67 wood-framed and 4 aluminum-framed windows. The total profit in this case is \$520.

(e)
$$\begin{array}{llrl} \text{maximize} & P = & 180W + 90A & \\ \text{subject to} & & 6W + 8A & \leq 48 \\ & & W & \leq 5 \\ & & A & \leq 4 \\ & & W, A & \geq 0 \end{array}$$

The optimal production schedule consists of 5 wood-framed and 2.25 aluminum-framed windows, with a total profit of \$1102.5.

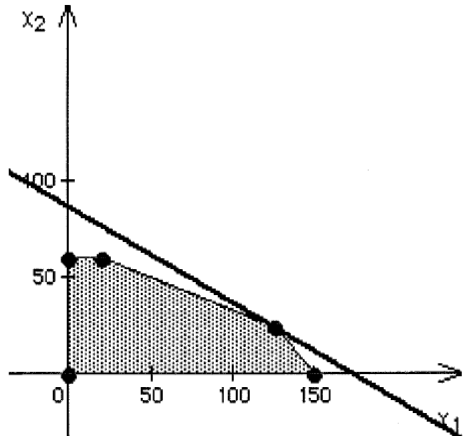
3.1-8.

(a) Let x_1 be the number of units of product 1 to produce and x_2 be the number of units of product 2 to produce. Then the problem can be formulated as follows:

$$\text{maximize } P = x_1 + 2x_2$$

$$\begin{array}{lrl} \text{subject to} & x_1 + 3x_2 & \leq 200 \\ & 2x_1 + 2x_2 & \leq 300 \\ & x_2 & \leq 60 \\ & x_1, x_2 & \geq 0 \end{array}$$

(b) Optimal Solution: $(x_1^*, x_2^*) = (125, 25)$ and $P^* = 175$

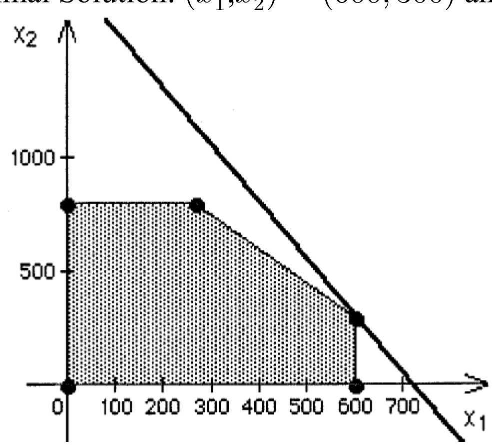


3.1-9.

(a) Let x_1 be the number of units on special risk insurance and x_2 be the number of units on mortgages.

$$\begin{array}{ll} \text{maximize} & z = 5x_1 + 2x_2 \\ \text{subject to} & 3x_1 + 2x_2 \leq 2400 \\ & x_2 \leq 800 \\ & 2x_1 \leq 1200 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

(b) Optimal Solution: $(x_1^*, x_2^*) = (600, 300)$ and $Z^* = 3600$



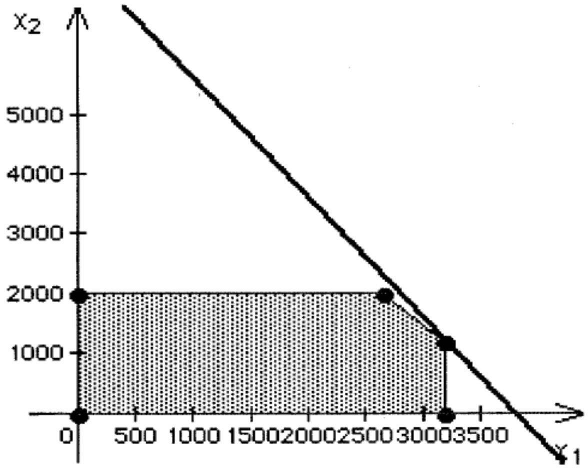
(c) The relevant two equations are $3x_1 + 2x_2 = 2400$ and $2x_1 = 1200$, so $x_1 = 600$ and $x_2 = \frac{1}{2}(2400 - 3x_1) = 300$, $z = 5x_1 + 2x_2 = 3600$

3.1-10.

(a) maximize $P = 0.8H + 0.3B$

subject to
$$\begin{array}{l} 0.1B \leq 200 \\ 0.25H \leq 800 \\ 3H + 2B \leq 12,000 \\ H, B \geq 0 \end{array}$$

(b) Optimal Solution: $(x_1^*, x_2^*) = (3200, 1200)$ and $P^* = 2920$



3.1-11.

(a) Let x_i be the number of units of product i produced for $i = 1, 2, 3$.

$$\begin{array}{ll} \text{maximize} & Z = 50x_1 + 20x_2 + 25x_3 \\ \text{subject to} & 9x_1 + 3x_2 + 5x_3 \leq 500 \\ & 5x_1 + 4x_2 \leq 350 \\ & 3x_1 + 2x_3 \leq 150 \\ & x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(b)

Solve Automatically by the Simplex Method:

Optimal Solution

Value of the
Objective Function: Z = 2904.7619

Variable	Value
X_1	26.1905
X_2	54.7619
X_3	20

Constraint	Slack or Surplus	Shadow Price
1	0	4.7619
2	0	1.42857
3	31.4286	0
4	0	1.19048

Sensitivity Analysis

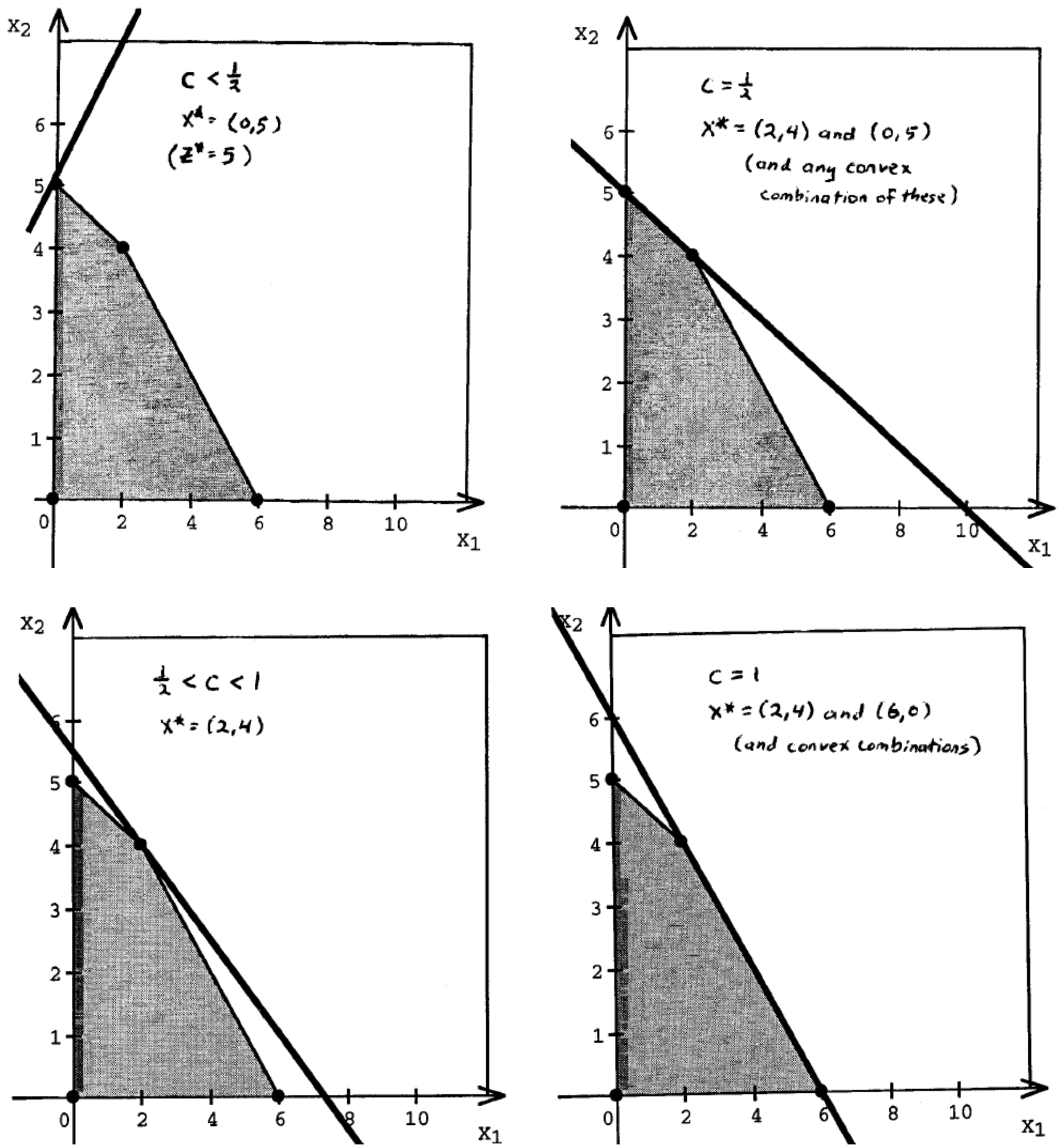
Objective Function Coefficient

Current Value	Allowable Range To Stay Optimal Minimum	Maximum
50	25	51.25
20	19	40
25	23.8095	$+\infty$

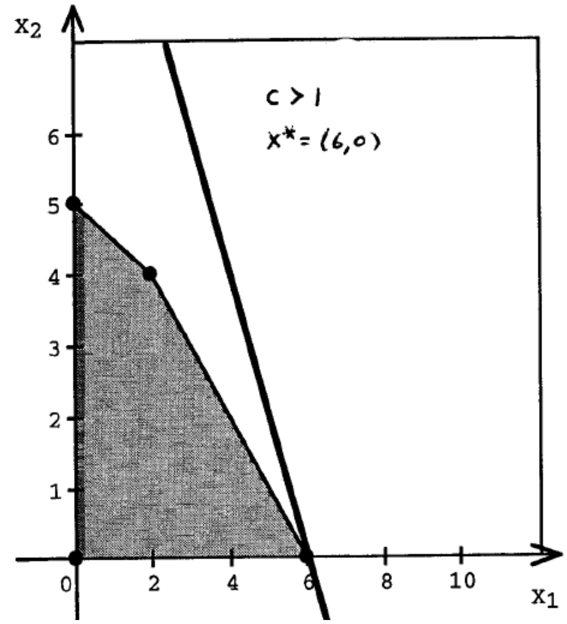
Right Hand Sides

Current Value	Allowable Range To Stay Feasible Minimum	Maximum
500	362.5	555
350	276.667	533.333
150	118.571	$+\infty$
20	0	47.5

3.1-12.

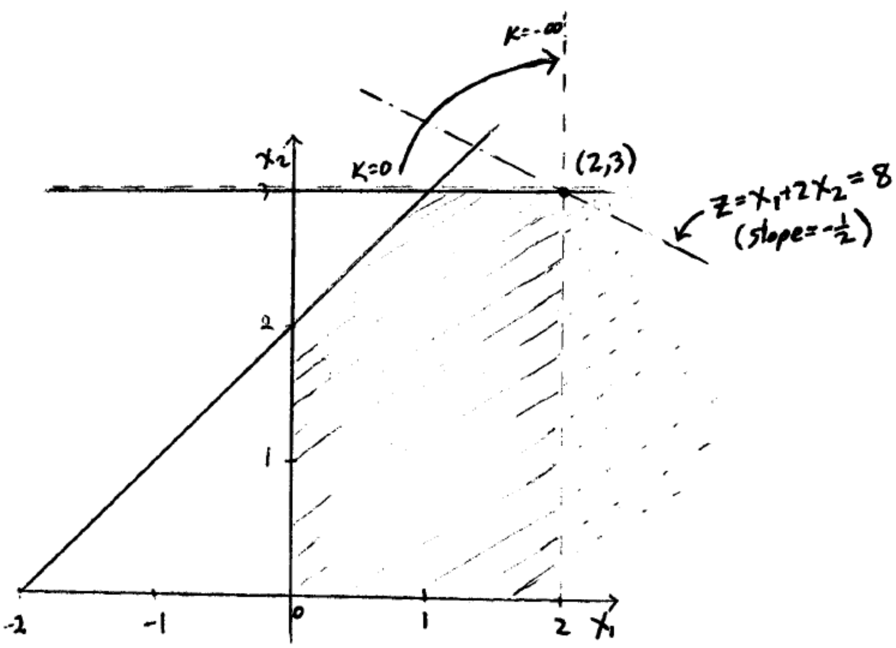


$c < \frac{1}{2}$
$x^* = (0,5)$
$(Z^* = 5)$
$c = \frac{1}{2}$
$x^* = (2,4)$ and $(0,5)$
(and any convex combination of these)
$\frac{1}{2} < c < 1$
$x^* = (2,4)$
$c = 1$
$x^* = (2,4)$ and $(6,0)$
(and convex combinations)



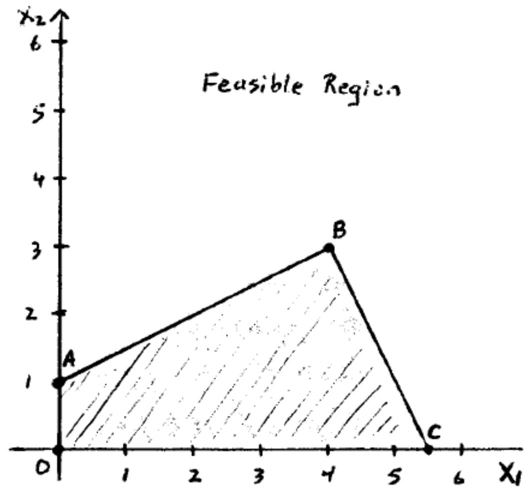
3.1-13.

First note that $(2,3)$ satisfies the three constraints, i.e., $(2,3)$ is always feasible for any value of k. Moreover, the third constraint is always binding at $(2,3)$, $kx_1 + x_2 = 2k + 3$. To check if $(2,3)$ is optimal, observe that changing k simply rotates the line that always passes through $(2,3)$. Rewriting this equation as $x_2 = -kx_1 + (2k+3)$, we see that the slope of the line is $-k$, and therefore, the slope ranges from 0 to $-\infty$.



As we can see, $(2,3)$ is optimal as long as the slope of the third constraint is less than the slope of the objective line, which is $-\frac{1}{2}$. If $k < \frac{1}{2}$, then we can increase the objective by traveling along the third constraint to the point $(2 + \frac{3}{k}, 0)$, which has an objective value of $2 + \frac{3}{k} > 8$ when $k < \frac{1}{2}$. For $k \geq \frac{1}{2}$, $(2,3)$ is optimal.

3.1-14.



Case 1: $c_2 = 0$ (vertical objective line)

If $c_1 > 0$, the objective value increases as x_1 increases, so $x^* = (\frac{11}{2}, 0)$, point C.

If $c_1 < 0$, the opposite is true so that all the points on the line from $(0,0)$ to $(0,1)$, line $\overline{OA}$, are optimal.

If $c_1 = 0$, the objective function is $0x_1 + 0x_2 = 0$ and every feasible point is optimal.

Case 2: $c_2 > 0$ (objective line with slope $-\frac{c_1}{c_2}$)

If $-\frac{c_1}{c_2} > \frac{1}{2}$ $x^* = (0,1)$, point A.

If $-\frac{c_1}{c_2} < -2$ $x^* = (\frac{11}{2}, 0)$, point C.

If $\frac{1}{2} > -\frac{c_1}{c_2} > -2$ $x^* = (4,3)$, point B.

If $-\frac{c_1}{c_2} = \frac{1}{2}$, any point on the line $\overline{AB}$ is optimal. Similarly, if $-\frac{c_1}{c_2} = -2$, any point on the line $\overline{BC}$ is optimal.

Case 3: $c_2 < 0$ (objective line with slope $-\frac{c_1}{c_2}$, objective value increases as the line is shifted down)

If $-\frac{c_1}{c_2} > 0$, i.e., $c_1 > 0$, $x^* = (\frac{11}{2}, 0)$, point C.

If $-\frac{c_1}{c_2} < 0$, i.e., $c_1 < 0$, $x^* = (0,0)$, point O.

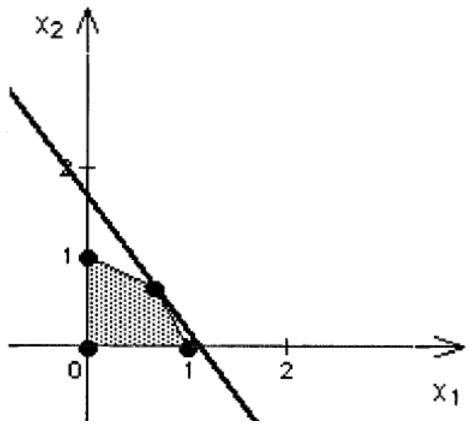
If $-\frac{c_1}{c_2} = 0$, i.e., $c_1 = 0$, x^* is any point on the line $\overline{OC}$.

3.2-1.

(a) maximize $P = 3A + 2B$

$$\begin{aligned} \text{subject to} \quad 2A + B &\leq 2 \\ A + 2B &\leq 2 \\ 3A + 3B &\leq 4 \\ A, B &\geq 0 \end{aligned}$$

(b) Optimal Solution: $(A, B) = (x_1^*, x_2^*) = (2/3, 2/3)$ and $P^* = 3.33$



(c) We have to solve $2A + B = 2$ and $A + 2B = 2$. By subtracting the second equation from the first one, we obtain $A - B = 0$, so $A = B$. Plugging this in the first equation, we get $2 = 2A + B = 3A$, hence $A = B = 2/3$.

3.2-2.

(a) TRUE (e.g., maximize $z = -x_1 + 4x_2$)

(b) TRUE (e.g., maximize $z = -x_1 + 3x_2$)

(c) FALSE (e.g., maximize $z = -x_1 - x_2$)

3.2-3.

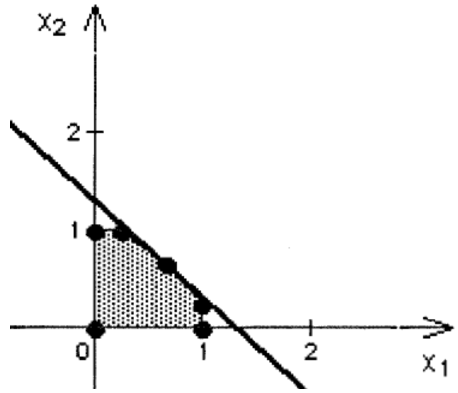
(a) As in the Wyndor Glass Co. problem, we want to find the optimal levels of two activities that compete for limited resources. Let x_1 and x_2 be the fraction purchased of the partnership in the first and second friends venture respectively.

	Resource Usage per Unit of Activity		
Resource	**1**	**2**	**Available Amount**
Fraction of partnership in 1st	1	0	1
Fraction of partnership in 2nd	0	1	1
Money	\$5000	\$4000	\$6000
Summer work hours	400	500	600
Unit Profit	\$4500	\$4500	

(b) maximize $P = 4500x_1 + 4500x_2$

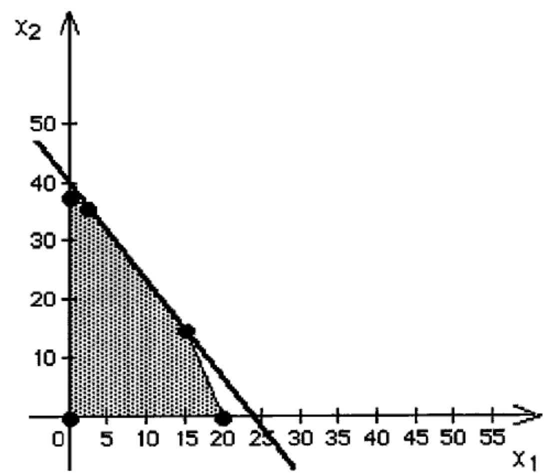
subject to
$$\begin{aligned} x_1 \quad\quad\quad\quad &\leq 1 \\ x_2 &\leq 1 \\ 5000x_1 + 4000x_2 &\leq 6000 \\ 400x_1 + \ \ 500x_2 &\leq 600 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(c) Optimal Solution: $(x_1^*, x_2^*) = (2/3, 2/3)$ and $P^* = 6000$

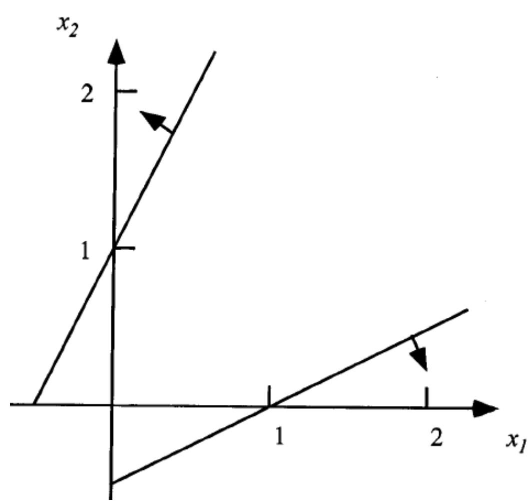


3.2-4.

Optimal Solutions: $(x_1^*, x_2^*) = (15, 15),\ (2.5, 35.833)$ and all points lying on the line connecting these two points, $Z^* = 12,000$

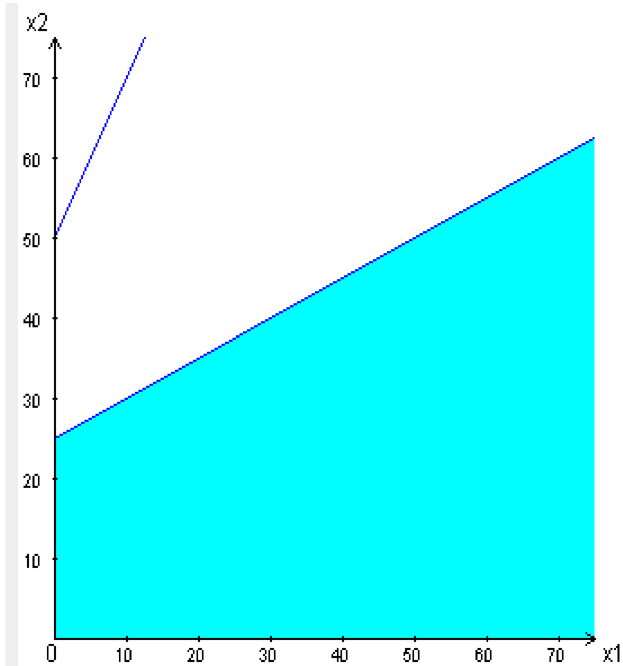


3.2-5.

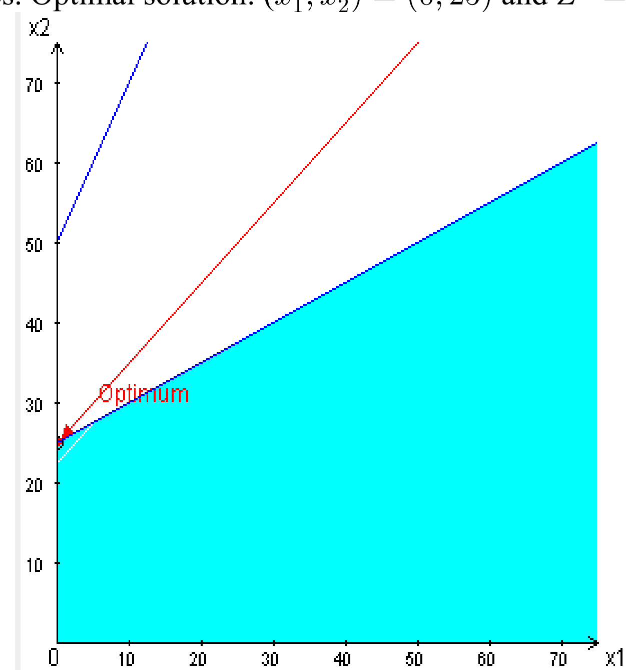


3.2-6.

(a)



(b) Yes. Optimal solution: $(x_1^*, x_2^*) = (0, 25)$ and $Z^* = 25$



(c) No. The objective function value rises as the objective line is slid to the right and since this can be done forever, there is no optimal solution.

(d) No, if there is no optimal solution even though there are feasible solutions, it means that the objective value can be made arbitrarily large. Such a case may arise if the data of the problem are not accurately determined. The objective coefficients may be chosen incorrectly or one or more constraints might have been ignored.

3.3-1.

Proportionality: It is fair to assume that the amount of work and money spent and the profit earned are directly proportional to the fraction of partnership purchased in either venture.

Additivity: The profit as well as time and money requirements for one venture should not affect neither the profit nor time and money requirements of the other venture. This assumption is reasonably satisfied.

Divisibility: Because both friends will allow purchase of any fraction of a full partnership, divisibility is a reasonable assumption.

Certainty: Because we do not know how accurate the profit estimates are, this is a more doubtful assumption. Sensitivity analysis should be done to take this into account.

3.3-2.

Proportionality: If either variable is fixed, the objective value grows proportionally to the increase in the other variable, so proportionality is reasonable.

Additivity: It is not a reasonable assumption, since the activities interact with each other. For example, the objective value at $(1,1)$ is not equal to the sum of the objective values at $(0,1)$ and $(1,0)$

Divisibility: It is not justified, since activity levels are not allowed to be fractional.

Certainty: It is reasonable, since the data provided is accurate.

3.4-1.

In this study, linear programming is used to improve prostate cancer treatments. The treatment planning problem is formulated as an MIP problem. The variables consist of binary variables that represent whether seeds were placed in a location or not and the continuous variables that denote the deviation of received dose from desired dose. The constraints involve the bounds on the dose to each anatomical structure and various physical constraints. Two models were studied. The first model aims at finding the maximum feasible subsystem with the binary variables while the second one minimizes a weighted sum of the dose deviations with the continuous variables.

With the new system, hundreds of millions of dollars are saved and treatment outcomes have been more reliable. The side effects of the treatment are considerably reduced and as a result of this, postoperation costs decreased. Since planning can now be done just before the operation, pretreatment costs decreased as well. The number of seeds required is reduced, so is the cost of procuring them. Both the quality of care and the quality of life after the operation are improved. The automated computerized system significantly eliminates the variability in quality. Moreover, the speed of the system allows the clinicians to efficiently handle disruptions.

3.4-2.

United Airlines used linear programming approach for scheduling. The purpose of this study was "to determine the needs for increased manpower, to identify excess manpower for reallocation, to reduce the time required for preparing schedules, to make manpower allocation more day- and time-sensitive, and to quantify the costs associated with scheduling" [p. 42]. The new system consisted of a mixed integer linear programming model, a continuous linear programming model, a heuristic rounding routine and report writer, and a network assignment model. The mixed integer LP model determines the times at which shifts can start. These are inputs to the continuous LP model, which, in turn, returns monthly schedules that minimize the labor costs. The constraints include employee and operating preferences. The solution is then rounded heuristically to obtain the final schedule.

"Benefits it has provided include significant labor cost savings, improved customer service, improved employee schedules, and quantified manpower planning and evaluation" [p. 48]. As a consequence of this, the revenues increased. The yearly savings in direct salary and benefit costs total to $6 million. "Unquantified capital benefits include additional revenue generated by improved service, benefits from the use of SMPS in contract negotiations, savings from reduced support staff requirements, savings from reduced manual scheduling efforts, cost reductions from additional smaller work groups, and reduced training requirements" [p. 48].

3.4-3.

(a) Proportionality: OK, since beam effects on tissue types are proportional to beam strength.

Additivity: OK, since effects from multiple beams are additive.

Divisibility: OK, since beam strength can be fractional.

Certainty: Due to the complicated analysis required to estimate the data about radiation absorption in different tissue types, sensitivity analysis should be employed.

(b) Proportionality: OK, provided there is no setup cost associated with planting a crop.

Additivity: OK, as long as crops do not interact.

Divisibility: OK, since acres are divisible.

Certainty: OK, since the data can be accurately obtained.

(c) Proportionality: OK, setup costs were considered.

Additivity: OK, since there is no interaction.

Divisibility: OK, since methods can be assigned fractional levels.

Certainty: Data is hard to estimate, it could easily be uncertain, so sensitivity analysis is useful.

3.4-4.

(a) Reclaiming solid wastes

Proportionality: The amalgamation and treatment costs are unlikely to be proportional. They are more likely to involve setup costs, e.g., treating 1,000 lbs. of material does not cost the same as treating 10 lbs. of material 100 times.

Additivity: OK, although it is possible to have some interaction between treatments of materials, e.g., if A is treated after B, the machines do not need to be cleaned out.

Divisibility: OK, unless materials can only be bought or sold in batches, say, of 100 lbs.

Certainty: The selling/buying prices may change. The treatment and amalgamation costs are, most likely, crude estimates and may change.

(b) Personnel scheduling

Proportionality: OK, although some costs need not be proportional to the number of agents hired, e.g., benefits and working space.

Additivity: OK, although some costs may not be additive.

Divisibility: One cannot hire a fraction of an agent.

Certainty: The minimum number of agents needed may be uncertain. For example, 45 agents may be sufficient rather than 48 for a nominal fee. Another uncertainty is whether an agent does the same amount of work in every shift.

(c) Distributing goods through a distribution network

Proportionality: There is probably a setup cost for delivery, e.g., delivering 50 units one by one does probably cost much more than delivering all together at once.

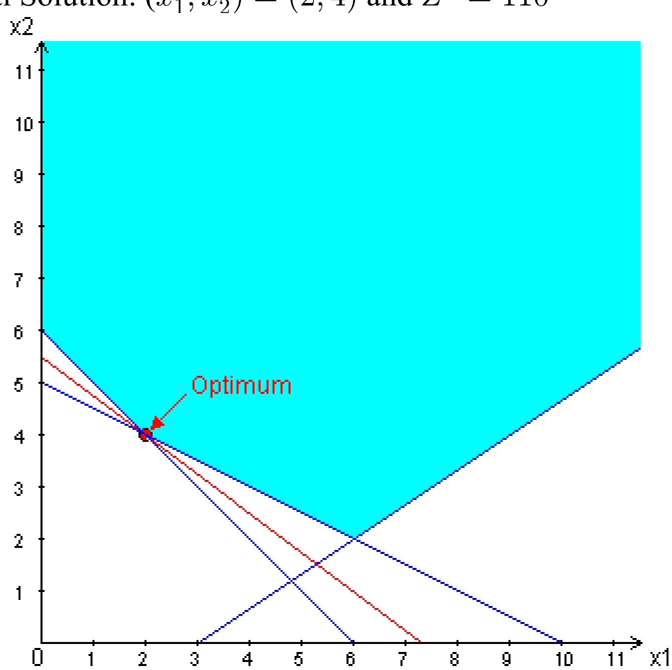
Additivity: OK, although it is possible to have two routes that can be combined to provide lower costs, e.g., $x_{\text{F2-DC}} = x_{\text{DC-W2}} = 50$, but the truck may be able to deliver 50 units directly from F2 to W2 without stopping at DC and hence saving some money. Another question is whether F1 and F2 produce equivalent units.

Divisibility: One cannot deliver a fraction of a unit.

Certainty: The shipping costs are probably approximations and are subject to change. The amounts produced may change as well.. Even the capacities may depend on available daily trucking force, weather and various other factors. Sensitivity analysis should be done to see the effects of uncertainty.

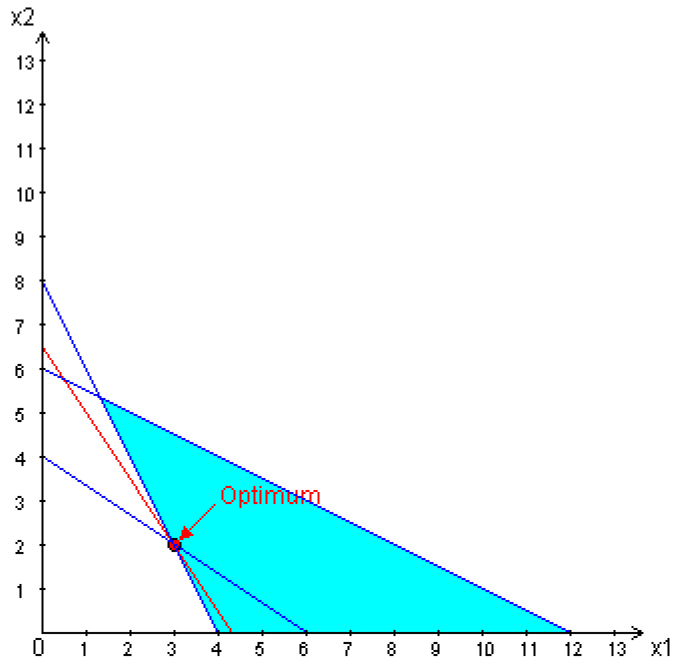
3.4-5.

Optimal Solution: $(x_1^*, x_2^*) = (2, 4)$ and $Z^* = 110$



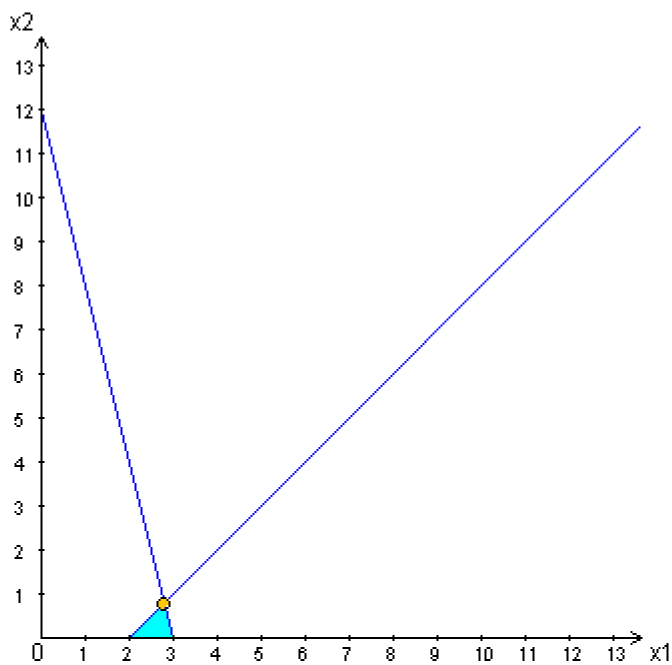
3.4-6.

Optimal Solution: $(x_1^*, x_2^*) = (3, 2)$ and $Z^* = 13$



3.4-7.

The feasible region can be represented as follows:

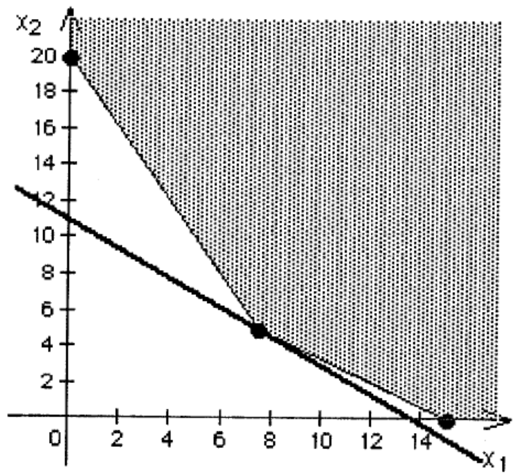


Given $c_2 = 2 > 0$, various cases that may arise are summarized in the following table:

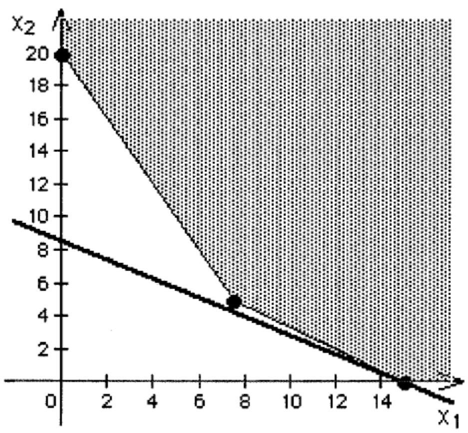
c_1	slope $= -\frac{c_1}{c_2}$	optimal solution (x_1^*, x_2^*)
$c_1 < -2$	$1 < -\frac{c_1}{c_2}$	$(2, 0)$
$c_1 = -2$	$-\frac{c_1}{c_2} = 1$	$(2, 0)$, $\left(\frac{14}{5}, \frac{4}{5}\right)$ and all points on the line connecting these two
$-2 < c_1 < 24$	$-12 < -\frac{c_1}{c_2} < 1$	$\left(\frac{14}{5}, \frac{4}{5}\right)$
$c_1 = 24$	$-\frac{c_1}{c_2} = -12$	$\left(\frac{14}{5}, \frac{4}{5}\right)$, $(3, 0)$ and all points on the line connecting these two
$24 < c_1$	$-\frac{c_1}{c_2} < -12$	$(3, 0)$

3.4-8.

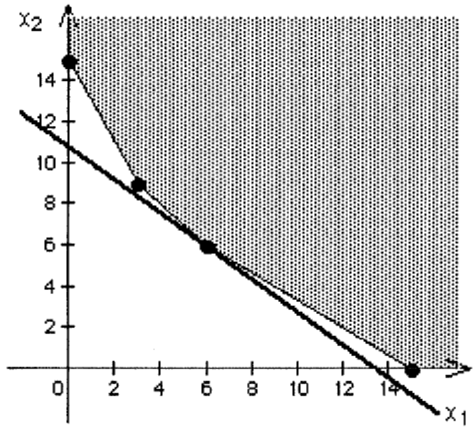
(a) Optimal Solution: $(x_1^*, x_2^*) = \left(7\frac{1}{2}, 5\right)$ and $C^* = 550$



(b) Optimal Solution: $(x_1^*, x_2^*) = (15, 0)$ and $C^* = 600$



(c) Optimal Solution: $(x_1^*, x_2^*) = (6, 6)$ and $C^* = 540$

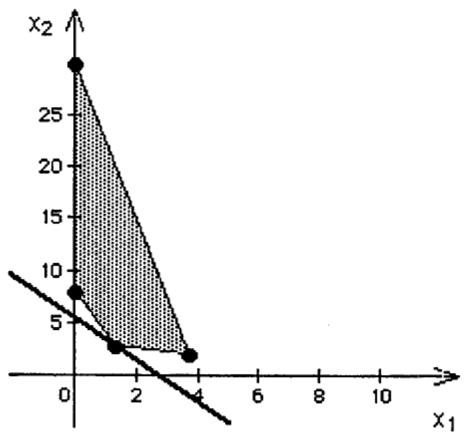


3.4-9.

(a) minimize $C = 4S + 2P$

$$\begin{aligned} \text{subject to} \quad 5S + 15P &\geq 50 \\ 20S + 5P &\geq 40 \\ 15S + 2P &\leq 60 \\ S, P &\geq 0 \end{aligned}$$

(b) Optimal Solution: $(S, P) = (x_1^*, x_2^*) = (1.3, 2.9)$ and $C^* = 10.91$



(c)

	Contribution Per Unit Steak	Potato	Totals		Level
Carbohydrate	5	15	50	≥	50
Protein	20	5	40	≥	40
Fat	15	2	24.91	≤	60
Unit Cost	4	2	**$ 10.91**		
Solution	**1.3**	**2.9**			

3.4-10.

(a) Let x_{ij} be the amount of space leased for $j = 1, \ldots, 6 - i$ months in month $i = 1, \ldots, 5$.

$$
\begin{aligned}
\text{minimize} \quad & C = 650(x_{11} + x_{21} + x_{31} + x_{41} + x_{51}) \\
& \quad + 1000(x_{12} + x_{22} + x_{32} + x_{42}) + 1350(x_{13} + x_{23} + x_{33}) \\
& \quad + 1600(x_{14} + x_{24}) + 1900x_{15} \\
\text{subject to} \quad & x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \geq 30,000 \\
& x_{12} + x_{13} + x_{14} + x_{15} + x_{21} + x_{22} + x_{23} + x_{24} \geq 20,000 \\
& x_{13} + x_{14} + x_{15} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} \geq 40,000 \\
& x_{14} + x_{15} + x_{23} + x_{24} + x_{32} + x_{33} + x_{41} + x_{42} \geq 10,000 \\
& x_{15} + x_{24} + x_{33} + x_{42} + x_{51} \geq 50,000 \\
& x_{ij} \geq 0,\ j = 1, \ldots, 6 - i \text{ and } i = 1, \ldots, 5
\end{aligned}
$$

(b)

	A	B	C	D	E	F	G	H	I
1		Contribution Toward Required Amount							
2	Month	1-1	1-2	1-3	1-4	1-5	2-1	2-2	2-3
3	1	1	1	1	1	1			
4	2		1	1	1	1	1	1	1
5	3			1	1	1		1	1
6	4				1	1			1
7	5					1			
8	Unit Cost	$ 650	$ 1,000	$ 1,350	$ 1,600	$ 1,900	$ 650	$ 1,000	$ 1,350
9	Solution	**0**	**0**	**0**	**0**	**30000**	**0**	**0**	**0**

	J	K	L	M	N	O	P	Q	R	S
1	Contribution Toward Required Amount									Resource
2	2-4	3-1	3-2	3-3	4-1	4-2	5-1	Totals		Available
3								30000	≥	30000
4	1							30000	≥	20000
5	1	1	1	1				40000	≥	40000
6	1		1	1	1	1		30000	≥	10000
7	1			1		1	1	50000	≥	50000
8	$ 1,600	$ 650	$ 1,000	$ 1,350	$ 650	$ 1,000	$ 650	**76499994**		
9	**0**	**10000**	**0**	**0**	**0**	**0**	**20000**			

Data cells:	B3:P8 and S3:S7
Changing cells:	B9:P9
Target cell:	Q8
Output cells:	Q3:Q7

	Q
3	=SUMPRODUCT(B3:P3,B9:P9)
4	=SUMPRODUCT(B4:P4,B9:P9)
5	=SUMPRODUCT(B5:P5,B9:P9)
6	=SUMPRODUCT(B6:P6,B9:P9)
7	=SUMPRODUCT(B7:P7,B9:P9)
8	**=SUMPRODUCT(B8:P8,B9:P9)**

3.4-11.

(a) Let f_1 = number of full-time consultants working the morning shift (8 a.m.-4 p.m.),
f_2 = number of full-time consultants working the afternoon shift (Noon-8 p.m.),
f_3 = number of full-time consultants working the evening shift (4 p.m.-midnight),
p_1 = number of part-time consultants working the first shift (8 a.m.-noon),
p_2 = number of part-time consultants working the second shift (Noon-4 p.m.),
p_3 = number of part-time consultants working the third shift (4 p.m.-8 p.m.),
p_4 = number of part-time consultants working the fourth shift (8 p.m.-midnight).

minimize $C = (40 \times 8)(f_1 + f_2 + f_3) + (30 \times 4)(p_1 + p_2 + p_3 + p_4)$

subject to
$$\begin{aligned}
&f_1 + p_1 \geq 4\\
&f_1 + f_2 + p_2 \geq 8\\
&f_2 + f_3 + p_3 \geq 10\\
&f_3 + p_4 \geq 6\\
&f_1 \geq 2p_1\\
&f_1 + f_2 \geq 2p_2\\
&f_2 + f_3 \geq 2p_3\\
&f_3 \geq 2p_4\\
&f_1, f_2, f_3, p_1, p_2, p_3, p_4 \geq 0
\end{aligned}$$

(b)

										Minimum
Time of Day	FT1	FT2	FT3	PT1	PT2	PT3	PT4	Total		Required
8 a.m. - Noon	1	0	0	1	0	0	0	4	≥	4
	1	0	0	-2	0	0	0	0	≥	0
Noon - 4 p.m.	1	1	0	0	1	0	0	8	≥	8
	1	1	0	0	-2	0	0	0	≥	0
4 p.m. - 8 p.m.	0	1	1	0	0	1	0	10	≥	10
	0	1	1	0	0	-2	0	0	≥	0
8 p.m. - Midnight	0	0	1	0	0	0	1	6	≥	6
	0	0	1	0	0	0	-2	0	≥	0
Unit Cost	$320	$320	$320	$120	$120	$120	$120	$4,107		
Solution	2.6667	2.6667	4	1.3333	2.6667	3.3333	2			

Note that the optimal solution has fractional components. If the number of consultants have to be integer, then the problem is an integer programming problem and the solution is $(3, 3, 4, 1, 2, 3, 2)$ with cost $\$4,160$.

3.4-12.

(a) Let x_{ij} be the number of units shipped from factory $i = 1, 2$ to customer $j = 1, 2, 3$.

minimize $C = 600x_{11} + 800x_{12} + 700x_{13} + 400x_{21} + 900x_{22} + 600x_{23}$

subject to
$$\begin{aligned}
&x_{11} + x_{12} + x_{13} = 400\\
&x_{21} + x_{22} + x_{23} = 500\\
&x_{11} + x_{21} = 300\\
&x_{12} + x_{22} = 200\\
&x_{13} + x_{23} = 400
\end{aligned}$$

and $x_{ij} \geq 0$, $i = 1, 2$ and $j = 1, 2, 3$

(b)

									Required
	F1-C1	F1-C2	F1-C3	F2-C1	F2-C2	F2-C3	Total		Amount
Factory 1	1	1	1	0	0	0	400	=	400
Factory 2	0	0	0	1	1	1	500	=	500
Customer 1	1	0	0	1	0	0	300	=	300
Customer 2	0	1	0	0	1	0	200	=	200
Customer 3	0	0	1	0	0	1	400	=	400
Unit Cost	$600	$800	$700	$400	$900	$600	$540,000		
Solution	0	200	200	300	0	200			

3.4-13.

(a)
$$A_1 + B_1 + R_1 = 60,000$$
$$A_2 + B_2 + C_2 + R_2 = R_1$$
$$A_3 + B_3 + R_3 = R_2 + 1.40A_1$$
$$A_4 + R_4 = R_3 + 1.40A_2 + 1.70B_1$$
$$D_5 + R_5 = R_4 + 1.40A_3 + 1.70B_2$$

(b) maximize $P = 1.40A_4 + 1.70B_3 + 1.90C_2 + 1.30D_5 + R_5$

subject to
$$A_1 + B_1 + R_1 = 60,000$$
$$A_2 + B_2 + C_2 - R_1 + R_2 = 0$$
$$-1.40A_1 + A_3 + B_3 - R_2 + R_3 = 0$$
$$-1.40A_2 + A_4 - 1.70B_1 - R_3 + R_4 = 0$$
$$-1.40A_3 - 1.70B_2 + D_5 - R_4 + R_5 = 0$$

and $A_t, B_t, C_t, D_t, R_t \geq 0$

(c)

	Contribution Toward Required Amount Per Unit Investment									Remainder							
Year	A1	A2	A3	A4	B1	B2	B3	C2	D5	R1	R2	R3	R4	R5	Totals		Required Amount
1	1	0	0	0	1	0	0	0	0	1	0	0	0	0	60000	=	60000
2	0	1	0	0	0	1	0	1	0	-1	1	0	0	0	0	=	0
3	-1.4	0	1	0	0	0	1	0	0	0	-1	1	0	0	0	=	0
4	0	-1.4	0	1	-1.7	0	0	0	0	0	0	-1	1	0	-1.33577E-12	=	0
5	0	0	-1.4	0	0	-1.7	0	0	1	0	0	0	-1	1	1.45519E-11	=	0
Unit Profit	0	0	0	1.4	0	0	1.7	1.9	1.3	0	0	0	0	1	**$ 152,880**		
Solution	**60000**	**0**	**84000**	**0**	**0**	**0**	**0**	**0**	**117600**	**0**	**0**	**0**	**0**	**0**			

3.4-14.

(a) Let x_i be the amount of Alloy i used for $i = 1, 2, 3, 4, 5$.

minimize $C = 77x_1 + 70x_2 + 88x_3 + 84x_4 + 94x_5$

subject to
$$60x_1 + 25x_2 + 45x_3 + 20x_4 + 50x_5 = 40$$
$$10x_1 + 15x_2 + 45x_3 + 50x_4 + 40x_5 = 35$$
$$30x_1 + 60x_2 + 10x_3 + 30x_4 + 10x_5 = 25$$
$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$

and $x_1, x_2, x_3, x_4, x_5 \geq 0$

(b)

	Contribution Toward Required Amount							Required
Requirement	Alloy 1	Alloy 2	Alloy 3	Alloy 4	Alloy 5	Total		Amount
% tin	60	25	45	20	50	40	=	40
% zinc	10	15	45	50	45	35	=	35
% lead	30	60	10	30	10	25	=	25
%total	1	1	1	1	1	1	=	1
Unit Cost	$77	$70	$88	$84	$94	$82.43		
Solution	0.0435	0.2826	0.6739	0	0			

3.4-15.

(a) Let x_{ij} be the number of tons of cargo type $i = 1, 2, 3, 4$ stowed in compartment $j =$ F (front), C (center), B (back).

$$\begin{aligned}
\text{maximize} \quad & P = 320(x_{1F} + x_{1C} + x_{1B}) + 400(x_{2F} + x_{2C} + x_{2B}) \\
& \quad + 360(x_{3F} + x_{3C} + x_{3B}) + 290(x_{4F} + x_{4C} + x_{4B}) \\
\text{subject to} \quad & x_{1F} + x_{2F} + x_{3F} + x_{4F} \leq 12 \\
& x_{1C} + x_{2C} + x_{3C} + x_{4C} \leq 18 \\
& x_{1B} + x_{2B} + x_{3B} + x_{4B} \leq 10 \\
& x_{1F} + x_{1C} + x_{1B} \leq 20 \\
& x_{2F} + x_{2C} + x_{2B} \leq 16 \\
& x_{3F} + x_{3C} + x_{3B} \leq 25 \\
& x_{4F} + x_{4C} + x_{4B} \leq 13 \\
& 500x_{1F} + 700x_{2F} + 600x_{3F} + 400x_{4F} \leq 7,000 \\
& 500x_{1C} + 700x_{2C} + 600x_{3C} + 400x_{4C} \leq 9,000 \\
& 500x_{1B} + 700x_{2B} + 600x_{3B} + 400x_{4B} \leq 5,000 \\
& \tfrac{1}{12}(x_{1F} + x_{2F} + x_{3F} + x_{4F}) - \tfrac{1}{18}(x_{1C} + x_{2C} + x_{3C} + x_{4C}) = 0 \\
& \tfrac{1}{12}(x_{1F} + x_{2F} + x_{3F} + x_{4F}) - \tfrac{1}{10}(x_{1B} + x_{2B} + x_{3B} + x_{4B}) = 0 \\
\text{and} \quad & x_{1F}, x_{2F}, x_{3F}, x_{4F}, x_{1C}, x_{2C}, x_{3C}, x_{4C}, x_{1B}, x_{2B}, x_{3B}, x_{4B} \geq 0
\end{aligned}$$

(b)

	Resource Usage Per Unit of Each Activity														Resource
Resource	1F	1C	1B	2F	2C	2B	3F	3C	3B	4F	4C	4B	Totals		Available
Front Wt.	1	0	0	1	0	0	1	0	0	1	0	0	12	≤	12
Center Wt.	0	1	0	0	1	0	0	1	0	0	1	0	18	≤	18
Back Wt.	0	0	1	0	0	1	0	0	1	0	0	1	10	≤	10
Cargo 1 Wt.	1	1	1	0	0	0	0	0	0	0	0	0	15	≤	20
Cargo 2 Wt.	0	0	0	1	1	1	0	0	0	0	0	0	12	≤	16
Cargo 3 Wt.	0	0	0	0	0	0	1	1	1	0	0	0	0	≤	25
Cargo 4 Wt.	0	0	0	0	0	0	0	0	0	1	1	1	13	≤	13
Space Front	500	0	0	700	0	0	600	0	0	400	0	0	7000	≤	7000
Space Center	0	500	0	0	700	0	0	600	0	0	400	0	9000	≤	9000
Space Back	0	0	500	0	0	700	0	0	600	0	0	400	5000	≤	5000

	Contribution Toward Required Amount														Required
Requirement	1F	1C	1B	2F	2C	2B	3F	3C	3B	4F	4C	4B	Totals		Amount
%F=%C	0.0833	-0.0556	0	0.0833	-0.0556	0	0.0833	-0.0556	0	0.08333	-0.0556	0	0	=	0
%F=%B	0.0833	0	-0.1	0.0833	0	-0.1	0.0833	0	-0.1	0.0833	0	-0.1	0	=	0
Unit Profit	320	320	320	400	400	400	360	360	360	290	290	290	$ 13,330		
Solution	**0**	**5**	**10**	**7.33333**	**4.167**	**0.000**	**0**	**0**	**0**	**4.66667**	**8.333**	**0.000**			

3.4-16.

(a) Let x_{ij} be the num ber of hours operator i is assigned to work on day j for $i = KC$, DH, HB SC, KS, NK and $j = M, Tu, W, Th, F$.

minimize $Z = 25(x_{KC,M} + x_{KC,W} + x_{KC,F}) + 26(x_{DH,Tu} + x_{DH,Th}) +$
$24(x_{HB,M} + x_{HB,Tu} + x_{HB,W} + x_{HB,F}) +$
$23(x_{SC,M} + x_{SC,Tu} + x_{SC,W} + x_{SC,F}) +$
$28(x_{KS,M} + x_{KS,W} + x_{KS,Th}) + 30(x_{NK,Th} + x_{NK,F})$

subject to
$x_{KC,M} \leq 6$, $x_{KC,W} \leq 6$, $x_{KC,F} \leq 6$
$x_{DH,Tu} \leq 6$, $x_{DH,Th} \leq 6$
$x_{HB,M} \leq 4$, $x_{HB,Tu} \leq 8$, $x_{HB,W} \leq 4$, $x_{HB,F} \leq 4$
$x_{SC,M} \leq 5$, $x_{SC,Tu} \leq 5$, $x_{SC,W} \leq 5$, $x_{SC,F} \leq 5$
$x_{KS,M} \leq 3$, $x_{KS,W} \leq 3$, $x_{KS,Th} \leq 8$
$x_{NK,Th} \leq 6$, $x_{NK,F} \leq 2$

$x_{KC,M} + x_{KC,W} + x_{KC,F} \geq 8$
$x_{DH,Tu} + x_{DH,Th} \geq 8$
$x_{HB,M} + x_{HB,Tu} + x_{HB,W} + x_{HB,F} \geq 8$
$x_{SC,M} + x_{SC,Tu} + x_{SC,W} + x_{SC,F} \geq 8$
$x_{KS,M} + x_{KS,W} + x_{KS,Th} \geq 7$
$x_{NK,Th} + x_{NK,F} \geq 7$

$x_{KC,M} + x_{HB,M} + x_{SC,M} + x_{KS,M} = 14$
$x_{DH,Tu} + x_{HB,Tu} + x_{SC,Tu} = 14$
$x_{KC,W} + x_{HB,W} + x_{SC,W} + x_{KS,W} = 14$
$x_{DH,Th} + x_{HB,Th} + x_{NK,Th} = 14$
$x_{KC,F} + x_{HB,F} + x_{SC,F} + x_{NK,F} = 14$

$x_{ij} \geq 0$ for all i, j.

(b)

	Resource Usage Per Unit of Each Activity																				Resource
Resource	KC,M	KC,W	KC,F	DH,Tu	DH,Th	HB,M	HB,Tu	HB,W	HB,F	SC,M	SC,Tu	SC,W	SC,F	KS,M	KS,W	KS,Th	NK,Th	NK,F	Total		Available
KC Knowledge	1	1	1																9	≥	8
DH Knowledge				1	1														8	≥	8
HB Knowledge						1	1	1	1										19	≥	8
SC Knowledge										1	1	1	1						20	≥	8
KS Knowledge														1	1	1			7	≥	7
NK Knowledge																	1	1	7	≥	7
Mon Hours	1					1				1				1					14	≥	14
Tues Hours				1			1				1								14	≥	14
Wed Hours		1						1				1			1				14	≥	14
Thurs Hours					1											1	1		14	≥	14
Fri Hours			1						1				1					1	14	≥	14
Availability KC,M	1																		4	≤	6
Availability KC,W		1																	2	≤	6
Availability KC,F			1																3	≤	6
Availability DH,Tu				1															2	≤	6
Availability DH,Th					1														6	≤	6
Availability HB,M						1													4	≤	4
Availability HB,Tu							1												7	≤	8
Availability HB,W								1											4	≤	4
Availability HB,F									1										4	≤	4
Availability SC,M										1									5	≤	5
Availability SC,Tu											1								5	≤	5
Availability SC,W												1							5	≤	5
Availability SC,F													1						5	≤	5
Availability KS,M														1					1	≤	3
Availability KS,W															1				3	≤	3
Availability KS,Th																1			3	≤	8
Availability NK,Th																	1		5	≤	6
Availability NK,F																		1	2	≤	2
Unit Cost	\$25	\$25	\$25	\$26	\$26	\$24	\$24	\$24	\$24	\$23	\$23	\$23	\$23	\$28	\$28	\$28	\$30	\$30	1755		
Solution	4	2	3	2	6	4	7	4	4	5	5	5	5	1	3	3	5	2			

3.4-17.

(a) Let $B =$ slices of bread, $P =$ tablespoons of peanut butter, $S =$ tablespoons of strawberry jelly, $G =$ graham crackers, $M =$ cups of milk, and $J =$ cups of juice.

minimize $\quad C = 5B + 4P + 7S + 8G + 15M + 35J$

subject to
$$
\begin{aligned}
&70B + 100P + 50S + 60G + 150M + 100J \geq 400 \\
&70B + 100P + 50S + 60G + 150M + 100J \leq 600 \\
&10B + 75P + 20G + 70M \leq 0.3(70B + 100P + 50S + 60G + 150M + 100J) \\
&3S + 2M + 120J \geq 60 \\
&3B + 4P + G + 8M + J \geq 12 \\
&B = 2 \\
&P \geq 2S \\
&M + J \geq 1
\end{aligned}
$$

and $\quad B, P, S, G, M, J \geq 0$

(b)

	Resource Usage Per Unit of Each Activity								Resource
Resource	Bread	PB	Jelly	Crackers	Milk	Juice	Total		Available
Min Calories	70	100	50	60	150	100	400	≥	400
Max Calories	70	100	50	60	150	100	400	≤	600
Fat	-11	45	-15	2	25	-30	0	≤	0
Vitamin C	0	0	3	0	2	120	60	≥	60
Protein	3	4	0	1	8	1	13.95	≥	12
Bread	1	0	0	0	0	0	2	=	2
PB&J	0	1	-2	0	0	0	0	≥	0
Liquid	0	0	0	0	1	1	1	≥	1
Unit Cost	5	4	7	8	15	35	47.31		
Solution	2	0.57	0.29	1.04	0.52	0.48			

3.5-1.

Upon facing problems about juice logistics, Welch's formulated the juice logistics model (JLM), which is "an application of LP to a single-commodity network problem . The decision variables deal with the cost of transfers between plants, the cost of recipes, and carrying cost- all cost that are key to the common planning unit of tons" [p. 20]. The goal is to find the optimal grape juice quantities shipped to customers and transferred between plants over a 12-month horizon. The optimal quantities minimize the total cost, i.e., the sum of transportation, recipe and storage costs. They satisfy balance equations, bounds on the ratio of grape juice sold, and limits on total grape juice sold.

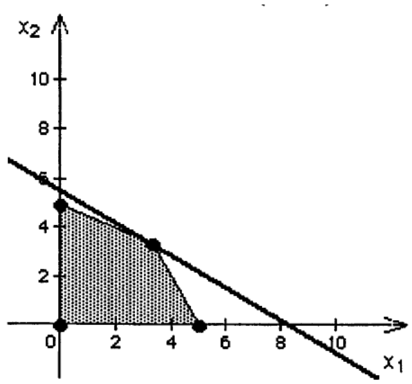
The JLM resulted in significant savings by preventing unprofitable decisions of the management. The savings in the first year of its implementation were over $130,000. Since the model can be run quickly, revising the decisions after observing the changes in the conditions is made easier. Thus, the flexibility of the system is improved. Moreover, the output helps the communication within the committee that is responsible for deciding on crop usage.

3.5-2.

(a) maximize $P = 20x_1 + 30x_2$

subject to
$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ 3x_1 + 3x_2 &\leq 20 \\ 2x_1 + 4x_2 &\leq 20 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) Optimal Solution: $(x_1^*, x_2^*) = \left(3\frac{1}{3}, 3\frac{1}{3}\right)$ and $P^* = 166.67$



(c) - (e)

Resource	Resource Usage Per Unit of Each Activity: Activity 1	Activity 2	Totals		Resource Available
1	2	1	10	≤	10
2	3	3	20	≤	20
3	2	4	20	≤	20
Unit Profit	20	30	**$ 166.67**		
Solution	**3.333**	**3.333**			

(d)

(x_1, x_2)	Feasible?	P
$(2, 2)$	Yes	\$100
$(3, 3)$	Yes	\$150
$(2, 4)$	Yes	\$160 Best
$(4, 2)$	Yes	\$140
$(3, 4)$	No	
$(4, 3)$	No	

3.5-3.

(a) maximize $P = 300A + 250B + 200C$

subject to
$$0.02A + 0.03B + 0.05C \leq 40$$
$$0.05A + 0.02B + 0.04C \leq 40$$

and $A, B, C \geq 0$

(b)

	Resource Usage Per Unit of Each Activity					Resource
Resource	Part A	Part B	Part C	Total		Available
Machine 1	0.02	0.03	0.05		≤	40
Machine 2	0.05	0.02	0.04		≤	40
Unit Profit	\$300	\$250	\$200			
Solution						

(c) Many answers are possible.

(A, B, C)	Feasible?	P
$(500, 500, 300)$	No	
$(350, 1000, 0)$	Yes	$\$355,000$
$(400, 1000, 0)$	Yes	$\$370,000$ Best

(d)

	Resource Usage Per Unit of Each Activity					Resource
Resource	Part A	Part B	Part C	Total		Available
Machine 1	0.02	0.03	0.05	40	≤	40
Machine 2	0.05	0.02	0.04	40	≤	40
Unit Profit	\$300	\$250	\$200	381818		
Solution	363.6363636	1090.909091	0			

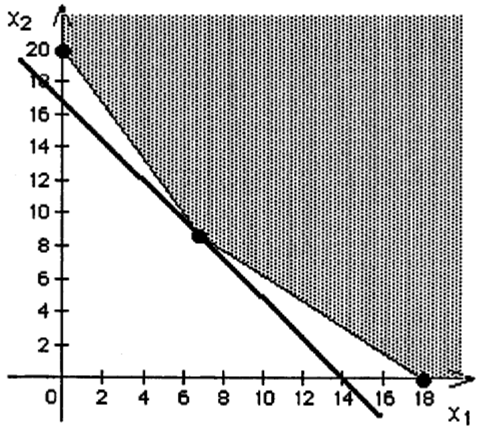
3.5-4.

(a) minimize $C = 60x_1 + 50x_2$

subject to
$$5x_1 + 3x_2 \geq 60$$
$$2x_1 + 2x_2 \geq 30$$
$$7x_1 + 9x_2 \geq 126$$

and $x_1, x_2 \geq 0$

(b) Optimal Solution: $(x_1^*, x_2^*) = (6.75, 8.75)$ and $C^* = 842.50$



(c) - (e)

Benefit	Benefit Contribution Per Unit of Each Activity Activity 1	Activity 2	Totals		Minimum Level
1	5	3	60	≥	60
2	2	2	31	≥	30
3	7	9	126	≥	126
Unit Cost	60	50	**$ 842.50**		
Solution	**6.75**	**8.75**			

(d)

(x_1, x_2)	Feasible?	C
$(7, 7)$	No	
$(7, 8)$	No	
$(8, 7)$	No	
$(8, 8)$	Yes	\$880 Best
$(8, 9)$	Yes	\$930
$(9, 8)$	Yes	\$940

3.5-5.

(a) m inimize $C = 84C + 72T + 60A$

subject to
$$90C + 20T + 40A \geq 200$$
$$30C + 80T + 60A \geq 180$$
$$10C + 20T + 60A \geq 150$$

and $C, T, A \geq 0$

(b) - (e)

Nutritional Ingredient	Corn	Kilogram of Tankage	Alfalfa	Totals		Minimum Level
Carbohydrates	90	20	40	200	≥	200
Proteins	30	80	60	180	≥	180
Vitamins	10	20	60	157	≥	150
Unit Cost	84	72	60	**$ 242**		
Solution	**1**	**0**	**2**			

(c) $(x_1, x_2, x_3) = (1, 2, 2)$ is a feasible solution with a daily cost of \$348. This diet will provide 210 kg of carbohydrates, 310 kg of protein, and 170 kg of vitamins daily.

(d) Answers will vary.

3.5-6.

(a) minimize $C = x_1 + x_2 + x_3$

subject to

$$2x_1 + x_2 + 0.5x_3 \geq 400$$
$$0.5x_1 + 0.5x_2 + x_3 \geq 100$$
$$1.5x_2 + 2x_3 \geq 300$$

and $x_1, x_2, x_3 \geq 0$

(b) - (e)

	Benefit Contribution Per Unit of Each Asset					
Year	Asset 1	Asset 2	Asset 3	Totals		Minimum Cash Flow Required
5	2	1	0.5	400	≥	400
10	0.5	0.5	1	150	≥	100
20	0	1.5	2	300	≥	300
Unit Cost	1	1	1	**\$ 300**		
Solution	**100**	**200**	**0**			

(c) $(x_1, x_2, x_3) = (100, 100, 200)$ is a feasible solution. This would generate \$400 million in 5 years, \$300 million in 10 years, and \$550 million in 20 years. The total investment will be \$400 million.

(d) Answers will vary.

3.6-1.

(a) In the following, the indices $i, j, k, l,$ and m refer to products, months, plants, processes and regions respectively. The decision variables are:

x_{ijklm} = amount of product i produced in month j in plant k using process l and to be sold in region m, and

s_{im} = amount of product i stored to be sold in March in region m.

The parameters of the problem are:

D_{ijm} = demand for product i in month j in region m,

c_{ikl} = unit production cost of product i in plant k using process l,

R_{ikl} = production rate of product i in plant k using process l,

p_i = selling price of product i,

T_{ikm} = transportation cost of product i product in plant k to be sold in region m,

A_j = days available for production in month j,

L = storage limit,

M_i = storage cost per unit of product i.

The objective is to maximize the total profit, which is the difference of the total revenue and the total cost. The total cost is the sum of the costs of production, inventory and transportation. Using the notation introduced, the objective is to maximize

$$\sum_i p_i\left(\sum_{j,k,l,m} x_{ijklm}\right) - \sum_{i,k,l} c_{ikl}\left(\sum_{j,m} x_{ijklm}\right) - \sum_i M_i\left(\sum_m s_{im}\right) - \sum_{i,k,m} T_{ikm}\left(\sum_{j,l} x_{ijklm}\right)$$

subject to the constraints

$$\sum_{k,l} x_{ijklm} - s_{im} \leq D_{ijm} \qquad \text{for } j = \text{February}; i = 1, 2; m = 1, 2$$

$$\sum_{k,l} x_{ijklm} + s_{im} \leq D_{ijm} \qquad \text{for } j = \text{March}; i = 1, 2; m = 1, 2$$

$$\sum_i s_{im} \leq L \qquad \text{for } m = 1, 2$$

$$\sum_{i,l} \frac{1}{R_{ikl}}\left(\sum_m x_{ijklm}\right) \leq A_j \qquad \text{for } j = \text{February, March}; k = 1, 2$$

$$x_{ijklm} \geq 0 \qquad \text{for } i, k, l, m = 1, 2 \text{ and } j = \text{February, March}$$

(b)

Quantity produced to be sold in the same region with process 1

Product	Plant 1 February	Plant 1 March	Plant2 February	Plant2 March
1	0	0	0	0
2	2400	2760	3200	3680

Quantity produced to be sold in the other region with process 1

Product	Plant 1 February	Plant 1 March	Plant2 February	Plant2 March
1	0	0	0	0
2	0	0	0	0

Quantity produced to be sold in the same region with process 2

Product	Plant 1 February	Plant 1 March	Plant2 February	Plant2 March
1	0	0	0	0
2	0	0	0	0

Quantity produced to be sold in the other region with process 2

Product	Plant 1 February	Plant 1 March	Plant2 February	Plant2 March
1	0	0	0	0
2	0	0	0	0

Days available	
February	20
March	23

Storage Limit
1000

		Total Profit
Revenue	1348480	$333,680
Prod Cost	1014800	
Transp Cost	0	
Storage Cost	0	

Demand

Product	Region1 February	Region1 March	Region2 February	Region2 March
1	3600	6300	4900	4200
2	4500	5400	5100	6000

Production costs

Product	Plant 1 Process 1	Plant 1 Process 2	Plant2 Process 1	Plant2 Process 2
1	$ 62	$ 59	$ 61	$ 65
2	$ 78	$ 85	$ 89	$ 86

Production rates

Product	Plant 1 Process 1	Plant 1 Process 2	Plant2 Process 1	Plant2 Process 2
1	100	140	130	110
2	120	150	160	130

Transportation Costs

Region	Product 1: 1	Product 1: 2	Product 2: 1	Product 2: 2
1	0	9	0	7
2	9	0	7	0

Product	Revenue
1	$83
2	$112

Product	Storage cost
1	$ 3
2	$ 4

Stored quantity

Region	Product 1	Product 2
1	0	0
2	0	0

Demand satisfied

Product	Plant 1 February	Plant 1 March	Plant2 February	Plant2 March
1	0	0	0	0
2	2400	2760	3200	3680

Capacity used

	Plant1	Plant2
Feb	20	20
Mar	23	23

	Amount stored
Plant1	0
Plant2	0

(c)

```
TITLE
    ManufacturingProblem;

INDEX
    product = (pr1,pr2);
    month = (feb,mar);
    plant = (pl1,pl2);
    process = (ps1,ps2);
    region = (r1,r2);

DATA
    demand[product,month,region] := (3600,4900,
                      6300,4200,
                      4500,5100,
                      5400,6000);
    days[month] := (20,23);
    storagecost[product] := (3,4);
    prodcost[product,plant,process] := (62,59,
                         61,65,
                         78,85,
                         89,86);
    rate[product,plant,process] := (100,140,
                      130,110,
                      120,150,
                      160,130);
    price[product] := (83,112);
    transpcost[product,plant,region] := (0,9,
                          9,0,
                          0,7,
                          7,0);

DECISION VARIABLES
    Volume[product,month,plant,process,region];
    Store[product,region];

MACRO

    Revenues := SUM(product,month,plant,process,region: price*Volume);
    ProductionCost := SUM(product,plant,process,month,region: prodcost*Volume);
    TransportationCost := SUM(product,plant,region,month,process: transpcost*Volume);
    StorageCost := SUM(product,region: storagecost*Store);

MODEL

    MAX  TotalProfit = Revenues - ProductionCost - TransportationCost - StorageCost;

SUBJECT TO
    SalesFeb[product,region,month] where(month=feb) : SUM(plant,process: Volume - Store) <= demand;
    SalesMar[product,region,month] where(month=mar) : SUM(plant,process: Volume + Store) <= demand;
    StorageLimit[region] : SUM(product: Store) <= 1000;
    Capacity[plant,month] : SUM(product,process,region: Volume/rate) <= days;

END
```

SOLUTION RESULT

Optimal solution found

MAX TotalPro = 333680.0000

MACROS

Macro Name	Values
Revenues	1348480.0000
ProductionCost	1014800.0000
TransportationCost	0.0000
StorageCost	0.0000

DECISION VARIABLES

VARIABLE Volume[product,month,plant,process,region] :

product	month	plant	process	region	Activity	Reduced Cost
pr1	feb	pl1	ps1	r1	0.0000	-19.8000
pr1	feb	pl1	ps1	r2	0.0000	-28.8000
pr1	feb	pl1	ps2	r1	0.0000	-5.1429
pr1	feb	pl1	ps2	r2	0.0000	-14.1429
pr1	feb	pl2	ps1	r1	0.0000	-15.3077
pr1	feb	pl2	ps1	r2	0.0000	-6.3077
pr1	feb	pl2	ps2	r1	0.0000	-24.4545
pr1	feb	pl2	ps2	r2	0.0000	-15.4545
pr1	mar	pl1	ps1	r1	0.0000	-19.8000
pr1	mar	pl1	ps1	r2	0.0000	-28.8000
pr1	mar	pl1	ps2	r1	0.0000	-5.1429
pr1	mar	pl1	ps2	r2	0.0000	-14.1429
pr1	mar	pl2	ps1	r1	0.0000	-15.3077
pr1	mar	pl2	ps1	r2	0.0000	-6.3077
pr1	mar	pl2	ps2	r1	0.0000	-24.4545
pr1	mar	pl2	ps2	r2	0.0000	-15.4545
pr2	feb	pl1	ps1	r1	2400.0000	0.0000
pr2	feb	pl1	ps1	r2	0.0000	-7.0000
pr2	feb	pl1	ps2	r1	0.0000	-0.2000
pr2	feb	pl1	ps2	r2	0.0000	-7.2000
pr2	feb	pl2	ps1	r1	0.0000	-7.0000
pr2	feb	pl2	ps1	r2	3200.0000	0.0000
pr2	feb	pl2	ps2	r1	0.0000	-9.3077
pr2	feb	pl2	ps2	r2	0.0000	-2.3077
pr2	mar	pl1	ps1	r1	2760.0000	0.0000
pr2	mar	pl1	ps1	r2	0.0000	-7.0000
pr2	mar	pl1	ps2	r1	0.0000	-0.2000
pr2	mar	pl1	ps2	r2	0.0000	-7.2000
pr2	mar	pl2	ps1	r1	0.0000	-7.0000
pr2	mar	pl2	ps1	r2	3680.0000	0.0000
pr2	mar	pl2	ps2	r1	0.0000	-9.3077
pr2	mar	pl2	ps2	r2	0.0000	-2.3077

VARIABLE Store[product,region] :

product	region	Activity	Reduced Cost
pr1	r1	0.0000	-3.0000
pr1	r2	0.0000	-3.0000
pr2	r1	0.0000	-4.0000
pr2	r2	0.0000	-4.0000

(d)

```
MODEL:

SETS:
PRODUCT/PR1 PR2/: PRICE, STORAGECOST;
MONTH/FEB MAR/: DAYS;
PLANT/PL1 PL2/;
PROCESS/PS1 PS2/;
REGION/R1 R2/;
LINK1(PRODUCT,MONTH,PLANT,PROCESS,REGION): VAR;
LINK2(PRODUCT,MONTH,REGION): DEMAND;
LINK3(PRODUCT,PLANT,PROCESS): PRODCOST;
LINK4(PRODUCT,PLANT,PROCESS): RATE;
LINK5(PRODUCT,REGION): STORE;
LINK6(PRODUCT,PLANT,REGION): TRANSPCOST;
ENDSETS

!OBJECTIVE FUNCTION;
MAX = @SUM(PRODUCT(I): PRICE(I)*@SUM(MONTH(J): @SUM(PLANT(K): @SUM(PROCESS(L):
@SUM(REGION(M): VAR(I,J,K,L,M)))))) - @SUM(LINK3(I,K,L): PRODCOST(I,K,L)*@SUM(MONTH(J):
@SUM(REGION(M): VAR(I,J,K,L,M)))) - @SUM(PRODUCT(I): STORAGECOST(I)*@SUM(REGION(M):
STORE(I,M))) - @SUM(LINK6(I,K,M): TRANSPCOST(I,K,M)*@SUM(MONTH(J): @SUM(PROCESS(L):
VAR(I,J,K,L,M))));

!CONSTRAINTS;
@FOR(PRODUCT(I): @FOR(REGION(M): @SUM(PLANT(K): @SUM(PROCESS(L): VAR(I,FEB,K,L,M))) -
STORE(I,M) <= DEMAND(I,FEB,M)));
@FOR(PRODUCT(I): @FOR(REGION(M): @SUM(PLANT(K): @SUM(PROCESS(L): VAR(I,MAR,K,L,M))) +
STORE(I,M) <= DEMAND(I,MAR,M)));
@FOR(REGION(M): @SUM(PRODUCT(I): STORE(I,M))<=1000);
@FOR(PLANT(K): @FOR(MONTH(J): @SUM(PRODUCT(I): @SUM(PROCESS(L):
(1/RATE(I,K,L))*@SUM(REGION(M): VAR(I,J,K,L,M)))) <= DAYS(J)));

!DATA PART;
DATA:
DEMAND = 3600 4900
         6300 4200
         4500 5100
         5400 6000;
DAYS = 20 23;
STORAGECOST = 3 4;
PRODCOST = 62 59
           61 65
           78 85
           89 86;
RATE = 100 140
       130 110
       120 150
       160 130;
PRICE = 83 112;
TRANSPCOST = 0 9
             9 0
             0 7
             7 0;
ENDDATA
END
```

```
Global optimal solution found at step:            8
Objective value:                           333680.0

                    Variable           Value

        VAR( P1, FEB, P1, P1, R1)      0.0000000
        VAR( P1, FEB, P1, P1, R2)      0.0000000
        VAR( P1, FEB, P1, P2, R1)      0.0000000
        VAR( P1, FEB, P1, P2, R2)      0.0000000
        VAR( P1, FEB, P2, P1, R1)      0.0000000
        VAR( P1, FEB, P2, P1, R2)      0.0000000
        VAR( P1, FEB, P2, P2, R1)      0.0000000
        VAR( P1, FEB, P2, P2, R2)      0.0000000
        VAR( P1, MAR, P1, P1, R1)      0.0000000
        VAR( P1, MAR, P1, P1, R2)      0.0000000
        VAR( P1, MAR, P1, P2, R1)      0.0000000
        VAR( P1, MAR, P1, P2, R2)      0.0000000
        VAR( P1, MAR, P2, P1, R1)      0.0000000
        VAR( P1, MAR, P2, P1, R2)      0.0000000
        VAR( P1, MAR, P2, P2, R1)      0.0000000
        VAR( P1, MAR, P2, P2, R2)      0.0000000
        VAR( P2, FEB, P1, P1, R1)       2400.000
        VAR( P2, FEB, P1, P1, R2)      0.0000000
        VAR( P2, FEB, P1, P2, R1)      0.0000000
        VAR( P2, FEB, P1, P2, R2)      0.0000000
        VAR( P2, FEB, P2, P1, R1)      0.0000000
        VAR( P2, FEB, P2, P1, R2)       3200.000
        VAR( P2, FEB, P2, P2, R1)      0.0000000
        VAR( P2, FEB, P2, P2, R2)      0.0000000
        VAR( P2, MAR, P1, P1, R1)       2760.000
        VAR( P2, MAR, P1, P1, R2)      0.0000000
        VAR( P2, MAR, P1, P2, R1)      0.0000000
        VAR( P2, MAR, P1, P2, R2)      0.0000000
        VAR( P2, MAR, P2, P1, R1)      0.0000000
        VAR( P2, MAR, P2, P1, R2)       3680.000
        VAR( P2, MAR, P2, P2, R1)      0.0000000
        VAR( P2, MAR, P2, P2, R2)      0.0000000

         STORE( P1, R1)        0.0000000
         STORE( P1, R2)        0.0000000
         STORE( P2, R1)        0.0000000
         STORE( P2, R2)        0.0000000
```

3.6-2.

(a)

```
MAX

    50x1+20x2+25x3 ;

SUBJECT TO

    9x1+3x2+5x3<=500 ;
    5x1+4x2<=350;
    3x1+2x3<=150;
    x3<=20;

END
```

Variable Name	Activity
x1	26.1905
x2	54.7619
x3	20.0000

(b)

```
max = 50*x1+20*x2+25*x3;
9*x1+3*x2+5*x3<=500;
5*x1+4*x2<=350;
3*x1+2*x3<=150;
x3<=20;
x1>=0; x2>=0; x3>=0;
```

```
Global optimal solution found at step:          4
Objective value:                         2904.762
```

Variable	Value
X1	26.19048
X2	54.76190
X3	20.00000

3.6-3.

(a)

```
TITLE
    TransportationProblem;

INDEX
    supply = (Wh1,Wh2);
    dest   = (C1,C2,C3);

DATA
    MaxCapacity[supply]  := (400,500);
    Required[dest]       := (300,200,400);

    ShippingCost[supply,dest] := (600,800,700,
                400,900,600);

 DECISION VARIABLES
     VolumeShipped[supply,dest]  ->  ""

 MODEL

     MIN  TotalCost = SUM(supply,dest: ShippingCost * VolumeShipped);

 SUBJECT TO

     Capacity[supply] : SUM(dest: VolumeShipped)    =  MaxCapacity ;
     Demand[dest]     : SUM(supply: VolumeShipped)  =  Required ;

 END
```

(b)

```
MODEL:

SETS:
      FACTORIES /F1 F2/: CAPACITY;
      CUSTOMERS /C1 C2 C3/: DEMAND;
      LINKS(FACTORIES, CUSTOMERS): COST, VOLUME;
ENDSETS
[OBJECTIVE] MIN = @SUM(LINKS(I,J):COST(I,J)*VOLUME(I,J));
!DEMAND CONSTRAINTS;
@FOR(CUSTOMERS(J): @SUM(FACTORIES(I): VOLUME(I,J))=DEMAND(J));
!SUPPLY CONSTRAINTS;
@FOR(FACTORIES(I): @SUM(CUSTOMERS(J):VOLUME(I,J))=CAPACITY(I));
!HERE IS THE DATA;
DATA:
CAPACITY = 400 500;
DEMAND = 300 200 400;
COST = 600 800 700
       400 200 400;
ENDDATA
END
```

```
Global optimal solution found at step:          2
Objective value:                         410000.0

                Variable           Value

         VOLUME( F1, C1)        300.0000
         VOLUME( F1, C2)       0.0000000
         VOLUME( F1, C3)        100.0000
         VOLUME( F2, C1)       0.0000000
         VOLUME( F2, C2)        200.0000
         VOLUME( F2, C3)        300.0000
```

3.6-4.

(a)

```
TITLE
    TransportationProblem;

INDEX
    student  = (KC,OH,HB,SC,KS,NK);
    day = (M,TU,W,TH,F);

DATA

    Wage[student]      :=(10,10.1,9.9,9.8,10.8,11.3);
    Gender[student]        := (0,0,0,0,1,1);
    Available[student,day] := (6,0,6,0,6,
                   0,6,0,6,0
                   4,8,4,0,4
                   5,5,5,0,5
                   3,0,3,8,0
                   0,0,0,6,2);
```

```
DECISION VARIABLES

    Work[student,day]  ->  ""

MODEL

    MIN  TotalCost = SUM(student,day: Wage * Work);

SUBJECT TO

    TimeConstraint[student,day] : Work <= Available ;
    MinimumWork0[student] where(Gender=0) : SUM(day: Work) >=8 ;
    MinimumWork1[student] where(Gender=1) : SUM(day: Work) >=7 ;
    AlwaysOpen[day] : SUM(student: Work) = 14 ;
END
□

                              MIN TotalCos =        709.6000

VARIABLE Work[student,day] :

   student  day          Activity
  -----------------------------------
   KC       M             4.0000
   KC       TU            0.0000
   KC       W             2.0000
   KC       TH            0.0000
   KC       F             3.0000
   OH       M             0.0000
   OH       TU            2.0000
   OH       W             0.0000
   OH       TH            6.0000
   OH       F             0.0000
   HB       M             4.0000
   HB       TU            7.0000
   HB       W             4.0000
   HB       TH            0.0000
   HB       F             4.0000
   SC       M             5.0000
   SC       TU            5.0000
   SC       W             5.0000
   SC       TH            0.0000
   SC       F             5.0000
   KS       M             1.0000
   KS       TU            0.0000
   KS       W             3.0000
   KS       TH            3.0000
   KS       F             0.0000
   NK       M             0.0000
   NK       TU            0.0000
   NK       W             0.0000
   NK       TH            5.0000
   NK       F             2.0000
  -----------------------------------
```

(b)

```
MODEL:

SETS:
      STUDENTS /KC OH HB SC KS NK/: WAGE, GENDER;
      DAYS /M TU W TH F/;
      LINKS(STUDENTS, DAYS): AVAILABLE, WORK;
ENDSETS

[OBJECTIVE] MIN = @SUM(LINKS(I,J):WAGE(I)*WORK(I,J));

!TIME CONSTRAINTS;
@FOR(LINKS(I,J): WORK(I,J)<=AVAILABLE(I,J));

!MINIMUM WORK CONSTRAINTS;
@FOR(STUDENTS(I)|GENDER(I) #EQ# 0: @SUM(LINKS(I,J):WORK(I,J))>=8);
@FOR(STUDENTS(I)|GENDER(I) #EQ# 1: @SUM(LINKS(I,J):WORK(I,J))>=7);

!ALWAYS OPEN CONSTRAINTS;
@FOR(DAYS(J): @SUM(LINKS(I,J): WORK(I,J))=14);
!HERE IS THE DATA;
DATA:
WAGE = 10 10.1 9.9 9.8 10.8 11.3;
GENDER = 0 0 0 0 1 1;
AVAILABLE=6 0 6 0 6
          0 6 0 6 0
          4 8 4 0 4
          5 5 5 0 5
          3 0 3 8 0
          0 0 0 6 2;
ENDDATA
END
```

```
   WORK( KC, M)        2.000000       WORK( SC, M)        5.000000
  WORK( KC, TU)       0.0000000      WORK( SC, TU)        5.000000
   WORK( KC, W)        3.000000       WORK( SC, W)        5.000000
  WORK( KC, TH)       0.0000000      WORK( SC, TH)       0.0000000
   WORK( KC, F)        4.000000       WORK( SC, F)        5.000000
   WORK( OH, M)       0.0000000       WORK( KS, M)        3.000000
  WORK( OH, TU)        2.000000      WORK( KS, TU)       0.0000000
   WORK( OH, W)       0.0000000       WORK( KS, W)        2.000000
  WORK( OH, TH)        6.000000      WORK( KS, TH)        2.000000
   WORK( OH, F)       0.0000000       WORK( KS, F)       0.0000000
   WORK( HB, M)        4.000000       WORK( NK, M)       0.0000000
  WORK( HB, TU)        7.000000      WORK( NK, TU)       0.0000000
   WORK( HB, W)        4.000000       WORK( NK, W)       0.0000000
  WORK( HB, TH)       0.0000000      WORK( NK, TH)        6.000000
   WORK( HB, F)        4.000000       WORK( NK, F)        1.000000
```

3.6-5.

(a)

```
MODEL

    MIN  84c+72t+60a;

SUBJECT TO

90c+20t+40a>=200;
30c+80t+60a>=180;
10c+20t+60a>=150;

END
```

```
SOLUTION RESULT

  Optimal solution found

    MIN Z        =        241.7143

DECISION VARIABLES

PLAIN VARIABLES

    Variable Name              Activity
  -------------------------------------
    c                            1.1429
    t                            0.0000
    a                            2.4286
  -------------------------------------
```

(b)

```
[OBJECTIVE] MIN = 84*C+72*T+60*A;

!CONSTRAINTS;

90*C+20*T+40*A>=200;
30*C+80*T+60*A>=180;
10*C+20*T+60*A>=150;
```

```
Global optimal solution found at step:              8
Objective value:                             241.7143

                      Variable           Value
                             C        1.142857
                             T       0.0000000
                             A        2.428571
```

3.6-6.

(a)

```
MODEL

    MIN  x1+x2+x3;

SUBJECT TO

2x1+x2+0.5x3>=400;
0.5x1+0.5x2+x3>=100;
1.5x2+2x3>=300;

END
```

```
SOLUTION RESULT

Optimal solution found

 MIN Z          =         300.0000

 Variable Name             Activity
-------------------------------------
 x1                        100.0000
 x2                        200.0000
 x3                          0.0000
-------------------------------------
```

(b)

```
[OBJECTIVE] MIN = X+Y+Z;

!CONSTRAINTS;

2*X+Y+0.5*Z>=400;
0.5*X+0.5*Y+Z>=100;
1.5*Y+2*Z>=300;
```

```
Global optimal solution found at step:          8
Objective value:                         300.0000

                  Variable           Value
                         X        100.0000
                         Y        200.0000
                         Z       0.0000000
```

```
Global optimal solution found at step:         21
Objective value:                         709.6000
```

3.6-7.

(a) The problem is to choose the amount of paper type k to be produced on machine type l at paper mill k and to be shipped to customer j, which we can represent as x_{ijkl} for $i = 1, \ldots, 10$; $j = 1, \ldots, 1000$; $k = 1, \ldots, 5$ and $l = 1, 2, 3$. The objective is to minimize

$$\sum_{i,k,l} P_{ikl}\left(\sum_j x_{ijkl}\right) + \sum_{i,j,k} T_{ijk}\left(\sum_l x_{ijkl}\right)$$

subject to

$\sum_{i,l} x_{ijkl}$	$\leq D_{jk}$	for $j = 1, \ldots, 1000$; $k = 1, \ldots, 5$	DEMAND
$\sum_{k,l} r_{klm}\left(\sum_j x_{ijkl}\right)$	$\leq R_{im}$	for $i = 1, \ldots, 10$; $m = 1, 2, 3, 4$	RAW MATERIAL
$\sum_k c_{kl}\left(\sum_j x_{ijkl}\right)$	$\leq C_{il}$	for $i = 1, \ldots, 10$; $l = 1, 2, 3$	CAPACITY
x_{ijkl}	≥ 0	for $i = 1, \ldots, 10$; $j = 1, \ldots, 1000$; $k = 1, \ldots, 5$; $l = 1, 2, 3$	

Note that $\sum_l x_{ijkl}$ is the total amount of paper type k shipped to customer j from paper mill i and $\sum_j x_{ijkl}$ is the total amount of paper type k made on machine type l at paper mill i

(b) $1000{*}5 + 10{*}4 + 10{*}3 = 5,070$ functional constraints

$10{*}1000{*}5{*}3 = 150,000$ decision variables

(c)

```
TITLE
    PaperManufacturing;

INDEX
    mill = 1..10;
    customer = 1..1000;
    machine = 1..3;
    material = 1..4;
    paper = 1..5;

DATA
    Required[customer,paper] = DATAFILE(Required.dat);
    Rate1[paper,machine,material] = DATAFILE(Rate1.dat);
    RawMaterial[mill,material] = DATAFILE(RawMaterial.dat);
    Rate2[paper,machine] = DATAFILE(Rate2.dat);
    MaxCapacity[mill,machine] = DATAFILE(MaxCapacity.dat);
    ProdCost[mill,paper,machine] = DATAFILE(ProdCost);
    TranspCost[mill,customer,paper] = DATAFILE(TranspCost);

DECISION VARIABLES
    Quantity[mill,customer,machine,paper]  ->  ""
```

```
MODEL

    MIN  TotalCost = SUM(mill,customer,machine,paper: ProdCost * Quantity)
        + SUM(mill,customer,machine,paper: TranspCost * Quantity);

SUBJECT TO

    Demand[customer,paper] : SUM(mill,machine: Quantity) >= Required ;
    Supply[mill,material] : SUM(customer,paper,machine: Rate1 * Quantity) <= RawMaterial;
    Capacity[mill,machine] : SUM(customer,paper: Rate2 * Quantity) < MaxCapacity ;
END
□
```

(d)

```
MODEL:

SETS:
MILLS /1..10/;
CUSTOMERS /1..1000/;
MACHINES /1..3/;
MATERIALS /1..4/;
PAPER /1..5/;
LINK1(CUSTOMERS,PAPER): DEMAND;
LINK2(PAPER, MACHINES, MATERIALS): RATE1;
LINK3(MILLS, MATERIALS): CAPACITY1;
LINK4(PAPER, MACHINES): RATE2;
LINK5(MILLS, MACHINES): CAPACITY2;
LINK6(MILLS, PAPER, MACHINES): PROD_COST;
LINK7(MILLS, CUSTOMERS, PAPER): TRANSP_COST;
LINK8(MILLS, CUSTOMERS, PAPER, MACHINES): QUANTITY;
ENDSETS

!OBJECTIVE IS TO MINIMIZE PRODUCTION COST + TRANSPORTATION COST;
MIN = @SUM(LINK6(I,K,L):PROD_COST(I,K,L) * @SUM(CUSTOMERS(J): QUANTITY(I,J,K,L))) +
      @SUM(LINK7(I,J,K):TRANSP_COST * @SUM(MACHINES(L): QUANTITY(I,J,K,L)));

!DEMAND CONSTRAINTS;
@FOR(LINK1(J,K): @SUM(MILLS(I): @SUM(MACHINES(L): QUANTITY(I,J,K,L)))>= DEMAND(J,K));

!RAW MATERIALS SUPPLY CONSTRAINTS;
@FOR(LINK3(I,M): @SUM(PAPER(K): @SUM(MACHINES(L): RATE1(K,L,M)*@SUM(CUSTOMERS(J):
QUANTITY(I,J,K,L)))) <= CAPACITY1(I,M));

!CAPACITY SUPPLY CONSTRAINTS;
@FOR(LINK5(I,L): @SUM(PAPER(K): RATE2(K,L) * @SUM(CUSTOMERS(J): QUANTITY(I,J,K,L))) <=
CAPACITY2(I,L));

!READ DATA FROM AN EXCEL FILE;
DATA:
DEMAND, RATE1, CAPACITY1, RATE2, CAPACITY2, PROD_COST, TRANSP_COST =
@WKX('C:\LINGO\DATA.WK4','DEMAND','RATE1','CAPACITY1','RATE2','CAPACITY2','PROD_COST','TRA
NSP_COST');
ENDDATA
END
```

3.7-1.

Answers will vary.

3.7-2.

Answers will vary.

Cases

3-1 a) In this case, we have two decision variables: one variable to determine the number of Family Thrillseekers we should assemble and one variable to determine the number of Classy Cruisers we should assemble. We also have the following three constraints:

1. The plant has a maximum of 48,000 labor hours. Each Thrillseeker requires six labor hours, and each Cruiser requires 10.5 labor hours. The sum of the total number of labor hours required to assemble all Thrillseekers and all Cruisers must be less than or equal to 48,000 hours.

2. The plant has a maximum of 20,000 doors available. Each Thrillseeker requires four doors, and each Cruiser requires two doors. The sum of the total number of doors required to assemble all Thrillseekers and all Cruisers must be less than or equal to 20,000 doors.

3. Because the demand for Cruisers is limited to 3,500 cars, the decision variable for the number of Cruisers we should assemble must be less than or equal to 3,500.

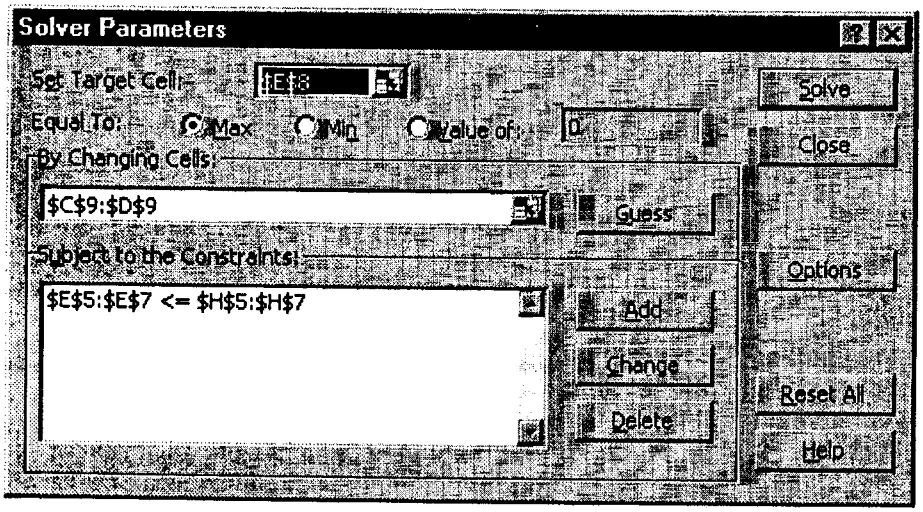
The formulas used in the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	=SUMPRODUCT(C5:D5,C9:D9)	<	=	48000
6		Doors	4	2	=SUMPRODUCT(C6:D6,C9:D9)	<	=	20000
7		Cruiser Demand	0	1	=SUMPRODUCT(C7:D7,C9:D9)	<	=	3500
8		Profit ($thousands)	3.6	5.4	**=SUMPRODUCT(C8:D8,C9:D9)**			
9		Solution	**3800**	**2400**				

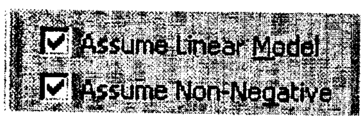
The values used in the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	48000	<	=	48000
6		Doors	4	2	20000	<	=	20000
7		Cruiser Demand	0	1	2400	<	=	3500
8		Profit ($thousands)	3.6	5.4	**26640**			
9		Solution	**3800**	**2400**				

We specify the following Solver settings.



Finally, throughout this case we use the following solver options.



Rachel's plant should assemble 3,800 Thrillseekers and 2,400 Cruisers to obtain a maximum profit of $26,640,000.

b) In part (a) above, we observed that the Cruiser demand constraint was not binding. Therefore, raising the demand for the Cruiser will not change the optimal solution. The marketing campaign should not be undertaken.

c) The new value of the right-hand side of the labor constraint becomes 48,000 * 1.25 = 60,000 labor hours. All formulas and Solver settings used in part (a) remain the same. The values for the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	56250	<	=	60000
6		Doors	4	2	20000	<	=	20000
7		Cruiser Demand	0	1	3500	<	=	3500
8		Profit ($thousands)	3.6	5.4	**30600**			
9		Solution	**3250**	**3500**				

Rachel's plant should now assemble 3,250 Thrillseekers and 3,500 Cruisers to achieve a maximum profit of $30,600,000.

d) Using over time labor increases the profit by $30,600,000 – $26,640,000 = $3,960,000. Rachel should therefore be willing to pay at most $3,960,000 extra for overtime labor beyond regular time rates.

e) The value of the right-hand side of the Cruiser demand constraint is 3,500 * 1.20 = 4,200 cars. The value of the right-hand side of the labor hour constraint is 48,000 * 1.25 = 60,000 hours. All formulas and Solver settings used in part (a) remain the same. Ignoring the costs of the advertising campaign and overtime labor, the values for the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	60000	<	=	60000
6		Doors	4	2	20000	<	=	20000
7		Cruiser Demand	0	1	4000	<	=	4200
8		Profit ($thousands)	3.6	5.4	**32400**			
9		Solution	**3000**	**4000**				

Rachel's plant should produce 3,000 Thrillseekers and 4,000 Cruisers for a maximum profit of $32,400,000. This profit excludes the costs of advertising and using overtime labor.

f) The advertising campaign costs $500,000. In the solution to part (e) above, we used the maximum overtime labor available, and the maximum use of overtime labor costs $1,600,000. Thus, our solution in part (e) required an extra $500,000 + $1,600,000 = $2,100,000. We perform the following cost/benefit analysis:

Profit in part (e):	$32,400,000
– Advertising and overtime costs:	$ 2,100,000
	$30,300,000

We compare the $30,300,000 profit with the $26,640,000 profit obtained in part (a) and conclude that the decision to run the advertising campaign and use overtime labor is a very wise, profitable decision.

g) Because we consider this question independently, the values of the right-hand sides for the Cruiser demand constraint and the labor hour constraint are the same as those in part (a). We now change the profit for the Thrillseeker from 3.6 to 2.8 in the problem formulation. All formulas and Solver settings used in part (a) remain the same. The values for the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	48000	<	=	48000
6		Doors	4	2	14500	<	=	20000
7		Cruiser Demand	0	1	3500	<	=	3500
8		Profit ($thousands)	2.8	5.4	**24150**			
9		Solution	**1875**	**3500**				

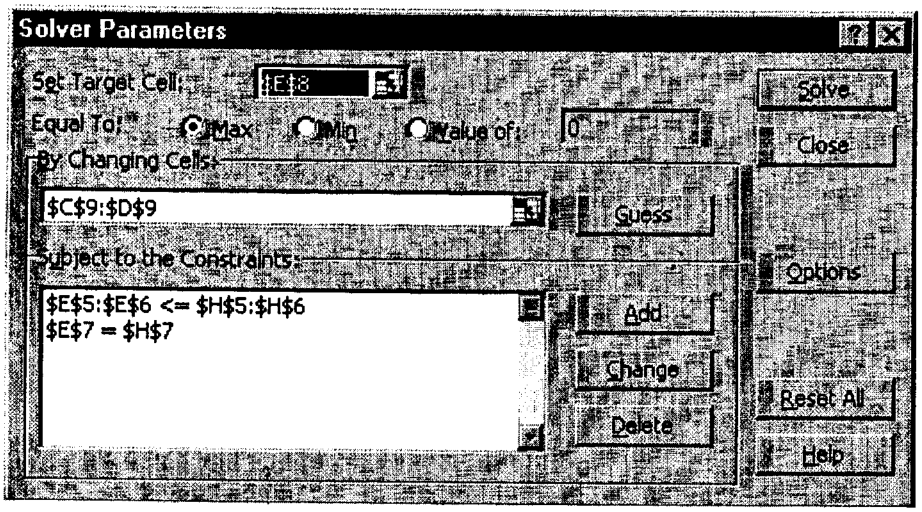
Rachel's plant should assemble 1,875 Thrillseekers and 3,500 Cruisers to obtain a maximum profit of $24,150,000.

h) Because we consider this question independently, the profit for the Thrillseeker remains the same as the profit specified in part (a). The labor hour constraint changes. Each Thrillseeker now requires 7.5 hours for assembly. All formulas and Solver settings used in part (a) remain the same. The values for the new problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	7.5	10.5	48000	<	=	48000
6		Doors	4	2	13000	<	=	20000
7		Cruiser Demand	0	1	3500	<	=	3500
8		Profit ($thousands)	3.6	5.4	**24300**			
9		Solution	**1500**	**3500**				

Rachel's plant should assemble 1,500 Thrillseekers and 3,500 Cruisers for a maximum profit of $24,300,000.

i) Because we consider this question independently, we use the problem formulation used in part (a). In this problem, however, the number of Cruisers assembled has to be strictly equal to the total demand. We use the following new Solver settings:



The formulas used in the problem formulation remain the same as those used in part (a). The values used in the problem follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	48000	<	=	48000
6		Doors	4	2	14500	<	=	20000
7		Cruiser Demand	0	1	3500		=	3500
8		Profit ($thousands)	3.6	5.4	**25650**			
9		Solution	**1875**	**3500**				

The new profit is $25,650,000, which is $26,640,000 – $25,650,000 = $990,000 less than the profit obtained in part (a). This decrease in profit is less than $2,000,000, so Rachel should meet the full demand for the Cruiser.

j) We now combine the new considerations described in parts (f), (g), and (h). In part (f), we decided to use both the advertising campaign and the overtime labor. The advertising campaign raises the demand for the Cruiser to 4,200 sedans, and the overtime labor increases the labor hour capacity of the plant to 60,000 labor hours. In part (g), we decreased the profit generated by a Thrillseeker to $2,800. In part (h), we increased the time to assemble a Thrillseeker to 7.5 hours. Including the increased demand for Cruisers, the increased plant capacity, the decreased unit profit for a Thrillseeker, and the increased time to assemble a Thrillseeker, the new problem is formulated as follows:

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	7.5	10.5	60000	<	=	60000
6		Doors	4	2	16880	<	=	20000
7		Cruiser Demand	0	1	4200	<	=	4200
8		Profit ($thousands)	2.8	5.4	**28616**			
9		Solution	**2120**	**4200**				

The formulas and Solver settings used for this problem are the same as those used in part (a). Rachel's plant should assemble 2,120 Thrillseekers and 4,200 Cruisers for a maximum profit of $28,616,000 – $2,100,000 = $26,516,000.

3-2 a) We want to determine the amount of potatoes and green beans Maria should purchase to minimize ingredient costs. We have two decision variables: one variable to represent the amount (in pounds) of potatoes Maria should purchase and one variable to represent the amount (in pounds) of green beans Maria should purchase. We also have constraints on nutrition, taste, and weight.

Nutrition Constraints
1. We first need to ensure that the dish has 180 grams of protein. We are told that 100 grams of potatoes have 1.5 grams of protein and 10 ounces of green beans have 5.67 grams of protein. Since we have decided to measure our decision variables in pounds, however, we need to determine the grams of protein in one pound of each ingredient.

We perform the following conversion for potatoes:

$$\text{100g of potatoes}\left(\frac{\text{1 lb}}{\text{453.6g}}\right) = \text{0.220459 lb of potatoes}$$

$$\frac{\text{1.5g of protein}}{\text{0.22046 lb of potatoes}} = \frac{\text{6.804g of protein}}{\text{1 lb of potatoes}}$$

We perform the following conversion for green beans:

$$\text{10 oz of green beans}\left(\frac{\text{28.35 g}}{\text{1 oz}}\right) = \text{283.5 g of green beans}$$

$$\text{283.5 g of green beans}\left(\frac{\text{1 lb}}{\text{453.6 g}}\right) = \text{0.625 lb of green beans}$$

$$\frac{\text{5.67 g of protein}}{\text{0.625 lb of green beans}} = \frac{\text{9.072 g of protein}}{\text{1 lb of green beans}}$$

The total grams of protein in the potatoes and green beans Maria purchases for the casserole must be greater than or equal to 180 grams.

2. We next need to ensure that the dish has 80 milligrams of iron. We are told that 100 grams of potatoes have 0.3 milligrams of iron and 10 ounces of green beans have 3.402 milligrams of iron. Since we have decided to measure our decision variables in pounds, however, we need to determine the milligrams of iron in one pound of each ingredient.

We perform the following conversion for potatoes:

$$\frac{\text{0.3 mg of iron}}{\text{0.22046 lb of potatoes}} = \frac{\text{1.3608 mg of iron}}{\text{1 lb of potatoes}}$$

We perform the following conversion for green beans:

$$\frac{\text{0.3402 mg of iron}}{\text{0.625 lb of green beans}} = \frac{\text{5.4432 mg of iron}}{\text{1 lb of green beans}}$$

The total milligrams of iron in the potatoes and green beans Maria purchases for the casserole must be greater than or equal to 80 milligrams.

3. We next need to ensure that the dish has 1,050 milligrams of vitamin C. We are told that 100 grams of potatoes have 12 milligrams of vitamin C and 10 ounces of green beans have 28.35 milligrams of vitamin C. Since we have decided to measure our decision variables in pounds, however, we need to determine the milligrams of vitamin C in one pound of each ingredient.

We perform the following conversion for potatoes:

$$\frac{12 \text{ mg of vitamin C}}{0.22046 \text{ lb of potatoes}} = \frac{54.432 \text{ mg of vitamin C}}{1 \text{ lb of potatoes}}$$

We perform the following conversion for green beans:

$$\frac{28.35 \text{ mg of vitamin C}}{0.625 \text{ lb of green beans}} = \frac{45.36 \text{ mg of vitamin C}}{1 \text{ lb of green beans}}$$

The total milligrams of vitamin C in the potatoes and green beans Maria purchases for the casserole must be greater than or equal to 1,050 milligrams.

<u>Taste Constraint</u>
Edson requires that the casserole contain at least a six to five ratio in the weight of potatoes to green beans. We have:

$$\frac{\text{pounds of potatoes}}{\text{pounds of green beans}} > \frac{6}{5}$$

$$5 \text{ (pounds of potatoes)} > 6 \text{ (pounds of green beans)}$$

<u>Weight Constraint</u>
Finally, Maria requires a minimum of 10 kilograms of potatoes and green beans together. Because we measure potatoes and green beans in pounds, we must perform the following conversion:

$$10 \text{ kg of potatoes and green beans}\left(\frac{1000 \text{ g}}{1 \text{ kg}}\right)\left(\frac{1 \text{ lb}}{453.6 \text{ g}}\right)$$

$$= 22.046 \text{ lb of potatoes and green beans}$$

The amount of potatoes and green beans Maria purchases must weigh 22.046 pounds or more.

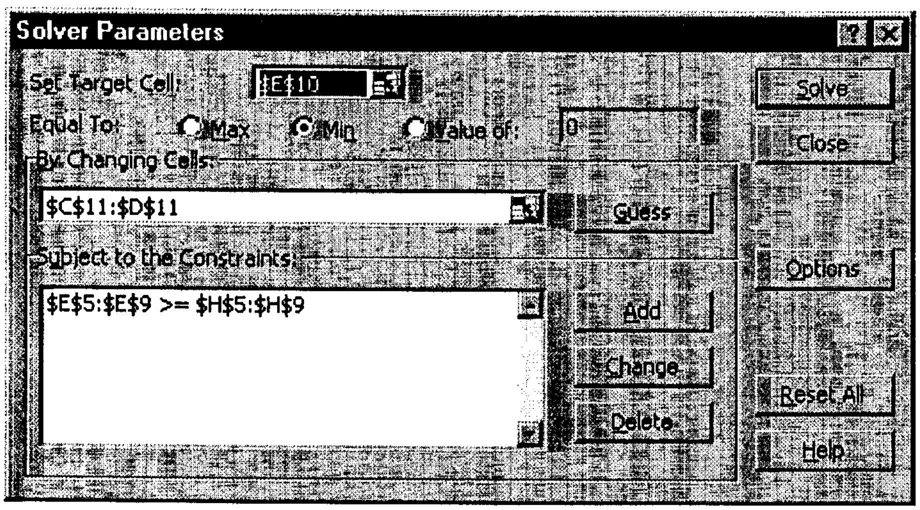
The formulas used in the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	=SUMPRODUCT(C5:D5,C11:D11)	>	=	180
6		Iron (mg)	1.3608	5.4432	=SUMPRODUCT(C6:D6,C11:D11)	>	=	80
7		Vitamin C (mg	54.432	45.36	=SUMPRODUCT(C7:D7,C11:D11)	>	=	1050
8		Taste	5	-6	=SUMPRODUCT(C8:D8,C11:D11)	>	=	0
9		Amount (lb)	1	1	=SUMPRODUCT(C9:D9,C11:D11)	>	=	22.046
10		Cost (per lb)	0.4	1	**=SUMPRODUCT(C10:D10,C11:D11)**			
11		Solution (lb)	**13.5667**	**11.3056**				

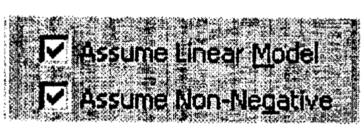
The values for the problem and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	194.8717949	>	=	180
6		Iron (mg)	1.3608	5.4432	80	>	=	80
7		Vitamin C (mg	54.432	45.36	1251282051	>	=	1050
8		Taste	5	-6	0	>	=	0
9		Amount (lb)	1	1	24.87224709	>	=	22.046
10		Cost (per lb)	0.4	1	**16.73223895**			
11		Solution (lb)	**13.567**	**11.306**				

The Solver settings used to solve the problem follow.



Finally, throughout this case we use the following Solver options.



Maria should purchase 13.567 lb of potatoes and 11.306 lb of green beans to obtain a minimum cost of $16.73.

b) The taste constrint changes. The new constraint is now.

$$\frac{\text{pounds of potatoes}}{\text{pounds of green beans}} > \frac{1}{2}$$

2 (pounds of potatoes) > 1 (pounds of green beans)

The formulas and Solver settings used to solve the problem remain the same as in part (a). The values for the problem and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	180	>	=	180
6		Iron (mg)	1.3608	5.4432	80	>	=	80
7		Vitamin C (mg	54.432	45.36	1110	>	=	1050
8		Taste	2	-1	8.450911229	>	=	0
9		Amount (lb)	1	1	22.4132863	>	=	22.046
10		Cost (per lb)	0.4	1	**16.2404468**			
11		Solution (lb)	**10.288**	**12.125**				

Maria should purchase 10.288 lb of potatoes and 12.125 lb of green beans to obtain a minimum cost of $16.24.

c) The right-hand side of the iron constraint changes from 80 mg to 65 mg. The formulas and Solver settings used in the problem remain the same as in part (a). The values for the new problem formulation and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	180	>	=	180
6		Iron (mg)	1.3608	5.4432	65	>	=	65
7		Vitamin C (mg	54.432	45.36	1222.5	>	=	1050
8		Taste	5	-6	31.0479129	>	=	0
9		Amount (lb)	1	1	23.7911522	>	=	22.046
10		Cost (per lb)	0.4	1	**14.31143445**			
11		Solution (lb)	**15.800**	**7.992**				

Maria should purchase 15.8 lb of potatoes and 7.992 lb of green beans to obtain a minimum cost of $14.31.

d) The iron requirement remains 65 mg. We need to change the price per pound of green beans from \$1.00 per pound to \$0.50 per pound. The formulas and Solver settings used in the problem remain the same as in part (a). The values for the new problem formulation and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	180	>	=	180
6		Iron (mg)	1.3608	5.4432	73.89473684	>	=	65
7		Vitamin C (mg	54.432	45.36	1155.789474	>	=	1050
8		Taste	5	-6	0	>	=	0
9		Amount (lb)	1	1	22.97410192	>	=	22.046
10		Cost (per lb)	0.4	0.5	**10.2339181**	3		
11		Solution (lb)	**12.531**	**10.443**				

Maria should purchase 12.531 lb of potatoes and 10.443 lb of green beans to obtain a minimum cost of \$10.23.

e) We still have two decision variables: one variable to represent the amount (in pounds) of potatoes Maria should purchase and one variable to represent the amount (in pounds) of lima beans Maria should purchase. To determine the grams of protein in one pound of lima beans, we perform the following conversion:

$$\frac{22.68 \text{ g of protein}}{0.625 \text{ lb of lima beans}} = \frac{36.288 \text{ g of protein}}{1 \text{ lb of lima beans}}$$

To determine the milligrams of iron in one pound of lima beans, we perform the following conversion:

$$\frac{6.804 \text{ mg of iron}}{0.625 \text{ lb of lima beans}} = \frac{10.8864 \text{ mg of iron}}{1 \text{ lb of lima beans}}$$

Lima beans contain no vitamin C, so we do not have to perform a measurement conversion for vitamin C.

We change the decision variable from green beans to lima beans and insert the new parameters for protein, iron, vitamin C, and cost. The formulas and Solver settings used in the problem remain the same as in part (a). The values for the new problem formulation and solution follows.

	A	B	C	D	E	F	G	H
3			Potatoes	Lima Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	36.288	260.4166667	>	=	180
6		Iron (mg)	1.3608	10.8864	65	>	=	65
7		Vitamin C (mg	54.432	0	1050	>	=	1050
8		Taste	5	-6	75.094	>	=	0
9		Amount (lb)	1	1	22.84961052	>	=	22.046
10		Cost (per lb)	0.4	0.6	**9.85174162**	3		
11		Solution (lb)	**19.290**	**3.559**				

Maria should purchase 19.29 lb of potatoes and 3.559 lb of lima beans to obtain a minimum cost of $9.85.

f) Edson takes pride in the taste of his casserole, and the optimal solution from above does not seem to preserve the taste of the casserole. First, Maria forces Edson to use lima beans instead of green beans, and lima beans are not an ingredient in Edson's original recipe. Second, although Edson places no upper limit on the ratio of potatoes to beans, the above recipe uses an over five to one ratio of potatoes to beans. This ratio seems unreasonable since such a large amount of potatoes will overpower the taste of beans in the recipe.

g) We only need to change the values on the right-hand side of the iron and vitamin C constraints. The formulas and Solver settings used in the problem remain the same as in part (a). The values used in the new problem formulation and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Lima Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	36.288	428.5718034	>	=	180
6		Iron (mg)	1.3608	10.8864	120	>	=	120
7		Vitamin C (mg)	54.432	0	685.7232823	>	=	500
8		Taste	5	-6	6.300	>	=	0
9		Amount (lb)	1	1	22.046	>	=	22.046
10		Cost (per lb)	0.4	0.6	**10.7080406**	1		
11		Solution (lb)	**12.598**	**9.448**				

Maria should purchase 12.598 lb of potatoes and 9.448 lb of lima beans to obtain a minimum cost of $10.71.

3-3 a) The number of operators that the hospital needs to staff the call center during each two-hour shift can be found in the following table:

	A	B	C	D	E	F
1	work	average number of	number of calls from	number of calls from	number of operators	number of operators
2	shift	calls per hour	English speakers	Spanish speakers	speaking English	speaking Spanish
3	7am to 9am	40	32	8	6	2
4	9am to 11am	85	68	17	12	3
5	11am to 1pm	70	56	14	10	3
6	1pm to 3pm	95	76	19	13	4
7	3pm to 5pm	80	64	16	11	3
8	5pm to 7pm	35	28	7	5	2
9	7pm to 9pm	10	8	2	2	1

For example, the average number of phone calls per hour during the shift from 7am to 9am equals 40. Since, on average, 80% of all phone calls are from English speakers, there is an average number of 32 phone calls per hour from English speakers during that shift. Since one operator takes, on average, 6 phone calls per hour, the hospital needs 32/6 = 5.333 English-speaking operators during that shift. The hospital cannot employ fractions of an operator and so needs 6 English-speaking operators for the shift from 7am to 9am.

b) The problems of determining how many Spanish-speaking operators and English-speaking operators Lenny needs to hire to begin each shift are independent. Therefore we can formulate two smaller linear programming models instead of one large model. We are going to have one model for the scheduling of the Spanish-speaking operators and another one for the scheduling of the English-speaking operators.

Lenny wants to minimize the operating costs while answering all phone calls. For the given scheduling problem we make the assumption that the only operating costs are the wages of the employees for the hours that they answer phone calls. The wages for the hours during which they perform paperwork are paid by other cost centers. Moreover, it does not matter for the callers whether an operator starts his or her work day with phone calls or with paperwork. For example, we do not need to distinguish between operators who start their day answering phone calls at 9am and operators who start their day with paperwork at 7am, because both groups of operators will be answering phone calls at the same time. And only this time matters for the analysis of Lenny's problem.

We define the decision variables according to the time when the employees have their first shift of answering phone calls. For the scheduling problem of the English-speaking operators we have 7 decision variables. First, we have 5 decision variables for full-time employees.

The number of operators having their first shift on the phone from 7am to 9am.
The number of operators having their first shift on the phone from 9am to 11am.
The number of operators having their first shift on the phone from 11am to 1pm.
The number of operators having their first shift on the phone from 1pm to 3pm.
The number of operators having their first shift on the phone from 3pm to 5pm.

In addition, we define 2 decision variables for part-time employees.

The number of part-time operators having their first shift from 3pm to 5pm.
The number of part-time operators having their first shift from 5pm to 7pm.

The unit cost coefficients in the objective function are the wages operators earn while they answer phone calls. All operators who have their first shift on the phone from 7am to 9am, 9am to 11am, or 11am to 1pm finish their work on the phone before 5pm. They earn 4*$10 = $40 during their time answering phone calls. All operators who have their first shift on the phone from 1pm to 3pm or 3pm to 5pm have one shift on the phone before 5pm and another one after 5pm. They earn 2*$10+2*$12 = $44 during their time answering phone calls. The second group of part-time operators, those having their first shift from 5pm to 7pm, earn 4*$12 = $48 during their time answering phone calls.

There are 7 constraints, one for each two-hour shift during which phone calls need to be answered. The right-hand sides for these constraints are the number of operators needed to ensure that all phone calls get answered in a timely manner. On the left-hand side we determine the number of operators on the phone during any given shift. For example, during the 11am to 1pm shift the total number of operators answering phone calls equals the sum of the number of operators who started answering calls at 7am and are currently in their second shift of the day and the number of operators who started answering calls at 11am.

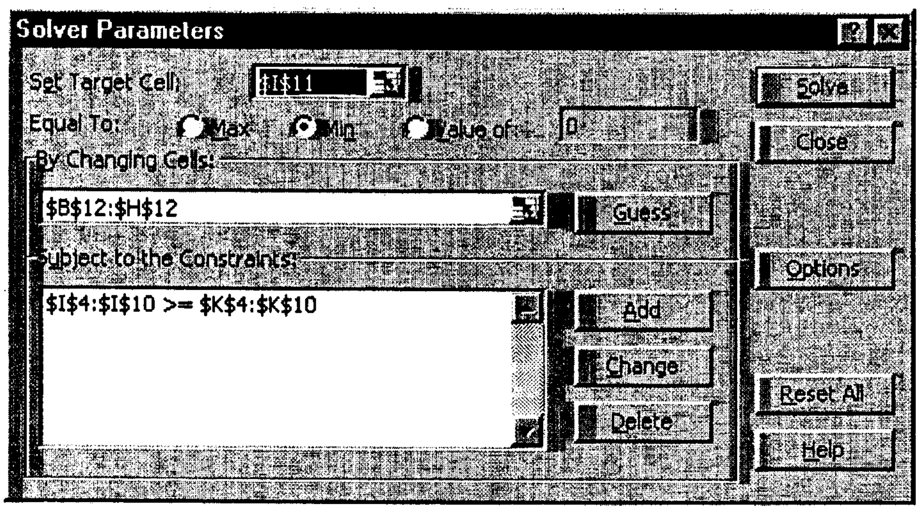
The following spreadsheet describes the entire problem formulation for the English-speaking employees:

	A	B	C	D	E	F	G	H	I	J	K
1											
2	Shifts of	Number of operators whose first shift of answering phone calls in English is from									Required number
3	phone operators	7am to 9am	9am to 11am	11am to 1 pm	1pm to 3pm	3pm to 5pm	3pm to 5pm (P)	5pm to 7pm (P)	Totals		of operators
4	7am to 9am	1	0	0	0	0	0	0	6	>=	6
5	9am to 11am	0	1	0	0	0	0	0	13	>=	12
6	11am to 1 pm	1	0	1	0	0	0	0	10	>=	10
7	1pm to 3pm	0	1	0	1	0	0	0	13	>=	13
8	3pm to 5pm	0	0	1	0	1	1	0	11	>=	11
9	5pm to 7pm	0	0	0	1	0	1	1	5	>=	5
10	7pm to 9pm	0	0	0	0	1	0	1	2	>=	2
11	Unit cost	40	40	40	44	44	44	48	1228	=	Total cost
12	Solution	**6**	**13**	**4**	**0**	**2**	**5**	**0**			

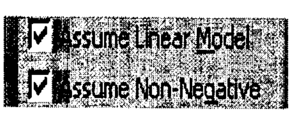
The following formulas are used in the problem formulation:

	I
1	
2	
3	Totals
4	=SUMPRODUCT(B4:H4,B12:H12)
5	=SUMPRODUCT(B5:H5,B12:H12)
6	=SUMPRODUCT(B6:H6,B12:H12)
7	=SUMPRODUCT(B7:H7,B12:H12)
8	=SUMPRODUCT(B8:H8,B12:H12)
9	=SUMPRODUCT(B9:H9,B12:H12)
10	=SUMPRODUCT(B10:H10,B12:H12)
11	**=SUMPRODUCT(B11:H11,B12:H12)**
12	

The solver appears as follows:



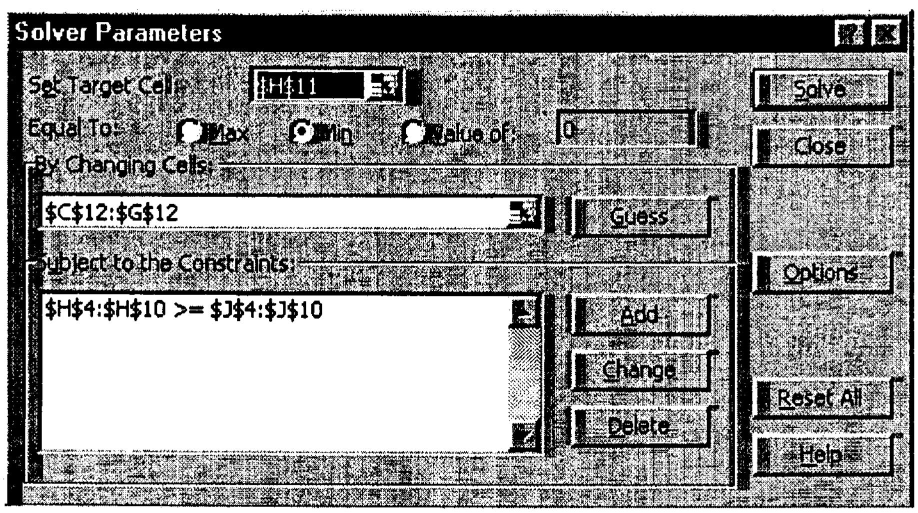
Throughout this analysis we use the following solver options:



The linear programming model for the Spanish-speaking employees can be developed in a similar fashion.

	A	B	C	D	E	F	G	H	I	J
1										
2		Shifts of	Number of operators whose first shift of answering phone calls in Spanish is from							Required number
3		phone operators	7am to 9am	9am to 11am	11am to 1 pm	1pm to 3pm	3pm to 5pm	Totals		of operators
4		7am to 9am	1	0	0	0	0	2	>=	2
5		9am to 11am	0	1	0	0	0	3	>=	3
6		11am to 1 pm	1	0	1	0	0	4	>=	3
7		1pm to 3pm	0	1	0	1	0	5	>=	4
8		3pm to 5pm	0	0	1	0	1	3	>=	3
9		5pm to 7pm	0	0	0	1	0	2	>=	2
10		7pm to 9pm	0	0	0	0	1	1	>=	1
11		Unit cost	40	40	40	44	44	412	=	Total cost
12		Solution	**2**	**3**	**2**	**2**	**1**			

	H
1	
2	
3	Totals
4	=SUMPRODUCT(C4:G4,C12:G12)
5	=SUMPRODUCT(C5:G5,C12:G12)
6	=SUMPRODUCT(C6:G6,C12:G12)
7	=SUMPRODUCT(C7:G7,C12:G12)
8	=SUMPRODUCT(C8:G8,C12:G12)
9	=SUMPRODUCT(C9:G9,C12:G12)
10	=SUMPRODUCT(C10:G10,C12:G12)
11	**=SUMPRODUCT(C11:G11,C12:G12)**
12	

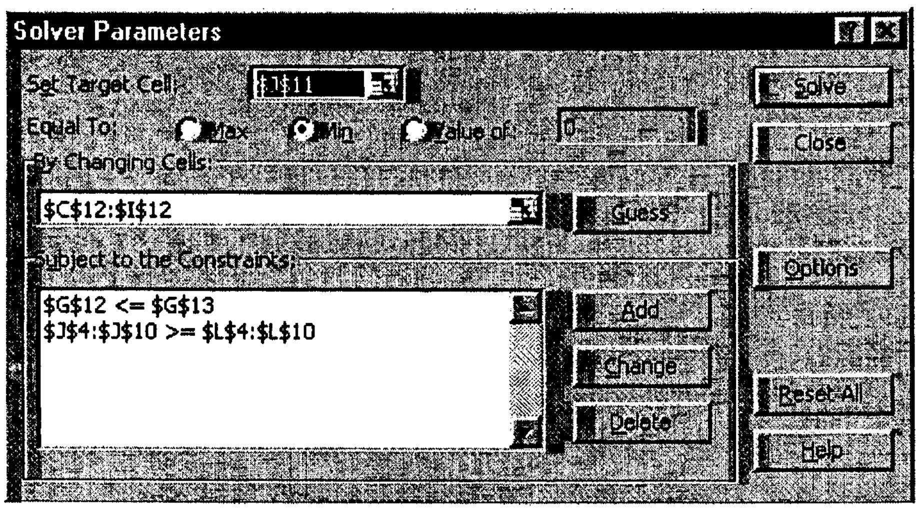


c) Lenny should hire 25 full-time English-speaking operators. Of these operators, 6 have their first phone shift from 7am to 9am, 13 from 9am to 11am, 4 from 11am to 1pm, and 2 from 3pm to 5pm. Lenny should also hire 5 part-time operators who start their work at 3pm. In addition, Lenny should hire 10 Spanish-speaking operators. Of these operators, 2 have their first shift on the phone from 7am to 9am, 3 from 9am to 11am, 2 from 11am to 1pm and 1pm to 3pm, and 1 from 3pm to 5pm. The total (wage) cost of running the calling center equals $1640 per day.

d) The restriction that Lenny can find only one English-speaking operator who wants to start work at 1pm affects only the linear programming model for English-speaking operators. This restriction does not put a bound on the number of operators who start their first phone shift at 1pm because those operators can start work at 11am with paperwork. However, this restriction does put an upper bound on the number of operators having their first phone shift from 3pm to 5pm. The new worksheet appears as follows.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		Shifts of	Number of operators whose first shift of answering phone calls in English is from									Required number
3		phone operators	7 am to 9am	9am to 11am	11am to 1 pm	1pm to 3pm	3pm to 5pm	3pm to 5pm (P)	5 pm to 7pm (P)	Totals		of operators
4		7am to 9am	1	0	0	0	0	0	0	6	>=	6
5		9am to 11am	0	1	0	0	0	0	0	13	>=	12
6		11am to 1 pm	1	0	1	0	0	0	0	12	>=	10
7		1pm to 3pm	0	1	0	1	0	0	0	13	>=	13
8		3pm to 5pm	0	0	1	0	1	1	0	11	>=	11
9		5pm to 7pm	0	0	0	1	0	1	1	5	>=	5
10		7pm to 9pm	0	0	0	0	1	0	1	2	>=	2
11		Unit cost	40	40	40	44	44	44	48	**1268**	=	Total cost
12		Solution	**6**	**13**	**6**	**0**	**1**	**4**	**1**			
13		Upper bounds					1					

The Solver dialogue box displays the additional constraint.



Lenny should hire 26 full-time English-speaking operators. Of these operators, 6 have their first phone shift from 7am to 9am, 13 from 9am to 11am, 6 from 11am to 1pm, and 1 from 3pm to 5pm. Lenny should also hire 4 part-time operators who start their work at 3pm and 1 part-time operator starting work at 5pm. The hiring of Spanish-speaking operators is unaffected. The new total (wage) costs equal $1680 per day.

e) For each hour, we need to divide the average number of calls per hour by the average processing speed, which is 6 calls per hour. The number of bilingual operators that the hospital needs to staff the call center during each two-hour shift can be found in the following table:

	A	B	C
1	work	average number of	number of operators
2	shift	calls per hour	speaking English
3	7am to 9am	40	7
4	9am to 11am	85	15
5	11am to 1pm	70	12
6	1pm to 3pm	95	16
7	3pm to 5pm	80	14
8	5pm to 7pm	35	6
9	7pm to 9pm	10	2

f) The linear programming model for Lenny's scheduling problem can be found in the same way as before, only that now all operators are bilingual.

	A	B	C	D	E	F	G	H	I	J	K
1											
2	Shifts of	Number of operators whose first shift of answering phone calls in both languages is from									Required number
3	phone operators	7am to 9am	9am to 11am	11 am to 1 pm	1pm to 3pm	3pm to 5pm	3pm to 5pm (P)	5pm to 7pm (P)	Totals		of operators
4	7am to 9am	1	0	0	0	0	0	0	7	>=	7
5	9 am to 11am	0	1	0	0	0	0	0	16	>=	15
6	11am to 1 pm	1	0	1	0	0	0	0	13	>=	12
7	1pm to 3pm	0	1	0	1	0	0	0	16	>=	16
8	3pm to 5pm	0	0	1	0	1	1	0	14	>=	14
9	5pm to 7pm	0	0	0	1	0	1	1	6	>=	6
10	7pm to 9pm	0	0	0	0	1	0	1	2	>=	2
11	Unit cost	40	40	40	44	44	44	48	**1512**	=	Total cost
12	Solution	**7**	**16**	**6**	**0**	**2**	**6**	**0**			

(The formulas and the solver dialogue box are identical to those in part (b).)

Lenny should hire 31 full-time bilingual operators. Of these operators, 7 have their first phone shift from 7am to 9am, 16 from 9am to 11am, 6 from 11am to 1pm, and 2 from 3pm to 5pm. Lenny should also hire 6 part-time operators who start their work at 3pm. The total (wage) cost of running the calling center equals $1512 per day.

g) The total cost of part (f) is $1512 per day; the total cost of part (b) is $1640. Lenny could pay an additional $1640-$1512 = $128 in total wages to the bilingual operators without increasing the total operating cost beyond those for the scenario with only monolingual operators. The increase of $128 represents a percentage increase of 128/1512 = 8.466%.

h) Creative Chaos Consultants has made the assumption that the number of phone calls is independent of the day of the week. But maybe the number of phone calls is very different on a Monday than it is on a Friday. So instead of using the same number of average phone calls for every day of the week, it might be more appropriate to determine whether the day of the week affects the demand for phone operators. As a result Lenny might need to hire more part-time employees for some days with an increased calling volume.

Similarly, Lenny might want to take a closer look at the length of the shifts he has scheduled. Using shorter shift periods would allow him to "fine tune" his calling centers and make it more responsive to demand fluctuations.

Lenny should investigate why operators are able to answer only 6 phone calls per hour. Maybe additional training of the operators could enable them to answer phone calls quicker and so increase the number of phone calls they are able to answer in an hour.

Finally, Lenny should investigate whether it is possible to have employees switching back and forth between paperwork and answering phone calls. During slow times phone operators could do some paperwork while they are sitting next to a phone, while in times of sudden large call volumes employees who are scheduled to do paperwork could quickly switch to answering phone calls.

Lenny might also want to think about the installation of an automated answering system that gives callers a menu of selections. Depending upon the caller's selection, the call is routed to an operator who specializes in answering questions about that selection.

Cases 3.4

a) In this case, the decisions to be made are

TV = number of commercials on television

M = Number of advertisements in magazines

SS = Number of advertisements in Sunday supplements

The resulting linear programming model is

Maximize Exposures = 1,300 TV + 600 M + 500 SS

subject to

(1) Resource Constraints:

$300\text{ TV} + 150\text{ M} + 100\text{ SS} \leq 4{,}000$ (ad budget in \$1,000s)

$90\text{ TV} + 30\text{ M} + 40\text{ SS} \leq 1{,}000$ (planning budget in \$1,000s)

$\text{TV} \leq 5$ (television spot avaiable)

(2) Benefit Constraints:

$1.2\text{ TV} + 0.1\text{ M} \geq 5$ (millions of young children)

$0.5\text{ TV} + 0.2\text{ M} + 0.2\text{ SS} \geq 5$ (millions of parents)

(3) Fixed-requirement Constraints:

$40\text{ TV} + 120\text{ SS} = 5$ (coupon budget in \$1,000s)

(4) Nonnegativity Constraints:

$\text{TV} \geq 0, \quad \text{M} \geq 0, \quad \text{SS} \geq 0$

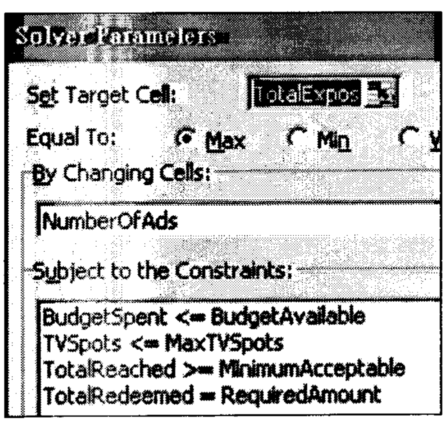
The linear programming spreadsheet model for this problem is shown below.

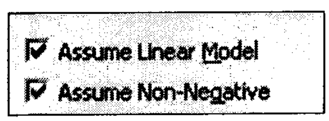
	A	B	C	D	E	F	G	H
1	Super Grain Corp. Advertising-Mix Problem							
2								
3			TV Spots	Magazine Ads	SS Ads			
4		Exposures per Ad	1,300	600	500			
5		(thousands)						
6				Cost per Ad ($thousands)		Budget Spent		Budget Available
7		Ad Budget	300	150	100	3,775	≤	4,000
8		Planning Budget	90	30	40	1,000	≤	1,000
9								
10				Number reached per Ad (millions)		Total Reached		Minimum Acceptable
11		Young Children	1.2	0.1	0	5	>=	5
12		Parents of Young Children	0.5	0.2	0.2	6	>=	5
13								
14			TV Spots	Magazine Ads	SS Ads	Total Redeemed		Required Amount
15		Coupon Redemption	0	40	120	1,490	=	1,490
16		Per Ad ($ thousand)						
17								Total Exposures
18			TV Spots	Magazine Ads	SS Ads			(thousands)
19		Number of Ads	3	14	7.75			16,175
20			≤					
21		Max TV Spots	5					

Range Name	Cells
BudgetAvaiable	H7:H8
BudgetSpent	F7:F8
CostPerAd	C7:E8
CouponRedemptionPerAd	C15:E15
ExposuresPerAD	C4:E4
MaxTVSpots	C21
MinimumAcceptable	H11:H12
NumberOfAds	C19:E19
NumberReachedPerAds	C11:E12
RequiredAmount	H15
TotalExposures	H19
TotalReached	F11:F12
TotalRedeemed	F15
TVSpots	C19

	H
17	Total Exposures
18	(thousands)
19	=SUMPRODUCT(ExposuresPerAd, NumberOfAds)

	F
6	Budget Spent
7	=SUMPRODUCT(C7:E7, NumberOfAds)
8	=SUMPRODUCT(C8:E8, NumberOfAds)
9	
10	Total Reached
11	=SUMPRODUCT(C11:E11, NumberOfAds)
12	=SUMPRODUCT(C12:E12, NumberOfAds)
13	
14	Total Redeemed
15	=SUMPRODUCT(CouponRedemptionPerAd, NumberOfAds)





After making all entries in the Solver dialouge box shown as above, plus selecting usual two Solver options, the Solver find the following optimal plan for the promotional campaign (given in row 19):

Run 3 TV commericals.

Run 14 advertisements in magazines.

Run 7.75 advertisements in Sunday supplements.

b) The violations of four assumptions of LP:

(1) **Proportationality assumption:** the advertisement cost may not be proportional to number of commercials on television or number of advertisements in magzines. The marginal cost for additional commercial can decrease.

(2) **Additivity assumption:** This assumption can be violated for benefit constraints because it states that there is no overlap between people who see the commercial on television or see the advertisements in magzine or Sunday suppleents

(3) **Divisibility assumption:** The decision variables in this case are number of commercial on TV or advertisements in magzines and Sunday supplements of major newspapers. Naturally, these variables should take on integer values.

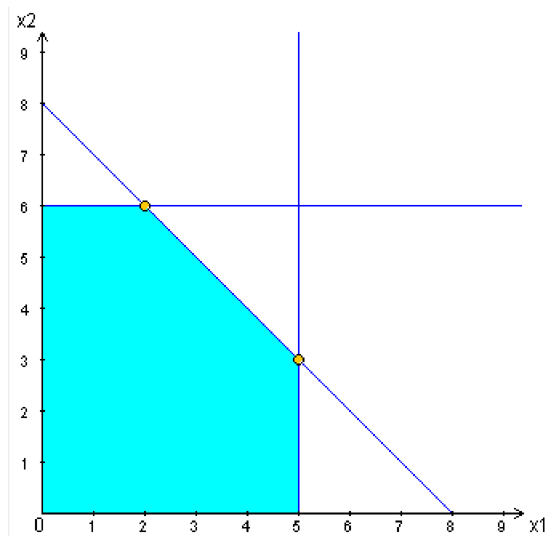
(4) **Certainty assumption:** Since this LP model is formulated to select some future courses of actions, the parameters used in this case, such as Exposures per Ad or Number Reached per Ad, are based on a prediction of future situation, which inevitably introduces some degree of uncertainty.

c) Since none of the assumptions appear to be badly violated, LP is reasonable at least as a first approximation. Later models, such as IP or NLP (as formulated in Case 12.3) can provide some refinement.

CHAPTER 4: SOLVING LINEAR PROGRAMMING PROBLEMS: THE SIMPLEX METHOD

4.1-1.

(a) Label the corner points as A, B, C, D, and E in the clockwise direction starting from $(0, 6)$.



(b)
A: $x_1 = 0$ and $x_2 = 6$
B: $x_2 = 6$ and $x_1 + x_2 = 8$
C: $x_1 + x_2 = 8$ and $x_1 = 5$
D: $x_1 = 5$ and $x_2 = 0$
E: $x_2 = 0$ and $x_1 = 0$

(c)
A: $(x_1,\ x_2) = (0, 6)$
B: $(x_1,\ x_2) = (6, 2)$
C: $(x_1,\ x_2) = (5, 3)$
D: $(x_1,\ x_2) = (5, 0)$
E: $(x_1,\ x_2) = (0, 0)$

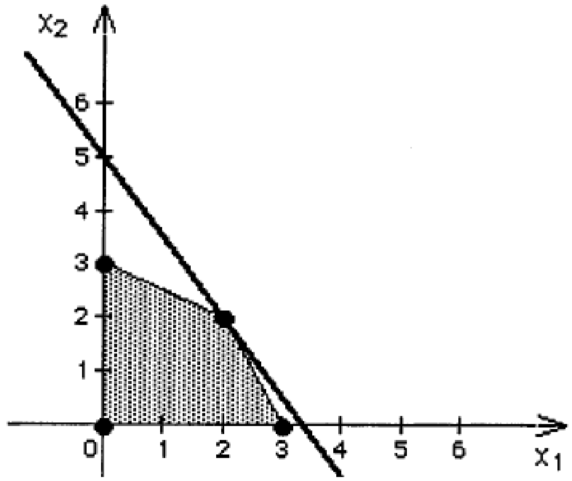
(d)

Corner Point	Adjacent Points
A	E, B
B	A, C
C	B, D
D	C, E
E	D, A

(e)
A and B: $x_2 = 6$
B and C: $x_1 + x_2 = 8$
C and D: $x_1 = 5$
D and E: $x_2 = 0$
E and A: $x_1 = 0$

4.1-2.

(a) Optimal solution: $(x_1^*, x_2^*) = (2, 2)$ with $Z^* = 10$



Label the corner points as A, B, C, and D in the clockwise direction starting from $(0, 3)$.

(b)

Corner Point	Corresponding Constraint Boundary Eq.s	
$A(0,3)$	$x_1 = 0$ and $x_1 + 2x_2 = 6$	$0 = 0$ and $0 + 2 \times 3 = 6$
$B(2,2)$	$x_1 + 2x_2 = 6$ and $2x_1 + x_2 = 6$	$2 + 2 \times 2 = 6$ and $2 \times 2 + 2 = 6$
$C(3,0)$	$2x_1 + x_2 = 6$ and $x_2 = 0$	$2 \times 3 + 0 = 6$ and $0 = 0$
$D(0,0)$	$x_1 = 0$ and $x_2 = 0$	$0 = 0$ and $0 = 0$

(c)

Corner Point	Adjacent Corner Points
$A(0,3)$	$D(0,0)$ and $B(2,2)$
$B(2,2)$	$A(0,3)$ and $C(3,0)$
$C(3,0)$	$B(2,2)$ and $D(0,0)$
$D(0,0)$	$C(3,0)$ and $A(0,3)$

(d) Optimal Solution: $(x_1^*, x_2^*) = (2, 2)$ with $Z^* = 10$

Corner Point (x_1, x_2)	Profit $= 3x_1 + 2x_2$
$A(0,3)$	6
$B(2,2)$	10
$C(3,0)$	9
$D(0,0)$	0

(e)

Corner Point	Profit	Next Step
$D(0,0)$	0	Check A and C.
$A(0,3)$	6	Move to C.
$C(3,0)$	9	Check B.
$B(2,2)$	10	Stop, B is optimal.*

* The next corner point is A, which has already been checked.

4.1-3.

(a)

Corner Point (A_1, A_2)	Profit $= 1,000A_1 + 2,000A_2$
$(0, 0)$	0
$(8, 0)$	$8,000$
$(6, 4)$	$14,000$
$(5, 5)$	$15,000$
$(0, 6.667)$	$13,333$

Optimal Solution: $(A_1^*, A_2^*) = (5, 5)$ with $Z^* = \$15,000$

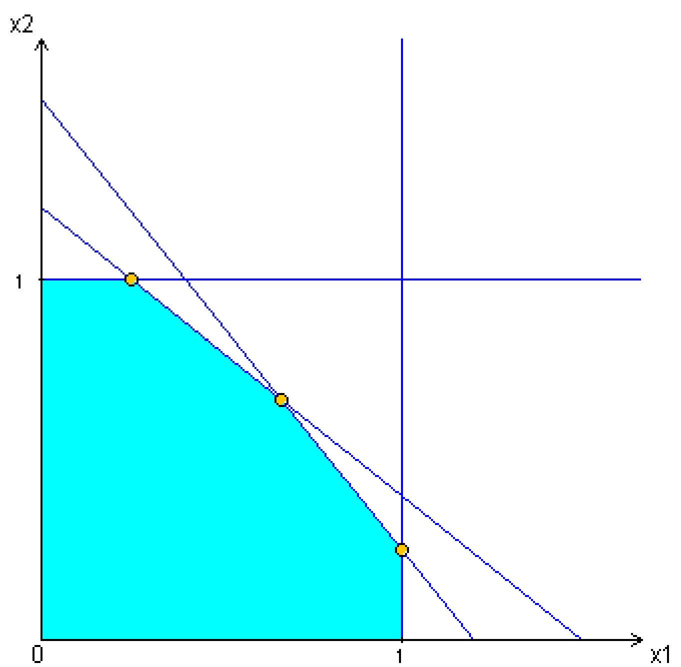
(b) Initiated at the origin, the simplex method can follow one of the two paths:

$(0, 0) \to (8, 0) \to (6, 4) \to (5, 5)$ or $(0, 0) \to (0, 6.7) \to (5, 5)$.

Consider the first path. The origin $(0, 0)$ is not optimal, since $(0, 6.7)$ and $(8, 0)$ are adjacent to $(0, 0)$, both are feasible and they have better objective values. $(8, 0)$ is not optimal because $(6, 4)$, which is adjacent to it, is feasible and better. $(5, 5)$ is optimal since both corner points that are adjacent to it are worse.

4.1-4.

(a)



(b)

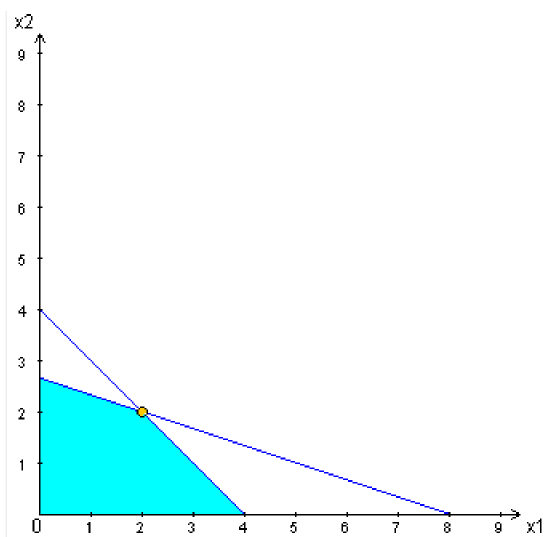
	CP Solution	Feasibility	Objective
A	$\left(0, \frac{3}{2}\right)$	Infeasible	6750
B	$\left(0, \frac{6}{5}\right)$	Infeasible	5400
C	$(0, 1)$	Feasible	4500
D	$\left(\frac{1}{4}, 1\right)$	Feasible	5625
E	$\left(\frac{2}{5}, 1\right)$	Infeasible	6300
F	$(1, 1)$	Infeasible	9000
G	$\left(\frac{2}{3}, \frac{2}{3}\right)$	Feasible	6000 *
H	$\left(1, \frac{2}{5}\right)$	Infeasible	6300
I	$\left(1, \frac{1}{4}\right)$	Feasible	5625
J	$(1, 0)$	Feasible	4500
K	$\left(\frac{6}{5}, 0\right)$	Infeasible	5400
L	$\left(\frac{3}{2}, 0\right)$	Infeasible	6750
M	$(0, 0)$	Feasible	0

The point G is optimal.

(c) Start at the origin M $= (0, 0)$. Both adjacent points C $= (1, 0)$ and J $= (0, 1)$ are feasible and have better objective values,so one can choose to move to either one of them. Suppose we choose C, which is not optimal since its adjacent CPF solution D is better. The other corner point that is adjacent to C is B, but it is infeasible, so move to D. Its adjacent G is feasible and better. The CPF solutions that are adjacent to G, namely D and I both have lower objective values. Hence, G is optimal. If one chooses to proceed to J instead of C after the starting point, then the simplex path follows the points M, J, I, G and using similar arguments, one obtains the optimality of G.

4.1-5.

(a)



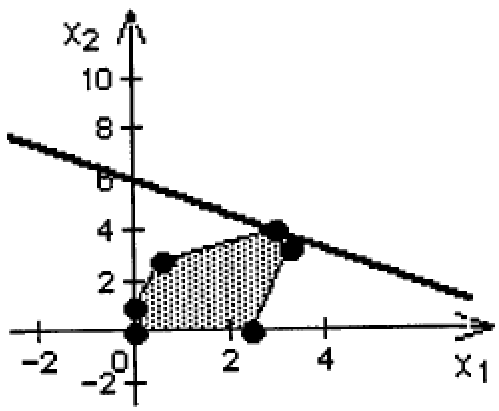
(b)

	CP Solution	Feasibility	Objective
A	$(0,4)$	Infeasible	8
B	$(0,\frac{8}{3})$	Feasible	$5\frac{1}{3}$
C	$(2,2)$	Feasible	6^*
D	$(4,0)$	Feasible	4
E	$(8,0)$	Infeasible	8
F	$(0,0)$	Feasible	0

The point C is optimal.

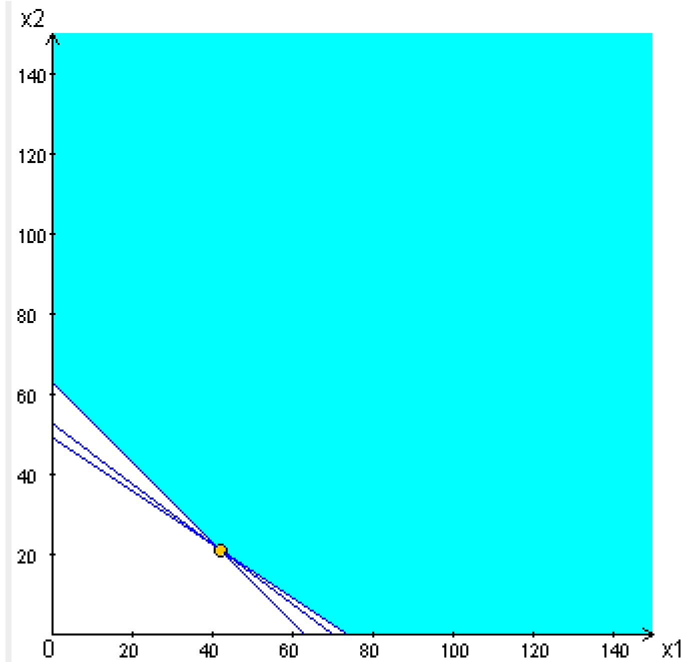
(c) The starting point F is not optimal, since B and D have better objective values. The objective value z increases faster along the edge FB ($5\frac{1}{3}/\frac{8}{3} = 2$) than along the edge FD ($4/4 = 1$), so we choose to move to point B. B is not optimal because the adjacent point C does better. Note that A is adjacent to B as well, but it is infeasible. C is optimal since the two CPF solutions adjacent to C, namely B and D have lower objective values.

4.1-6.



Corner Point	Profit $= 2x_1 + 3x_2$	Next Step
$(0,0)$	0	Check $(2.5,0)$ and $(0,1)$.
$(2.5,0)$	5	Move to $(2.5,0)$.
$(0,1)$	3	Check $(3.3333,3.333)$.
$(3.3333,3.333)$	16.667	Move to $(3,4)$. Check $(3,4)$.
$(3,4)$	18	Move to $(3,4)$. Check $(0.6,2.8)$.
$(0.6,2.8)$	9.6	Stop, $(3,4)$ is optimal.

4.1-7.



Corner Point	Cost $= 5x_1 + 7x_2$	Next Step
$(42, 21)$	357	Check $(73.5, 0)$ and $(0, 63)$.
$(73.5, 0)$	367.5	Stop, $(42, 21)$ is optimal.
$(0, 63)$	441	

4.1-8.

(a) TRUE. Use optimality test. In minimization problems, "better" means smaller. To see this, note that min $Z = -\max(-Z)$.

(b) FALSE. CPF solutions are not the only possible optimal solutions, there can be infinitely many optimal solutions. This is indeed the case when there are more than one optimal solution. For example, consider the problem

$$\begin{array}{lll} \text{maximize} & Z = & x_1 + x_2 \\ \text{subject to} & & x_1 + x_2 \leq 10 \\ & & x_1, x_2 \quad \geq 0 \end{array}$$

where $Z^* = 10$, $x_1^* = k$ and $x_2^* = 10 - k$ with $k \in [0, 10]$ are all optimal solutions.

(c) TRUE. However, this is not always true. It is possible to have an unbounded feasible region where an entire ray with only one CPF solution is optimal.

4.1-9.

(a) The problem may not have an optimal solution.

(b) The optimality test checks whether the current corner point is optimal. The iterative step only moves to a new corner point.

(c) The simplex method can choose the origin as the initial corner point only when it is feasible.

(d) One of the adjacent points is likely to be better, not necessarily optimal.

(e) The simplex method only identifies the rate of improvement, not all the adjacent corner points.

4.2-1.

(a) Augmented form:

$$\begin{array}{ll} \text{maximize} & 4500x_1 + 4500x_2 \\ \text{subject to} & \begin{array}{rcl} x_1 \qquad\qquad + x_3 \qquad\qquad\qquad\qquad & = & 1 \\ x_2 \qquad\quad + x_4 \qquad\qquad\quad & = & 1 \\ 5000x_1 + 4000x_2 \qquad\qquad + x_5 \qquad & = & 6000 \\ 400x_1 + 500x_2 \qquad\qquad\qquad + x_6 & = & 600 \\ x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 \end{array} \end{array}$$

(b)

	CPF Solution	BF Solution	Nonbasic Variables	Basic Variables
A	$(0,1)$	$(0,1,1,0,2000,100)$	x_1, x_4	x_2, x_3, x_5, x_6
B	$(\frac{1}{4},1)$	$(\frac{1}{4},1,\frac{3}{4},0,750,0)$	x_4, x_6	x_1, x_2, x_3, x_5
C	$(\frac{2}{3},\frac{2}{3})$	$(\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3},0,0)$	x_5, x_6	x_1, x_2, x_3, x_4
D	$(1,\frac{1}{4})$	$(1,\frac{1}{4},0,\frac{3}{4},0,75)$	x_3, x_5	x_1, x_2, x_4, x_6
E	$(1,0)$	$(1,0,0,1,1000,200)$	x_2, x_3	x_1, x_4, x_5, x_6
F	$(0,0)$	$(0,0,1,1,6000,600)$	x_1, x_2	x_3, x_4, x_5, x_6

(c) <u>BF Solution A:</u> Set $x_1 = x_4 = 0$ and solve

$$\begin{aligned} &x_3 = 1 \\ &x_2 = 1 \\ &4000x_2 + x_5 = 6000 \;\Rightarrow\; x_5 = 2000 \\ &500x_2 + x_6 = 600 \;\Rightarrow\; x_6 = 100 \end{aligned}$$

<u>BF Solution B:</u> Set $x_4 = x_6 = 0$ and solve

$$\begin{aligned} &x_1 + x_3 = 1 \;\Rightarrow\; x_3 = 3/4 \\ &x_2 = 1 \\ &5000x_1 + 4000x_2 + x_5 = 6000 \;\Rightarrow\; x_5 = 750 \\ &400x_1 + 500x_2 = 600 \;\Rightarrow\; x_1 = 1/4 \end{aligned}$$

<u>BF Solution C:</u> Set $x_5 = x_6 = 0$ and solve

$$\begin{aligned} &x_1 + x_3 = 1 \\ &x_2 + x_4 = 1 \\ &5000x_1 + 4000x_2 = 6000 \\ &400x_1 + 500x_2 = 600 \end{aligned}$$

From the last two equations, $x_1 = x_2 = 2/3$ and from the first two, $x_3 = x_4 = 1/3$.

BF Solution D: Set $x_3 = x_5 = 0$ and solve

$$x_1 = 1$$
$$x_2 + x_4 = 1 \;\Rightarrow\; x_4 = 3/4$$
$$5000x_1 + 4000x_2 = 6000 \;\Rightarrow\; x_2 = 1/4$$
$$400x_1 + 500x_2 + x_6 = 600 \;\Rightarrow\; x_6 = 75$$

BF Solution E: Set $x_2 = x_3 = 0$ and solve

$$x_1 = 1$$
$$x_4 = 1$$
$$5000x_1 + x_5 = 6000 \;\Rightarrow\; x_5 = 1000$$
$$400x_1 + x_6 = 600 \;\Rightarrow\; x_6 = 200$$

BF Solution F: Set $x_1 = x_2 = 0$ and solve

$$x_3 = 1$$
$$x_4 = 1$$
$$x_5 = 6000$$
$$x_6 = 600$$

4.2-2.

(a) Augmented form:

maximize $\quad x_1 + 2x_2$

subject to
$$\begin{aligned} x_1 + 3x_2 + x_3 \quad\quad &= 8 \\ x_1 + \;\; x_2 \quad\quad + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(b)

	CPF Solution	BF Solution	Nonbasic Variables	Basic Variables
A	$(0,0)$	$(0,0,8,4)$	x_1, x_2	x_3, x_4
B	$(0,\frac{8}{3})$	$(0,\frac{8}{3},0,\frac{4}{3})$	x_1, x_3	x_2, x_4
C	$(2,2)$	$(2,2,0,0)$	x_3, x_4	x_1, x_2
D	$(4,0)$	$(4,0,4,0)$	x_2, x_4	x_1, x_3

(c) BF Solution A: Set $x_1 = x_2 = 0$ and solve

$$x_3 = 8$$
$$x_4 = 4$$

BF Solution B: Set $x_1 = x_3 = 0$ and solve

$$3x_2 = 8 \;\Rightarrow\; x_2 = 8/3$$
$$x_2 + x_4 = 4 \;\Rightarrow\; x_4 = 4/3$$

BF Solution C: Set $x_3 = x_4 = 0$ and solve

$$x_1 + 3x_2 = 8$$
$$x_1 + x_2 = 4$$

From these two equations, $x_1 = x_2 = 2$.

BF Solution D: Set $x_2 = x_4 = 0$ and solve

$$x_1 + x_3 = 8 \Rightarrow x_3 = 4$$
$$x_1 = 4$$

(d)

	CP Infeasible Sol.'n	Basic Infeasible Sol.'n	Nonbasic Var.'s	Basic Var.'s
E	$(0,4)$	$(0,4,-4,0)$	x_1, x_4	x_2, x_3
F	$(8,0)$	$(8,0,0,-4)$	x_2, x_3	x_1, x_4

(e) Basic Infeasible Solution E: Set $x_1 = x_4 = 0$ and solve

$$3x_2 + x_3 = 8 \Rightarrow x_3 = -4$$
$$x_2 = 4$$

Basic Infeasible Solution F: Set $x_2 = x_3 = 0$ and solve

$$x_1 = 8$$
$$x_1 + x_4 = 4 \Rightarrow x_4 = -4$$

4.3-1.

After the sudden decline of prices at the end of 1995, Samsung Electronics faced the urgent need to improve its noncompetitive cycle times. The project called SLIM (short cycle time and low inventory in manufacturing) was initiated to address this problem. As part of this project, floor-scheduling problem is formulated as a linear programming model. The goal is to identify the optimal values "for the release of new lots into the fab and for the release of initial WIP from every major manufacturing step in discrete periods, such as work days, out to a horizon defined by the user" [p. 71]. Additional variables are included to determine the route of these through alternative machines. The optimal values "minimize back-orders and finished-goods inventory" [p. 71] and satisfy capacity constraints and material flow equations. CPLEX was used to solved the linear programs.

With the implementation of SLIM, Samsung significantly reduced its cycle times and as a result of this increased its revenue by $1 billion (in five years) despite the decrease in selling prices. The market share increased from 18 to 22 percent. The utilization of machines was improved. The reduction in lead times enabled Samsung to forecast sales more accurately and so to carry less inventory. Shorter lead times also meant happier customers and a more efficient feedback mechanism, which allowed Samsung to respond to customer needs. Hence, SLIM did not only help Samsung to survive a crisis that drove many out of the business, but it did also provide a competitive advantage in the business.

4.3-2.

Optimal Solution: $(x_1^*, x_2^*) = \left(\frac{2}{3}, \frac{2}{3}\right)$, $Z^* = 6000$

$$\text{Max } Z = 4500\, X_1 + 4500\, X_2$$

subject to

$$
\begin{array}{lrcrl}
1) & 1\, X_1 & + & 0\, X_2 & \le 1 \\
2) & 0\, X_1 & + & 1\, X_2 & \le 1 \\
3) & 5000\, X_1 & + & 4000\, X_2 & \le 6000 \\
4) & 400\, X_1 & + & 500\, X_2 & \le 600
\end{array}
$$

and

$$X_1 \ge 0,\ X_2 \ge 0.$$

Solve Interactively by the Simplex Method:

$$
\begin{array}{ll}
0) & Z - 4500\, X_1 - 4500\, X_2 + 0\, X_3 + 0\, X_4 + 0\, X_5 + 0\, X_6 = 0 \\
1) & 1\, X_1 + 0\, X_2 + 1\, X_3 + 0\, X_4 + 0\, X_5 + 0\, X_6 = 1 \\
2) & 0\, X_1 + 1\, X_2 + 0\, X_3 + 1\, X_4 + 0\, X_5 + 0\, X_6 = 1 \\
3) & 5000\, X_1 + 4000\, X_2 + 0\, X_3 + 0\, X_4 + 1\, X_5 + 0\, X_6 = 6000 \\
4) & 400\, X_1 + 500\, X_2 + 0\, X_3 + 0\, X_4 + 0\, X_5 + 1\, X_6 = 600
\end{array}
$$

$$X_1 \ge 0,\ X_2 \ge 0,\ X_3 \ge 0,\ X_4 \ge 0,\ X_5 \ge 0,\ X_6 \ge 0.$$

$$
\begin{array}{ll}
0) & Z + 0\, X_1 - 4500\, X_2 + 4500\, X_3 + 0\, X_4 + 0\, X_5 + 0\, X_6 = 4500 \\
1) & 1\, X_1 + 0\, X_2 + 1\, X_3 + 0\, X_4 + 0\, X_5 + 0\, X_6 = 1 \\
2) & 0\, X_1 + 1\, X_2 + 0\, X_3 + 1\, X_4 + 0\, X_5 + 0\, X_6 = 1 \\
3) & 0\, X_1 + 4000\, X_2 - 5000\, X_3 + 0\, X_4 + 1\, X_5 + 0\, X_6 = 1000 \\
4) & 0\, X_1 + 500\, X_2 - 400\, X_3 + 0\, X_4 + 0\, X_5 + 1\, X_6 = 200
\end{array}
$$

$$X_1 \ge 0,\ X_2 \ge 0,\ X_3 \ge 0,\ X_4 \ge 0,\ X_5 \ge 0,\ X_6 \ge 0.$$

$$
\begin{array}{ll}
0) & Z + 0\, X_1 + 0\, X_2 - 1125\, X_3 + 0\, X_4 + 1.12\, X_5 + 0\, X_6 = 5625 \\
1) & 1\, X_1 + 0\, X_2 + 1\, X_3 + 0\, X_4 + 0\, X_5 + 0\, X_6 = 1 \\
2) & 0\, X_1 + 0\, X_2 + 1.25\, X_3 + 1\, X_4 - 2e{-}4\, X_5 + 0\, X_6 = 0.75 \\
3) & 0\, X_1 + 1\, X_2 - 1.25\, X_3 + 0\, X_4 + 2e{-}4\, X_5 + 0\, X_6 = 0.25 \\
4) & 0\, X_1 + 0\, X_2 + 225\, X_3 + 0\, X_4 - 0.12\, X_5 + 1\, X_6 = 75
\end{array}
$$

$$X_1 \ge 0,\ X_2 \ge 0,\ X_3 \ge 0,\ X_4 \ge 0,\ X_5 \ge 0,\ X_6 \ge 0.$$

$$
\begin{array}{ll}
0) & Z + 0\, X_1 + 0\, X_2 + 0\, X_3 + 0\, X_4 + 0.5\, X_5 + 5\, X_6 = 6000 \\
1) & 1\, X_1 + 0\, X_2 + 0\, X_3 + 0\, X_4 + 6e{-}4\, X_5 - 4e{-}3\, X_6 = 0.66667 \\
2) & 0\, X_1 + 0\, X_2 + 0\, X_3 + 1\, X_4 + 4e{-}4\, X_5 - 6e{-}3\, X_6 = 0.33333 \\
3) & 0\, X_1 + 1\, X_2 + 0\, X_3 + 0\, X_4 - 4e{-}4\, X_5 + 6e{-}3\, X_6 = 0.66667 \\
4) & 0\, X_1 + 0\, X_2 + 1\, X_3 + 0\, X_4 - 6e{-}4\, X_5 + 4e{-}3\, X_6 = 0.33333
\end{array}
$$

$$X_1 \ge 0,\ X_2 \ge 0,\ X_3 \ge 0,\ X_4 \ge 0,\ X_5 \ge 0,\ X_6 \ge 0.$$

4.3-3.

(a) maximize $Z = x_1 + 2x_2$

subject to
$$\begin{aligned} x_1 + 3x_2 + x_3 \quad &= 8 \\ x_1 + x_2 \quad + x_4 &= 4 \\ x_1, x_2, x_3, x_4 \quad &\geq 0 \end{aligned}$$

Initialization: $x_1 = x_2 = 0 \;\Rightarrow x_3 = 8,\ x_4 = 4,\ z = x_1 + 2x_2 = 0$, is not optimal since the improvement rates are positive. Since it offers a rate of improvement of 2, choose to increase x_2, which becomes the entering basic variable for Iteration 1. Given $x_1 = 0$, the highest possible increase in x_2 is found by looking at:

$$x_3 = 8 - 3x_2 \geq 0 \;\Rightarrow\; x_2 \leq 8/3$$

$$x_4 = 4 - x_2 \geq 0 \;\Rightarrow\; x_2 \leq 4$$

The minimum of these two bounds is $8/3$, so x_2 can be raised to $8/3$ and $x_3 = 0$ leaves the basis. Using Gaussian elimination, we obtain:

$$\begin{aligned} Z = \quad & \tfrac{1}{3}x_1 \qquad - \tfrac{2}{3}x_3 \qquad + \tfrac{16}{3} \\ & \tfrac{1}{3}x_1 + x_2 + \tfrac{1}{3}x_3 \qquad = \tfrac{8}{3} \\ & \tfrac{2}{3}x_1 \qquad - \tfrac{1}{3}x_3 + x_4 = \tfrac{4}{3} \\ & x_1, x_2, x_3, x_4 \qquad \geq 0 \end{aligned}$$

Again $\left(0, \frac{8}{3}, 0, \frac{4}{3}\right)$ is not optimal since the rate of improvement for x_1 is $\frac{1}{3} > 0$ and x_1 can be increased to 2. Consequently, x_4 becomes 0. By Gaussian elimination:

$$\begin{aligned} Z = \quad & \qquad - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_4 + 6 \\ & \quad x_2 + \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 = 2 \\ & x_1 \qquad - \tfrac{1}{2}x_3 + \tfrac{3}{2}x_4 = 2 \\ & x_1, x_2, x_3, x_4 \qquad \geq 0 \end{aligned}$$

The current solution is optimal, since increasing x_3 or x_4 would decrease the objective value. Hence $x^* = (2, 2, 0, 0)$, $Z^* = 6$.

(b) Optimal Solution: $\left(x_1^*, x_2^*\right) = \left(2, 2\right)$, $Z^* = 6$

Solve Interactively by the Simplex Method:

```
0) Z-  1 X1-  2 X2+  0 X3+  0 X4 = 0
1)     1 X1+  3 X2+  1 X3+  0 X4 = 8
2)     1 X1+  1 X2+  0 X3+  1 X4 = 4

X1 ≥ 0, X2 ≥ 0, X3 ≥ 0, X4 ≥ 0.
```

$$
\begin{aligned}
&0)\ Z - 0.33\ X_1 + 0\ X_2 + 0.67\ X_3 + 0\ X_4 = 5.33333\\
&1)\ \boxed{0.333\ X_1} + 1\ X_2 + 0.33\ X_3 + 0\ X_4 = 2.66667\\
&2)\ \boxed{0.667\ X_1} + 0\ X_2 - 0.33\ X_3 + 1\ X_4 = 1.33333
\end{aligned}
$$

$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0,\ X_4 \geq 0.$

$$
\begin{aligned}
&0)\ Z + 0\ X_1 + 0\ X_2 + 0.5\ X_3 + 0.5\ X_4 = 6\\
&1)\ 0\ X_1 + 1\ X_2 + 0.5\ X_3 - 0.5\ X_4 = 2\\
&2)\ 1\ X_1 + 0\ X_2 - 0.5\ X_3 + 1.5\ X_4 = 2
\end{aligned}
$$

$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0,\ X_4 \geq 0.$

(c) The solution is the same.

Value of the Objective Function: Z = 6

Variable	Value
X_1	2
X_2	2

Objective Function Coefficient

Current Value	Allowable Range To Stay Optimal: Minimum	Maximum
1	0.66667	2
2	1	3

4.3-4.

Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, 10, 6\frac{2}{3})$, $Z^* = 70$

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	Right Side
Z	0	1	-4	-3	-6	0	0	0
X_4	1	0	3	1	3	1	0	30
X_5	2	0	2	2	3	0	1	40

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	Right Side
Z	0	1	2	-1	0	2	0	60
X_3	1	0	1	0.3333	1	0.3333	0	10
X_5	2	0	-1	1	0	-1	1	10

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	Right Side
Z	0	1	1	0	0	1	1	70
X_3	1	0	1.3333	0	1	0.6667	-0.333	6.66667
X_2	2	0	-1	1	0	-1	1	10

4.3-5.

Optimal Solution: $\left(x_1^*, x_2^*, x_3^*\right) = \left(0, 23.68, 25.26\right)$, $Z^* = 221.1$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-3	-4	-5	0	0	0	0
X4	1	0	3	1	5*	1	0	0	150
X5	2	0	1	4	1	0	1	0	120
X6	3	0	2	0	2	0	0	1	105

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	-3	0	1	0	0	150
X3	1	0	0.6	0.2	1	0.2	0	0	30
X5	2	0	0.4	3.8*	0	-0.2	1	0	90
X6	3	0	0.8	-0.4	0	-0.4	0	1	45

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0.316	0	0	0.842	0.789	0	221.1
X3	1	0	0.579	0	1	0.211	-0.05	0	25.26
X2	2	0	0.105	1	0	-0.05	0.263	0	23.68
X6	3	0	0.842	0	0	-0.42	0.105	1	54.47

4.3-6.

(a) The simplest adaptation of the simplex method is to force x_2 and x_3 into the basis at the earliest opportunity. One can also find the optimal solution directly by using Gaussian elimination.

(b) $Z = 5x_1 + 3x_2 + 4x_3$
$2x_1 + x_2 + x_3 + x_4 = 20$
$3x_1 + x_2 + 2x_3 + x_5 = 30$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

(i) Increase x_2 setting $x_1 = x_3 = 0$.
$x_4 = 20 - x_2 \geq 0 \Rightarrow x_2 \leq 20 \leftarrow$ minimum
$x_5 = 30 - x_2 \geq 0 \Rightarrow x_2 \leq 30$
Let $x_2 = 20$ and $x_4 = 0$.
$Z = -x_1 + x_3 - 3x_4 + 60$
$2x_1 + x_2 + x_3 + x_4 = 20$
$x_1 + x_3 - x_4 + x_5 = 10$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

(ii) Increase x_3 setting $x_1 = x_4 = 0$.
$x_2 = 20 - x_3 \geq 0 \Rightarrow x_3 \leq 20$
$x_5 = 10 - x_3 \geq 0 \Rightarrow x_2 \leq 10 \leftarrow$ minimum
Let $x_3 = 10$ and $x_5 = 0$.
$Z = -2x_1 - 2x_4 - x_5 + 70$
$x_1 + x_2 + 2x_4 - x_5 = 10$
$x_1 + x_3 - x_4 + x_5 = 10$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, 10, 10)$ and $Z^* = 70$

4.3-7.

(a) Because $x_2 = 0$ in the optimal solution, the problem can be reduced to:

$$
\begin{array}{lll}
\text{maximize} & Z = & 2x_1 + 3x_3 \\
\text{subject to} & & x_1 + 2x_3 \leq 30 \\
& & x_1 + x_3 \leq 24 \\
& & 3x_1 + 3x_3 \leq 60 \\
& & x_1, x_3 \quad \geq 0
\end{array}
$$

or equivalently

$$
\begin{array}{lll}
\text{maximize} & z = & 2x_1 + 3x_3 \\
\text{subject to} & & x_1 + 2x_3 \leq 30 \\
& & x_1 + x_3 \leq 20 \\
& & x_1, x_3 \quad \geq 0
\end{array}
$$

Since $x_1 > 0$ and $x_3 > 0$ in the optimal solution, they should be basic variables in the optimal solution. Choosing these two as the first two entering basic variables will lead to an optimal solution. The leaving basic variables will be determined by the minimum ratio test.

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (10, 0, 10)$ and $Z^* = 50$

Basic			Coefficient of				
Variable	Eq.	Z	X1	X3	X4	X5	RHS
Z	0	1	-2	-3	0	0	0
X4	1	0	1	2	1	0	30
X5	2	0	1	1	0	1	20

Basic			Coefficient of				
Variable	Eq.	Z	X1	X3	X4	X5	RHS
Z	0	1	-0.5	0	1.5	0	45
X3	1	0	0.5	1	0.5	0	15
X5	2	0	0.5	0	-0.5	1	5

Basic Variable	Eq.	Z	Coefficient of X1	X3	X4	X5	RHS
Z	0	1	0	0	1	1	50
X3	1	0	0	1	1	-1	10
X1	2	0	1	0	-1	2	10

4.3-8.

(a) FALSE. The simplex method's rule for choosing the entering basic variable is used because it gives the best rate of improvement for the objective value at the given corner point.

(b) TRUE. The simplex method's rule for choosing the leaving basic variable determines which basic variable drops to zero first as the entering basic variable is increased. Choosing any other one can cause this variable to become negative, so infeasible.

(c) FALSE. When the simplex method solves for the next BF solution, elementary algebraic operations are used to eliminate each basic variable from all but one equation (its equation) and to give it a coefficient of one in that equation.

4.4-1.

Optimal Solution: $(x_1^*, x_2^*) = (2/3, 2/3)$ and $Z^* = 6,000$

Solve Interactively by the Simplex Method:

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	-4500	-4500	0	0	0	0	0
X_3	1	0	1	0	1	0	0	0	1
X_4	2	0	0	1	0	1	0	0	1
X_5	3	0	5000	4000	0	0	1	0	6000
X_6	4	0	400	500	0	0	0	1	600

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	0	-4500	4500	0	0	0	4500
X_1	1	0	1	0	1	0	0	0	1
X_4	2	0	0	1	0	1	0	0	1
X_5	3	0	0	4000	-5000	0	1	0	1000
X_6	4	0	0	500	-400	0	0	1	200

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	0	0	-1125	0	1.125	0	5625
X_1	1	0	1	0	1	0	0	0	1
X_4	2	0	0	0	1.25	1	-2e-4	0	0.75
X_2	3	0	0	1	-1.25	0	0.0002	0	0.25
X_6	4	0	0	0	225	0	-0.125	1	75

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	0	0	0	0	0.5	5	6000
X_1	1	0	1	0	0	0	0.0006	-0.004	0.66667
X_4	2	0	0	0	0	1	0.0004	-0.006	0.33333
X_2	3	0	0	1	0	0	-4e-4	0.0056	0.66667
X_3	4	0	0	0	1	0	-6e-4	0.0044	0.33333

4.4-2.

Optimal Solution: $(x_1^*, x_2^*) = (2, 2)$ and $Z^* = 6$

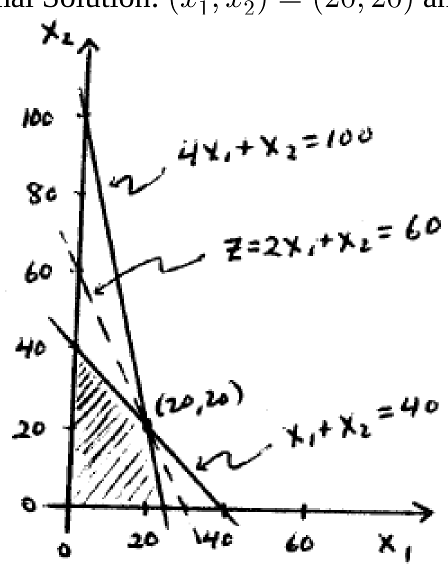
Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	-1	-2	0	0	0
X_3	1	0	1	3	1	0	8
X_4	2	0	1	1	0	1	4

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	-0.333	0	0.6667	0	5.33333
X_2	1	0	0.3333	1	0.3333	0	2.66667
X_4	2	0	0.6667	0	-0.333	1	1.33333

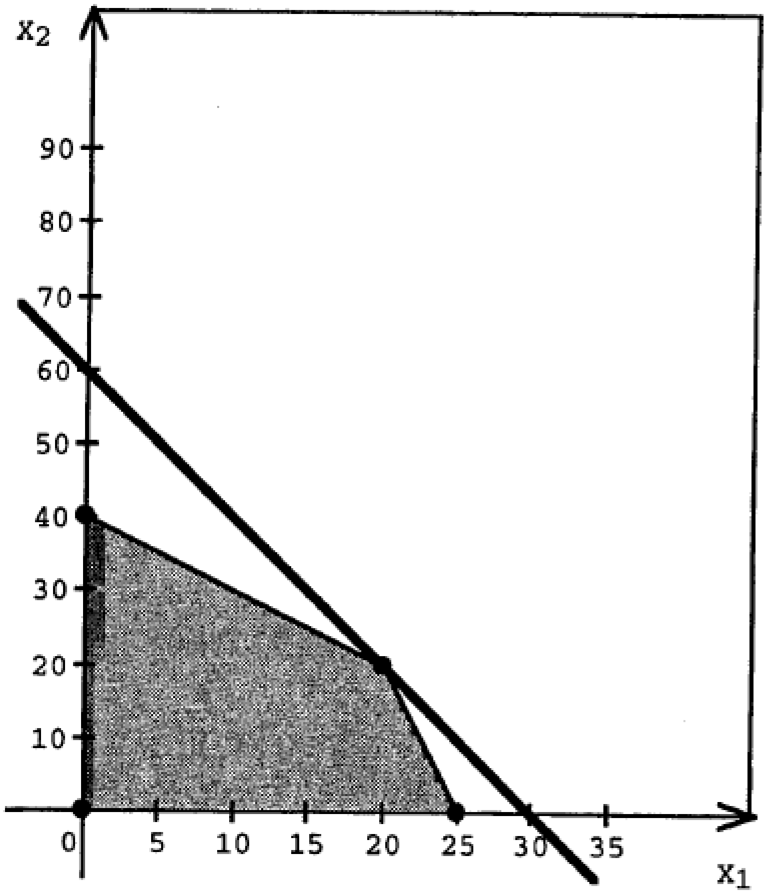
Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	0	0	0.5	0.5	6
X_2	1	0	0	1	0.5	-0.5	2
X_1	2	0	1	0	-0.5	1.5	2

4.4-3.

(a) Optimal Solution: $(x_1^*, x_2^*) = (20, 20)$ and $Z^* = 60$



(b) Optimal Solution: $(x_1^*, x_2^*) = (20, 20)$ and $Z^* = 60$



Corner Point	Z
$(20, 20)$	60^*
$(\ 0, 40)$	40
$(25,\ \ 0)$	50
$(\ 0,\ \ 0)$	0

(c) <u>Iteration 1:</u> $x_1 = x_2 = 0 \Rightarrow x_3 = 40$ and $x_4 = 100$ (slack variables)

Increase x_1, set $x_2 = 0$.

$x_3 = 40 - x_1 \geq 0 \Rightarrow x_1 \leq 40$

$x_4 = 100 - 4x_1 \geq 0 \Rightarrow x_1 \leq 25 \leftarrow$ minimum

Let $x_1 = 25$ and $x_4 = 0$.

$Z = \frac{1}{2}x_2 - \frac{1}{2}x_4 + 50$

$\frac{3}{4}x_2 + x_3 - \frac{1}{4}x_4 = 15$

$x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_4 = 25$

$x_1, x_2, x_3, x_4 \geq 0$

Iteration 2: $(25, 0, 15, 0)$ is not optimal so increase x_2, set $x_4 = 0$.

$x_3 = 15 - \frac{3}{4}x_2 \geq 0 \Rightarrow x_2 \leq 20 \leftarrow$ minimum

$x_1 = 25 - \frac{1}{4}x_2 \geq 0 \Rightarrow x_2 \leq 100$

Let $x_2 = 20$ and $x_3 = 0$.

$Z = -\frac{2}{3}x_3 - \frac{1}{3}x_4 + 60$

$x_2 + \frac{4}{3}x_3 - \frac{1}{3}x_4 = 20$

$x_1 - \frac{1}{3}x_3 + \frac{1}{3}x_4 = 20$

$x_1, x_2, x_3, x_4 \geq 0$

Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (20, 20, 0, 0)$ and $Z^* = 60$

(d) Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (20, 20, 0, 0)$ and $Z^* = 60$

```
0) Z-    2 X1-    1 X2+    0 X3+    0 X4 = 0
1)       1 X1+    1 X2+    1 X3+    0 X4 = 40
2)       4 X1+    1 X2+    0 X3+    1 X4 = 100

X1 ≥ 0, X2 ≥ 0, X3 ≥ 0, X4 ≥ 0.
```

```
0) Z+    0 X1- 0.5 X2+    0 X3+ 0.5 X4 = 50
1)       0 X1+0.75 X2+    1 X3-0.25 X4 = 15
2)       1 X1+0.25 X2+    0 X3+0.25 X4 = 25

X1 ≥ 0, X2 ≥ 0, X3 ≥ 0, X4 ≥ 0.
```

```
0) Z+    0 X1+    0 X2+0.67 X3+0.33 X4 = 60
1)       0 X1+    1 X2+1.33 X3-0.33 X4 = 20
2)       1 X1+    0 X2-0.33 X3+0.33 X4 = 20

X1 ≥ 0, X2 ≥ 0, X3 ≥ 0, X4 ≥ 0.
```

(e) - (f)

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	Right Side
Z	0	1	-2	-1	0	0	0
X_3	1	0	1	1	1	0	40
X_4	2	0	4	1	0	1	100

The coefficients for x_1 and x_2 are negative so this solution is not optimal. Let x_1 enter the basis, since it offers largest improvement rate, so the column lying under x_1 will be the pivot column. To find out how much x_1 can be increased, use the ratio test:

x_3: $40/1 = 40$
x_4: $100/4 = 25 \leftarrow$ minimum,

so x_4 leaves the basis and its row is the pivot row.

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	0	-0.5	0	0.5	50
X_3	1	0	0	0.75	1	-0.25	15
X_1	2	0	1	0.25	0	0.25	25

The coefficient of x_2 is still negative, so this solution is not optimal. Let x_2 enter the basis, its column is the pivot column. To find out how much x_2 can be increased, use the ratio test:

x_3: $15/0.75 = 20 \leftarrow$ minimum
x_1: $25/0.25 = 100$,

so x_3 leaves the basis and its row is the pivot row.

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	0	0	0.6667	0.3333	60
X_2	1	0	0	1	1.3333	-0.333	20
X_1	2	0	1	0	-0.333	0.3333	20

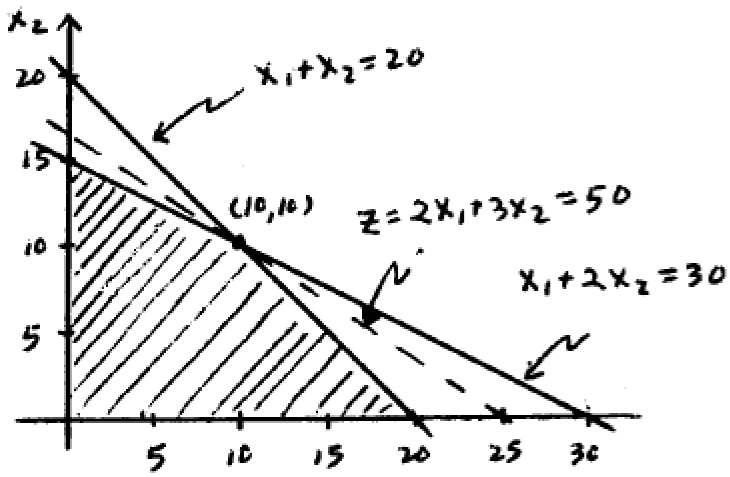
All the coefficients in the objective row are nonnegative, so the solution $(20, 20, 0, 0)$ is optimal with an objective value of 60.

(g)

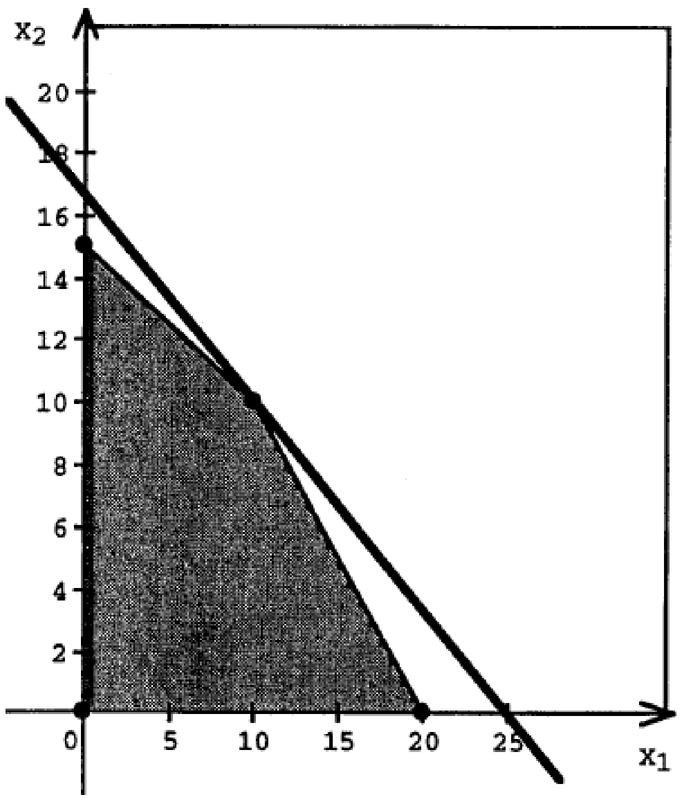
2	1	60		
20	**20**	← solution		
1	1	40	≤	40
4	1	100	≤	100

4.4-4.

(a) Optimal Solution: $(x_1^*, x_2^*) = (10, 10)$ and $Z^* = 50$



(b) Optimal Solution: $(x_1^*, x_2^*) = (10, 10)$ and $Z^* = 50$



Corner Point	Z
$(10, 10)$	50^*
$(\ 0, 15)$	45
$(20,\ \ 0)$	40
$(\ 0,\ \ 0)$	0

(c) Iteration 1: $x_1 = x_2 = 0 \Rightarrow x_3 = 30$ and $x_4 = 20$ (slack variables)

Increase x_2 and set $x_1 = 0$.

$x_3 = 30 - 2x_2 \geq 0 \Rightarrow x_2 \leq 15 \leftarrow$ minimum

$x_4 = 20 - x_2 \geq 0 \Rightarrow x_1 \leq 20$

Let $x_2 = 15$ and $x_3 = 0$.

$Z = \frac{1}{2}x_1 - \frac{3}{2}x_3 + 45$

$\frac{1}{2}x_1 + x_2 + \frac{1}{2}x_3 = 15$

$\frac{1}{2}x_1 - \frac{1}{2}x_3 + x_4 = 5$

$x_1, x_2, x_3, x_4 \geq 0$

Iteration 2: $(0, 15, 0, 5)$ is not optimal so increase x_1, set $x_3 = 0$.

$x_2 = 15 - \frac{1}{2}x_1 \geq 0 \Rightarrow x_1 \leq 30$

$x_4 = 5 - \frac{1}{2}x_1 \geq 0 \Rightarrow x_1 \leq 10 \leftarrow$ minimum

Let $x_1 = 10$ and $x_3 = 0$.

$Z = -x_3 - x_4 + 50$

$x_2 + x_3 - x_4 = 10$

$x_1 - x_3 + 2x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (10, 10, 0, 0)$ and $Z^* = 50$

(d) Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (10, 10, 0, 0)$ and $Z^* = 50$

```
0) Z-   2 X1-   3 X2+   0 X3+   0 X4 = 0
1)      1 X1+   2 X2+   1 X3+   0 X4 = 30
2)      1 X1+   1 X2+   0 X3+   1 X4 = 20

X1 ≥ 0, X2 ≥ 0, X3 ≥ 0, X4 ≥ 0.

0) Z- 0.5 X1+   0 X2+ 1.5 X3+   0 X4 = 45
1)    0.5 X1+   1 X2+ 0.5 X3+   0 X4 = 15
2)    0.5 X1+   0 X2- 0.5 X3+   1 X4 = 5

X1 ≥ 0, X2 ≥ 0, X3 ≥ 0, X4 ≥ 0.

0) Z+   0 X1+   0 X2+   1 X3+   1 X4 = 50
1)      0 X1+   1 X2+   1 X3-   1 X4 = 10
2)      1 X1+   0 X2-   1 X3+   2 X4 = 10

X1 ≥ 0, X2 ≥ 0, X3 ≥ 0, X4 ≥ 0.
```

(e) - (f)

Bas Var	Eq No	Coefficient of Z	x_1	x_2	x_3	x_4	Right Side
Z	0	1	-2	-3	0	0	0
x_3	1	0	1	2	1	0	30
x_4	2	0	1	1	0	1	20

The coefficients for x_1 and x_2 are negative so this solution is not optimal. Let x_2 enter the basis, since it offers largest improvement rate, so the column lying under x_2 will be the pivot column. To find out how much x_1 can be increased, use the ratio test:

x_3: $30/2 = 15 \leftarrow$ minimum
x_4: $20/1 = 20$,

so x_3 leaves the basis and its row is the pivot row.

Bas Var	Eq No	Coefficient of Z	x_1	x_2	x_3	x_4	Right Side
Z	0	1	-0.5	0	1.5	0	45
x_2	1	0	0.5	1	0.5	0	15
x_4	2	0	0.5	0	-0.5	1	5

The coefficient of x_1 is still negative, so this solution is not optimal. Let x_1 enter the basis, its column is the pivot column. To find out how much x_1 can be increased, use the ratio test:

x_2: $15/0.5 = 30$
x_4: $5/0.5 = 10 \leftarrow$ minimum,

so x_4 leaves the basis and its row is the pivot row.

Bas Var	Eq No	Coefficient of Z	x_1	x_2	x_3	x_4	Right Side
Z	0	1	0	0	1	1	50
x_2	1	0	0	1	1	-1	10
x_1	2	0	1	0	-1	2	10

All the coefficients in the objective row are nonnegative, so the solution $(10, 10, 0, 0)$ is optimal with an objective value of 50.

(g)

2	3			
10	**10**	**50**		
1	2	30	≤	30
1	1	20	≤	20

4.4-5.

(a) Set $x_1 = x_2 = x_3 = 0$.

(0) $Z - 5x_1 - 9x_2 - 7x_3 = 0$

(1) $x_1 + 3x_2 + 2x_3 + x_4 = 10 \Rightarrow x_4 = 10$

(2) $3x_1 + 4x_2 + 2x_3 + x_5 = 12 \Rightarrow x_5 = 12$

(3) $2x_1 + x_2 + 2x_3 + x_6 = 8 \Rightarrow x_6 = 8$

Optimality Test: The coefficients of all nonbasic variables are positive, so the solution $(0,0,0,10,12,8)$ is not optimal.

Choose x_2 as the entering basic variable, since it has the largest coefficient.

(1) $x_1 + 3x_2 + 2x_3 + x_4 = 10 \Rightarrow x_4 = 10 - 3x_2 \Rightarrow x_2 \leq 10/3$

(2) $3x_1 + 4x_2 + 2x_3 + x_5 = 12 \Rightarrow x_5 = 12 - 4x_2 \Rightarrow x_2 \leq 3 \leftarrow$ minimum

(3) $2x_1 + x_2 + 2x_3 + x_6 = 8 \Rightarrow x_6 = 8 - x_2 \Rightarrow x_2 \leq 8$

We choose x_5 as the leaving basic variable. Set $x_1 = x_5 = x_3 = 0$.

(0) $Z + 1.75x_1 - 2.5x_3 + 2.25x_5 = 27$

(1) $-1.25x_1 + 0.5x_3 + x_4 - 0.75x_5 = 1 \Rightarrow x_4 = 1$

(2) $0.75x_1 + x_2 + 0.5x_3 + 0.25x_5 = 3 \Rightarrow x_2 = 3$

(3) $1.25x_1 + 1.5x_3 - 0.25x_5 + x_6 = 5 \Rightarrow x_6 = 5$

Optimality Test: The coefficient of x_3 is positive, so the solution $(0,3,0,1,0,5)$ is not optimal.

Let x_3 be the entering basic variable.

(1)
$-1.25x_1 + 0.5x_3 + x_4 - 0.75x_5 = 1 \Rightarrow x_4 = 1 - 0.5x_3 \Rightarrow x_3 \leq 2 \leftarrow$ minimum

(2) $0.75x_1 + x_2 + 0.5x_3 + 0.25x_5 = 3 \Rightarrow x_2 = 3 - 0.5x_3 \Rightarrow x_3 \leq 6$

(3) $1.25x_1 + 1.5x_3 - 0.25x_5 + x_6 = 5 \Rightarrow x_6 = 5 - 1.5x_3 \Rightarrow x_3 \leq 10/3$

We choose x_4 as the leaving basic variable. Set $x_1 = x_5 = x_4 = 0$.

(0) $Z - 4.5x_1 + 5x_4 - 1.5x_5 = 32$

(1) $-2.5x_1 + x_3 + 2x_4 - 1.5x_5 = 2 \Rightarrow x_3 = 2$

(2) $2x_1 + x_2 - x_4 + x_5 = 2 \Rightarrow x_2 = 2$

(3) $5x_1 - 3x_4 + 2x_5 + x_6 = 2 \Rightarrow x_6 = 2$

Optimality Test: The coefficient of x_1 is positive, so the solution $(0,2,2,0,0,2)$ is not optimal.

Let x_1 be the entering basic variable.

(1) $\quad -2.5x_1 + x_3 + 2x_4 - 1.5x_5 = 2 \Rightarrow x_3 = 2 + 2.5x_1$

(2) $\quad 2x_1 + x_2 - x_4 + x_5 = 2 \Rightarrow x_2 = 2 - 2x_1 \Rightarrow x_1 \leq 1$

(3) $\quad 5x_1 - 3x_4 + 2x_5 + x_6 = 2 \Rightarrow x_6 = 2 - 5x_1 \Rightarrow x_1 \leq 0.4 \leftarrow$ minimum

We choose x_6 as the leaving basic variable. Set $x_6 = x_5 = x_4 = 0$.

(0) $\quad Z + 2.3x_4 + 0.3x_5 + 0.9x_6 = 33.8$

(1) $\quad x_3 + 0.5x_4 - 0.5x_5 + 0.5x_6 = 3 \Rightarrow x_3 = 3$

(2) $\quad x_2 + 0.2x_4 + 0.2x_5 - 0.4x_5 = 1.2 \Rightarrow x_2 = 1.2$

(3) $\quad x_1 - 0.6x_4 + 0.4x_5 + 0.2x_6 = 0.4 \Rightarrow x_1 = 0.4$

Optimality Test: The coefficients of all nonbasic variables are nonpositive, so the solution $(0.4, 1.2, 3, 0, 0, 0)$ is optimal.

(b) Optimal solution: $(x_1^*, x_2^*, x_3^*) = (0.4, 1.2, 3)$ and $Z^* = 33.8$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-5	-9	-7	0	0	0	0
X4	1	0	1	3	2	1	0	0	10
X5	2	0	3	4*	2	0	1	0	12
X6	3	0	2	1	2	0	0	1	8

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	1.75	0	-2.5	0	2.25	0	27
X4	1	0	-1.25	0	0.5*	1	-0.75	0	1
X2	2	0	0.75	1	0.5	0	0.25	0	3
X6	3	0	1.25	0	1.5	0	-0.25	1	5

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-4.5	0	0	5	-1.5	0	32
X3	1	0	-2.5	0	1	2	-1.5	0	2
X2	2	0	2	1	0	-1	1	0	2
X6	3	0	5*	0	0	-3	2	1	2

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	2.3	0.3	0.9	33.8
X3	1	0	0	0	1	0.5	-0.5	0.5	3
X2	2	0	0	1	0	0.2	0.2	-0.4	1.2
X1	3	0	1	0	0	-0.6	0.4	0.2	0.4

(c) Excel Solver

	Coefficient of					
	X1	X2	X3	Total		
Constraint 1	1	3	2	10	≤	10
Constraint 2	3	4	2	12	≤	12
Constraint 3	2	1	2	8	≤	8
Objective	5	9	7	33.8		
Solution	0.4	1.2	3			

4.4-6.

(a) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, \frac{4}{3}, \frac{4}{3})$ and $Z^* = 14\frac{2}{3}$

$$
\begin{aligned}
&0)\ Z - 3\,X_1 - 5\,X_2 - 6\,X_3 + 0\,X_4 + 0\,X_5 + 0\,X_6 + 0\,X_7 = 0\\
&1)\ 2\,X_1 + 1\,X_2 + 1\,X_3 + 1\,X_4 + 0\,X_5 + 0\,X_6 + 0\,X_7 = 4\\
&2)\ 1\,X_1 + 2\,X_2 + 1\,X_3 + 0\,X_4 + 1\,X_5 + 0\,X_6 + 0\,X_7 = 4\\
&3)\ 1\,X_1 + 1\,X_2 + 2\,X_3 + 0\,X_4 + 0\,X_5 + 1\,X_6 + 0\,X_7 = 4\\
&4)\ 1\,X_1 + 1\,X_2 + 1\,X_3 + 0\,X_4 + 0\,X_5 + 0\,X_6 + 1\,X_7 = 3
\end{aligned}
$$

$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0,\ X_4 \geq 0,\ X_5 \geq 0,\ X_6 \geq 0,\ X_7 \geq 0.$

$$
\begin{aligned}
&0)\ Z + 0\,X_1 - 2\,X_2 + 0\,X_3 + 0\,X_4 + 0\,X_5 + 3\,X_6 + 0\,X_7 = 12\\
&1)\ 1.5\,X_1 + 0.5\,X_2 + 0\,X_3 + 1\,X_4 + 0\,X_5 - 0.5\,X_6 + 0\,X_7 = 2\\
&2)\ 0.5\,X_1 + 1.5\,X_2 + 0\,X_3 + 0\,X_4 + 1\,X_5 - 0.5\,X_6 + 0\,X_7 = 2\\
&3)\ 0.5\,X_1 + 0.5\,X_2 + 1\,X_3 + 0\,X_4 + 0\,X_5 + 0.5\,X_6 + 0\,X_7 = 2\\
&4)\ 0.5\,X_1 + 0.5\,X_2 + 0\,X_3 + 0\,X_4 + 0\,X_5 - 0.5\,X_6 + 1\,X_7 = 1
\end{aligned}
$$

$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0,\ X_4 \geq 0,\ X_5 \geq 0,\ X_6 \geq 0,\ X_7 \geq 0.$

$$
\begin{aligned}
&0)\ Z + 0.67\,X_1 + 0\,X_2 + 0\,X_3 + 0\,X_4 + 1.33\,X_5 + 2.33\,X_6 + 0\,X_7 = 14.6667\\
&1)\ 1.333\,X_1 + 0\,X_2 + 0\,X_3 + 1\,X_4 - 0.33\,X_5 - 0.33\,X_6 + 0\,X_7 = 1.33333\\
&2)\ 0.333\,X_1 + 1\,X_2 + 0\,X_3 + 0\,X_4 + 0.67\,X_5 - 0.33\,X_6 + 0\,X_7 = 1.33333\\
&3)\ 0.333\,X_1 + 0\,X_2 + 1\,X_3 + 0\,X_4 - 0.33\,X_5 + 0.67\,X_6 + 0\,X_7 = 1.33333\\
&4)\ 0.333\,X_1 + 0\,X_2 + 0\,X_3 + 0\,X_4 - 0.33\,X_5 - 0.33\,X_6 + 1\,X_7 = 0.33333
\end{aligned}
$$

$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0,\ X_4 \geq 0,\ X_5 \geq 0,\ X_6 \geq 0,\ X_7 \geq 0.$

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, \frac{4}{3}, \frac{4}{3})$ and $Z^* = 14\frac{2}{3}$

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Right Side
Z	0	1	-3	-5	-6	0	0	0	0	0
X_4	1	0	2	1	1	1	0	0	0	4
X_5	2	0	1	2	1	0	1	0	0	4
X_6	3	0	1	1	2	0	0	1	0	4
X_7	4	0	1	1	1	0	0	0	1	3

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Right Side
Z	0	1	0	-2	0	0	0	3	0	12
X_4	1	0	1.5	0.5	0	1	0	-0.5	0	2
X_5	2	0	0.5	1.5	0	0	1	-0.5	0	2
X_3	3	0	0.5	0.5	1	0	0	0.5	0	2
X_7	4	0	0.5	0.5	0	0	0	-0.5	1	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right Side
Z	0	1	0.6667	0	0	0	1.3333	2.3333	0	14.6667
X4	1	0	1.3333	0	0	1	-0.333	-0.333	0	1.33333
X2	2	0	0.3333	1	0	0	0.6667	-0.333	0	1.33333
X3	3	0	0.3333	0	1	0	-0.333	0.6667	0	1.33333
X7	4	0	0.3333	0	0	0	-0.333	-0.333	1	0.33333

(c)

3	5	6			14.7
0	**1.3**	**1.3** ← solution			
2	1	1	2.7	≤	4
1	2	1	4	≤	4
1	1	2	4	≤	4

4.4-7.

Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (1.5, 0.5, 0)$ and $Z^* = 2.5$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right Side
Z	0	1	-2	1	-1	0	0	0	0
X4	1	0	3	1	1	1	0	0	6
X5	2	0	1	-1	2	0	1	0	1
X6	3	0	1	1	-1	0	0	1	2

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right Side
Z	0	1	0	-1	3	0	2	0	2
X4	1	0	0	4	-5	1	-3	0	3
X1	2	0	1	-1	2	0	1	0	1
X6	3	0	0	2	-3	0	-1	1	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right Side
Z	0	1	0	0	1.5	0	1.5	0.5	2.5
X4	1	0	0	0	1	1	-1	-2	1
X1	2	0	1	0	0.5	0	0.5	0.5	1.5
X2	3	0	0	1	-1.5	0	-0.5	0.5	0.5

4.4-8.

Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (6\frac{2}{3}, 0, 36\frac{2}{3})$ and $Z^* = 66\frac{2}{3}$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right Side
Z	0	1	1	-1	-2	0	0	0	0
X4	1	0	1	2	-1	1	0	0	20
X5	2	0	-2	4	2	0	1	0	60
X6	3	0	2	3	1	0	0	1	50

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right Side
Z	0	1	-1	3	0	0	1	0	60
X4	1	0	0	4	0	1	0.5	0	50
X3	2	0	-1	2	1	0	0.5	0	30
X6	3	0	3	1	0	0	-0.5	1	20

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right Side
Z	0	1	0	3.3333	0	0	0.8333	0.3333	66.6667
X4	1	0	0	4	0	1	0.5	0	50
X3	2	0	0	2.3333	1	0	0.3333	0.3333	36.6667
X1	3	0	1	0.3333	0	0	-0.167	0.3333	6.66667

4.5-1.

(a) TRUE. The ratio test tells how far the entering basic variable can be increased before one of the current basic variables drops below zero. If there is a tie for which variable should leave the basis, then both variables drop to zero at the same value of the entering basic variable. Since only one variable can become nonbasic in any iteration, the other will remain in the basis even though it will be zero.

(b) FALSE. If there is no leaving basic variable, then the solution is unbounded and the entering basic variable can be increased indefinitely.

(c) FALSE. All basic variables always have a coefficient of zero in row 0 of the final tableau.

(d) FALSE.

Example 1: maximize $x_1 - x_2$

subject to $x_1 - x_2 \leq 1$

$x_1, x_2 \quad \geq 0$

Clearly, any solution $(x_1^*, x_2^*) = (k+1, k)$ for $k \in [0, \infty)$ with $z^* = 1$ is optimal. The problem has infinitely many optimal solutions and the feasible region is not bounded.

Example 2: maximize $-x_1$

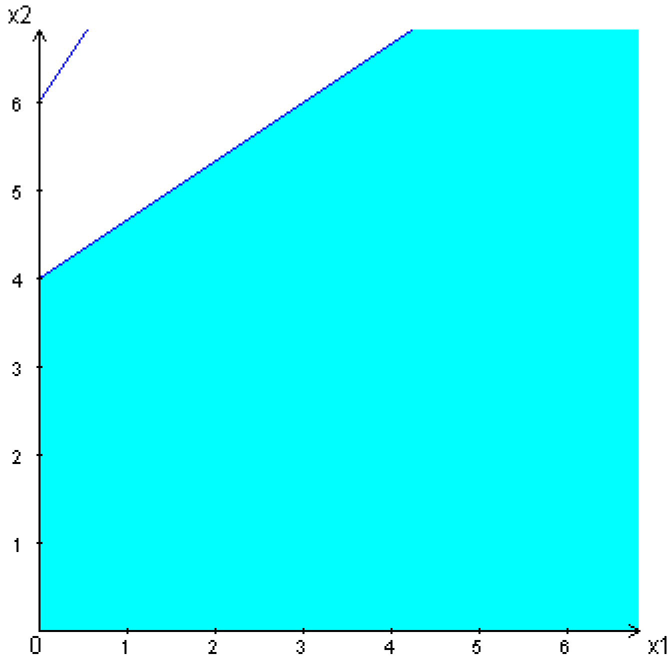
subject to $-x_1 - x_2 \leq 1$

$x_1, x_2 \quad \geq 0$

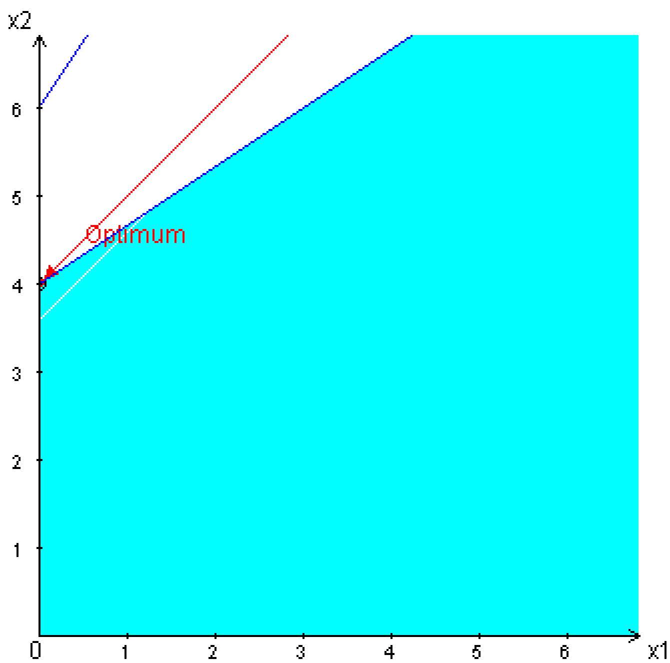
Any solution $(0, x_2^*)$ with $x_2 \geq 0$ is optimal.

4.5-2.

(a)



(b) Yes, the optimal solution is $(x_1^*, x_2^*) = (0, 4)$ with $Z^* = 4$.



(c) No, the objective function value is maximized by sliding the objective function line to the right. This can be done forever, so there is no optimal solution.

(d) No, there exist solutions that make the objective value arbitrarily large. This usually occurs when a constraint is left out of the model.

(e) Let the objective function be $Z = x_1 - x_2$. Then, the initial tableau is:

		Coefficient of					
BV	Eq.	Z	x_1	x_2	x_3	x_4	Right Side
Z	(0)	1	-1	1	0	0	0
x_3	(1)	0	-2	3	1	0	12
x_4	(2)	0	-3	2	0	1	12

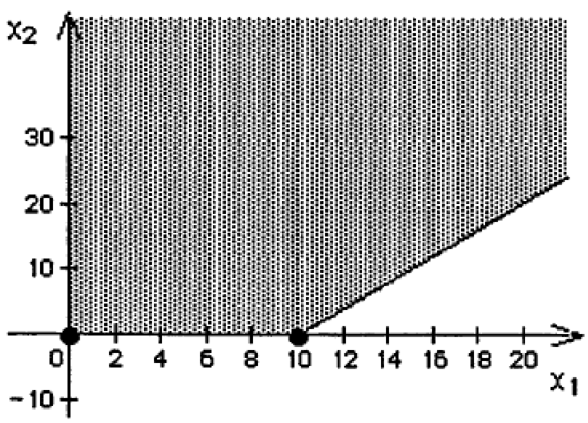
The pivot column, the column of x_1, has all negative elements, so Z is unbounded.

(f) The Solver tells that the Set Cell values do not converge. There is no optimal solution because a better solution can always be found.

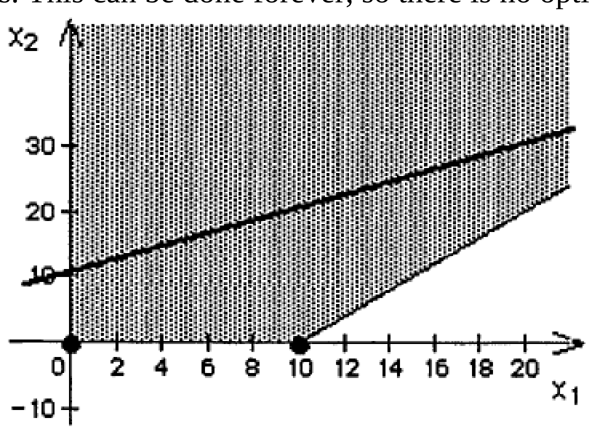
	Coefficient of				
	X1	X2	Total		
Constraint 1	-2	3	0	≤	12
Constraint 2	-3	2	0	≤	12
Objective	1	-1	0		
Solution	0	0			

4.5-3.

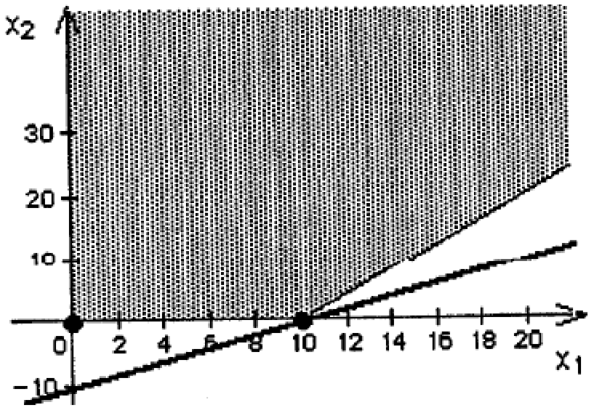
(a)



(b) No. the objective function value is maximized by sliding the objective function line upwards. This can be done forever, so there is no optimal solution.



(c) Yes, the optimal solution is $(x_1^*, x_2^*) = (10, 0)$ with $Z^* = 10$.



(d). No, there exist solutions that make z arbitrarily large. This usually occurs when a constraint is left out of the model.

(e) Let the objective function be $Z = -x_1 + x_2$. Then, the initial tableau is:

		Coefficient of					
BV	Eq.	Z	x_1	x_2	x_3	x_4	Right Side
Z	(0)	1	1	-1	0	0	0
x_3	(1)	0	2	-1	1	0	20
x_4	(2)	0	1	-2	0	1	20

The pivot column, the column of x_2, has all elements negative, so Z is unbounded.

(f) The Solver tells that the Set Cell values do not converge. There is no optimal solution because a better solution can always be found.

	Contribution Per Unit of Each Activity				Resource
Resource	Activity 1	Activity 2	Totals		Available
1	2	-1	0	≤	20
2	1	-2	0	≤	20
Unit Profit	-1	1	$ -		
Solution	**0**	**0**			

4.5-4.

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Right Side
Z	0	1	-5	-1	-3	-4	0	0	0	0
X_5	1	0	1	-2	4	3	1	0	0	20
X_6	2	0	-4	6	5	-4	0	1	0	40
X_7	3	0	2	-3	3	8	0	0	1	50

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Right Side
Z	0	1	0	-11	17	11	5	0	0	100
X_1	1	0	1	-2	4	3	1	0	0	20
X_6	2	0	0	-2	21	8	4	1	0	120
X_7	3	0	0	1	-5	2	-2	0	1	10

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Right Side
Z	0	1	0	0	-38	33	-17	0	11	210
X_1	1	0	1	0	-6	7	-3	0	2	40
X_6	2	0	0	0	11	12	0	1	2	140
X_2	3	0	0	1	-5	2	-2	0	1	10

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Right Side
Z	0	1	0	0	0	74.455	-17	3.4545	17.909	693.636
X_1	1	0	1	0	0	13.545	-3	0.5455	3.0909	116.364
X_3	2	0	0	0	1	1.0909	0	0.0909	0.1818	12.7273
X_2	3	0	0	1	0	7.4545	-2	0.4545	1.9091	73.6364

We can see from either the second or third iteration that because all of the constraint coefficients of x_5 are nonpositive, it can be increased without forcing any basic variable to zero. From the third iteration, $(116.364 + 3\theta, 73.6364 + 2\theta, 12.7273, 0)$ is feasible for any $\theta \geq 0$ and $Z = 693.636 + 17\theta$ is unbounded.

4.5-5.

(a) The constraints of any LP problem can be expressed in matrix notation as:

$$Ax = b, x \geq 0.$$

If $x^1, x^2, \ldots, x^N$ are feasible solutions and $x = \sum_{k=1}^{N} \alpha_k x^k$ with $\sum_{k=1}^{N} \alpha_k = 1$ and $\alpha_k \geq 0$ for $k = 1, \ldots, N$, then

$$Ax = A\sum_{k=1}^{N} \alpha_k x^k = \sum_{k=1}^{N} \alpha_k A x^k = \sum_{k=1}^{N} \alpha_k b = b,\ x = \sum_{k=1}^{N} \alpha_k x^k \geq 0,$$

so x is also a feasible solution.

(b) This follows immediately from (a), since basic feasible solutions are feasible solutions.

4.5-6.

(a) Suppose Z^* is the value of the objective function for an optimal solution and $x^1, x^2, \ldots, x^N$ are optimal BF solutions. From Problem 4.5-5, $x = \sum_{k=1}^{N} \alpha_k x^k$ is feasible for any choice of $\alpha_k \geq 0$ $(k = 1, \ldots, N)$ satisfying $\sum_{k=1}^{N} \alpha_k = 1$. The objective function value at x is:

$$c^T x = c^T \sum_{k=1}^{N} \alpha_k x^k = \sum_{k=1}^{N} \alpha_k c^T x^k = \sum_{k=1}^{N} \alpha_k Z^* = Z^*,$$

so x is also an optimal solution.

(b) Consider any feasible solution x that is not a weighted average of the optimal BF solutions. Since x is feasible, it must be a weighted average of the basic feasible solutions, which are not all optimal by assumption. Let $\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^L$ are the basic feasible solutions that are not optimal. Then,

$$x = \sum_{k=1}^{N} \alpha_k x^k + \sum_{i=1}^{L} \beta_i \overline{x}^i$$

where $\sum_{k=1}^{N} \alpha_k + \sum_{i=1}^{L} \beta_i = 1$, $\alpha_k \geq 0$ $(k = 1, \ldots, N)$, $\beta_i \geq 0$ $(i = 1, \ldots, L)$ and $\beta_i \neq 0$ for some i. The objective function value at x is:

$$c^T x = c^T \sum_{k=1}^{N} \alpha_k x^k + c^T \sum_{i=1}^{L} \beta_i \overline{x}^i = \sum_{k=1}^{N} \alpha_k c^T x^k + \sum_{i=1}^{L} \beta_i c^T \overline{x}^i.$$

Since $\overline{x}^i$ is not optimal, $c^T \overline{x}^i < Z^*$ for every i. Because there is at least one positive β_i and $c^T x^k = Z^*$,

$$c^T x < \left(\sum_{k=1}^{N} \alpha_k + \sum_{i=1}^{L} \beta_i \right) Z^* = Z^*.$$

Hence, x cannot be optimal.

4.5-7.

(a) $x_1 \leq 6$
$x_2 \leq 3$
$-x_1 + 3x_2 \leq 6$

(b)

Unit Profit (Prod.1)	Unit Profit (Prod.2)	Objective	Multiple Opt. Solutions
-1	3	$-x_1 + 3x_2$	line segment between $(0,2)$ & $(3,3)$
0	1	x_2	line segment between $(3,3)$ & $(6,3)$
1	0	x_1	line segment between $(6,3)$ & $(6,0)$
0	-1	$-x_2$	line segment between $(0,0)$ & $(6,0)$
-1	0	$-x_1$	line segment between $(0,0)$ & $(0,2)$

(c)

Corner Point (x_1, x_2)	Profit $= -x_1 + 2x_2$
$(0,0)$	0
$(0,2)$	4
$(3,3)$	3
$(6,3)$	0
$(6,0)$	-6

Optimal Solution: $(x_1^*, x_2^*) = (0, 2)$ with $Z^* = 4$

(d)

Z	x_1	x_2	x_3	x_4	x_5	RS
1	1	-2	0	0	0	0
0	1	0	1	0	0	6
0	0	1	0	1	0	3
0	-1	$[3]$	0	0	1	6

$\longrightarrow$

Z	x_1	x_2	x_3	x_4	x_5	RS
1	$1/3$	0	0	0	$2/3$	4
0	-1	0	1	0	0	6
0	$1/3$	0	0	1	$-1/3$	1
0	$-1/3$	1	0	0	$1/3$	2

So the unique optimal solution is $(x_1^*, x_2^*) = (0, 2)$ with V$^* = 4$.

4.5-8.

```
Bas|Eq|                Coefficient of                | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6   |  side
___|__|__|___________________________________________|_______
   |  |  |                                           |
 Z | 0| 1|  -50    -25    -20    -40      0      0   |      0
 X5| 1| 0|    2*     1      0      0      1      0   |     30
 X6| 2| 0|    0      0      1      2      0      1   |     20

Bas|Eq|                Coefficient of                | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6   |  side
___|__|__|___________________________________________|_______
   |  |  |                                           |
 Z | 0| 1|    0      0    -20    -40     25      0   |    750
 X1| 1| 0|    1    0.5      0      0    0.5      0   |     15
 X6| 2| 0|    0      0      1      2*     0      1   |     20

Bas|Eq|                Coefficient of                | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6   |  side
___|__|__|___________________________________________|_______
   |  |  |                                           |
 Z | 0| 1|    0      0      0      0     25     20   |   1150
 X1| 1| 0|    1    0.5*     0      0    0.5      0   |     15
 X4| 2| 0|    0      0    0.5      1      0    0.5   |     10
```

Since the objective coefficients (row Z) for x_2 and x_3 are zero, we can pivot to get other optimal BF solutions.

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	0	25	20	1150
X2	1	0	2	1	0	0	1	0	30
X4	2	0	0	0	0.5*	1	0	0.5	10

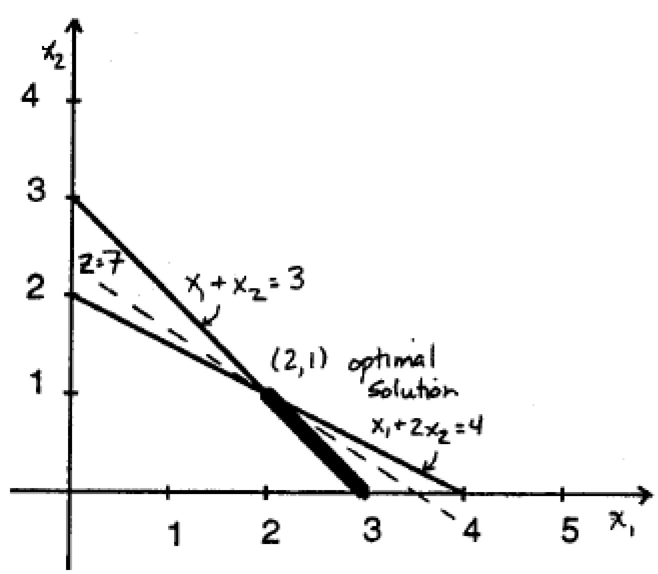
Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	0	25	20	1150
X2	1	0	2*	1	0	0	1	0	30
X3	2	0	0	0	1	2	0	1	20

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	0	25	20	1150
X1	1	0	1	0.5	0	0	0.5	0	15
X3	2	0	0	0	1	2	0	1	20

Hence, the optimal BF solutions are $(15, 0, 0, 10)$, $(0, 30, 0, 10)$, $(0, 30, 20, 0)$, and $(15, 0, 20, 0)$, all with objective function value 1150.

4.6-1.

(a) Optimal Solution: $(x_1^*, x_2^*) = (2, 1)$ and $Z^* = 7$



(b) Initial artificial BF solution: $(0, 0, 4, 3)$

Bas Var	Eq No	Z	X_1	X_2	X_3	$\bar{X}_4$	Right Side
			-1M	-1M			
Z	0	1	-2	-3	0	0	-3M
X_3	1	0	1	2	1	0	4
$\bar{X}_4$	2	0	1	1	0	1	3

(c) Optimal Solution: $(x_1^*, x_2^*) = (2, 1)$ and $Z^* = 7$

Bas Var	Eq No	Z	X_1	X_2	X_3	$\bar{X}_4$	Right Side
			-0.5M		0.5M		-1M
Z	0	1	-0.5	0	+1.5	0	+6
X_2	1	0	0.5	1	0.5	0	2
$\bar{X}_4$	2	0	0.5	0	-0.5	1	1

Bas Var	Eq No	Z	X_1	X_2	X_3	$\bar{X}_4$	Right Side
						1M	
Z	0	1	0	0	1	+1	7
X_2	1	0	0	1	1	-1	1
X_1	2	0	1	0	-1	2	2

4.6-2.

(a) - (b) Initial artificial BF solution: $(0, 0, 0, 0, 300, 300)$

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
			-10M	-4M	-5M	-7M			
Z	0	1	-4	-2	-3	-5	0	0	-600M
$\bar{X}_5$	1	0	2	3	4	2	1	0	300
$\bar{X}_6$	2	0	8	1	1	5	0	1	300

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
				-2.75M	-3.75M	-0.75M		1.25M	-225M
Z	0	1	0	-1.5	-2.5	-2.5	0	+0.5	+150
$\bar{X}_5$	1	0	0	2.75	3.75	0.75	1	-0.25	225
X_1	2	0	1	0.125	0.125	0.625	0	0.125	37.5

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
							1M	1M	
Z	0	1	0	0.3333	0	-2	+0.667	+0.333	300
X_3	1	0	0	0.7333	1	0.2	0.2667	-0.067	60
X_1	2	0	1	0.0333	0	0.6	-0.033	0.1333	30

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
							1M	1M	
Z	0	1	3.3333	0.4444	0	0	+0.556	+0.778	400
X_3	1	0	-0.333	0.7222	1	0	0.2778	-0.111	50
X_4	2	0	1.6667	0.0556	0	1	-0.056	0.2222	50

Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (0, 0, 50, 50)$ and $Z^* = 400$

(c) - (d) - (e) - (f) Initial artificial BF solution: $(0, 0, 0, 0, 300, 300)$

Phase 1:

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
$\bar{Z}$	0	1	-10	-4	-5	-7	0	0	-600
$\bar{X}_5$	1	0	2	3	4	2	1	0	300
$\bar{X}_6$	2	0	8	1	1	5	0	1	300

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
$\bar{Z}$	0	1	0	-2.75	-3.75	-0.75	0	1.25	-225
$\bar{X}_5$	1	0	0	2.75	3.75	0.75	1	-0.25	225
X_1	2	0	1	0.125	0.125	0.625	0	0.125	37.5

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	0	0	0	0	1	1	0
X_3	1	0	0	0.7333	1	0.2	0.2667	-0.067	60
X_1	2	0	1	0.0333	0	0.6	-0.033	0.1333	30

Phase 2:

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	0	0.3333	0	-2	300
X_3	1	0	0	0.7333	1	0.2	60
X_1	2	0	1	0.0333	0	0.6	30

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	3.3333	0.4444	0	0	400
X_3	1	0	-0.333	0.7222	1	0	50
X_4	2	0	1.6667	0.0556	0	1	50

Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (0, 0, 50, 50)$ and $Z^* = 400$

(g) The basic solutions of the two methods coincide. They are artificial BF solutions for the revised problem until both artificial variables x_5 and x_6 are driven out of the basis, which in the two-phase method is the end of Phase 1.

(h)

	4	2	3	5		400	Maximum value
Variables	0	0	50	50			
						RHS	
Constraints	2	3	4	2	300 "="	300	
	8	1	1	5	300 "="	300	

	2	3	1		7	Minimum value
Variables	0	3	-2			
					RHS	
Constraints	1	4	2	8 >=	8	
	3	2	0	6 >=	6	

4.6-3.

(a) maximize $-Z = -2x_1 - 3x_2 - x_3$

subject to

$$\begin{aligned} -x_1 - 4x_2 - 2x_3 &\leq -8 \\ -3x_1 - 2x_2 &\leq -6 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0.8, 1.8, 0)$ and $Z^* = 7$

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	$\bar{X}_6$	$\bar{X}_7$	Right Side
Z	0	1	-4M +2	-6M +3	-2M +1	1M	1M	0	0	-14M
$\bar{X}_6$	1	0	1	4	2	-1	0	1	0	8
$\bar{X}_7$	2	0	3	2	0	0	-1	0	1	6

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	$\bar{X}_6$	$\bar{X}_7$	Right Side
Z	0	1	-2.5M +1.25	0	1M -0.5	-0.5M +0.75	1M	1.5M -0.75	0	-2M -6
X_2	1	0	0.25	1	0.5	-0.25	0	0.25	0	2
$\bar{X}_7$	2	0	2.5	0	-1	0.5	-1	-0.5	1	2

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	$\bar{X}_6$	$\bar{X}_7$	Right Side
Z	0	1	0	0	0	0.5	0.5	1M -0.5	1M -0.5	-7
X_2	1	0	0	1	0.6	-0.3	0.1	0.3	-0.1	1.8
X_1	2	0	1	0	-0.4	0.2	-0.4	-0.2	0.4	0.8

Pivoting x_3 for x_2 gives an alternate optimal BF solution, $(2, 0, 3)$.

(c) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0.8, 1.8, 0)$ and $Z^* = 7$

Phase 1:

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	$\bar{X}_6$	$\bar{X}_7$	Right Side
Z	0	1	-4	-6	-2	1	1	0	0	-14
$\bar{X}_6$	1	0	1	4	2	-1	0	1	0	8
$\bar{X}_7$	2	0	3	2	0	0	-1	0	1	6

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	$\bar{X}_6$	$\bar{X}_7$	Right Side
Z	0	1	-2.5	0	1	-0.5	1	1.5	0	-2
X_2	1	0	0.25	1	0.5	-0.25	0	0.25	0	2
$\bar{X}_7$	2	0	2.5	0	-1	0.5	-1	-0.5	1	2

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	$\bar{X}_6$	$\bar{X}_7$	Right Side
Z	0	1	0	0	0	0	0	1	1	0
X_2	1	0	0	1	0.6	-0.3	0.1	0.3	-0.1	1.8
X_1	2	0	1	0	-0.4	0.2	-0.4	-0.2	0.4	0.8

Phase 2:

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	Right Side
Z	0	1	0	0	-5e-20 ≈0	0.5	0.5	-7
X_2	1	0	0	1	0.6	-0.3	0.1	1.8
X_1	2	0	1	0	-0.4	0.2	-0.4	0.8

Pivoting x_3 for x_2 gives an alternate optimal BF solution, $(2, 0, 3)$.

(d) The basic solutions of the two methods coincide. They are artificial BF solutions for the revised problem until both artificial variables x_6 and x_7 are driven out of the basis, which in the two-phase method is the end of Phase 1.

(e)

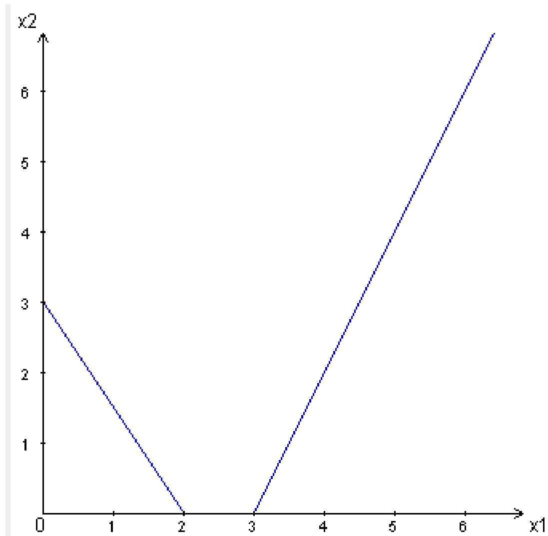
	2	3	1		7	Optimal Value
Variables	0	3	-2			
					RHS	
Constraints	1	4	2	8 >=	8	
	3	2	0	6 >=	6	

4.6-4.

Once all artificial variables are driven out of the basis in a maximization (minimization) problem. Choosing an artificial variable to reenter the basis can only lower (raise) the objective function value by an arbitrarily large amount depending on M.

4.6-5.

(a)



(b) The Solver could not find a feasible solution.

	Coefficient of				
	X1	X2	Total		
Constraint 1	3	2	6	≤	6
Constraint 2	-2	1	-4	≤	-6
Objective	5	4	10		
Solution	2	0			

(c)

BV	Eq.	Z	X1	X2	X3	X4	X5	RHS
Z	0	1	-5	-4	0	0	1M	0
X3	1	0	3	2	1	0	0	6
X5	2	0	2	-1	0	-1	1	6

BV	Eq.	Z	X1	X2	X3	X4	X5	RHS
Z	0	1	-5-2M	-4+1M	0	1M	0	-6M
X3	1	0	3	2	1	0	0	6
X5	2	0	2	-1	0	-1	1	6

BV	Eq.	Z	X1	X2	X3	X4	X5	RHS
Z	0	1	0	-2/3+(7/3)M	5/3+(2/3)M	1M	0	10-2M
X1	1	0	1	2/3	1/3	0	0	2
X5	2	0	0	-7/3	-2/3	-1	1	2

In the optimal solution, the artificial variable X_5 is basic and takes a positive value, so the problem has no feasible solutions.

(d)

BV	Eq.	Z	X1	X2	X3	X4	X5	RHS
Z	0	1	0	0	0	0	1	0
X3	1	0	3	2	1	0	0	6
X5	2	0	2	-1	0	-1	1	6

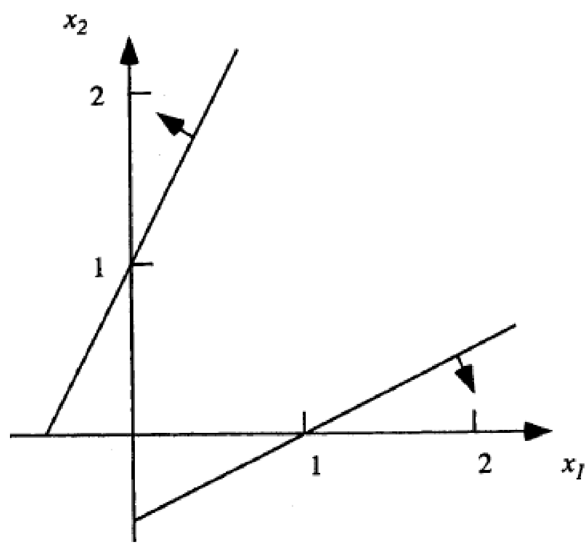
BV	Eq.	Z	X1	X2	X3	X4	X5	RHS
Z	0	1	-2	1	0	1	0	-6
X3	1	0	3	2	1	0	0	6
X5	2	0	2	-1	0	-1	1	6

BV	Eq.	Z	X1	X2	X3	X4	X5	RHS
Z	0	1	0	7/3	2/3	1	0	-2
X1	1	0	1	2/3	1/3	0	0	2
X5	2	0	0	-7/3	-2/3	-1	1	2

Since the artificial variable X_5 is not zero in the optimal solution of Phase I Problem, the original model must have no feasible solutions.

4.6-6.

(a)



(b) The Solver could not find a feasible solution.

	Benefit Contribution Per Unit of Each Activity				Minimum
Benefit	Activity 1	Activity 2	Totals		Level
1	-2	1	0	≥	1
2	1	-2	0	≥	1
Unit Cost	5000	7000	$ -		
Solution	**0**	**0**			

(c)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	5000	7000	0	0	1.0e6	1.0e6	0
X1	1	0	−2	1	−1	0	1	0	1
X1	2	0	1*	−2	0	−1	0	1	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	17000	0	5000	1.0e6	1.0e6	−5000
X1	1	0	0	−3	−1	−2	1	2	3
X1	2	0	1	−2	0	−1	0	1*	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	−1e6	2.0e6	0	1.0e6	1.0e6	0	−1e6
X1	1	0	−2	1	−1	0	1*	0	1
X6	2	0	1	−2	0	−1	0	1	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	1.0e6	1.0e6	1.0e6	1.0e6	0	0	−2e6
X5	1	0	−2	1	−1	0	1	0	1
X6	2	0	1	−2	0	−1	0	1	1

(d)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	0	1	1	0
X1	1	0	−2	1	−1	0	1	0	1
X1	2	0	1*	−2	0	−1	0	1	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	0	1	1	0
X1	1	0	0	−3	−1	−2	1*	2	3
X1	2	0	1	−2	0	−1	0	1	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	3	1	2	0	−1	−3
X5	1	0	0	−3	−1	−2	1	2	3
X1	2	0	1	−2	0	−1	0	1*	1

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	1	1	1	1	0	0	−2
X5	1	0	−2	1	−1	0	1	0	1
X6	2	0	1	−2	0	−1	0	1	1

4.6-7.

(a) Initial artificial BF solution: $(0, 0, 0, 0, 20, 50)$

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
			-3M	-2M	-2M				
Z	0	1	-2	-5	-3	1M	0	0	-70M
$\bar{X}_5$	1	0	1	-2	1	-1	1	0	20
$\bar{X}_6$	2	0	2	4	1	0	0	1	50

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, 0, 50)$ and $Z^* = 150$

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
				-8M	1M	-2M	3M		-10M
Z	0	1	0	-9	-1	-2	+2	0	+40
X_1	1	0	1	-2	1	-1	1	0	20
$\bar{X}_6$	2	0	0	8	-1	2	-2	1	10

Bas Var	Eq No	Coefficient of: Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	0	0	-2.125	0.25	1M -0.25	1M +1.125	51.25
X_1	1	0	1	0	0.75	-0.5	0.5	0.25	22.5
X_2	2	0	0	1	-0.125	0.25	-0.25	0.125	1.25

Bas Var	Eq No	Coefficient of: Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	2.8333	0	0	-1.167	1M +1.167	1M +1.833	115
X_3	1	0	1.3333	0	1	-0.667	0.6667	0.3333	30
X_2	2	0	0.1667	1	0	0.1667	-0.167	0.1667	5

Bas Var	Eq No	Coefficient of: Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	4	7	0	0	1M	1M +3	150
X_3	1	0	2	4	1	0	0	1	50
X_4	2	0	1	6	0	1	-1	1	30

(c) Initial artificial BF solution: $(0, 0, 0, 0, 20, 50)$

Phase 1:

Bas Var	Eq No	Coefficient of: Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	-3	-2	-2	1	0	0	-70
$\bar{X}_5$	1	0	1	-2	1	-1	1	0	20
$\bar{X}_6$	2	0	2	4	1	0	0	1	50

(d)

Bas Var	Eq No	Coefficient of: Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	0	-8	1	-2	3	0	-10
X_1	1	0	1	-2	1	-1	1	0	20
$\bar{X}_6$	2	0	0	8	-1	2	-2	1	10

Bas Var	Eq No	Coefficient of: Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	0	0	0	0	1	1	0
X_1	1	0	1	0	0.75	-0.5	0.5	0.25	22.5
X_2	2	0	0	1	-0.125	0.25	-0.25	0.125	1.25

(e) - (f) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, 0, 50)$ and $Z^* = 150$

Phase 2:

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	0	0	-2.125	0.25	51.25
X_1	1	0	1	0	0.75	-0.5	22.5
X_2	2	0	0	1	-0.125	0.25	1.25

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	2.8333	0	0	-1.167	115
X_3	1	0	1.3333	0	1	-0.667	30
X_2	2	0	0.1667	1	0	0.1667	5

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	4	7	0	0	150
X_3	1	0	2	4	1	0	50
X_4	2	0	1	6	0	1	30

(g) The basic solutions of the two methods coincide. They are artificial basic feasible solutions for the revised problem until both artificial variables x_5 and x_6 are driven out of the basis, which in the two-phase method is the end of Phase 1.

(h)

	2	5	3		150	Optimal Value
Solution	0	0	50			
					Right Hand Side	
Constraints	1	-2	1	50 >=	20	
	2	4	1	50 "="	50	

4.6-8.

(a)

Phase 1:

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
$\underline{Z}$	0	1	-8	-4	-12	1	0	0	-700
$\bar{X}_5$	1	0	5	2	7	0	1	0	420
$\bar{X}_6$	2	0	3	2	5	-1	0	1	280

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	-0.8	0.8	0	-1.4	0	2.4	-28
$\bar{X}_5$	1	0	0.8	-0.8	0	1.4	1	-1.4	28
X_3	2	0	0.6	0.4	1	-0.2	0	0.2	56

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	5e-20	1e-19	0	0	1	1	2e-18
X_4	1	0	0.5714	-0.571	0	1	0.7143	-1	20
X_3	2	0	0.7143	0.2857	1	0	0.1429	0	60

(b)

	0	0	0	0	1	1		0	Minimum Value
Variables	0	0	60	20	0	0			
								RHS	
Constraints	5	2	7	0	1	0	420 "="	420	
	3	2	5	-1	0	1	280 "="	280	

(c) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (35, 0, 35)$ and $Z^* = 175$

Phase 2:

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	-0.143	0.1429	0	0	-180
X_4	1	0	0.5714	-0.571	0	1	20
X_3	2	0	0.7143	0.2857	1	0	60

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	Right Side
Z	0	1	-1e-20 ≈0	8e-20 ≈0	0	0.25	-175
X_1	1	0	1	-1	0	1.75	35
X_3	2	0	0	1	1	-1.25	35

Pivoting x_2 into the basis for x_3 provides the alternative optimal BF solution $(70, 35, 0)$.

(d)

	2	1	3		175	Minimum Value
Variables	36.6	1.6	33.4			
					RHS	
Constraints	5	2	7	420 "="	420	
	3	2	5	280 >=	280	

4.6-9.

(a) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, 15, 15)$ and $Z^* = 90$

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
$\underline{Z}$	0	1	-5M +3	-4M +2	-8M +4	1M	0	0	-180M
$\bar{X}_5$	1	0	2	1	3	0	1	0	60
$\bar{X}_6$	2	0	3	3	5	-1	0	1	120

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	0.333M +0.333	-1.33M +0.667	0	1M	2.667M -1.333	0	-20M -80
$\underline{X}_3$	1	0	0.6667	0.3333	1	0	0.3333	0	20
$\bar{X}_6$	2	0	-0.333	1.3333	0	-1	-1.667	1	20

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	0.5	0	0	0.5	1M -0.5	1M -0.5	-90
X_3	1	0	0.75	0	1	0.25	0.75	-0.25	15
X_2	2	0	-0.25	1	0	-0.75	-1.25	0.75	15

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, 15, 15)$ and $Z^* = 90$

Phase 1:

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
$\underline{Z}$	0	1	-5	-4	-8	1	0	0	-180
$\bar{X}_5$	1	0	2	1	3	0	1	0	60
$\bar{X}_6$	2	0	3	3	5	-1	0	1	120

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	0.3333	-1.333	0	1	2.6667	0	-20
$\underline{X}_3$	1	0	0.6667	0.3333	1	0	0.3333	0	20
$\bar{X}_6$	2	0	-0.333	1.3333	0	-1	-1.667	1	20

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	$\bar{X}_6$	Right Side
Z	0	1	-3e-20	0	0	0	1	1	0
X_3	1	0	0.75	0	1	0.25	0.75	-0.25	15
X_2	2	0	-0.25	1	0	-0.75	-1.25	0.75	15

Phase 2:

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	Right Side
Z	0	1	0.5	0	0	0.5	-90
x_3	1	0	0.75	0	1	0.25	15
x_2	2	0	-0.25	1	0	-0.75	15

(c) In both the Big-M method and the two-phase method, only the final tableau represents a feasible solution for the original problem.

(d)

	3	2	4		90	Optimal Value
Solution	0	15	15			
					Right Hand Side	
Constraints	2	1	3	60 "="	60	
	3	3	5	120 >=	120	

4.6-10.

(a) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (20, 30, 0)$ and $Z^* = 120$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	Right Side
Z	0	1	-1M +3	2	-1M +7	1M	0	0	-20M
$\bar{x}_5$	1	0	-1	1	0	0	1	0	10
$\bar{x}_6$	2	0	2	-1	1	-1	0	1	10

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	Right Side
Z	0	1	0	-0.5M +3.5	-0.5M +5.5	0.5M +1.5	0	0.5M -1.5	-15M -15
$\bar{x}_5$	1	0	0	0.5	0.5	-0.5	1	0.5	15
x_1	2	0	1	-0.5	0.5	-0.5	0	0.5	5

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	Right Side
Z	0	1	0	0	2	5	1M -7	1M -5	-120
x_2	1	0	0	1	1	-1	2	1	30
x_1	2	0	1	0	1	-1	1	1	20

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (20, 30, 0)$ and $Z^* = 120$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	Right Side
Z	0	1	-1	0	-1	1	0	0	-20
$\bar{x}_5$	1	0	-1	1	0	0	1	0	10
$\bar{x}_6$	2	0	2	-1	1	-1	0	1	10

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	Right Side
Z	0	1	0	-0.5	-0.5	0.5	0	0.5	-15
$\bar{x}_5$	1	0	0	0.5	0.5	-0.5	1	0.5	15
x_1	2	0	1	-0.5	0.5	-0.5	0	0.5	5

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	Right Side
Z	0	1	0	0	0	0	1	1	0
x_2	1	0	0	1	1	-1	2	1	30
x_1	2	0	1	0	1	-1	1	1	20

Phase 2:

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	Right Side
Z	0	1	0	0	2	5	-120
x_2	1	0	0	1	1	-1	30
x_1	2	0	1	0	1	-1	20

(c) Only the final tableau for the Big-M method and the two-phase method represent feasible solutions to the original problem.

(d)

	3	2	7		120	Optimal Value
Solution	30	20	0			
					Right Hand Side	
Constraints	-1	1	0	10 "="	10	
	2	-1	1	10 >=	10	

4.6-11.

(a) FALSE. The initial basic solution for the artificial model is not feasible for the original model.

(b) FALSE. If at least one of the artificial variables is not zero, then the real problem is infeasible.

(c) FALSE. The two methods are basically equivalent, so they should take the same number of iterations.

4.6-12.

(a) Substitute $x_1 = x_1^+ - x_1^-$, where both x_1^+ and x_1^- are nonnegative.

maximize $Z = 3x_1^+ - 3x_1^- + 7x_2 + 5x_3$

$$\text{subject to} \quad \begin{aligned} 3x_1^+ - 3x_1^- + \ x_2 + 2x_3 &\leq 9 \\ -2x_1^+ + 2x_1^- + \ x_2 + 3x_3 &\leq 12 \\ x_1^+, x_1^-, x_2, x_3 &\geq 0 \end{aligned}$$

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (-0.6, 10.8, 0)$ and $Z^* = 73.8$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-3	3	-7	-5	0	0	0
X5	1	0	3	-3	1*	2	1	0	9
X6	2	0	-2	2	1	3	0	1	12

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	18	-18	0	9	7	0	63
X3	1	0	3	-3	1	2	1	0	9
X6	2	0	-5	5*	0	1	-1	1	3

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	12.6	3.4	3.6	73.8
X3	1	0	0	0	1	2.6	0.4	0.6	10.8
X2	2	0	-1	1	0	0.2	-0.2	0.2	0.6

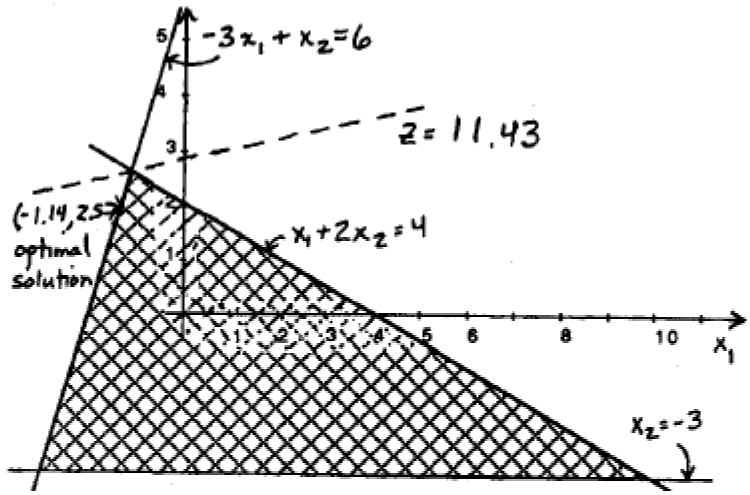
Note that x_1^+, x_1^-, x_2, and x_3 are renamed as X_1, X_2, X_3 and X_4 respectively.

(c)

	Coefficient of						
	X1	X2	X3	X4	Total		
Constraint 1	3	-3	1	2	9	≤	9
Constraint 2	-2	2	1	3	12	≤	12
Objective	3	-3	7	5	73.8		
Solution	0	0.6	10.8	0			

4.6-13.

(a) Optimal Solution: $(x_1^*, x_2^*) = (-1.14, 2.57)$ and $Z^* = 11.43$



(b) Let $x_{1,\text{OLD}} = x_1 - x_2$ and $x_{2,\text{OLD}} + 3 = x_3$.

$$\begin{array}{lrl} \text{maximize } Z = & -x_1 + x_2 + 4x_3 - 12 & \\ \text{subject to} & -3x_1 + 3x_2 + x_3 & \leq 9 \\ & x_1 - x_2 + 2x_3 & \leq 10 \\ & x_1, x_2, x_3 & \geq 0 \end{array}$$

(c) Optimal Solution: $(x_1^*, x_2^*) = (-1.14, 2.57)$ and $Z^* = 11.43$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right Side
Z	0	1	1	-1	-4	0	0	0
X4	1	0	-3	3	1	1	0	9
X5	2	0	1	-1	2	0	1	10

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right Side
Z	0	1	3	-3	0	0	2	20
X4	1	0	-3.5	3.5	0	1	-0.5	4
X3	2	0	0.5	-0.5	1	0	0.5	5

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right Side
Z	0	1	0	0	0	0.8571	1.5714	23.4286
X2	1	0	-1	1	0	0.2857	-0.143	1.14286
X3	2	0	0	0	1	0.1429	0.4286	5.57143

Optimal solution for the revised problem: $(0, 1.14, 5.57)$ with $Z^* = 23.43$

4.6-14.

(a) Let $x_{1,\text{OLD}} = x_1 - x_2$, $x_{2,\text{OLD}} = x_3 - x_4$, and $x_{3,\text{OLD}} = x_5 - x_6$.

$$\text{maximize } Z = -x_1 + x_2 + 2x_3 - 2x_4 + x_5 - x_6$$

$$\text{subject to} \quad \begin{aligned} 3x_3 - 3x_4 + x_5 - x_6 &\leq 120 \\ x_1 - x_2 - x_3 + x_4 - 4x_5 + 4x_6 &\leq 80 \\ -3x_1 + 3x_2 + x_3 - x_4 + 2x_5 - 2x_6 &\leq 100 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

(b)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	Right Side
Z	0	1	1	-1	-2	2	-1	1	0	0	0	0
X7	1	0	0	0	3	-3	1	-1	1	0	0	120
X8	2	0	1	-1	-1	1	-4	4	0	1	0	80
X9	3	0	-3	3	1	-1	2	-2	0	0	1	100

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	Right Side
Z	0	1	1	-1	0	0	-0.33	0.333	0.667	0	0	80
X3	1	0	0	0	1	-1	0.333	-0.33	0.333	0	0	40
X8	2	0	1	-1	0	0	-3.67	3.667	0.333	1	0	120
X9	3	0	-3	3	0	0	1.667	-1.67	-0.33	0	1	60

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	Right Side
Z	0	1	0	0	0	0	0.222	-0.22	0.556	0	0.333	100
X3	1	0	0	0	1	-1	0.333	-0.33	0.333	0	0	40
X8	2	0	0	0	0	0	-3.11	3.111	0.222	1	0.333	140
X2	3	0	-1	1	0	0	0.556	-0.56	-0.11	0	0.333	20

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	Right Side
Z	0	1	0	0	0	0	0	0	0.571	0.071	0.357	110
X3	1	0	0	0	1	-1	3e-20	-3e-2	0.357	0.107	0.036	55
X6	2	0	0	0	0	0	-1	1	0.071	0.321	0.107	45
X2	3	0	-1	1	0	0	0	0	-0.07	0.179	0.393	45

Optimal solution for the revised problem: $(0, 45, 55, 0, 0, 45)$

Optimal solution for the original problem: $(x_1^*, x_2^*, x_3^*) = (-45, 55, -45)$ and $Z^* = 110$

(c)

	-1	2	1		**110**	Optimal Value
Solution	**-45**	**55**	**-45**			
					Right Hand Side	
Constraints	0	3	1	120 <=	120	
	1	-1	-4	80 <=	80	
	-3	1	2	100.0 <=	100	

4.6-15.

(a) In order to decrease the objective function value in the simplex method, choose the nonbasic variable that has the (largest) positive coefficient in the objective row, as the entering basic variable. The ratio test is conducted the same way as in the maximization problem to determine the leaving basic variable.

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (11.67, 0, 17.5)$ and $Z^* = 122$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
			3M	8M	6M	-1M	-1M			140
Z	0	1	-3	-8	-5	0	0	0	0	0
X6	1	0	0	3	4	-1	0	1	0	70
X7	2	0	3	5*	2	0	-1	0	1	70

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
			-1.8M		2.8M	-1M	0.6M		-1.6M	28 M
Z	0	1	1.8	0	-1.8	0	-1.6	0	1.6	112
X6	1	0	-1.8	0	2.8*	-1	0.6	1	-0.6	28
X2	2	0	0.6	1	0.4	0	-0.2	0	0.2	14

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
								-1M	-1M	
Z	0	1	0.64	0	0	-0.643	-1.214	0.64	1.21	130
X3	1	0	-0.64	0	1	-0.36	0.214	0.357	-0.21	10
X2	2	0	0.857*	1	0	0.143	-0.29	-0.14	0.286	10

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
								-1M	-1M	
Z	0	1	0	-0.75	0	-0.75	1	0.75	1	122
X3	1	0	0	0.75	1	-0.25	0	0.25	0	17.5
X1	2	0	1	1.167	0	0.167	-0.33	-0.17	0.333	11.67

4.6-16.

(a) maximize $Z = -2x_1 + 2x_2 + x_3 - 4x_4 + 3x_5$

subject to

$$
\begin{aligned}
x_1 - x_2 + x_3 + 3x_4 - x_5 &\le 4 \\
-x_1 + x_2 + x_4 - x_5 &\le 1 \\
2x_1 - 2x_2 + x_3 &\le 2 \\
x_1 - x_2 + 2x_3 + x_4 + 2x_5 &= 2 \\
x_1, x_2, x_3, x_4, x_5 &\ge 0
\end{aligned}
$$

(b)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	Right side
			-1M	1M	-2M	-1M	-2M					-2 M
Z	0	1	2	-2	-1	4	-3	0	0	0	0	0
X6	1	0	1	-1	1	3	2	1	0	0	0	4
X7	2	0	-1	1	0	1	-1	0	1	0	0	1
X8	3	0	2	-2	1	0	0	0	0	1	0	2
X9	4	0	1	-1	2	1	2*	0	0	0	1	2

(c)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	Right side
Z	0	1	-1	1	-2	-1	-2	0	0	0	0	-2

(d)

	-2	1	-4	3		17 Optimal Value
Solution	-4	0	0	3		
						Right Hand Side
Constraints	1	1	3	2	2 <=	4
	1	0	-1	1	-1 >=	-1
	2	1	0	0	-8 <=	2
	1	2	1	2	2 "="	2

4.6-17.

Reformulation:

maximize $Z = 4x_1 + 5x_2 + 3x_3$

subject to
$$x_1 + x_2 + 2x_3 - x_4 + \overline{x}_7 = 20$$
$$15x_1 + 6x_2 - 5x_3 + x_5 = 50$$
$$x_1 + 3x_2 + 5x_3 + x_6 = 30$$
$$x_1, x_2, x_3, x_4, x_5, x_6, \overline{x}_7 \geq 0$$

Phase 1:

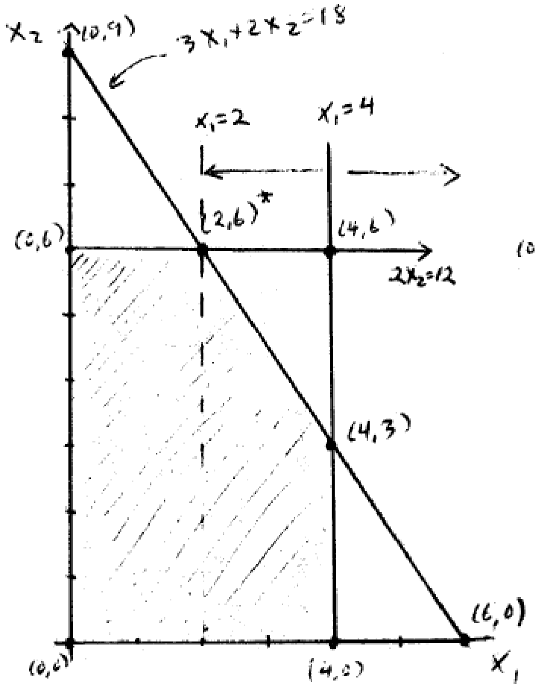
Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	$\overline{x}_7$	Right Side
Z	0	1	-1	-1	-2	1	0	0	0	-20
$\overline{x}_7$	1	0	1	1	2	-1	0	0	1	20
x_5	2	0	15	6	-5	0	1	0	0	50
x_6	3	0	1	3	5	0	0	1	0	30

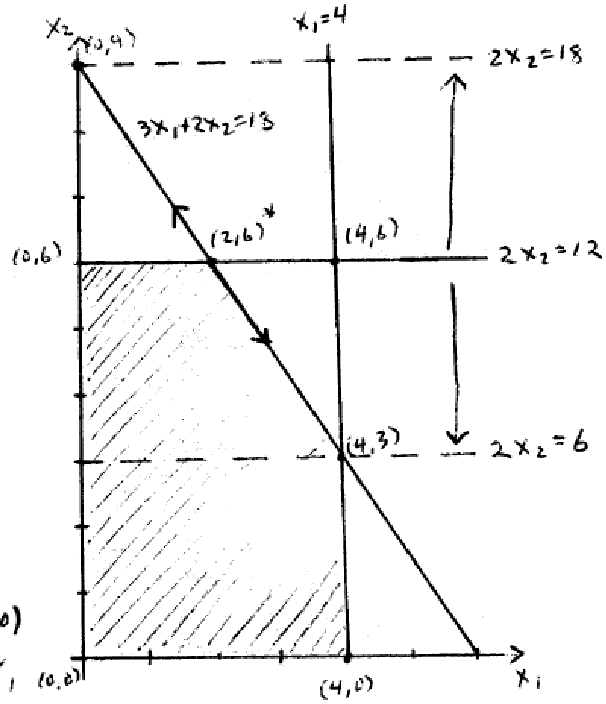
Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	$\overline{x}_7$	Right Side
Z	0	1	-0.6	0.2	0	1	0	0.4	0	-8
$\overline{x}_7$	1	0	0.6	-0.2	0	-1	0	-0.4	1	8
x_5	2	0	16	9	0	0	1	1	0	80
x_3	3	0	0.2	0.6	1	0	0	0.2	0	6

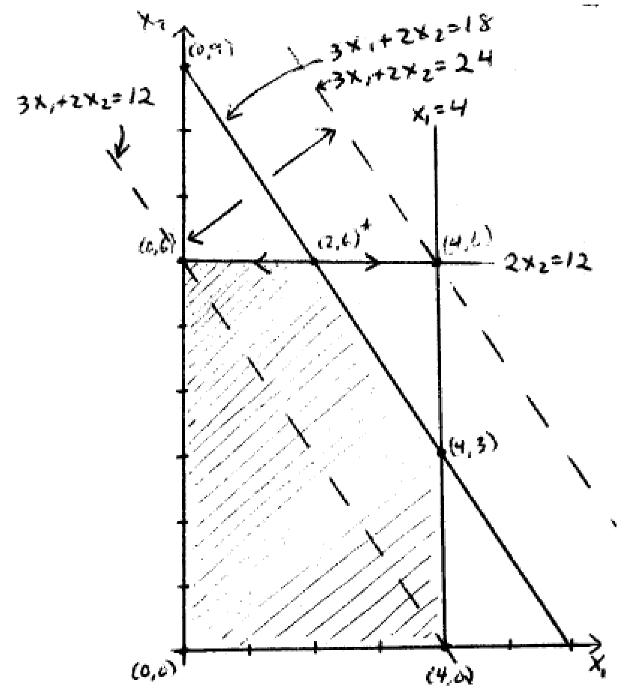
Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	$\overline{x}_7$	Right Side
Z	0	1	0	0.5375	0	1	0.0375	0.4375	0	-5
$\overline{x}_7$	1	0	0	-0.538	0	-1	-0.038	-0.438	1	5
x_1	2	0	1	0.5625	0	0	0.0625	0.0625	0	5
x_3	3	0	0	0.4875	1	0	-0.013	0.1875	0	5

Since this is the optimal tableau for Phase 1 and the artificial variable $\overline{x}_7 = 5 > 0$, the problem is infeasible.

4.7-1.







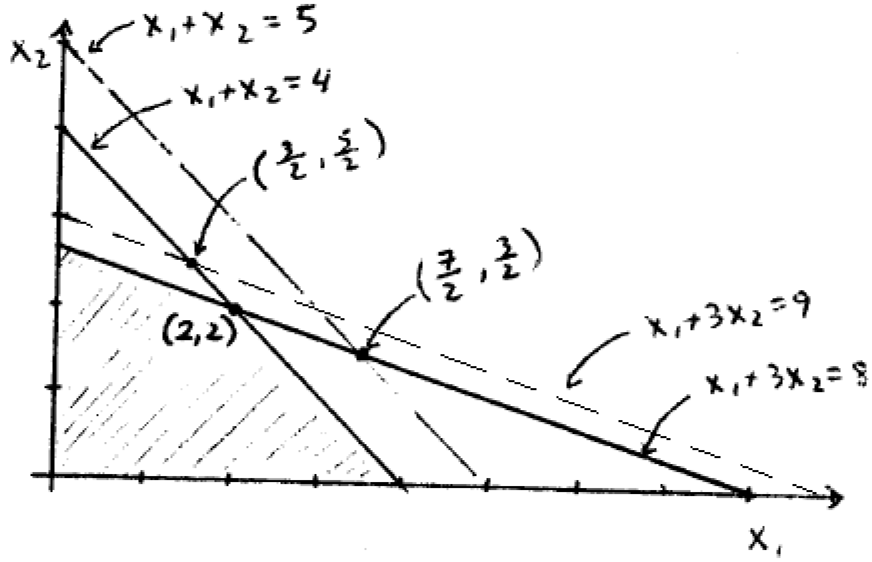
The CP solution $(2,6)$ remains feasible and optimal if the constraint $x_1 \leq 4$ is changed to $x_1 \leq k$ with $2 \leq k < \infty$. However, if $k < 2$, then this solution ceases to be feasible and the optimal solution becomes $(k,6)$. This agrees with the allowable range (allowable increase: 1E$+30$, allowable decrease: 4) for this constraint given in Figure 4.10.

Now, suppose instead that the constraint $2x_2 \leq 12$ is replaced by $2x_2 \leq k$. Then, the intersection of the lines $2x_2 = k$ and $3x_1 + 2x_2 = 18$ can be expressed as $((18-k)/3, k/2)$. This CP solution is feasible as long as $0 \leq x_1 \leq 4$ or equivalently $6 \leq k \leq 18$. In that case, provided that the objective function is the same, this solution is optimal. Hence, the right-hand side of this constraint can be increased or decreased by 6.

If the third constraint is $3x_1 + 2x_2 \leq k$, then the CP solution determined by this and $2x_2 \leq 12$ becomes $((k-12)/3, 6)$. This point is feasible and optimal as long as $0 \leq x_1 \leq 4$ or equivalently $12 \leq k \leq 24$, so the allowable change for this constraint is also ± 6, as given in Figure 4.10.

4.7-2.

(a)

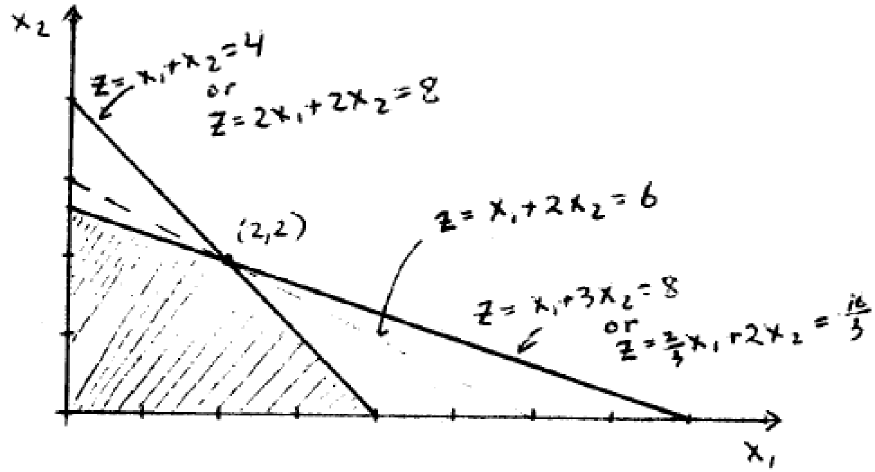


Constraint (1): $x_1 + 3x_2 \leq 8$: $x_1 + 3x_2 = 8 \Rightarrow x_1 = x_2 = 2$ and $Z = 6$
$x_1 + 3x_2 = 9 \Rightarrow x_1 = 3/2$, $x_2 = 5/2$ and $Z = 13/2$
$\Delta Z = 13/2 - 6 = 1/2 = y_1^*$

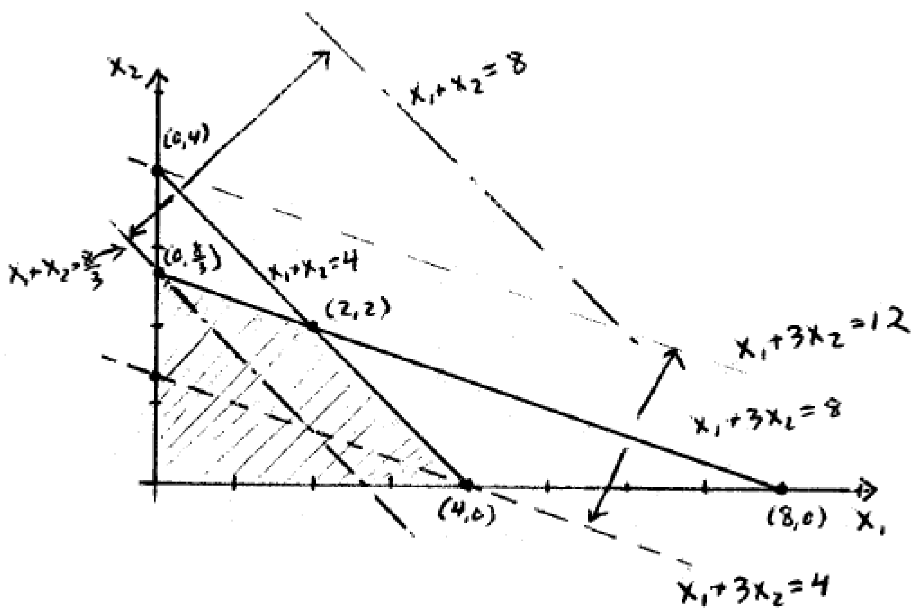
Constraint (2): $x_1 + x_2 \leq 4$: $x_1 + x_2 = 4 \Rightarrow x_1 = x_2 = 2$ and $Z = 6$
$x_1 + x_2 = 5 \Rightarrow x_1 = 7/2$, $x_2 = 3/2$ and $Z = 13/2$
$\Delta Z = 13/2 - 6 = 1/2 = y_2^*$

(b) From (a), we see that the right-hand sides $b_1 = 8$ and $b_2 = 4$ are sensitive parameters. The graph in part (a) shows that both constraints are active (binding) at the optimal solution, so all the coefficients $a_{11} = 1$, $a_{12} = 3$, $a_{21} = 1$, and $a_{22} = 1$ are sensitive parameters, too. As will be seen in (c), the objective coefficients $c_1 = 1$ and $c_2 = 2$ are not sensitive parameters.

(c) Observe that the optimal solution remains the same for $2/3 \leq c_1 \leq 2$ (with $c_2 = 2$ fixed) and $1 \leq c_2 \leq 3$ (with $c_1 = 1$ fixed)



(d) The dashed lines "- - -" in the graph below suggest that the CP solution ranges from $(4,0)$ to $(0,4)$ when $4 \leq b_1 \leq 12$. Outside this range, the CP solution becomes infeasible. The dashed lines "- · -" represent the second constraint for different right-hand side values. They suggest that the CP solution ranges from $(0,8/3)$ to $(0,8)$ when $8/3 \leq b_2 \leq 8$. Hence, the allowable ranges are $4 \leq b_1 \leq 12$ and $8/3 \leq b_2 \leq 8$.



(e)

	1	2		6	Optimal Value
Variables	**2**	**2**			
				RHS	
Constraints	1	3	8 <=	8	
	1	1	4 <=	4	

Adjustable Cells

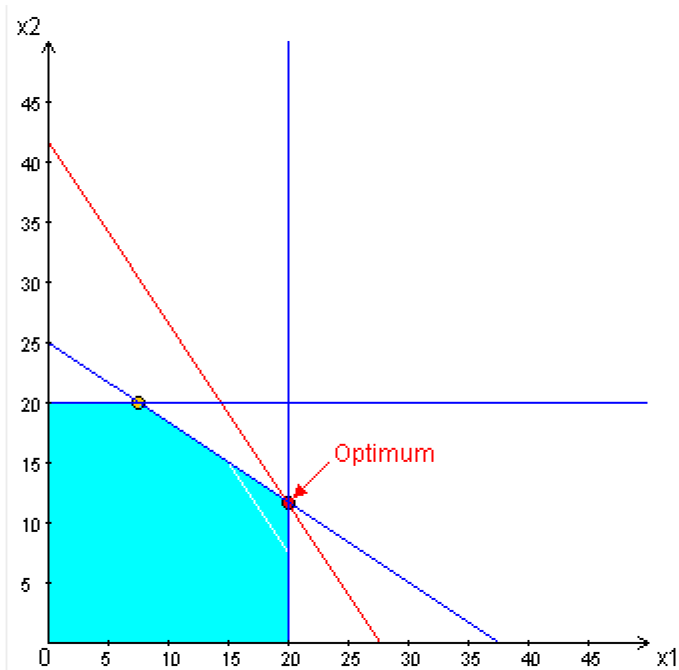
Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B2		2	0	1	1	0.333333
C2		2	0	2	1	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E4		8	0.5	8	4	4
E5		4	0.5	4	4	1.333333

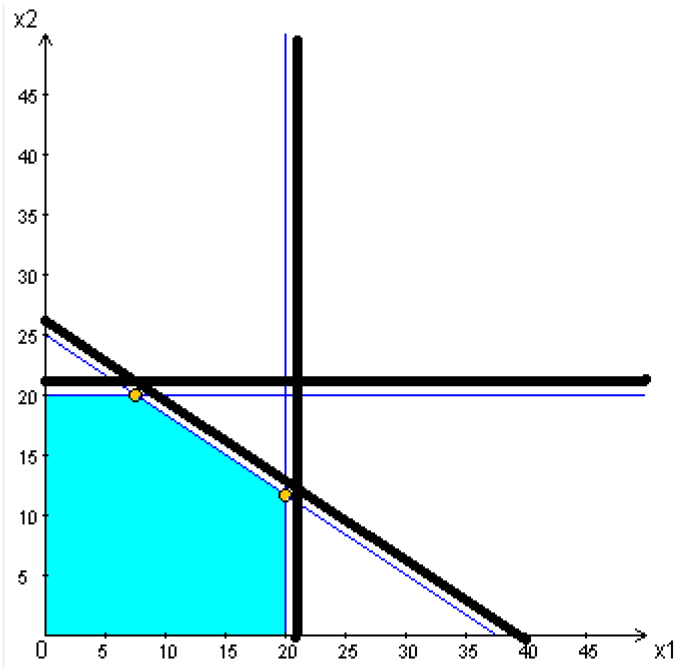
4.7-3.

(a) Optimal Solution: $(x_1^*, x_2^*) = (20, 11.67)$ and $Z^* = 83.33$



Corner Point	Z
$(0, 20)$	40
$(7.5, 20)$	62.5
$(20, 11.67)$	83.33*
$(20, 0)$	60
$(0, 0)$	0

(b)



Increasing resource 1 to 61 units increases Z to $3(20.33) + 2(11.44) = 83.89$, so $\Delta Z = y_1^* = 0.56$.
Increasing resource 2 to 76 units increases Z to $4(20) + 2(12) = 84$, so $\Delta Z = y_2^* = 0.67$.
The third constraint is not binding, so $y_3^* = 0$.

(c) To increase Z by 15, resource 1 should be increased by $\frac{15}{y_1^*} = \frac{15}{0.56} \approx 27$. Solving the LP problem with resource 1 set to $60 + 27 = 87$ returns the result $Z = 98.33$.

4.7-4.

(a) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0.5, 0, 4.5)$ and $Z^* = 14$

Bas Var	Eq No	Coefficient of: Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	-1	7	-3	0	0	0	0
X_4	1	0	2	1	-1	1	0	0	4
X_5	2	0	4	-3	0	0	1	0	2
X_6	3	0	-3	2	1	0	0	1	3

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	0	1	-10	13	0	0	0	3	9
x_4	1	0	-1	3	0	1	0	1	7
x_5	2	0	4	-3	0	0	1	0	2
x_3	3	0	-3	2	1	0	0	1	3

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	0	1	0	5.5	0	0	2.5	3	14
x_4	1	0	0	2.25	0	1	0.25	1	7.5
x_1	2	0	1	-0.75	0	0	0.25	0	0.5
x_3	3	0	0	-0.25	1	0	0.75	1	4.5

(b) The shadow prices for the three resources are given by the reduced costs (in the objective function) for the corresponding slack variables. These values are circled in the table above. The shadow prices for resources 1, 2 and 3 are 0, 2.5 and 3 respectively. They represent the rate at which the objective function value z increases as the corresponding resource is increased. For instance, increasing resource 3 by one unit increases Z by 3, provided that no other constraints cause any trouble.

(c)

	1	-7	3			14	Optimal Val
Variables	**0.5**	**0**	**4.5**				
					RHS		
Constraints	2	1	-1	-3.5	<=	4	
	4	-3	0	2	<=	2	
	-3	2	1	3	<=	3	

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$3	Variables	0.5	0	1	7.3333333	10
\$C\$3	Variables	0	-5.5	-7	5.5	1E+30
\$D\$3	Variables	4.5	0	3	22	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$F\$5	Constraints	-3.5	0	4	1E+30	7.5
\$F\$6		2	2.5	2	1E+30	2
\$F\$7		3	3	3	1E+30	4.5

4.7-5.

(a) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, 1, 3)$ and $Z^* = 7$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	0	1	-2	2	-3	0	0	0	0
x_4	1	0	-1	1	1	1	0	0	4
x_5	2	0	2	-1	1	0	1	0	2
x_6	3	0	1	1	3	0	0	1	12

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	0	1	4	-1	0	0	3	0	6
x_4	1	0	-3	2	0	1	-1	0	2
x_3	2	0	2	-1	1	0	1	0	2
x_6	3	0	-5	4	0	0	-3	1	6

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	0	1	2.5	0	0	0.5	2.5	0	7
x_2	1	0	-1.5	1	0	0.5	-0.5	0	1
x_3	2	0	0.5	0	1	0.5	0.5	0	3
x_6	3	0	1	0	0	-2	-1	1	2

(b) The shadow prices are $y_1^* = 0.5$, $y_2^* = 2.5$ and $y_3^* = 0$. They are the marginal values of resources 1, 2 and 3 respectively.

(c)

	2	-2	3			7	Optimal Value
Variables	**0**	**1**	**3**				
						RHS	
Constraints	-1	1	1	4	<=	4	
	2	-1	1	2	<=	2	
	1	1	3	10	<=	12	

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B3	Variables	0	-2.5	2	2.5	1E+30
C3	Variables	1	0	-2	1.6666667	1
D3	Variables	3	0	3	1E+30	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F5	Constraints	4	0.5	4	1	2
F6		2	2.5	2	2	6
F7		10	0	12	1E+30	2

4.7-6.

(a) Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (11, 0, 3, 0)$ and $Z^* = 52$

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	-5	-2	1	-3	0	0	0
X_5	1	0	3	2	-3	1	1	0	24
X_6	2	0	3	3	1	3	0	1	36

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	0	1.3333	-4	-1.333	1.6667	0	40
X_1	1	0	1	0.6667	-1	0.3333	0.3333	0	8
X_6	2	0	0	1	4	2	-1	1	12

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	0	2.3333	0	0.6667	0.6667	1	52
X_1	1	0	1	0.9167	0	0.8333	0.0833	0.25	11
X_3	2	0	0	0.25	1	0.5	-0.25	0.25	3

(b) The shadow prices are $y_1^* = 0.6667$ and $y_2^* = 1$. They are the marginal values of resources 1 and 2 respectively.

(c)

	5	4	-1	3			
Variables	11	0	3	0			52 Optimal Value
Constraints							RHS
	3	2	-3	1	24	<=	24
	3	3	1	3	36	<=	36

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C3	Variables	11	0	5	1E+30	0.363636
D3	Variables	0	-0.33333333	4	0.33333333	1E+30
E3	Variables	3	0	-1	2.66666667	1.333333
F3	Variables	0	-0.66666667	3	0.66666667	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
H5	Constraints	24	0.66666667	24	12	132
H6		36	1	36	1E+30	12

4.9-1.

```
Linear Programming Model:
Number of Decision Variables:      2
Number of Functional Constraints:  4
Max Z =   4500 X1 + 4500 X2
subject to
 1)           1 X1 +    0 X2 <=       1
 2)           0 X1 +    1 X2 <=       1
 3)        5000 X1 + 4000 X2 <=    6000
 4)         400 X1 +  500 X2 <=     600
and
        X1 >= 0, X2 >= 0.
Solve Automatically by the Interior Point Algorithm:
(X1, X2) = (0.1, 0.2) and Alpha = 0.5
```

It.	X1	X2	Z
0	0.1	0.2	1350
1	0.1999	0.58008	3509.91
2	0.26144	0.76085	4600.3
3	0.33761	0.81491	5186.35
4	0.40279	0.82027	5503.76
5	0.4661	0.79837	5690.12
6	0.56345	0.73487	5842.42
7	0.62351	0.69021	5911.71
8	0.6511	0.67092	5949.09
9	0.66172	0.66525	5971.35
10	0.66487	0.66511	5984.91
11	0.66582	0.66582	5992.4
12	0.66624	0.66624	5996.2
13	0.66646	0.66646	5998.1
14	0.66656	0.66656	5999.05
15	0.66661	0.66661	5999.52

4.9-2.

The linear programming problem is:

Number of Decision Variables: 2

Number of Functional Constraints: 2

Max Z = 1 X1 + 2 X2

subject to

1) 1 X1 + 3 X2 <= 8

2) 1 X1 + 1 X2 <= 4

and

X1 >= 0, X2 >= 0.

Solve Automatically by the Interior Point Algorithm:

(X1, X2) = (0.1, 0.2) and Alpha = 0.5

It.	X1	X2	Z
0	0.1	0.2	0.5
1	0.24587	1.36804	2.98196
2	0.25651	1.97283	4.20217
3	0.26482	2.27423	4.81327
4	0.28233	2.42047	5.12328
5	0.32398	2.48263	5.28924
6	0.43489	2.48368	5.40225
7	0.82513	2.37261	5.57036
8	1.4229	2.17597	5.77485
9	1.72185	2.07758	5.87702
10	1.86959	2.03012	5.92984
11	1.94077	2.00909	5.95894
12	1.97327	2.00166	5.97659
13	1.98735	2.00011	5.98758
14	1.99373	2	5.99373
15	1.99687	2	5.99687

Cases

4-1 a) The fixed design and fashion costs are sunk costs and therefore should not be considered when setting the production now in July. Since the velvet shirts have a positive contribution to covering the sunk costs, they should be produced or at least considered for production according to the linear programming model. Had Ted raised these concerns before any fixed costs were made, then he would have been correct to advise against designing and producing the shirts. With a contribution of $22 and a demand of 6000 units, maximum expected profit will be only $132,000. This amount will not be enough to cover the $500,000 in fixed costs directly attributable to this product.

b) The following insight greatly simplifies the analysis of the problem. The production processes of the various clothing items are not all linked together. We can separate the clothing items according to the materials that are used in their production and instead of one large linear programming problem we can formulate 4 smaller problems.

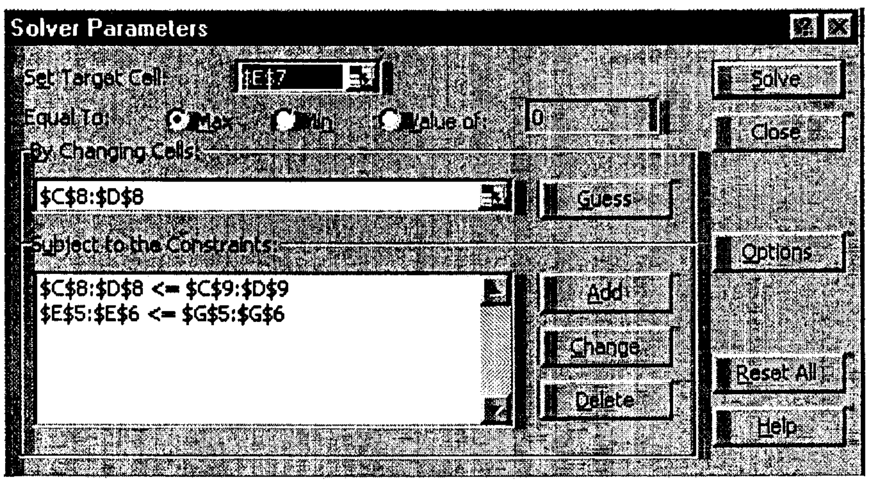
We use the term net contribution of a sales item to describe the difference between its total revenues and variable costs. The net contribution does not reflect any part of the fixed costs.

The cashmere sweater is the only item consisting of cashmere. The net contribution of one cashmere sweater equals $450 - $150 – 1.5*$60 = $210. TrendLines can sell at most 4000 sweaters and has 9000 yards of cashmere as raw material. It is optimal to produce 4000 sweaters using 6000 yards of cashmere yielding a net contribution of 4000*$210 = $840,000.

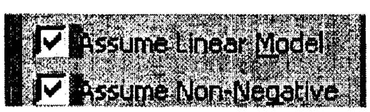
The silk blouse and camisole are the only items using silk and no other materials are used for these items. We can determine the optimal production amounts of these two items through a simple linear program. The first constraint models the resource limitation in the production process that Katherine has ordered 18,000 yards of silk. The second constraint models the production condition that whenever a silk blouse is produced automatically also a silk camisole is produced. Finally we must include the stated upper bounds on the number of silk items we can sell.

	A	B	C	D	E	F	G
1							
2							
3			Activity				
4		Constraint	silk blouse	silk camisole	Totals		Constraint RHS
5		silk	1.5	0.5	18000	<=	18000
6		production	1	-1	-8000	<=	0
7		unit profit	60.5	53.5	**1226000**		
8		Solution	**7000**	**15000**			
9		Maximum	12000	15000			

	E
3	
4	Totals
5	=SUMPRODUCT(C5:D5,C8:D8)
6	=SUMPRODUCT(C6:D6,C8:D8)
7	**=SUMPRODUCT(C7:D7,C8:D8)**
8	



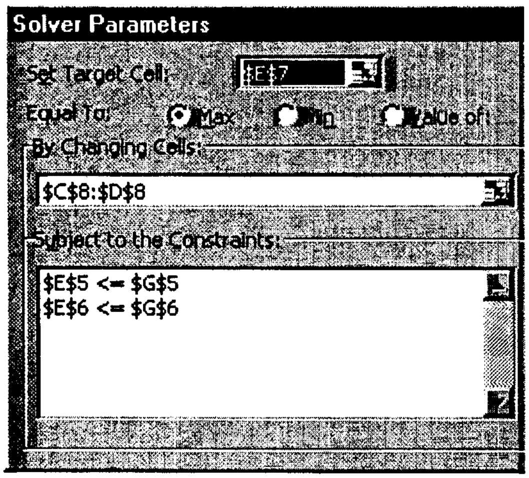
Throughout this case we use the following solver options.



TrendLines should produce 7000 silk blouses and 15000 silk camisoles yielding a net contribution of $1,226,000.

We can determine the optimal production plan for the items made from cotton in a similar fashion. There are no demand limitations for the cotton items.

	A	B	C	D	E	F	G
1							
2							
3			Activity				
4		Constraint	cotton sweater	cotton m-s	Totals		Constraint RHS
5		wool	1.5	0.5	30000	<=	30000
6		production	1	-1	-60000	<=	0
7		unit profit	66.25	33.75	**2025000**		
8		Solution	**0**	**60000**			

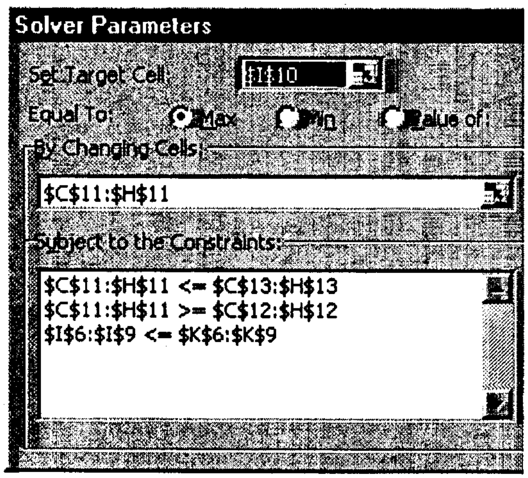


TrendLines should produce 60000 cotton mini-skirts but no cotton sweaters yielding a net contribution of $2,025,000.

It remains to develop a linear programming problem for determining the optimal production quantities of the tailored wool slacks, the tailored skirt, the wool blazer, the velvet pants and shirts, and the button-down blouse. We include four constraints for the resource limitations on wool, velvet, rayon, and acetate. Upper and lower bounds are given for many items. When there is no lower bound, we insert 0, when there is no upper bound, we determine a safe upper bound as a consequence of the resource limitations.

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3			Resource Usage Per Unit of Each Activity								
4			Activity								Resource
5		Resource	tail.wool slacks	tail.skirt	wool blazer	velvet pants	velvet shirt	b.-d. blouse	Totals		Available
6		wool	3	0	2.5	0	0	0	25100	<=	45000
7		acetate	2	1.5	1.5	2	0	0	28000	<=	28000
8		rayon	0	2	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	3	1.5	0	9000	<=	20000
10		unit profit	110	143.25	155.25	136	22	26.625	**2771933.333**		
11		Solution	**4200**	**8066.666667**	**5000**	**0**	**6000**	**9244.444444**			
12		Minimum	4200	2800	3000	0	0	0			
13		Maximum	7000	20000	5000	5500	6000	20000			

	I
4	
5	Totals
6	=SUMPRODUCT(C6:H6,C11:H11)
7	=SUMPRODUCT(C7:H7,C11:H11)
8	=SUMPRODUCT(C8:H8,C11:H11)
9	=SUMPRODUCT(C9:H9,C11:H11)
10	**=SUMPRODUCT(C10:H10,C11:H11)**
11	



TrendLines should produce 4200 wool slacks, 8066.67 skirts, 5000 wool blazers, no velvet pants, 6000 velvet shirts, and 9244.44 button-down blouses. The net contribution of these items equals $2,771,933.33. (Of course, TrendLines cannot produce two-thirds of a skirt, so the actual solution should be integer. You will learn about integer programming in chapter 8.)

The net contribution of all clothing items equals $840,000 + $1,226,00 + $ 2,025,000 + $2,771,933.33 = $6,862,933.33. So far we have not considered the sunk costs for the three fashion shows and the designers which total $8,960,000. The total profit equals $6,862,933.33 - $8,960,000 = -$2,097,066.67. So, TrendLines actually loses almost $2.1 million.

c) If velvet cannot be sent back to the textile wholesaler, then the whole quantity will be considered as a sunk cost and therefore added to the fixed costs. The objective function coefficients of items using velvet will no longer include the material cost. The objective function coefficients of the velvet pants and shirts are now \$175 and \$40, respectively.

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3			Resource Usage Per Unit of Each Activity								
4			Activity								Resource
5		Resource	tail.wool slacks	tail.skirt	wool blazer	velvet pants	velvet shirt	b.-d. blouse	Totals		Available
6		wool	3	0	2.5	0	0	0	25100	<=	45000
7		acetate	2	1.5	1.5	2	0	0	28000	<=	28000
8		rayon	0	2	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	3	1.5	0	20000	<=	20000
10		unit profit	110	143.25	155.25	172	40	26.625	2983822.22		
11		Solution	**4200**	**3177.777778**	**5000**	**3666.666667**	**6000**	**15762.96296**			
12		Minimum	4200	2800	3000	0	0	0			
13		Maximum	7000	20000	5000	5500	6000	20000			

The production plan changes considerably. TrendLines should produce 4200 wool slacks, 3177.77 skirts, 5000 wool blazers, 3666.67 velvet pants, 6000 velvet shirts, and 15762.92 button-down blouses. The production decisions for all other items are unaffected by the change. The net contribution of all clothing items equals \$840,000 + \$1,226,00 + \$ 2,025,000 + \$2,983,822.22 = \$7,074,822.22. The sunk costs now include the material cost for velvet and total \$9,200,000. The loss equals \$9,200,000 - \$7,074,822.22 = \$2,125,177.78.

d) When TrendLines cannot return the velvet to the wholesaler, the costs for velvet cannot be recovered. These cost are no longer variable cost but now are sunk cost. As a consequence the increased net contribution of the velvet items makes them more attractive to produce. This way the revenues from selling these items can contribute to the recovery of at least some of the fixed costs. Instead of zero TrendLines produces now 3666.67 velvet pants. These pants also require some acetate and thus their production affects the production plan for all other items. Since it is not optimal to make full use of the ordered velvet in part (b) it comes as no surprise that the loss in part (c) is even bigger than in part (b).

e) The unit contribution of a wool blazer changes to \$75.25.

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3			Resource Usage Per Unit of Each Activity								
4			Activity								Resource
5		Resource	tail.wool slacks	tail.skirt	wool blazer	velvet pants	velvet shirt	b.-d. blouse	Totals		Available
6		wool	3	0	2.5	0	0	0	20100	<=	45000
7		acetate	2	1.5	1.5	2	0	0	28000	<=	28000
8		rayon	0	2	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	3	1.5	0	9000	<=	20000
10		unit profit	110	143.25	75.25	136	22	26.625	2436933.333		
11		Solution	**4200**	**10066.66667**	**3000**	**0**	**6000**	**6577.777778**			
12		Minimum	4200	2800	3000	0	0	0			
13		Maximum	7000	20000	5000	5500	6000	20000			

TrendLines should produce 4200 wool slacks, 10066.67 skirts, the minimum of 3000 wool blazers, no velvet pants, 6000 velvet shirts, and 6577.78 button-down blouses. The production decisions for all other items are unaffected by the change. The net contribution of all clothing items equals \$840,000 + \$1,226,00 + \$ 2,025,000 + \$2,436,933.33 = \$6,527,933.33. The loss equals \$8,960,000 - \$6,527,933.33= \$2,432,066.67.

f) The right-hand-side of the acetate constraint changes.

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3			Resource Usage Per Unit of Each Activity								
4					Activity						Resource
5		Resource	tail.wool slacks	tail.skirt	wool blazer	velvet pants	velvet shirt	b.-d. blouse	Totals		Available
6		wool	3	0	2.5	0	0	0	25100	<=	45000
7		acetate	2	1.5	1.5	2	0	0	38000	<=	38000
8		rayon	0	2	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	3	1.5	0	9000	<=	20000
10		unit profit	110	143.25	155.25	136	22	26.625	3490266.667		
11		Solution	**4200**	**14733.33333**	**5000**	**0**	**6000**	**355.5555556**			
12		Minimum	4200	2800	3000	0	0	0			
13		Maximum	7000	15000	5000	5500	6000	20000			

TrendLines should produce 4200 wool slacks, 14733.33 skirts, the minimum of 5000 wool blazers, no velvet pants, 6000 velvet shirts, and 355.55 button-down blouses. The production decisions for all other items are unaffected by the change. The net contribution of all clothing items equals \$840,000 + \$1,226,00 + \$ 2,025,000 + \$3,490,266.67 = \$7,581,266.67. The loss equals \$8,960,000 - \$7,581,266.67 = \$1,378,733.33.

g) The net contribution of one cashmere sweater sold in the November sale equals 0.6*$450 - $150 – 1.5*$60 = $30. After producing 4000 sweaters to be sold in September and October TrendLines has 3000 yards of cashmere as raw material left. It is optimal to produce 2000 more sweaters using the remaining 3000 yards of cashmere yielding an additional contribution of 2000*$30 = $60,000.

For the three linear programming problems determining the production plans for all other clothing items we need to include new decision variables representing the number of clothing items that are sold during the November sale. Clearly TrendLines does not want to produce items with a negative net contribution. Therefore, we need to consider only those clothing items that have a positive net contribution after taking the sales price into account.

	A	B	C	D	E	F	G	H
1								
2								
3				Activity				
4		Constraint	silk blouse	silk camisole	silk camisole(sale)	Totals		Constraint RHS
5		silk	1.5	0.5	0.5	18000	<=	18000
6		production	1	-1	-1	-8000	<=	0
7		unit profit	60.5	53.5	5.5	**1226000**		
8		Solution	**7000**	**15000**	**0**			
9		Maximum	12000	15000	36000			

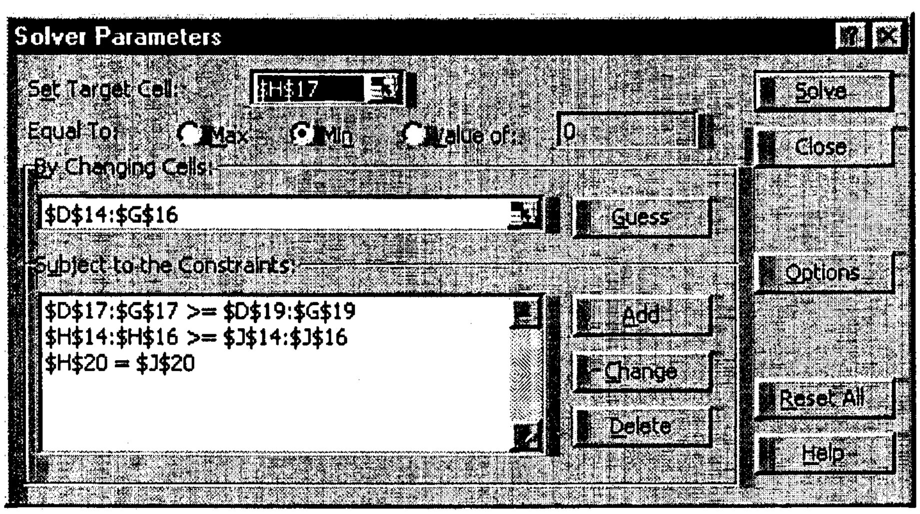
	A	B	C	D	E	F	G	H	I
1									
2									
3				Activity					
4		Constraint	cotton sweater	sweater(sale)	cotton m-s	m-s (sale)	Totals		Constraint RHS
5		wool	1.5	1.5	0.5	0.5	30000	<=	30000
6		production	1	1	-1	-1	-60000	<=	0
7		unit profit	66.25	14.25	33.75	3.75	**2025000**		
8		Solution	**0**	**0**	**60000**	**0**			

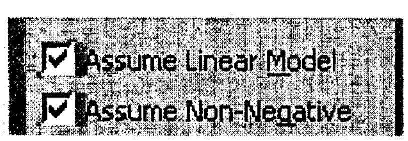
	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2													
3						Resource Usage Per Unit of Each Activity							
4						Activity							Resource
5		Resource	tail.wool slacks	tail.skirt	skirt (sale)	wool blazer	blazer (sale)	velvet pants	velvet shirt	b.-d. blouse	Totals		Available
6		wool	3	0	0	2.5	2.5	0	0	0	25100	<=	45000
7		acetate	2	1.5	1.5	1.5	1.5	2	0	0	28000	<=	28000
8		rayon	0	2	2	0	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	0	0	3	1.5	0	9000	<=	20000
10		unit profit	110	143.25	35.25	155.25	27.25	136	22	26.625	**2771933.333**		
11		Solution	**4200**	**8066.666667**	**0**	**5000**	**0**	**0**	**6000**	**9244.444444**			
12		Minimum	4200	2800	0	3000	0	0	0	0			
13		Maximum	7000	15000	15000	5000	20000	5500	6000	20000			

It only pays to produce 2000 more Cashmere sweaters. The production plan for all other items is the same as in part (b). The sale of the Cashmere sweaters reduces the loss by $60,000 to $2,037,066.67.

4-2 a) We define 12 decision variables, one for each age group surveyed in each region. Rob's restrictions are easily modeled as constraints. For example, his condition that at least 20 percent of the surveyed customers have to be from the first age group requires that the sum of the variables for the age group "18 to 25" across all three regions is at least 400. All his other requirements are modeled similarly. Finally, the sum of all variables has to equal 2000, because that is the number of customers Rob wants to have interviewed.

	A	B	C	D	E	F	G	H	I	J	K
1											
2				Cost per Person							
3				Age Group							
4				18 to 25	26 to 40	41 to 50	51 and over				
5			Silicon Valley	$4.75	$6.50	$6.50	$5.00				
6		Region	Big Cities	$5.25	$5.75	$6.25	$6.25				
7			Small Towns	$6.50	$7.50	$7.50	$7.25				
8											
9											
10											
11				Number of People Surveyed							
12				Age Group							
13				18 to 25	26 to 40	41 to 50	51 and over	Totals	Survey restrictions		
14			Silicon Valley	**600**	**0**	**0**	**300**	900	>=	300	
15		Region	Big Cities	**0**	**550**	**150**	**0**	700	>=	700	
16			Small Towns	**250**	**0**	**150**	**0**	400	>=	400	
17		Totals		850	550	300	300	**$11,200**	=	Total Cost	
18				>=	>=	>=	>=	$12,880.00	=	Bid	
19	Survey restrictions			400	550	300	300	Total Surveys			
20								2000	=	2000	
21											
22			Formula in cell H14:		=SUM(D14:G14)						
23			Formula in cell H15:		=SUM(D15:G15)						
24			Formula in cell H16:		=SUM(D16:G16)						
25			Formula in cell D17:		=SUM(D14:D16)						
26			Formula in cell E17:		=SUM(E14:E16)						
27			Formula in cell F17:		=SUM(F14:F16)						
28			Formula in cell G17:		=SUM(G14:G16)						
29			Formula in cell H20:		=SUM(D14:G16)						
30			Formula in cell H17:		=SUMPRODUCT(D5:G7,D14:G16)						
31			Formula in cell H18:		=1.15*H17						



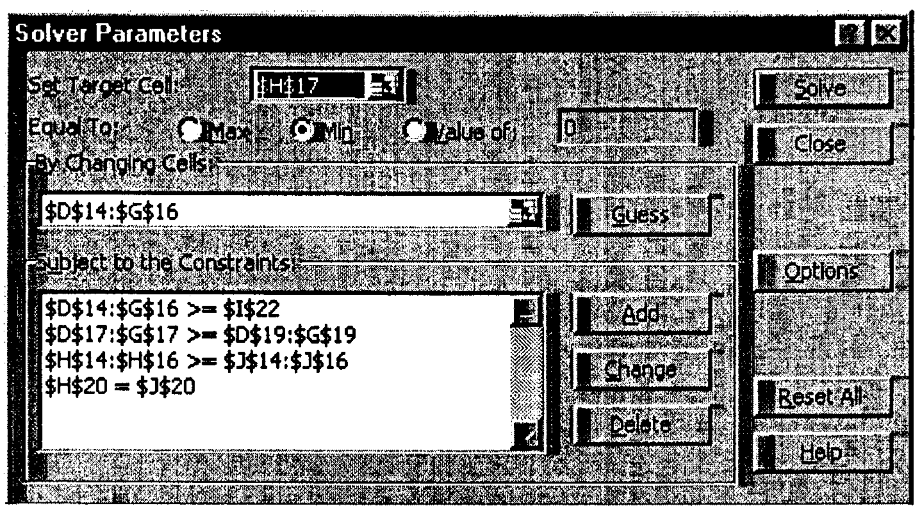


The cost of conducting the survey meeting all constraints imposed by AmeriBank incurs cost of $11,200. The mix of customers is displayed in the spreadsheet above.

b) Sophisticated Surveys will submit a bid of 1.15*$11200 = $12,880.

c) We need to include the new lower-bound constraint on all variables.

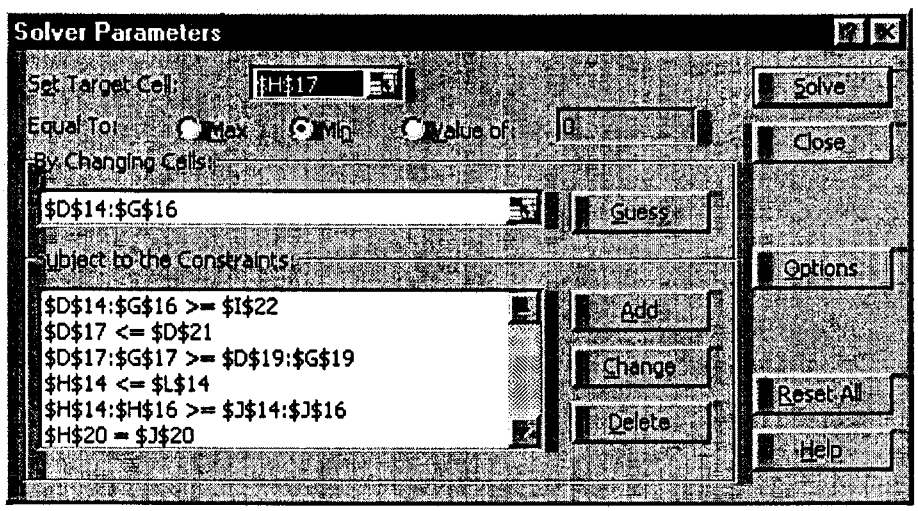
	A	B	C	D	E	F	G	H	I	J	K
1											
2				Cost per Person							
3				Age Group							
4				18 to 25	26 to 40	41 to 50	51 and over				
5			Silicon Valley	$4.75	$6.50	$6.50	$5.00				
6		Region	Big Cities	$5.25	$5.75	$6.25	$6.25				
7			Small Towns	$6.50	$7.50	$7.50	$7.25				
8											
9											
10											
11				Number of People Surveyed							
12				Age Group							
13				18 to 25	26 to 40	41 to 50	51 and over	Totals	Survey restrictions		
14			Silicon Valley	**600**	**50**	**50**	**200**	900	>=	300	
15		Region	Big Cities	**150**	**450**	**50**	**50**	700	>=	700	
16			Small Towns	**100**	**50**	**200**	**50**	400	>=	400	
17		Totals		850	550	300	300	**11387.5**	=	Total Cost	
18				>=	>=	>=	>=	$13,095.62	=	Bid	
19	Survey restrictions			400	550	300	300		Total Surveys		
20								2000	=	2000	
21								Minimum value for each variable			
22			Formula in cell H14:		"=SUM(D14:G14)"				50		
23			Formula in cell H15:		"=SUM(D15:G15)"						
24			Formula in cell H16:		"=SUM(D16:G16)"						
25			Formula in cell D17:		"=SUM(D14:D16)"						
26			Formula in cell E17:		"=SUM(E14:E16)"						
27			Formula in cell F17:		"=SUM(F14:F16)"						
28			Formula in cell G17:		"=SUM(G14:G16)"						
29			Formula in cell H20:		"=SUM(D14:G16)"						
30			Formula in cell H17:		"=SUMPRODUCT(D5:G7,D14:G16)"						
31			Formula in cell H18:		"=1.15*H17"						



The new requirement increases the bid to $13,095.62.

d) We include upper bounds on the total number of people surveyed in Silicon Valley and from the age group of 18 to 25 year-olds.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2				Cost per Person								
3				Age Group								
4				18 to 25	26 to 40	41 to 50	51 and over					
5			Silicon Valley	$4.75	$6.50	$6.50	$5.00					
6		Region	Big Cities	$5.25	$5.75	$6.25	$6.25					
7			Small Towns	$6.50	$7.50	$7.50	$7.25					
8												
9												
10												
11				Number of People Surveyed								
12				Age Group								
13				18 to 25	26 to 40	41 to 50	51 and over	Totals	Survey restrictions			
14			Silicon Valley	100	50	50	450	650	>=	300	<=	650
15		Region	Big Cities	400	450	50	50	950	>=	700		
16			Small Towns	100	50	200	50	400	>=	400		
17		Totals		600	550	300	550	$11,575	=	Total Cost		
18				>=	>=	>=	>=	$13,311.25	=	Bid		
19	Survey restrictions			400	550	300	300		Total Surveys			
20				<=				2000	=	2000		
21				600				Minimum value for each variable				
22									50			
23												
24			Formula in cell H14:		"=SUM(D14:G14)"							
25			Formula in cell H15:		"=SUM(D15:G15)"							
26			Formula in cell H16:		"=SUM(D16:G16)"							
27			Formula in cell D17:		"=SUM(D14:D16)"							
28			Formula in cell E17:		"=SUM(E14:E16)"							
29			Formula in cell F17:		"=SUM(F14:F16)"							
30			Formula in cell G17:		"=SUM(G14:G16)"							
31			Formula in cell H20:		"=SUM(D14:G16)"							
32			Formula in cell H17:		"=SUMPRODUCT(D5:G7,D14:G16)"							
33			Formula in cell H18:		"=1.15*H17"							



The new requirements increase the bid to $13,311.25.

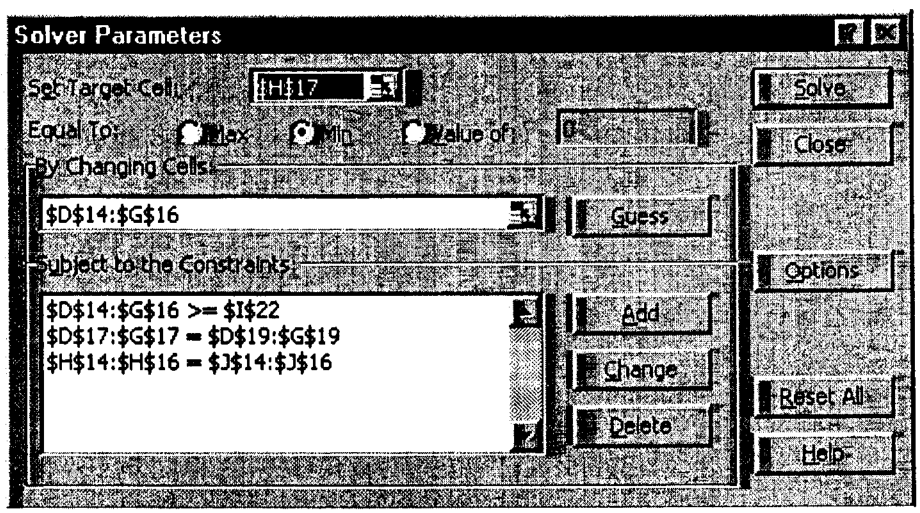
e) The three cost factors for the age group "18 to 25" are changed.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2				Cost per Person								
3				Age Group								
4				18 to 25	26 to 40	41 to 50	51 and over					
5			Silicon Valley	$6.50	$6.50	$6.50	$5.00					
6		Region	Big Cities	$6.75	$5.75	$6.25	$6.25					
7			Small Towns	$7.00	$7.50	$7.50	$7.25					
8												
9												
10												
11				Number of People Surveyed								
12				Age Group								
13				18 to 25	26 to 40	41 to 50	51 and over	Totals	Survey restrictions			
14			Silicon Valley	**50**	**50**	**50**	**500**	650	>=	300	<=	650
15		Region	Big Cities	**100**	**600**	**200**	**50**	950	>=	700		
16			Small Towns	**250**	**50**	**50**	**50**	400	>=	400		
17		Totals		400	700	300	600	**$ 12,025**	=	Total Cost		
18				>=	>=	>=	>=	$13,828.75	=	Bid		
19	Survey restrictions			400	550	300	300	Total Surveys				
20				<=				2000	=	2000		
21				600				Minimum value for each variable				
22									50			
23												
24			Formula in cell H14:	"=SUM(D14:G14)"								
25			Formula in cell H15:	"=SUM(D15:G15)"								
26			Formula in cell H16:	"=SUM(D16:G16)"								
27			Formula in cell D17:	"=SUM(D14:D16)"								
28			Formula in cell E17:	"=SUM(E14:E16)"								
29			Formula in cell F17:	"=SUM(F14:F16)"								
30			Formula in cell G17:	"=SUM(G14:G16)"								
31			Formula in cell H20:	"=SUM(D14:G16)"								
32			Formula in cell H17:	"=SUMPRODUCT(D5:G7,D14:G16)"								
33			Formula in cell H18:	"=1.15*H17"								

With the new cost factors the bid increases to $13,828.75.

f) We eliminate all lower and upper bounds on the age groups and regions and replace them with Rob's strict requirements. These requirements also ensure that exactly 2000 people are surveyed so that we can drop that constraint too.

	A	B	C	D	E	F	G	H	I	J	K
1											
2				Cost per Person							
3				Age Group							
4				18 to 25	26 to 40	41 to 50	51 and over				
5			Silicon Valley	$6.50	$6.50	$6.50	$5.00				
6		Region	Big Cities	$6.75	$5.75	$6.25	$6.25				
7			Small Towns	$7.00	$7.50	$7.50	$7.25				
8											
9											
10											
11				Number of People Surveyed							
12				Age Group							
13				18 to 25	26 to 40	41 to 50	51 and over	Totals	Survey restrictions		
14			Silicon Valley	**50**	**50**	**50**	**250**	400	=	400	
15		Region	Big Cities	**50**	**600**	**300**	**50**	1000	=	1000	
16			Small Towns	**400**	**50**	**50**	**100**	600	=	600	
17		Totals		500	700	400	400	**$ 12,475**	=	Total Cost	
18				=	=	=	=	$14,346.25	=	Bid	
19		Survey restrictions		500	700	400	400				
20											
21								Minimum value for each variable			
22			Formula in cell H14:		"=SUM(D14:G14)"				50		
23			Formula in cell H15:		"=SUM(D15:G15)"						
24			Formula in cell H16:		"=SUM(D16:G16)"						
25			Formula in cell D17:		"=SUM(D14:D16)"						
26			Formula in cell E17:		"=SUM(E14:E16)"						
27			Formula in cell F17:		"=SUM(F14:F16)"						
28			Formula in cell G17:		"=SUM(G14:G16)"						
29			Formula in cell H20:		"=SUM(D14:G16)"						
30			Formula in cell H17:		"=SUMPRODUCT(D5:G7,D14:G16)"						
31			Formula in cell H18:		"=1.15*H17"						



Rob's strict requirements increase the cost of the survey by $450. The new bid of Sophisticated Surveys is $14,346.25.

4-3

a & b)

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	422.222222	177.777778	600	=	600
Area 3	0	227.777778	322.222222	550	=	550
Area 4	350	0	0	350	=	350
Area 5	366.666667	0	133.333333	500	=	500
Area 6	83.3333333	0	366.666667	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 555,555.56

Grade Constraints:

	School 1	School 2	School 3
6th Graders	269.333333	368.555556	339.111111
7th Graders	288	362.111111	300.888889
8th Graders	242.666667	369.333333	360
30% of Total	240	330	300
36% of Total	288	396	360

c) The recommendation to the school board is to assign students to schools as shown in the above solution section of the spreadsheet. Quantities that are not integers must be rounded since partial students cannot be sent.

d) The following solution decreases total bussing costs by over $135,000 but violates the grade constraints that were imposed. Solutions will vary and those than satisfy the grade constraints will be likely to increase the total bussing costs.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	0	0	550	550	=	550
Area 4	350	0	0	350	=	350
Area 5	500	0	0	500	=	500
Area 6	0	0	450	450	=	450
Total	850	1050	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 420,000.00

Grade Constraints:

	School 1	School 2	School 3
6th Graders	293	366	318
7th Graders	310	339	302
8th Graders	247	345	380
30% of Total	255	315	300
36% of Total	306	378	360

e) The number of students assigned from each area to each school changes to the solution shown below and the total bussing cost is reduced by almost $162,000.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	0
4	350	0.28	0.4	0.32	0	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	0	0	550	550	=	550
Area 4	350	0	0	350	=	350
Area 5	318.181818	0	181.818182	500	=	500
Area 6	131.818182	50	268.181818	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 393,636.36

Grade Constraints:

	School 1	School 2	School 3
6th Graders	266.909091	383	327.090909
7th Graders	285.090909	353	312.909091
8th Graders	248	364	360
30% of Total	240	330	300
36% of Total	288	396	360

f) The number of students assigned from each area to each school changes to the solution shown below and the total bussing cost is reduced by over $215,000.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	0	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	0	0
4	350	0.28	0.4	0.32	0	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	0	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	38.7096771	411.290323	0	450	=	450
Area 2	0	236.559139	363.440861	600	=	600
Area 3	0	77.95699	472.04301	550	=	550
Area 4	350	0	0	350	=	350
Area 5	435.483871	0	64.5161288	500	=	500
Area 6	75.8064517	374.193548	0	450	=	450
Total	900	1100	900			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 340,053.76

Grade Constraints:

	School 1	School 2	School 3
6th Graders	306	369.752688	301.247312
7th Graders	324	352.247312	274.752688
8th Graders	270	378	324
30% of Total	270	330	270
36% of Total	324	396	324

g)

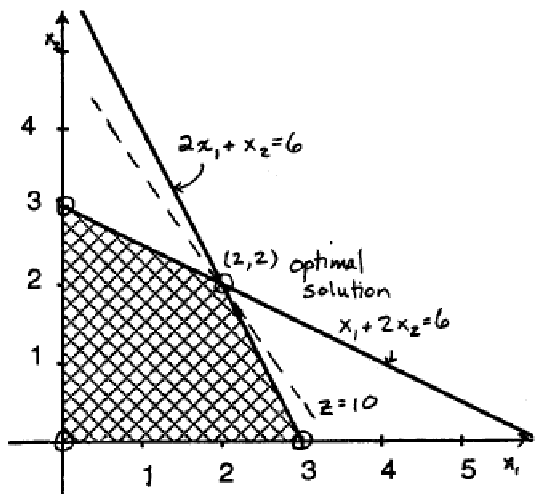
Option	Cost	# students walking 1 to 1.5 miles	# students walking more than 1.5 miles
current	$555,555.56	0	0
1	$393,636.36	900	0
2	$340,053.76	900	491

h) Answers will vary.

CHAPTER 5: THE THEORY OF THE SIMPLEX METHOD

5.1-1.

(a) Optimal Solution: $(x_1^*, x_2^*) = (2, 2)$ and $Z^* = 10$



(c) maximize $Z = 3x_1 + 2x_2$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 6 \\ x_1 + 2x_2 + x_4 &= 6 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(b) - (d)

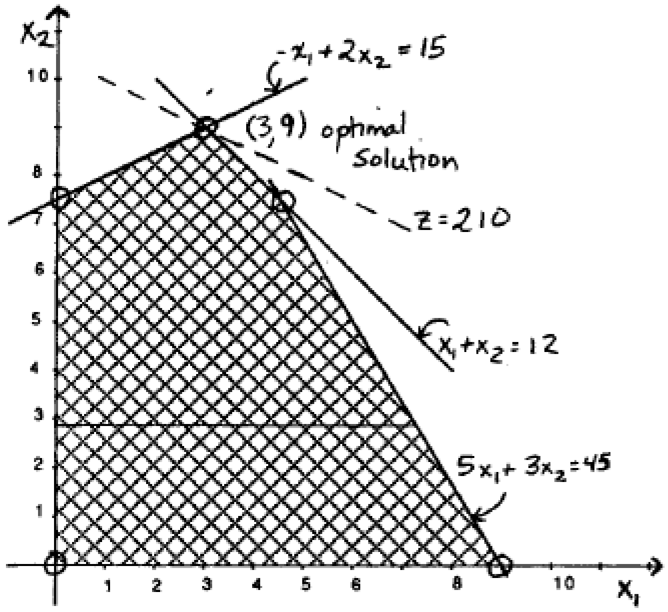
Defining Equations	CP	Feasible?	Basic Solution	Indicating Variables	Equations
$x_1 = 0$ $x_2 = 0$	(0 , 0)	Yes	(0 , 0 , 6 , 6)	x_1 x_2	$x_3 = 6$ $x_4 = 6$
$x_1 = 0$ $2x_1 + x_2 = 6$	(0 , 6)	No	(0 , 6 , 0 , -6)	x_1 x_3	$x_2 = 6$ $2x_2 + x_4 = 6$
$x_1 = 0$ $x_1 + 2x_2 = 6$	(0 , 3)	Yes	(0 , 3 , 3 , 0)	x_1 x_4	$x_2 + x_3 = 6$ $2x_2 = 6$
$x_2 = 0$ $2x_1 + x_2 = 6$	(3 , 0)	Yes	(3 , 0 , 0 , 3)	x_2 x_3	$2x_1 = 6$ $x_1 + x_4 = 6$
$x_2 = 0$ $x_1 + 2x_2 = 6$	(6 , 0)	No	(6 , 0 , -6 , 0)	x_2 x_4	$2x_1 + x_3 = 6$ $x_1 = 6$
$2x_1 + x_2 = 6$ $x_1 + 2x_2 = 6$	(2 , 2)	Yes	(2 , 2 , 0 , 0)	x_3 x_4	$2x_1 + x_2 = 6$ $x_1 + 2x_2 = 6$

(e)

Step	CPF Sol.'n	Deleted Defining Eq.	Added Defining Eq.	Deleted Ind.Var.	Added Ind.Var.
1	$(0, 0)$	$x_1 = 0$	$2x_1 + x_2 = 6$	x_1	x_3
2	$(3, 0)$	$x_2 = 0$	$x_1 + 2x_2 = 6$	x_2	x_4
3	$(2, 2)$ OPTIMAL				

5.1-2.

(a) Optimal Solution: $(x_1^*, x_2^*) = (3, 9)$ and $Z^* = 210$



(c)

$$
\begin{array}{lll}
\text{maximize} & Z = & 10x_1 + 20x_2 \\
\text{subject to} & & -x_1 + 2x_2 + x_3 = 15 \\
& & x_1 + x_2 + x_4 = 12 \\
& & 5x_1 + 3x_2 + x_5 = 45 \\
& & x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

(b) - (d)

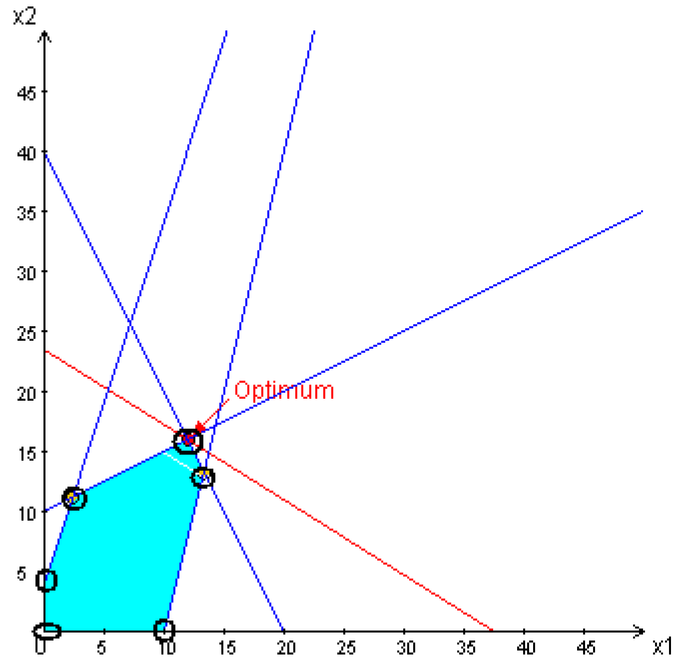
Defining Equations	CP	Feasible?	Basic Solution	Indicating Variables	Equations
$x_1 = 0$ $x_2 = 0$	$(0, 0)$	Yes	$(0,0,15,12,45)$	x_1 x_2	$x_3 = 15$ $x_4 = 12$ $x_5 = 45$
$x_1 = 0$ $-x_1+2x_2 = 15$	$(0, 7.5)$	Yes	$(0,7.5,0,4.5,22.5)$	x_1 x_3	$2x_2 = 15$ $x_2+x_4 = 12$ $3x_2+x_5 = 45$
$x_1 = 0$ $x_1+x_2 = 12$	$(0, 12)$	No	$(0,12,-9,0,9)$	x_1 x_4	$2x_2+x_3 = 15$ $x_2+x_4 = 12$ $3x_2+x_5 = 45$
$x_1 = 0$ $5x_1+3x_2 = 45$	$(0, 15)$	No	$(0,15,-15,-3,0)$	x_1 x_5	$2x_2+x_3 = 15$ $x_2+x_4 = 12$ $3x_2 = 45$
$x_2 = 0$ $-x_1+2x_2 = 15$	$(-15, 0)$	No	$(-15,0,0,3,120)$	x_2 x_3	$x_1+x_4 = 12$ $-x_1 = 15$ $5x_1+x_5 = 45$
$x_2 = 0$ $x_1+x_2 = 12$	$(12, 0)$	No	$(12,0,27,0,-15)$	x_2 x_4	$x_1 = 12$ $-x_1+x_3 = 15$ $5x_1+x_5 = 45$
$x_2 = 0$ $5x_1+3x_2 = 45$	$(9, 0)$	Yes	$(9,0,24,3,0)$	x_2 x_5	$x_1+x_4 = 12$ $-x_1+x_3 = 15$ $5x_1 = 45$
$x_1+x_2 = 12$ $-x_1+2x_2 = 15$	$(3, 9)$	Yes	$(3,9,0,0,3)$	x_3 x_4	$x_1+x_2 = 12$ $-x_1+2x_2 = 15$ $5x_1+3x_2+x_5 = 45$
$5x_1+3x_2 = 45$ $-x_1+2x_2 = 15$	$(45/13, 120/13)$	No	$(45/13,120/13,0,-19/13,0)$	x_3 x_5	$x_1+x_2+x_4 = 12$ $-x_1+2x_2 = 15$ $5x_1+3x_2 = 45$
$x_1+x_2 = 12$ $5x_1+3x_2 = 45$	$(4.5, 7.5)$	Yes	$(4.5,7.5,3.5,0,0)$	x_4 x_5	$x_1+x_2 = 12$ $-x_1+2x_2+x_4 = 15$ $5x_1+3x_2 = 45$

(e)

Step	CPF Sol.'n	Deleted Defining Eq.	Added Defining Eq.	Deleted Ind.Var.	Added Ind.Var.
1	$(0, 0)$	$x_2 = 0$	$-x_1 + 2x_2 = 15$	x_2	x_3
2	$(0, 7.5)$	$x_1 = 0$	$x_1 + x_2 = 12$	x_1	x_4
3	$(3, 9)$ OPTIMAL				

5.1-3.

(a) Optimal Solution: $(x_1^*, x_2^*) = (12, 16)$ and $Z^* = 188$



(b) The corner point $(12, 16)$ has the best objective value 188, so is optimal.

CPF Sol.'n	Defining Equations	BF Solution	NB Var.'s	z
$(0, 0)$	$x_1 = 0, x_2 = 0$	$(0, 0, 80, 4, 20, 40)$	x_1, x_2	0
$(0, 4)$	$x_1 = 0, -3x_1 + x_2 = 4$	$(0, 4, 72, 0, 12, 44)$	x_1, x_4	32
$(2.4, 11.2)$	$-3x_1 + x_2 = 4, -x_1 + 2x_2 = 20$	$(2.4, 11.2, 48, 0, 0, 41.6)$	x_4, x_5	101.6
$(12, 16)$	$-x_1 + 2x_2 = 20, 4x_1 + 2x_2 = 80$	$(12, 16, 0, 24, 0, 8)$	x_3, x_5	188
$(13.3, 13.3)$	$4x_1 + 2x_2 = 80, 4x_1 - x_2 = 40$	$(13.3, 13.3, 0, 16.7, 6.7, 0)$	x_3, x_6	173.3
$(10, 0)$	$4x_1 - x_2 = 40, x_2 = 0$	$(10, 0, 40, 34, 30, 0)$	x_2, x_6	50

(c) All sets yield a solution.

CP Infeas. Sol.'n	Defining Equations	Basic Infeas. Solutions	NB Var.'s
$(-\frac{4}{3}, 0)$	$-3x_1 + x_2 = 4, x_2 = 0$	$(-\frac{4}{3}, 0, 85\frac{1}{3}, 0, 18\frac{2}{3}, 45\frac{1}{3})$	x_2, x_4
$(-20, 0)$	$-x_1 + 2x_2 = 20, x_2 = 0$	$(-20, 0, 160, -56, 0, 120)$	x_2, x_5
$(0, 40)$	$4x_1 + 2x_2 = 80, x_1 = 0$	$(0, 40, 0, -36, -60, 80)$	x_1, x_3
$(0, 10)$	$-x_1 + 2x_2 = 20, x_1 = 0$	$(0, 10, 60, -6, 0, 50)$	x_1, x_5
$(7.2, 25.6)$	$4x_1 + 2x_2 = 80, -3x_1 + x_2 = 4$	$(7.2, 25.6, 0, 0, -24, 36.8)$	x_3, x_4
$(44, 136)$	$-3x_1 + x_2 = 4, 4x_1 - x_2 = 40$	$(44, 136, -368, 0, -208, 0)$	x_4, x_6
$(\frac{100}{7}, \frac{120}{7})$	$4x_1 - x_2 = 40, -x_1 + 2x_2 = 20$	$(\frac{100}{7}, \frac{120}{7}, -\frac{80}{7}, \frac{208}{7}, 0, 0)$	x_5, x_6
$(20, 0)$	$4x_1 + 2x_2 = 80, x_2 = 0$	$(20, 0, 0, 64, 40, -40)$	x_2, x_3
$(0, -40)$	$4x_1 - x_2 = 40, x_1 = 0$	$(0, -40, 160, 44, 100, 0)$	x_1, x_6

5.1-4.

(a) $(x_1, x_2, x_3) = (10, 0, 0)$

(b) $x_2 = 0, x_3 = 0, x_1 - x_2 + 2x_3 = 10$

5.1-5.

(a)

CPF Sol.'n	Defining Equations
$(0,0,0)$	$x_1=0, x_2=0, x_3=0$
$(4,0,0)$	$x_1=4, x_2=0, x_3=0$
$(4,2,0)$	$x_1=4, x_1+x_2=6, x_3=0$
$(2,4,0)$	$x_2=4, x_1+x_2=6, x_3=0$
$(0,4,0)$	$x_1=0, x_2=4, x_3=0$
$(0,4,2)$	$x_1=0, x_2=4, -x_1+2x_3=4$
$(2,4,3)$	$x_1+x_2=6, x_2=4, -x_1+2x_3=4$
$(4,2,4)$	$x_1+x_2=6, x_1=4, -x_1+2x_3=4$
$(4,0,4)$	$x_2=0, x_1=4, -x_1+2x_3=4$
$(0,0,2)$	$x_2=0, x_1=0, -x_1+2x_3=4$

(b) $x_1+x_2=6, x_2=4, -x_1+2x_3=4$

(c) $x_1=4, x_1=0, x_2=0 \Rightarrow$ inconsistent system

5.1-6.

(a) - (b)

Defining Equations	CP	Feas.?	Basic Solution	NB Var.'s
$x_1=0, x_2=0$	$(0,0)$	No	$(0,0,30,-50,-30)$	x_1, x_2
$x_1=0, -3x_1+2x_2=30$	$(0,15)$	No	$(0,15,0,-35,-15)$	x_1, x_3
$x_1=0, 2x_1+x_2=50$	$(0,50)$	No	$(0,50,-70,0,20)$	x_1, x_4
$x_1=0, x_1+x_2=30$	$(0,30)$	No	$(0,30,-30,-20,0)$	x_1, x_5
$x_2=0, -3x_1+2x_2=30$	$(-10,0)$	No	$(-10,0,0,-70,-40)$	x_2, x_3
$x_2=0, 2x_1+x_2=50$	$(25,0)$	No	$(25,0,105,0,-5)$	x_2, x_4
$x_2=0, x_1+x_2=30$	$(30,0)$	Yes	$(30,0,120,10,0)$	x_2, x_5
$-3x_1+2x_2=30, 2x_1+x_2=50$	$(10,30)$	Yes	$(10,30,0,0,10)$	x_3, x_4
$-3x_1+2x_2=30, x_1+x_2=30$	$(6,24)$	No	$(6,24,0,-14,0)$	x_3, x_5
$2x_1+x_2=50, x_1+x_2=30$	$(20,10)$	Yes	$(20,10,70,0,0)$	x_4, x_5

5.1-7.

(a) - (b)

Defining Equations	CP	Feas.?	Basic Solution	NB Var.'s
$x_1=0, x_2=0$	$(0,0)$	Yes	$(0,0,10,60,18,44)$	x_1, x_2
$x_1=0, x_2=10$	$(0,10)$	Yes	$(0,10,0,10,8,34)$	x_1, x_3
$x_1=0, 2x_1+5x_2=60$	$(0,12)$	No	$(0,12,-2,0,6,32)$	x_1, x_4
$x_1=0, x_1+x_2=18$	$(0,18)$	No	$(0,18,-8,-30,0,26)$	x_1, x_5
$x_1=0, 3x_1+x_2=44$	$(0,44)$	No	$(0,44,-34,-160,-26,0)$	x_1, x_6
$x_2=0, x_2=10$	No Solution			x_2, x_3
$x_2=0, 2x_1+5x_2=60$	$(30,0)$	No	$(30,0,10,0,-12,-46)$	x_2, x_4
$x_2=0, x_1+x_2=18$	$(18,0)$	No	$(18,0,10,24,0,-10)$	x_2, x_5
$x_2=0, 3x_1+x_2=44$	$(14.67,0)$	Yes	$(14.67,0,10,30.67,3.33,0)$	x_2, x_6
$x_2=10, 2x_1+5x_2=60$	$(5,10)$	Yes	$(5,10,0,0,3,19)$	x_3, x_4
$x_2=10, x_1+x_2=18$	$(8,10)$	No	$(8,10,0,-6,0,10)$	x_3, x_5
$x_2=10, 3x_1+x_2=44$	$(11.33,10)$	No	$(11.33,10,0,-12.67,-3.33,0)$	x_3, x_6
$2x_1+5x_2=60, x_1+x_2=18$	$(10,8)$	Yes	$(10,8,2,0,0,6)$	x_4, x_5
$2x_1+5x_2=60, 3x_1+x_2=44$	$(12.31,7.08)$	No	$(12.31,7.08,2.92,0,-1.38,0)$	x_4, x_6
$x_1+x_2=18, 3x_1+x_2=44$	$(13,5)$	Yes	$(13,5,5,9,0,0)$	x_5, x_6

5.1-8.

(a) If the feasible region is unbounded, then there may be no optimal solution.

(b) There may be multiple optimal solutions, in which case the weighted average of any optimal CPF solutions is optimal, too.

(c) An adjacent CPF solution may have an equal objective function value, then all the points that lie on the line segment between these two corner points are optimal.

5.1-9.

(a) FALSE. (p.5-10) Property 1: (a) If there is exactly one optimal solution, then it must be a CPF solution. (b) If there are multiple optimal solutions, then at least two of them must be adjacent CPF solutions. An optimal solution that is not a CPF solution can be obtained by taking a convex combination of two optimal CPF solutions.

(b) FALSE. (p.5-12) The number of CPF solutions is at most $\begin{pmatrix} m+n \\ n \end{pmatrix} = \frac{(m+n)!}{m!n!}$.

(c) FALSE. (p.5-13) The adjacent CPF solution that has a better objective function value than the initial CPF solution may be adjacent to another CPF solution that has an even better objective function value.

5.1-10.

(a) TRUE. By Property 1(a), there must be multiple solutions, since this optimal solution is not a CPF solution. But then, there must be infinitely many optimal solutions, namely any convex combination of optimal solutions.

(b) TRUE. Any point x on the line segment connecting x^* and x^{**} can be expressed as $x = \alpha x^* + (1-\alpha)x^{**}$ with $\alpha \in [0,1]$. Both x^* and x^{**} have the optimal objective value Z^*. The objective function value at x is

$$Z = c^T(\alpha x^* + (1-\alpha)x^{**}) = \alpha Z^* + (1-\alpha)Z^* = Z^*,$$

so x is optimal. Since the feasible region is convex, any such point is feasible.

(c) FALSE. The simultaneous solution of any set of n constraint boundary equations may be infeasible or may not even exist.

5.1-11.

(a) TRUE. If there are no optimal solutions, then either the problem is infeasible or the objective value is unbounded (Chapter 3). The former is not the case by assumption of the problem. Also by assumption again, the feasible region is bounded, so the objective value is bounded, so the latter cannot be the case. Hence, there must be at least one optimal solution.

(b) FALSE. If a solution is optimal, it need not be a BF solution. A convex combination of two optimal BF solutions is optimal even though it is not a BF solution. This follows from Property 1, since BF solutions are CPF solutions.

(c) TRUE. Since BF solutions correspond to CPF solutions, this follows directly from Property 2.

5.1-12.

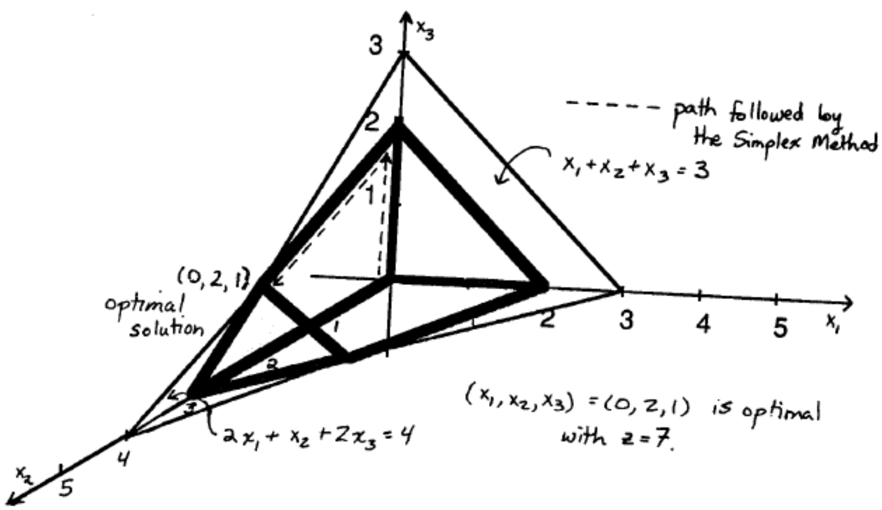
$x_1 = 0, 2x_1 + x_2 + 3x_3 = 60, 3x_1 + 3x_2 + 5x_3 = 120 \Rightarrow (x_1, x_2, x_3) = (0, 15, 15)$

5.1-13.

Since $x_2 > 0$ and $x_3 > 0$, $x_2 = 0$ and $x_3 = 0$ cannot be part of the three boundary equations, so the boundary equations are $x_1 = 0, 2x_1 + x_2 + x_3 = 20, 3x_1 + x_2 + 2x_3 = 30$. Then, the optimal solutions is $(x_1, x_2, x_3) = (0, 10, 10)$.

5.1-14.

(a)



(b) The simplex method follows this path because moving along the chosen edges provides the greatest increase in the objective value for a unit move in the chosen direction among all possible edges at each vertex/decision point.

(c)

Edge	Constraint Boundary Equations	End Points	Additional Constraints
1	$x_2 = 0, x_1 = 0$	$(0,0,0), (0,0,2)$	$x_3 = 0, 2x_1 + x_2 + 2x_3 = 4$
2	$2x_1 + x_2 + 2x_3 = 4, x_1 = 0$	$(0,0,2), (0,2,1)$	$x_2 = 0, x_1 + x_2 + x_3 = 3$

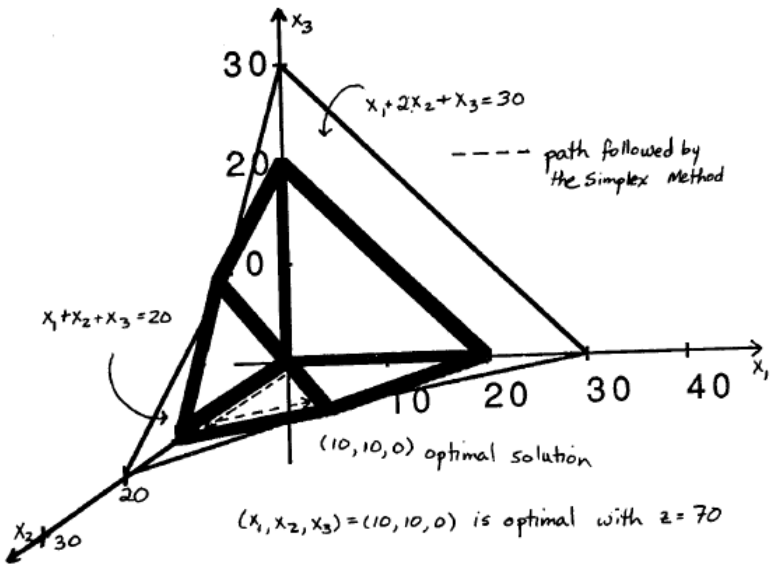
(d) - (e)

CP	Defining Equations	BF Solution	NB Var.'s
$(0,0,0)$	$x_1 = 0, x_2 = 0, x_3 = 0$	$(0,0,0,4,3)$	x_1, x_2, x_3
$(0,0,2)$	$x_1 = 0, x_2 = 0, 2x_1 + x_2 + 2x_3 = 4$	$(0,0,2,0,1)$	x_1, x_2, x_4
$(0,2,1)$	$x_1 = 0, 2x_1 + x_2 + 2x_3 = 4, x_1 + x_2 + x_3 = 3$	$(0,0,2,0,1)$	x_1, x_4, x_5

The nonbasic variables having value zero are equivalent to indicating variables. They indicate that their associated inequality constraints are actually equalities. The associated equalities are the defining equations.

5.1-15.

(a)



(b) The simplex method follows this path because moving along the chosen edges provides the greatest increase in the objective value for a unit move in the chosen direction among all possible edges at each vertex/decision point.

(c)

Edge	Constraint Boundary Equations	End Points	Additional Constraints
1	$x_1=0, x_3=0$	$(0,0,0),(0,15,0)$	$x_2=0, x_1+2x_2+x_3=30$
2	$x_3=0, x_1+2x_2+x_3=30$	$(0,15,0),(10,10,0)$	$x_1=0, x_1+x_2+x_3=20$

(d) - (e)

CP	Defining Equations	BF Solution	NB Var.'s
$(0,0,0)$	$x_1=0, x_2=0, x_3=0$	$(0,0,0,20,30)$	x_1,x_2,x_3
$(0,15,0)$	$x_1=0, x_3=0, x_1+2x_2+x_3=30$	$(0,15,0,5,0)$	x_1,x_3,x_5
$(10,10,0)$	$x_3=0, x_1+2x_2+x_3=30, x_1+x_2+x_3=20$	$(10,10,0,0,0)$	x_3,x_4,x_5

The nonbasic variables having value zero are equivalent to indicating variables. They indicate that their associated inequality constraints are actually equalities. The associated equalities are the defining equations.

5.1-16.

(a) When the objective is to maximize $Z=x_3$, both corner points $(4,2,4)$ and $(4,0,4)$ are optimal, with $Z^*=4$.

(b) When the objective is to maximize $Z=-x_1+2x_3$, all the corner points $(0,0,2)$, $(4,0,4)$, $(4,2,4)$, $(2,4,3)$ and $(0,4,2)$ are optimal, with $Z^*=4$.

5.1-17.

(a) Geometrically, each constraint is a plane and the points that are feasible for a given (inequality) constraint form a half-space. The line segment defined by any two feasible points must lie entirely on the feasible side of the plane and therefore, all the points on the line segment are feasible, implying that the set of solutions for any one constraint is a convex set.

(b) Because the points in the feasible region of the LP problem satisfy all the constraints simultaneously, it must be the case that for any two feasible points, the points on the line segment joining them must also satisfy each constraint (from (a)). Hence, the set of solutions that satisfy all the constraints simultaneously is a convex set.

5.1-18.

To maximize $Z = 3x_1 + 4x_2 + 3x_3$, starting at the origin $(0,0,0)$, one first chooses to move to $(0,4,0)$ because this edge offers the best rate of improvement among all edges at the origin. From $(0,4,0)$, the edge that increases the objective function fastest is the one that connects to either $(0,4,2)$ or $(2,4,0)$. From either one these, the edge that gives the best rate of increase connects to $(2,4,3)$. Then, the only edge that provides an improvement in Z connects to the optimal solution $(4,2,4)$.

5.1-19.

(a)

Original Constraint	Boundary Equation	Indicating Variable
$x_1 \geq 0$	$x_1 = 0$	x_1
$x_2 \geq 0$	$x_2 = 0$	x_2
$x_3 \geq 0$	$x_3 = 0$	x_3
$x_1 + x_4 = 4$	$x_1 = 4$	x_4
$x_2 + x_5 = 4$	$x_2 = 4$	x_5
$x_1 + x_2 + x_6 = 6$	$x_1 + x_2 = 6$	x_6
$-x_1 + 2x_3 + x_7 = 4$	$-x_1 + 2x_3 = 4$	x_7

(b)

CPF Sol.'n	Defining Equations	BF Solution	NB Var.'s
$(2,4,3)$	$x_1 + x_2 = 6, x_2 = 4, -x_1 + 2x_3 = 4$	$(2,4,3,2,0,0,0)$	x_5, x_6, x_7
$(4,2,4)$	$x_1 + x_2 = 6, -x_1 + 2x_3 = 4, x_1 = 4$	$(4,2,4,0,2,0,0)$	x_4, x_6, x_7
$(0,4,2)$	$x_1 = 0, x_2 = 4, -x_1 + 2x_3 = 4$	$(0,4,2,4,0,2,0)$	x_1, x_5, x_7
$(2,4,0)$	$x_3 = 0, x_1 + x_2 = 6, x_2 = 4$	$(2,4,0,2,0,0,6)$	x_3, x_5, x_6

(c) Because the sets of defining equations of $(4,2,4)$, $(0,4,2)$ and $(2,4,0)$ differ from the set of defining equations of $(2,4,3)$ by only one equation, they are adjacent to $(2,4,3)$. On the other hand, the sets of defining equations of $(4,2,4)$, $(0,4,2)$ and $(2,4,0)$ differ by more than one equation, they are not adjacent to each other. The same statement is true if we substitute "nonbasic variables" for "defining equations" and "variable" for "equation."

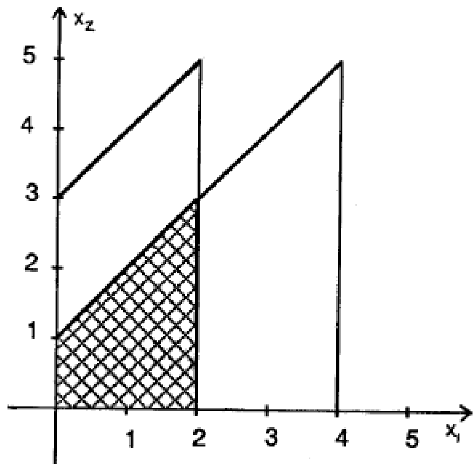
5.1-20.

(a) x_5 enters.

(b) x_4 leaves.

(c) $(4, 2, 4, 0, 2, 0, 0)$

5.1-21.



5.2-1.

(a) Optimal Solution: $\begin{pmatrix} x_3 \\ x_1 \\ x_5 \end{pmatrix} = B^{-1}b = \frac{1}{27}\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix}\begin{pmatrix} 180 \\ 270 \\ 180 \end{pmatrix} = \begin{pmatrix} 50 \\ 30 \\ 50 \end{pmatrix}$

$$Z = cx = (\,8 \quad 4 \quad 6 \quad 3 \quad 9\,)\begin{pmatrix} 30 \\ 0 \\ 50 \\ 0 \\ 50 \end{pmatrix} = 990$$

(b) Shadow prices: $c_B B^{-1} = \frac{1}{27}(\,6 \quad 8 \quad 9\,)\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix} = \begin{pmatrix} 1.33 \\ 1 \\ 2.67 \end{pmatrix}$

5.2-2.

$c = (\,5 \quad 8 \quad 7 \quad 4 \quad 6 \quad 0 \quad 0\,)$, $A = \begin{pmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $x_B = \begin{pmatrix} x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$

$c_B = (\,0 \quad 0\,)$, $-c = (\,-5 \quad -8 \quad -7 \quad -4 \quad -6 \quad 0 \quad 0\,)$, so x_2 enters.

Revised x_2 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, so x_7 leaves.

Iteration 1: $B_{\text{new}}^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3/5 \\ 0 & 1/5 \end{pmatrix}$,

$$x_B = \begin{pmatrix} x_6 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -3/5 \\ 0 & 1/5 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix},\ c_B = \begin{pmatrix} 0 & 8 \end{pmatrix}$$

Revised row 0: $\begin{pmatrix} 0 & 8/5 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{pmatrix}$-$\begin{pmatrix} 5 & 8 & 7 & 4 & 6 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} -1/5 & 0 & -3/5 & -4/5 & -2/5 & 0 & 8/5 \end{pmatrix}$, so x_4 enters.

Revised x_4 coefficients: $\begin{pmatrix} 1 & -3/5 \\ 0 & 1/5 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix}$, so x_6 leaves.

Iteration 2: $B_{\text{new}}^{-1} = \begin{pmatrix} 2 & 3 \\ 2 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5/4 & -3/4 \\ -1/2 & 1/2 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5/4 & -3/4 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 5 \end{pmatrix},\ c_B = \begin{pmatrix} 4 & 8 \end{pmatrix}$$

Revised row 0: $\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{pmatrix}$-$\begin{pmatrix} 5 & 8 & 7 & 4 & 6 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$, so the current solution is optimal.

Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (0, 5, 0, 5/2, 0)$ and $Z^* = 50$

5.2-3.

$c = \begin{pmatrix} 3 & 2 & 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$

CP $(0,0)$: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$

Row 0: $\begin{pmatrix} -3 & -2 & 0 & 0 \end{pmatrix}$

CP $(3,0)$: $B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix},\ c_B = \begin{pmatrix} 3 & 0 \end{pmatrix}$$

Row 0: $\begin{pmatrix} 3/2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/2 & 3/2 & 0 \end{pmatrix}$

CP $(2,2)$: $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix},\ c_B = \begin{pmatrix} 3 & 2 \end{pmatrix}$$

Row 0: $\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/3 & 1/3 \end{pmatrix}$

Optimal Solution: $(x_1^*, x_2^*) = (2, 2)$ and $Z^* = 10$

5.2-4.

$c = (\,1 \quad 2 \quad 0 \quad 0\,)$, $A = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

$c_B = (\,0 \quad 0\,)$, Row 0: $(\,-1 \quad -2 \quad 0 \quad 0\,)$, so x_2 enters the basis.

Revised x_2 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, so x_3 leaves the basis.

Iteration 1: $B_{\text{new}}^{-1} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix}$

$x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix}$, $c_B = (\,2 \quad 0\,)$

Revised row 0: $(\,2/3 \quad 0\,)\begin{pmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} - (\,1 \quad 2 \quad 0 \quad 0\,)$

$= (\,-1/3 \quad 0 \quad 2/3 \quad 0\,)$, so x_1 enters the basis.

Revised x_1 coefficients: $\begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$, so x_4 leaves.

Iteration 2: $B_{\text{new}}^{-1} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$

$x_B = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $c_B = (\,2 \quad 1\,)$

Revised row 0: $(\,1/2 \quad 1/2\,)\begin{pmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} - (\,1 \quad 2 \quad 0 \quad 0\,)$

$= (\,0 \quad 0 \quad 1/2 \quad 1/2\,)$, so the current solution is optimal.

Optimal Solution: $(x_1^*, x_2^*) = (2, 2)$ and $Z^* = 6$

5.2-5.

$c = (\,5 \quad 4 \quad -1 \quad 3 \quad 0 \quad 0\,)$, $A = \begin{pmatrix} 3 & 2 & -3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 3 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 24 \\ 36 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $x_B = \begin{pmatrix} x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 24 \\ 36 \end{pmatrix}$

$c_B = (\,0 \quad 0\,)$, Row 0: $(\,-5 \quad -4 \quad 1 \quad -3 \quad 0 \quad 0\,)$, so x_1 enters the basis.

Revised x_1 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, so x_5 leaves the basis.

Iteration 1: $B_{\text{new}}^{-1} = \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_1 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}, \; c_B = (\, 5 \quad 0 \,)$$

Revised row 0: $(\, 5/3 \quad 0 \,) \begin{pmatrix} 3 & 2 & -3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 3 & 0 & 1 \end{pmatrix} - (\, 5 \quad 4 \quad -1 \quad 3 \quad 0 \quad 0 \,)$

$= (\, 0 \quad -2/3 \quad -4 \quad -4/3 \quad 5/3 \quad 0 \,)$, so x_3 enters the basis.

Revised x_3 coefficients: $\begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, so x_6 leaves.

Iteration 2: $B_{\text{new}}^{-1} = \begin{pmatrix} 3 & -3 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/12 & 1/4 \\ -1/4 & 1/4 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/12 & 1/4 \\ -1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix}, \; c_B = (\, 5 \quad -1 \,)$$

Revised row 0: $(\, 2/3 \quad 1 \,) \begin{pmatrix} 3 & 2 & -3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 3 & 0 & 1 \end{pmatrix} - (\, 5 \quad 4 \quad -1 \quad 3 \quad 0 \quad 0 \,)$

$= (\, 0 \quad 1/3 \quad 0 \quad 2/3 \quad 2/3 \quad 1 \,)$, so current solution is optimal.

Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (11, 0, 3, 0)$ and $Z^* = 52$

5.3-1.

(a) $B^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$

Final constraint columns for (x_1, x_2, x_3):

$$B^{-1}A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

$c_B = (\, -1 \quad 0 \quad 2 \,)$

Final objective coefficients for (x_1, x_2, x_3):

$$c_B B^{-1} A - c = (\, -1 \quad 0 \quad 2 \,) \begin{pmatrix} 5 & 1 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 1 \end{pmatrix} - (\, 1 \quad -1 \quad 2 \,) = (\, 2 \quad 0 \quad 0 \,)$$

Right-hand side:

$$B^{-1}b = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ 5 \\ 11 \end{pmatrix} \text{ and } z = (\, -1 \quad 0 \quad 2 \,) \begin{pmatrix} 14 \\ 5 \\ 11 \end{pmatrix} = 8$$

Final tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	2	0	0	1	1	0	8
X2	1	0	5	1	0	1	3	0	14
X6	2	0	2	0	0	0	1	1	5
X3	3	0	4	0	1	1	2	0	11

(b) Defining equations: $2x_1 - 2x_2 + 3x_3 = 5, x_1 + x_2 - x_3 = 3, x_1 = 0$

5.3-2.

(a) $B^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

Final constraint columns for (x_1, x_2, x_3, x_4):

$$B^{-1}A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 3 & 1 \end{pmatrix}$$

$c_B = (3 \quad 2)$

Final objective coefficients for (x_1, x_2, x_3, x_4):

$$c_B B^{-1}A - c = (3 \quad 2)\begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 3 & 1 \end{pmatrix} - (4 \quad 3 \quad 1 \quad 2) = (3 \quad 0 \quad 2 \quad 0)$$

Right-hand side:

$$B^{-1}b = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } Z = (3 \quad 2)\begin{pmatrix} 1 \\ 3 \end{pmatrix} = 9$$

Final tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	3	0	2	0	1	1	9
X2	1	0	1	1	-1	0	1	-1	1
X4	2	0	2	0	3	1	-1	2	3

(b) Defining equations: $4x_1 + 2x_2 + x_3 + x_4 = 5, 3x_1 + x_2 + 2x_3 + x_4 = 4, x_1 = 0,$ $x_3 = 0$

5.3-3.

$$B^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix}$$

Final constraint columns for (x_1, x_2, x_3):

$$B^{-1}A = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1/2 \\ -4 & -2 & -3/2 \\ 1 & 2 & 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 4 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$c_B = (\,0 \quad 2 \quad 6\,)$

Final objective coefficients for (x_1, x_2, x_3):

$$c_B B^{-1}A - c = (\,0 \quad 2 \quad 6\,) \begin{pmatrix} 0 & 4 & 0 \\ 0 & 4 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (\,6 \quad 1 \quad 2\,) = (\,0 \quad 7 \quad 0\,)$$

Right-hand side:

$$B^{-1}b = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix} \text{ and } Z = (\,0 \quad 2 \quad 6\,) \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix} = 6$$

Final tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	7	0	2	0	2	6
X5	1	0	0	4	0	1	1	2	7
X3	2	0	0	4	1	-2	0	4	0
X1	3	0	1	0	0	1	0	-1	1

5.3-4.

(a) $B^{-1} = \begin{pmatrix} 3/16 & -1/8 & 0 & 0 \\ -1/4 & 1/2 & 0 & 0 \\ -3/8 & 1/4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Current constraint columns for (x_1, x_2, x_3):

$$B^{-1}A = \begin{pmatrix} 3/16 & -1/8 & 0 & 0 \\ -1/4 & 1/2 & 0 & 0 \\ -3/8 & 1/4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8 & 2 & 3 \\ 4 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 9/16 \\ 0 & 1 & -3/4 \\ 0 & 0 & -1/8 \\ 0 & 0 & 1 \end{pmatrix}$$

$c_B = (\,20 \quad 6 \quad 0 \quad 0\,)$

Current objective coefficients for (x_1, x_2, x_3):

$$c_B B^{-1} A - c = (\,20 \quad 6 \quad 0 \quad 0\,) \begin{pmatrix} 1 & 0 & 9/16 \\ 0 & 1 & -3/4 \\ 0 & 0 & -1/8 \\ 0 & 0 & 1 \end{pmatrix} - (\,20 \quad 6 \quad 8\,) = (\,0 \quad 0 \quad -5/4\,)$$

Right-hand side:

$$B^{-1}b = \begin{pmatrix} 3/16 & -1/8 & 0 & 0 \\ -1/4 & 1/2 & 0 & 0 \\ -3/8 & 1/4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 200 \\ 100 \\ 50 \\ 20 \end{pmatrix} = \begin{pmatrix} 25 \\ 0 \\ 0 \\ 20 \end{pmatrix} \text{ and } Z = (\,20 \quad 6 \quad 0 \quad 0\,) \begin{pmatrix} 25 \\ 0 \\ 0 \\ 20 \end{pmatrix} = 500$$

Current tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	-1.25	2.25	0.5	0	0	500
X1	1	0	1	0	0.563	0.188	-0.13	0	0	25
X2	2	0	0	1	-0.75	-0.25	0.5	0	0	0
X6	3	0	0	0	-0.13	-0.38	0.25	1	0	0
X7	4	0	0	0	1	0	0	0	1	20

(b) The revised simplex method would generate the reduced costs for row 0 and then the revised column for x_3.

(c) Defining equations: $8x_1 + 2x_2 + 3x_3 = 200, 4x_1 + 3x_2 = 100$, $x_3 = 0$

Note that $2x_1 + x_3 = 50$ is also binding at the current solution.

5.3-5.

(a)

$$(\,-c_1 \quad -c_2 \quad -c_3 \;\vdots\; 0 \quad 0 \;\vdots\; 0\,) + (\,\tfrac{3}{5} \quad \tfrac{4}{5}\,) \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & \vdots & b \\ 2 & 1 & 3 & 0 & 1 & \vdots & 2b \end{pmatrix} = (\,\tfrac{7}{16} \quad 0 \quad 0 \quad \tfrac{3}{5} \quad \tfrac{4}{5} \quad z^*\,)$$

$-c_1 + \frac{11}{5} = \frac{7}{16} \Rightarrow c_1 = \frac{3}{2}, -c_2 + 2 = 0 \Rightarrow c_2 = 2, -c_3 + 3 = 0 \Rightarrow c_3 = 3$

(b) $B^{-1} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}, B^{-1}b = b^* \Leftrightarrow \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} \begin{pmatrix} b \\ 2b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Leftrightarrow b = 5$

(c) Using (a): $Z^* = c_B b^* = (\,c_2 \quad c_3\,) \begin{pmatrix} 1 \\ 3 \end{pmatrix} = (\,2 \quad 3\,) \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 11$

Using (b): $Z^* = \overline{c}_B b = (\,3/5 \quad 4/5\,) \begin{pmatrix} b \\ 2b \end{pmatrix} = (\,3/5 \quad 4/5\,) \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 11$

5.3-6.

Iteration 1: Multiply row 2 by $5/2$ and add to row 0, i.e., premultiply A_0 by $(\,0 \quad 5/2 \quad 0\,)$ and add to row 0, where

$$A_0 = \begin{pmatrix} 1 & 0 & \vdots & 1 & 0 & 0 & \vdots & 4 \\ 0 & 2 & \vdots & 0 & 1 & 0 & \vdots & 12 \\ 3 & 2 & \vdots & 0 & 0 & 1 & \vdots & 18 \end{pmatrix}.$$

Iteration 2: Add row 3 to row 0, i.e., premultiply A_1 by $(\,0 \quad 0 \quad 1\,)$ and add to row 0, where

$$A_1 = \begin{pmatrix} 1 & 0 & \vdots & 1 & 0 & 0 & \vdots & 4 \\ 0 & 1 & \vdots & 0 & 1/2 & 0 & \vdots & 6 \\ 3 & 0 & \vdots & 0 & -1 & 1 & \vdots & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1 & 1 \end{pmatrix} A_0.$$

Therefore, the final row 0 is: initial row $0 + (\,0 \quad 5/2 \quad 0\,)A_0 + (\,0 \quad 0 \quad 1\,)A_1$,

$$= (\,-3 \quad -5 \;\; \vdots \;\; 0 \quad 0 \quad 0 \;\; \vdots \;\; 0\,) + \left[\begin{pmatrix} 0 & \frac{5}{2} & 0 \end{pmatrix} + (\,0 \quad 0 \quad 1\,)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1 & 1 \end{pmatrix}\right] A_0$$

$$= (\,-3 \quad -5 \;\; \vdots \;\; 0 \quad 0 \quad 0 \;\; \vdots \;\; 0\,) + \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix} A_0$$

5.3-7.

(a) Use the columns corresponding to artificial variables in exactly the same way as a slack variable would have been used. Note that the shadow price of this column may be positive or negative.

(b) For the reversed inequalities, use the negative of the column corresponding to the slack variable in exactly the same formulae. The artificial column may be discarded.

(c) Same as (b).

(d) No change, use slack and artificial variables as above.

5.3-8.

$$\begin{array}{lll} \text{maximize} & Z = & 5x_1 + 4x_2 \qquad\qquad - Mx_5 \\ \text{subject to} & & 3x_1 + 2x_2 + x_3 \qquad\qquad\quad = 6 \\ & & 2x_1 - \ x_2 \qquad - x_4 + \quad x_5 = 6 \\ & & \ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

Initial Tableau:

			Coefficient of					
BV	Eq	Z	x_1	x_2	x_3	x_4	x_5	RS
Z	0	1	$-5-2M$	$-4+M$	0	M	0	$-6M$
x_3	1	0	3	2	1	0	0	6
x_5	2	0	2	-1	0	-1	1	6

The columns that will contain S^* for applying the fundamental insight in the final tableau are those associated with x_3 and x_5, since those columns form the 2×2 identity matrix in the initial tableau.

Final Tableau:

			Coefficient of					
BV	Eq	Z	x_1	x_2	x_3	x_4	x_5	RS
Z	0	1	0	$-\frac{2}{3}+\frac{7}{3}M$	$\frac{5}{3}+\frac{2}{3}M$	M	0	$10-2M$
x_1	1	0	1	$\frac{2}{3}$	$\frac{1}{3}$	0	0	2
x_5	2	0	0	$-\frac{7}{3}$	$-\frac{2}{3}$	-1	1	2

5.3-9.

(a) $B^{-1} = \begin{pmatrix} 3/10 & -1/10 \\ -2/10 & 2/5 \end{pmatrix}$

Final constraint columns for $(x_1, x_2, x_3, x_4, x_6)$:

$$B^{-1}A = \begin{pmatrix} 3/10 & -1/10 \\ -2/10 & 2/5 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 & -1 & 0 \\ 3 & 2 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3/5 & -3/10 & 1/10 \\ 1 & 0 & -2/5 & 2/10 & -2/5 \end{pmatrix}$$

$$c_B = (\,-6M+3 \quad -4M+2\,)$$

Final objective coefficients for $(x_1, x_2, x_3, x_4, x_6)$:

$$-c_B B^{-1}A + c = -\,(\,-6M+3 \quad -4M+2\,)\begin{pmatrix} 0 & 1 & 3/5 & -3/10 & 1/10 \\ 1 & 0 & -2/5 & 2/10 & -2/5 \end{pmatrix}$$

$$+\,(\,-4M{+}2 \quad -6M{+}3 \quad -2M{+}2 \quad M \quad M\,) = (\,0 \quad 0 \quad 1 \quad 1/2 \quad 1/2\,)$$

Right-hand side:

$$B^{-1}b = \begin{pmatrix} 3/10 & -1/10 \\ -2/10 & 2/5 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 9/5 \\ 4/5 \end{pmatrix}$$

$$z = -14M + c_B x_B = -14M + (\,-6M+3 \quad -4M+2\,)\begin{pmatrix} 9/5 \\ 4/5 \end{pmatrix} = 7$$

Final tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X6	$\bar{X}5$	$\bar{X}7$	Right side
								1M	1M	
Z	0	-1	0	0	1	0.5	0.5	-0.5	-0.5	-7
X2	1	0	0	1	0.6	-0.3	0.1	0.3	-0.1	1.8
X1	2	0	1	0	-0.4	0.2	-0.4	-0.2	0.4	0.8

(b) The constraints in the original tableau can be expressed as $(\,A \;\vdots\; I \;\vdots\; I \;\vdots\; b\,)$ with the second identity matrix corresponding to the artificial variables. Premultiply this matrix by M to get:

$$(\,A^* \;\vdots\; S^* \;\vdots\; L^* \;\vdots\; b^*\,) = \mathrm{M}(\,A \;\vdots\; I \;\vdots\; I \;\vdots\; b\,) = (\,\mathrm{M}A \;\vdots\; \mathrm{M} \;\vdots\; \mathrm{M} \;\vdots\; \mathrm{M}b\,),$$

where $\mathrm{M} = S^* = L^* = \begin{pmatrix} 3/10 & -1/10 \\ -1/5 & 2/5 \end{pmatrix}$.

Original row 0: $t = (\, c{+}e^T A\mathrm{M} \;\vdots\; \mathrm{M}e^T \;\vdots\; 0 \;\vdots\; -\mathrm{M}e^T b\,)$

Final tableau: $t^* = t + v^T(\, A \;\vdots\; I \;\vdots\; I \;\vdots\; b\,)$

$= (\, Z^*{+}c \;\vdots\; -y^* \;\vdots\; \mathrm{M}e^T{-}y^* \;\vdots\; Z^*\,)$

$= (\, c{+}e^T A\mathrm{M} \;\vdots\; \mathrm{M}e^T \;\vdots\; 0 \;\vdots\; -\mathrm{M}e^T b\,) + v^T(\, A \;\vdots\; I \;\vdots\; I \;\vdots\; b\,)$

$= (\, c{+}e^T A\mathrm{M}{+}v^T A \;\vdots\; \mathrm{M}e^T{+}v \;\vdots\; v \;\vdots\; -\mathrm{M}e^T b + v^T b\,)$

Hence, $v = -y^* + \mathrm{M}e^T = \begin{pmatrix} -\frac{1}{2}{+}\mathrm{M} & -\frac{1}{2}{+}\mathrm{M} \end{pmatrix}$.

(c) $t = (\, 2 \quad 3 \quad 2 \quad 0 \quad \mathrm{M} \quad 0 \quad \mathrm{M} \quad 0\,) = (\, c \;\vdots\; 0 \;\vdots\; \mathrm{M}e^T \;\vdots\; \mathrm{M}e^T b\,)$

$t^* = t + v^T(\, A \;\vdots\; I \;\vdots\; I \;\vdots\; b\,)$

$= (\, Z^*{+}c \;\vdots\; -y^* \;\vdots\; \mathrm{M}e^T{-}y^* \;\vdots\; Z^*\,)$

$= (\, c \;\vdots\; 0 \;\vdots\; \mathrm{M}e^T \;\vdots\; \mathrm{M}e^T b\,) + v^T(\, A \;\vdots\; I \;\vdots\; I \;\vdots\; b\,)$

$= (\, c{+}v^T A \;\vdots\; v \;\vdots\; \mathrm{M}e^T{+}v \;\vdots\; \mathrm{M}e^T b + v^T b\,)$

Hence, $v = -y^* = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

(d) Defining equations: $\quad x = \mathrm{M}b \Leftrightarrow \mathrm{M}^{-1}x = b$

$$x_1 + 4x_2 + 2x_3 = 8, 3x_1 + 2x_2 = 6, x_3 = 0$$

5.3-10.

(a) $\quad -2x_1 + 2x_2 + x_3 + x_4 \qquad = 10$ (i)

$\qquad 3x_1 + \;\; x_2 - x_3 \qquad + x_5 = 20$ (ii)

Multiply (i) by 1.5 and add to (ii).

$\qquad 4x_2 + \frac{1}{2}x_3 + \frac{3}{2}x_4 + x_5 = 35$ (iii)

Divide (*) by -2 and add to (iii).

$\qquad x_1 + 3x_2 + x_4 + x_5 = 30$ (iv)

Multiply (iii) by 2.

$\qquad 8x_2 + x_3 + 3x_4 + 2x_5 = 70$ (v)

Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (30, 0, 70)$ and $Z^* = 230$

(b) $\quad$ (original objective)-3(iv) $-$ 2(v)

$$\begin{array}{lrrrr} & 3x_1 + 7x_2 + 2x_3 & & & \\ & -3x_1 - 9x_2 - 3x_3 & & & - 3x_5 \\ & & -16x_2 - 2x_3 & - 6x_4 & - 4x_5 \\ \Rightarrow & & -18x_2 - 3x_3 & - 6x_4 & - 7x_5 \end{array}$$

Hence, the shadow prices are 9 and 7.

(c) Defining equations: $-2x_1 + 2x_2 + x_3 = 10, 3x_1 + x_2 - x_3 = 20, x_2 = 0$

(d) $B = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$, $x_B = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 30 \\ 70 \end{pmatrix}$

$$y^* = \begin{pmatrix} 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 7 \end{pmatrix}$$

Revised row 0: $\begin{pmatrix} 9 & 7 \end{pmatrix}\begin{pmatrix} -2 & 2 & 1 & 1 & 0 \\ 3 & 1 & -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 7 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 18 & 0 & 9 & 7 \end{pmatrix}$,

so the current solution is optimal.

(e) Final tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	18	0	9	7	230
X1	1	0	1	3	0	1	1	30
X3	2	0	0	8	1	3	2	70

5.4-1.

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Revised x_2 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

x_2 enters and x_7 leaves.

Iteration 1: $\eta = \begin{pmatrix} -\frac{a_{12}}{a_{22}} \\ \frac{1}{a_{22}} \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} \\ \frac{1}{5} \end{pmatrix}$

$$B_{\text{new}}^{-1} = \begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{pmatrix}$$

Revised x_4 coefficients: $\begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{2}{5} \end{pmatrix}$

x_4 enters and x_6 leaves.

Iteration 2: $\eta = \begin{pmatrix} \frac{1}{a'_{11}} \\ -\frac{a'_{24}}{a'_{11}} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} \\ -\frac{1}{2} \end{pmatrix}$

$$B_{\text{new}}^{-1} = \begin{pmatrix} \frac{5}{4} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

5.4-2.

$$c = \begin{pmatrix} 1 & 2 & 4 & 0 & 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} 3 & 1 & 5 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}$$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}$$

$c_B = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$, Row 0: $\begin{pmatrix} -1 & -2 & -4 & 0 & 0 & 0 \end{pmatrix}$

x_3 enters the basis.

Revised x_3 coefficients: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

x_4 leaves the basis.

Iteration 1: $\eta = \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{5} \\ -\frac{2}{5} \end{pmatrix}$

$$B_{\text{new}}^{-1} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ -\frac{2}{5} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ -\frac{2}{5} & 0 & 1 \end{pmatrix}$$

$$x_B = \begin{pmatrix} x_3 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ -\frac{2}{5} & 0 & 1 \end{pmatrix}\begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

$c_B = \begin{pmatrix} 4 & 0 & 0 \end{pmatrix}$

Revised row 0:

$$\begin{pmatrix} \frac{4}{5} & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 & 5 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 4 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{5} & -\frac{6}{5} & 0 & \frac{4}{5} & 0 & 0 \end{pmatrix}$$

x_2 enters the basis.

Revised x_2 coefficients: $\begin{pmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ -\frac{2}{5} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{19}{5} \\ -\frac{2}{5} \end{pmatrix}$

x_5 leaves.

Iteration 2: $\eta = \begin{pmatrix} -\frac{1}{19} \\ \frac{5}{19} \\ \frac{2}{19} \end{pmatrix}$

$$B_{\text{new}}^{-1} = \begin{pmatrix} 1 & -\frac{1}{19} & 0 \\ 0 & \frac{5}{19} & 0 \\ 0 & \frac{2}{19} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ -\frac{2}{5} & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{19} & -\frac{1}{19} & 0 \\ -\frac{1}{19} & \frac{5}{19} & 0 \\ -\frac{8}{19} & \frac{2}{19} & 1 \end{pmatrix}$$

$$x_B = \begin{pmatrix} x_3 \\ x_2 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{4}{19} & -\frac{1}{19} & 0 \\ -\frac{1}{19} & \frac{5}{19} & 0 \\ -\frac{8}{19} & \frac{2}{19} & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{32}{19} \\ \frac{30}{19} \\ \frac{69}{19} \end{pmatrix}$$

$c_B = \begin{pmatrix} 4 & 2 & 0 \end{pmatrix}$

Revised row 0:

$$\begin{pmatrix} \frac{14}{19} & \frac{6}{19} & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 & 5 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 4 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{29}{19} & 0 & 0 & \frac{14}{19} & \frac{6}{19} & 0 \end{pmatrix}$$

The current solution is optimal.

Optimal Solution: $(x_1^*, x_2^*, x_3^*) = \left(0, \frac{30}{19}, \frac{32}{19}\right)$ and $Z^* = \frac{188}{19}$

5.4-3.

$$c = \begin{pmatrix} 2 & -2 & 3 & 0 & 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix}$$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix}$$

$c_B = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$, Row 0: $\begin{pmatrix} -2 & 2 & -3 & 0 & 0 & 0 \end{pmatrix}$

x_3 enters the basis.

Revised x_3 coefficients: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

x_5 leaves the basis.

Iteration 1: $\eta = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$, $B_{\text{new}}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_4 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$$

$c_B = \begin{pmatrix} 0 & 3 & 0 \end{pmatrix}$

Revised row 0:

$$\begin{pmatrix} 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -2 & 3 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1 & 0 & 0 & 3 & 0 \end{pmatrix}$$

x_2 enters the basis.

Revised x_2 coefficients: $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

x_4 leaves.

Iteration 2: $\eta = \begin{pmatrix} 1/2 \\ 1/2 \\ -2 \end{pmatrix}$, $B_{\text{new}}^{-1} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ -2 & -1 & 1 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$c_B = \begin{pmatrix} -2 & 3 & 0 \end{pmatrix}$

Revised row 0:

$$\begin{pmatrix} 1/2 & 5/2 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -2 & 3 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5/2 & 0 & 0 & 1/2 & 5/2 & 0 \end{pmatrix}$$

The current solution is optimal.

Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0, 1, 3)$ and $Z^* = 7$

5.4-4.

$c = (\,10 \quad 20 \quad 0 \quad 0 \quad 0\,)$, $A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 5 & 3 & 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix}$$

$c_B = (\,0 \quad 0 \quad 0\,)$, Row 0: $(\,-10 \quad -20 \quad 0 \quad 0 \quad 0 \quad 0\,)$

x_2 enters the basis.

Revised x_2 coefficients: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

x_3 leaves the basis.

Iteration 1: $\eta = \begin{pmatrix} 1/2 \\ -1/2 \\ -3/2 \end{pmatrix}$, $B_{\text{new}}^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix} = \begin{pmatrix} 7.5 \\ 4.5 \\ 22.5 \end{pmatrix}$$

$c_B = (\,20 \quad 0 \quad 0\,)$

Revised row 0:

$$(\,10 \quad 0 \quad 0\,) \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 5 & 3 & 0 & 0 & 1 \end{pmatrix} - (\,10 \quad 20 \quad 0 \quad 0 \quad 0\,)$$

$= (\,-20 \quad 0 \quad 10 \quad 0 \quad 0\,)$

x_1 enters the basis.

Revised x_1 coefficients: $\begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 3/2 \\ 13/2 \end{pmatrix}$

x_4 leaves.

Iteration 2: $\eta = \begin{pmatrix} 1/3 \\ 2/3 \\ -13/3 \end{pmatrix}$, $B_{\text{new}}^{-1} = \begin{pmatrix} 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 2/3 & -13/3 & 1 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_2 \\ x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 2/3 & -13/3 & 1 \end{pmatrix} \begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 3 \end{pmatrix}$$

$c_B = \begin{pmatrix} 20 & 10 & 0 \end{pmatrix}$

Revised row 0:

$$\begin{pmatrix} 10/3 & 40/3 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 5 & 3 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 10 & 20 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 10/3 & 40/3 & 0 \end{pmatrix}$$

The current solution is optimal.

Optimal Solution: $(x_1^*, x_2^*) = (3, 9)$ and $Z^* = 210$

CHAPTER 6: DUALITY THEORY AND SENSITIVITY ANALYSIS

6.1-1.

(a) minimize $15y_1 + 12y_2 + 45y_3$
subject to
$$\begin{aligned} -y_1 + y_2 + 5y_3 &\geq 10 \\ 2y_1 + y_2 + 3y_3 &\geq 20 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

(b) minimize $4y_1 + 2y_2 + 12y_3$
subject to
$$\begin{aligned} -y_1 + 2y_2 + y_3 &\geq 2 \\ y_1 - y_2 + y_3 &\geq -2 \\ y_1 + y_2 + 3y_3 &\geq 3 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

6.1-2.

(a)

	x_1	x_2	x_3	x_4	
y_1	1	-2	4	3	≤ 20
y_2	-4	6	5	-4	≤ 40
y_3	2	-3	3	8	≤ 50
	$5 \leq$	$1 \leq$	$3 \leq$	$4 \leq$	

minimize $20y_1 + 40y_2 + 50y_3$
subject to
$$\begin{aligned} y_1 - 4y_2 + 2y_3 &\geq 5 \\ -2y_1 + 6y_2 - 3y_3 &\geq 1 \\ 4y_1 + 5y_2 + 3y_3 &\geq 3 \\ 3y_1 - 4y_2 + 8y_3 &\geq 4 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

(b) The dual problem has no feasible solution.

6.1-3.

(a) Apply the simplex method to the dual of the problem, since the dual has fewer constraints (not including nonnegativity constraints). We expect that the simplex method will go through fewer basic feasible solutions.

(b) Apply the simplex method to the primal problem, since it has fewer constraints (not including nonnegativity constraints). We expect that the simplex method will go through fewer basic feasible solutions.

6.1-4.

(a) minimize $12y_1 + y_2$
subject to
$$\begin{aligned} y_1 + y_2 &\geq -1 \\ y_1 + y_2 &\geq -2 \\ 2y_1 - y_2 &\geq -1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

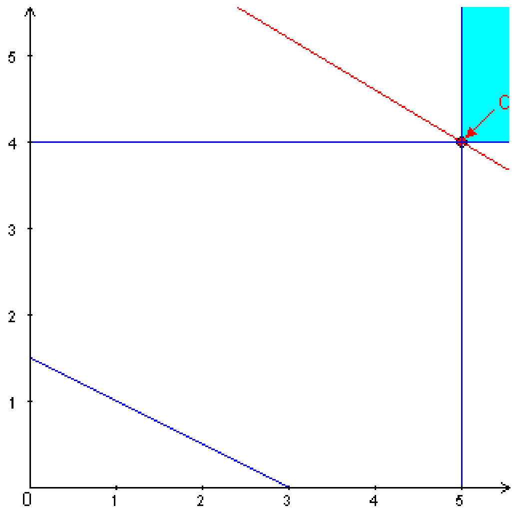
(b) It is clear from the dual problem that $(y_1, y_2) = (0, 0)$ is the optimal dual solution. By strong duality, $Z = 0 \leq 0$.

6.1-5.

(a)

$$
\begin{array}{llll}
\text{minimize} & 15y_1 + 25y_2 & & \\
\text{subject to} & y_1 & & \geq 5 \\
 & & y_2 & \geq 4 \\
 & y_1 + & 2y_2 & \geq 3 \\
 & y_1, y_2 & & \geq 0
\end{array}
$$

(b) Optimal Solution: $(y_1^*, y_2^*) = (5, 4)$, so shadow prices for resources 1 and 2 are 5 and 4 respectively.



(c)

Optimal Solution
Objective Function: Z = 175

Variable	Value
X1	15
X2	25
X3	0

Objective Function Coefficients
Allowable Range to Stay Optimal

Current Value	Minimum	Maximum
5	0	infin
4	0	infin
3	-infin	13

Constraint	Slack or Surplus	Shadow Price
1	0	5
2	0	4

Allowable Range for Right-Hand Side

Current Value	Minimum	Maximum
15	0	infin
25	0	infin

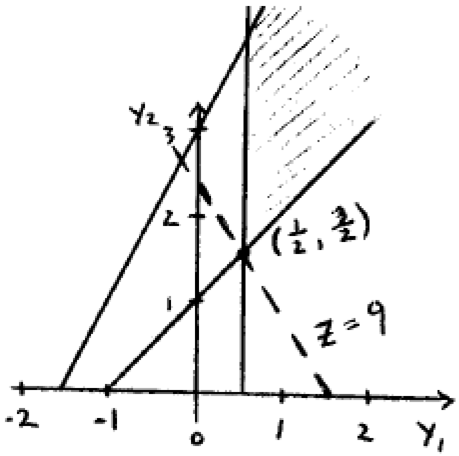
6.1-6.

(a) minimize $6y_1 + 4y_2$

subject to

$$\begin{aligned} 2y_1 \qquad\quad &\geq 1 \\ 2y_1 - \quad y_2 &\geq -3 \\ -2y_1 + 2y_2 &\geq 2 \\ y_1, y_2 &\geq 0 \end{aligned}$$

(b) Optimal Solution: $(y_1^*, y_2^*) = (1/2, 3/2)$, so shadow prices for resources 1 and 2 are $1/2$ and $3/2$ respectively.



(c)

Optimal Solution

Value of the
Objective Function: Z = 9

Variable	Value
X_1	5
X_2	0
X_3	2

Constraint	Slack or Surplus	Shadow Price
1	0	0.5
2	0	1.5

Sensitivity Analysis

Objective Function Coefficient

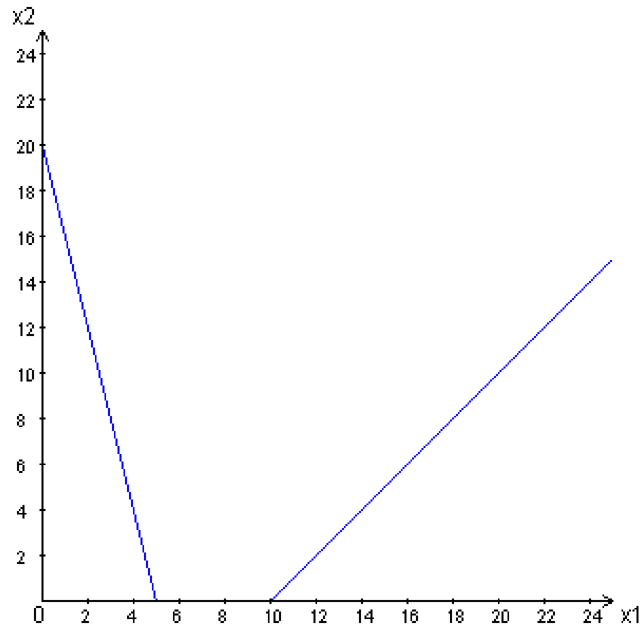
Current Value	Allowable Range To Stay Optimal Minimum	Maximum
1	0	$+\infty$
-3	$-\infty$	-0.5
2	-1	7

Right Hand Sides

Current Value	Allowable Range To Stay Feasible Minimum	Maximum
6	-4	$+\infty$
4	0	$+\infty$

6.1-7.

(a) The feasible region is empty.

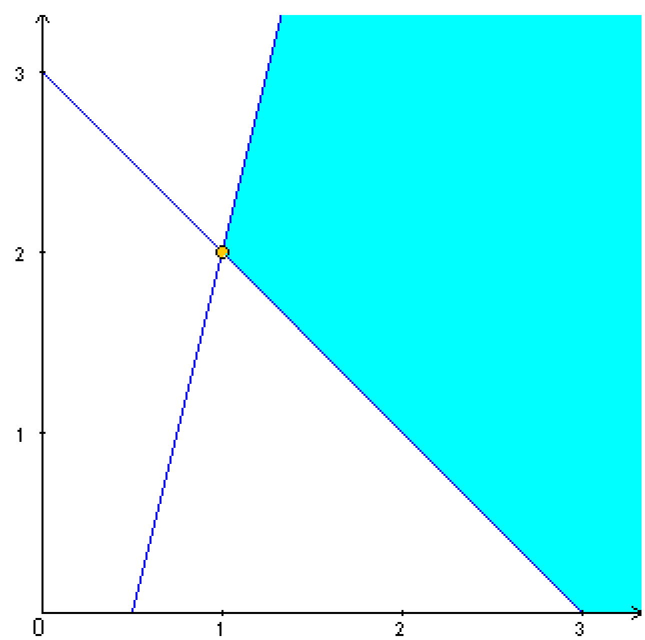


(b) minimize $20y_1 - 10y_2$

subject to

$$\begin{aligned} 4y_1 - y_2 &\geq 2 \\ y_1 + y_2 &\geq 3 \\ y_1, y_2 &\geq 0 \end{aligned}$$

(c) Note that the dual objective function can be expressed as $5(4y_1 - y_2) - 5y_2$. If for any y_2, y_1 is chosen such that $4y_1 - y_2 = 2$, then the objective function equals $10 - 5y_2$. Hence, by choosing y_2 properly, the dual objective can be made arbitrarily small.



6.1-8.

Primal: maximize $x_1 + x_2$

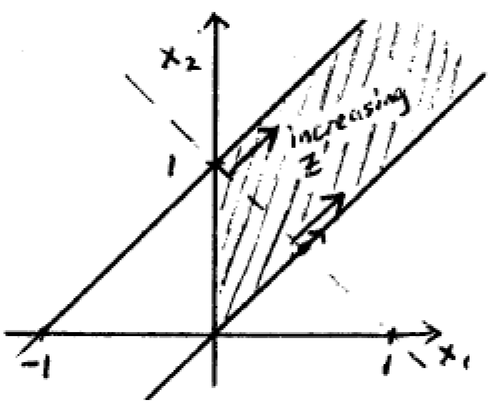
subject to
$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 - x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

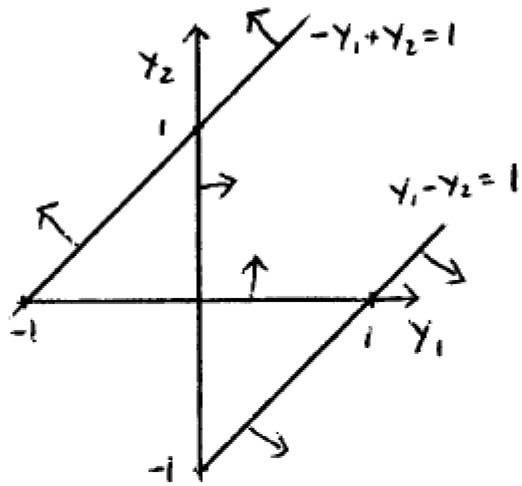
Let $x_1 = x_2 = c \to \infty$, $Z = 2c$ is unbounded.

Dual: minimize y_1

subject to
$$\begin{aligned} -y_1 + y_2 &\geq 1 \\ y_1 - y_2 &\geq 1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

The dual problem is infeasible.





6.1-9.

Primal: maximize x_1

subject to

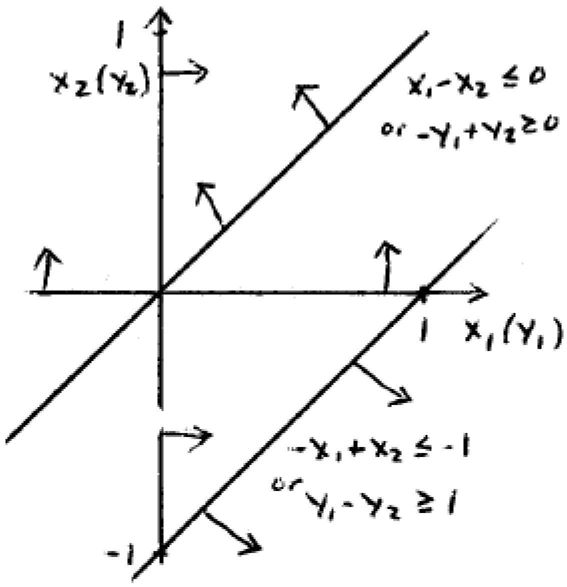
$$\begin{aligned} x_1 - x_2 &\leq 0 \\ -x_1 + x_2 &\leq -1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Dual: minimize $-y_2$

subject to

$$\begin{aligned} y_1 - y_2 &\geq 1 \\ -y_1 + y_2 &\geq 0 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Neither the primal nor the dual is feasible. They have the same two constraints, which contradict each other, so their feasible region is empty.



6.1-10.

Primal: maximize $x_1 + x_2$

subject to

$$\begin{aligned} x_1 &\leq -1 \\ x_1 + x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The primal problem is clearly infeasible.

Dual: minimize $-y_1$

subject to

$$\begin{aligned} y_1 + y_2 &\geq 1 \\ y_2 &\geq 1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Let $c \to \infty$ in the feasible solution $(c, 1)$, so the objective function value is unbounded.

6.1-11.

Let x^0 and y^0 be a primal and a dual feasible point respectively. By weak duality,

$$-\infty < cx^0 \leq y^0 b < \infty.$$

Furthermore, for any primal feasible point x and any dual feasible point y,

$$cx \leq y^0 b \text{ and } cx^0 \leq yb.$$

This means that the primal problem cannot be unbounded, as it is bounded above by $y^0 b$ and similarly, the dual problem cannot be bounded as it is bounded below by cx^0. Therefore, since the primal problem (and the dual problem) has a feasible solution and the objective function value is bounded, it must have an optimal solution.

6.1-12.

(a) From the primal, $Ax \leq b$, $x \geq 0$ and from the dual, $y^T A \geq c^T$, $y \geq 0$, so

$$y^T A - c^T \geq 0, x \geq 0 \Rightarrow (y^T A - c^T)x \geq 0$$

$$b - Ax \geq 0, y \geq 0 \Rightarrow y^T(b - Ax) \geq 0.$$

In other words, $y^T Ax \geq c^T x$ and $y^T b \geq y^T Ax$, so $y^T b \geq y^T Ax \geq c^T x$, which is weak duality.

(b) There are many ways to prove this. The simplest is by contradiction. Assume the primal objective Z can be increased indefinitely and the dual does have a feasible solution. By weak duality, $c^T x \leq y^T b$ for all primal feasible x, given y is a dual feasible solution. This means that Z is bounded above, which contradicts the assumption. Hence, if Z is unbounded, then the dual must be infeasible.

6.1-13.

Primal:	maximize	$Z = cx$	Dual:	minimize	$W = yb$
	subject to	$Ax \leq b$		subject to	$yA \geq c$
		$x \geq 0$			$y \geq 0$

Since changing b to $\overline{b}$ keeps the dual feasible region unchanged, y^* must be feasible for the new problem. Let $\overline{y}$ be the optimal solution for the new dual, then clearly $\overline{y}\overline{b} \leq y^*\overline{b}$, since $\overline{y}$ is optimal. Furthermore, by strong duality, $c\overline{x} = \overline{y}\overline{b} \leq y^*\overline{b}$.

6.1-14.

(a) TRUE. If A is an $n \times m$ matrix, then in standard form, the number of functional constraints is n for the primal and m for the dual. The number of variables is m in the primal and n in the dual. Hence, for both, the sum of the number of constraints and variables is $m + n$.

(b) FALSE. This cannot be true since the weak and strong duality theorems imply that the primal and the dual objective function values are the same only at optimality.

(c) FALSE. If the primal problem has an unbounded objective function value, the dual problem must be infeasible, since by weak duality, if the dual has a feasible solution $\overline{y}$, the primal objective value is $Z = cx \leq \overline{y}b$.

6.2-1.

(a) Iteration 0: Since all coefficients are zero, at the current solution $(0, 0)$, the three resources (production time per week at plant 1, 2 and 3) are free goods. This means increasing them does not improve the objective value.

Iteration 1: $(0, 5/2, 0)$. Now resource 2 has been entirely used up and contributes $5/2$ to profit per unit of resource. Since this is positive, it is worthwhile to continue fully using this resource.

Iteration 2: $(0, 3/2, 1)$. Resources 2 and 3 are used up and contribute a positive amount to profit. Resource 1 is a free good while resources 2 and 3 contribute $3/2$ and 1 per unit of resource respectively.

(b) Iteration 0: $(-3, -5)$. Both activities 1 and 2 (number of batches of product 1 and 2 produced per week) can be initiated to give a more profitable allocation of the resources. The current contribution of the resources required to produce one batch of product 1 or 2 to the profit is smaller than the unit profit per batch of product 1 or 2 respectively.

Iteration 1: $(-3, 0)$. Again activity 1 can be initiated to give a more profitable use of resources, but activity 2 is already being produced (or the resources are being used just as well in other activities).

Iteration 2: $(0, 0)$. Both activities are being produced (or the resources are being used just as profitably elsewhere).

(c) Iteration 1: Since activities 1 and 2 can be initiated to increase the profit (give the same amount of resources), we choose to increase one of these. We choose activity 2 as the entering activity (basic variable), since it increases the profit by 5 for every unit of product 2 produced (as opposed to 3 for product 1).

Iteration 2: Only activity 1 can be initiated for more profit, so we do so.

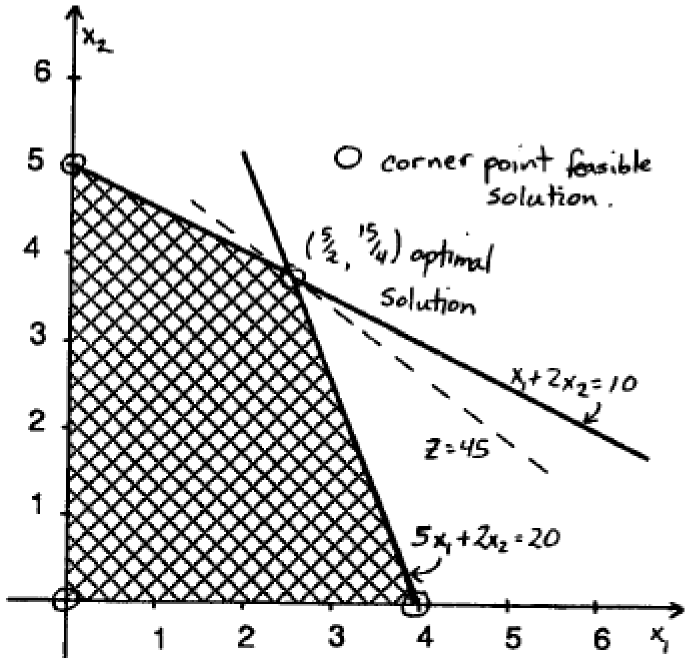
Iteration 3: Both activity 1 and 2 are being used. Furthermore, since the coefficients for x_3, x_4 and x_5 are nonnegative, it is not worthwhile to cut back on the use of any of the resources. Thus, we must be at the optimal solution.

6.3-1.

(a) minimize $W = 20y_1 + 10y_2$

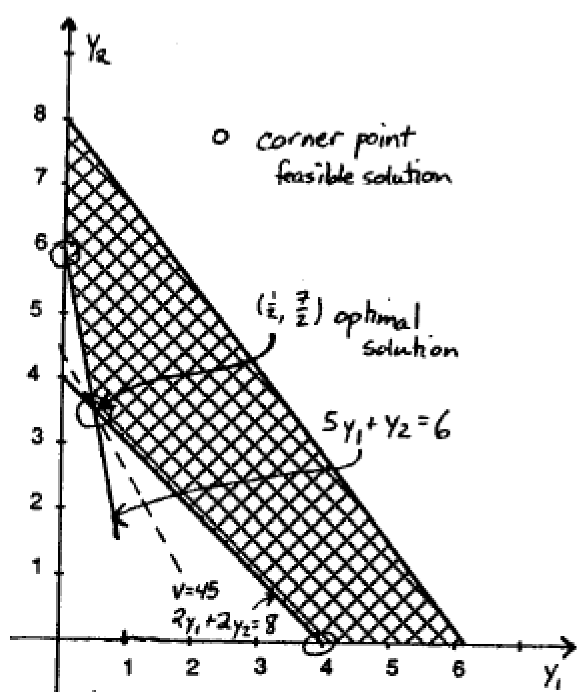
subject to
$$\begin{aligned} 5y_1 + y_2 &\geq 6 \\ 2y_1 + 2y_2 &\geq 8 \\ y_1, y_2 &\geq 0 \end{aligned}$$

(b) Primal:



$(x_1, x_2) = (5/2, 15/4)$ is optimal with $Z = 45$. Infeasible corner point solutions are $(0, 10)$ and $(10, 0)$.

Dual:



$(y_1, y_2) = (1/2, 7/2)$ is optimal with $W = 45$. Infeasible corner point solutions are $(0, 4)$, $(0, 0)$ and $(6/5, 0)$.

(c)

Primal BS	Feasible?	Z	Dual BS	Feasible?
$(0,5,10,0)$	Yes	40	$(0,4,-2,0)$	No
$(0,0,20,10)$	Yes	0	$(0,0,-6,-8)$	No
$(4,0,0,6)$	Yes	24	$(6/5,0,0,-28/5)$	No
$(5/2,15/4,0,0)$	Yes	45	$(1/2,7/2,0,0)$	Yes
$(0,10,0,-10)$	No	80	$(4,0,14,0)$	Yes
$(10,0,-30,0)$	No	60	$(0,6,0,4)$	Yes

(d)

Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
Z	0	1	-6	-8	0	0	0
X3	1	0	5	2	1	0	20
X4	2	0	1	2*	0	1	10

Primal: $(0,0,20,10)$
Dual: $(0,0,-6,-8)$

Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
Z	0	1	-2	0	0	4	40
X3	1	0	4*	0	1	-1	10
X2	2	0	0.5	1	0	0.5	5

Primal: $(0,5,10,0)$
Dual: $(0,4,-2,0)$

Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
Z	0	1	0	0	0.5	3.5	45
X1	1	0	1	0	0.25	-0.25	2.5
X2	2	0	0	1	-0.13	0.625	3.75

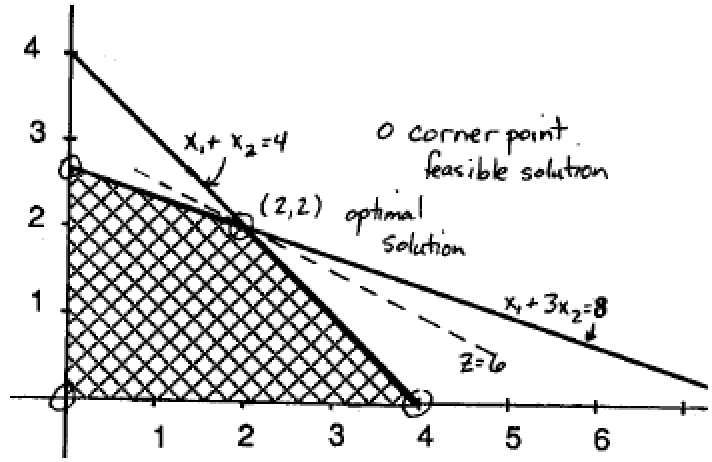
Primal: $(5/2,15/4,0,0)$
Dual: $(1/2,7/2,0,0)$

6.3-2.

(a) minimize $W = 8y_1 + 4y_2$

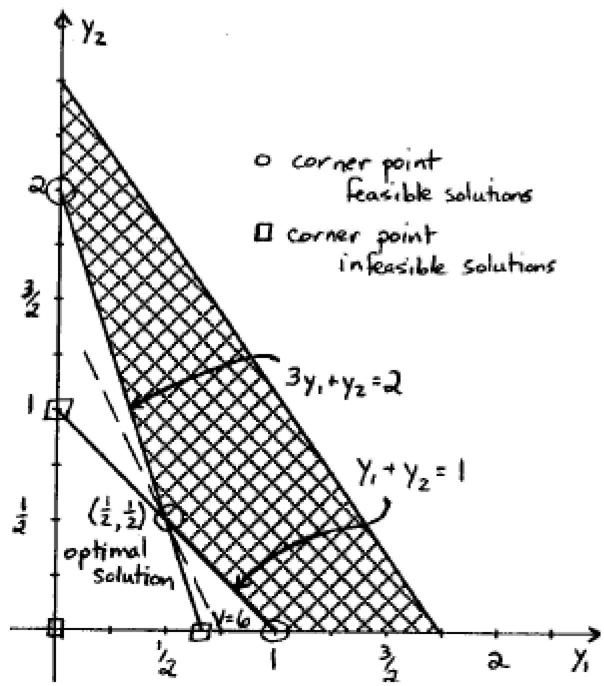
subject to
$$y_1 + y_2 \geq 1$$
$$3y_1 + y_2 \geq 2$$
$$y_1, y_2 \geq 0$$

(b) Primal:



$(x_1, x_2) = (2, 2)$ is optimal with $Z = 6$. Infeasible corner point solutions are $(8, 0)$ and $(0, 4)$.

Dual:



$(y_1, y_2) = (1/2, 1/2)$ is optimal with $W = 6$.

(c)

Primal BS	Feasible?	Z	Dual BS	Feasible?
$(4,0,4,0)$	Yes	4	$(0,1,0,-1)$	No
$(0,0,8,4)$	Yes	0	$(0,0,-1,-2)$	No
$(0,8/3,0,4/3)$	Yes	$16/3$	$(2/3,0,-1/3,0)$	No
$(2,2,0,0)$	Yes	6	$(1/2,1/2,0,0)$	Yes
$(0,4,-4,0)$	No	8	$(0,2,1,0)$	Yes
$(8,0,0,-4)$	No	8	$(1,0,0,1)$	Yes

(d)

```
Bas|Eq|        Coefficient of        | Right
Var|No| Z|   X1     X2     X3     X4 | side
___|__|__|___________________________|______
   |  |  |                           |
 Z | 0| 1|   -1     -2      0      0 | 0
 X3| 1| 0|    1     3*      1      0 | 8
 X4| 2| 0|    1      1      0      1 | 4
```

Primal: $(0,0,8,4)$

Dual: $(0,0,-1,-2)$

```
Bas|Eq|        Coefficient of        | Right
Var|No| Z|   X1     X2     X3     X4 | side
___|__|__|___________________________|______
   |  |  |                           |
 Z | 0| 1|-0.33      0 0.667       0 | 5.333
 X2| 1| 0|0.333      1 0.333       0 | 2.667
 X4| 2| 0|0.667*     0 -0.33       1 | 1.333
```

Primal: $(0,8/3,0,4/3)$

Dual: $(2/3,0,-1/3,0)$

```
Bas|Eq|        Coefficient of        | Right
Var|No| Z|   X1     X2     X3     X4 | side
___|__|__|___________________________|______
   |  |  |                           |
 Z | 0| 1|    0      0    0.5    0.5 | 6
 X2| 1| 0|    0      1    0.5   -0.5 | 2
 X1| 2| 0|    1      0   -0.5    1.5 | 2
```

Primal: $(2,2,0,0)$

Dual: $(1/2,1/2,0,0)$

6.3-3.

NB Primal Var.	Assoc. Dual Var.	NB Dual Var.
x_1, x_2	y_4, y_5	y_1, y_2, y_3
x_1, x_4	y_4, y_2	y_1, y_3, y_5
x_4, x_5	y_2, y_3	y_1, y_4, y_5
x_3, x_5	y_1, y_3	y_2, y_4, y_5
x_2, x_3	y_5, y_1	y_2, y_3, y_4
x_1, x_5	y_4, y_3	y_1, y_2, y_5
x_3, x_4	y_1, y_2	y_3, y_4, y_5
x_2, x_5	y_5, y_3	y_1, y_2, y_4

In all cases, complementary slackness holds: $x_1y_4 = x_2y_5 = x_3y_1 = x_4y_2 = x_5y_3 = 0$.

6.3-4.

If either the primal or the dual has a degenerate optimal basic feasible solution, then the other may have multiple solutions. For example, consider the problem:

$$\begin{array}{ll} \text{maximize} & 3x_1 \\ \text{subject to} & a_{11}x_1 + x_2 = 0 \\ & -2x_1 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

If $a_{11} > 0$, we can pivot and get an alternative optimal solution to the dual problem. If $a_{11} \leq 0$, we cannot.

The converse is true, however. If a problem has multiple optimal solutions, then two of them must be adjacent corner points. To move from the tableau of one solution to that of the other requires exactly one pivot. Suppose x_j enters and x_k leaves. A partial tableau is:

		x_j		RS
		$\bar{c}_j$		
x_k		$\bar{a}_{kj}$		$\bar{b}_k$

$\bar{a}_{kj}$ must be positive and $\bar{b}_k \geq 0$. If $\bar{b}_k > 0$, then $\bar{c}_j$ or Z would change with the pivot. If $\bar{b}_k = 0$, then x_j pivots in at value zero and the resulting tableau represents the same corner point, contradicting the assumption that the two optimal solutions are distinct.

6.3-5.

(a)

$$\begin{array}{ll} \text{minimize} & W = 10y_1 \\ \text{subject to} & y_1 \geq 3 \\ & -2y_1 \geq -8 \\ & y_1 \geq 0 \end{array}$$

The optimal solution is $y_1 = 3$ and $W = 30$.

(b) $(y_1, y_2, y_3) = (3, 0, 2)$ is the optimal basic feasible solution for the dual. By complementary slackness, $y_1 x_3 = y_2 x_1 = y_3 x_2 = 0$, so $x_2 = x_3 = 0$. Since $x_1 - 2x_2 + x_3 = 10$, $(x_1, x_2, x_3) = (10, 0, 0)$ is optimal for the primal.

(c) The constraints for the dual problem can be expressed as:

$$c_1 \leq y_1 \leq \tfrac{-c_2}{2} = 4,$$

so if $c_1 > 4$, the dual is infeasible and the primal objective function is unbounded.

6.3-6.

(a) minimize $W = 10y_1 + 10y_2$

subject to
$$\begin{aligned} y_1 + 3y_2 &\geq 2 \\ 2y_1 + 3y_2 &\geq 7 \\ y_1 + 2y_2 &\geq 4 \\ y_1, y_2 &\geq 0 \end{aligned}$$

(b) $(0, 5/2)$ is feasible for the dual problem. By weak duality,

$$W = 10 \cdot 0 + 10 \cdot 5/2 = 25 \geq z,$$

so the optimal primal objective function value is less than 25.

(c)

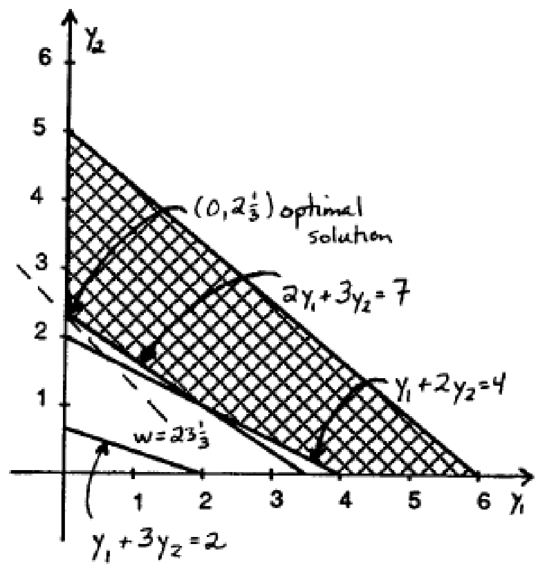
Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	-2	-7	-4	0	0	0
X4	1	0	1	2	1	1	0	10
X5	2	0	3	3*	2	0	1	10

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	5	0	0.667	0	2.333	23.33
X4	1	0	-1	0	-0.33*	1	-0.67	3.333
X2	2	0	1	1	0.667	0	0.333	3.333

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	3	0	0	2	1	30
X3	1	0	3	0	1	-3	2	-10
X2	2	0	-1	1	0	2	-1	10

The primal basic solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 10, -10, 0, 0)$, which is not feasible. The dual basic solution is $(y_1, y_2, z_1 - c_1, z_2 - c_2, z_3 - c_3) = (2, 1, 3, 0, 0)$.

(d)



$(y_1, y_2) = (0, 7/3)$ is optimal with $W = 70/3$. From the dual solution, y_2, y_3 and y_5 are basic; therefore, x_3, x_5 and x_1 are nonbasic primal variables, x_2 and x_4 are basic.

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	-2	-7	-4	0	0	0
X4	1	0	1	2	1	1	0	10
X5	2	0	3	3*	2	0	1	10

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	5	0	0.667	0	2.333	23.33
X4	1	0	-1	0	-0.33*	1	-0.67	3.333
X2	2	0	1	1	0.667	0	0.333	3.333

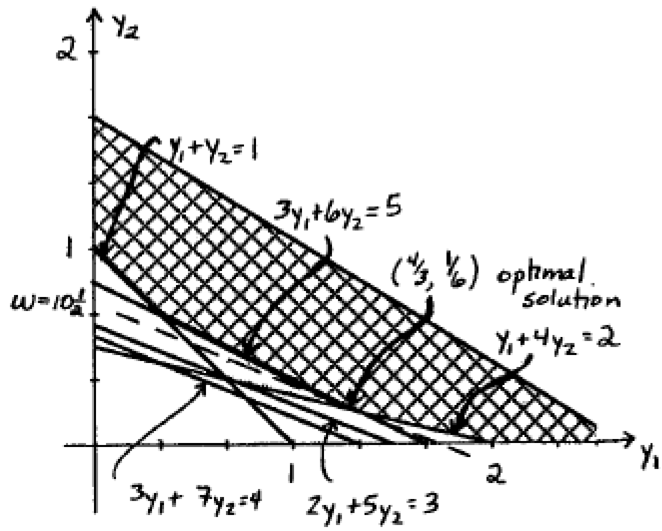
$(x_1, x_2, x_3, x_4, x_5) = (0, 10/3, 0, 10/3, 0)$ is the primal optimal basic solution with $Z = 70/3$.

6.3-7.

(a) minimize $W = 6y_1 + 15y_2$

subject to
$$
\begin{aligned}
&y_1 + 4y_2 \geq 2 \\
&3y_1 + 6y_2 \geq 5 \\
&2y_1 + 5y_2 \geq 3 \\
&3y_1 + 7y_2 \geq 4 \\
&y_1 + y_2 \geq 1 \\
&y_1, y_2 \geq 0
\end{aligned}
$$

(b) $(y_1, y_2) = (4/3, 1/6)$ is optimal with $W = 21/2$.



(c) $(z_1 - c_1)$ and $(z_2 - c_2)$ are nonbasic in the dual, so x_1 and x_2 must be basic in the optimal primal solution.

(d)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	-2	-5	-3	-4	-1	0	0	0
X6	1	0	1	3*	2	3	1	1	0	6
X7	2	0	4	6	5	7	1	0	1	15

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	-0.33	0	0.333	1	0.667	1.667	0	10
X2	1	0	0.333	1	0.667	1	0.333	0.333	0	2
X7	2	0	2*	0	1	1	-1	-2	1	3

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	0.5	1.167	0.5	1.333	0.167	10.5
X2	1	0	0	1	0.5	0.833	0.5	0.667	-0.17	1.5
X1	2	0	1	0	0.5	0.5	-0.5	-1	0.5	1.5

$(x_1, x_2) = (3/2, 3/2)$ is optimal with $Z = 21/2$.

(e) The defining equations are:

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 &= 6 \\ 4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 &= 15 \\ x_3 \qquad\qquad &= 0 \\ x_4 \qquad &= 0 \\ x_5 &= 0, \end{aligned}$$

which have the solution $(x_1, x_2, x_3, x_4, x_5) = (3/2, 3/2, 0, 0, 0)$.

6.3-8.

(a) minimize $W = 10y_1 + 20y_2$

subject to

$$\begin{aligned} -2y_1 + 3y_2 &\geq 3 \\ 2y_1 + \ \ y_2 &\geq 7 \\ y_1 - \ \ y_2 &\geq 2 \\ y_1, y_2 &\geq 0 \end{aligned}$$

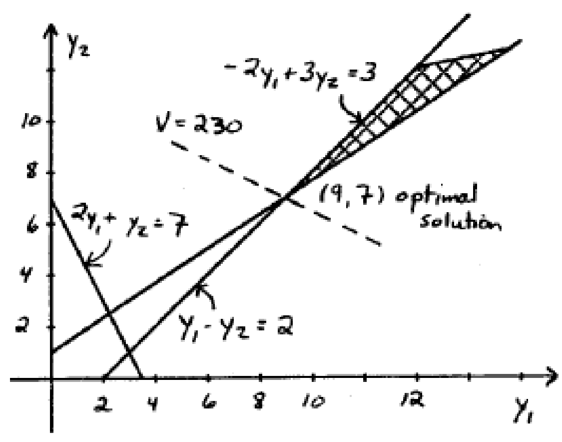
(b) Because x_2, x_4 and x_5 are nonbasic in the optimal primal solution, y_1, y_2 and y_4 will be basic in the optimal dual solution.

(c) The defining equations are:

$$\begin{aligned} -2y_1 + 3y_2 - y_3 \qquad\qquad &= 3 \\ 2y_1 + \ \ y_2 \qquad - y_4 \qquad &= 7 \\ y_1 - \ \ y_2 \qquad\qquad - y_5 &= 2 \\ y_3 \qquad\qquad &= 0 \\ y_5 &= 0, \end{aligned}$$

which have the solution $(y_1, y_2, y_3, y_4, y_5) = (9, 7, 0, 18, 0)$.

(d) $(y_1, y_2) = (9, 7)$ is optimal with $W = 230$.



6.3-9.

(a) minimize $W = 10y_1 + 60y_2 + 18y_3 + 44y_4$

subject to

$$\begin{aligned} 2y_2 + \ \ y_3 + \ 3y_4 &\geq 2 \\ y_1 + \ 5y_2 + \ \ y_3 + \ \ y_4 &\geq 1 \\ y_1, y_2, y_3, y_4 &\geq 0 \end{aligned}$$

(b) The defining equations for $(x_1, x_2) = (13, 5)$ are:

$$x_1 + x_2 = 18 \text{ and } 3x_1 + x_2 = 44.$$

Then y_3 and y_4 must be basic in the optimal dual solution whereas y_1, y_2 and y_3 are non-basic.

(c) The basic variables in the primal optimal solution are x_1, x_2, x_3 and x_4. Introduce x_1 and x_2 into the basis.

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-2	-1	0	0	0	0	0
X3	1	0	0	1	1	0	0	0	10
X4	2	0	2	5	0	1	0	0	60
X5	3	0	1	1	0	0	1	0	18
X6	4	0	3*	1	0	0	0	1	44

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	-0.33	0	0	0	0.667	29.33
X3	1	0	0	1	1	0	0	0	10
X4	2	0	0	4.333	0	1	0	-0.67	30.67
X5	3	0	0	0.667*	0	0	1	-0.33	3.333
X1	4	0	1	0.333	0	0	0	0.333	14.67

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	0	0.5	0.5	31
X3	1	0	0	0	1	0	-1.5	0.5	5
X4	2	0	0	0	0	1	-6.5	1.5	9
X2	3	0	0	1	0	0	1.5	-0.5	5
X1	4	0	1	0	0	0	-0.5	0.5	13

$(x_1, x_2, x_3, x_4, x_5, x_6) = (13, 5, 5, 9, 0, 0)$ is optimal with $Z = 31$. The dual solution is $(y_1, y_2, y_3, y_4, y_5, y_6) = (0, 0, 1/2, 1/2, 0, 0)$.

(d) The defining equations are:

$$\begin{aligned}
2y_2 + y_3 + 3y_4 &= 2 \\
y_1 + 5y_2 + y_3 + y_4 &= 1 \\
y_1 &= 0 \\
y_2 &= 0,
\end{aligned}$$

which are satisfied by $(0, 0, 1/2, 1/2, 0, 0)$.

6.3-10.

(a) The optimal dual solution corresponds to row 0 computed by the simplex method to determine optimality.

(b) The complementary basic solution corresponds to row 0 as well.

6.4-1.

(a)

$$\begin{array}{ll} \text{minimize} & W = 10y_1 + 20y_2 \\ \text{subject to} & 2y_1 + \ \ y_2 = 5 \\ & 3y_1 + 2y_2 \geq 4 \\ & y_1 \leq 0 \text{ } (y_2 \text{ unconstrained in sign}) \end{array}$$

(b) Standard form:

$$\begin{array}{ll} \text{maximize} & Z = 5x_1^+ - 5x_1^- + 4x_2 \\ \text{subject to} & -2x_1^+ + 2x_1^- - 3x_2 \leq -10 \\ & \ \ x_1^+ - \ \ x_1^- + 2x_2 \leq \ \ 20 \\ & -\ x_1^+ + \ \ x_1^- - 2x_2 \leq -20 \\ & \ \ x_1^+, x_1^-, x_2 \geq 0 \end{array}$$

Dual:

$$\begin{array}{ll} \text{minimize} & W = -10y_1 + 20y_2 - 20y_3 \\ \text{subject to} & -2y_1 + \ \ y_2 - \ \ y_3 \geq 5 \\ & \ \ 2y_1 - \ \ y_2 + \ \ y_3 \geq -5 \\ & -3y_1 + 2y_2 - 2y_3 \geq \ 4 \\ & \ \ y_1, y_2, y_3 \geq 0 \end{array}$$

Let $y_2' = y_2 - y_3$ and $y_1' = -y_1$. Then the dual is:

$$\begin{array}{ll} \text{minimize} & W' = 10y_1' + 20y_2' \\ \text{subject to} & 2y_1' + \ \ y_2' = 5 \\ & 3y_1' + 2y_2' \geq 4 \\ & y_1' \leq 0 \text{ } (y_2' \text{ unconstrained in sign}) \end{array}$$

as given in part (a).

6.4-2.

(a) Since $\{Ax = b\}$ is equivalent to

$$\left\{ \begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix} \right\},$$

changing the primal functional constraints from $Ax \leq b$ to $Ax = b$ changes the dual to:

$$\begin{array}{ll} \text{minimize} & W = (\,\overline{y}^T \quad \overline{u}^T\,) \begin{pmatrix} b \\ -b \end{pmatrix} \\ \text{subject to} & (\,\overline{y}^T \quad \overline{u}^T\,) \begin{pmatrix} A \\ -A \end{pmatrix} \geq c \\ & \overline{y}, \overline{u} \geq 0. \end{array}$$

Let $y = \overline{y} - \overline{u}$.

$$\begin{array}{ll} \text{minimize} & W = yb \\ \text{subject to} & yA \geq c \\ & y \text{ unrestricted in sign} \end{array}$$

Hence, the only change is the deletion of the nonnegativity constraints.

(b) $\{Ax \geq b\}$ is equivalent to $\{-Ax \leq -b\}$, so the dual of

$$\begin{array}{ll} \text{maximize} & Z = cx \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

is

$$\begin{array}{ll} \text{minimize} & W = \overline{y}(-b) \\ \text{subject to} & \overline{y}(-A) \geq c \\ & \overline{y} \geq 0. \end{array}$$

Let $y = -\overline{y}$.

$$\begin{array}{ll} \text{minimize} & W = yb \\ \text{subject to} & yA \geq c \\ & y \leq 0 \end{array}$$

Hence, $y \geq 0$ is replaced by $y \leq 0$ in the dual.

(c)

$$\begin{array}{llllll} \text{Primal:} & \text{maximize} & Z = cx & \Leftrightarrow & \text{maximize} & Z = cx^+ - cx^- \\ & \text{subject to} & Ax \leq b & & \text{subject to} & Ax^+ - Ax^- \leq b \\ & & x \text{ unrestricted in sign} & & & x^+, x^- \geq 0 \\ \text{Dual:} & \text{minimize} & W = yb & \Leftrightarrow & \text{minimize} & W = yb \\ & \text{subject to} & yA \geq c & & \text{subject to} & yA = c \\ & & y(-A) \geq -c & & & y \geq 0 \\ & & y \geq 0 & & & \end{array}$$

Hence, $yA \geq c$ is replaced by $yA = c$.

6.4-3.

$$\begin{array}{ll} \text{maximize} & W = 8y_1 + 6y_2 \\ \text{subject to} & y_1 + 3y_2 \leq 2 \\ & 4y_1 + 2y_2 \leq 3 \\ & 2y_1 \leq 1 \\ & y_1, y_2 \geq 0 \end{array}$$

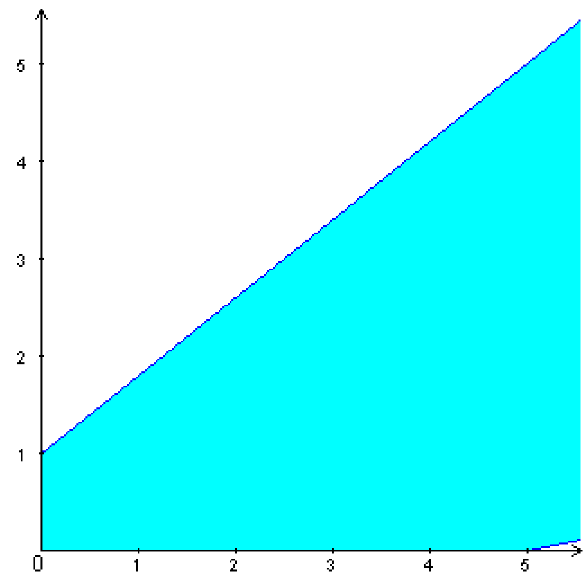
6.4-4.

(a) maximize $W = 4y_1 + 10y_2$

subject to

$$\begin{aligned} -4y_1 + \ \ 5y_2 &\le 5 \\ 2y_1 - 10y_2 &\le 10 \\ y_1, y_2 &\ge 0 \end{aligned}$$

(b)



Since v can be increased indefinitely, the primal problem is infeasible, by weak duality.

6.4-5.

minimize $W = 2.7y_1 + \ 6y_2 + \ 6y_3'$

subject to

$$\begin{aligned} 0.3y_1 + 0.5y_2 + 0.6y_3' &\ge -0.4 \\ 0.1y_1 + 0.5y_2 + 0.4y_3' &\ge -0.5 \end{aligned}$$

$y_1 \ge 0, y_3' \le 0, y_2$ unrestricted in sign

$\Leftrightarrow$ maximize $-W = -2.7y_1 - \ 6y_2 - \ 6y_3'$

subject to

$$\begin{aligned} 0.3y_1 + 0.5y_2 + 0.6y_3' &\ge -0.4 \\ 0.1y_1 + 0.5y_2 + 0.4y_3' &\ge -0.5 \end{aligned}$$

$y_1 \ge 0, y_3' \le 0, y_2$ unrestricted in sign

$\Leftrightarrow$ maximize $W' = 2.7y_1' + \ 6y_2' + \ 6y_3$

subject to

$$\begin{aligned} -0.3y_1' - 0.5y_2' - 0.6y_3 &\ge -0.4 \\ -0.1y_1' - 0.5y_2' - 0.4y_3 &\ge -0.5 \end{aligned}$$

$y_1' \le 0, y_3 \ge 0, y_2'$ unrestricted in sign

$\Leftrightarrow$ maximize $W' = 2.7y_1' + \ 6y_2' + \ 6y_3$

subject to

$$\begin{aligned} 0.3y_1' + 0.5y_2' + 0.6y_3 &\le 0.4 \\ 0.1y_1' + 0.5y_2' + 0.4y_3 &\le 0.5 \end{aligned}$$

$y_1' \le 0, y_3 \ge 0, y_2'$ unrestricted in sign

6.4-6.

(a) maximize $Z = 2x_1 + 5x_2 + 3x_3$

subject to
$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &\geq 20 \\ 2x_1 + 4x_2 + \ \ x_3 &= 50 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Dual: minimize $W = 20y_1 + 50y_2$

subject to
$$\begin{aligned} y_1 + \ \ 2y_2 &\geq 2 \\ -2y_1 + \ \ 4y_2 &\geq 5 \\ y_1 + \ \ \ \ y_2 &\geq 3 \end{aligned}$$
$y_1 \leq 0, y_2$ unconstrained in sign

(b) maximize $Z = -2x_1 + \ \ x_2 - 4x_3 + 3x_4$

subject to
$$\begin{aligned} x_1 + \ \ x_2 + 3x_3 + 2x_4 &\leq \ \ 4 \\ x_1 \qquad\quad - \ \ x_3 + \ \ x_4 &\geq -1 \\ 2x_1 + \ \ x_2 \qquad\qquad\qquad &\leq \ \ 2 \\ x_1 + 2x_2 + \ \ x_3 + 2x_4 &= \ \ 2 \end{aligned}$$
$x_1$unconstrained in sign, $x_2, x_3, x_4 \geq 0$

Dual: minimize $W = 4y_1 - y_2 + 2y_3 + 2y_4$

subject to
$$\begin{aligned} y_1 + y_2 + 2y_3 + \ \ y_4 &= -2 \\ y_1 \qquad + \ \ y_3 + 2y_4 &\geq \ \ 1 \\ 3y_1 - y_2 \qquad + \ \ y_4 &\geq -4 \\ 2y_1 + y_2 \qquad + 2y_4 &\geq \ \ 3 \end{aligned}$$
$y_1, y_3 \geq 0, y_2 \leq 0, y_4$ unconstrained in sign

6.4-7.

(a) minimize $W = 300y_1 + 300y_2$

subject to
$$\begin{aligned} 2y_1 + \ \ 8y_2 &\geq 4 \\ 3y_1 + \ \ \ \ y_2 &\geq 2 \\ 4y_1 + \ \ \ \ y_2 &\geq 3 \\ 2y_1 + \ \ 5y_2 &\geq 5 \end{aligned}$$
y_1, y_2 unconstrained in sign

(b) maximize $Z = 4x_1 + 2x_2 + 3x_3 + 5x_4$

subject to
$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 2x_4 &= 300 \\ 8x_1 + \ \ x_2 + \ \ x_3 + 5x_4 &= 300 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Standard form: maximize $Z = \ \ 4x_1 + 2x_2 + 3x_3 + 5x_4$

subject to
$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 2x_4 &\leq \ \ 300 \\ -2x_1 - 3x_2 - 4x_3 - 2x_4 &\leq -300 \\ 8x_1 + \ \ x_2 + \ \ x_3 + 5x_4 &\leq \ \ 300 \\ -8x_1 - \ \ x_2 - \ \ x_3 - 5x_4 &\leq -300 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual: minimize $W = 300y_1 - 300y_2 + 300y_3 - 300y_4$

subject to

$$\begin{aligned}
2y_1 - 2y_2 + 8y_3 - 8y_4 &\geq 4\\
3y_1 - 3y_2 + y_3 - y_4 &\geq 2\\
4y_1 - 4y_2 + y_3 - y_4 &\geq 3\\
2y_1 - 2y_2 + 5y_3 - 5y_4 &\geq 5\\
y_1, y_2, y_3, y_4 &\geq 0
\end{aligned}$$

Let $y_1' = y_1 - y_2$ and $y_2' = y_3 - y_4$.

minimize $W = 300y_1' + 300y_2'$

subject to

$$\begin{aligned}
2y_1' + 8y_2' &\geq 4\\
3y_1' + y_2' &\geq 2\\
4y_1' + y_2' &\geq 3\\
2y_1' + 5y_2' &\geq 5
\end{aligned}$$

y_1', y_2' unconstrained in sign

6.4-8.

(a) minimize $W = 120y_1 + 80y_2 + 100y_3$

subject to

$$\begin{aligned}
y_2 - 3y_3 &= -1\\
3y_1 - y_2 + y_3 &= 2\\
y_1 - 4y_2 + 2y_3 &= 1\\
y_1, y_2, y_3 &\geq 0
\end{aligned}$$

(b)Standard form:

maximize $Z = -x_1' + x_1'' + 2x_2' - 2x_2'' + x_3' - x_3''$

subject to

$$\begin{aligned}
3x_2' - 3x_2'' + x_3' - x_3'' &\leq 120\\
x_1' - x_1'' - x_2' + x_2'' - 4x_3' + 4x_3'' &\leq 80\\
-3x_1' + 3x_1'' + x_2' - x_2'' + 2x_3' - 2x_3'' &\leq 100\\
x_1', x_1'', x_2', x_2'', x_3', x_3'' &\geq 0
\end{aligned}$$

Dual: minimize $W = 120y_1 + 80y_2 + 100y_3$

subject to

$$\begin{aligned}
y_2 - 3y_3 &\geq -1\\
-y_2 + 3y_3 &\geq 1\\
3y_1 - y_2 + y_3 &\geq 2\\
-3y_1 + y_2 - y_3 &\geq -2\\
y_1 - 4y_2 + 2y_3 &\geq 1\\
-y_1 + 4y_2 - 2y_3 &\geq -1\\
y_1, y_2, y_3 &\geq 0
\end{aligned}$$

minimize $W = 120y_1 + 80y_2 + 100y_3$

subject to

$$\begin{aligned}
y_2 - 3y_3 &= -1\\
3y_1 - y_2 + y_3 &= 2\\
y_1 - 4y_2 + 2y_3 &= 1\\
y_1, y_2, y_3 &\geq 0
\end{aligned}$$

6.4-9.

The dual problem for the Wyndor Glass Co. example:

$$\begin{array}{ll} \text{maximize} & -W = -4y_1 - 12y_2 - 18y_3 \\ \text{subject to} & \begin{array}{rcrcrl} -y_1 & & & - & 3y_3 & \leq -3 \\ & - & 2y_2 & - & 2y_3 & \leq -5 \end{array} \\ & \quad y_1, y_2, y_3 \geq 0 \end{array}$$

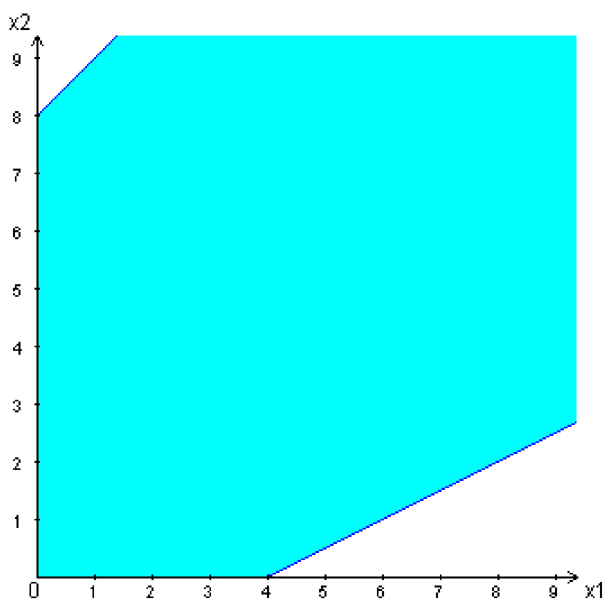
The dual of the dual:

$$\begin{array}{ll} \text{minimize} & -Z = -3x_1 - 5x_2 \\ \text{subject to} & \begin{array}{rcrl} -x_1 & & & \geq -4 \\ & & -2x_2 & \geq -12 \\ -3x_1 & - & 2x_2 & \geq -18 \end{array} \\ & \quad x_1, x_2 \geq 0 \end{array}$$

$$\Leftrightarrow \quad \begin{array}{ll} \text{maximize} & Z = 3x_1 + 5x_2 \\ \text{subject to} & \begin{array}{rcrl} x_1 & & & \leq 4 \\ & & x_2 & \leq 12 \\ 3x_1 & + & 2x_2 & \leq 18 \end{array} \\ & \quad x_1, x_2 \geq 0 \end{array}$$

6.4-10.

(a) The objective is unbounded below.

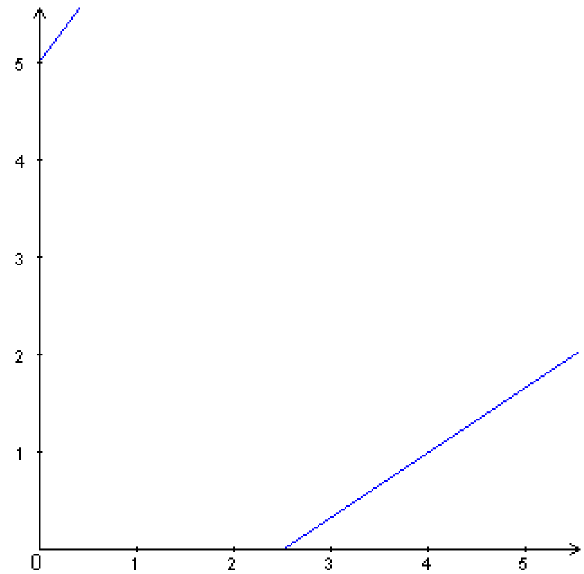


(b) $$\begin{array}{ll} \text{maximize} & 8y_1 + 24y_2 \\ \text{subject to} & \begin{array}{rcrl} 2y_1 & - & 3y_2 & \leq -5 \\ -4y_1 & + & 3y_2 & \leq -15 \end{array} \\ & \quad y_1, y_2 \leq 0 \end{array}$$

Equivalently:

$$\begin{array}{ll} \text{minimize} & 8y_1 + 24y_2 \\ \text{subject to} & 2y_1 - \ \ 3y_2 \geq 5 \\ & -4y_1 + \ \ 3y_2 \geq 15 \\ & y_1, y_2 \geq 20 \end{array}$$

(c) The dual has no feasible solution.



6.5-1.

(a) Since x_1 was nonbasic, changing its coefficients does not affect feasibility. To check optimality, we need to check dual feasibility. The first dual constraint becomes

$$0y_1 + 5y_2 \geq -2,$$

which is always true, since $y_2 \geq 0$. Hence the current basic solution remains optimal.

(b) Adding a new variable does not affect primal feasibility, simply let $x_6 = 0$. To check optimality, check dual feasibility. The constraint that corresponds to x_6 in the dual is

$$3y_1 + 5y_2 \geq 10,$$

assuming $x_6 \geq 0$. $(y_1, y_2) = (5, 0)$ satisfies this constraint, so the current basic solution with $x_6 = 0$ is optimal.

6.5-2.

(a) Since x_3 is nonbasic, the primal solution is still feasible. The dual constraint associated with x_3, $3y_1 - 2y_2 \geq -2$ is violated by $(y_1, y_2) = (0, 2)$, so the current basic solution is not optimal.

(b) Letting $x_6 = 0$, primal feasibility still holds. The dual constraint associated with this variable, $y_1 + 2y_2 \geq 3$ is satisfied by $(y_1, y_2) = (0, 2)$, so the current basic solution remains optimal.

6.5-3.

Since x_3 was nonbasic, changing its coefficients does not affect primal feasibility. To see whether the solution remains optimal, check if the complementary basic solution remains feasible for the dual problem. The third dual constraint becomes

$$3y_1 + 2y_2 + y_3 \geq 4,$$

which is satisfied by $(y_1, y_2, y_3) = (1, 1, 0)$, so the current basic solution remains optimal.

6.6-1.

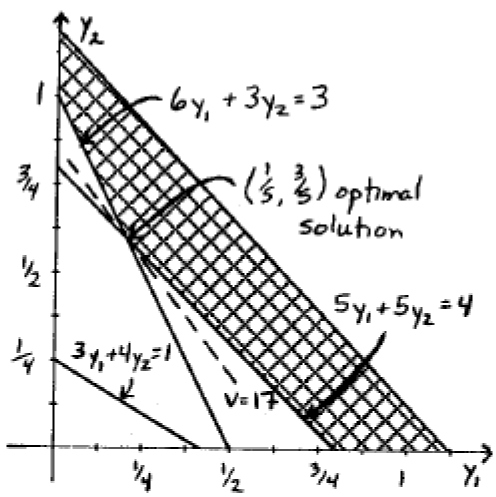
(a) $(x_1, x_2, x_3) = (5/3, 0, 3)$, $Z = 17$

(b) minimize $W = 25y_1 + 20y_2$

subject to

$$\begin{aligned} 6y_1 + \ 3y_2 &\geq 3 \\ 3y_1 + \ 4y_2 &\geq 1 \\ 5y_1 + \ 5y_2 &\geq 4 \\ y_1, y_2 &\geq 0 \end{aligned}$$

(c) Optimal Solution: $(y_1, y_2) = (1/5, 3/5)$, $W = 17$



(d) Since the new dual constraint $2y_1 + 3y_2 \geq 3$ is violated by $(y_1, y_2) = (1/5, 3/5)$, the current solution is no longer optimal.

(e) New x_2 column:

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{4}{5} \end{pmatrix}$$

(f) The new primal variable adds a constraint to the dual, $3y_1 + 2y_2 \geq 2$, which is not satisfied by $(y_1, y_2) = (1/5, 3/5)$, so the current solution is no longer optimal.

(g) $\overline{c}_{\text{new}} = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} - 2 = -\frac{1}{5}$, new column: $\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{5} \end{pmatrix}$

6.6-2.

(a) $\Delta b_1 = -15$, $\Delta b_2 = 0$

$$\Rightarrow \Delta Z^* = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -15 \\ 0 \end{pmatrix} = -3$$

$$\Delta b_1^* = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} -15 \\ 0 \end{pmatrix} = -5$$

$$\Delta b_2^* = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} -15 \\ 0 \end{pmatrix} = 3$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	2	0	0.2	0.6	14
X1	1	0	1	-0.33	0	0.333	-0.33	-3.33
X3	2	0	0	1	1	-0.2	0.4	6

The current basic solution $(-10/3, 0, 6, 0, 0)$ is infeasible and superoptimal.

(b) $\Delta b_1 = 0$, $\Delta b_2 = -10$

$$\Rightarrow \Delta Z^* = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 0 \\ -10 \end{pmatrix} = -6$$

$$\Delta b_1^* = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ -10 \end{pmatrix} = 10/3$$

$$\Delta b_2^* = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 0 \\ -10 \end{pmatrix} = -4$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	2	0	0.2	0.6	11
X1	1	0	1	-0.33	0	0.333	-0.33	5
X3	2	0	0	1	1	-0.2	0.4	-1

The current basic solution $(5, 0, -1, 0, 0)$ is infeasible and superoptimal.

(c) $\Delta c_2 = 2 \Rightarrow \Delta(z_2^* - c_2) = -2$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	0	0.2	0.6	17
X1	1	0	1	-0.33	0	0.333	-0.33	1.667
X3	2	0	0	1	1	-0.2	0.4	3

The current basic solution $(5/3, 0, 3, 0, 0)$ stays optimal.

(d) $\Delta c_3 = -2 \Rightarrow \Delta(z_3^* - c_3) = 2$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	2	2	0.2	0.6	17
X1	1	0	1	-0.33	0	0.333	-0.33	1.667
X3	2	0	0	1	1	-0.2	0.4	3

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	0	0.6	-0.2	11
X1	1	0	1	-0.33	0	0.333	-0.33	1.667
X3	2	0	0	1	1	-0.2	0.4	3

The current basic solution $(5/3, 0, 3, 0, 0)$ stays optimal.

(e) $\Delta a_{12} = 0$, $\Delta a_{22} = -2$

$$\Rightarrow \Delta(z_2^* - c_2) = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = -\frac{6}{5}$$

$$\Delta a_{12}^* = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \frac{2}{3}$$

$$\Delta a_{22}^* = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = -\frac{4}{5}$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0.8	0	0.2	0.6	17
X1	1	0	1	0.333	0	0.333	-0.33	1.667
X3	2	0	0	0.2	1	-0.2	0.4	3

The current basic solution $(5/3, 0, 3, 0, 0)$ is feasible and optimal.

(f) $\Delta a_{11} = 2$, $\Delta a_{21} = 0$

$$\Rightarrow \Delta(z_1^* - c_1) = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \frac{2}{5}$$

$$\Delta a_{11}^* = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \frac{2}{3}$$

$$\Delta a_{21}^* = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = -\frac{2}{5}$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0.4	2	0	0.2	0.6	17
X1	1	0	1.667	-0.33	0	0.333	-0.33	1.667
X3	2	0	-0.4	1	1	-0.2	0.4	3

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	2.08	0	0.12	0.68	16.6
X1	1	0	1	-0.2	0	0.2	-0.2	1
X3	2	0	0	0.92	1	-0.12	0.32	3.4

The current basic solution $(0.71, 0, 3.57, 0, 0)$ is feasible and optimal.

6.6-3.

(a) $\Delta b_1 = -2$, $\Delta b_2 = 1$

$$\Rightarrow \Delta Z^* = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -1$$

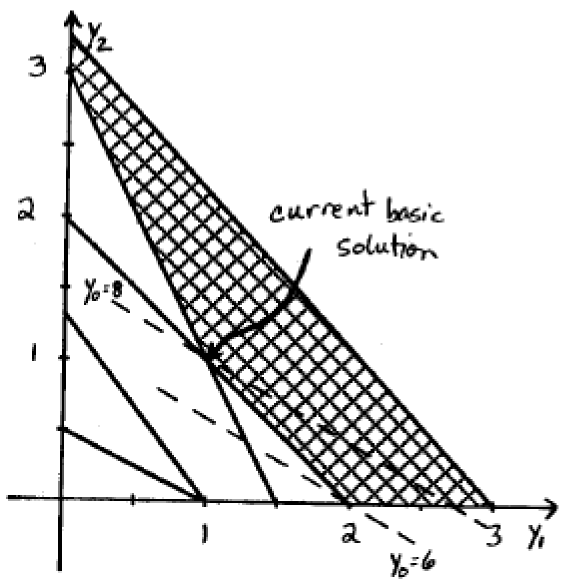
$$\Delta b_1^* = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -3$$

$$\Delta b_2^* = (-1 \quad 2)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 4$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	3	0	2	0	1	1	8
X2	1	0	1	1	-1	0	1	-1	-2
X4	2	0	2	0	3	1	-1	2	7

From the tableau, we see that the primal basic solution is feasible, but not optimal.



From the graph, we can see the current basic solution is feasible, but not optimal.

(b) $\Delta c_1 = -1 \Rightarrow \Delta(z_1^* - c_1) = 1$

$\Delta c_2 = 2 \Rightarrow \Delta(z_2^* - c_2) = -2$

$\Delta c_3 = 1 \Rightarrow \Delta(z_3^* - c_3) = -1$

$\Delta c_4 = 1 \Rightarrow \Delta(z_4^* - c_4) = -1$

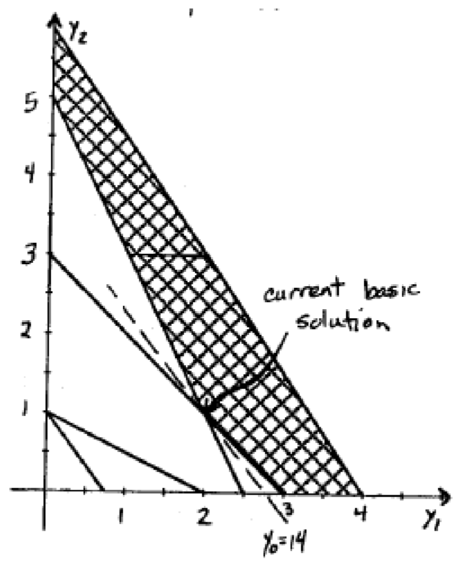
New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	4	-2	1	-1	1	1	9
X2	1	0	1	1	-1	0	1	-1	1
X4	2	0	2	0	3	1	-1	2	3

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	8	0	2	0	2	1	14
X2	1	0	1	1	-1	0	1	-1	1
X4	2	0	2	0	3	1	-1	2	3

The primal basic solution is both feasible and optimal.



From the graph, we see that the current basic solution is feasible and optimal.

(c) $\Delta a_{11} = -2, \Delta a_{21} = 1$

$$\Delta c_1 = 3 \Rightarrow \Delta(z_1^* - c_1) = -3 + (\,1 \quad 1\,)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -4$$

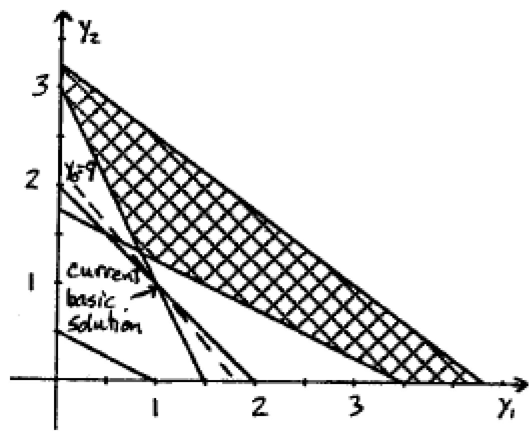
$$\Delta a_{11}^* = (\,1 \quad -1\,)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -3$$

$$\Delta a_{21}^* = (\,-1 \quad 2\,)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 4$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-1	0	2	0	1	1	9
X2	1	0	-2	1	-1	0	1	-1	1
X4	2	0	6	0	3	1	-1	2	3

The primal basic solution is infeasible, but satisfies the optimality criterion.



From the graph, the current basic solution is infeasible and superoptimal.

(d) $\Delta a_{12} = 3$, $\Delta a_{22} = 1$

$$\Delta c_2 = 7 \Rightarrow \Delta(z_2^* - c_2) = -7 + (\,1 \quad 1\,)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = -3$$

$$\Delta a_{12}^* = (\,1 \quad -1\,)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2$$

$$\Delta a_{22}^* = (\,-1 \quad 2\,)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = -1$$

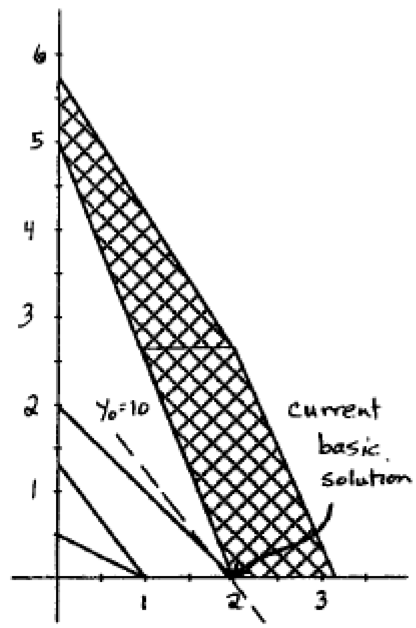
New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	3	-3	2	0	1	1	9
X2	1	0	1	3	-1	0	1	-1	1
X4	2	0	2	-1	3	1	-1	2	3

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	4	0	1	0	2	0	10
X2	1	0	0.333	1	-0.33	0	0.333	-0.33	0.333
X4	2	0	2.333	0	2.667	1	-0.67	1.667	3.333

The primal basic solution is feasible and optimal.



From the graph, the current basic solution is feasible and optimal.

6.7-1.

The model Ep(x) is developed to identify a long-term management plan that satisfies the legal requirements and optimizes PALCO's operations and profitability. The model consists of a linear program with the objective of maximizing present net worth subject to harvest-flow constraints, political and environmental constraints. Detailed sensitivity analysis is performed to "determine the optimal mix of habitat types within each of individual watersheds" [p. 93]. Many instances of the LP problem are run with varying parameters.

The financial benefits of this study include an increase of over $398 million in present net worth and of over $29 million in average yearly net revenues. Sustained-yield annual-harvest levels have increased. The habitat mix is improved in accordance with political and environmental regulations. A more profitable long-term plan paved the way for improved short- and mid-term plans. Sensitivity analysis enabled PALCO to improve its knowledge base of the ecosystem and to adjust its plans quickly when a change in costs or in regulations occurs. Since its decisions are now justified through a systematic approach, PALCO is able to obtain better terms from banks. The study did not only affect PALCO and the habitat controlled by PALCO. It has also "shown that the forest product industries can coexist with wildlife and contribute to their habitats" [p. 104] and "increased quality of life for future generations" [p. 105].

6.7-2.

(a) $\Delta b_1 = 10$, $\Delta b_2 = 0$

$$\Rightarrow \Delta Z^* = (5 \quad 0)\begin{pmatrix} 10 \\ 0 \end{pmatrix} = 50$$

$$\Delta b_1^* = (1 \quad 0)\begin{pmatrix} 10 \\ 0 \end{pmatrix} = 10$$

$$\Delta b_2^* = (-4 \quad 1)\begin{pmatrix} 10 \\ 0 \end{pmatrix} = -40$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	150
X2	1	0	-1	1	3	1	0	30
X5	2	0	16	0	-2	-4	1	-30

The current basic solution is infeasible and superoptimal.

(b) $\Delta b_1 = 0$, $\Delta b_2 = -20$

$$\Rightarrow \Delta Z^* = (5 \quad 0)\begin{pmatrix} 0 \\ -20 \end{pmatrix} = 0$$

$$\Delta b_1^* = (1 \quad 0)\begin{pmatrix} 0 \\ -20 \end{pmatrix} = 0$$

$$\Delta b_2^* = (-4 \quad 1)\begin{pmatrix} 0 \\ -20 \end{pmatrix} = -20$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	100
X2	1	0	-1	1	3	1	0	20
X5	2	0	16	0	-2	-4	1	-10

The current basic solution is infeasible and superoptimal.

(c) $\Delta b_1 = -10$, $\Delta b_2 = 10$

$$\Rightarrow \Delta Z^* = (5 \quad 0)\begin{pmatrix} -10 \\ 10 \end{pmatrix} = -50$$

$$\Delta b_1^* = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -10 \\ 10 \end{pmatrix} = -10$$

$$\Delta b_2^* = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} -10 \\ 10 \end{pmatrix} = 50$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	50
X2	1	0	-1	1	3	1	0	10
X5	2	0	16	0	-2	-4	1	60

The current basic solution is feasible and optimal.

(d) $\Delta c_3 = -5 \Rightarrow \Delta(z_3^* - c_3) = 5$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	7	5	0	100
X2	1	0	-1	1	3	1	0	20
X5	2	0	16	0	-2	-4	1	10

The current basic solution is feasible and optimal.

(e) $\Delta a_{11} = 1, \Delta a_{21} = -7$

$$\Delta c_1 = 3 \Rightarrow \Delta(z_1^* - c_1) = -3 + \begin{pmatrix} 5 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -7 \end{pmatrix} = 2$$

$$\Delta a_{11}^* = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -7 \end{pmatrix} = 1$$

$$\Delta a_{21}^* = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -7 \end{pmatrix} = -11$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	2	0	2	5	0	100
X2	1	0	0	1	3	1	0	20
X5	2	0	5	0	-2	-4	1	10

The current basic solution is feasible and optimal.

(f) $\Delta a_{12} = 1$, $\Delta a_{22} = 1$

$$\Delta c_2 = 1 \Rightarrow \Delta(z_2^* - c_2) = -1 + \begin{pmatrix} 5 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4$$

$$\Delta a_{12}^* = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$$

$$\Delta a_{22}^* = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -3$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	4	2	5	0	100
X2	1	0	-1	2	3	1	0	20
X5	2	0	16	-3	-2	-4	1	10

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	2	0	-4	3	0	60
X2	1	0	-0.5	1	1.5	0.5	0	10
X5	2	0	14.5	0	2.5	-2.5	1	40

The current basic solution is feasible, but not optimal.

(g) $\Delta a_{16} = 3$, $\Delta a_{26} = 5$

$$\Delta c_6 = 10 \Rightarrow \Delta(z_6^* - c_6) = -10 + \begin{pmatrix} 5 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 5$$

$$\Delta a_{16}^* = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3$$

$$\Delta a_{26}^* = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = -7$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
z	0	1	0	0	2	5	0	5	100
X2	1	0	-1	1	3	1	0	3	20
X5	2	0	16	0	-2	-4	1	-7	10

The current basic solution is feasible and optimal.

(h) New Tableau and Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	2	5	0	0	100
X2	1	0	-1	1	3	1	0	0	20
X5	2	0	16	0	-2	-4	1	0	10
X6	3	0	2	3	5	0	0	1	50

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	2	5	0	0	100
X2	1	0	-1	1	3	1	0	0	20
X5	2	0	16	0	-2	-4	1	0	10
X6	3	0	5	0	-4	-3	0	1	-10

The current basic solution is infeasible and superoptimal.

(i) $\Delta a_{11} = 0$, $\Delta a_{21} = -2$

$$\Delta c_1 = 0 \Rightarrow \Delta(z_1^* - c_1) = 0 + (\,5 \quad 0\,)\begin{pmatrix} 0 \\ -2 \end{pmatrix} = 0$$

$$\Delta a_{11}^* = (\,1 \quad 0\,)\begin{pmatrix} 0 \\ -2 \end{pmatrix} = 0$$

$$\Delta a_{21}^* = (\,-4 \quad 1\,)\begin{pmatrix} 0 \\ -2 \end{pmatrix} = -2$$

$\Delta a_{12} = 0$, $\Delta a_{22} = 1$

$$\Delta c_2 = 0 \Rightarrow \Delta(z_2^* - c_2) = 0 + (\,5 \quad 0\,)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Delta a_{12}^* = (\,1 \quad 0\,)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Delta a_{22}^* = (\,-4 \quad 1\,)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

$\Delta b_1 = 0$, $\Delta b_2 = 10$

$$\Rightarrow \Delta Z^* = (\,5 \quad 0\,)\begin{pmatrix} 0 \\ 10 \end{pmatrix} = 0$$

$$\Delta b_1^* = (\,1 \quad 0\,)\begin{pmatrix} 0 \\ 10 \end{pmatrix} = 0$$

$$\Delta b_2^* = (\,-4 \quad 1\,)\begin{pmatrix} 0 \\ 10 \end{pmatrix} = 10$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	100
X2	1	0	-1	1	3	1	0	20
X5	2	0	14	1	-2	-4	1	20

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	100
X2	1	0	-1	1	3	1	0	20
X5	2	0	15	0	-5	-5	1	0

6.7-3.

$$\Delta b_1 = 2\theta, \Delta b_2 = -\theta$$

$$\Rightarrow \Delta Z^* = \begin{pmatrix} 5 & 0 \end{pmatrix} \begin{pmatrix} 2\theta \\ -\theta \end{pmatrix} = 10\theta$$

$$\Delta b_1^* = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2\theta \\ -\theta \end{pmatrix} = 2\theta$$

$$\Delta b_2^* = \begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} 2\theta \\ -\theta \end{pmatrix} = -9\theta$$

$$\Rightarrow Z = 100 + 10\theta$$

$$b_1^* \geq 0 \Leftrightarrow 20 + 2\theta \geq 0$$

$$b_2^* \geq 0 \Leftrightarrow 10 - 9\theta \geq 0$$

$$\Leftrightarrow -10 \leq \theta \leq 10/9$$

6.7-4.

Original Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	1	0	2	20
X4	1	0	0	-1	5	1	-1	20
X1	2	0	1	4	-1	0	1	10

(a) $\Delta b_1 = -10$, $\Delta b_2 = 20$

$$\Rightarrow \Delta Z^* = (0 \quad 2)\begin{pmatrix}-10\\20\end{pmatrix} = 40$$

$$\Delta b_1^* = (1 \quad -1)\begin{pmatrix}-10\\20\end{pmatrix} = -30$$

$$\Delta b_2^* = (0 \quad 1)\begin{pmatrix}-10\\20\end{pmatrix} = 20$$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	1	0	2	60
X4	1	0	0	-1	5	1	-1	-10
X1	2	0	1	4	-1	0	1	30

The current basic solution is superoptimal, but infeasible.

Revised Final Tableau After Reoptimization (Dual Simplex Method):

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0.333	0	12.33	2.333	0	46.67
X2	1	0	0.333	1	1.333	0.333	0	6.667
X5	2	0	-0.33	0	-6.33	-1.33	1	3.333

(b) $\Delta a_{13} = -1$, $\Delta a_{23} = -1$

$$\Delta c_3 = 1 \Rightarrow \Delta(z_3^* - c_3) = -1 + (0 \quad 2)\begin{pmatrix}-1\\-1\end{pmatrix} = -3$$

$$\Delta a_{13}^* = (1 \quad -1)\begin{pmatrix}-1\\-1\end{pmatrix} = 0$$

$$\Delta a_{23}^* = (0 \quad 1)\begin{pmatrix}-1\\-1\end{pmatrix} = -1$$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	-2	0	2	20
X4	1	0	0	-1	5	1	-1	20
X1	2	0	1	4	-2	0	1	10

The current basic solution is feasible, but not optimal.

Revised Final Tableau After Reoptimization (Simplex Method):

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0.6	0	0.4	1.6	28
X3	1	0	0	-0.2	1	0.2	-0.2	4
X1	2	0	1	3.6	0	0.4	0.6	18

(c) $\Delta a_{11} = 2$, $\Delta a_{21} = 1$

$$\Delta c_1 = 2 \Rightarrow \Delta(z_2^* - c_2) = -2 + (0 \quad 2)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$$

$$\Delta a_{11}^* = (1 \quad -1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1$$

$$\Delta a_{21}^* = (0 \quad 1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1$$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	1	0	2	20
X4	1	0	1	-1	5	1	-1	20
X1	2	0	2	4	-1	0	1	10

Revised Final Tableau After Converting to Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	1	0	2	20
X4	1	0	0	-3	5.5	1	-1.5	15
X1	2	0	1	2	-0.5	0	0.5	5

The current basic solution is feasible and optimal.

(d) $\Delta a_{16} = 1$, $\Delta a_{26} = 2$

$$\Delta c_6 = -3 \Rightarrow \Delta(z_6^* - c_6) = 3 + (0 \quad 2)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 7$$

$$\Delta a_{16}^* = (1 \quad -1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1$$

$$\Delta a_{26}^{*} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2$$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	1	1	0	2	7	20
X4	1	0	0	-1	5	1	-1	-1	20
X1	2	0	1	4	-1	0	1	2	10

The current basic solution is feasible and optimal.

(e) $\Delta c_1 = -1 \Rightarrow \Delta(z_1^* - c_1) = 1$

$\Delta c_2 = -2 \Rightarrow \Delta(z_2^* - c_2) = 2$

$\Delta c_3 = 1 \Rightarrow \Delta(z_3^* - c_3) = -1$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	1	3	0	0	2	20
X4	1	0	0	-1	5	1	-1	20
X1	2	0	1	4	-1	0	1	10

Revised Final Tableau After Converting to Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	-1	1	0	1	10
X4	1	0	0	-1	5	1	-1	20
X1	2	0	1	4	-1	0	1	10

The current basic solution is feasible, but not optimal.

Revised Final Tableau After Reoptimization (Simplex Method):

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0.25	0	0.75	0	1.25	12.5
X4	1	0	0.25	0	4.75	1	-0.75	22.5
X2	2	0	0.25	1	-0.25	0	0.25	2.5

(f) New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	1	1	0	2	0	20
X4	1	0	0	-1	5	1	-1	0	20
X1	2	0	1	4	-1	0	1	0	10
X6	3	0	3	2	3	0	0	1	25

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	1	1	0	2	0	20
X4	1	0	0	-1	5	1	-1	0	20
X1	2	0	1	4	-1	0	1	0	10
X6	3	0	0	-10	6	0	-3	1	-5

The current basic solution is infeasible and superoptimal.

Tableau After Reoptimization:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	1.6	0	1.7	0.1	19.5
X4	1	0	0	0	4.4	1	-0.7	-0.1	20.5
X2	2	0	0	1	-0.6	0	0.3	-0.1	0.5
X1	3	0	1	0	1.4	0	-0.2	0.4	8

(g) $\Delta a_{22} = -2$, $\Delta a_{23} = 3$

$$\Rightarrow \Delta(z_2^* - c_2) = \begin{pmatrix} 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = -4$$

$$\Delta a_{12}^* = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = 2$$

$$\Delta a_{22}^* = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = -2$$

$$\Rightarrow \Delta(z_3^* - c_3) = \begin{pmatrix} 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 6$$

$$\Delta a_{13}^* = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = -3$$

$$\Delta a_{23}^* = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3$$

$$\Delta b_1 = 0,\, \Delta b_2 = 25$$

$$\Rightarrow \Delta Z^* = \begin{pmatrix} 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 25 \end{pmatrix} = 50$$

$$\Delta b_1^* = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 25 \end{pmatrix} = -25$$

$$\Delta b_2^* = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 25 \end{pmatrix} = 25$$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	-3	7	0	2	70
X4	1	0	0	1	2	1	-1	-5
X1	2	0	1	2	2	0	1	35

The current basic solution is neither feasible nor optimal.

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0.333	0	12.33	2.333	0	70
X2	1	0	0.333	1	1.333	0.333	0	10
X5	2	0	0.333	0	-0.67	-0.67	1	15

6.7-5.

$$\Delta b_1 = 3\theta,\, \Delta b_2 = -\theta$$

$$\Rightarrow \Delta Z^* = \begin{pmatrix} 0 & 2 \end{pmatrix} \begin{pmatrix} 3\theta \\ -\theta \end{pmatrix} = -2\theta$$

$$\Delta b_1^* = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 3\theta \\ -\theta \end{pmatrix} = 4\theta$$

$$\Delta b_2^* = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 3\theta \\ -\theta \end{pmatrix} = -\theta$$

$$Z^*(\theta) = 20 - 2\theta$$

$(x_1, x_2, x_3, x_4, x_5) = (10 - \theta, 0, 0, 20 + 4\theta, 0)$ is feasible if $-5 \le \theta \le 10$.

6.7-6.

Original Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	2	1	1	0	18
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

(a) $\Delta b_1 = -5$, $\Delta b_2 = 1$, $\Delta b_3 = -2$

$$\Rightarrow \Delta Z^* = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ -2 \end{pmatrix} = -4$$

$$\Delta b_1^* = \begin{pmatrix} 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ -2 \end{pmatrix} = -2$$

$$\Delta b_2^* = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ -2 \end{pmatrix} = -1$$

$$\Delta b_3^* = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ -2 \end{pmatrix} = -3$$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	2	1	1	0	14
X2	1	0	0	1	5	1	3	0	22
X6	2	0	0	0	2	0	1	1	6
X1	3	0	1	0	4	1	2	0	18

The current basic solution is feasible and optimal.

(b) $\Delta c_3 = 1 \Rightarrow \Delta(z_3^* - c_3) = -1$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	1	1	1	0	18
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

The current basic solution remains feasible and optimal.

(c) $\Delta c_1 = 3 \Rightarrow \Delta(z_1^* - c_1) = -3$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-1	0	2	1	1	0	18
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

Revised Final Tableau After Converting to Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	6	2	3	0	39
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

The current basic solution is feasible and optimal.

(d) $\Delta a_{13} = 1,\ \Delta a_{23} = 1,\ \Delta a_{33} = 0$

$$\Delta c_3 = 3 \Rightarrow \Delta(z_3^* - c_3) = -3 + \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -1$$

$$\Delta a_{13}^* = \begin{pmatrix} 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 4$$

$$\Delta a_{23}^* = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1$$

$$\Delta a_{33}^* = (1 \quad 2 \quad 0)\begin{pmatrix}1\\1\\0\end{pmatrix} = 3$$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	1	1	1	0	18
X2	1	0	0	1	9	1	3	0	24
X6	2	0	0	0	3	0	1	1	7
X1	3	0	1	0	7	1	2	0	21

The current basic solution remains feasible and optimal.

(e) $\Delta a_{11} = -2,\ \Delta a_{21} = -1,\ \Delta a_{31} = 2$

$$\Delta c_1 = -1 \Rightarrow \Delta(z_1^* - c_1) = 1 + (1 \quad 1 \quad 0)\begin{pmatrix}-2\\-1\\2\end{pmatrix} = -2$$

$$\Delta a_{11}^* = (1 \quad 3 \quad 0)\begin{pmatrix}-2\\-1\\2\end{pmatrix} = -5$$

$$\Delta a_{21}^* = (0 \quad 1 \quad 1)\begin{pmatrix}-2\\-1\\2\end{pmatrix} = 1$$

$$\Delta a_{31}^* = (1 \quad 2 \quad 0)\begin{pmatrix}-2\\-1\\2\end{pmatrix} = -4$$

$\Delta a_{12} = 0,\ \Delta a_{22} = 2,\ \Delta a_{32} = 3$

$$\Delta c_2 = -1 \Rightarrow \Delta(z_2^* - c_2) = 1 + (1 \quad 1 \quad 0)\begin{pmatrix}0\\2\\3\end{pmatrix} = 3$$

$$\Delta a_{12}^* = (1 \quad 3 \quad 0)\begin{pmatrix}0\\2\\3\end{pmatrix} = 6$$

$$\Delta a_{22}^* = (0 \quad 1 \quad 1)\begin{pmatrix}0\\2\\3\end{pmatrix} = 5$$

$$\Delta a_{32}^* = (1 \quad 2 \quad 0)\begin{pmatrix}0\\2\\3\end{pmatrix} = 4$$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-2	3	2	1	1	0	18
X2	1	0	-5	7	5	1	3	0	24
X6	2	0	1	5	2	0	1	1	7
X1	3	0	-3	4	4	1	2	0	21

Revised Final Tableau After Converting to Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	1	1	0	0	15
X2	1	0	0	1	-5	-2	-1	0	-33
X6	2	0	0	0	35	13	8	1	223
X1	3	0	1	0	-8	-3	-2	0	-51

The current basic solution is superoptimal, but infeasible.

Revised Final Tableau After Reoptimization (Dual Simplex Method):

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	4.4	0	0	0.4	0.6	3.6
X1	1	0	1	-0.2	0	0	-0.2	0.2	0.2
X3	2	0	0	2.6	1	0	0.6	0.4	3.4
X4	3	0	0	-7	0	1	-1	-1	8

(f) $\Delta c_1 = 3 \Rightarrow \Delta(z_1^* - c_1) = -3$

$\Delta c_2 = 2 \Rightarrow \Delta(z_2^* - c_2) = -2$

$\Delta c_3 = 2 \Rightarrow \Delta(z_3^* - c_3) = -2$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-3	-2	0	1	1	0	18
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

Revised Final Tableau After Converting to Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	22	6	13	0	129
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

The current basic solution is feasible and optimal.

(g) $\Delta a_{11} = -1,\ \Delta a_{21} = 0,\ \Delta a_{31} = 0$

$$\Rightarrow \Delta(z_1^* - c_1) = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -1$$

$$\Delta a_{11}^* = \begin{pmatrix} 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -1$$

$$\Delta a_{21}^* = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\Delta a_{31}^* = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -1$$

$\Delta a_{12} = 1,\ \Delta a_{22} = 0,\ \Delta a_{32} = 0$

$$\Rightarrow \Delta(z_2^* - c_2) = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$\Delta a_{12}^* = \begin{pmatrix} 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$\Delta a_{22}^* = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\Delta a_{32}^* = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$\Delta a_{13} = 2,\ \Delta a_{23} = 0,\ \Delta a_{33} = 0$

$$\Rightarrow \Delta(z_3^* - c_3) = (1 \quad 1 \quad 0)\begin{pmatrix}2\\0\\0\end{pmatrix} = 2$$

$$\Delta a_{13}^* = (1 \quad 3 \quad 0)\begin{pmatrix}2\\0\\0\end{pmatrix} = 2$$

$$\Delta a_{23}^* = (0 \quad 1 \quad 1)\begin{pmatrix}2\\0\\0\end{pmatrix} = 0$$

$$\Delta a_{33}^* = (1 \quad 2 \quad 0)\begin{pmatrix}2\\0\\0\end{pmatrix} = 2$$

$\Delta b_1 = -3$, $\Delta b_2 = 0$, $\Delta b_3 = 0$

$$\Rightarrow \Delta Z^* = (1 \quad 1 \quad 0)\begin{pmatrix}-3\\0\\0\end{pmatrix} = -3$$

$$\Delta b_1^* = (1 \quad 3 \quad 0)\begin{pmatrix}-3\\0\\0\end{pmatrix} = -3$$

$$\Delta b_2^* = (0 \quad 1 \quad 1)\begin{pmatrix}-3\\0\\0\end{pmatrix} = 0$$

$$\Delta b_3^* = (1 \quad 2 \quad 0)\begin{pmatrix}-3\\0\\0\end{pmatrix} = -3$$

Revised Final Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-1	1	4	1	1	0	15
X2	1	0	-1	2	7	1	3	0	21
X6	2	0	0	0	2	0	1	1	7
X1	3	0	0	1	6	1	2	0	18

Revised Final Tableau After Converting to Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	3	1	0	0	12
X2	1	0	0	1	6	1	2	0	18
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	5	1	1	0	15

The current basic solution is feasible and optimal.

(h)

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	2	1	1	0	0	18
X2	1	0	0	1	5	1	3	0	0	24
X6	2	0	0	0	2	0	1	1	0	7
X1	3	0	1	0	4	1	2	0	0	21
X7	4	0	2	1	3	0	0	0	1	60

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	2	1	1	0	0	18
X2	1	0	0	1	5	1	3	0	0	24
X6	2	0	0	0	2	0	1	1	0	7
X1	3	0	1	0	4	1	2	0	0	21
X7	4	0	0	0	-10	-3	-7	0	1	-6

The current basic solution is infeasible and superoptimal.

Tableau After Reoptimization:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	0.571	0.571	0	0	0.143	17.14
X2	1	0	0	1	0.714	-0.29	0	0	0.429	21.43
X5	2	0	0	0	1.429	0.429	1	0	-0.14	0.857
X1	3	0	1	0	1.143	0.143	0	0	0.286	19.29
X6	4	0	0	0	0.571	-0.43	0	1	0.143	6.143

6.7-7.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B11	Solution F1-DC	50	-200	300	200	1E+30
C11	Solution F2-DC	30	0	400	100	1E+30
D11	Solution F1-W1	30	0	700	1E+30	200
E11	Solution F2-W1	40	0	900	1E+30	100
F11	Solution DC-W1	30	0	200	200	1E+30
G11	Solution DC-W2	50	-100	400	100	1E+30

(a) F2-DC, F2-W1 and DC-W2 have the smallest margins for error (100). The greatest effort in estimating the unit shipping costs should be placed on these lanes.

(b)

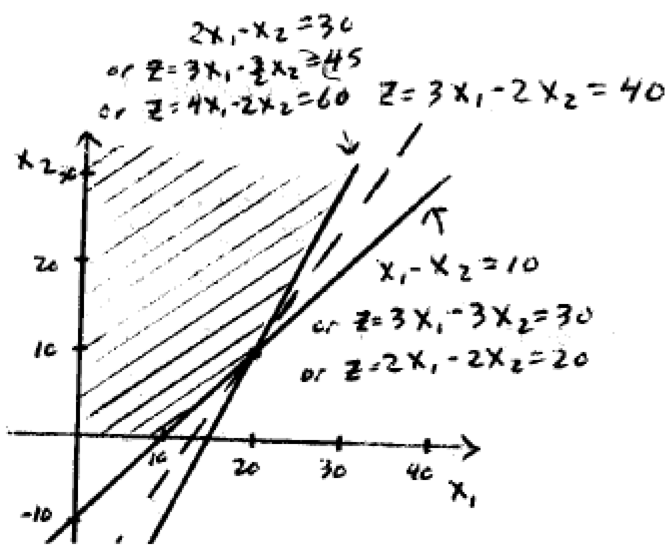
Cost	Allowable Range
$C_{\text{F1-DC}}$	≤ 500
$C_{\text{F2-DC}}$	≤ 500
$C_{\text{F1-W1}}$	≥ 500
$C_{\text{F2-W1}}$	≥ 800
$C_{\text{DC-W1}}$	≤ 400
$C_{\text{DC-W2}}$	≤ 500

(c) The range of optimality for each unit shipping cost indicates how much that shipping cost can change before the optimal shipping quantities change.

(d) Use the 100% rule for simultaneous changes in the objective function coefficients. If the sum of the percentage changes does not exceed 100%, the optimal solution will remain optimal. If it exceeds 100%, then it may or may not be optimal for the new problem.

6.7-8.

(a)



The allowable range for c_1 is $2 \leq c_1 \leq 4$ and the one for c_2 is $-3 \leq c_2 \leq -3/2$.

(b) Increasing c_1 by Δc_1 ($c_1 = 3 + \Delta c_1$)causes the coefficient of x_1 in row 0 of the final tableau to become $-\Delta c_1$. To make it 0, add Δc_1 times row 2 to row 0:

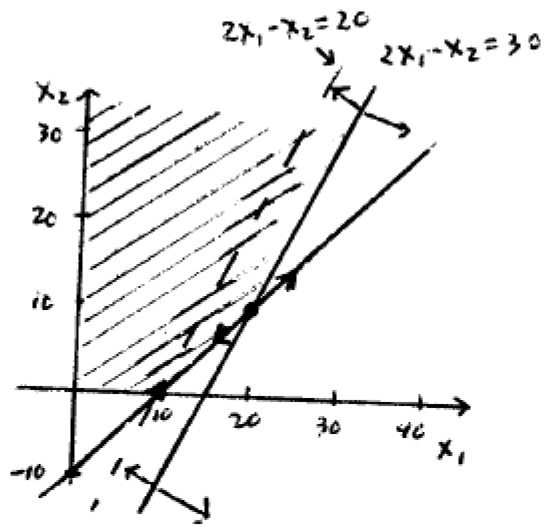
$$(\,-\Delta c_1 \quad 0 \quad 1 \quad 1\,) + \Delta c_1(\,1 \quad 0 \quad 1 \quad -1\,) = (\,0 \quad 0 \quad 1+\Delta c_1 \quad 1-\Delta c_1\,).$$

For optimality, we need $1 + \Delta c_1 \geq 0$ and $1 - \Delta c_1 \geq 0$, so $-1 \leq \Delta c_1 \leq 1$. Hence, the allowable range for c_1 is $3 - 1 = 2 \leq c_1 \leq 3 + 1 = 4$. Similarly, increasing c_2 by Δc_2 ($c_2 = -2 + \Delta c_2$)causes the coefficient of x_2 in row 0 of the final tableau to become $-\Delta c_2$. To make it 0, add Δc_2 times row 1 to row 0:

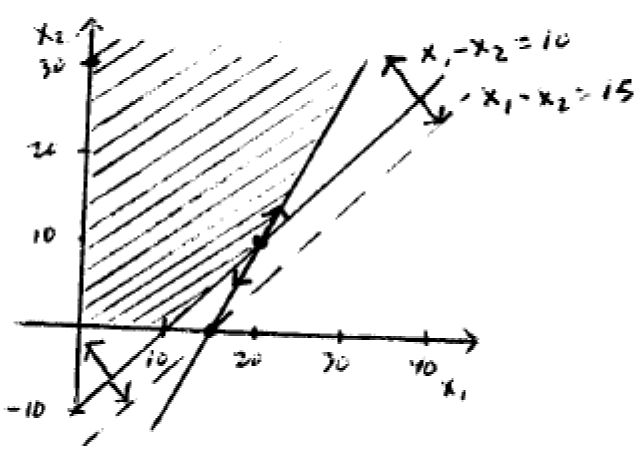
$$(\,0 \quad -\Delta c_2 \quad 1 \quad 1\,) + \Delta c_2(\,0 \quad 1 \quad 1 \quad -2\,) = (\,0 \quad 0 \quad 1+\Delta c_2 \quad 1-2\Delta c_2\,).$$

For optimality, we need $1 + \Delta c_2 \geq 0$ and $1 - 2\Delta c_2 \geq 0$, so $-1 \leq \Delta c_2 \leq 1/2$. Hence, the allowable range for c_2 is $-2 - 1 = -3 \leq c_2 \leq -2 + 1/2 = -3/2$.

(c)



The allowable range for b_1 is $b_1 \geq 20$.



The allowable range for b_2 is $b_2 \leq 15$.

(d) If we increase b_1 by Δb_1, the final right-hand side becomes:

$$S^*\overline{b} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 30 + \Delta b_1 \\ 10 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Delta b_1.$$

In order to preserve feasibility, $\Delta b_1 \geq -10$, so the allowable range for b_1 is $b_1 \geq 20$. Similarly, if b_2 is increased by Δb_2, the final right-hand side becomes:

$$S^*\overline{b} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 30 \\ 10 + \Delta b_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} \Delta b_2.$$

In order to preserve feasibility, $\Delta b_2 \leq 5$, so the allowable range for b_2 is $b_2 \leq 15$.

(e) (in MPL)

```
MAX 3x1-2x2;

SUBJECT TO
2x1-x2<=30;
x1-x2<=10;
```

Variable Name	Coefficient	Lower Range	Upper Range
x1	3.0000	2.0000	4.0000
x2	-2.0000	-3.0000	-1.5000

RANGES RHS

PLAIN CONSTRAINTS

Constraint Name	RHS Value	Lower Bound	Upper Bound
c1	30.0000	20.0000	1E+020
c2	10.0000	-1E+020	15.0000

6.7-9.

If we increase b_i by Δb_i, the final right-hand side becomes:

$$b^* = S^*\overline{b} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} \\ \frac{9}{4} & 1 & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} 4 + \Delta b_1 \\ 24 + \Delta b_2 \\ 18 + \Delta b_3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ \frac{3}{2} \\ \frac{39}{2} \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{3}{4} \\ \frac{9}{4} \end{pmatrix} \Delta b_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Delta b_2 + \begin{pmatrix} 0 \\ \frac{1}{4} \\ -\frac{3}{4} \end{pmatrix} \Delta b_3.$$

Assuming $\Delta b_2 = \Delta b_3 = 0$, Δb_1 must satisfy:

$$4 + \Delta b_1 \geq 0 \Leftrightarrow \Delta b_1 \geq -4$$

$$\tfrac{3}{2} - \tfrac{3}{4} \Delta b_1 \geq 0 \Leftrightarrow \Delta b_1 \leq 2$$

$\frac{39}{2} + \frac{9}{4}\Delta b_1 \geq 0 \Leftrightarrow \Delta b_1 \geq -\frac{78}{9}$

$\Leftrightarrow\ -4 \leq \Delta b_1 \leq 2 \Leftrightarrow 0 \leq b_1 \leq 6.$

Assuming $\Delta b_1 = \Delta b_3 = 0$, Δb_2 must satisfy:

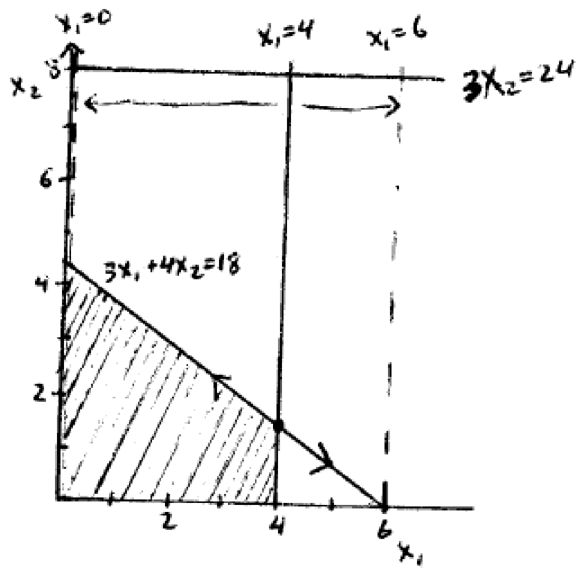
$\frac{39}{2} + \Delta b_2 \geq 0 \Leftrightarrow \Delta b_2 \geq -\frac{39}{2} \Leftrightarrow b_2 \geq \frac{9}{2}.$

Assuming $\Delta b_1 = \Delta b_2 = 0$, Δb_3 must satisfy:

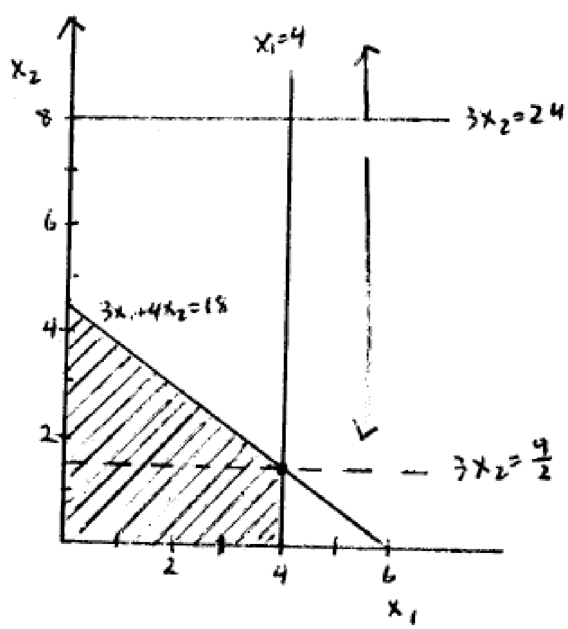
$\frac{3}{2} + \frac{1}{4}\Delta b_3 \geq 0 \Leftrightarrow \Delta b_3 \geq -6$

$\frac{39}{2} - \frac{3}{4}\Delta b_3 \geq 0 \Leftrightarrow \Delta b_3 \leq 26$

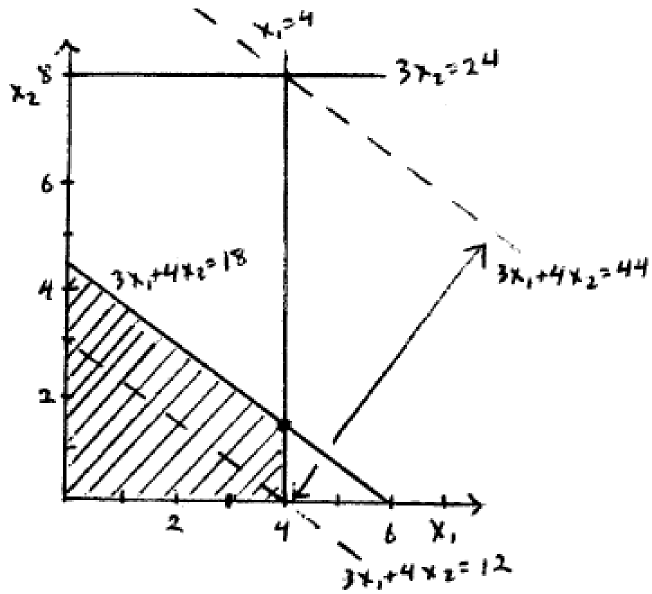
$\Leftrightarrow 12 \leq b_3 \leq 44.$



The allowable range for b_1 is $0 \leq b_1 \leq 6$.



The allowable range for b_2 is $9/2 \leq b_2$.



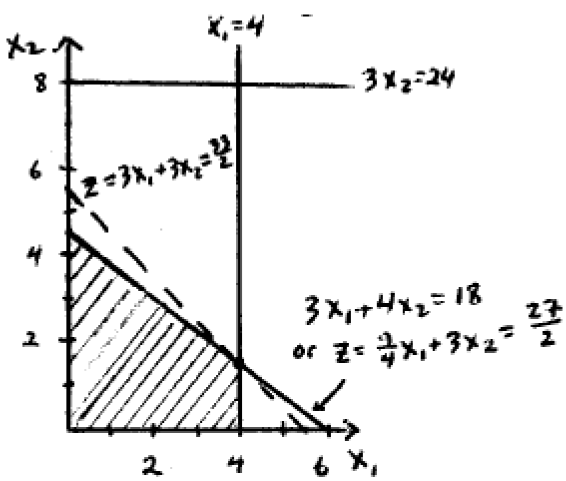
The allowable range for b_3 is $12 \leq b_3 \leq 44$.

6.7-10.

If we increment c_1 by Δc_1 ($c_1 = 3 + \Delta c_1$), the coefficient of x_1 in row 0 of the final tableau becomes $-\Delta c_1$. Add Δc_1 times row 1 to row 0 to get:

$$\begin{pmatrix} -\Delta c_1 & 0 & \frac{3}{4} & 0 & \frac{3}{4} \end{pmatrix} + \Delta c_1 \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{3}{4} + \Delta c_1 & 0 & \frac{3}{4} \end{pmatrix}.$$

For optimality, we need $(3/4) + \Delta c_1 \geq 0$, so $\Delta c_1 \geq -3/4$. Hence, the allowable range for c_1 is $c_1 \geq 9/4$.



The allowable range for c_1 is $c_1 \geq 9/4$. No matter how large c_1 gets, $(4, 3/2)$ stays optimal as long as $c_1 \geq 9/4$.

6.7-11.

If we increment c_2 by Δc_2 ($c_2 = 5 + \Delta c_2$), the coefficient of x_2 in row 0 of the final tableau becomes $-\Delta c_2$. Add Δc_2 times row 2 to row 0 to get:

$$\begin{pmatrix} \frac{9}{2} & -\Delta c_2 & 0 & 0 & \frac{5}{2} \end{pmatrix} + \Delta c_2 \begin{pmatrix} \frac{3}{2} & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{9}{2} + \frac{3}{2}\Delta c_2 & 0 & 0 & 0 & \frac{5}{2} + \frac{1}{2}\Delta c_2 \end{pmatrix}.$$

For optimality, we need $(9/2) + (3/2)\Delta c_2 \geq 0$ and $(5/2) + (1/2)\Delta c_2 \geq 0$, so $\Delta c_2 \geq -3$, so the allowable range for c_2 is $c_2 \geq 2$. Looking at Figure 6.3, we see that if $c_2 = 2$, $Z = 3x_1 + 2x_2 = 18$ lies exactly on the constraint boundary. Thus, if c_2 is decreased any more, $(0, 9)$ does not remain optimal and the optimal solution becomes $(4, 3)$. On the other hand, as c_2 increases, the objective function gets closer to the horizontal line $Z = x_2 = 9$, so for any $c_2 \geq 2$, $(0, 9)$ stays optimal.

6.7-12.

(a) $$b^* = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \Delta b_1 \\ 0 \\ 0 \end{pmatrix} \geq 0 \Leftrightarrow \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Delta b_1 \geq 0$$

$$\Leftrightarrow \Delta b_1 \geq -2 \Leftrightarrow b_1 \geq 2$$

$$b^* = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ \Delta b_2 \\ 0 \end{pmatrix} \geq 0 \Leftrightarrow \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{3} \end{pmatrix} \Delta b_2 \geq 0$$

$$\Leftrightarrow -6 \leq \Delta b_2 \leq 6 \Leftrightarrow 6 \leq b_2 \leq 18$$

$$b^* = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \Delta b_3 \end{pmatrix} \geq 0 \Leftrightarrow \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix} \Delta b_3 \geq 0$$

$$\Leftrightarrow -6 \leq \Delta b_3 \leq 6 \Leftrightarrow 12 \leq b_3 \leq 24$$

(b) $(\text{Row } 0) + \Delta c_1 (\text{Row } 3) \geq 0 \Leftrightarrow \frac{3}{2} - \frac{1}{3}\Delta c_1 \geq 0$ and $1 + \frac{1}{3}\Delta c_1 \geq 0$

$$\Leftrightarrow -3 \leq \Delta c_1 \leq \frac{9}{2} \Leftrightarrow 0 \leq c_1 \leq \frac{15}{2}$$

$$(\text{Row } 0) + \Delta c_2 (\text{Row } 2) \geq 0 \Leftrightarrow \frac{3}{2} + \frac{1}{2}\Delta c_2 \geq 0$$

$$\Leftrightarrow -3 \leq \Delta c_2 \Leftrightarrow 2 \leq c_2$$

(c) (in MPL)

```
MAX 3x1+5x2;

SUBJECT TO
x1<=4;
2x2<=12;
3x1+2x2<=18;
```

PLAIN CONSTRAINTS

Constraint Name	RHS Value	Lower Bound	Upper Bound
c1	4.0000	2.0000	1E+020
c2	12.0000	6.0000	18.0000
c3	18.0000	12.0000	24.0000

PLAIN VARIABLES

Variable Name	Coefficient	Lower Range	Upper Range
x1	3.0000	0.0000	7.5000
x2	5.0000	2.0000	1E+020

6.7-13.

(a) $$b^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 4+\Delta b_1 \\ 24 \\ 18 \\ 24 \end{pmatrix} \geq 0 \Leftrightarrow \begin{pmatrix} 4 \\ 8 \\ 8 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Delta b_1 \geq 0$$

$\Leftrightarrow \Delta b_1 \geq -4 \Leftrightarrow b_1 \geq 0$

$$b^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 24+\Delta b_2 \\ 18 \\ 24 \end{pmatrix} \geq 0 \Leftrightarrow \begin{pmatrix} 4 \\ 8 \\ 8 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Delta b_2 \geq 0$$

$\Leftrightarrow \Delta b_2 \geq -8 \Leftrightarrow b_2 \geq 16$

$$b^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 24 \\ 18+\Delta b_3 \\ 24 \end{pmatrix} \geq 0 \Leftrightarrow \begin{pmatrix} 4 \\ 8 \\ 8 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Delta b_3 \geq 0$$

$\Leftrightarrow \Delta b_3 \geq -2 \Leftrightarrow b_3 \geq 16$

$$b^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 24 \\ 18 \\ 24+\Delta b_4 \end{pmatrix} \geq 0 \Leftrightarrow \begin{pmatrix} 4 \\ 8 \\ 8 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \Delta b_4 \geq 0$$

$\Leftrightarrow \Delta b_4 \geq -24$, $\Delta b_4 \leq 12$, $\Delta b_4 \leq 3 \Leftrightarrow 0 \leq b_4 \leq 27$

(b) Incrementing c_1 by Δc_1, the coefficient of x_1 in row 0 of the final tableau becomes $(1/3) - \Delta c_1$. In order for the solution to remain optimal, $(1/3) - \Delta c_1 \geq 0$, so

$$c_1 \leq 3 + \tfrac{1}{3} = \tfrac{10}{3}.$$

Incrementing c_2 by Δc_2, the coefficient of x_2 in row 0 of the final tableau becomes $-\Delta c_2$. Using row 2 to eliminate this coefficient, we get:

$$\left(\tfrac{1}{3} \quad -\Delta c_2 \quad 0 \quad 0 \quad 0 \quad \tfrac{5}{3}\right) + \Delta c_2\left(\tfrac{2}{3} \quad 1 \quad 0 \quad 0 \quad 0 \quad \tfrac{1}{3}\right)$$
$$= \left(\tfrac{1}{3} + \tfrac{2}{3}\Delta c_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad \tfrac{5}{3} + \tfrac{1}{3}\Delta c_2\right).$$

To keep optimality, we need:

$$\tfrac{1}{3} + \tfrac{2}{3}\Delta c_2 \geq 0 \text{ and } \tfrac{5}{3} + \tfrac{1}{3}\Delta c_2 \geq 0 \Leftrightarrow \Delta c_2 \geq -\tfrac{1}{2} \Leftrightarrow c_2 \geq \tfrac{9}{2}.$$

(c) (in MPL)

```
MAX 3x1+5x2;

SUBJECT TO
x1<=4;
2x2<=12;
3x1+2x2<=18;
2x1+3x2<=24;
```

PLAIN CONSTRAINTS

Constraint Name	RHS Value	Lower Bound	Upper Bound
c1	4.0000	2.0000	1E+020
c2	12.0000	6.0000	14.4000
c3	18.0000	12.0000	21.0000
c4	24.0000	22.0000	1E+020

PLAIN VARIABLES

Variable Name	Coefficient	Lower Range	Upper Range
x1	3.0000	0.0000	7.5000
x2	5.0000	2.0000	1E+020

6.7-14.

$\Delta c_1 = \theta \Rightarrow \Delta(z_1^* - c_1) = -\theta$

$\Delta c_2 = 2\theta \Rightarrow \Delta(z_2^* - c_2) = -2\theta$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
$Z(\theta)$	0	1	$-\theta$	-2θ	$3/4$	0	$3/4$	$33/2$
X1	1	0	1	0	1	0	0	4
X2	2	0	0	1	$-3/4$	0	$1/4$	$3/2$
X4	3	0	0	0	$9/4$	1	$-3/4$	$39/2$

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3			Right side
$Z(\theta)$	0	1	0	0	$3/4 - \theta/2$	0	$3/4 + \theta/2$	$33/2 + 7\theta$
X1	1	0	1	0	1	0	0	4
X2	2	0	0	1	$-3/4$	0	$1/4$	$3/2$
X4	3	0	0	0	$9/4$	1	$-3/4$	$39/2$

The current basic solution is optimal if $\frac{3}{4} - \frac{\theta}{2} \geq 0$ and $\frac{3}{4} + \frac{\theta}{2} \geq 0$, so $-\frac{3}{2} \leq \theta \leq \frac{3}{2}$.

6.7-15.

$\Delta c_1 = \theta \Rightarrow \Delta(z_1^* - c_1) = -\theta$

$\Delta c_2 = -\theta \Rightarrow \Delta(z_2^* - c_2) = \theta$

New Tableau:

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right side
Z	0	1	$-\theta$	θ	2	1	1	0	18
x_2	1	0	0	1	5	1	3	0	24
x_6	2	0	0	0	2	0	1	1	7
x_1	3	0	1	0	4	1	2	0	21

Proper Form:

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right side
Z	0	1	0	0	$2-\theta$	1	$1-\theta$	0	$18-3\theta$
x_2	1	0	0	1	5	1	3	0	24
x_6	2	0	0	0	2	0	1	1	7
x_1	3	0	1	0	4	1	2	0	21

The current basic solution is optimal if $2-\theta \geq 0$ and $1-\theta \geq 0$, so $\theta \leq 1$. Clearly, $Z(\theta) = 18 - 3\theta$ is maximized when θ is as small as possible. Since θ is restricted to be nonnegative, $\theta = 0$ is optimal.

6.7-16.

(a) Row 0 of the final tableau is: $(\,4\theta \quad \theta \quad 3-\theta \quad 2 \quad 2 \quad 24\,)$. Use row 1 and 2 to eliminate x_1 and x_2. We get:

$$(\,4\theta \quad \theta \quad 3-\theta \quad 2 \quad 2 \quad 24\,) - 4\theta(\,1 \quad 0 \quad -1 \quad 1 \quad -1 \quad 2\,) - \theta(\,0 \quad 1 \quad 5 \quad -2 \quad 3 \quad 1\,)$$

$$= (\,0 \quad 0 \quad 3-2\theta \quad 2-2\theta \quad 2+\theta \quad 24-9\theta\,).$$

To preserve optimality, we need: $3 - 2\theta \geq 0 \Leftrightarrow \theta \leq \frac{3}{2}$

$$2 - 2\theta \geq 0 \Leftrightarrow \theta \leq 1$$

$$2 + \theta \geq 0 \Leftrightarrow \theta \geq -2,$$

so the range of values over which the solution stays optimal is $-2 \leq \theta \leq 1$. Since $Z - 9\theta$ is decreasing in θ, the best choice of θ is -2, then $Z = 42$.

(b) $$S^*\bar{b} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 7+\Delta b_1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}\Delta b_1$$

$2 + \Delta b_1 \geq 0$ and $1 - 2\Delta b_1 \geq 0 \Leftrightarrow -2 \leq \Delta b_1 \leq \frac{1}{2} \Leftrightarrow 5 \leq b_1 \leq \frac{15}{2}$

$$S^*\bar{b} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 5+\Delta b_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\Delta b_2$$

$2 - \Delta b_2 \geq 0$ and $1 + 3\Delta b_2 \geq 0 \Leftrightarrow -\frac{1}{3} \leq \Delta b_2 \leq 2 \Leftrightarrow \frac{14}{3} \leq b_2 \leq 7$

(c) From the final row 0 in part (a), we get $y_1^* = 2 - 2\theta$ and $y_2^* = 2 + \theta$. Decreasing the first resource (b_1) by one and increasing the second one (b_2) by one gives us a new objective function value $\overline{Z} = Z - (2 - 2\theta) + (2 + \theta) = Z + \theta$, so the objective function value increases by θ.

(d) Dual: minimize $W(\theta) = 7y_1 + 5y_2$

subject to
$$\begin{aligned} 3y_1 + 2y_2 &\geq 10 - 4\theta \\ y_1 + y_2 &\geq 4 - \theta \\ 2y_1 + 3y_2 &\geq 7 + \theta \\ y_1, y_2 &\geq 0 \end{aligned}$$

Starting Tableau:

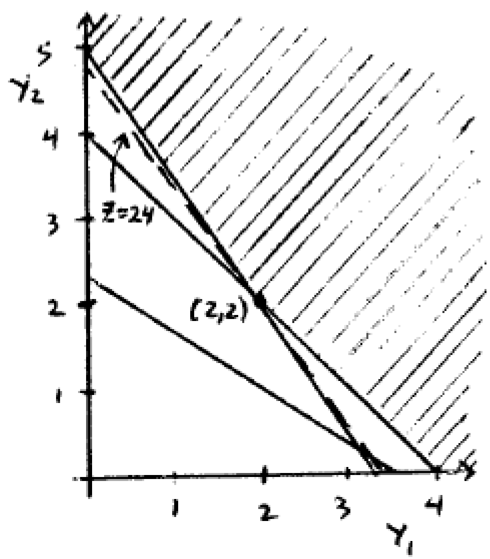
-7	-5	0	0	0	0
-3	-2	1	0	0	$-10 + 4\theta$
-1	-1	0	1	0	$-\ 4 + \theta$
-2	-3	0	0	1	$-\ 7 - \theta$

Force y_1 and y_2 into the basis and y_3 and y_4 out of the basis.

0	0	-2	-1	0	$24 - 9\theta$
0	1	1	-3	0	$2 + \theta$
1	0	-1	2	0	$2 - 2\theta$
0	0	1	-5	1	$3 - 2\theta$

The shadow prices are $(y_1^*, y_2^*) = (2 - 2\theta, 2 + \theta)$ as found in part (c).

Graphically: $(y_1^*, y_2^*) = (2, 2)$ when $\theta = 0$



6.7-17.

(a) $$S^*\bar{b} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5+\theta \\ 6+2\theta \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \theta$$

$(3-\theta, 0, 1+\theta, 0, 0)$ is feasible if $3-\theta \geq 0$ and $1+\theta \geq 0$, so $-1 \leq \theta \leq 3$. The new objective function value is then:

$$Z(\theta) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 5+\theta \\ 6+2\theta \end{pmatrix} = 11 + 3\theta,$$

which is increasing in θ, so the best choice of θ is 3 and $Z = 20$.

(b) Incrementing c_1 by Δc_1 and adding Δc_1 times row 1 to row 0, we get:

$$\begin{aligned} &\begin{pmatrix} -\Delta c_1 & 1 & 0 & 1 & 1 & 11+3\theta \end{pmatrix} + \Delta c_1 \begin{pmatrix} 1 & 5 & 0 & 3 & -2 & 3-\theta \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1+5\Delta c_1 & 0 & 1+3\Delta c_1 & 1-2\Delta c_1 & 11+3\Delta c_1+(3-\Delta c_1)\theta \end{pmatrix}. \end{aligned}$$

To preserve optimality, we need:

$$1 + 5\Delta c_1 \geq 0 \Leftrightarrow \Delta c_1 \geq -\tfrac{1}{5}$$

$$1 + 3\Delta c_1 \geq 0 \Leftrightarrow \Delta c_1 \geq -\tfrac{1}{3}$$

$$1 - 2\Delta c_1 \geq 0 \Leftrightarrow \Delta c_1 \leq \tfrac{1}{2},$$

so $-\frac{1}{5} \leq \Delta c_1 \leq \frac{1}{2}$ and $\frac{9}{5} \leq c_1 \leq \frac{5}{2}$.

6.7-18.

(a) $$\Delta c_1 = -2\theta,\ \Delta a_{11} = 1 \Rightarrow \Delta(z_1^* - c_1) = 2\theta + \begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\theta + 2$$

$$\Delta a_{11}^* = \begin{pmatrix} -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2$$

$$\Delta a_{21}^* = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$\Delta c_2 = \theta \Rightarrow \Delta(z_2^* - c_2) = -\theta$$

$$\Delta b_1 = 10 \Rightarrow \Delta Z^* = \begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 10 \\ 0 \end{pmatrix} = 20$$

$$\Delta b_1^* = \begin{pmatrix} -2 & 3 \end{pmatrix} \begin{pmatrix} 10 \\ 0 \end{pmatrix} = -20$$

$$\Delta b_2^* = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 10 \\ 0 \end{pmatrix} = 10$$

New Tableau:

Bas Var	Eq No	Z	Coefficient of				Right side
			$X1$	$X2$	$X3$	$X4$	
Z	0	1	$2\theta+2$	$-\theta$	2	2	130
x2	1	0	-2	1	-2	3	-5
X1	2	0	2	0	1	-1	15

Proper Form:

Bas Var	Eq No	Z	Coefficient of				Right side
			$X1$	$X2$	$X3$	$X4$	
Z	0	1	0	0	$-3\theta+1$	$4\theta+3$	$-20\theta+115$
X2	1	0	0	1	-1	2	10
X1	2	0	1	0	$1/2$	$-1/2$	$15/2$

For θ near 0, the optimal solution is $(x_1, x_2, x_3, x_4) = (15/2, 10, 0, 0)$ with $Z = -20\theta + 115$.

(b) The solution in (a) remains optimal if $3\theta + 1 \geq 0$ and $4\theta + 3 \geq 0$, so the allowable range for θ is $-3/4 \leq \theta \leq 1/3$.

(c) $Z(\theta) = -20\theta + 115$ attains its largest value when θ is smallest, so $\theta = 0$.

6.7-19.

(a) $\Delta c_1 = 9$, $\Delta a_{11} = 1$, $\Delta a_{21} = 1 \Rightarrow \Delta(z_1^* - c_1) = -9 + \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -6$

$$\Delta a_{11}^* = \begin{pmatrix} 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$$

$$\Delta a_{21}^* = \begin{pmatrix} -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -3$$

New Tableau:

Bas Var	Eq No	Z	Coefficient of					Right side
			$X1$	$X2$	$X3$	$X4$	$X5$	
Z	0	1	-6	2	0	2	1	19
X1	1	0	3	5	0	3	-1	1
X3	2	0	-3	-7	1	-5	2	2

Proper Form:

Bas Var	Eq No	Z	Coefficient of					Right side
			$X1$	$X2$	$X3$	$X4$	$X5$	
Z	0	1	0	12	0	8	-1	21
X1	1	0	1	$5/3$	0	1	$-1/3$	$1/3$
X3	2	0	0	-2	1	-2	1^*	3

Optimal Tableau:

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	0	10	1	6	0	24
x_1	1	0	1	1	1/3	1/3	0	4/3
x_5	2	0	0	-2	1	2	1	3

With the new technology, $(x_1, x_2, x_3, x_4, x_5) = (4/3, 0, 0, 0, 3)$ is optimal with $Z = 24$.

(b) The changes in $z_1^* - c_1$, a_{11}^* and a_{21}^* are θ times the values in part (a).

New Tableau:

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	-6θ	2	0	2	1	19
x_1	1	0	$1+2\theta$	5	0	3	-1	1
x_3	2	0	-3θ	7	1	-5	2	2

Proper Form:

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	0	$\frac{34\theta+2}{2\theta+1}$	0	$\frac{22\theta+2}{2\theta+1}$	$\frac{-4\theta+1}{2\theta+1}$	$\frac{44\theta+19}{2\theta+1}$
x_1	1	0	1	$\frac{5}{2\theta+1}$	0	$\frac{3}{2\theta+1}$	$\frac{-1}{2\theta+1}$	$\frac{1}{2\theta+1}$
x_3	2	0	0	$\frac{29\theta+7}{2\theta+1}$	1	$\frac{19\theta+5}{2\theta+1}$	$\frac{\theta+2}{2\theta+1}$	$\frac{7\theta+2}{2\theta+1}$

Since $2\theta + 1 > 0$ for all choices of $\theta \in [0, 1]$, the right-hand side always remains positive, so the current solution is always feasible for $\theta \in [0, 1]$. For optimality, we need

$$34\theta + 2 \geq 0,\ 22\theta + 2 \geq 0 \text{ and } -4\theta + 1 \geq 0,$$

so $\theta \leq 1/4$. Hence, the current basis is optimal for $\theta \in [0, 1/4]$.

6.7-20.

$$\Delta c_1 = 2\theta \Rightarrow \Delta(z_1^* - c_1) = -2\theta$$

$$\Delta c_2 = \theta \Rightarrow \Delta(z_2^* - c_2) = -\theta$$

$$\Delta c_3 = -\theta \Rightarrow \Delta(z_3^* - c_3) = \theta$$

$$\Delta b_1 = 6\theta,\ \Delta b_2 = -8\theta \Rightarrow \Delta Z^* = \begin{pmatrix} 9 & 7 \end{pmatrix} \begin{pmatrix} 6\theta \\ -8\theta \end{pmatrix} = -2\theta$$

$$\Delta b_1^* = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 6\theta \\ -8\theta \end{pmatrix} = -2\theta$$

$$\Delta b_2^* = \begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 6\theta \\ -8\theta \end{pmatrix} = 2\theta$$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	X_5	Right side
Z	0	1	-2θ	$20-\theta$	θ	9	7	$115-2\theta$
X_1	1	0	1	3	0	1	1	$15-2\theta$
X_3	2	0	0	8	1	3	2	$35+2\theta$

Proper Form:

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	X_5	Right side
Z	0	1	0	$20-3\theta$	0	$9-\theta$	7	$115-7\theta-8\theta^2$
X_1	1	0	1	3	0	1	1	$15-2\theta$
3	2	0	0	8	1	3	2	$35+2\theta$

For $\theta \geq 0$, the current basic solution is feasible if $15-2\theta \geq 0$ and $35+2\theta \geq 0$, so $\theta \leq 15/2$. It is also optimal if $20-3\theta \geq 0$ and $9-\theta \geq 0$, so $\theta \leq 20/3$. Hence, for the current solution to be optimal, we need $\theta \leq 20/3$. For $0 \leq \theta \leq 20/3$,

$$Z(\theta) = 15 - 7\theta - 8\theta^2,$$

which is maximized when $\theta = 0$.

6.7-21.

(a)

$$B^{-1} = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\Rightarrow b = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 4-\theta \\ 12-4\theta \\ 18-3\theta \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{4}{3} \\ 2 \\ -\frac{1}{3} \end{pmatrix} \theta$$

$$Z(\theta) = 3\left(2 + \tfrac{1}{3}\theta\right) + 5(6 - 2\theta) = 36 - 9\theta$$

To keep feasibility: $2 - \frac{4}{3}\theta \geq 0 \Leftrightarrow \theta \leq \frac{3}{2}$

$$6 - 2\theta \geq 0 \Leftrightarrow \theta \leq 3$$

$$2 + \tfrac{1}{3}\theta \geq 0 \Leftrightarrow \theta \geq -6$$

Hence, if $-6 \leq \theta \leq 3/2$, $(x_1^*, x_2^*) = (2 + \theta/3, 6 - 2\theta)$ and $Z^*(\theta) = 36 - 9\theta$.

(b) Since $Z^*(\theta) = 36 - 9\theta$, every unit of change (increase) in the production of the old product results in a change (decrease) in the profit (of the optimal production of the two new products) of 9 (\$9,000 per batch). Thus, θ should be positive if the unit profit of the old product is more than this and negative if less. The break-even point is \$9,000 per batch of the old product.

(c) As shown in part (a), $\theta \leq 3/2$ is needed to keep feasibility, so the production rate of the old product cannot be increased by more than 1.5 units without changing the final basic feasible solution.

(d) From part (a), $\theta \geq -6$, so the production rate of the old product cannot be decreased by more than 6 units without changing the final basic feasible solution.

6.7-22.

$$\Delta c_2 = 4 \Rightarrow \Delta(z_2^* - c_2) = -4$$

$$\Delta c_3 = 1 \Rightarrow \Delta(z_3^* - c_3) = -1$$

$$\Delta b_3 = -1 \Rightarrow \Delta Z^* = \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -1$$

$$\Delta b_1^* = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 1$$

$$\Delta b_2^* = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$\Delta b_3^* = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -1$$

New Tableau:

Bas Var	Eq No	Z	Coefficient of x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	x_7	Right side
Z	0	1	0	1	-1	M+2	0	M	1	7
x_1	1	0	1	-1	0	1	0	0	-1	2
x_5	2	0	0	3	0	1	1	-1	0	2
x_3	3	0	0	2	1	0	0	0	1	1

Proper Form:

Bas Var	Eq No	Z	Coefficient of x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	x_7	Right side
Z	0	1	0	3	0	M+2	0	M	2	8
x_1	1	0	1	-1	0	1	0	0	-1	2
x_5	2	0	0	3	0	1	1	-1	0	2
x_3	3	0	0	2	1	0	0	0	1	1

The current basic solution is feasible and optimal.

6.8-1.

(a)

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$2	$5			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	2	10	≤	10
6	Resource 2	1	3	12	≤	12
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	6	2			$22

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B9	Solution Activity 1	6	0	2	0.5	0.33333
C9	Solution Activity 2	2	0	5	1	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D5	Resource 1 Used	10	1	10	2	2
D6	Resource 2 Used	12	1	12	3	2

(b) The optimal solution is $(0, 4)$ if the unit profit for Activity 1 is $1.

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$1	$5			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	2	8	≤	10
6	Resource 2	1	3	12	≤	12
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	0	4			$20

The optimal solution is $(10, 0)$ if the unit profit for Activity 1 is $3.

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$3	$5			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	2	10	≤	10
6	Resource 2	1	3	10	≤	12
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	10	0			$30

(c) The optimal solution is $(10, 0)$ if the unit profit for Activity 2 is \$2.50.

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	\$2	\$2.50			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	2	10	≤	10
6	Resource 2	1	3	10	≤	12
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	10	0			\$20

The optimal solution is $(0, 4)$ if the unit profit for Activity 2 is \$7.50.

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	\$2	\$7.50			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	2	8	≤	10
6	Resource 2	1	3	12	≤	12
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	0	4			\$30

(d)

	A	B	C	D
11	Unit Profit for	Solution		Total
12	Activity 1	Activity 1	Activity 2	Profit
13		6	2	\$22.00
14	\$1.00	0	4	\$20.00
15	\$1.20	0	4	\$20.00
16	\$1.40	0	4	\$20.00
17	\$1.60	0	4	\$20.00
18	\$1.80	6	2	\$20.80
19	\$2.00	6	2	\$22.00
20	\$2.20	6	2	\$23.20
21	\$2.40	6	2	\$24.40
22	\$2.60	10	0	\$26.00
23	\$2.80	10	0	\$28.00
24	\$3.00	10	0	\$30.00

27	Unit Profit for	Solution		Total
28	Activity 2	Activity 1	Activity 2	Profit
29		6	2	$22.00
30	$2.50	10	0	$20.00
31	$3.00	10	0	$20.00
32	$3.50	10	0	$20.00
33	$4.00	6	2	$20.00
34	$4.50	6	2	$21.00
35	$5.00	6	2	$22.00
36	$5.50	6	2	$23.00
37	$6.00	0	4	$24.00
38	$6.50	0	4	$26.00
39	$7.00	0	4	$28.00
40	$7.50	0	4	$30.00

The allowable range for the unit profit of Activity 1 is approximately between $1.60 and $1.80 up to between $2.40 and $2.60. The allowable range for the unit profit of Activity 2 is between $3.50 and $4 up to between $5.50 and $6.

(e) The allowable range for the unit profit of Activity 1 is approximately between $1.67 and $2.50. The allowable range for the unit profit of Activity 2 is between $4 and $6.

Objective Coefficient

Current Value	Allowable Range to Stay Optimal	
	Minimum	Maximum
2	1.67	2.5
5	4	6

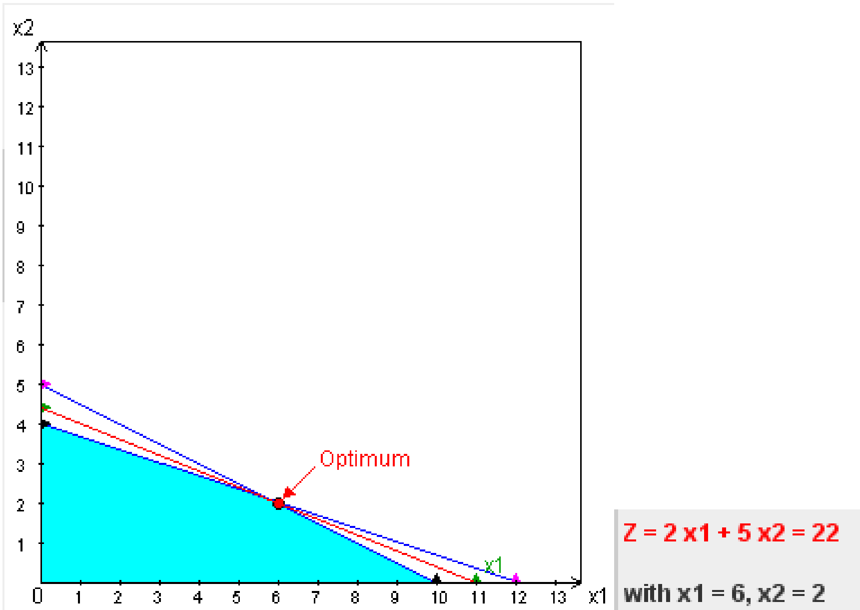
(f) The allowable range for the unit profit of Activity 1 is approximately between $1.67 and $2.50. The allowable range for the unit profit of Activity 2 is between $4 and $6.

(g)

	A	B	C	D	E	F	G	H	I	J	K	L	M
11	Total Profit						Unit Profit for Activity 2						
12		$22	$2.50	$3.00	$3.50	$4.00	$4.50	$5.00	$5.50	$6.00	$6.50	$7.00	$7.50
13		$1.00	$11.00	$12.00	$14.00	$16.00	$18.00	$20.00	$22.00	$24.00	$26.00	$28.00	$30.00
14		$1.20	$12.20	$13.20	$14.20	$16.00	$18.00	$20.00	$22.00	$24.00	$26.00	$28.00	$30.00
15		$1.40	$14.00	$14.40	$15.40	$16.40	$18.00	$20.00	$22.00	$24.00	$26.00	$28.00	$30.00
16	Unit Profit	$1.60	$16.00	$16.00	$16.60	$17.60	$18.60	$20.00	$22.00	$24.00	$26.00	$28.00	$30.00
17	for	$1.80	$18.00	$18.00	$18.00	$18.80	$19.80	$20.80	$22.00	$24.00	$26.00	$28.00	$30.00
18	Activity 1	$2.00	$20.00	$20.00	$20.00	$20.00	$21.00	$22.00	$23.00	$24.00	$26.00	$28.00	$30.00
19		$2.20	$22.00	$22.00	$22.00	$22.00	$22.20	$23.20	$24.20	$25.20	$26.20	$28.00	$30.00
20		$2.40	$24.00	$24.00	$24.00	$24.00	$24.00	$24.40	$25.40	$26.40	$27.40	$28.40	$30.00
21		$2.60	$26.00	$26.00	$26.00	$26.00	$26.00	$26.00	$26.60	$27.60	$28.60	$29.60	$30.60
22		$2.80	$28.00	$28.00	$28.00	$28.00	$28.00	$28.00	$28.00	$28.80	$29.80	$30.80	$31.80
23		$3.00	$30.00	$30.00	$30.00	$30.00	$30.00	$30.00	$30.00	$30.00	$31.00	$32.00	$33.00
24													

25	Solution						Unit Profit for Activity 2						
26		(6,2)	$2.50	$3.00	$3.50	$4.00	$4.50	$5.00	$5.50	$6.00	$6.50	$7.00	$7.50
27		$1.00	(6,2)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)
28		$1.20	(6,2)	(6,2)	(6,2)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)
29		$1.40	(10,0)	(6,2)	(6,2)	(6,2)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)
30	Unit Profit	$1.60	(10,0)	(10,0)	(6,2)	(6,2)	(6,2)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)
31	for	$1.80	(10,0)	(10,0)	(10,0)	(6,2)	(6,2)	(6,2)	(0,4)	(0,4)	(0,4)	(0,4)	(0,4)
32	Activity 1	$2.00	(10,0)	(10,0)	(10,0)	(6,2)	(6,2)	(6,2)	(6,2)	(0,4)	(0,4)	(0,4)	(0,4)
33		$2.20	(10,0)	(10,0)	(10,0)	(10,0)	(6,2)	(6,2)	(6,2)	(6,2)	(6,2)	(0,4)	(0,4)
34		$2.40	(10,0)	(10,0)	(10,0)	(10,0)	(10,0)	(6,2)	(6,2)	(6,2)	(6,2)	(6,2)	(0,4)
35		$2.60	(10,0)	(10,0)	(10,0)	(10,0)	(10,0)	(10,0)	(6,2)	(6,2)	(6,2)	(6,2)	(6,2)
36		$2.80	(10,0)	(10,0)	(10,0)	(10,0)	(10,0)	(10,0)	(10,0)	(6,2)	(6,2)	(6,2)	(6,2)
37		$3.00	(10,0)	(10,0)	(10,0)	(10,0)	(10,0)	(10,0)	(10,0)	(6,2)	(6,2)	(6,2)	(6,2)

(h) Keeping the unit profit of Activity 2 fixed, the unit profit of Activity 1 cannot be changed to less than 1.67 or more than 2.5 without changing the optimal solution. Similarly if the unit profit of Activity 1 is fixed at 1, the unit profit of Activity 2 needs to stay between 4 and 6 so that the optimal solution remains the same. Otherwise, the objective function line becomes either too flat or too steep and the optimal solution becomes $(0, 4)$ or $(10, 0)$.



6.8-2.

(a) The original model:

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$2	$5			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	2	10	≤	10
6	Resource 2	1	3	12	≤	12
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	6	2			$22.00

With one additional unit of resource 1:

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$2	$5			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	2	11	≤	11
6	Resource 2	1	3	12	≤	12
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	9	1			$23.00

The shadow price (the increase in total profit) is $1.

(b) The shadow price of $1 is valid in the range of 8 to 12.

	A	B	C	D	E
12	Available	Solution		Total	Incremental
13	Resource 1	Activity 1	Activity 2	Profit	Profit
14		6	2	$22.00	
15	5	0	2.5	$12.50	
16	6	0	3	$15.00	$2.50
17	7	0	3.5	$17.50	$2.50
18	8	0	4	$20.00	$2.50
19	9	3	3	$21.00	$1.00
20	10	6	2	$22.00	$1.00
21	11	9	1	$23.00	$1.00
22	12	12	0	$24.00	$1.00
23	13	12	0	$24.00	$0.00
24	14	12	0	$24.00	$0.00
25	15	12	0	$24.00	$0.00

(c) With one additional unit of resource 2:

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$2	$5			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	2	10	≤	10
6	Resource 2	1	3	13	≤	13
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	4	3			$23.00

The shadow price (the increase in total profit) is $1.

(d) The shadow price of $1 is valid in the range of 10 to 15.

	A	B	C	D	E
12	Available	Solution		Total	Incremental
13	Resource 2	Activity 1	Activity 2	Profit	Profit
14		6	2	$22.00	
15	6	6	0	$12.00	
16	7	7	0	$14.00	$2.00
17	8	8	0	$16.00	$2.00
18	9	9	0	$18.00	$2.00
19	10	10	0	$20.00	$2.00
20	11	8	1	$21.00	$1.00
21	12	6	2	$22.00	$1.00
22	13	4	3	$23.00	$1.00
23	14	2	4	$24.00	$1.00
24	15	0	5	$25.00	$1.00
25	16	0	5	$25.00	$0.00
26	17	0	5	$25.00	$0.00
27	18	0	5	$25.00	$0.00

(e) From the sensitivity report, the shadow prices for both constraints are $1. According to the allowable increase and decrease, the allowable range for the right-hand side of the first constraint is 8 to 12. Similarly, the allowable range for the right-hand side of the second constraint is 10 to 15.

Adjustable Cells

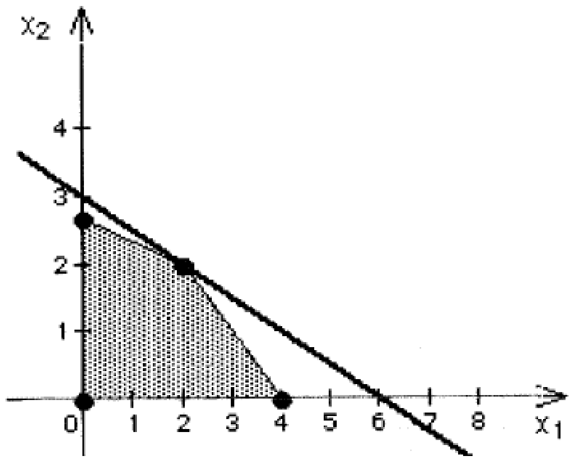
Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B9	Solution Activity 1	6	0	2	0.5	0.333
C9	Solution Activity 2	2	0	5	1	1

Constraints

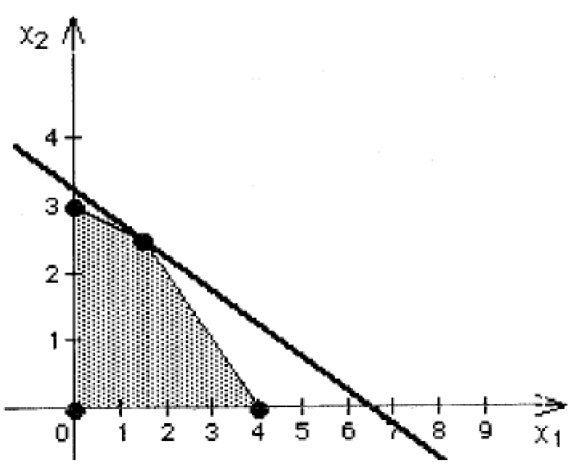
Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D5	Resource 1 Used	10	1	10	2	2
D6	Resource 2 Used	12	1	12	3	2

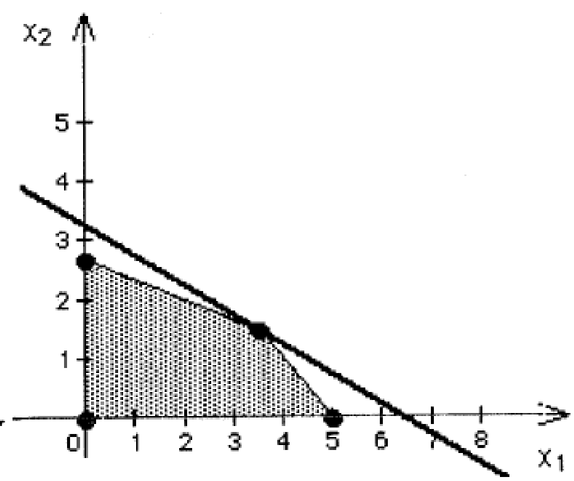
6.8-3.

(a) Optimal Solution: $(x_1, x_2) = (2, 2)$, with profit $6



(b)





(c) The original model:

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$1	$2			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	3	8	≤	8
6	Resource 2	1	1	4	≤	4
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	2	2			$6.00

The shadow price for resource 1 is $0.50.

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$1	$2			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	3	9	≤	9
6	Resource 2	1	1	4	≤	4
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	1.5	2.5			$6.50

The shadow price for resource 2 is $0.50.

	A	B	C	D	E	F
1		Activity 1	Activity 2			
2	Unit Profit	$1	$2			
3						
4		Resource Usage		Used		Available
5	Resource 1	1	3	8	≤	8
6	Resource 2	1	1	5	≤	5
7						
8		Activity 1	Activity 2			Total Profit
9	Solution	3.5	1.5			$6.50

(d) The allowable range for the right-hand side of the resource 1 constraint is approximately between 4 (or less) and 12.

	A	B	C	D	E
12	Available	Solution		Total	Incremental
13	Resource 1	Activity 1	Activity 2	Profit	Profit
14		2	2	$6.00	
15	4	4	0	$4.00	
16	5	3.5	0.5	$4.50	$0.50
17	6	3	1	$5.00	$0.50
18	7	2.5	1.5	$5.50	$0.50
19	8	2	2	$6.00	$0.50
20	9	1.5	2.5	$6.50	$0.50
21	10	1	3	$7.00	$0.50
22	11	0.5	3.5	$7.50	$0.50
23	12	0	4	$8.00	$0.50
24	13	0	4	$8.00	$0.00
25	14	0	4	$8.00	$0.00

The allowable range for the right-hand side of the resource 2 constraint is approximately between 3 and 8.

	A	B	C	D	E
28	Available	Solution		Total	Incremental
29	Resource 2	Activity 1	Activity 2	Profit	Profit
30		2	2	$6.00	
31	0	0	0	$0.00	
32	1	0	1	$2.00	$2.00
33	2	0	2	$4.00	$2.00
34	3	0.5	2.5	$5.50	$1.50
35	4	2	2	$6.00	$0.50
36	5	3.5	1.5	$6.50	$0.50
37	6	5	1	$7.00	$0.50
38	7	6.5	0.5	$7.50	$0.50
39	8	8	0	$8.00	$0.50
40	9	8	0	$8.00	$0.00
41	10	8	0	$8.00	$0.00

(e) The shadow price for both resources is $0.50. The allowable range for the right-hand side of the first resource is between 4 and 12 and that of the second resource is between 2.667 and 8.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B9	Solution Activity 1	2	0	1	1	0.333
C9	Solution Activity 2	2	0	2	1	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D5	Resource 1 Used	8	0.5	8	4	4
D6	Resource 2 Used	4	0.5	4	4	1.333

(f) These shadow prices tell management that for each additional unit of resource, the profit increases by $0.50 (for small changes). Management is then able to evaluate whether or not to change the available amount of resources.

6.8-4.

(a)

	A	B	C	D	E	F
1		Toys	Subassemblies			
2	Unit Profit	$3.00	-$2.50			
3						
4		Resource Usage		Used		Available
5	Subassembly A	2	-1	3,000	≥	3,000
6	Subassembly B	1	-1	1,000	≥	1,000
7						
8		Toys	Subassemblies			Total Profit
9	Production	2,000	1,000			$3,500

(b)

Unit Profit for Toys	Optimal Production Rates		Total
	Toys	Subassemblies	Profit
$2.00	1000	0	$2000
$2.50	1000	0	$2500
$3.00	2000	1000	$3500
$3.50	2000	1000	$4500
$4.00	2000	1000	$5500

The estimate of the unit profit for toys can be off by something between 0 and 0.50 before the optimal solution changes. There is no change in the solution for an increase in the unit profit for toys, at least for an increase up to $1.

(c)

Unit Profit for Subassemblies	Optimal Production Rates		Total
	Toys	Subassemblies	Profit
-$3.50	1000	0	$3000
-$3.00	1000	0	$3000
-$2.50	2000	1000	$3500
-$2.00	2000	1000	$4000
-$1.50	2000	1000	$4500

The estimate of the unit profit for subassemblies can be off by something between 0 and 0.50 before the optimal solution changes. There is no change in the solution for an increase in the unit profit for subassemblies, at least for an increase up to $1.

(d) Solver Table for change in unit profit for toys as in (b):

	A	B	C	D
11	Unit Profit	Production		
12	for Toys	Toys	Subassemblies	Total Profit
13		2,000	1,000	$3,500
14	$2.00	1000	0	$2,000
15	$2.25	1000	0	$2,250
16	$2.50	1000	0	$2,500
17	$2.75	2000	1000	$3,000
18	$3.00	2000	1000	$3,500
19	$3.25	2000	1000	$4,000
20	$3.50	2000	1000	$4,500
21	$3.75	2000	1000	$5,000
22	$4.00	2000	1000	$5,500

Solver Table for change in unit profit for subassemblies as in (c):

	A	B	C	D
11	Unit Profit	Production		
12	for Subassemblies	Toys	Subassemblies	Total Profit
13		2,000	1,000	$3,500
14	-$3.50	1000	0	$3,000
15	-$3.25	1000	0	$3,000
16	-$3.00	1000	0	$3,000
17	-$2.75	2000	1000	$3,250
18	-$2.50	2000	1000	$3,500
19	-$2.25	2000	1000	$3,750
20	-$2.00	2000	1000	$4,000
21	-$1.75	2000	1000	$4,250
22	-$1.50	2000	1000	$4,500

(e) The unit profit for toys can vary between $2.50 and $5 before the solution changes. For subassemblies, the unit profit can change between -$3 and -1.50 before the solution changes.

(f) The allowable range of the unit profit for toys is $2.50 to $5 whereas that for subassemblies is -$3 to -$1.50.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B9	Production Toys	2,000	0	3	2	0.5
C9	Production Subassemblies	1,000	0	-2.5	1	0.5

(g)

	A	B	C	D	E	F	G	H	I	J	K
11	Total Profit					Unit Profit for Subassemblies					
12		$3,500	-$3.50	-$3.25	-$3.00	-$2.75	-$2.50	-$2.25	-$2.00	-$1.75	-$1.50
13		$2.00	$2,000	$2,000	$2,000	$2,000	$2,000	$2,000	$2,000	$2,250	$2,500
14		$2.25	$2,250	$2,250	$2,250	$2,250	$2,250	$2,250	$2,500	$2,750	$3,000
15		$2.50	$2,500	$2,500	$2,500	$2,500	$2,500	$2,750	$3,000	$3,250	$3,500
16	Unit Profit	$2.75	$2,750	$2,750	$2,750	$2,750	$3,000	$3,250	$3,500	$3,750	$4,000
17	for Toys	$3.00	$3,000	$3,000	$3,000	$3,250	$3,500	$3,750	$4,000	$4,250	$4,500
18		$3.25	$3,250	$3,250	$3,500	$3,750	$4,000	$4,250	$4,500	$4,750	$5,000
19		$3.50	$3,500	$3,750	$4,000	$4,250	$4,500	$4,750	$5,000	$5,250	$5,500
20		$3.75	$4,000	$4,250	$4,500	$4,750	$5,000	$5,250	$5,500	$5,750	$6,000
21		$4.00	$4,500	$4,750	$5,000	$5,250	$5,500	$5,750	$6,000	$6,250	$6,500

(h) As long as the sum of the percentage change of the unit profit for subassemblies does not exceed 100% (where the allowable range is given in part (f)), the solution does not change.

6.8-5.

(a)

	A	B	C	D	E	F
1		Toys	Subassemblies			
2	Unit Profit	$3.00	-$2.50			
3						
4		Resource Usage		Used		Available
5	Subassembly A	2	-1	3,000	≥	3,000
6	Subassembly B	1	-1	1,000	≥	1,000
7						
8		Toys	Subassemblies			Total Profit
9	Production	2,000	1,000			$3,500.00
10		≥				
11		2,500				

(b)

	A	B	C	D	E	F
1		Toys	Subassemblies			
2	Unit Profit	$3.00	-$2.50			
3						
4		Resource Usage		Used		Available
5	Subassembly A	2	-1	3,001	≥	3,001
6	Subassembly B	1	-1	1,000	≥	1,000
7						
8		Toys	Subassemblies			Total Profit
9	Production	2,001	1,001			$3,500.50
10		≥				
11		2,500				

The shadow price for subassembly A is $0.50, which is the maximum premium that the company should be willing to pay.

(c)

	A	B	C	D	E	F
1		Toys	Subassemblies			
2	Unit Profit	$3.00	-$2.50			
3						
4		Resource Usage		Used		Available
5	Subassembly A	2	-1	3,000	≥	3,000
6	Subassembly B	1	-1	1,001	≥	1,001
7						
8		Toys	Subassemblies			Total Profit
9	Production	1,999	998			$3,502.00
10		≥				
11		2,500				

The shadow price for subassembly B is $2, which is the maximum premium that the company should be willing to pay.

(d)

	A	B	C	D	E
14	Available	Production		Total	Incremental
15	Subassembly A	Toys	Subassemblies	Profit	Profit
16		2,000	1,000	$3,500.00	
17	3,000	2,000	1,000	$3,500.00	
18	3,100	2,100	1,100	$3,550.00	$50.00
19	3,200	2,200	1,200	$3,600.00	$50.00
20	3,300	2,300	1,300	$3,650.00	$50.00
21	3,400	2,400	1,400	$3,700.00	$50.00
22	3,500	2,500	1,500	$3,750.00	$50.00
23	3,600	2,500	1,500	$3,750.00	$0.00
24	3,700	2,500	1,500	$3,750.00	$0.00
25	3,800	2,500	1,500	$3,750.00	$0.00
26	3,900	2,500	1,500	$3,750.00	$0.00
27	4,000	2,500	1,500	$3,750.00	$0.00

The shadow price is still valid until the maximum supply of subassembly A is at least 3,500.

(e)

	A	B	C	D	E
14	Available	Production		Total	Incremental
15	Subassembly B	Toys	Subassemblies	Profit	Profit
16		2,000	1,000	$3,500.00	
17	1,000	2,000	1,000	$3,500.00	
18	1,100	1,900	800	$3,700.00	$200.00
19	1,200	1,800	600	$3,900.00	$200.00
20	1,300	1,700	400	$4,100.00	$200.00
21	1,400	1,600	200	$4,300.00	$200.00
22	1,500	1,500	0	$4,500.00	$200.00
23	1,600	1,500	0	$4,500.00	$0.00
24	1,700	1,500	0	$4,500.00	$0.00
25	1,800	1,500	0	$4,500.00	$0.00
26	1,900	1,500	0	$4,500.00	$0.00
27	2,000	1,500	0	$4,500.00	$0.00

The shadow price is still valid until the maximum supply of subassembly A is at least 1,500.

(f)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B9	Production Toys	2,000	0	3	2	0.5
C9	Production Subassemblies	1,000	0	-2.5	1	0.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D5	Subassembly A Used	3,000	0.5	3000	500	1000
D6	Subassembly B Used	1,000	2	1000	500	500

As shown in the sensitivity report, the shadow price is $0.50 for subassembly A and $2 for subassembly B. According to the allowable increase and decrease, the allowable range for the right-hand side of the subassembly A constraint is 2,000 to 3,500. The allowable range for the right-hand side of the subassembly B constraint is 500 to 1,500.

6.8-6.

(a) The optimal solution does not change.

(b) The optimal solution changes.

	B	C	D	E	F	G	H	I	J
3		6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am			
4		Shift	Shift	Shift	Shift	Shift			
5	Cost per Shift	$170	$160	$175	$170	$195			
6							Total		Minimum
7	Time Period		Shift Works Time Period? (1=yes, 0=no)				Working		Needed
8	6am-8am	1	0	0	0	0	48	≥	48
9	8am-10am	1	1	0	0	0	79	≥	79
10	10am- 12pm	1	1	0	0	0	79	≥	65
11	12pm-2pm	1	1	1	0	0	112	≥	87
12	2pm-4pm	0	1	1	0	0	64	≥	64
13	4pm-6pm	0	0	1	1	0	82	≥	73
14	6pm-8pm	0	0	1	1	0	82	≥	82
15	8pm-10pm	0	0	0	1	0	49	≥	43
16	10pm-12am	0	0	0	1	1	64	≥	52
17	12am-6am	0	0	0	0	1	15	≥	15
18									
19		6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am			
20		Shift	Shift	Shift	Shift	Shift			Total Cost
21	Number Working	48	31	33	49	15			$30,150

(c) The optimal solution changes.

	B	C	D	E	F	G	H	I	J
3		6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am			
4		Shift	Shift	Shift	Shift	Shift			
5	Cost per Shift	$170	$165	$175	$170	$195			
6							Total		Minimum
7	Time Period		Shift Works Time Period? (1=yes, 0=no)				Working		Needed
8	6am-8am	1	0	0	0	0	48	≥	48
9	8am-10am	1	1	0	0	0	79	≥	79
10	10am- 12pm	1	1	0	0	0	79	≥	65
11	12pm-2pm	1	1	1	0	0	112	≥	87
12	2pm-4pm	0	1	1	0	0	64	≥	64
13	4pm-6pm	0	0	1	1	0	82	≥	73
14	6pm-8pm	0	0	1	1	0	82	≥	82
15	8pm-10pm	0	0	0	1	0	49	≥	43
16	10pm-12am	0	0	0	1	1	64	≥	52
17	12am-6am	0	0	0	0	1	15	≥	15
18									
19		6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am			
20		Shift	Shift	Shift	Shift	Shift			Total Cost
21	Number Working	48	31	33	49	15			$30,305

(d) The optimal solution does not change.

(e) The optimal solution does not change.

(f)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C21	Number Working Shift	48	0	170	1E+30	10
D21	Number Working Shift	31	0	160	10	160
E21	Number Working Shift	39	0	175	5	175
F21	Number Working Shift	43	0	180	1E+30	5
G21	Number Working Shift	15	0	195	1E+30	195

Part (a): The optimal solution does not change (within the allowable increase of $10).

Part (b): The optimal solution does change (outside the allowable decrease of $5).

Part (c): Percentage of allowable increase for shift 2: $(165-160)/10 = 50\%$

Percentage of allowable decrease for shift 4: $(180-170)/5 = 200\%$

Sum: 250%

The optimal solution may or may not change.

Part (d): Percentage of allowable decrease for shift 1: $(170-166)/10 = 40\%$

Percentage of allowable increase for shift 2: $(164-160)/10 = 40\%$

Percentage of allowable decrease for shift 3: $(175-171)/175 = 2\%$

Percentage of allowable increase for shift 4: $(184-180)/\infty = 0\%$

Percentage of allowable increase for shift 5: $(199-194)/\infty = 0\%$

Sum: 82%

The optimal solution does not change.

Part (e): Percentage of allowable increase for shift 1: $(173.40-170)/\infty = 0\%$

Percentage of allowable increase for shift 2: $(163.20-160)/10 = 32\%$

Percentage of allowable increase for shift 3: $(178.50-175)/5 = 70\%$

Percentage of allowable increase for shift 4: $(183.60-180)/\infty = 0\%$

Percentage of allowable increase for shift 5: $(198.90-195)/\infty = 0\%$

Sum: 102%

The optimal solution may or may not change.

(g)

	B	C	D	E	F	G	H
24	Cost per Shift	6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am	Total
25	6am-2pm	Shift	Shift	Shift	Shift	Shift	Cost
26		48	31	39	43	15	$30,610
27	$155	54	25	39	43	15	$29,860
28	$158	54	25	39	43	15	$30,022
29	$161	48	31	39	43	15	$30,178
30	$164	48	31	39	43	15	$30,322
31	$167	48	31	39	43	15	$30,466
32	$170	48	31	39	43	15	$30,610
33	$173	48	31	39	43	15	$30,754
34	$176	48	31	39	43	15	$30,898
35	$179	48	31	39	43	15	$31,042
36	$182	48	31	39	43	15	$31,186
37	$185	48	31	39	43	15	$31,330

	B	C	D	E	F	G	H
40	Cost per Shift	6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am	Total
41	8am-4pm	Shift	Shift	Shift	Shift	Shift	Cost
42		48	31	39	43	15	$30,610
43	$145	48	31	39	43	15	$30,145
44	$148	48	31	39	43	15	$30,238
45	$151	48	31	39	43	15	$30,331
46	$154	48	31	39	43	15	$30,424
47	$157	48	31	39	43	15	$30,517
48	$160	48	31	39	43	15	$30,610
49	$163	48	31	39	43	15	$30,703
50	$166	48	31	39	43	15	$30,796
51	$169	48	31	39	43	15	$30,889
52	$172	54	25	39	43	15	$30,970
53	$175	54	25	39	43	15	$31,045

	B	C	D	E	F	G	H
56	Cost per Shift	6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am	Total
57	Noon-8pm	Shift	Shift	Shift	Shift	Shift	Cost
58		48	31	39	43	15	$30,610
59	$160	48	31	39	43	15	$30,025
60	$163	48	31	39	43	15	$30,142
61	$166	48	31	39	43	15	$30,259
62	$169	48	31	39	43	15	$30,376
63	$172	48	31	39	43	15	$30,493
64	$175	48	31	39	43	15	$30,610
65	$178	48	31	39	43	15	$30,727
66	$181	48	31	33	49	15	$30,838
67	$184	48	31	33	49	15	$30,937
68	$187	48	31	33	49	15	$31,036
69	$190	48	31	33	49	15	$31,135

	B	C	D	E	F	G	H
88	Cost per Shift	6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am	Total
89	10pm-6am	Shift	Shift	Shift	Shift	Shift	Cost
90		48	31	39	43	15	$30,610
91	$180	48	31	39	43	15	$30,385
92	$183	48	31	39	43	15	$30,430
93	$186	48	31	39	43	15	$30,475
94	$189	48	31	39	43	15	$30,520
95	$192	48	31	39	43	15	$30,565
96	$195	48	31	39	43	15	$30,610
97	$198	48	31	39	43	15	$30,655
98	$201	48	31	39	43	15	$30,700
99	$204	48	31	39	43	15	$30,745
100	$207	48	31	39	43	15	$30,790
101	$210	48	31	39	43	15	$30,835

6.8-7.

	B	C	D	E	F	G	H	I	J
3		6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am			
4		Shift	Shift	Shift	Shift	Shift			
5	Cost per Shift	$170	$160	$175	$180	$195			
6							Total		Minimum
7	Time Period	Shift Works Time Period? (1=yes, 0=no)					Working		Needed
8	6am-8am	1	0	0	0	0	48	≥	48
9	8am-10am	1	1	0	0	0	79	≥	79
10	10am- 12pm	1	1	0	0	0	79	≥	65
11	12pm-2pm	1	1	1	0	0	118	≥	87
12	2pm-4pm	0	1	1	0	0	70	≥	64
13	4pm-6pm	0	0	1	1	0	82	≥	73
14	6pm-8pm	0	0	1	1	0	82	≥	82
15	8pm-10pm	0	0	0	1	0	43	≥	43
16	10pm-12am	0	0	0	1	1	58	≥	52
17	12am-6am	0	0	0	0	1	15	≥	15
18									
19		6am-2pm	8am-4pm	Noon-8pm	4pm-midnight	10pm-6am			
20		Shift	Shift	Shift	Shift	Shift			Total Cost
21	Number Working	48	31	39	43	15			$30,610

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C21	Number Working Shift	48	0	170	1E+30	10
D21	Number Working Shift	31	0	160	10	160
E21	Number Working Shift	39	0	175	5	175
F21	Number Working Shift	43	0	180	1E+30	5
G21	Number Working Shift	15	0	195	1E+30	195

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
H8	6am-8am Working	48	10	48	6	48
H9	8am-10am Working	79	160	79	1E+30	6
H10	10am- 12pm Working	79	0	65	14	1E+30
H11	12pm-2pm Working	118	0	87	31	1E+30
H12	2pm-4pm Working	70	0	64	6	1E+30
H13	4pm-6pm Working	82	0	73	9	1E+30
H14	6pm-8pm Working	82	175	82	1E+30	6
H15	8pm-10pm Working	43	5	43	6	6
H16	10pm-12am Working	58	0	52	6	1E+30
H17	12am-6am Working	15	195	15	1E+30	6

(a) The following shifts can be increased by the indicated amounts without increasing the total cost: Serve 10-12 a.m. → 14

Serve 12-2 p.m. → 31

Serve 2-4 p.m. → 6

Serve 4-6 p.m. → 9

Serve 10-12 p.m. → 6.

(b) For each of the following shifts, the total cost increases by the amount indicated per unit increase. These costs hold for the indicated increases.

Shift	Increased Cost	Valid for Increase
Serve 6-8 a.m.	\$10	6
Serve 8-10 a.m.	\$160	8
Serve 6-8 p.m.	\$175	8
Serve 8-10 p.m.	\$5	6
Serve 12-6 a.m.	\$195	8

(c) Percentage of allowable increase for 6-8 a.m.: $(49-48)/6 = 16.7\%$
Percentage of allowable increase for 8-10 a.m.: $(80-79)/\infty = 0\%$
Percentage of allowable increase for 6-8 p.m.: $(83-82)/\infty = 0\%$
Percentage of allowable increase for 8-10 p.m.: $(44-43)/6 = 16.7\%$
Percentage of allowable increase for 12-6 a.m.: $(16-15)/\infty = 0\%$
Sum: 33.4%

The shadow prices are still valid.

(d) Percentage of allowable increase for 6-8 a.m.: $(49-48)/6 = 16.7\%$
Percentage of allowable increase for 8-10 a.m.: $(80-79)/\infty = 0\%$
Percentage of allowable increase for 10-12 a.m.: $(66-65)/14 = 7.1\%$
Percentage of allowable increase for 12-2 p.m.: $(88-87)/31 = 3.2\%$
Percentage of allowable increase for 2-4 p.m.: $(65-64)/6 = 16.7\%$
Percentage of allowable increase for 4-6 p.m.: $(74-73)/9 = 11.1\%$
Percentage of allowable increase for 6-8 p.m.: $(83-82)/\infty = 0\%$
Percentage of allowable increase for 8-10 p.m.: $(44-43)/6 = 16.7\%$
Percentage of allowable increase for 10-12 p.m.: $(53-52)/6 = 16.7\%$
Percentage of allowable increase for 12-6 a.m.: $(16-15)/\infty = 0\%$
Sum: 88.2%

The shadow prices are still valid.

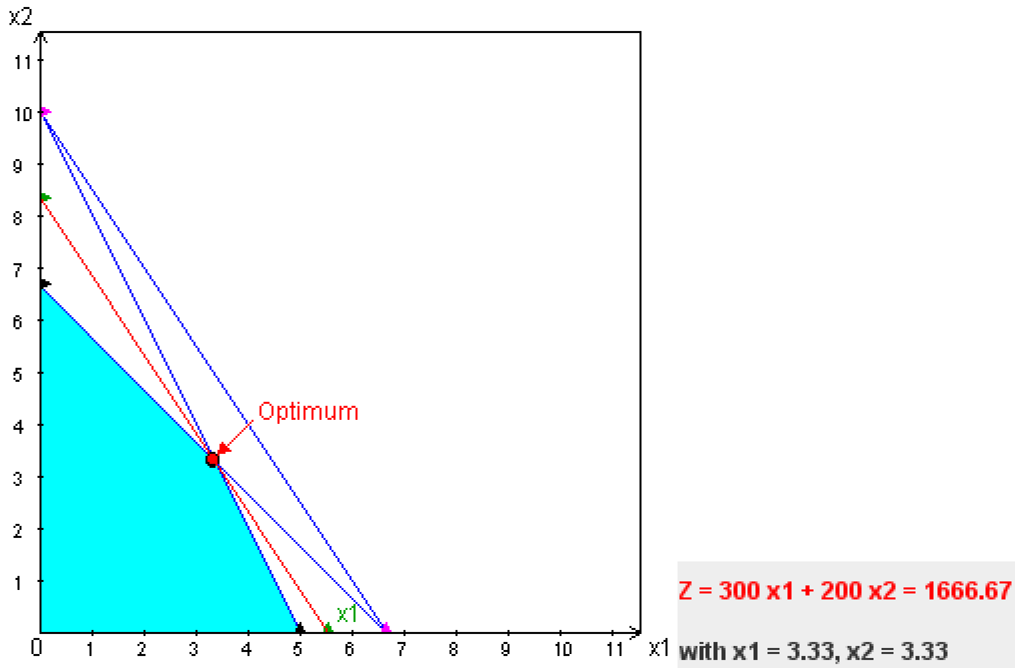
(e) All numbers can be increased by $100/88.2 \approx 1.13$ hours before it is no longer definite that the shadow prices remain valid.

6.8-8.

(a) Let x_G and x_W be the number of grandfather and wall clocks produced respectively.

$$\begin{array}{lrl}
\text{maximize} & 300x_G + 200x_W & \\
\text{subject to} & 6x_G + 4x_W & \leq 40 \\
& 8x_G + 4x_W & \leq 40 \\
& 3x_G + 3x_W & \leq 20 \\
\text{and} & x_G, x_W & \geq 0
\end{array}$$

(b) Optimal Solution: $(x_G, x_W) = (3.33, 3.33)$, $Z^* = 1666.67$



Objective Coefficient

Current Value	Allowable Range to Stay Optimal	
	Minimum	Maximum
300	200	400
200	150	300

The unit profit for grandfather clocks is allowed to vary between \$200 and \$400, so if it changed from \$300 to \$375, the optimal solution would remain the same, provided that there are no other changes in the model. However, if in addition to this, the unit profit for wall clocks is changed to \$175, the optimal solution becomes $(5, 0)$.

(c) Using Excel Solver:

	Grandfather Clocks	Wall Clocks			
Unit Profit	\$300	\$200			
			Hours		Hours
	Hours Required per Clock		Used		Available
David	6	4	33.33	≤	40
LaDeana	8	4	40	≤	40
Lydia	3	3	20	≤	20
	Grandfather Clocks	Wall Clocks			Total Profit
Clocks Produced	3.33	3.33			\$1,667

(d)

	Grandfather Clocks	Wall Clocks			
Unit Profit	$375	$200			
			Hours		Hours
	Hours Required per Clock		Used		Available
David	6	4	33.33	≤	40
LaDeana	8	4	40	≤	40
Lydia	3	3	20	≤	20
	Grandfather Clocks	Wall Clocks			Total Profit
Clocks Produced	3.33	3.33			$1,917

Changing the unit profit of grandfather clocks to $375 does not change the optimal solution.

	Grandfather Clocks	Wall Clocks			
Unit Profit	$375	$175			
			Hours		Hours
	Hours Required per Clock		Used		Available
David	6	4	30.00	≤	40
LaDeana	8	4	40	≤	40
Lydia	3	3	15	≤	20
	Grandfather Clocks	Wall Clocks			Total Profit
Clocks Produced	5.00	0.00			$1,875

If we also change the unit profit of wall clocks to $175, then the optimal solution changes to reflect the fact that it is now more profitable to produce only grandfather clocks.

(e)

Unit Profit	Optimal Clocks Produced		Total
for Grandfather Clocks	Grandfather Clocks	Wall Clocks	Profit
	3.33	3.33	$1,666.67
$150	0.00	6.67	$1,333.33
$170	0.00	6.67	$1,333.33
$190	0.00	6.67	$1,333.33
$210	3.33	3.33	$1,366.67
$230	3.33	3.33	$1,433.33
$250	3.33	3.33	$1,500.00
$270	3.33	3.33	$1,566.67
$290	3.33	3.33	$1,633.33
$310	3.33	3.33	$1,700.00
$330	3.33	3.33	$1,766.67
$350	3.33	3.33	$1,833.33
$370	3.33	3.33	$1,900.00
$390	3.33	3.33	$1,966.67
$410	5.00	0.00	$2,050.00
$430	5.00	0.00	$2,150.00
$450	5.00	0.00	$2,250.00

From the Solver Table, the allowable range to stay optimal for the unit profit of grandfather clocks is the interval $[210 - \Delta_1, 390 + \Delta_2]$, where $\Delta_1 > 20$ and $0 \leq \Delta_2 < 20$.

Unit Profit	Optimal Clocks Produced		Total
for Wall Clocks	Grandfather Clocks	Wall Clocks	Profit
	3.33	3.33	$1,666.67
$50	5.00	0.00	$1,500.00
$70	5.00	0.00	$1,500.00
$90	5.00	0.00	$1,500.00
$110	5.00	0.00	$1,500.00
$130	5.00	0.00	$1,500.00
$150	5.00	0.00	$1,500.00
$170	3.33	3.33	$1,566.67
$190	3.33	3.33	$1,633.33
$210	3.33	3.33	$1,700.00
$230	3.33	3.33	$1,766.67
$250	3.33	3.33	$1,833.33
$270	3.33	3.33	$1,900.00
$290	3.33	3.33	$1,966.67
$310	0.00	6.67	$2,066.67
$330	0.00	6.67	$2,200.00
$350	0.00	6.67	$2,333.33

From the Solver Table, the allowable range to stay optimal for the unit profit of wall clocks is the interval $[170 - \Delta_3, 290 + \Delta_4]$, where $\Delta_3 > 20$ and $0 \leq \Delta_4 < 20$.

(f)

Total Profit			Unit Profit for Wall Clocks						
		$1,666.67	$50	$100	$150	$200	$250	$300	$350
		$150	$750.00	$833.33	$1,000.00	$1,333.33	$1,666.67	$2,000.00	$2,333.33
		$200	$1,000.00	$1,000.00	$1,166.67	$1,333.33	$1,666.67	$2,000.00	$2,333.33
Unit Profit for		$250	$1,250.00	$1,250.00	$1,333.33	$1,500.00	$1,666.67	$2,000.00	$2,333.33
Grandfather Clocks		$300	$1,500.00	$1,500.00	$1,500.00	$1,666.67	$1,833.33	$2,000.00	$2,333.33
		$350	$1,750.00	$1,750.00	$1,750.00	$1,833.33	$2,000.00	$2,166.67	$2,333.33
		$400	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,166.67	$2,333.33	$2,500.00
		$450	$2,250.00	$2,250.00	$2,250.00	$2,250.00	$2,333.33	$2,500.00	$2,666.67

Clocks Produced			Unit Profit for Wall Clocks						
		(3.33,3.33)	$50	$100	$150	$200	$250	$300	$350
		$150	(5,0)	(3.33,3.33)	(3.33,3.33)	(6.67,6.67)	(6.67,6.67)	(6.67,6.67)	(6.67,6.67)
		$200	(5,0)	(5,0)	(3.33,3.33)	(3.33,3.33)	(6.67,6.67)	(6.67,6.67)	(6.67,6.67)
Unit Profit for		$250	(5,0)	(5,0)	(3.33,3.33)	(3.33,3.33)	(3.33,3.33)	(6.67,6.67)	(6.67,6.67)
Grandfather Clocks		$300	(5,0)	(5,0)	(5,0)	(3.33,3.33)	(3.33,3.33)	(3.33,3.33)	(6.67,6.67)
		$350	(5,0)	(5,0)	(5,0)	(3.33,3.33)	(3.33,3.33)	(3.33,3.33)	(3.33,3.33)
		$400	(5,0)	(5,0)	(5,0)	(5,0)	(3.33,3.33)	(3.33,3.33)	(3.33,3.33)
		$450	(5,0)	(5,0)	(5,0)	(5,0)	(3.33,3.33)	(3.33,3.33)	(3.33,3.33)

(g) If David is available to work a maximum of 45 hours, the optimal solution and the total profit do not change. Even when he is available for 40 hours, he is required to use less.

	Grandfather Clocks	Wall Clocks			
Unit Profit	$300	$200			
			Hours		Hours
	Hours Required per Clock		Used		Available
David	6	4	33.33	≤	45
LaDeana	8	4	40	≤	40
Lydia	3	3	20	≤	20
	Grandfather Clocks	Wall Clocks			Total Profit
Clocks Produced	3.33	3.33			$1,667

If LaDeana is available for 5 more hours every week, the optimal number of grandfather clocks to be produced increases whereas the optimal number of wall clocks to be produced decreases. The total profit increases by $125.

	Grandfather Clocks	Wall Clocks			
Unit Profit	$300	$200			
			Hours		Hours
	Hours Required per Clock		Used		Available
David	6	4	35.83	≤	40
LaDeana	8	4	45	≤	45
Lydia	3	3	20	≤	20
	Grandfather Clocks	Wall Clocks			Total Profit
Clocks Produced	4.58	2.08			$1,792

Finally, if Lydia increases her availability by 5 hours, the optimal number of grandfather clocks to be produced decreases whereas the optimal number of wall clocks to be produced increases. The optimal total profit increases by $166, which is more than the increase caused by increasing LaDeana's working hours by the same amount.

	Grandfather Clocks	Wall Clocks			
Unit Profit	$300	$200			
			Hours		Hours
	Hours Required per Clock		Used		Available
David	6	4	36.67	≤	40
LaDeana	8	4	40	≤	40
Lydia	3	3	25	≤	25
	Grandfather Clocks	Wall Clocks			Total Profit
Clocks Produced	1.67	6.67			$1,833

Note that in each case, the binding constraints remain the same.

(h)

Available Hours	Optimal Clocks Produced		Total
for David	Grandfather Clocks	Wall Clocks	Profit
	3.33	3.33	$1,666.67
35	3.33	3.33	$1,666.67
37	3.33	3.33	$1,666.67
39	3.33	3.33	$1,666.67
41	3.33	3.33	$1,666.67
43	3.33	3.33	$1,666.67
45	3.33	3.33	$1,666.67

Available Hours	Optimal Clocks Produced		Total
for LaDeana	Grandfather Clocks	Wall Clocks	Profit
	3.33	3.33	$1,666.67
35	2.08	4.58	$1,541.67
37	2.58	4.08	$1,591.67
39	3.08	3.58	$1,641.67
41	3.58	3.08	$1,691.67
43	4.08	2.58	$1,741.67
45	4.58	2.08	$1,791.67

Available Hours	Optimal Clocks Produced		Total
for Lydia	Grandfather Clocks	Wall Clocks	Profit
	3.33	3.33	$1,666.67
15	5.00	0.00	$1,500.00
17	4.33	1.33	$1,566.67
19	3.67	2.67	$1,633.33
21	3.00	4.00	$1,700.00
23	2.33	5.33	$1,766.67
25	1.67	6.67	$1,833.33

(i)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D25	GrandfatherClocksProduced	3.33	0.00	300	100	100
E25	WallClocksProduced	3.33	0.00	200	100	50

The unit profit for grandfather clocks should stay in the interval $[200, 400]$ and that for wall clocks should stay in $[150, 300]$ for the optimal solution to remain unchanged.

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F20	DavidHoursUsed	33.33	0.00	40	1E+30	6.666666667
F21	LadeanaHoursUsed	40	25	40	13.33333333	13.33333333
F22	LydiaHoursUsed	20	33.33333333	20	10	5

Provided that the maximum number of hours David is available is more than 33.334, the binding constraints stay the same. LaDeana's number of available hours can differ from 40 only by 13.333. Lydia's maximum number of hours is allowed to vary between 15 and 30.

(j) The constraint associated with Lydia has the highest shadow price, so Lydia should be the one to increase the maximum number of hours available to work per week.

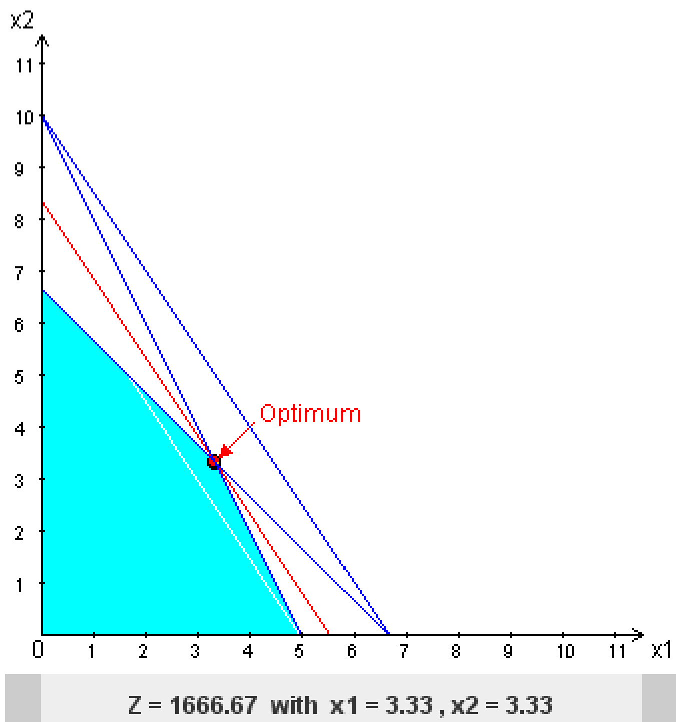
(k) The constraint associated with David is not binding in the optimal solution. In other words, David is required to work less than the maximum number of hours he is available. Hence increasing his availability does not improve the profit unless the other partners offer more time as well, so the shadow price of his constraint is equal to zero.

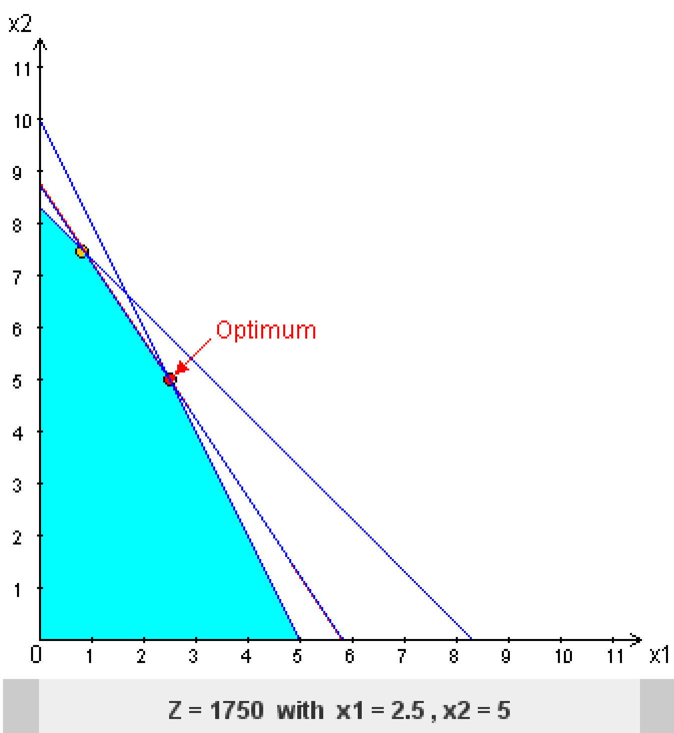
(l) The allowable increase for Lydia's hours is 10, so this shadow price can be used for an increase of 5. If Lydia increases her available hours from 20 to 25, the total profit is improved by approximately $5 \times 33.333 = \$166.665$, which is pretty close to what was found in part (g). The difference is due to rounding.

(m) When David changes his maximum of hours to 35 and Lydia changes hers to 25, the constraints that are binding in the optimal solution change to that of David and LaDeana. The constraint of Lydia becomes unbinding. The total profit increases by $83, which is half of the change resulting from Lydia alone. The change suggested by the shadow prices would be $5 \times 33.333 + 5 \times 0 = \$166.665$. The individual changes fall in the allowable range; however, they change simultaneously, so we cannot use the shadow prices in this case.

	Grandfather Clocks	Wall Clocks			
Unit Profit	$300	$200			
			Hours		Hours
	Hours Required per Clock		Used		Available
David	6	4	35.00	≤	35
LaDeana	8	4	40	≤	40
Lydia	3	3	22.5	≤	25
	Grandfather Clocks	Wall Clocks			Total Profit
Clocks Produced	2.50	5.00			$1,750

(n)




x2
11
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
11
x1
Optimum
Z = 1750 with x1 = 2.5 , x2 = 5

Cases

6-1 a) The decisions to be made are how which types of abatement methods will be used and at what fractions of their abatement capacities for the blast furnaces and the open-hearth furnaces. The constraints on these decisions are the technological limits on how heavily each method can be used and the required reductions in the annual emission rate. The overall measure of performance is cost, which is to be minimized.

	Benefit Contribution Per Unit of Abatement Method								
	Taller Smokestacks		Filters		Better Fuels				Acceptable
Constraints	Blast	Open-hearth	Blast	Open-hearth	Blast	Open-hearth	Totals		Level
reduce particulates	12	9	25	20	17	13	60	≥	60
reduce sulfer oxides	35	42	18	31	56	49	150	≥	150
reduce hydrocarbons	37	53	28	24	29	20	125	≥	125
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	0.6226975	≤	1
filters - blast	0	0	1	0	0	0	0.3434794	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.0475728	≤	1
fuels - open-hearth	0	0	0	0	0	1	1	≤	1
Unit Cost	8	10	7	6	11	9	$ 32.155		
Solution	**1**	**0.6227**	**0.3435**	**1**	**0.0476**	**1**			

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B14	Solution Blast	1	0	8	0.336210968	1E+30
C14	Solution Open-hearth	0.6227	0.0000	10	0.429446289	0.666961637
D14	Solution Blast	0.3435	0.0000	7	0.381632655	2.011459969
E14	Solution Open-hearth	1	0	6	1.816085017	1E+30
F14	Solution Blast	0.0476	0.0000	11	2.975225225	0.044638358
G14	Solution Open-hearth	1	0	9	0.044161638	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
H4	reduce particulates Totals	60	0.111046969	60	14.29714286	7.48
H5	reduce sulfer oxides Totals	150	0.126817108	150	20.453125	1.689655172
H6	reduce hydrocarbons Totals	125	0.069325636	125	2.041666667	21.69195612
H7	smokestacks - blast Totals	1	-0.336210968	1	0.246231156	0.748477435
H8	smokestacks - open-hearth Totals	0.622697455	0	1	1E+30	0.377302545
H9	filters - blast Totals	0.343479402	0	1	1E+30	0.656520598
H10	filters - open-hearth Totals	1	-1.816085017	1	0.110609481	1
H11	fuels - blast Totals	0.047572816	0	1	1E+30	0.952427184
H12	fuels - open-hearth Totals	1	-0.044161638	1	0.048086359	0.962708538

b) The right-hand-side of each constraint with a non-zero shadow price is sensitive, since changing its value will impact the total cost. All three required reductions in emission rates are sensitive parameters. All of the objective coefficients have an allowable range to stay optimal around them, and thus are not as sensitive. However, for some, the allowable change is small—in particular, the cost of the two better fuel options (with an allowable increase of only 0.045 and an allowable decrease of 0.044, respectively) are fairly sensitive. Thus, all five of these parameters should be estimated more closely, if possible.

c) The following table shows in which cases the optimal solution will change:

Current Value	10% Less Value	Solution Changes?	10% More Value	Solution Changes?
8	7.2	No	8.8	Yes
10	9	Yes	11	Yes
7	6.3	No	7.7	Yes
6	5.4	No	6.6	No
11	9.9	Yes	12.1	No
9	8.1	No	9.9	Yes

This suggests that focus should be put on estimating all of the costs except the one that is currently $6 million since it's optimal solution will not change with a 10% increase or decrease. Special consideration should be given to the estimate of the current $10 million cost since it affects the optimal solution for both an increase and a decrease.

d) Here is the corresponding dual problem.

	Benefit Contribution Per Unit of Abatement Method								
	Taller Smokestacks		Filters		Better Fuels				Accepatabl
Constraints	Blast	Open-earth	Blast	Open-earth	Blast	Open-earth	Dual Variables		Level
reduce particulates	12	9	25	20	17	13	y1	<= 0	60
reduce sulfer-oxides	35	42	18	31	56	49	y2	<= 0	150
reduce hydrocarbons	37	53	28	24	29	20	y3	<= 0	125
smokestacks - blast	1	0	0	0	0	0	y4	>= 0	1
smokestacks - open-earth	0	1	0	0	0	0	y5	>= 0	1
filters - blast	0	0	1	0	0	0	y6	>= 0	1
filters - open-earth	0	0	0	1	0	0	y7	>= 0	1
fuels - blast	0	0	0	0	1	0	y8	>= 0	1
fuels - open-earth	0	0	0	0	0	1	y9	>= 0	1
Totals									
	≥	≥	≥	≥	≥	≥			
Unit Cost	-8	-10	-7	-6	-11	-9			

This is the sensitivity report of primal (maximization problem)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C16	Solution Blast	1.00	0.00	-8	1E+30	0.336210968
D16	Solution Open-earth	0.62	0.00	-10	0.666961637	0.429446294
E16	Solution Blast	0.34	0.00	-7	2.01145997	0.381632659
F16	Solution Open-earth	1.00	0.00	-6	1E+30	1.816085017
G16	Solution Blast	0.05	0.00	-11	0.044638358	2.975225225
H16	Solution Open-earth	1.00	0.00	-9	1E+30	0.044161638

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
I6	reduce particulates Totals	60	-0.111046969	60	14.29714286	7.48
I7	reduce sulfer-oxides Totals	150	-0.126817108	150	20.453125	1.689655172
I8	reduce hydrocarbons Totals	125	-0.069325636	125	2.041666667	21.69195612
I9	smokestacks - blast Totals	1	0.336210968	1	0.246231156	0.748477435
I10	smokestacks - open-earth Totals	0.623	0.000	1	1E+30	0.377302545
I11	filters - blast Totals	0.343	0.000	1	1E+30	0.656520598
I12	filters - open-earth Totals	1	1.816085017	1	0.110609481	1
I13	fuels - blast Totals	0.0476	0.0000	1	1E+30	0.952427184
I14	fuels - open-earth Totals	1	0.044161638	1	0.048086359	0.962708538

The dual variables are the shadow prices of the constraints.

If the primal had been left in minimization form, the sign of the dual variables would be the opposite.

The dual would be the same except that the "sign" constraints on the dual variables changes from ≥ to ≤ and visceversa, and the dual functional constraints all change from ≥ to ≤. (CONT'D)

d) (CONT'D) It would also be a maximization problem, instead of minimization.

e)

Pollutant	Rate that cost changes	Maximum increase before rate changes	Maximum decrease before rate changes
Particulates	0.111	14.297	7.48
Sulfur oxides	0.127	20.453	1.69
Hydrocarbons	0.069	2.042	21.692

f) Particulates and sulfur oxides:

For each unit increase in particulate reduction, cost will increase by \$0.111 million.
For each unit decrease in sulfur oxide reduction, cost will decrease by \$0.127 million.
Thus, cost will remain equal if for each unit increase in particulate reduction, the sulfur oxide reduction is reduced by \$0.111 / \$0.127 = 0.874 units.

Particulates and hydrocarbons:

For each unit increase in particulate reduction, cost will increase by \$0.111 million.
For each unit decrease in hydrocarbon reduction, cost will decrease by \$0.069 million.
Thus, cost will remain equal if for each unit increase in particulate reduction, the hydrocarbon reduction is reduced by \$0.111 / \$0.069 = 1.609 units.

Particulates and both sulfur oxides and hydrocarbons:

For each unit increase in particulate reduction, cost will increase by \$0.111 million.
For each simultaneous unit decrease in sulfur oxide and hydrocarbon reduction, cost will decrease by \$0.127 + \$0.069 = \$0.196.
Thus, cost will remain equal if for each unit increase in particulate reduction, the sulfur oxide and hydrocarbon reduction are each reduced by \$0.111 / \$0.196 = 0.566 units.

g) The formulation is the same except that the right hand side of the constraints corresponding to Table 3.12 become

$60 + \theta \frac{60}{100}$; $150 + \theta \frac{150}{100}$; and $125 + \theta \frac{125}{100}$.

The rate at which the optimal cost of an optimal solution would increase with a small increase in θ from zero is given by:

$$y^* \Delta b = \underbrace{[0.111;\ 0.127;\ 0.0693]}_{\text{These are the shadow prices for the first 3 constraints.}} \begin{bmatrix} 0.6\theta \\ 1.5\theta \\ 1.25\theta \end{bmatrix} = 0.344\theta$$

So the rate of increase is 0.344θ.

h) 10% increase

Constraints	Taller Smokestacks Blast	Taller Smokestacks Open-hearth	Filters Blast	Filters Open-hearth	Better Fuels Blast	Better Fuels Open-hearth	Totals		Acceptable Level
reduce particulates	12	9	25	20	17	13	66	≥	66
reduce sulfer oxides	35	42	18	31	56	49	165	≥	165
reduce hydrocarbons	37	53	28	24	29	20	137.5	≥	137.5
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	0.7188402	≤	1
filters - blast	0	0	1	0	0	0	0.4359748	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.2135922	≤	1
fuels - open-hearth	0	0	0	0	0	1	1	≤	1
Unit Cost	8	10	7	6	11	9	**$ 35.590**		
Solution	**1**	**0.7188**	**0.4360**	**1**	**0.2136**	**1**			

Benefit Contribution Per Unit of Abatement Method

20% increase

Constraints	Taller Smokestacks Blast	Taller Smokestacks Open-hearth	Filters Blast	Filters Open-hearth	Better Fuels Blast	Better Fuels Open-hearth	Totals		Acceptable Level
reduce particulates	12	9	25	20	17	13	72	≥	72
reduce sulfer oxides	35	42	18	31	56	49	180	≥	180
reduce hydrocarbons	37	53	28	24	29	20	150	≥	150
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	0.8149829	≤	1
filters - blast	0	0	1	0	0	0	0.5284702	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.3796117	≤	1
fuels - open-hearth	0	0	0	0	0	1	1	≤	1
Unit Cost	8	10	7	6	11	9	**$ 39.025**		
Solution	**1**	**0.8150**	**0.5285**	**1**	**0.3796**	**1**			

Benefit Contribution Per Unit of Abatement Method

30% increase

Constraints	Taller Smokestacks Blast	Taller Smokestacks Open-hearth	Filters Blast	Filters Open-hearth	Better Fuels Blast	Better Fuels Open-hearth	Totals		Acceptable Level
reduce particulates	12	9	25	20	17	13	78	≥	78
reduce sulfer oxides	35	42	18	31	56	49	195	≥	195
reduce hydrocarbons	37	53	28	24	29	20	162.5	≥	162.5
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	0.9111257	≤	1
filters - blast	0	0	1	0	0	0	0.6209656	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.5456311	≤	1
fuels - open-hearth	0	0	0	0	0	1	1	≤	1
Unit Cost	8	10	7	6	11	9	**$ 42.460**		
Solution	**1**	**0.9111**	**0.6210**	**1**	**0.5456**	**1**			

Benefit Contribution Per Unit of Abatement Method

40% increase

Benefit Contribution Per Unit of Abatement Method

	Taller Smokestacks		Filters		Better Fuels				Acceptable
Constraints	Blast	Open-hearth	Blast	Open-hearth	Blast	Open-hearth	Totals		Level
reduce particulates	12	9	25	20	17	13	84	≥	84
reduce sulfer oxides	35	42	18	31	56	49	210	≥	210
reduce hydrocarbons	37	53	28	24	29	20	175	≥	175
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	1	≤	1
filters - blast	0	0	1	0	0	0	0.705282	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.7815769	≤	1
fuels - open-hearth	0	0	0	0	0	1	0.9293187	≤	1
Unit Cost	8	10	7	6	11	9	**$ 45.898**		
Solution	**1**	**1.0000**	**.7053**	**1**	**0.7816**	**0.9293187**			

50% increase

Benefit Contribution Per Unit of Abatement Method

	Taller Smokestacks		Filters		Better Fuels				Acceptable
Constraints	Blast	Open-hearth	Blast	Open-hearth	Blast	Open-hearth	Totals		Level
reduce particulates	12	9	25	20	17	13	93.378953	≥	90
reduce sulfer oxides	35	42	18	31	56	49	225	≥	225
reduce hydrocarbons	37	53	28	24	29	20	187.5	≥	187.5
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	1	≤	1
filters - blast	0	0	1	0	0	0	0.9491107	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	1	≤	1
fuels - open-hearth	0	0	0	0	0	1	0.8962451	≤	1
Unit Cost	8	10	7	6	11	9	**$ 49.710**		
Solution	**1**	**1.0000**	**0.9491**	**1**	**1.0000**	**0.8962451**			

Subtracting \$3.5 for each 10% reduction gives the following costs:

10% - \$32.090
20% - \$32.025
30% - \$31.960
40% - \$31.898
50% - \$32.210

To minimize the total cost of both pollution abatement and taxes, a 40% reduction should be chosen.

i) The sensitivity report for a 40% reduction is shown below.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B14	Solution Blast	1	0	8	0.552692013	1E+30
C14	Solution Open-hearth	1.0000	0.0000	10	0.429446287	1E+30
D14	Solution Blast	0.7053	0.0000	7	0.381632653	1.292358804
E14	Solution Open-hearth	1	0	6	1.789231947	1E+30
F14	Solution Blast	0.7816	0.0000	11	0.384387352	0.044638358
G14	Solution Open-hearth	0.929318704	0	9	0.044161637	0.371892925

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
H4	reduce particulates Totals	84	0.099260015	84	0.264818356	0.845849802
H5	reduce sulfer oxides Totals	210	0.124011227	210	1.112449799	6.294117647
H6	reduce hydrocarbons Totals	175	0.081653483	175	0.863773966	0.253199269
H7	smokestacks - blast Totals	1	-0.552692013	1	0.014418823	0.043313262
H8	smokestacks-open-hearth Totals	1	-0.429446287	1	0.007268433	0.022703764
H9	filters - blast Totals	0.70528196	0	1	1E+30	0.29471804
H10	filters - open-hearth Totals	1	-1.789231947	1	1.174670633	0.116240034
H11	fuels - blast Totals	0.781576933	0	1	1E+30	0.218423067
H12	fuels - open-hearth Totals	0.929318704	0	1	1E+30	0.070681296

Pollutant	Rate that cost changes	Maximum increase before rate changes	Maximum decrease before rate changes
Particulates	0.099	0.265	0.846
Sulfur oxides	0.124	1.112	6.294
Hydrocarbons	0.082	0.864	0.253

<u>Particulates and sulfur oxides:</u>
For each unit increase in particulate reduction, cost will increase by $0.099 million.
For each unit decrease in sulfur oxide reduction, cost will decrease by $0.124 million.
Thus, cost will remain equal if for each unit increase in particulate reduction, the sulfur oxide reduction is reduced by $0.099 / $0.124 = 0.798 units.

<u>Particulates and hydrocarbons:</u>
For each unit increase in particulate reduction, cost will increase by $0.099 million.
For each unit decrease in hydrocarbon reduction, cost will decrease by $0.082 million.
Thus, cost will remain equal if for each unit increase in particulate reduction, the hydrocarbon reduction is reduced by $0.099 / $0.082 = 1.207 units.

<u>Particulates and both sulfur oxides and hydrocarbons:</u>
For each unit increase in particulate reduction, cost will increase by $0.099 million.
For each simultaneous unit decrease in sulfur oxide and hydrocarbon reduction, cost will decrease by $0.124 + $0.082 = $0.206.
Thus, cost will remain equal if for each unit increase in particulate reduction, the sulfur oxide and hydrocarbon reduction are each reduced by $0.099 / $0.206 = 0.481 units.

6-.2 a) The decisions to be made are how much acreage should be planted in each of the crops and how many cows and hens to have for the coming year. The constraints on these decisions are amount of labor hours available, the investment funds available, the number of acres available, the space available in the barn and chicken house, the minimum requirements for feed to be planted. The overall measure of performance is monetary worth, which is to be maximized.

b & c)

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
	Resource Usage Per Unit of Activity											
acreage	1	1	1	2	2	0	0	0	0	640	≤	640
barn space	0	0	0	1	1	0	0	0	0	30	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	0	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	0	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	0	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 46,817		
Net Value	$70	$60	$40	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 72,550		
Remaining Investment										$ 20,000		
Living Expenses										$ (40,000)		
										$ 99,367		Total Monetary Worth
Solution	450	30	100	30	0	2000	0	1063	1364			

Note that the cells for current cows and current hens are not changing cells but fixed amounts.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B17	Solution soybean	450	0	70	1E+30	8.400000002
C17	Solution corn	30	0	60	8.400000002	1E+30
D17	Solution wheat	100	0	40	17.15005129	1E+30
F17	Solution cows	0	-53.00000097	699.9999983	53.00000097	1E+30
H17	Solution hens	0	-0.857502564	3.499997547	0.857502564	1E+30
I17	Solution W/S Labor	1063	0	5	57.3	0.915371347
J17	Solution S/F Labor	1364	0	5.5	34.5	0.929824579

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
K4	acreage Totals	640	57.3	640	974.2857143	450
K5	barn space Totals	30	0	42	1E+30	12
K6	chicken house space Totals	2000	0	5000	1E+30	3000
K7	winter/spring hours Totals	4000	5	4000	1E+30	1063
K8	summer/fall hours Totals	4500	5.5	4500	1E+30	1364
K9	investment fund Totals	0	0	20000	1E+30	20000
K10	feed for cows Totals	0	8.400000002	0	30	450
K11	feed for chickens Totals	0	24.15	0	100	450

This model predicts that the family's monetary worth at the end of the coming year will be $99, 367.

d) Range of optimality -

	soybeans:	$61.6 \leq$ soybeans value $\leq \infty$
	corn:	$-\infty \leq$ corn value ≤ 68.4
	wheat:	$-\infty \leq$ wheat value ≤ 57.15

e) Drought

Resource Usage Per Unit of Activity

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	259.33333	≤	640
barn space	0	0	0	1	1	0	0	0	0	42	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2666.6667	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	20000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	4.547E-13	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	1.705E-11	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 55,544		
Net Value	-$10	-$15	$0	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 52,320		
Remaining Investment										$		
Living Expenses										$(40,000)		
										$ 67,864		Total Monetary Worth
Solution	0	42	133	30	12	2000	667	562	1036			

Flood

Resource Usage Per Unit of Activity

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	640	≤	640
barn space	0	0	0	1	1	0	0	0	0	42	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2666.6667	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	20000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	-380.6667	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	-7.11E-15	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 51,318		
Net Value	$15	$20	$10	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 62,737		
Remaining Investment										$		
Living Expenses										$(40,000)		
										$ 74,055		Total Monetary Worth
Solution	0	423	133	30	12	2000	667	220	579			

Early Frost

Resource Usage Per Unit of Activity

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	640	≤	640
barn space	0	0	0	1	1	0	0	0	0	30	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	0	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	6.673E-11	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	7.104E-11	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 46,817		
Net Value	$50	$40	$30	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 61,950		
Remaining Investment										$ 20,000		
Living Expenses										$(40,000)		
										$ 88,767		Total Monetary Worth
Solution	450	30	100	30	0	2000	0	1063	1364			

Drought and Early Frost

Resource Usage Per Unit of Activity

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	226	≤	640
barn space	0	0	0	1	1	0	0	0	0	42	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	18000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	-1.28E-12	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	0	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 55,039		
Net Value	-$15	-$20	-$10	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 49,610		
Remaining Investment										$ 2,000		
Living Expenses										$(40,000)		Total Monetary Worth
Solution	0	42	100	30	12	2000	0	782	1260			

Flood and Early Frost

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
	Resource Usage Per Unit of Activity											
acreage	1	1	1	2	2	0	0	0	0	362	≤	640
barn space	0	0	0	1	1	0	0	0	0	37.333333	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	5000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	20000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	-2.69E-12	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	7.671E-11	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 56,336		
Net Value	$10	$10	$5	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 53,523		
Remaining Investment										(0)		
Living Expenses										(40,000)		
Solution	0	37.3	250	30	7.333	2000	3000	76	540	$ 69,860		Total Monetary Worth

f)

Opt. Sol. Used	Good Weather	Drought	Flood	Early Frost	Drought & Early Frost	Flood & Early Frost
	Family's monetary worth at year's end if the scenario is actually:					
Good Weather	99,367	57,117	70,417	88,767	53,717	67,367
Drought	76,348	67,864	70,668	74,174	66,321	69,581
Flood	94,962	57,929	74,055	85,175	54,482	69,162
Early Frost	99,367	57,117	70,417	88,767	53,717	67,367
Drought & Early Frost	75,009	67,859	70,329	73,169	66,649	69,409
Flood & Early Frost	80,476	67,676	71,483	77,230	64,990	69,860

The "Flood & Early Frost" solution looks like the best conservative option. The "Flood" option looks good for those who would like more risk.

g and h)

The expected net value for each of the crops is calculated as follows:

Soybeans: (\$70)(0.4) + (-\$10)(0.2) + (\$15)(0.1) + (\$50)(0.15) + (-\$15)(0.1) + (\$10)(0.05) = \$34 million,

Corn: (\$60)(0.4) + (-\$15)(0.2) + (\$20)(0.1) + (\$40)(0.15) + (-\$20)(0.1) + (\$10)(0.05) = \$27.5 million,

Wheat: (\$40)(0.4) + (\$0)(0.2) + (\$10)(0.1) + (\$30)(0.15) + (-\$10)(0.1) + (\$5)(0.05) = \$20.75 million.

The resulting spreadsheet solution is shown below:

	Resource Usage Per Unit of Activity											
Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	640	≤	640
barn space	0	0	0	1	1	0	0	0	0	42	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	18000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	0	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	0	≤	0
Net Income	\$0	\$0	\$0	\$850	\$850	\$4.25	\$4.25	\$5	\$5.50	\$ 49,781		
Net Value	\$34.0	\$27.5	\$20.8	\$1,050	\$1,350	\$1.88	\$2.25	\$0	\$0	\$ 68,756		
Remaining Investment										\$ 2,000		
Living Expenses										\$ (40,000)		
										\$ 80,537		Total Monetary Worth
Solution	414	42	100	30	12	2000	0	368	680			

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$17	Solution soybean	414	0	34	7.499999997	0.400002814
\$C\$17	Solution corn	42	0	27.5	4.899999997	22.49999999
\$D\$17	Solution wheat	100	0	20.75	0.400002814	1E+30
\$F\$17	Solution cows	12	0	700	1E+30	22.49999999
\$H\$17	Solution hens	0	-0.020000141	3.499999875	0.020000141	1E+30
\$I\$17	Solution W/S Labor	368	0	5	0.388601036	0.071429073
\$J\$17	Solution S/F Labor	680	0	5.5	0.394736842	0.075472229

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$K\$4	acreage Totals	640	21.3	640	368.2	414
\$K\$5	barn space Totals	42	22.49999999	42	1.333333333	12
\$K\$6	chicken house space Totals	2000	0	5000	1E+30	3000
\$K\$7	winter/spring hours Totals	4000	5	4000	1E+30	368.2
\$K\$8	summer/fall hours Totals	4500	5.5	4500	1E+30	680
\$K\$9	investment fund Totals	18000	0	20000	1E+30	2000
\$K\$10	feed for cows Totals	0	4.899999997	0	42	414
\$K\$11	feed for chickens Totals	0	7.399999998	0	100	414

This model predicts that the family's monetary worth at the end of the coming year will be \$80,537.

i) The shadow price for the investment constraint is zero, indicating that additional investment funds will not increase their total monetary worth at all. Thus, it is not worthwhile to obtain a bank loan. The shadow price would need to be at least \$1.10 before a loan at 10% interest would be worthwhile.

j) The expected net value for soybeans can increase up to \$7.50 or decrease up to \$0.40; for corn can increase up to \$4.90 or decrease up to \$22.50; for wheat can increase up to \$0.40 or decrease any amount without changing the optimal solution. The expected net value for soybeans and wheat should be estimated most carefully.

The solution is sensitive to decreases in the expected value of soybeans and increases in the expected value of wheat. If the *cumulative* decrease in the expected value of soybeans *and* increase in the expected value of wheat exceeds \$0.40, then the 100% rule will be violated, and the solution might change.

k) Answers will vary.

6-3 a)

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost (\$/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	422.222222	177.777778	600	=	600
Area 3	0	227.777778	322.222222	550	=	550
Area 4	350	0	0	350	=	350
Area 5	366.666667	0	133.333333	500	=	500
Area 6	83.3333333	0	366.666667	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = \$ 555,555.56

Grade Constraints:

	School 1	School 2	School 3
6th Graders	269.333333	368.555556	339.111111
7th Graders	288	362.111111	300.888889
8th Graders	242.666667	369.333333	360
30% of Total	240	330	300
36% of Total	288	396	360

b)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B14	Area 1 School 1	0	177.77778	300	1E+30	177.77778
C14	Area 1 School 2	450	0	0	177.77778	1.554E+17
D14	Area 1 School 3	0	266.66667	700	1E+30	266.66667
B15	Area 2 School 1	0	-800	0	1E+30	800
C15	Area 2 School 2	422.22222	0	400	34.210526	4.5454555
D15	Area 2 School 3	177.77778	0	500	4.5454555	34.210526
B16	Area 3 School 1	0	11.111114	600	1E+30	11.111114
C16	Area 3 School 2	227.77778	0	300	4.5454555	34.210526
D16	Area 3 School 3	322.22222	0	200	34.210526	7.6923092
B17	Area 4 School 1	350	0	200	366.66667	2.339E+16
C17	Area 4 School 2	0	366.66667	500	1E+30	366.66667
D17	Area 4 School 3	0	-433.3333	0	1E+30	433.33333
B18	Area 5 School 1	366.66667	0	0	16.66667	108.33333
C18	Area 5 School 2	0	233.33333	0	1E+30	233.33333
D18	Area 5 School 3	133.33333	0	400	108.33333	16.66667
B19	Area 6 School 1	83.333333	0	500	33.333342	166.66667
C19	Area 6 School 2	0	200	300	1E+30	200
D19	Area 6 School 3	366.66667	0	0	166.66667	33.333342

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B30	8th Graders School 1	242.66667	0	0	1E+30	45.333333
C30	8th Graders School 2	369.33333	0	0	1E+30	26.666667
D30	8th Graders School 3	360	-6666.667	0	5.3333333	0.6666667
B20	Total School 1	800	0	900	1E+30	100
C20	Total School 2	1100	-177.7778	1100	36.363636	3.7735849
D20	Total School 3	1000	-144.4444	1000	42.105263	3.8834951
B28	6th Graders School 1	269.33333	0	0	29.333333	1E+30
C28	6th Graders School 2	368.55556	0	0	38.555556	1E+30
D28	6th Graders School 3	339.11111	0	0	39.111111	1E+30
B28	6th Graders School 1	269.33333	0	0	1E+30	18.666667
C28	6th Graders School 2	368.55556	0	0	1E+30	27.444444
D28	6th Graders School 3	339.11111	0	0	1E+30	20.888889
B29	7th Graders School 1	288	0	0	48	1E+30
C29	7th Graders School 2	362.11111	0	0	32.111111	1E+30
D29	7th Graders School 3	300.88889	0	0	0.8888889	1E+30
B29	7th Graders School 1	288	-2777.778	0	0.2580645	2.9090909
C29	7th Graders School 2	362.11111	0	0	1E+30	33.888889
D29	7th Graders School 3	300.88889	0	0	1E+30	59.111111
B30	8th Graders School 1	242.66667	0	0	2.6666667	1E+30
C30	8th Graders School 2	369.33333	0	0	39.333333	1E+30
D30	8th Graders School 3	360	0	0	60	1E+30
E14	Area 1 Total	450	177.77778	450	3.7735849	36.363636
E15	Area 2 Total	600	577.77778	600	3.7735849	36.363636
E16	Area 3 Total	550	477.77778	550	3.7735849	36.363636
E17	Area 4 Total	350	311.11111	350	72.727273	6.4516129
E18	Area 5 Total	500	-55.55556	500	12.903226	145.45455
E19	Area 6 Total	450	277.77778	450	3.2258065	36.363636

c) The bussing cost from area 6 to school 1 can increase $33.33 before the current optimal solution would no longer be optimal. The new solution with a 10% increase ($50) is shown below.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	550	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	72.7272726	50	427.272727	550	=	550
Area 4	350	0	0	350	=	350
Area 5	318.181818	0	181.818182	500	=	500
Area 6	59.0909093	0	390.909091	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 559,318.18

Grade Constraints:

	School 1	School 2	School 3
6th Graders	264	381	332
7th Graders	288	355	308
8th Graders	248	364	360
30% of Total	240	330	300
36% of Total	288	396	360

d) The bussing cost from area 6 to school 2 can increase any amount and the current optimal solution will still be optimal.

e) According to the 100% rule, the bussing cost from area 6 can increase uniformly up to 6.67% ($33 for school 1, and $20 for school 2) without changing the solution. Beyond that the solution might change. This calculation is shown below.

School 1: $500 → $533.33. % of allowable increase $= 100\left(\frac{533.33-500}{33.33}\right) = 100\%$

School 2: $300 → $320. % of allowable increase $= 100\left(\frac{320-300}{\infty}\right) = \underline{0\%}$

Sum = 100%.

The new spreadsheet solution is shown below.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	550	330	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	72.7272727	50	427.272727	550	=	550
Area 4	350	0	0	350	=	350
Area 5	318.181818	0	181.818182	500	=	500
Area 6	59.0909091	0	390.909091	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 559,318.18

Grade Constraints:

	School 1	School 2	School 3
6th Graders	264	381	332
7th Graders	288	355	308
8th Graders	248	364	360
30% of Total	240	330	300
36% of Total	288	396	360

f) The shadow price for school 1 is zero. Thus, adding a temporary classroom at school 1 would not save any money, and thus would not be worthwhile.

The shadow price for school 2 is -$177.78. Thus, adding a temporary classroom at school 2 would save ($177.78)(20) = $3555.60 in bussing cost. This is worthwhile, since it exceeds the $2500 leasing cost.

The shadow price for school 3 is -$144.44. Thus, adding a temporary classroom at school 3 would save ($144.44)(20) = $2888.80 in bussing cost. This is also worthwhile, since it exceeds the $2500 leasing cost.

g) For school 2, the allowable increase for school capacity is 36. This means the shadow price is only valid for a single additional portable classroom.

For school 3, the allowable increase for school capacity is 42. This means the shadow price is valid for up to two additional portable classrooms.

h) The following combinations do not violate the 100% rule:

Portables to add to school 2	Portables to add to school 3	100%-rule calculation	Bussing Cost Savings
1	0	(20/36) + (0/42) = 55.6%	($177.78)(20)=$2888.80
0	1	(0/36) + (20/42) = 47.6%	
0	2	(0/36) + (40/42) = 95.23%	

Each combination yields the following total savings

Portables to add to school 2	Portables to add to school 3	Bussing Cost Savings	Lease Cost	Total Savings
1	0	($177.78)(20) = $3555.60	$2500	$1055.60
0	1	($144.44)(20) = $2888.80	$2500	$388.80
0	2	($144.44)(40) = $5777.60	$5000	$777.60

Of these combinations, adding one portable to school 2 is best in terms of minimizing total cost. The spreadsheet solution is shown below.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	520	80	600	=	600
Area 3	0	150	400	550	=	550
Area 4	350	0	0	350	=	350
Area 5	340	0	160	500	=	500
Area 6	90	0	360	450	=	450
Total	780	1120	1000			
	≤	≤	≤			
Capacity	900	1120	1000			

Total Bussing Cost = $ 552,000.00
Leasing Cost = $ 2,500.00
Total Cost = $ 554,500.00

Grade Constraints:

	School 1	School 2	School 3
6th Graders	261.2	381.4	334.4
7th Graders	280.8	364.6	305.6
8th Graders	238	374	360
30% of Total	234	336	300
36% of Total	280.8	403.2	360

i) Adding two portables to school 2 yields the following solution. This is the best plan.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	0	90	460	550	=	550
Area 4	350	0	0	350	=	350
Area 5	318.947368	0	181.052632	500	=	500
Area 6	95.2631579	0	354.736842	450	=	450
Total	764.210526	1140	995.789474			
	≤	≤	≤			
Capacity	900	1140	1000			

Total Bussing Cost =	**$**	**549,052.63**
Leasing Cost =	**$**	**5,000.00**
Total Cost =	**$**	**554,052.63**

Grade Constraints:

	School 1	School 2	School 3
6th Graders	254.778947	393	329.221053
7th Graders	275.115789	367.8	308.084211
8th Graders	234.315789	379.2	358.484211
30% of Total	229.263158	342	298.736842
36% of Total	275.115789	410.4	358.484211

Cases

6.4 In this case, the decisions to be made are

TV = number of units of advertising on television

PM = number of units of advertising in the printed media

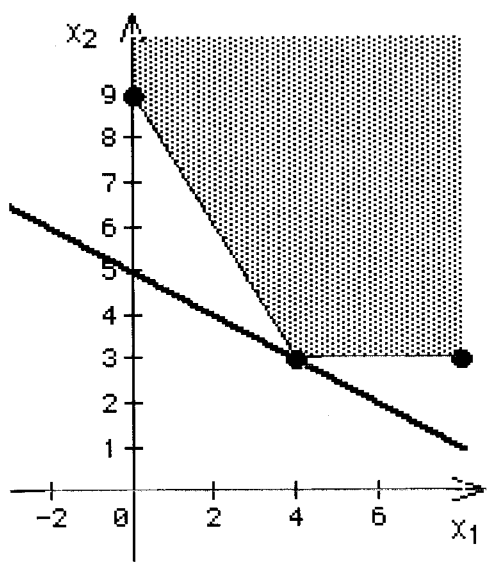
The resulting linear programming model is

Maximize Cost = 1 TV + 2 PM (in millions of dollars)
subject to

Stain remover:	1 PM	=	3	(in %)
Liquid detergent:	3 TV + 2 PM	=	18	(in %)
Powder detergent:	-1 TV + 4 PM	=	4	(in %)

TV = 0, PM = 0

a) Optimal Solution: 4 units of television advertising and 3 units of print media advertising, with a total cost of $10 million.



b) The Solver find the following optimal advertising plan:

C14 = 4 (Undertake 4 units of advertising on televcision)

D14 = 3 (Undertake 3 units of advertising in the printed mdia)

The target cell G14 indicates that the total cost of this advertising plan would be $10 million.

The linear programming spreadsheet model for this problem is shown below.

	A	B	C	D	E	F	G
1	**Profit & Gambit Co. Advertising-Mix Problem**						
2							
3			**Television**	**Print Media**			
4		Unit Cost ($millions)	1	2			
5							
6					Increased		Minimum
7			Increase in Sales per Unit of Advertising		Sales		Increase
8		Stain remover	0%	1%	0	>=	3%
9		Liquid detergent	3%	2%	0	>=	18%
10		Powder detergent	-1%	4%	0	>=	4%
11							
12							**Total Cost**
13			**Television**	**Print Media**			**($millions)**
14		Advertising Units	4	3			10

	E
6	Increased
7	Sales
8	=SUMPRODUCT(C8:D8, AdvertisingUnits)
9	=SUMPRODUCT(C9:D9, AdvertisingUnits)
10	=SUMPRODUCT(C10:D10, AdvertisingUnits)

	G
12	**Total Cost**
13	**($millions)**
14	=SUMPRODUCT(UnitCost, AdvertisingUnits)

Range Name	**Cells**
AdvertisingUnits	C14:D14
IncreasedSales	E8:E10
IncreasedSalesPerUnitOfAdvertising	C8:D10
MinimumIncrease	G8:G10
TotalCost	G14
UnitCost	C4:D4

Solver Parameters

Set Target Cell: TotalCost

Equal To: ( ) Max (•) Min ( ) Va

By Changing Cells:

AdvertisingUnits

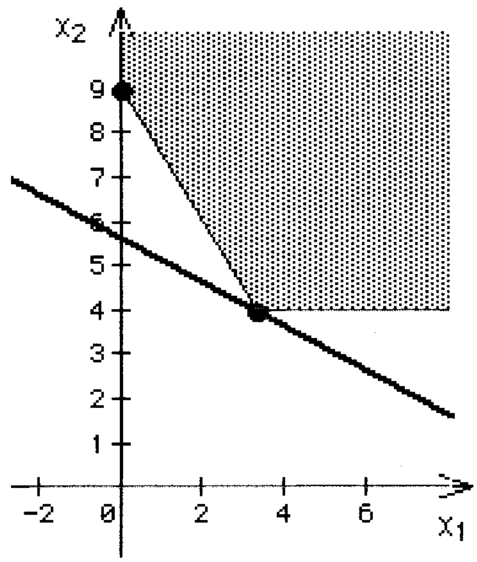
Subject to the Constraints:

IncreasedSales >= MinimumIncrease

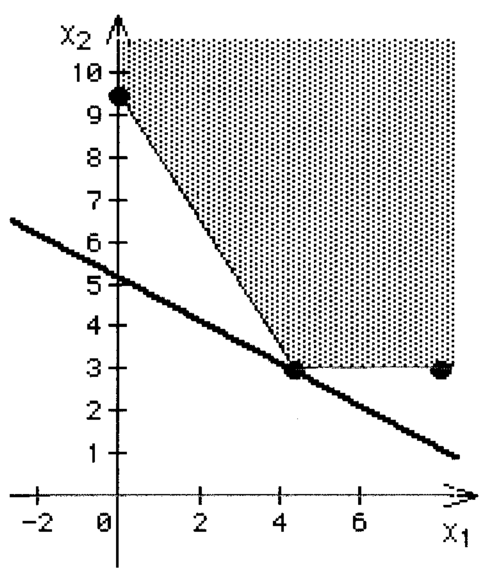
[x] Assume Linear Model

[x] Assume Non-Negative

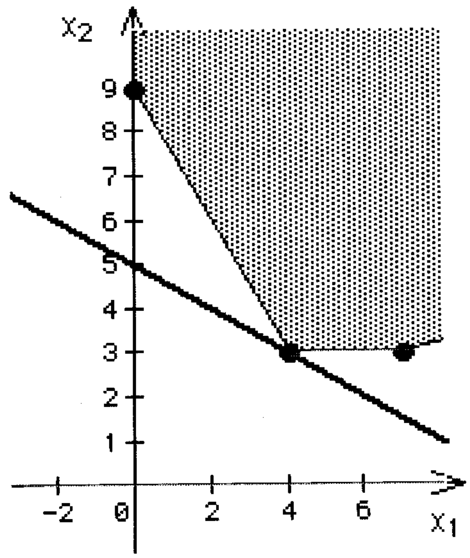
c) Increasing the required minimum increase in sales for Stain Remover by 1% changes the solution to 3.33 units of television advertising and 4 units of print media advertising, and increases the total cost by $1.33 million to $11.33 million.



Increasing the required minimum increase in sales for Liquid Detergent by 1% changes the solution to 4.33 units of television advertising and 3 units of print media advertising,, and increases the total cost by $0.33 million to $10.33 million.



Increasing the required minimmum increase in sales for Powder Detergent by 1% has no impact on the solution nor the total cost.



d) Original Solution:

	B	C	D	E	F	G
3		Television	Print Media			
4	Unit Cost ($millions)	1	2			
5						
6				Increased		Minimum
7		Increase in Sales per Unit of Advertising		Sales		Increase
8	Stain Remover	0%	1%	3%	3	3%
9	Liquid Detergent	3%	2%	18%	3	18%
10	Powder Detergent	-1%	4%	8%	3	4%
11						
12						Total Cost
13		Television	Print Media			($millions)
14	Advertising Units	4	3			10

Increasing the required minimum increase in sales for Stain Remover by 1% increases the total cost by $1.333 million.

	B	C	D	E	F	G
3		Television	Print Media			
4	Unit Cost ($millions)	1	2			
5						
6				Increased		Minimum
7		Increase in Sales per Unit of Advertising		Sales		Increase
8	Stain Remover	0%	1%	4%	3	4%
9	Liquid Detergent	3%	2%	18%	3	18%
10	Powder Detergent	-1%	4%	13%	3	4%
11						
12						Total Cost
13		Television	Print Media			($millions)
14	Advertising Units	3.333	4			11.333

Increasing the required minimum increase in sales for Liquid Detergent by 1% increases the total cost by $0.333 million.

	B	C	D	E	F	G
3		Television	Print Media			
4	Unit Cost ($millions)	1	2			
5						
6				Increased		Minimum
7		Increase in Sales per Unit of Advertising		Sales		Increase
8	Stain Remover	0%	1%	3%	≥	3%
9	Liquid Detergent	3%	2%	19%	≥	19%
10	Powder Detergent	-1%	4%	8%	≥	4%
11						
12						Total Cost
13		Television	Print Media			($millions)
14	Advertising Units	4.333	3			10.333

Increasing the required minimmum increase in sales for Powder Detergent by 1% has no impact on the total cost.

	B	C	D	E	F	G
3		Television	Print Media			
4	Unit Cost ($millions)	1	2			
5						
6				Increased		Minimum
7		Increase in Sales per Unit of Advertising		Sales		Increase
8	Stain Remover	0%	1%	3%	≥	3%
9	Liquid Detergent	3%	2%	18%	≥	18%
10	Powder Detergent	-1%	4%	8%	≥	5%
11						
12						Total Cost
13		Television	Print Media			($millions)
14	Advertising Units	4	3			10

e)

	B	C	D	E	F
17	Minimum Increase	Advertising Units		Total Cost	Incremental
18	Stain Remover	Television	Print Media	($millions)	Cost ($million)
19		4	3	10	
20	0%	4.571	2.143	8.857	
21	1%	4.571	2.143	8.857	0.000
22	2%	4.571	2.143	8.857	0.000
23	3%	4	3	10.000	1.143
24	4%	3.333	4	11.333	1.333
25	5%	2.667	5	12.667	1.333
26	6%	2	6	14.000	1.333

	B	C	D	E	F
29	Minimum Increase	Advertising Units		Total Cost	Incremental
30	Liquid Detergent	Television	Print Media	($millions)	Cost ($million)
31		4	3	10.000	
32	0%	0	3	6.000	
33	1%	0	3	6.000	0.000
34	2%	0	3	6.000	0.000
35	3%	0	3	6.000	0.000
36	4%	0	3	6.000	0.000
37	5%	0	3	6.000	0.000
38	6%	0	3	6.000	0.000
39	7%	0.333	3	6.333	0.333
40	8%	0.667	3	6.667	0.333
41	9%	1	3	7.000	0.333
42	10%	1.333	3	7.333	0.333
43	11%	1.667	3	7.667	0.333
44	12%	2	3	8.000	0.333
45	13%	2.333	3	8.333	0.333
46	14%	2.667	3	8.667	0.333
47	15%	3	3	9.000	0.333
48	16%	3.333	3	9.333	0.333
49	17%	3.667	3	9.667	0.333
50	18%	4	3	10.000	0.333
51	19%	4.333	3	10.333	0.333
52	20%	4.667	3	10.667	0.333
53	21%	5	3	11.000	0.333
54	22%	5.333	3	11.333	0.333
55	23%	5.667	3	11.667	0.333
56	24%	6	3	12.000	0.333
57	25%	6.333	3	12.333	0.333
58	26%	6.667	3	12.667	0.333
59	27%	7	3	13.000	0.333
60	28%	7.333	3	13.333	0.333
61	29%	7.667	3	13.667	0.333
62	30%	8	3	14.000	0.333
63	31%	8.286	3.071	14.429	0.429
64	32%	8.571	3.143	14.857	0.429
65	33%	8.857	3.214	15.286	0.429
66	34%	9.143	3.286	15.714	0.429
67	35%	9.429	3.357	16.143	0.429
68	36%	9.714	3.429	16.571	0.429

	B	C	D	E	F
71	Minimum Increase	Advertising Units		Total Cost	Incremental
72	Powder Detergent	Television	Print Media	($millions)	Cost ($million)
73		4	3	10	
74	0%	4	3	10	
75	1%	4	3	10	0.000
76	2%	4	3	10	0.000
77	3%	4	3	10	0.000
78	4%	4	3	10	0.000
79	5%	4	3	10	0.000
80	6%	4	3	10	0.000
81	7%	4	3	10	0.000
82	8%	4	3	10	0.000

f) Sensitivity Report:

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C14	Advertising Units Television	4	0	1	2	1
D14	Advertising Units Print Media	3	0	2	1E+30	1.333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E8	Stain Remover Sales	3%	133.33	0.03	0.06	0.008571429
E9	Liquid Detergent Sales	18%	33.33	0.18	0.12	0.12
E10	Powder Detergent Sales	8%	0	0.04	0.04	1E+30

The shadow price indicates the increase in total cost (in $millions) per unit increase in the right hand side (i.e., per 100% increase). Thus, a 1% increase in the minimum required increase in sales will only increase the total cost by one hundredth of the shadow price, or $1.33 million for the Stain Remover, $0.33 million for the Liquid Detergent, and $0 million for the Powder Detergent.

The allowable range for the required minimum increase in sales constraint for Stain Remover is 2.15% to 9%.

The allowable range for the required minimum increase in sales constraint for Liquid Detergent is 6% to 30%.

The allowable range for the required minimum increase in sales constraint for Powder Detergent is -8% to 8%.

These allowable ranges can also be seen in the results from part (c). For Stain Remover, the incremental cost remains $1.33 million for each 1% change above 3%. Similarly, for Liquid Detergent, the incremental cost remains $0.33 million for each 1% change above between 6% and 30%. For Powder Detergent, the incremental cost remains $0 million for each 1% change throughout the Solver Table.

g) Suppose that each of the original numbers in MinimumIncrease (G8:G10) is increased by 1%.

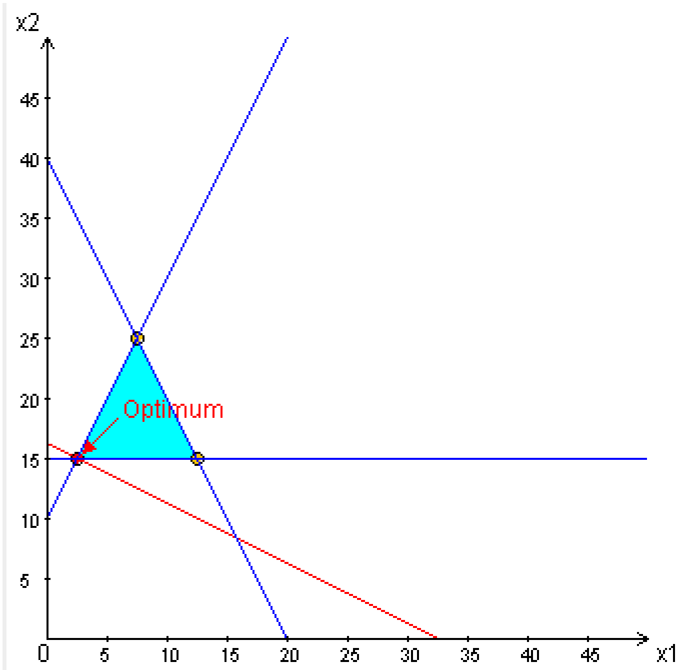
Percent of allowable increase for Stain Remover used = (4% – 3%) / 6% = 16.7%.
Percent of allowable increaes for Liquid Detergent used = (19% – 18%) / 12% = 8.3%.
Percent of allowable increase for Powder Detergent used = (5% – 4%) / 4% = 25%.
Sum = 50%.

Thus, if each of the original numbers in MinimumIncrease (G8:G10) is increased by 2%, the sum will be 100%. By the 100% rule, this is the most they can be increased before the shadow prices may no longer be valid.

h) Answers will vary.

7.1-1.

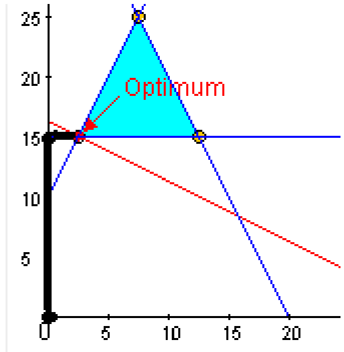
(a)



(b) Optimal Solution: $(x_1, x_2) = (2.5, 15)$, $Z = -32.5$

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	1	1	2	0	0	0	0
	x_3	1	0	2	1	1	0	0	40
	x_4	2	0	0	-1	0	1	0	-15
	x_5	3	0	-2	1	0	0	1	10
1	Z	0	1	0	0	0	2	0	-30
	x_3	1	0	2	0	1	1	0	25
	x_2	2	0	0	1	0	-1	0	15
	x_5	3	0	-2	0	0	1	1	-5
2	Z	0	1	0	0	0	2.5	0.5	-32.5
	x_3	1	0	0	0	1	2	1	20
	x_2	2	0	0	1	0	-1	0	15
	x_1	3	0	1	0	0	-0.5	-0.5	2.5

(c) The path taken by the dual simplex method is $(0,0) \to (0,15) \to (2.5,15)$.



7.1-2.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	-1	5	2	4	0	0	0
	x_4	1	0	-3	-1	-2	1	0	-4
	x_5	2	0	-6	-3	-5	0	1	-10
1	Z	0	-1	1	0	$\frac{2}{3}$	0	$\frac{2}{3}$	$-\frac{20}{3}$
	x_4	1	0	-1	0	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{2}{3}$
	x_2	2	0	2	1	$\frac{5}{3}$	0	$-\frac{1}{3}$	$\frac{10}{3}$
2	Z	0	-1	0	0	$\frac{1}{3}$	1	$\frac{1}{3}$	$-\frac{22}{3}$
	x_1	1	0	1	0	$\frac{1}{3}$	-1	$\frac{1}{3}$	$\frac{2}{3}$
	x_2	2	0	0	1	1	2	-1	2

Optimal Solution: $(x_1, x_2, x_3) = (2/3, 2, 0)$, $Z = 22/3$

7.1-3.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RS
0	Z	0	-1	7	2	5	4	0	0	0	0
	x_5	1	0	-2	-4	-7	-1	1	0	0	-5
	x_6	2	0	8	-4	-6	-4	0	1	0	-8
	x_7	3	0	-3	-8	-1	-4	0	0	1	-4
1	Z	0	-1	3	0	2	2	0	$\frac{1}{2}$	0	-4
	x_5	1	0	6	0	-1	3	1	-1	0	3
	x_2	2	0	2	1	$\frac{3}{2}$	1	0	$-\frac{1}{4}$	0	2
	x_7	3	0	13	0	11	4	0	-2	1	12

Optimal Solution: $(x_1, x_2, x_3, x_4) = (0, 2, 0, 0)$, $Z = 4$

7.1-4.

(a) Optimal Solution: $(x_1, x_2) = (10, 10)$, $Z = 250$

Iter.	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS	Primal Solution	Dual Solution
0	Z	0	1	-15	-10	0	0	0	0	$(0, 0, 40.20, 90)$	$(0, 0, 0, -15, -10)$
	x_3	1	0	3	1	1	0	0	40		
	x_4	2	0	1	1	0	1	0	20		
	x_5	3	0	5	3	0	0	1	90		
1	Z	0	1	0	-5	5	0	0	200	$(\frac{40}{3}, 0, 0, \frac{20}{3}, \frac{70}{3})$	$(5, 0, 0, 0, -5)$
	x_1	1	0	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{40}{3}$		
	x_4	2	0	0	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{20}{3}$		
	x_5	3	0	0	$\frac{4}{3}$	$-\frac{5}{3}$	0	1	$\frac{70}{3}$		
2	Z	0	1	0	0	$\frac{5}{2}$	$\frac{15}{2}$	0	250	$(10, 10, 0, 0, 10)$	$(\frac{5}{2}, \frac{15}{2}, 0, 0, 0)$
	x_1	1	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	10		
	x_2	2	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	10		
	x_5	3	0	0	0	-1	-2	1	10		

(b) The dual problem is:

$$\begin{array}{lll} \text{minimize} & 40y_1 + 20y_2 + 90y_3 & \\ \text{subject to} & 3y_1 + \ \ y_2 + \ 5y_3 & \geq 15 \\ & \ \ y_1 + \ \ y_2 + \ 3y_3 & \geq 10 \\ & y_1, y_2, y_3 & \geq 0. \end{array}$$

Iter.	BV	Eq. #	Z	y_1	y_2	y_3	y_4	y_5	RS	Primal Solution	Dual Solution
0	Z	0	-1	40	20	90	0	0	0	$(0, 0, 40, 20, 90)$	$(0, 0, 0, -15, -10)$
	y_4	1	0	-3	-1	-5	1	0	-15		
	y_5	2	0	-1	-1	-3	0	1	-10		
1	Z	0	-1	0	$\frac{20}{3}$	$\frac{70}{3}$	$\frac{40}{3}$	0	-200	$(\frac{40}{3}, 0, 0, \frac{20}{3}, \frac{70}{3})$	$(5, 0, 0, 0, -5)$
	y_1	1	0	1	$\frac{1}{3}$	$\frac{5}{3}$	$-\frac{1}{3}$	0	5		
	y_5	2	0	0	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	1	-5		
2	Z	0	-1	0	0	10	10	10	-250	$(10, 10, 0, 0, 10)$	$(\frac{5}{2}, \frac{15}{2}, 0, 0, 0)$
	y_1	1	0	1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}$		
	y_2	2	0	0	1	2	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{15}{2}$		

Optimal Solution: $(y_1, y_2, y_3) = (5/2, 15/2, 0)$, $Z = 250$

The sequence of basic and complementary basic solutions is identical to that in part (a).

7.1-5.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	1	0	0	0	$\frac{3}{2}$	1	54
	x_3	1	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
	x_2	2	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	3	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	-2
1	Z	0	1	$\frac{3}{2}$	0	0	0	$\frac{5}{2}$	45
	x_3	1	0	1	0	1	0	0	4
	x_2	2	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
	x_4	3	0	-3	0	0	1	-1	6

Optimal Solution: $(x_1, x_2, x_3, x_4, x_5) = (0, 9, 4, 6, 0)$, $Z = 45$

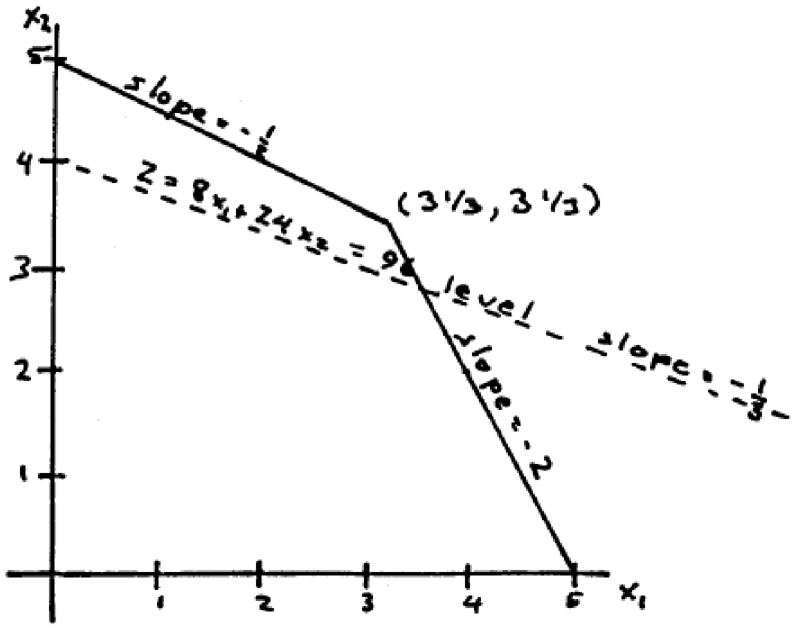
7.1-6.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	1	0	0	2	5	0	150
	x_2	1	0	-1	1	3	1	0	30
	x_5	2	0	16	0	-2	-4	1	-30
1	Z	0	1	16	0	0	1	1	120
	x_2	1	0	23	1	0	-5	$\frac{3}{2}$	-15
	x_3	2	0	-8	0	1	2	$-\frac{1}{2}$	15
2	Z	0	1	$\frac{103}{5}$	$\frac{1}{5}$	0	0	$\frac{13}{10}$	117
	x_4	1	0	$-\frac{23}{5}$	$-\frac{1}{5}$	0	1	$-\frac{3}{10}$	3
	x_3	2	0	$\frac{6}{5}$	$\frac{2}{5}$	1	0	$\frac{1}{10}$	9

Optimal Solution: $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 9, 3, 0)$, $Z = 117$

7.2-1.

(a)



The solution $(0,5)$ is optimal with $Z = 120$. It remains optimal as long as

$$-\frac{8+\theta}{24-2\theta} \leq -\frac{1}{2} \Leftrightarrow \theta \leq 2,$$

at which point $(10/3, 10/3)$ becomes optimal. In turn, this solution remains optimal until

$$-\frac{8+\theta}{24-2\theta} \leq -2 \Leftrightarrow \theta \leq 8,$$

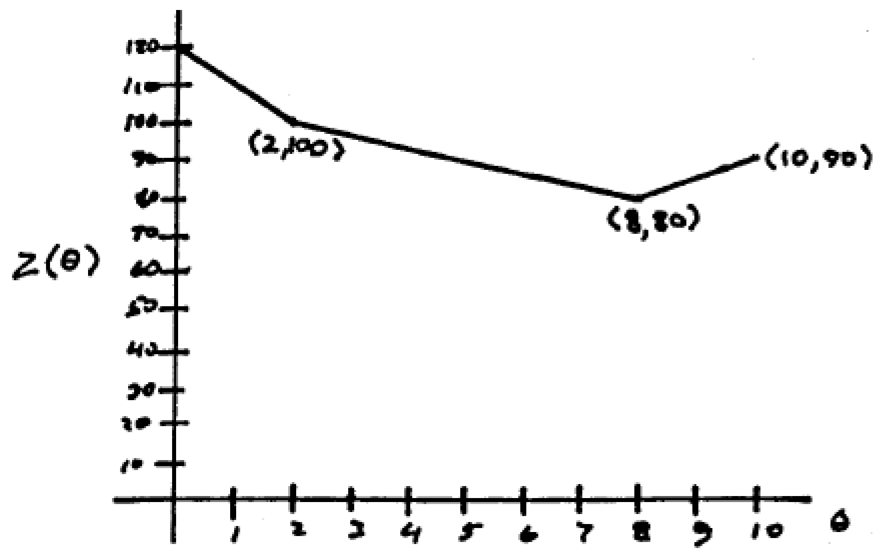
at which point $(5,0)$ becomes optimal.

θ	(x_1^*, x_2^*)	$Z^*(\theta)$
$0 \leq \theta \leq 2$	$(0,5)$	$120 - 10\theta$
$2 \leq \theta \leq 8$	$(10/3, 10/3)$	$(320 - 10\theta)/3$
$8 \leq \theta \leq 10$	$(5,0)$	$40 + 5\theta$

(b)

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	RS
0	Z	0	1	$-8-\theta$	$-24+2\theta$	0	0	0
	x_3	1	0	1	2	1	0	10
	x_4	2	0	2	1	0	1	10
1	Z	0	1	$4-2\theta$	0	$12-\theta$	0	$120-10\theta$
	x_2	1	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	5
	x_4	2	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	5
2	Z	0	1	0	0	$\frac{40-5\theta}{3}$	$\frac{8-4\theta}{3}$	$\frac{320-10\theta}{3}$
	x_2	1	0	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{10}{3}$
	x_1	2	0	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{10}{3}$
3	Z	0	1	0	$\frac{-40+5\theta}{2}$	0	$\frac{8+\theta}{2}$	$40+5\theta$
	x_3	1	0	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	5
	x_1	2	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	5

The solutions found in iterations 1, 2 and 3 are optimal for $0 \leq \theta \leq 2$, $2 \leq \theta \leq 8$ and $8 \leq \theta \leq 10$ respectively.



(c) The graph in part (b) suggests that $\theta = 0$ is optimal. Since $Z(\theta)$ is convex in θ, the maximum is attained at $\theta = 0$ or $\theta = 10$. Thus, only the linear programming problems corresponding to $\theta = 0$ and $\theta = 10$ need to be solved.

7.2-2.

Iteration	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	x_6	RS
0	Z	0	1	$-20-4\theta$	$-30+3\theta$	-5	0	0	0	0
	x_4	1	0	3	3	1	1	0	0	10
	x_5	2	0	8	6	4	0	1	0	25
	x_6	3	0	6	1	1	0	0	1	15
1	Z	0	1	$10-7\theta$	0	$5-\theta$	$10-\theta$	0	0	$100-10\theta$
	x_2	1	0	1	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{10}{3}$
	x_5	2	0	2	0	2	-2	1	0	5
	x_6	3	0	5	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	1	$\frac{35}{3}$
2	Z	0	1	0	0	$\frac{55-\theta}{15}$	$\frac{160-22\theta}{15}$	0	$\frac{-10+7\theta}{5}$	$\frac{230+19\theta}{3}$
	x_2	1	0	0	1	$\frac{1}{5}$	$\frac{2}{5}$	0	$-\frac{1}{5}$	1
	x_5	2	0	0	0	$\frac{26}{15}$	$-\frac{28}{15}$	1	$-\frac{2}{5}$	$\frac{1}{3}$
	x_1	3	0	1	0	$\frac{2}{15}$	$-\frac{1}{15}$	0	$\frac{1}{5}$	$\frac{7}{3}$
3	Z	0	1	0	$\frac{-80+11\theta}{3}$	$\frac{-5+2\theta}{3}$	0	0	$\frac{10+2\theta}{3}$	$50+10\theta$
	x_4	1	0	0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$\frac{5}{2}$
	x_5	2	0	0	$\frac{14}{3}$	$\frac{8}{3}$	0	1	$-\frac{4}{3}$	5
	x_1	3	0	1	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$	$\frac{5}{2}$

θ	(x_1^*, x_2^*, x_3^*)	$Z^*(\theta)$
$0 \leq \theta \leq 10/7$	$(0, \frac{10}{3}, 0)$	$100-10\theta$
$10/7 \leq \theta \leq 80/11$	$(\frac{7}{3}, 1, 0)$	$\frac{230+19\theta}{3}$
$80/11 \leq \theta$	$(\frac{5}{2}, 0, 0)$	$50+10\theta$

7.2-3.

(a) Starting with the optimal tableau for $\theta = 0$, after two iterations, we get:

Iter.	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RS
0	Z	0	1	0	0	$5-\theta$	$2+2\theta$	$8-3\theta$	220
	x_2	1	0	0	1	1	1	-1	10
	x_1	2	0	1	0	0	-1	2	10
1	Z	0	1	$\frac{-8+3\theta}{2}$	0	$5-\theta$	$\frac{12+\theta}{2}$	0	$180+15\theta$
	x_2	1	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$	0	15
	x_5	2	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	5
2	Z	0	1	$\frac{-13+4\theta}{2}$	0	0	$\frac{7+2\theta}{2}$	0	$105+30\theta$
	x_3	1	0	$\frac{1}{2}$	$-5+\theta$	1	$\frac{1}{2}$	0	15
	x_5	2	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	5

θ	(x_1^*, x_2^*, x_3^*)	$Z^*(\theta)$
$0 \leq \theta \leq 8/3$	$(10, 10, 0)$	220
$8/3 \leq \theta \leq 5$	$(0, 15, 0)$	$180+15\theta$
$5 \leq \theta$	$(0, 0, 15)$	$105+30\theta$

(b) The dual problem is:

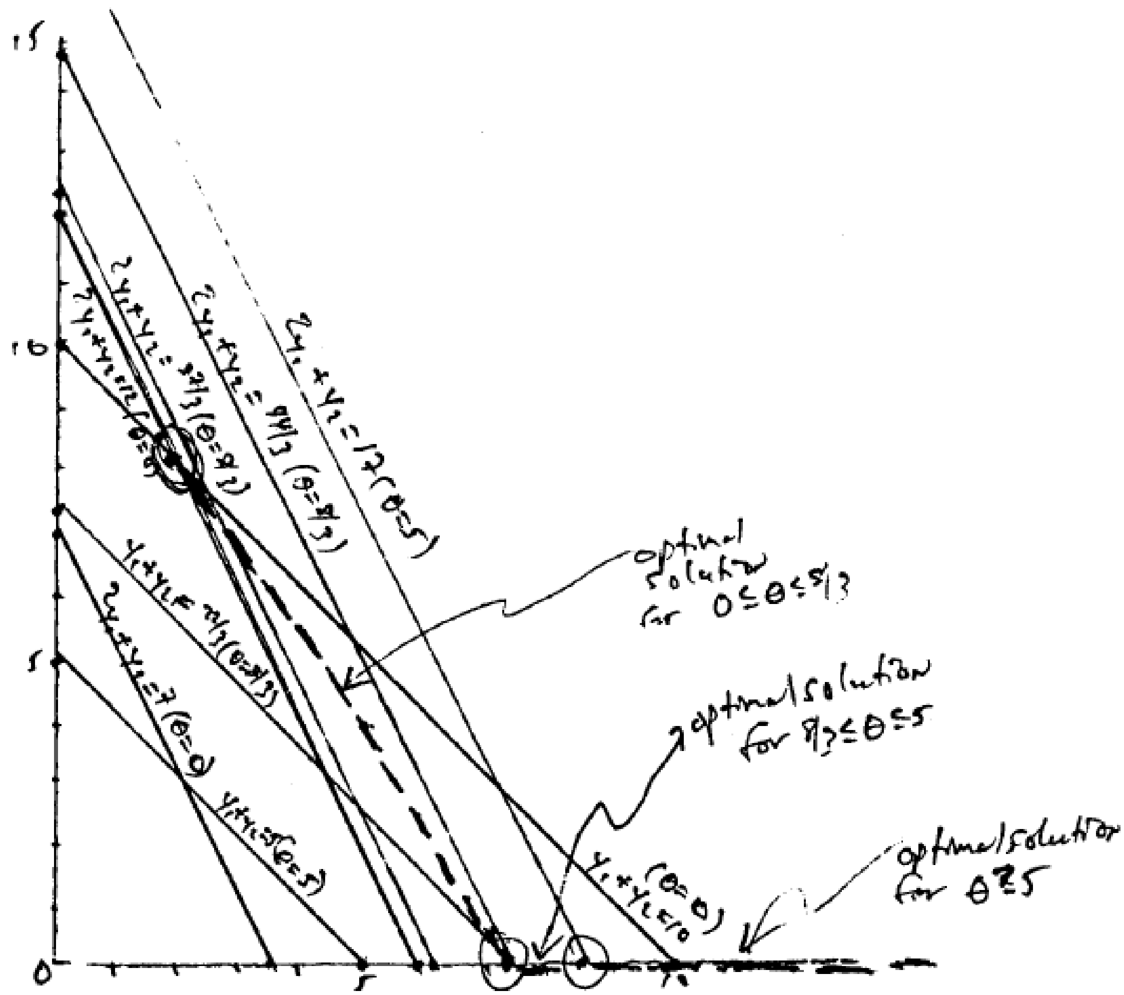
$$\begin{array}{lrl} \text{minimize} & 30y_1 + 20y_2 & \\ \text{subject to} & y_1 + \quad y_2 & \geq 10 - \theta \\ & 2y_1 + \quad y_2 & \geq 12 + \theta \\ & 2y_1 + \quad y_2 & \geq 7 + 2\theta \\ & y_1, y_2 & \geq 0. \end{array}$$

Starting with the optimal tableau for $\theta = 0$, after two iterations, we get:

Iter.	BV	Eq. #	Z	y_1	y_2	y_3	y_4	y_5	RS
0	Z	0	-1	0	0	10	10	0	-220
	y_2	1	0	0	1	-2	1	0	$8 - 3\theta$
	y_1	2	0	1	0	1	-1	0	$2 + 2\theta$
	y_5	3	0	0	0	0	-1	1	$5 - \theta$
1	Z	0	-1	0	5	0	15	0	$-180 - 15\theta$
	y_3	1	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	$-4 + 1.5\theta$
	y_1	2	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$6 + 0.5\theta$
	y_5	3	0	0	0	0	-1	1	$5 - \theta$
2	Z	0	-1	0	5	0	0	15	$-105 - 30\theta$
	y_3	1	0	0	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-6.5 + 2\theta$
	y_1	2	0	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$3.5 + \theta$
	y_4	3	0	0	0	0	1	-1	$-5 + \theta$

θ	(y_1^*, y_2^*)	$Z^*(\theta)$
$0 \leq \theta \leq 8/3$	$(2 + 2\theta, 8 - 3\theta)$	220
$8/3 \leq \theta \leq 5$	$(6 + 0.5\theta, 0)$	$180 + 15\theta$
$5 \leq \theta$	$(3.5 + \theta, 0)$	$105 + 30\theta$

The basic solutions are the same as those in part (a).



$0 \leq \theta \leq 8/3$: y^* from $(2, 8)$ to $(22/3, 0)$
$8/3 \leq \theta \leq 5$: y^* from $(22/3, 0)$ to $(17/2, 0)$
$5 \leq \theta$: $y^* = (3.5 + \theta, 0)$

7.2-4.

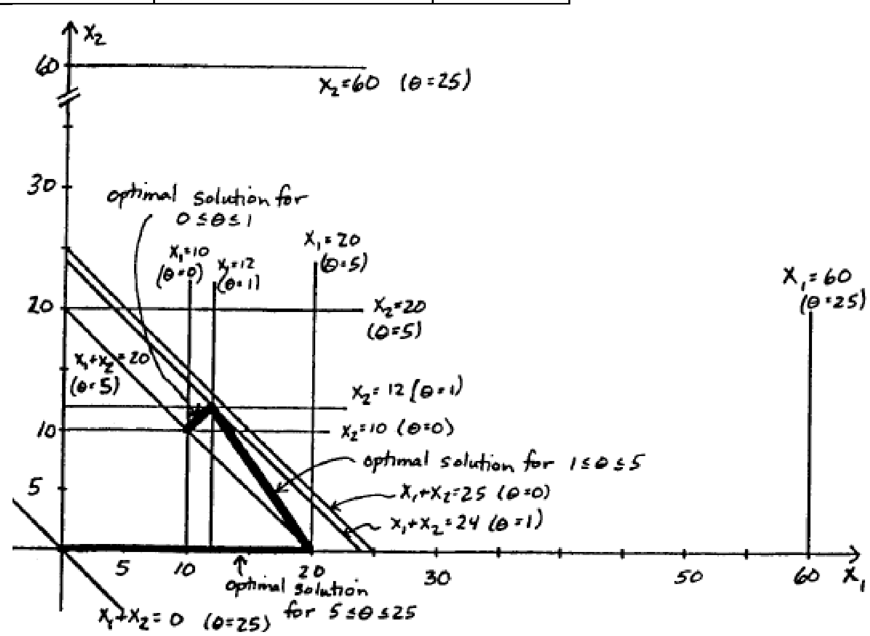
Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	-2	-1	0	0	0	0
x_3	1	0	1	0	1	0	0	$10+2\theta$
x_4	2	0	1*	1	0	1	0	$25-\theta$
x_5	3	0	0	1	0	0	1	$10+2\theta$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	0	1	0	2	0	$50-2\theta$
x_3	1	0	0	-1*	1	-1	0	$15+3\theta$
x_1	2	0	1	1	0	1	0	$25-\theta$
x_5	3	0	0	1	0	0	1	$10+2\theta$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	0	0	1	1	0	$35+\theta$
x_2	1	0	0	1	-1	1	0	$15-3\theta$
x_1	2	0	1	0	1	0	0	$10+2\theta$
x_5	3	0	0	0	1	-1	1	$-5+5\theta$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	0	0	2	0	1	$30+6\theta$
x_2	1	0	0	1	0	0	1	$10+2\theta$
x_1	2	0	1	0	1	0	0	$10+2\theta$
x_4	3	0	0	0	-1	1	-1	$5-5\theta$

θ	(x_1^*, x_2^*)	$Z^*(\theta)$
$0 \leq \theta \leq 1$	$(10+2\theta, 10+2\theta)$	$30+6\theta$
$1 \leq \theta \leq 5$	$(10+2\theta, 15-3\theta)$	$35+\theta$
$5 \leq \theta \leq 25$	$(25-\theta, 0)$	$50-2\theta$



7.2-5.

Starting with the optimal tableau for $\theta = 0$, after two iterations, we get:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	28	7	21	0	0	35	1050+35t
X5	1	0	0	-8	-2	-3	1	0	-3	45-5t
X6	2	0	0	0	-3*	-2	0	1	-2	18-3t
X1	3	0	1	2	1	2	0	0	1	30+t

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	28	0	49/3	0	7/3	91/3	1092+28t
X5	1	0	0	-8	0	-5/3	1	-2/3*	-5/3	33-3t
X3	2	0	0	0	1	2/3	0	-1/3	2/3	-6+t
X1	3	0	1	2	0	4/3	0	1/3	1/3	36

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	0	21/2	7/2	0	49/2	1207.5+17.5t
X6	1	0	0	12	0	5/2	-3/2	1	5/2	-49.5+4.5t
X3	2	0	0	4	1	3/2	-1/2	0	3/2	-22.5+2.5t
X1	3	0	1	-2	0	1/2	1/2	0	-1/2	52.5-1.5t

θ	$(x_1^*, x_2^*, x_3^*, x_4^*)$	$Z^*(\theta)$
$0 \leq \theta \leq 6$	$(30 + \theta, 0, 0, 0)$	$1050 + 35\theta$
$6 \leq \theta \leq 11$	$(36, 0, -6 + \theta, 0)$	$1092 + 28\theta$
$11 \leq \theta \leq 35$	$(52.5 - 1.5\theta, 0, -22.5 + 2.5\theta, 0)$	$1207.5 + 17.5\theta$

$\theta = 30$ provides the largest value of the objective function: $x^*(30) = (7.5, 0, 52.5, 0)$, $Z^*(30) = 1732.5$.

7.2-6.

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	0	0	2	5	0	$100+10\theta$
x_2	1	0	-1	1	3	1	0	$20+2\theta$
x_5	2	0	16	0	-2^*	-4	1	$10-9\theta$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	16	0	0	1	1	$110+\theta$
x_2	1	0	23	1	0	-5^*	$3/2$	$35-\frac{23}{2}\theta$
x_3	2	0	-8	0	1	2	$-1/2$	$-5+\frac{9}{2}\theta$

Bas Var	Eq No	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	0	1	$103/5$	$1/5$	0	0	$13/10$	$117-\frac{13}{10}\theta$
x_4	1	0	$-23/5$	$-1/5$	0	1	$-3/10$	$-7+\frac{23}{10}\theta$
x_3	2	0	$6/5$	$2/5$	1	0	$1/10$	$9-\frac{\theta}{10}$

θ	(x_1^*, x_2^*, x_3^*)	$Z^*(\theta)$
$0 \le \theta \le 10/9$	$(0, 20+2\theta, 0)$	$100+10\theta$
$10/9 \le \theta \le 70/23$	$(0, 35-11.5\theta, -5+4.5\theta)$	$110+\theta$
$70/23 \le \theta \le 90$	$(0, 0, 9-0.1\theta)$	$117+1.3\theta$

7.2-7.

(a) Let $x^{(k)}$ be the kth optimal solution obtained as θ is increased from 0. Each $x^{(k)}$ is optimal for some θ-interval, say $\theta_k \le \theta \le \theta_{k+1}$, and the objective function value $Z(\theta) = \alpha_k + \beta_k\theta$ for some α_k and β_k, so $Z(\theta)$ is linear in this interval. As the interval changes, α_k and β_k change so that a different linear function is obtained for each interval.

(b) The problem is:

$$\begin{aligned} &\text{maximize} && Z(\theta) = \sum_{j=1}^{n} (c_j + \alpha_j\theta)x_j \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij}x_j \le b_i,\ i = 1, 2, \ldots, m \\ &&& x_j \ge 0,\ j = 1, 2, \ldots, n. \end{aligned}$$

Note that the feasible region does not depend on θ. Consider $\theta_1 < \theta_2$ and let $\theta_3 = \lambda\theta_1 + (1-\lambda)\theta_2$ for some $0 \le \lambda \le 1$. Let $x_j^{(1)}$, $x_j^{(2)}$ and $x_j^{(3)}$ be the optimal values of x_j $(j = 1, 2, \ldots, n)$ for θ_1, θ_2 and θ_3 respectively. Let $Z(\theta, x) = \sum_{j=1}^{n}(c_j + \alpha_j\theta)x_j$.

$$Z^*(\theta_1) = Z(\theta_1, x^{(1)}) \ge Z(\theta_1, x^{(3)}) \Rightarrow \lambda Z^*(\theta_1) \ge \lambda Z(\theta_1, x^{(3)})$$

$$Z^*(\theta_2) = Z(\theta_2, x^{(2)}) \ge Z(\theta_2, x^{(3)}) \Rightarrow (1-\lambda) Z^*(\theta_2) \ge (1-\lambda) Z(\theta_2, x^{(3)})$$

$$\begin{aligned}\Rightarrow \lambda Z^*(\theta_1) + (1-\lambda)Z^*(\theta_2) &\geq \lambda Z(\theta_1, x^{(3)}) + (1-\lambda)Z(\theta_2, x^{(3)}) \\ &= \lambda\sum_{j=1}^{n}(c_j + \alpha_j\theta_1)x_j^{(3)} + (1-\lambda)\sum_{j=1}^{n}(c_j + \alpha_j\theta_2)x_j^{(3)} \\ &= \sum_{j=1}^{n}\left[c_j + \alpha_j(\lambda\theta_1 + (1-\lambda)\theta_2)\right]x_j^{(3)} \\ &= \sum_{j=1}^{n}(c_j + \alpha_j\theta_3)x_j^{(3)} = Z(\theta_3, x^{(3)}) = Z^*(\theta_3)\end{aligned}$$

Hence, $Z^*(\theta)$ is convex in θ.

7.2-8.

(a) The same argument as in part (a) of problem 7.2-7 holds.

(b) The problem is:

$$\begin{array}{ll} \text{maximize} & Z(\theta) = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \sum_{j=1}^{n} a_{ij}x_j \leq b_i + \alpha_i\theta,\ i = 1, 2, \ldots, m \\ & x_j \geq 0,\ j = 1, 2, \ldots, n. \end{array}$$

Consider $\theta_1 < \theta_2$ and let $\theta_3 = \lambda\theta_1 + (1-\lambda)\theta_2$ for some $0 \leq \lambda \leq 1$. Let $x_j^{(1)}$, $x_j^{(2)}$ and $x_j^{(3)}$ be the optimal values of x_j $(j = 1, 2, \ldots, n)$ for θ_1, θ_2 and θ_3 respectively.

$$\begin{aligned}\lambda Z^*(\theta_1) + (1-\lambda)Z^*(\theta_2) &= \lambda\sum_{j=1}^{n} c_j x_j^{(1)} + (1-\lambda)\sum_{j=1}^{n} c_j x_j^{(2)} \\ &= \sum_{j=1}^{n} c_j(\lambda x_j^{(1)} + (1-\lambda)x_j^{(2)})\end{aligned}$$

If $x'_j = \lambda x_j^{(1)} + (1-\lambda)x_j^{(2)}$ $(j = 1, 2, \ldots, n)$, then x' is feasible for $\theta = \theta_3$, since

$$\begin{aligned}\sum_{j=1}^{n} a_{ij}x'_j = \lambda\sum_{j=1}^{n} a_{ij}x_j^{(1)} + (1-\lambda)\sum_{j=1}^{n} a_{ij}x_j^{(2)} &= \lambda(b_i + \alpha_i\theta) + (1-\lambda)(b_i + \alpha_i\theta) \\ &= b_i + \alpha_i\theta,\ i = 1, 2, \ldots, m.\end{aligned}$$

Since $x^{(3)}$ is optimal for θ_3,

$$\sum_{j=1}^{n} c_j(\lambda x_j^{(1)} + (1-\lambda)x_j^{(2)}) \leq \sum_{j=1}^{n} c_j x_j^{(3)} = Z^*(\theta_3).$$

Hence, $Z^*(\theta)$ is concave in θ.

7.2-9.

From duality theory,

$$Z^{**} = \text{minimum} \quad \sum_{i=1}^{m}(b_i + k_i)y_i$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij}y_i \geq c_j,\ j = 1, 2, \ldots, n$$

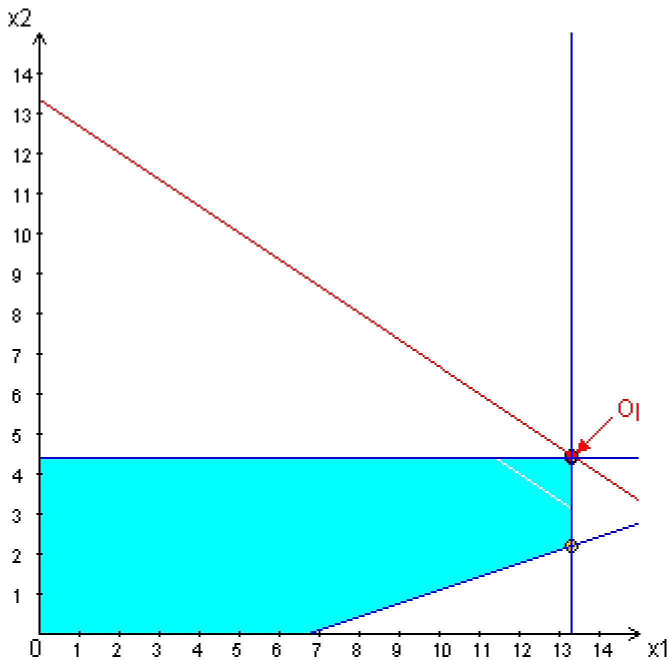
$$y_i \geq 0,\ i = 1, 2, \ldots, m.$$

$(y_1^*, y_2^*, \ldots, y_m^*)$ is feasible for this problem, so

$$Z^{**} \leq \sum_{i=1}^{m}(b_i + k_i)y_i^* = Z^* + \sum_{i=1}^{m} k_i y_i^*.$$

7.3-1.

(a) Optimal Solution: $(x_1^*, x_2^*) = (13.33, 4.44)$ and $Z^* = 40$



(b) $u_1 = \frac{40}{3}, u_2 = \frac{40}{9}, y_1 = \frac{40}{3} - x_1, y_2 = \frac{40}{9} - x_2$

Start with the initial solution $x_1 = x_2 = 0$ and $x_3 = 20$.

Iteration	Basic Variable	Eq	Coefficient of: Z	X_1	X_2	X_3	Right Side
0	Z	(0)	1	-2	-3	0	0
	X_3	(1)	0	3	-9	1	20

Since x_2 has the smallest coefficient in row 0, let it be the entering basic variable. It has no upper bound from Equation (1), so x_2 reaches its upper bound and we replace it by y_2.

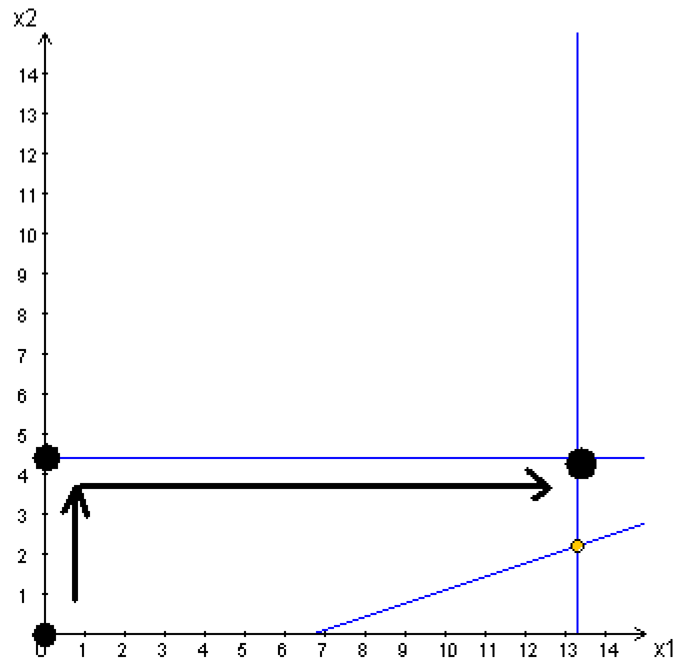
Iteration	Basic Variable	Eq	Coefficient of:				Right Side
			Z	x_1	y_2	x_3	
1	Z	(0)	1	-2	3	0	120/9
	x_3	(1)	0	3	9	1	60

Because it has a negative coefficient, x_1 enters the basis. From Equation (1), $x_1 \leq 20$, but this is greater than u_1, so x_1 reaches its upper bound and we replace it by y_1.

Iteration	Basic Variable	Eq	Coefficient of:				Right Side
			Z	y_1	y_2	x_3	
2	Z	(0)	1	2	3	0	40
	x_3	(1)	0	-3	9	1	20

There are no variables with negative coefficients, hence, the optimal solution is $x_1 = 40/3$, $x_2 = 40/9$ and $Z = 40$

(c)



7.3-2.

BV	Eq.	Z	x_1	x_2	x_3	x_4	x_5	RS	
Z	0	1	-1	-3	2	0	0	0	$x_2 \leq 3$
x_4	1	0	0	1	-2	1	0	1	$x_2 \leq 1$
x_5	2	0	2	1	2	0	1	8	$x_2 \leq 8$

BV	Eq.	Z	x_1	x_2	x_3	x_4	x_5	RS	
Z	0	1	-1	0	-4	3	0	3	$x_3 \leq 2$
x_2	1	0	0	1	-2	1	0	1	$x_3 \leq 1$
x_5	2	0	2	0	4	-1	1	7	$x_3 \leq 1\frac{3}{4}$

BV	Eq.	Z	x_1	y_2	x_3	x_4	x_5	RS	
Z	0	1	-1	0	-4	3	0	3	$x_3 \leq 2$
y_2	1	0	0	1	2	-1	0	2	$x_3 \leq 1$
x_5	2	0	2	0	4	-1	1	7	$x_3 \leq 1\frac{3}{4}$

BV	Eq.	Z	x_1	y_2	x_3	x_4	x_5	RS	
Z	0	1	-1	2	0	1	0	7	
x_3	1	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1	$x_1 \leq 1$
x_5	2	0	2	-2	0	1	1	3	$x_1 \leq 1\frac{1}{2}$

BV	Eq.	Z	y_1	y_2	x_3	x_4	x_5	RS
Z	0	1	-1	2	0	1	0	8
x_3	1	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1
x_5	2	0	2	-2	0	1	1	1

$(x_1, x_2, x_3) = (1, 3, 1)$ is optimal with $Z = 8$.

7.3-3.

Initial Tableau

BV	Eq.	Z	x_1	x_2	x_3	x_4	x_5	x_6	RS
Z	0	1	-2	-3	2	-5	0	0	0
x_5	1	0	2	2	1	2	1	0	5
x_6	2	0	1	2	-3	4	0	1	5

Final Tableau (after five iterations)

BV	Eq.	Z	x_1	y_2	x_3	y_4	x_5	x_6	RS
Z	0	1	0	$\frac{1}{7}$	0	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{54}{7}$
x_1	1	0	1	$-\frac{8}{7}$	0	$-\frac{10}{7}$	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{2}{7}$
x_3	2	0	0	$\frac{2}{7}$	1	$\frac{6}{7}$	$\frac{1}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$

$(x_1, x_2, x_3, x_4) = (2/7, 1, 3/7, 1)$ is optimal with $Z = 54/7$.

7.3-4.

Initial Tableau

BV	Eq.	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RS
Z	0	1	-2	-5	-3	-4	-1	0	0	0
x_6	1	0	1	3	2	3	1	1	0	6
x_7	2	0	4	6	5	7	1	0	1	15

Final Tableau (after seven iterations)

BV	Eq.	Z	y_1	y_2	y_3	y_4	x_5	x_6	x_7	RS
Z	0	1	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	0	10
y_4	1	0	$\frac{1}{3}$	1	$\frac{2}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	1
x_7	2	0	$-\frac{5}{3}$	1	$-\frac{1}{3}$	0	$-\frac{4}{3}$	$-\frac{7}{3}$	1	0

$(x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, 0, 0)$ is optimal with $Z = 10$.

7.3-5.

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side	
-Z	0	1	3	4	2	0	0	0	$X_1 \le 25$
X4	1	0	-1*	-1	0	1	0	-15	$X_1 \le 15$
X5	2	0	0	-1	-1	0	1	-10	

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side	
-Z	0	1	0	1	2	3	0	-45	$X_2 \le 5$
X1	1	0	1	1	0	1	0	15	$X_2 \le 15$
X5	2	0	0	-1	1	0	1	-10	$X_2 \le 10$

Bas Var	Eq No	Z	Coefficient of X1	Y2	X3	X4	X5	Right side	
-Z	0	1	0	-1	2	3	0	-50	$X_3 \le 15$
X1	1	0	1	-1	0	1	0	10	
X5	2	0	0	1	-1*	0	1	-5	$X_3 \le 5$

Bas Var	Eq No	Z	Coefficient of X1	Y2	X3	X4	X5	Right side
-Z	0	1	0	1	0	3	2	-60
X1	1	0	1	-1	0	1	0	10
X3	2	0	0	-1	1	0	-1	5

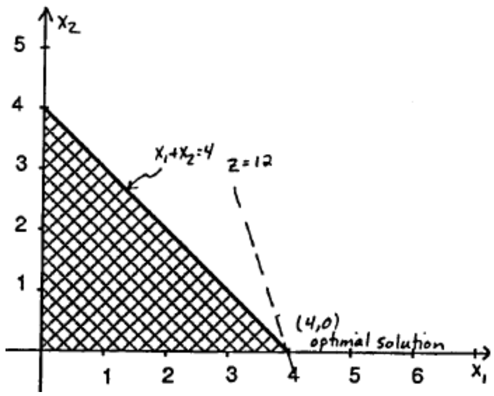
$(x_1, x_2, x_3) = (10, 5, 5)$ is optimal with $Z = 60$.

7.4-1.

It.	X_1	X_2	X_3
0	1	3	7
1	1.04605	4.95395	10.9539
2	0.93406	6.06594	13.0659

7.4-2.

(a)



The feasible corner point solutions are $(0,0)$, $(0,4)$ and $(4,0)$. The last one is optimal with $Z=12$.

(b)

Iter.	x_1	x_2	Z
0	1	1	4
1	1.875	1.125	6.75
2	2.6981	0.8019	8.89621
3	3.34396	0.40095	10.4328
4	3.6671	0.20047	11.2018

(c)

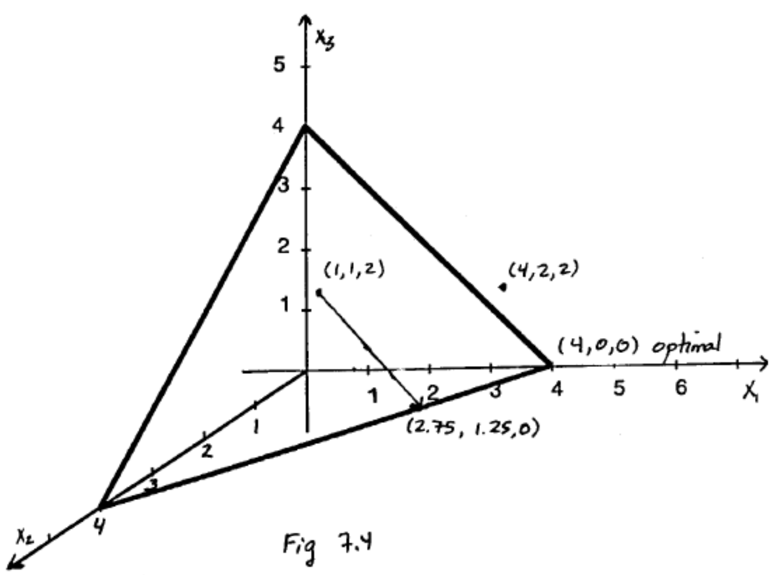
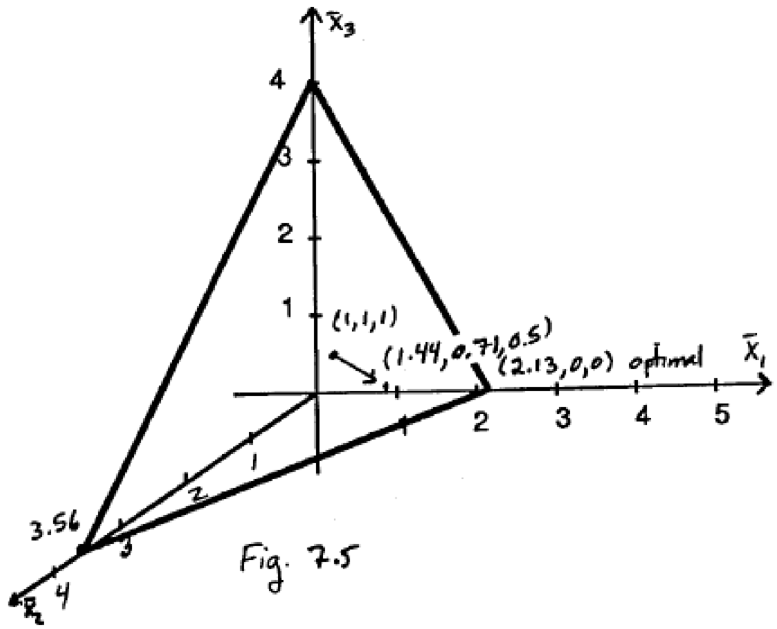
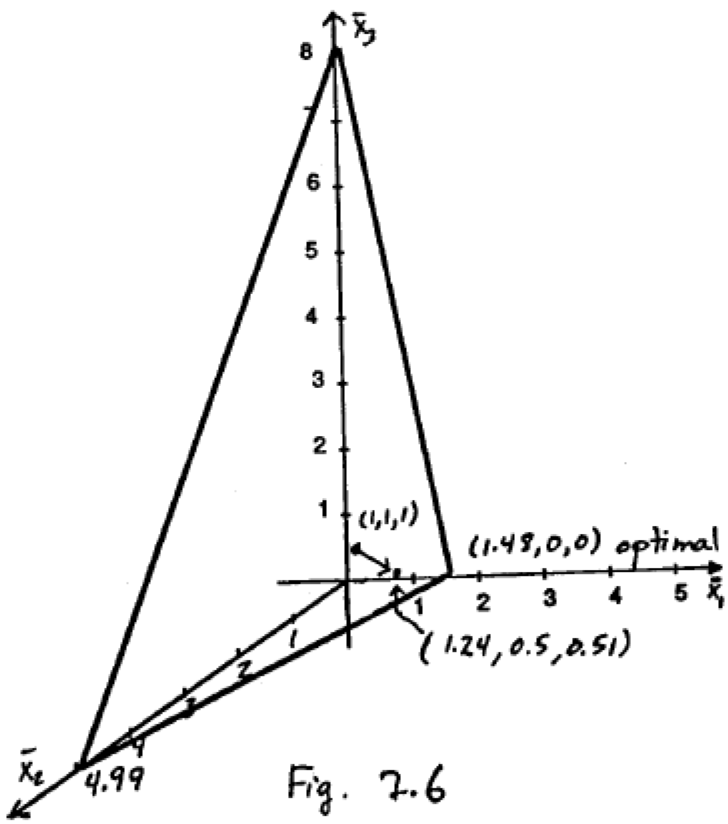


Fig 7.4



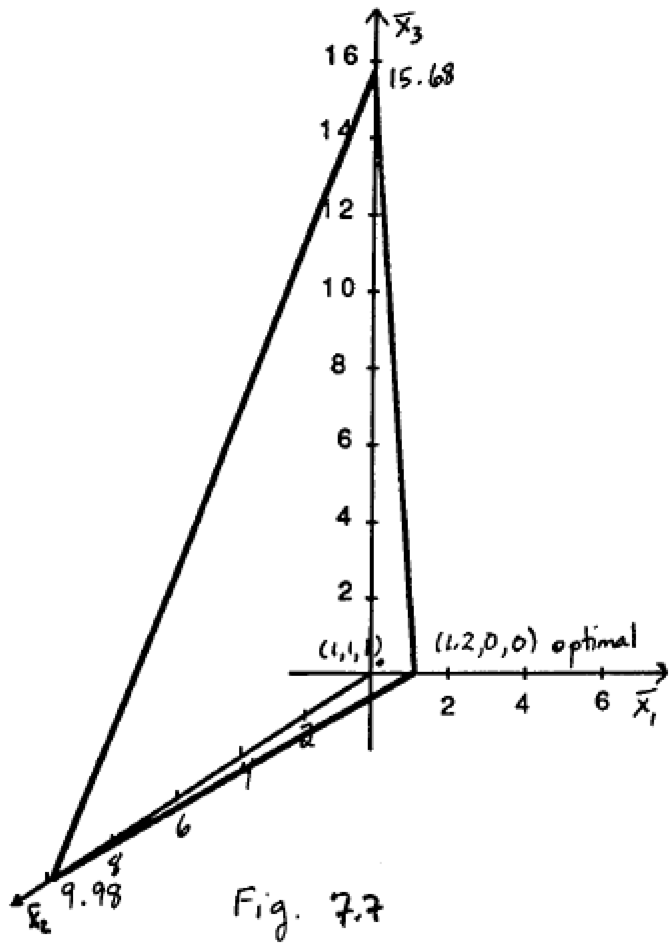
$\bar{x}_3$
4
3
2
1
(1,1,1)
(1.44,0.71,0.5)
(2.13,0,0) optimal
$\bar{x}_1$
2
3
4
5
1
2
3
3.56
4
$\bar{x}_2$

Fig. 7.5



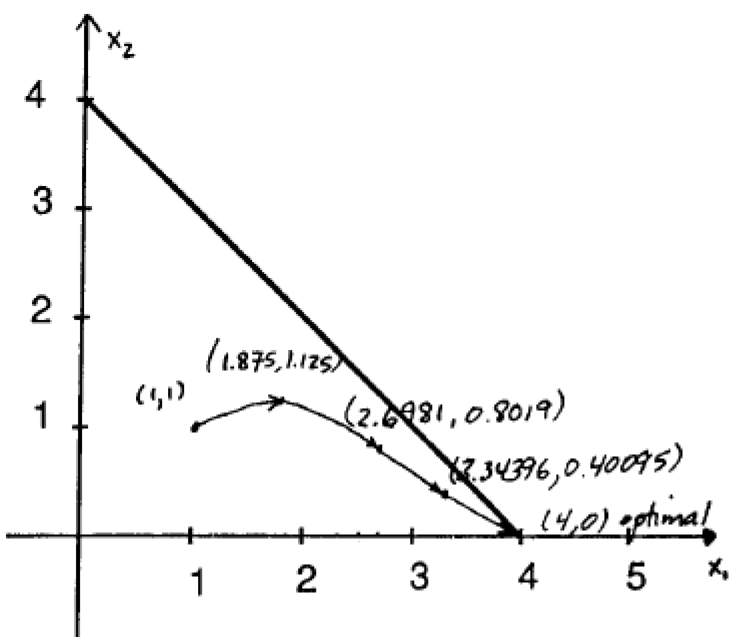
$\bar{x}_3$
8
6
5
4
3
2
1
(1,1,1)
(1.48,0,0) optimal
1
2
3
4
5
$\bar{x}_1$
(1.24,0.5,0.51)
1
2
3
4
$\bar{x}_2$
4.99

Fig. 7.6



$\bar{x}_3$
16
15.68
14
12
10
8
6
4
2
(1,1,1)
(1.2,0,0) optimal
2
4
6
$\bar{x}_1$
2
4
6
8
9.98
$\bar{x}_2$

Fig. 7.7



x_2
4
3
2
1
(1.875,1.125)
(1,1)
(2.6981, 0.8019)
(3.34396, 0.40095)
(4,0) optimal
1
2
3
4
5
x_1

Fig. 7-8

7.4-3.

(a)

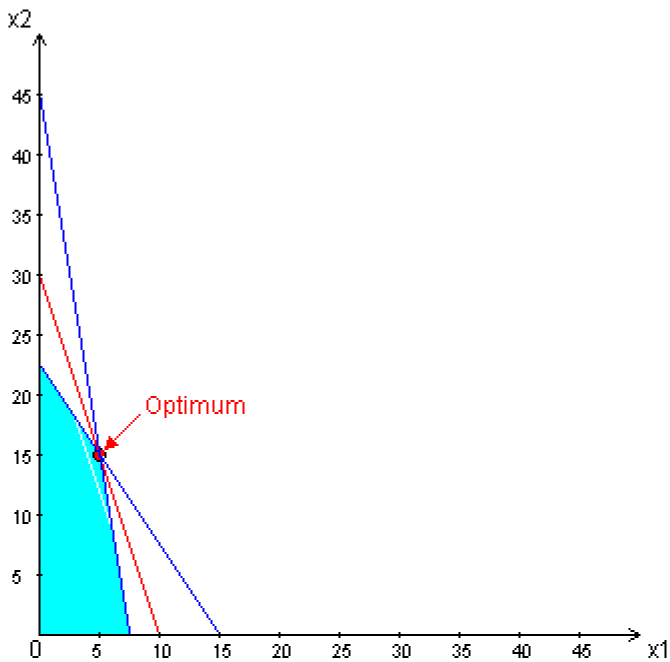
Iter.	x_1	x_2	Z
0	4	4	12
1	2	6	14
2	1	7	15
3	0.5	7.5	15.5
4	0.25	7.75	15.75
5	0.125	7.875	15.875
6	0.0625	7.9375	15.9375
7	0.03125	7.96875	15.9688
8	0.01562	7.98438	15.9844
9	0.00781	7.99219	15.9922

(b) The value of x_1 is halved at each step so subsequent trial solutions should be of the form $(x_1, x_2) = (2^{-i}, 8 - 2^{-i})$ for $i = 1, 2, \ldots$.

(c) The smallest integer i such that $2^{-i} - 2^{-(i+1)} = 2^{-(i+1)} \leq 0.01$ is 6, so $(x_1, x_2) = (2^{-7}, 8 - 2^{-7}) = (0.0078, 7.9922)$ in iteration 9.

7.4-4.

(a) Optimal Solution: $(x_1, x_2) = (5, 15)$, $Z = 30$



(b) The gradient is $(3, 1)$. Moving from the origin in the direction $(3, 1)$, the first boundary point encountered is the optimal solution $(5, 15)$.

(c) $\alpha = 0.5$

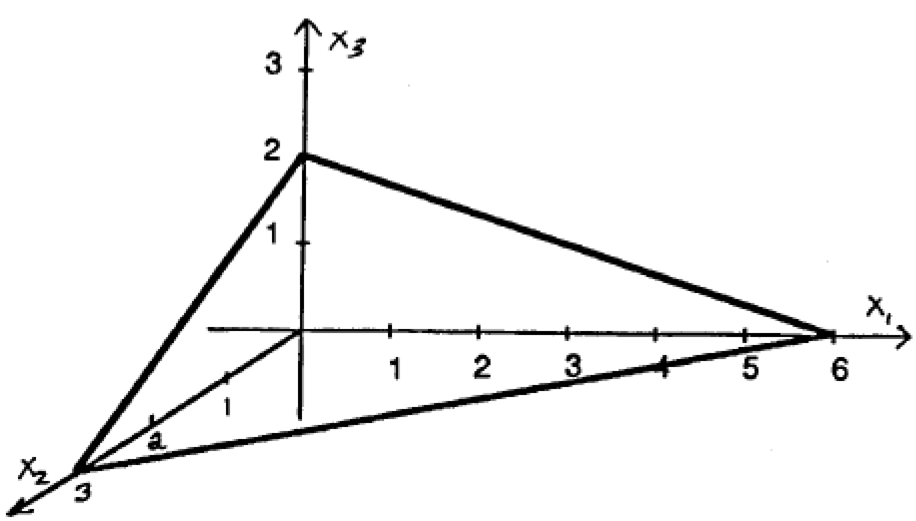
Iter.	X1	X2	Z
0	1	1	4
1	3.999	2.006	14.003
2	5.547	2.217	18.859
3	6.293	2.492	21.371
4	6.582	3.131	22.878
5	6.454	5.089	24.451
6	5.668	10.133	27.137
7	5.254	12.686	28.449
8	5.059	13.946	29.122
9	4.98	14.547	29.487
10	4.964	14.812	29.705

(d) $\alpha = 0.9$

Iter.	X1	X2	Z
0	1	1	4
1	6.398	2.811	22.005
2	6.668	4.614	24.617
3	5.107	14.051	29.372
4	4.962	14.979	29.863
5	5.002	14.962	29.969
6	4.997	15.001	29.992
7	5	14.998	29.998
8	5	15	29.999
9	5	15	30
10	5	15	30

7.4-5.

(a)



(b) Gradient: $\begin{pmatrix} 2 & 5 & 7 \end{pmatrix}$

Projected Gradient: $P\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} = \left[I - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \left(\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \right] \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$

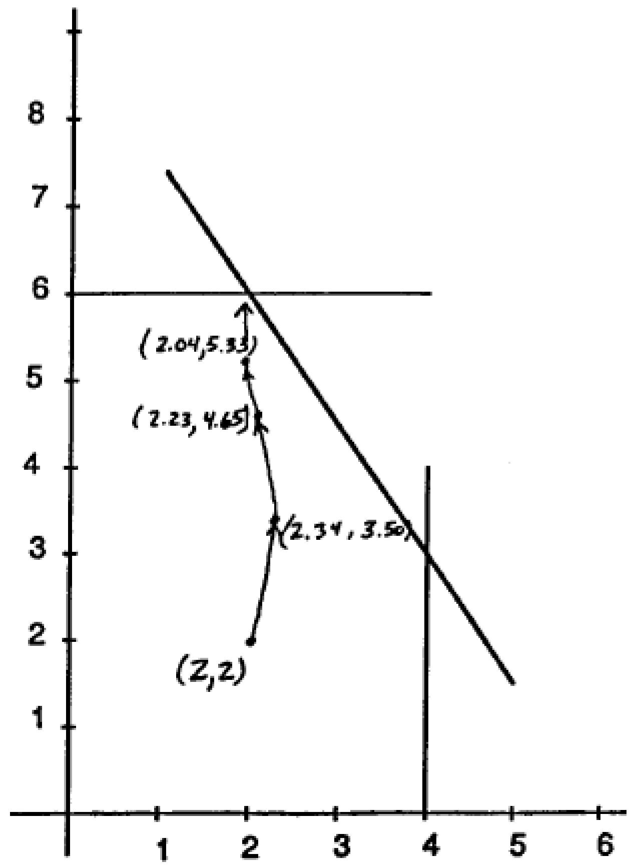
$$= \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} - \frac{1}{14} \begin{pmatrix} 33 \\ 66 \\ 99 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} -5 \\ 4 \\ -1 \end{pmatrix}$$

(c) - (d)

Iter.	x_1	x_2	x_3	Z
0	1	1	1	14
1	0.5	1.4	0.9	14.3
2	0.25969	2.19516	0.45	14.6452
3	0.17947	2.57276	0.225	14.7978
4	0.1069	2.7778	0.1125	14.8903
5	0.05595	2.88765	0.05625	14.9439
6	0.0281	2.94376	0.02812	14.9719
7	0.01406	2.97188	0.01406	14.9859
8	0.00703	2.98594	0.00703	14.993
9	0.00352	2.99297	0.00352	14.9965
10	0.00176	2.99648	0.00176	14.9982

7.4-6.

Iter.	x_1	x_2	Z
0	2	2	16
1	2.336	3.496	24.488
2	2.23067	4.65399	29.962
3	2.03597	5.32699	32.7429
4	1.95211	5.6635	34.1738
5	1.95054	5.83175	35.0104
6	1.97169	5.91587	35.4944
7	1.98588	5.95788	35.7471
8	1.99296	5.97891	35.8734
9	1.99648	5.98945	35.9367
10	1.99824	5.99473	35.9684
11	1.99912	5.99736	35.9842
12	1.99956	5.99868	35.9921
13	1.99978	5.99934	35.996
14	1.99989	5.99967	35.998
15	1.99995	5.99984	35.999



8
7
6
5
4
3
2
1
1
2
3
4
5
6
(2.04, 5.33)
(2.23, 4.65)
(2.34, 3.50)
(2,2)

SUPPLEMENT TO CHAPTER 7

LINEAR GOAL PROGRAMMING AND ITS SOLUTION PROCEDURES

7S-1.

(a) $3x_1 + 4x_2 + 2x_3 - y^+ + y^- = 60$

(b) Let c^+ be the coefficient of y^+ and c^- be the one for y^-, so $c^+ = 2c^-$.

7S-2.

(a)

minimize sum of amounts under market share for product 1 and 2

subject to
$$x_1 + x_2 + x_3 \le 55$$
$$x_3 \ge 10$$
$$x_1, x_2 \ge 0$$

(b) $y_1 = 0.5x_1 + 0.2x_3 - 15, y_1 = y_1^+ - y_1^-, y_2 = 0.3x_2 + 0.2x_3 - 10, y_2 = y_2^+ - y_2^-$

minimize $y_1^- + y_2^-$

subject to
$$0.5x_1 + 0.2x_3 - y_1^+ + y_1^- = 15$$
$$0.3x_2 + 0.2x_3 - y_2^+ + y_2^- = 10$$
$$x_1 + x_2 + x_3 \le 55$$
$$x_3 \ge 10$$
$$x_1, x_2, y_1^+, y_1^-, y_2^+, y_2^- \ge 0$$

(c)

Goals	Unit Contribution Per Unit of Each Activity Campaign 1	Campaign 2	Campaign 3	Level Achieved		Goal	Amount Over	Amount Under	Totals		Right-Hand Side
Market Share 1	0.5	0	0.2	15	≥	15	0	0	15	=	15
Market Share 2	0	0.3	0.2	8.33333	≥	10	0	1.667	10	=	10
Budget	1	1	1	55	≤	55			55	≤	55
Campaign 3 budget	0	0	1	41.6667	≥	10			41.667	≥	10
Solution	13.333	0	41.667								

Weighted Sum of Deviations 1.667

7S-3.

(a)
$$6x_1 + 4x_2 + 5x_3 - y_1^+ + y_1^- = 50$$
$$8x_1 + 7x_2 + 5x_3 - y_2^+ + y_2^- = 75$$
$$P = 20x_1 + 15x_2 + 25x_3$$

(b) $Z = 20x_1 + 15x_2 + 25x_3 - 6y_1^+ - 6y_1^- - 3y_2^-$

(c)

maximize $20x_1 + 15x_2 + 25x_3 - 6y_1^+ - 6y_1^- - 3y_2^-$

subject to
$$6x_1 + 4x_2 + 5x_3 - y_1^+ + y_1^- = 50$$
$$8x_1 + 7x_2 + 5x_3 - y_2^+ + y_2^- = 75$$
$$x_1, x_2, x_3, y_1^+, y_1^-, y_2^+, y_2^- \ge 0$$

(d)

Goals	Unit Contribution Per Unit of Each Activity Product 1	Product 2	Product 3	Level Achieved		Goal	Amount Over	Amount Under	Totals		Right-Hand Side
Profit	20	15	25	375							
Employment	6	4	5	75	=	50	25	0	50	=	50
Earnings	8	7	5	75	≥	75	0	0	75	=	75
Solution	0	0	15								

Weighted Sum of Deviations = 225

7S-4.

(a) No, we would not expect the optimal solution to change. Goal 1 is already met, so increasing the weight on that goal would not change anything. Goal 2 is already exceeded, so decreasing the penalty weight for this goal would only decrease our desire to avoid exceeding this goal.

(b)

Goals	Product 1	Product 2	Product 3	Achieved		Goal	Over	Under	Totals		Side
Profit	12	9	15	140	≥	140	0	0	140	=	140
Employment	5	3	4	58.3333	=	40	18.3333	0	40	=	40
Investment	5	7	8	58.3333	≤	55	3.33333	0	55	=	55
Solution	11.6667	0	0								

Weighted Sum of Deviations = 46.6667

(c)

Goals	Unit Contribution Per Unit of Each Act Product 1	Product 2	Product 3	Level Achieved		Goal	Amount Over	Amount Under	Totals		Right-Hand Side
Profit	12	9	15	140	≥	140	0	0	140	=	140
Employment	5	3	4	58.3333	=	40	18.3333	0	40	=	40
Investment	5	7	8	58.3333	≤	55	3.33333	0	55	=	55
Solution	11.6667	0	0								

Weighted Sum of Deviations = 28.3333

7S-5.

(a)

minimize 0.01(amount under foreign capital goal)
+ (amount under citizens fed goal)
+ (amount under goal for citizens employed)
+ (amount over goal for citizens employed)

(b)

minimize $0.01y_1^- + y_2^- + y_3^+ + y_3^-$

subject to
$$1000x_1 + 1000x_2 + 1000x_3 + x_4 = 15\text{M}$$
$$3000x_1 + 5000x_2 + 4000x_3 - y_1^+ + y_1^- = 70\text{M}$$
$$150x_1 + 75x_2 + 100x_3 - y_2^+ + y_2^- = 1.75\text{M}$$
$$10x_1 + 15x_2 + 12x_3 - y_3^+ + y_3^- = 0.2\text{M}$$
$$x_1, x_2, x_3, x_4, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \geq 0$$

(c)

Goals	Unit Contribution Per Unit of Each Activity Product 1	Product 2	Product 3	Level Achieved		Goal	Amount Over	Amount Under	Totals		Right-Hand Side
Foreign Capital	3000	5000	4000	58333333	≥	70000000	0	11666666.7	70000000	=	70000000
Citizens Fed	150	75	100	1750000	≥	1750000	0	0	1750000	=	1750000
Citizens Employed	10	15	12	183333.3	=	200000	0	16666.7	200000	=	200000
Acres	1000	1000	1000	15000000	≤	15000000					
Solution	8333.333	6666.667	0								

<-- Weighted Sum of Deviations = 133333.3

(d) minimize $M_2y_1^- + M_1y_2^- + y_3^+ + y_3^-$

subject to

$$\begin{aligned}
&1000x_1 + 1000x_2 + 1000x_3 + x_4 = 15\text{M} \\
&3000x_1 + 5000x_2 + 4000x_3 - y_1^+ + y_1^- = 70\text{M} \\
&150x_1 + 75x_2 + 100x_3 - y_2^+ + y_2^- = 1.75\text{M} \\
&10x_1 + 15x_2 + 12x_3 - y_3^+ + y_3^- = 0.2\text{M} \\
&x_1, x_2, x_3, x_4, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \geq 0
\end{aligned}$$

(e) Optimal Solution: $(x_1, x_2, x_3) = (50000/6, 20000/6, 0)$ thousand acres

$$Z = (35 \cdot 10^6/3)M_2 + 50000/3.$$

BV	Eq#	Z	x_1	x_2	x_3	x_4	y_1^+	y_1^-	y_2^+	y_2^-	y_3^+	y_3^-	RHS
Z	0	-1	$-150M_1$ $-3000M_2$ -10	$-75M_1$ $-5000M_2$ -15	$-100M_1$ $-4000M_2$ -12	0	M_2	0	M_1	0	2	0	$-175M_1$ $-7000M_2$ -20
x_4	1	0	1000	1000	1000	1	0	0	0	0	0	0	1500
y_1^-	2	0	3000	5000	4000	0	-1	1		0	0	0	7000
y_2^-	3	0	150*	75	100	0	0	0	-1	1	0	0	175
y_3^-	4	0	10	15	12	0	0	0	0	0	-1	1	20
Z	0	-1	0	$-3500M_2$ -10	$-2000M_2$ -12	0	M_2	0	$-20M_2$ $-1/15$	$M_1 + 20M_2$ $1/15$	2	0	$3500M_2$ $-25/3$
x_4	1	0	0	500*	1000/3	1	0	0	20/3	-20/3	0	0	1000/3
y_1^-	2	0	0	3500	2000	0	-1	1	20	-20	0	0	3500
x_1	3	0	1	1/2	2/3	0	0	0	-1/150	1/150	0	0	7/6
y_3^-	4	0	0	10	12	0	0	0	1/15	-1/15	-1	1	25/3
Z	0	-1	0	0	$1000/3M_2 + 4/3$	$7M_2 + 1/100$	M_2	0	$80/3M_2 - 1/15$	$M_1 - 80/3M_2 - 1/15$	2	0	$\frac{3500}{3}M_2$ $-8/3$
x_2	1	0	0	1	2/3	1/500	0	0	1/75	-1/75	0	0	2/3
y_1^-	2	0	0	0	-1000/3	-7	-1	1	-80/3	80/3	0	0	3500/3
x_1	3	0	1	0	1/3	-1/1000	0	0	0	0	0	0	5/6
y_3^-	4	0	0	0	-4/3	-1/50	0	0	-1/15	1/15	-1	1	6/3

(f) With only $M_1y_2^-$ in the objective function, we get $y_2^- = Z = 0$, so fix $y_2^- = 0$ and bring $M_2y_1^-$ into the objective function. Now $y_1^- = 11,666,666\frac{2}{3}$. Fix y_1^- at this value (remembering subtract from RHS) and optimize for the third priority. Then the solution in part (c) is obtained: $(x_1, x_2, y_1^-, y_3^-) = \left(8333\frac{1}{3}, 6666\frac{2}{3}, 11666666\frac{2}{3}, 16666\frac{2}{3}\right)$.

7S-6.

(a) minimize $M_1y_1^+ + M_2y_2^+ + M_2y_2^- + y_3^-$

subject to

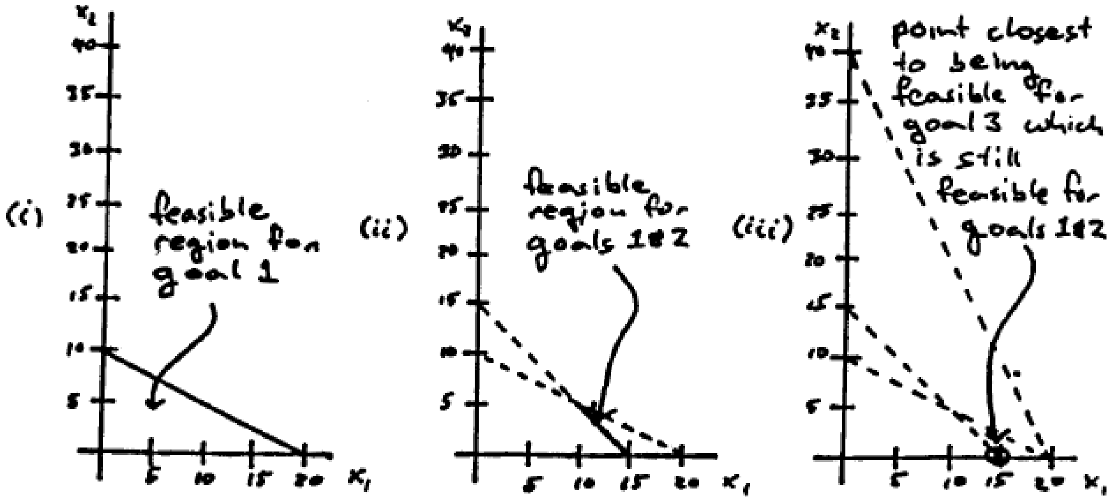
$$\begin{aligned}
&x_1 + 2x_2 - y_1^+ + y_1^- = 20 \\
&x_1 + x_2 - y_2^+ + y_2^- = 15 \\
&2x_1 + x_2 - y_3^+ + y_3^- = 40 \\
&x_1, x_2, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \geq 0
\end{aligned}$$

(b) - (c)

Optimal Solution: $(x_1, x_2) = (15, 0)$, $Z = 10$

	BV	E	Z	x_1	x_2	y_1^+	y_1^-	y_2^+	y_2^-	y_3^+	y_3^-	RHS
0	Z	0	-1	$-M_2-2$	$-M_2-1$	M_1	0	$2M_2$	0	1	0	$-15M_2-40$
	y_1^-	1	0	1	2	-1	1	0	0	0	0	20
	y_2^-	2	0	1	1	0	0	-1	1	0	0	15
	y_3^-	3	0	2	1	0	0	0	0	-1	1	40
1	Z	0	-1	0	1	M_1	0	M_2-2	M_2+2	1	0	-10
	y_1^-	1	0	0	1	-1	1	1	-1	0	0	5
	x_1	2	0	1	1	0	0	-1	1	0	0	15
	y_3^-	3	0	0	-1	0	0	2	-2	-1	1	10

(d)



(e) minimize $Z_1 = M_1 y_1^+$

subject to

$$x_1 + 2x_2 - y_1^+ + y_1^- = 20$$
$$[x_1 + x_2 - y_2^+ + y_2^- = 15]$$
$$[2x_1 + x_2 - y_3^+ + y_3^- = 40]$$
$$x_1, x_2 \geq 0$$

The feasible region is a shown in figure (i) of part (d). Fix $y_1^+ = 0$.

minimize $Z_2 = M_2 y_2^+ + M_2 y_2^-$

subject to

$$x_1 + 2x_2 - y_1^+ + y_1^- = 20$$
$$x_1 + x_2 - y_2^+ + y_2^- = 15$$
$$[2x_1 + x_2 - y_3^+ + y_3^- = 40]$$
$$x_1, x_2 \geq 0$$

The feasible region is a shown in figure (ii) of part (d). Fix $y_1^+ = y_2^+ = y_2^- = 0$.

minimize $Z_3 = y_3^-$

subject to

$$x_1 + 2x_2 - y_1^+ + y_1^- = 20$$
$$x_1 + x_2 - y_2^+ + y_2^- = 15$$
$$2x_1 + x_2 - y_3^+ + y_3^- = 40$$
$$x_1, x_2 \geq 0$$

The solution is $(15, 0)$ with $Z_3 = 10$.

7S-7.

(a) minimize $M_1 y_1^+ + M_2 y_2^- + y_3^-$

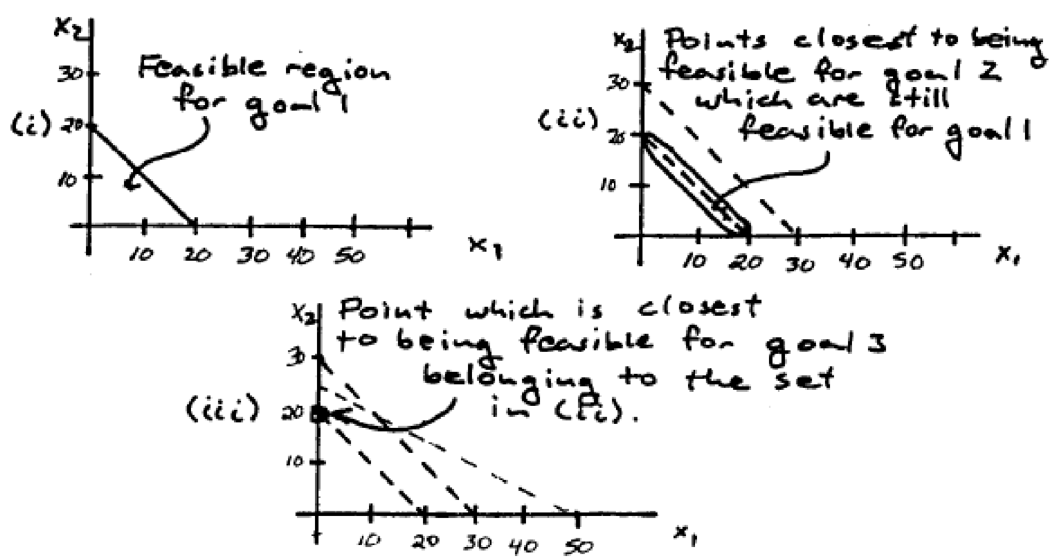
subject to
$$x_1 + x_2 - y_1^+ + y_1^- = 20$$
$$x_1 + x_2 - y_2^+ + y_2^- = 30$$
$$x_1 + 2x_2 - y_3^+ + y_3^- = 50$$
$$x_1, x_2, y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^- \geq 0$$

(b) - (c)

Optimal Solution: $(x_1, x_2) = (0, 20)$, $Z = 10M_2 + 10$

	BV	E	Z	x_1	x_2	y_1^+	y_1^-	y_2^+	y_2^-	y_3^+	y_3^-	RHS
0	Z	0	-1	$-M_2-1$	$-M_2-2$	M_1	0	M_2	0	1	0	$-30M_2-50$
	y_1^-	1	0	1	1	-1	1	0	0	0	0	20
	y_2^-	2	0	1	1	0	0	-1	1	0	0	30
	y_3^-	3	0	1	2	0	0	0	0	-1	1	50
1	Z	0	-1	1	0	M_1-M_2-2	M_2+2	M_2	0	1	0	$-10M_2-10$
	x_2	1	0	1	1	-1	1	0	0	0	0	20
	y_2^-	2	0	0	0	1	-1	-1	1	0	0	10
	y_3^-	3	0	-1	0	2	-2	0	0	-1	1	10

(d)



7S-8.

If $z_i = z_i^+ - z_i^-$, where $z_i^+, z_i^- \geq 0$, then $|z_i| = z_i^+ + z_i^-$.

(a) minimize $\sum_{i=1}^{n} (z_i^+ + z_i^-)$

subject to
$$z_i^+ - z_i^- = y_i - (a + bx_i),\ i = 1, 2, \ldots, n$$
$$z_i^+, z_i^- \geq 0,\ i = 1, 2, \ldots, n$$

(b) minimize z

subject to
$$z_i^+ - z_i^- = y_i - (a + bx_i),\ i = 1, 2, \ldots, n$$
$$0 \leq z_i^+ \leq z,\ i = 1, 2, \ldots, n$$
$$0 \leq z_i^- \leq z,\ i = 1, 2, \ldots, n$$

Cases

7S.1 a) We need to develop a goal programming problem whose solution characterizes Mr. Baker's shipping policy. The decision variables are the number (in 1000's) of basic, advanced, and supreme packages to send, and the number of doctors to send. Note: measuring most variables in 1000's greatly improves the reliability of the Excel Solver.

Mr. Baker faces three hard constraints. Because of the size limitation, the total number of package must not exceed 40,000. Second, the total weight can not exceed 6 million pounds. Finally, the total number of Supreme packages cannot exceed 100 times the number of doctors. These constraints are included in the spreadsheet as follows.

TotalPackages (E14) = SizeLimit (E16)
TotalWeight (E10) = WeightRestriction (G10)
SupremePackages (D14) = SafetyRestriction (D16)

In addition, we need to include three constraints for Mr. Baker's goals. We measure the deviations from the goals using changing cells (Deviations in I4:J6), and enforce the correct value of these changing cells with the constraints in columns L through N.

Finally, the penalty weights are entered in I15:J17, and the weight sum of deviations calculated in L15.

The spreadsheet follows.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1					Goals				Deviations			Constraints			
2					Level				Amount	Amount		Balance			
3		Basic	Advanced	Supreme	Achieved		Goal		Over	Under		(Level-Over+Under)		Goal	
4	Goal 1 (Cost)	$300	$350	$720	21,000	≤	20,000		1,000	0		20,000	=	20,000	$thousand
5	Goal 2 (Packages Sent)	1	1	1	40	≥	3		37	0		3	=	3	thousand
6	Goal 3 (Population Reached)	30	35	54	1,488	≥	2,200		0	712		2,200	=	2,200	thousand
7															
8					Total		Weight								
9					Weight		Restriction								
10	Weight	120	180	220	6,000	≤	6,000	thousand pounds							
11															
12					Total										
13		Basic	Advanced	Supreme	Packages			Penalty	Over	Under		Weighted Sum			
14	Packages Sent (thousands)	28	0	12	40			Weights	Goal	Goal		of Deviations			
15				≤	≤			Goal 1	0.001			50.84			
16	Doctors	120	Safety	12	40			Goal 2		1					
17			Restriction	0.1	Size Limit			Goal 3		0.07					
18				per Doctor											
19	Cost per Doctor ($thousand)	33													

Range Name	Cells
AmountOver	I4:I6
AmountUnder	J4:J6
Balance	L4:L6
CostPerDoctor	B19
Deviations	I4:J6
Doctors	B16
Goal	G4:G6
LevelAchieved	E4:E6
PackagesSent	B14:D14
PenaltyWeights	I15:J17
SafetyRestriction	D16
SizeLimit	E16
SumOfDeviations	L15
SupremePackages	D14
TotalPackages	E14
TotalWeight	E10
Weight	B10:D10
WeightRestriction	G10

	E
2	Level
3	Achieved
4	=SUMPRODUCT(B4:D4,PackagesSent)+Doctors*CostPerDoctor
5	=SUMPRODUCT(B5:D5,PackagesSent)
6	=SUMPRODUCT(B6:D6,PackagesSent)
7	
8	Total
9	Weight
10	=SUMPRODUCT(Weight,PackagesSent)
11	
12	Total
13	Packages
14	=SUM(PackagesSent)

	L	M	N
2	Balance		
3	(Level-Over+Under)		Goal
4	=LevelAchieved-AmountOver+AmountUnder	=	=Goal
5	=LevelAchieved-AmountOver+AmountUnder	=	=Goal
6	=LevelAchieved-AmountOver+AmountUnder	=	=Goal

	C	D
16	Safety	=D17*Doctors
17	Restriction	0.1

	L
13	Weighted Sum
14	of Deviations
15	=SUMPRODUCT(PenaltyWeights,Deviations)

Mr. Baker should send 28,000 basic packages and 12,000 supreme packages along with 120 doctors to Cuba.

b) The penalty weight for being under goal 3 changes. One-half percent of the population is 55,000. Therefore, the new penalty weight is 10 points / 55 (thousand people) = 0.182. The new solution follows.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1					Goals				Deviations			Constraints			
2					Level				Amount	Amount		Balance			
3		Basic	Advanced	Supreme	Achieved		Goal		Over	Under		(Level-Over+Under)		Goal	
4	Goal 1 (Cost)	$300	$350	$720	21,000	≤	20,000		1,000	0		20,000	=	20,000	$thousand
5	Goal 2 (Packages Sent)	1	1	1	40	≥	3		37	0		3	=	3	thousand
6	Goal 3 (Population Reached)	30	35	54	1,488	≥	2,200		0	712		2,200	=	2,200	thousand
7															
8					Total		Weight								
9					Weight		Restriction								
10	Weight	120	180	220	6,000	≤	6,000	thousand pounds							
11															
12					Total										
13		Basic	Advanced	Supreme	Packages		Penalty		Over	Under		Weighted Sum			
14	Packages Sent (thousands)	28	0	12	40		Weights		Goal	Goal		of Deviations			
15				≤	≤		Goal 1		0.001			130.45			
16	Doctors	120	Safety	12	40		Goal 2			1					
17			Restriction	0.1	Size Limit		Goal 3			0.182					
18				per Doctor											
19	Cost per Doctor ($thousand)	33													

The optimal shipping policy did not change. The plan appears to be insensitive to increases in the penalty weight for violating the goal to reach at least 20% of the Cuban population.

c) The doctors needed per thousand supreme packages changes from 0.1 to 0.075. The new solution follows.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1					Goals				Deviations			Constraints			
2					Level				Amount	Amount		Balance			
3		Basic	Advanced	Supreme	Achieved		Goal		Over	Under		(Level-Over+Under)		Goal	
4	Goal 1 (Cost)	$300	$350	$720	22,320	≤	20,000		2,320	0		20,000	=	20,000	$thousand
5	Goal 2 (Packages Sent)	1	1	1	40	≥	3		37	0		3	=	3	thousand
6	Goal 3 (Population Reached)	30	35	54	1,488	≥	2,200		0	712		2,200	=	2,200	thousand
7															
8					Total		Weight								
9					Weight		Restriction								
10	Weight	120	180	220	6,000	≤	6,000	thousand pounds							
11															
12					Total										
13		Basic	Advanced	Supreme	Packages		Penalty		Over	Under		Weighted Sum			
14	Packages Sent (thousands)	28	0	12	40		Weights		Goal	Goal		of Deviations			
15				≤	≤		Goal 1		0.001			131.77			
16	Doctors	160	Safety	12	40		Goal 2			1					
17			Restriction	0.075	Size Limit		Goal 3			0.182					
18				per Doctor											
19	Cost per Doctor ($thousand)	33													

While the number of packages Mr. Baker should ship has not changed, the number of doctors is now 160.

d) The budget restriction is now a hard constraint and the penalty variables for the cost goal can be eliminated.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1					Goals				Deviations			Constraints			
2					Level				Amount	Amount		Balance			
3		Basic	Advanced	Supreme	Achieved		Goal		Over	Under		(Level-Over+Under)		Goal	
4	Cost (Hard Constraint)	$300	$350	$720	20,000	≤	20,000								$thousand
5	Goal 2 (Packages Sent)	1	1	1	40	≥	3		37	0		3	=	3	thousand
6	Goal 3 (Population Reached)	30	35	54	1,465	≥	2,200		0	735.5		2,200	=	2,200	thousand
7															
8					Total		Weight								
9					Weight		Restriction								
10	Weight	120	180	220	6,000	≤	6,000	thousand pounds							
11															
12					Total										
13		Basic	Advanced	Supreme	Packages										
14	Packages Sent (thousands)	27	2.5	10.5	40			Penalty	Over	Under		Weighted Sum			
15				≤	≤			Weights	Goal	Goal		of Deviations			
16	Doctors	105	Safety	10.5	40			Goal 2		1		51.49			
17			Restriction	0.1	Size Limit			Goal 3		0.07					
18				per Doctor											
19	Cost per Doctor ($thousand)	33													

Mr. Baker should send 27,000 basic packages, 2,500 advanced packages, and 10,500 supreme packages along with 105 doctors to Cuba.

e) We start by minimizing the amount over goal 1 (total cost = $20 million).

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1					Goals				Deviations			Constraints			
2					Level				Amount	Amount		Balance			
3		Basic	Advanced	Supreme	Achieved		Goal		Over	Under		(Level-Over+Under)		Goal	
4	Goal 1 (Cost)	$300	$350	$720	20,000	≤	20,000		0	0		20,000	=	20,000	$thousand
5	Goal 2 (Packages Sent)	1	1	1	19.024	≥	3		16.024	0		3	=	3	thousand
6	Goal 3 (Population Reached)	30	35	54	1,027	≥	2,200		0	1,173		2,200	=	2,200	thousand
7															
8					Total		Weight		Minimize Over Goal 1						
9					Weight		Restriction								
10	Weight	120	180	220	4,185	≤	6,000								
11															
12					Total										
13		Basic	Advanced	Supreme	Packages										
14	Packages Sent (thousands)	0	0	19.024	19.02361111										
15				≤	≤										
16	Doctors	191	Safety	19.1	40										
17			Restriction	0.1	Size Limit										
18				per Doctor											
19	Cost per Doctor ($thousand)	33													

Then, since goal 2 is already met, we move on to goal 3. We minimize the amount under goal 3 (population reached = 20%), while constraining (amount over goal 1 = 0) and (amount under goal 2 = 0).

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1					Goals				Deviations			Constraints			
2					Level				Amount	Amount		Balance			
3		Basic	Advanced	Supreme	Achieved		Goal		Over	Under		(Level-Over+Under)		Goal	
4	Goal 1 (Cost)	$300	$350	$720	20,000	≤	20,000		0	0		20,000	=	20,000	$thousand
5	Goal 2 (Packages Sent)	1	1	1	40	≥	3		37	0		3	=	3	thousand
6	Goal 3 (Population Reached)	30	35	54	1,464	≥	2,200		0	736		2,200	=	2,200	thousand
7															
8					Total		Weight		Minimize Under Goal 3						
9					Weight		Restriction		(Over Goal 1 = 0)						
10	Weight	120	180	220	6,000	≤	6,000		(Under Goal 2 = 0)						
11															
12					Total										
13		Basic	Advanced	Supreme	Packages										
14	Packages Sent (thousands)	27	2.5	10.5	40										
15				≤	≤										
16	Doctors	105	Safety	10.5	40										
17			Restriction	0.1	Size Limit										
18				per Doctor											
19	Cost per Doctor ($thousand)	33													

Mr. Baker should send 27 thousand basic packages, 2,500 advanced packages, and 10,500 supreme packages, along with 105 doctors.

7S.2 a) The two decisions to be made are how much to spend on the two security systems. Hence, we define the following two variables.

Let PS = thousands of dollars spent per portal system
SS = thousands of dollars spent per screening system.

b) Preemptive goal programming is appropriate because there is a clear order of priorities.

Priority 1 is met by all possible systems.

Priority 2 (hereafter referred to as goal 1) is that the false alarm rate should not exceed 10%. The false alarm rate of the two systems is as follows:

Portal System: $10\% - (1\%)(PS - 90) / 15$

Screening System: $6\% - (1\%)(SS - 60) / 30$

Goal 1 is thus

$$[10\% - (1\%)(PS - 90) / 15] + [6\% - (1\%)(SS - 60) / 30] = 10\%$$

Priority 3 (hereafter referred to as goal 2) is that the first budgetary guideline should be met (total expenditures = \$250,000). Goal 2 is thus

$$PS + SS = 250$$

Priority 4 (hereafter referred to as goal 3) is that the second budgetary guideline should be met (average total maintenance cost \$30,000). The maintenance cost of the two systems is as follows:

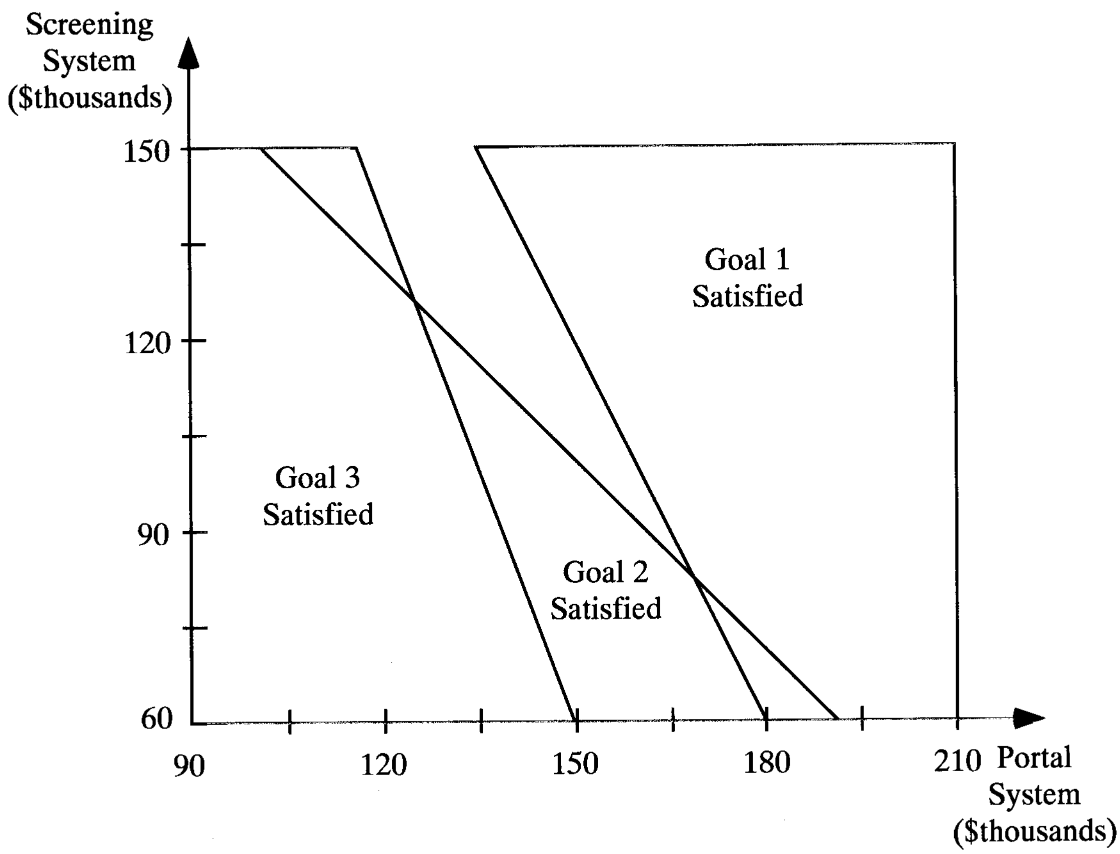
Portal System: $15 + (PS - 90) / 10$

Screening System: $9 + (SS - 60) / 25$

Goal 3 is thus

$$[15 + (PS - 90) / 10] + [9 + (SS - 60) / 25] = 30$$

c)



Goal 1 is satisfied inside the rightmost polygon. Goal 2 is satisfied in the polygon in the middle. The small triangle with vertices at (180, 60), (170, 80), (190, 60) is the only area where both goal 1 and goal 2 are satisfied.

Applying preemptive goal programming, the first solution will be somewhere inside the region where goal 1 is satisfied.

The second solution (minimizing the amount over goal 2 while constraining goal 1 to be met) will give a solution inside the small triangle where both goal 1 and goal 2 are met.

The third solution (minimizing the amount over goal 3 while constraining goal 1 and 2 to be met) will pick the solution inside the small triangle (since goal 1 and 2 must remain to be met) that is closest to meeting goal 3. This occurs at (170, 80). That is, they should spend $170 thousand on the portal system and $80 thousand on the screening system.

d) We start by minimizing the amount over goal 1 (false alarm rate = 10%).

	A	B	C	D	E	F	G	H	I	J	K	L
1			Goals			Deviations			Constraints			
2		Level				Amount	Amount		Balance			
3		Achieved		Goal		Over	Under		(Level-Over+Under)		Goal	
4	Goal 1 (False Alarm Rate)	10%	²	10%		0	0		10%	=	10%	
5	Goal 2 (Total Expenditure)	250	²	250		0	0		250	=	250	($thousand)
6	Goal 3 (Maintenance Cost)	32.8	²	30		2.8	0		30	=	30	($thousand)
7												
8		Portal	Screening			Minimize Over Goal 1						
9		System	System									
10	Minimum	90	60									
11		²	²									
12	Expenditure ($thousand/system)	170	80									
13		²	²									
14	Maximum	210	150									
15												
16	False Alarm Rate	5%	5%									
17	Base Rate	10%	6%									
18	Minus 1% per ($x thousand)	15	30									
19												
20	Maintenance Cost ($thousand)	23	9.8									
21	Base Rate	15	9									
22	Plus $1 per $x	10	25									

	B
2	Level
3	Achieved
4	=SUM(FalseAlarmRate)
5	=SUM(Expenditure)
6	=SUM(MaintenanceCost)

	I	J	K
2	Balance		
3	(Level-Over+Under)		Goal
4	=LevelAchieved-AmountOver+AmountUnder	=	=Goal
5	=LevelAchieved-AmountOver+AmountUnder	=	=Goal
6	=LevelAchieved-AmountOver+AmountUnder	=	=Goal

	A	B	C
16	False Alarm Rate	=B17-(1%)*(Expenditure-Minimum)/B18	=C17-(1%)*(Expenditure-Minimum)/C18
17	Base Rate	0.1	0.06
18	Minus 1% per ($x thousand)	15	30
19			
20	Maintenance Cost ($thousand)	=B21+(Expenditure-Minimum)/B22	=C21+(Expenditure-Minimum)/C22
21	Base Rate	15	9
22	Plus $1 per $x	10	=30/1.2

Range Name	Cells
AmountOver	F4:F6
AmountUnder	G4:G6
Balance	I4:I6
Deviations	F4:G6
Expenditure	B12:C12
FalseAlarmRate	B16:C16
Goal	D4:D6
LevelAchieved	B4:B6
MaintenanceCost	B20:C20
Maximum	B14:C14
Minimum	B10:C10

Since goal 2 is already met, we move on to minimizing the amount over goal 3 (maintenance cost = $30,000), while constraining (amount over goal 1 = 0) and (amount over goal 2 = 0).

	A	B	C	D	E	F	G	H	I	J	K	L
1		Goals				Deviations			Constraints			
2		Level				Amount	Amount		Balance			
3		Achieved		Goal		Over	Under		(Level-Over+Under)		Goal	
4	Goal 1 (False Alarm Rate)	10%	2	10%		0	0		10%	=	10%	
5	Goal 2 (Total Expenditure)	250	2	250		0	0		250	=	250	($thousand)
6	Goal 3 (Maintenance Cost)	32.8	2	30		2.8	0		30	=	30	($thousand)
7												
8		Portal	Screening			Minimize Over Goal 3						
9		System	System			(Over Goal 1 = 0)						
10	Minimum	90	60			(Over Goal 2 = 0)						
11		2	2									
12	Expenditure ($thousand/system)	170	80									
13		2	2									
14	Maximum	210	150									
15												
16	False Alarm Rate	5%	5%									
17	Base Rate	10%	6%									
18	Minus 1% per ($x thousand)	15	30									
19												
20	Maintenance Cost ($thousand)	23	9.8									
21	Base Rate	15	9									
22	Plus $1 per $x	10	25									

e) The first two goals are now hard constraints, and we minimize the amount over goal 3.

	A	B	C	D	E	F	G	H	I	J	K	L
1		Goals				Deviations			Constraints			
2		Level				Amount	Amount		Balance			
3		Achieved		Goal		Over	Under		(Level-Over+Under)		Goal	
4	Goal 1 (False Alarm Rate)	10%	2	10%	Hard Constraint							
5	Goal 2 (Total Expenditure)	250	2	250	Hard Constraint							($thousand)
6	Goal 3 (Maintenance Cost)	32.8	2	30		2.8	0		30	=	30	($thousand)
7												
8		Portal	Screening			Minimize Over Goal 1						
9		System	System									
10	Minimum	90	60									
11		2	2									
12	Expenditure ($thousand/system)	170	80									
13		2	2									
14	Maximum	210	150									
15												
16	False Alarm Rate	5%	5%									
17	Base Rate	10%	6%									
18	Minus 1% per ($x thousand)	15	30									
19												
20	Maintenance Cost ($thousand)	23	9.8									
21	Base Rate	15	9									
22	Plus $1 per $x	10	25									

If the linear program had no feasible solution, this would imply that it is not possible to meet all of the higher priority goals that were turned into hard constraints.

f) We no longer use goal programming. The goal is to minimize the total false alarm rate subject to meeting the first budgetary guideline (total expenditure), but ignoring the second budgetary guideline (maintenance cost). The spreadsheet model follows.

	A	B	C	D
1		Level		
2		Achieved		Maximum
3	Total False Alarm Rate	9%		Expenditure
4	Total Expenditure	250	≤	250
5	Maintenance Cost	34		
6				
7		Portal	Screening	
8		System	System	
9	Minimum	90	60	
10		≤	≤	
11	Expenditure ($thousand/system)	190	60	
12		≤	≤	
13	Maximum	210	150	
14				
15	False Alarm Rate	3%	6%	
16	Base Rate	10%	6%	
17	Minus 1% per ($x thousand)	15	30	
18				
19	Maintenance Cost ($thousand)	25	9	
20	Base Rate	15	9	
21	Plus $1 per $x	10	25	

The total false alarm rate can be lowered to 9% by ignoring the second budgetary guideline (maintenance cost).

g) Further what-if analysis might look at how low the false-alarm rate can be lowered by ignoring the first budgetary guideline, but meeting the second. Also, it would be interesting to look at how the minimum false alarm rate changes as both of the budgetary guidelines are varied.

CHAPTER 8: THE TRANSPORTATION AND ASSIGNMENT PROBLEMS

8.1-1.

While growing continuously as a global company, Procter & Gamble faced the need to restructure for enhanced effectiveness. The goal was to optimize work processes and to minimize expenses while maintaining customer satisfaction. Lowered transportation costs due to changes in the trucking industry and reduced product packages suggested that the total transportation costs could be decreased. In the meantime, shorter product life cycles justified smaller number of plants. Consequently, P&G had to decide on where to locate the plants, what and how much to produce in each. This would be impossible without reviewing the distribution system. Hence, two problems for each product category needed to be solved: a distribution-location problem and a product-sourcing problem.

First, optimal distribution center (DC) locations and optimal customer assignments are found by solving an uncapacitated facility-location model. The objective in this problem is to minimize the total cost of transportation and supply while the primary restriction is to satisfy customer demand. Fixed costs involved in locating DCs are ignored. The total number of DCs is determined beforehand subjectively. The solution of this problem is an input to the product sourcing problem.

With fixed DC locations and their capacities, product sourcing is modeled as a transportation problem. Sources are plants, destinations are DCs and customers. The location and capacity of the plants are specified by the product-strategy teams. Decision variables are the amounts of demand at each destination to be met from each source. The objective is to minimize the total cost while satisfying the demand at each destination without exceeding the capacity of each source. The costs consist of manufacturing, warehousing and transportation costs. An out-of-kilter algorithm is used to solve this problem for each product category.

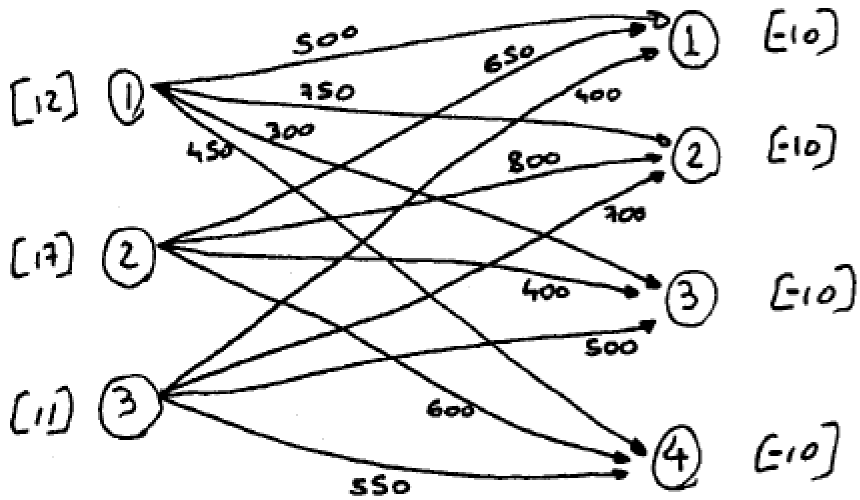
The benefits of this study included a reduction in the number of plants in North America by 20% and savings of over $200 million per year. The reduction in manufacturing costs, due to lowered number of plants and personnel coupled with improved efficiency of the supply chain, outweighs the increase in delivery costs. The gains from this study led P&G to making OR/MS a part of its decision-making process.

8.1-2.

(a)

		Unit Cost ($) Destination (Distribution Center)				
		1	2	3	4	Supply
Source (Plant)	1	500	750	300	450	12
	2	650	800	400	600	17
	3	400	700	500	550	11
Demand		10	10	10	10	

(b)



(c)

		Shipment Quantities Destination (Distribution Center)						
		1	2	3	4	Totals		Supply
Source (Plant)	1	0	0	2	10	12	=	12
	2	0	9	8	0	17	=	17
	3	10	1	0	0	11	=	11
Totals		10	10	10	10			
		=	=	=	=	Total Cost	=	$ 20,200
Demand		10	10	10	10			

8.1-3.

(a) Let x_1 and x_2 be the number of pints purchased from Dick today and tomorrow respectively, x_3 and x_4 be the number of pints purchased from Harry today and tomorrow respectively.

```
Min Z =     3 X1 + 2.7 X2 + 2.9 X3 + 2.8 X4

subject to

            1 X1 +   1 X2 +   0 X3 +   0 X4  <= 5
            0 X1 +   0 X2 +   1 X3 +   1 X4  <= 4
            1 X1 +   0 X2 +   1 X3 +   0 X4  = 3
            0 X1 +   1 X2 +   0 X3 +   1 X4  >= 4
and
      X1 >= 0, X2 >= 0, X3 >= 0, X4 >= 0.
```

INITIAL TABLEAU

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	Right side
			-1M	-1M	-1M	-1M	1M					-7M
Z	0	-1	3	2.7	2.9	2.8	0	0	0	0	0	0
X6	1	0	1	1	0	0	0	1	0	0	0	5
X7	2	0	0	0	1	1	0	0	1	0	0	4
X8	3	0	1	0	1	0	0	0	0	1	0	3
X9	4	0	0	1	0	1	-1	0	0	0	1	4

(b)

		Destination 1 Today	2 Tomorrow	3 Dummy	Supply
Dick	1	3	2.7	0	5
Harry	2	2.9	2.8	0	4
Demand		3	4	2	

(c)

		Destination 1	2	3	Supply
	1		4	1	5
Source	2	3		1	4
Demand		3	4	2	Cost is 19.5

8.1-4.

(a)

Cost Per Unit Distributed

		Product 1	2	3	Dummy	Supply
	1	41	55	48	0	400
	2	39	51	45	0	600
Plant	3	42	56	50	0	400
	4	38	52	1M	0	600
	5	39	53	1M	0	1000
Demand		700	1000	900	400	

(b)

	D1	D2	D3	D4	Supply
S1			400		400
S2		100	500		600
S3				400	400
S4		600			600
S5	700	300			1000
Demand	700	1000	900	400	Cost is 121200

8.1-5.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D15	Bellingham Sacramento	0	14.9999999	464	1E+30	15
E15	Bellingham Salt Lake City	20	0	513	15	21
F15	Bellingham Rapid City	0	83.9999996	654	1E+30	84
G15	Bellingham Albuquerque	55	0	867	21	351
D16	Eugene Sacramento	80	0	352	15	1E+30
E16	Eugene Salt Lake City	45	0	416	21	15
F16	Eugene Rapid City	0	217	690	1E+30	217
G16	Eugene Albuquerque	0	20.9999997	791	1E+30	21
D17	Albert Lea Sacramento	0	728	995	1E+30	728
E17	Albert Lea Salt Lake City	0	351	682	1E+30	351
F17	Albert Lea Rapid City	70	0	388	84	1E+30
G17	Albert Lea Albuquerque	30	0	685	351	84

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D18	Totals Sacramento	80	267	80	0	20
E18	Totals Salt Lake City	65	331	65	0	20
F18	Totals Rapid City	70	388	70	0	70
G18	Totals Albuquerque	85	685	85	0	30
H15	Bellingham Totals	75	182	75	30	0
H16	Eugene Totals	125	85	125	20	0
H17	Albert Lea Totals	100	0	100	0	1E+30

		Range of Optimality Destination			
		Sacramento	Salt Lake City	Rapid City	Albuquerque
Source	Bellingham	449 to ∞	492 to 528	570 to ∞	516 to 888
	Eugene	-∞ to 367	401 to 437	473 to ∞	770 to ∞
	Albert Lea	267 to ∞	331 to ∞	-∞ to 472	601 to 1036

These ranges tell the management how much each individual cost can be changed without changing the optimal solution.

8.1-6.

(a) Introduce a dummy customer 5 to represent the excess amount sent to customer 3 and a dummy plant 4 to represent the units that are sold to, but not received by customers 4 and 5.

(a) - (c) - (d)

Profit per unit distributed

		Customer					
		1	2	3	4	5	Supply
Plant	1	800	700	500	200	500	60
	2	500	200	100	300	100	80
	3	600	400	300	500	300	40
	4	1.00E+06	1.00E+06	1.00E+06	0	0	60
	Demand	40	60	20	60	60	

Shipments

		Customer					
		1	2	3	4	5	Supply
Plant	1		60				60
	2	40			40		80
	3			20	20		40
	4					60	60
	Demand	40	60	20	60	60	Profit is 90000

(b) - (e)

Cost per unit distributed

		Customer					
		1	2	3	4	5	Supply
Plant	1	-800	-700	-500	-200	-500	60
	2	-500	-200	-100	-300	-100	80
	3	-600	-400	-300	-500	-300	40
	4	1.00E+06	1.00E+06	1.00E+06	0	0	60
	Demand	40	60	20	60	60	

Shipments

		Customer					
		1	2	3	4	5	Supply
Plant	1		60				60
	2	40			40		80
	3			20	20		40
	4					60	60
	Demand	40	60	20	60	60	Cost is -90000

The profit is $\$90,000$.

8.1-7.

(a) - (b)

		Distribution center				
		1	2	3	4 Dummy	Supply
Plant	1	800	700	400	0	50
	2	600	800	500	0	50
	Demand	20	20	20	40	

(c)

		Distribution center				
		1	2	3	4 Dummy	Supply
Plant	1		20	20	10	50
	2	20			30	50
	Demand	20	20	20	40	Cost is 34000

8.1-8.

(a) - (b) Let destination $2i-1$ represent the demand of 10 at center i and destination $2i$ represent the extra demand up to 20 shipped to center $i=1,2,3$.

		Cost per unit distributed Destination							
		1	2	3	4	5	6	7 Dummy	Supply
plant	1	800	800	700	700	400	400	0	50
plant	2	600	600	800	800	500	500	0	50
dummy	3	1.00E+06	0	1.00E+06	0	1.00E+06	0	1.00E+06	30
	Demand	10	20	10	20	10	20	40	

(c)

		Cost per unit distributed Destination							
		1	2	3	4	5	6	7 Dummy	Supply
plant	1			10		10	20	10	50
plant	2	10	10					30	50
dummy	3		10		20				30
	Demand	10	20	10	20	10	20	40	Cost is 31000

8.1-9.

(a) Let source $2i-1$ be regular time production and $2i$ be overtime production in month $i=1,2,3$. Let destination $2i-1$ represent the contracted sales for product 1 and $2i$ represent the contracted sales for product 2 in month $i=1,2,3$. Destination 7 is dummy.

		Cost Per Unit Distributed (in $1,000's) Destination							
		1	2	3	4	5	6	7	Supply
Source	1	15	16	16	18	18	19	0	10
	2	18	20	19	22	21	23	0	3
	3	1M	1M	17	15	19	16	0	8
	4	1M	1M	20	18	22	19	0	2
	5	1M	1M	1M	1M	19	17	0	10
	6	1M	1M	1M	1M	22	22	0	3
	Demand	5	3	3	5	4	4	12	

(b)

		Destination							
		1	2	3	4	5	6	7	Supply
	1	5	3	2					10
	2							3	3
	3			1	5		2		8
Source	4							2	2
	5					4	2	4	10
	6							3	3
Demand		5	3	3	5	4	4	12	Cost is 389,000

Hence, the total cost is $\$389,000$ and no overtime is necessary.

8.2-1.

(a) Vogel's approximation method would choose x_{21} as the first basic variable.

Cost Per Unit Distributed

		Destination				
		1	2	3	Supply	Row Difference
	1	15	9	13	7	4
Source	2	11	1M	17	5	6 <-- Maximum
	3	9	11	9	3	0
Demand		7	3	5		
Column Difference		2	2	4		

(b) Russell's approximation method would choose x_{12} as the first basic variable.

Cost Per Unit Distributed

		Destination				
		1	2	3	Supply	Row Maximum
	1	15	9	13	7	15
Source	2	11	1M	17	5	1M
	3	9	11	9	3	11
Demand		7	3	5		
Column Maximum		15	1M	17		

Δ_{ij}	1	2	3
1	-15	$-1M-6$	-19
2	$-1M-4$	$-1M$	$-1M$
3	-17	$-1M$	-19

(c) Initial BF solution using northwest corner rule:

		Destination			
		1	2	3	Supply
	1	7	-	-	7
Source	2	-	3	2	5
	3	-	-	3	3
Demand		7	3	5	

8.2-2.

(a) Northwest Corner Rule

	Destination					
	1	2	3	4	5	Supply
1	2 / B / 4	4 / B / 0	6 / 0	5 / 0	7 / 0	4
2	7 / 0	6 / B / 4	3 / B / 2	M / B / 0	4 / 0	6
3	8 / 0	7 / 0	5 / 0	2 / B / 5	5 / B / 1	6
4	0 / 0	0 / 0	0 / 0	0 / 0	0 / B / 4	4
Demand	4	4	2	5	5	

Cost: 53

(b) Vogel's Approximation Method

```
        |              Destination            |
        |  1   |  2   |  3   |  4   |  5   |Supply
________|______|______|______|______|______|______
        |   2| |   4| |   6| |   5| |   7| |
      1 |---- B|----  |----  |----  |----  |
        |     4|     0|     0|     0|     0|    4
        |______|______|______|______|______|
        |   7| |   6| |   3| |   M| |   4| |
      2 |----  |----  |---- B|----  |---- B|
        |     0|     0|     2|     0|     4|    6
        |______|______|______|______|______|
        |   8| |   7| |   5| |   2| |   5| |
      3 |---- B|---- B|----  |---- B|---- B|
        |     0|     0|     0|     5|     1|    6
        |______|______|______|______|______|
        |   0| |   0| |   0| |   0| |   0| |
      4 |----  |---- B|----  |----  |----  |
        |     0|     4|     0|     0|     0|    4
        |______|______|______|______|______|______
 Demand |     4      4      2      5      5|
```

Cost: 45

(c) Russell's Approximation Method

```
        |              Destination            |
        |  1   |  2   |  3   |  4   |  5   |Supply  u[i]
________|______|______|______|______|______|____________
        |   2| |   4| |   6| |   5| |   7| |      |
      1 |---- B|----  |----  |----  |----  |      |
        |     4|     2|     7|     8|     7|     4|    -5
        |______|______|______|______|______|      |
        |   7| |   6| |   3| |   M| |   4| |      |
      2 |----  |----  |---- B|----  |---- B|      |
        |     1|     0|     2| 1M- 1|     4|     6|    -1
        |______|______|______|______|______|      |
        |   8| |   7| |   5| |   2| |   5| |      |
      3 |----  |---- B|----  |---- B|---- B|      |
        |     1|     0|     1|     5|     1|     6|     0
        |______|______|______|______|______|      |
        |   0| |   0| |   0| |   0| |   0| |      |
      4 |---- B|---- B|----  |----  |----  |      |
        |     0|     4|     3|     5|     2|     4|    -7
        |______|______|______|______|______|______|_____
 Demand |     4      4      2      5      5|
        |__________________________________|
   v[j] |     7      7      4      2      5|
```

Cost: 45

Note that Vogel's and Russell's approximation methods return an optimal solution.

8.2-3.

(a) Northwest Corner Rule

Source \ Destination	1	2	3	4	5	6	Supply
1	13 (3)	10 (2)	22	29	18	0	5
2	14	13 (3)	16 (3)	21	M	0	6
3	3	0	M (1)	11 (5)	6 (1)	0	7
4	18	9	19	23	11 (4)	0	4
5	30	24	34	36	28 (1)	0 (2)	3
Demand	3	5	4	5	6	2	

Cost: $M + 279$

(b) Vogel's Approximation Method

Source \ Destination	1	2	3	4	5	6	Supply
1	13	10 (3)	22	29	18 (2)	0	5
2	14 (1)	13 (1)	16 (4)	21	M	0	6
3	3 (2)	0	M	11 (5)	6	0	7
4	18	9	19	23	11 (4)	0	4
5	30	24 (1)	34	36	28	0 (2)	3
Demand	3	5	4	5	6	2	

Cost: 286

Arbitrarily breaking the tie differently returns the solution below with cost $M + 260$.

Source \ Destination	1	2	3	4	5	6	Supply
1	13	10 (5)	22	29	18	0	5
2	14	13 (0)	16 (4)	21 (1)	M (1)	0	6
3	3 (3)	0	M	11 (4)	6	0	7
4	18	9	19	23	11 (4)	0	4
5	30	24	34	36	28 (1)	0 (2)	3
Demand	3	5	4	5	6	2	

(c) Russell's Approximation Method

Source \ Destination	1	2	3	4	5	6	Supply
1	13 2	10 3	22	29	18	0	5
2	14	13	16 4	21 2	M	0	6
3	3 1	0	M	11	6 6	0	7
4	18	9 2	19	23 2	11	0	4
5	30	24	34	36 1	28	0 2	3
Demand	3	5	4	5	6	2	

Cost: 301

8.2-4.

(a) All the supply and demand values are integers. By the integer solutions property, the resulting basic feasible solutions will be integral. All the supplies and demands are one, so the only possible values of the variables in a basic feasible solution are 0 and 1. The 1's indicate the assignment of a source to a destination.

(b) There are 7 basic variables in every basic feasible solution and 3 of them are degenerate.

(d) The variables are chosen in the order $x_{13}, x_{24}, x_{44}, x_{32}, x_{41}, x_{43}, x_{42}$.

	Destination 1	2	3	4	Supply	u[i]
1	7 0	4 0	1 B 1	4 0	1	0
2	4 0	6 0	7 0	2 B 1	1	0
3	8 0	5 B 1	4 0	6 0	1	0
4	6 B 1	7 B 0	6 B 0	3 B 0	1	0
Demand	1	1	1	1		
v[j]	0	0	0	0		

(c) - (e)

(0)

	Destination 1	2	3	4	Supply	u[i]
1	7 / L / 1	4 / / -5	1 / E / -7	4 / / -1	1	2
2	4 / P / 0	6 / P / 1	7 / / 2	2 / / 0	1	-1
3	8 / / 5	5 / P / 0	4 / P / 1	6 / / 5	1	-2
4	6 / / 1	7 / / 0	6 / B / 0	3 / B / 1	1	0
Demand	1	1	1	1		
v[j]	5	7	6	3		

(1)

	Destination 1	2	3	4	Supply	u[i]
1	7 / / 7	4 / / 2	1 / B / 1	4 / / 6	1	1
2	4 / B / 1	6 / B / 0	7 / / 2	2 / / 0	1	5
3	8 / / 5	5 / B / 1	4 / B / 0	6 / / 5	1	4
4	6 / / 1	7 / / 0	6 / B / 0	3 / B / 1	1	6
Demand	1	1	1	1		
v[j]	-1	1	0	-3		

Optimal assignment (source,destination): $(1,3), (2,1), (3,2), (4,4)$, cost: 13

8.2-5.

				u_i
464 / 15	513 / 20	654 / 84	867 / 55	182
352 / 80	416 / 45	690 / 217	791 / 21	85
995 / 728	682 / 351	388 / 70	685 / 30	0
v_j: 267	331	388	685	

Cost: $\$152,535$, $c_{ij} - u_i - v_j \geq 0$ for all i and j, so the solution is optimal.

8.2-6.

(0)	Destination 1	2	3	4	5	Supply	u[i]
1	8 / ---- / 1	6 / ---- / 3	3 / ---- B / 20	7 / ---- / 1	5 / ---- / 2	20	2
2	5 / ---- P / 25	M / ---- / 1M- 1	8 / ---- / 7	4 / ---- P / 5	7 / ---- / 6	30	0
3	6 / ---- / -1	3 / ---- P / 25	9 / ---- / 6	6 / ---- P / 5	8 / ---- / 5	30	2
4	0 / ---- E / -4	0 / ---- L / 0	0 / ---- B / 0	0 / ---- / -3	0 / ---- B / 20	20	-1
Demand	25	25	20	10	20		
v[j]	5	1	1	4	1		

(1)

	Destination 1	2	3	4	5	Supply	u[i]
1	8 5	6 7	3 B 20	7 5	5 2	20	3
2	5 P 25	M 1M- 1	8 3	4 P 5	7 2	30	5
3	6 E -1	3 B 25	9 2	6 L 5	8 1	30	7
4	0 B 0	0 4	0 B 0	0 1	0 B 20	20	0
Demand	25	25	20	10	20		
v[j]	0	-4	0	-1	0		

(2)

	Destination 1	2	3	4	5	Supply	u[i]
1	8 5	6 6	3 B 20	7 5	5 2	20	3
2	5 B 20	M 1M- 2	8 3	4 B 10	7 2	30	5
3	6 B 5	3 B 25	9 3	6 1	8 2	30	6
4	0 B 0	0 3	0 B 0	0 1	0 B 20	20	0
Demand	25	25	20	10	20		
v[j]	0	-3	0	-1	0		

The current solution is optimal: $x_{13} = 20, x_{21} = 20, x_{24} = 10, x_{31} = 5, x_{32} = 25$ and $x_{45} = 20$, with cost 305. The optimality condition $c_{ij} - u_i - v_j \geq 0$ for all i and j is met.

8.2-7.

(a) Northwest Corner Rule

(0)

Source \ Destination	1	2	3	4	Supply	u_i
1	3: 3	7: 2	6: 0	4: 1	5	0
2	2: 2	4: 1 −	3: 1 +	2: 2	2	-3
3	4: -1	3: +	8: 1 −	5: 2	3	2
Demand	3	3	2	2		
v_j	3	7	6	3	Z = 48	

(1)

Source \ Destination	1	2	3	4	Supply	u_i
1	3: 3	7: 2 −	6: 0	4: +	5	7
2	2: 2	4: 0	3: 2	2: -4	2	4
3	4: 5	3: 1 +	8: 6	5: 2 −	3	3
Demand	3	3	2	2		
v_j	-4	0	-1	2	Z = 42	

(2)

Source \ Destination	1	2	3	4	Supply	u_i
1	3: 3	7: 5	6: 5	4: 2	5	0
2	2: -3	4: 0 −	3: 2	2: +	2	2
3	4: 0	3: 3 +	8: 6	5: 0 −	3	1
Demand	3	3	2	2		
v_j	3	2	1	4	Z = 32	

(3)

Source \ Destination	1	2	3	4	Supply	u_i
1	3: 3	7: 1	6: 1	4: 2	5	2
2	2: 1	4: 0	3: 2	2: 0	2	0
3	4: 4	3: 3	8: 6	5: 4	3	-1
Demand	3	3	2	2		
v_j	1	4	3	2	Z = 32	

3 iterations are required to reach optimality.

(b) Vogel's Approximation Method

Source	Destination 1	2	3	4	Supply	u_i
1	3 / 3	7 / 0	6 / 0	4 / 2	5	0
2	2 / 2	4 / 0	3 / 2	2 / 1	2	-3
3	4 / 5	3 / 3	8 / 6	5 / 5	3	-4
Demand	3	3	2	2		
v_j	3	7	6	4	Z = 32	

The solution is optimal, no iteration of network simplex is needed.

(c) Russell's Approximation Method

Source	Destination 1	2	3	4	Supply	u_i
1	3 / 3	7 / 0	6 / 0	4 / 2	5	0
2	2 / 2	4 / 0	3 / 2	2 / 1	2	-3
3	4 / 5	3 / 3	8 / 6	5 / 5	3	-4
Demand	3	3	2	2		
v_j	3	7	6	4	Z = 32	

The solution is optimal, no iteration of network simplex is needed.

8.2-8.

(a)

Unit Shipping Cost

		Retail Outlet 1	2	3	4	Supply
Plant	1	700	800	500	200	10
	2	200	900	100	400	20
	3	400	500	300	100	20
	4	200	100	400	300	10
Demand		20	10	10	20	

(b)

```
          |           Destination          |
          |   1   |   2   |   3   |   4    |Supply  u[i]
__________|_______|_______|_______|________|______________
          | 700|  | 800|  | 500|  | 200|   |        |
      1   |---- B|----    |----   |----    |        |
          |    10|      0|      0|      0 |     10|      0
          |_______|_______|_______|________|        |
          | 200|  | 900|  | 100|  | 400|   |        |
      2   |---- B|---- B|----    |----    |        |
          |    10|     10|      0|      0 |     20|      0
          |_______|_______|_______|________|        |
          | 400|  | 500|  | 300|  | 100|   |        |
      3   |----   |---- B|---- B|---- B   |        |
          |      0|      0|     10|     10 |     20|      0
          |_______|_______|_______|________|        |
          | 200|  | 100|  | 400|  | 300|   |        |
      4   |----   |----   |----   |---- B   |        |
          |      0|      0|      0|     10 |     10|      0
          |_______|_______|_______|________|________|______
 Demand   |    20      10      10      20 |
__________|________________________________|
   v[j]   |     0       0       0       0 |
          |                                |  Z = 25000
```

(c)

```
          |           Destination          |
          |   1   |   2   |   3   |   4    |Supply  u[i]
__________|_______|_______|_______|________|______________
          | 700|  | 800|  | 500|  | 200|   |        |
      1   |---- L|----    |----   |---- E  |        |
          |    10|   -600|   -700|   -800 |     10|    700
          |_______|_______|_______|________|        |
          | 200|  | 900|  | 100|  | 400|   |        |
      2   |---- P|---- P|----    |----    |        |
          |    10|     10|   -600|   -100 |     20|    200
          |_______|_______|_______|________|        |
          | 400|  | 500|  | 300|  | 100|   |        |
      3   |----   |---- P|---- B|---- P   |        |
          |   600|      0|     10|     10 |     20|   -200
          |_______|_______|_______|________|        |
          | 200|  | 100|  | 400|  | 300|   |        |
      4   |----   |----   |----   |---- B   |        |
          |   200|   -600|   -100|     10 |     10|      0
          |_______|_______|_______|________|________|______
 Demand   |    20      10      10      20 |
__________|________________________________|
   v[j]   |     0     700     500     300 |
          |                                |  Z = 25000
```

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|_____________
          | 700|  | 800|  | 500|  | 200|  |       |
        1 |----   |----   |----   |---- B|       |
          |   800|   200|    100|     10|     10|   100
          |_______|_______|_______|_______|       |
          | 200|  | 900|  | 100|  | 400|  |       |
        2 |---- B|---- L|---- E|----   |       |
          |    20|      0|   -600|   -100|     20|   400
          |_______|_______|_______|_______|       |
          | 400|  | 500|  | 300|  | 100|  |       |
        3 |----   |---- P|---- P|---- B|       |
          |   600|     10|     10|      0|     20|     0
          |_______|_______|_______|_______|       |
          | 200|  | 100|  | 400|  | 300|  |       |
        4 |----   |----   |----   |---- B|       |
          |   200|   -600|   -100|     10|     10|   200
          |_______|_______|_______|_______|_______|______
   Demand |    20      10      10      20|
__________|_______________________________|
     v[j] |  -200     500     300     100|
          |                               | Z = 17000
```

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|_____________
          | 700|  | 800|  | 500|  | 200|  |       |
        1 |----   |----   |----   |---- B|       |
          |   200|   200|    100|     10|     10|   100
          |_______|_______|_______|_______|       |
          | 200|  | 900|  | 100|  | 400|  |       |
        2 |---- B|----   |---- B|----   |       |
          |    20|    600|      0|    500|     20|  -200
          |_______|_______|_______|_______|       |
          | 400|  | 500|  | 300|  | 100|  |       |
        3 |----   |---- L|---- B|---- P|       |
          |     0|     10|     10|      0|     20|     0
          |_______|_______|_______|_______|       |
          | 200|  | 100|  | 400|  | 300|  |       |
        4 |----   |---- E|----   |---- P|       |
          |  -400|   -600|   -100|     10|     10|   200
          |_______|_______|_______|_______|_______|______
   Demand |    20      10      10      20|
__________|_______________________________|
     v[j] |   400     500     300     100|
          |                               | Z = 17000
```

```
           |          Destination          |
           |   1    |   2    |   3    |   4    |Supply  u[i]
___________|________|________|________|________|______________
           | 700|   | 800|   | 500|   | 200|   |       |
     1     |----    |----    |----    |---- B  |       |
           |    200 |    800 |    100 |     10 |     10|    200
           |________|________|________|________|       |
           | 200|   | 900|   | 100|   | 400|   |       |
     2     |---- P  |----    |---- P  |----    |       |
           |     20 |   1200 |      0 |    500 |     20|   -100
           |________|________|________|________|       |
           | 400|   | 500|   | 300|   | 100|   |       |
     3     |----    |----    |---- P  |---- P  |       |
           |      0 |    600 |     10 |     10 |     20|    100
           |________|________|________|________|       |
           | 200|   | 100|   | 400|   | 300|   |       |
     4     |---- E  |---- B  |----    |---- L  |       |
           |   -400 |     10 |   -100 |      0 |     10|    300
           |________|________|________|________|_______|______
 Demand    |     20       10       10       20 |
___________|___________________________________|
  v[j]     |    300     -200      200        0 |
           |                                   | Z = 11000
```

```
           |          Destination          |
           |   1    |   2    |   3    |   4    |Supply  u[i]
___________|________|________|________|________|______________
           | 700|   | 800|   | 500|   | 200|   |       |
     1     |----    |----    |----    |---- B  |       |
           |    200 |    400 |    100 |     10 |     10|    300
           |________|________|________|________|       |
           | 200|   | 900|   | 100|   | 400|   |       |
     2     |---- B  |----    |---- B  |----    |       |
           |     20 |    800 |      0 |    500 |     20|      0
           |________|________|________|________|       |
           | 400|   | 500|   | 300|   | 100|   |       |
     3     |----    |----    |---- B  |---- B  |       |
           |      0 |    200 |     10 |     10 |     20|    200
           |________|________|________|________|       |
           | 200|   | 100|   | 400|   | 300|   |       |
     4     |---- B  |---- B  |----    |----    |       |
           |      0 |     10 |    300 |    400 |     10|      0
           |________|________|________|________|_______|______
 Demand    |     20       10       10       20 |
___________|___________________________________|
  v[j]     |    200      100      100     -100 |
           |                                   | Z = 11000
```

Optimal Solution: $x_{14} = 10, x_{21} = 20, x_{33} = 10, x_{34} = 10, x_{42} = 10$, cost: $\$11,000$

8.2-9.

(a) Since there is no limit on the electricity and natural gas available, let the supply of electricity be the sum of demands for electricity, water and space heating and the supply of natural gas be the sum of demands for water and space heating.

	Electricity (1)	Product Water (2)	Space (3)	Dummy (4)	Supply
Electricity (1)	50	50	140	0	100
Natural Gas (2)	M	110	100	0	70
Solar Heater (3)	M	70	90	0	40
Demand	30	20	50	110	

(b) Northwest Corner Rule

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|______________
          |  50|  |  50|  | 140|  |   0|  |        |
     1    |---- B|---- B|---- B|----    |        |
          |    30|    20|    50|     0|   100|       0
          |_______|_______|_______|_______|        |
          |   M|  | 110|  | 100|  |   0|  |        |
     2    |----   |----   |---- B|---- B|        |
          |     0|     0|     0|    70|    70|       0
          |_______|_______|_______|_______|        |
          |   M|  |  70|  |  90|  |   0|  |        |
     3    |----   |----   |----   |---- B|        |
          |     0|     0|     0|    40|    40|       0
          |_______|_______|_______|_______|________|______
  Demand  |    30      20      50     110|
__________|_______________________________|
    v[j]  |     0       0       0       0|
          |                               |  Z = 9500
```

(c)

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|______________
          |  50|  |  50|  | 140|  |   0|  |        |
     1    |---- B|---- B|---- L|---- E|        |
          |    30|    20|    50|   -40|   100|       0
          |_______|_______|_______|_______|        |
          |   M|  | 110|  | 100|  |   0|  |        |
     2    |----   |----   |---- P|---- P|        |
          | 1M-10|   100|     0|    70|    70|     -40
          |_______|_______|_______|_______|        |
          |   M|  |  70|  |  90|  |   0|  |        |
     3    |----   |----   |----   |---- B|        |
          | 1M-10|    60|   -10|    40|    40|     -40
          |_______|_______|_______|_______|________|______
  Demand  |    30      20      50     110|
__________|_______________________________|
    v[j]  |    50      50     140      40|
          |                               |  Z = 9500
```

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|_____________
          |  50|  |  50|  | 140|  |   0|  |       |
    1     |---- B|---- B|----   |---- B|       |
          |    30|    20|    40|    50|   100|      0
          |_______|_______|_______|_______|       |
          |   M|  | 110|  | 100|  |   0|  |       |
    2     |----   |----   |---- P|---- P|       |
          | 1M-50|    60|    50|    20|    70|      0
          |_______|_______|_______|_______|       |
          |   M|  |  70|  |  90|  |   0|  |       |
    3     |----   |----   |---- E|---- L|       |
          | 1M-50|    20|   -10|    40|    40|      0
          |_______|_______|_______|_______|_______|______
Demand    |    30      20      50     110|
__________|_______________________________|
   v[j]   |    50      50     100       0|
          |                               |  Z = 7500
```

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|_____________
          |  50|  |  50|  | 140|  |   0|  |       |
    1     |---- B|---- B|----   |---- B|       |
          |    30|    20|    40|    50|   100|      0
          |_______|_______|_______|_______|       |
          |   M|  | 110|  | 100|  |   0|  |       |
    2     |----   |----   |---- B|---- B|       |
          | 1M-50|    60|    10|    60|    70|      0
          |_______|_______|_______|_______|       |
          |   M|  |  70|  |  90|  |   0|  |       |
    3     |----   |----   |---- B|----   |       |
          | 1M-40|    30|    40|    10|    40|    -10
          |_______|_______|_______|_______|_______|______
Demand    |    30      20      50     110|
__________|_______________________________|
   v[j]   |    50      50     100       0|
          |                               |  Z = 7100
```

The optimal solution is to meet 30 units of electricity, 20 units of water heating and 40 units of space heating with electricity, 10 units of space heating with natural gas and 40 units of space heating with solar heater. This costs \$7,100.

(d) Vogel's Approximation Method

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|______________
          |  50|  |  50|  | 140|  |   0|  |       |
     1    |---- B|---- B|---- B|----    |       |
          |    30|    20|    50|     0|   100|      0
          |_______|_______|_______|_______|       |
          |   M|  | 110|  | 100|  |   0|  |       |
     2    |----   |----   |----   |---- B|       |
          |     0|     0|     0|    70|    70|      0
          |_______|_______|_______|_______|       |
          |   M|  |  70|  |  90|  |   0|  |       |
     3    |----   |----   |---- B|---- B|       |
          |     0|     0|     0|    40|    40|      0
          |_______|_______|_______|_______|_______|______
 Demand   |    30      20      50     110|
__________|_______________________________|
   v[j]   |     0       0       0       0|
          |                               |  Z = 9500
```

(e)

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|______________
          |  50|  |  50|  | 140|  |   0|  |       |
     1    |---- B|---- B|---- P|---- E|       |
          |    30|    20|    50|   -50|   100|      0
          |_______|_______|_______|_______|       |
          |   M|  | 110|  | 100|  |   0|  |       |
     2    |----   |----   |----   |---- B|       |
          | 1M+ 0|   110|    10|    70|    70|    -50
          |_______|_______|_______|_______|       |
          |   M|  |  70|  |  90|  |   0|  |       |
     3    |----   |----   |---- P|---- L|       |
          | 1M+ 0|    70|     0|    40|    40|    -50
          |_______|_______|_______|_______|_______|______
 Demand   |    30      20      50     110|
__________|_______________________________|
   v[j]   |    50      50     140      50|
          |                               |  Z = 9500
```

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|_____________
          |  50|  |  50|  | 140|  |   0|  |       |
       1  |---- B|---- B|---- L|---- P|       |
          |     30|     20|     10|     40|   100|       0
          |_______|_______|_______|_______|       |
          |    M| | 110|  | 100|  |   0|  |       |
       2  |----   |----   |---- E|---- P|       |
          | 1M-50|     60|    -40|     70|    70|       0
          |_______|_______|_______|_______|       |
          |    M| |  70|  |  90|  |   0|  |       |
       3  |----   |----   |---- B|----    |       |
          | 1M+ 0|     70|     40|     50|    40|     -50
          |_______|_______|_______|_______|_______|______
 Demand   |     30      20      50     110|
__________|_______________________________|
   v[j]   |     50      50     140       0|
          |                               |  Z = 7500

          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|_____________
          |  50|  |  50|  | 140|  |   0|  |       |
       1  |---- B|---- B|----    |---- B|       |
          |     30|     20|     40|     50|   100|       0
          |_______|_______|_______|_______|       |
          |    M| | 110|  | 100|  |   0|  |       |
       2  |----   |----   |---- B|---- B|       |
          | 1M-50|     60|     10|     60|    70|       0
          |_______|_______|_______|_______|       |
          |    M| |  70|  |  90|  |   0|  |       |
       3  |----   |----   |---- B|----    |       |
          | 1M-40|     30|     40|     10|    40|     -10
          |_______|_______|_______|_______|_______|______
 Demand   |     30      20      50     110|
__________|_______________________________|
   v[j]   |     50      50     100       0|
          |                               |  Z = 7100
```

(f) Russell's Approximation Method

```
          |         Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|______________
          |  50|  |  50|  | 140|  |   0|  |       |
        1 |---- B|---- B|---- B|----    |       |
          |     30|     20|     50|      0|   100|      0
          |_______|_______|_______|_______|       |
          |   M|  | 110|  | 100|  |   0|  |       |
        2 |----   |----   |---- B|---- B|       |
          |      0|      0|      0|     70|    70|      0
          |_______|_______|_______|_______|       |
          |   M|  |  70|  |  90|  |   0|  |       |
        3 |----   |----   |----   |---- B|       |
          |      0|      0|      0|     40|    40|      0
          |_______|_______|_______|_______|_______|______
  Demand  |     30      20      50     110|
__________|_______________________________|
    v[j]  |      0       0       0       0|
          |                               |  Z = 9500
```

Note that different solutions may be obtained, since ties are broken arbitrarily.

(g) Russell's approximation method returns the same initial solution as the northwest corner rule, so the solution is the same as in (c). The initial BF solution using Vogel's and Russell's methods provides the same optimal solution as in (c). The optimal solution obtained starting from each of the three rules is the same. Also, in each case, the number of iterations required by the transportation simplex method is two.

8.2-10.

Vogel's Approximation Method

```
       |              Destination                 |
  (0)  |   1   |   2   |   3   |   4   |   5   |Supply  u[i]
  _____|_______|_______|_______|_______|_______|____________
       |1.08|  |1.09|  |1.11|  |1.13|  |     0||       |
     1 |---- B|---- B|---- B|----   |----    |       |
       |    10|    15|     0|  0.01|  0.02|     25|  -0.02
       |______|______|______|______|______|       |
       |    M| |1.11|  |1.13|  |1.14|  |     0||       |
     2 |----   |----   |----   |---- P|---- P|       |
       | 1M- 1|     0|     0|    15|    20|     35|     0
       |______|______|______|______|______|       |
       |    M| |    M| | 1.1|  |1.11|  |     0||       |
     3 |----   |----   |---- B|---- B|----    |       |
       | 1M- 1| 1M- 1|    25|     5|  0.03|     30|  -0.03
       |______|______|______|______|______|       |
       |    M| |    M| |    M| |1.13|  |     0||       |
     4 |----   |----   |----   |---- E|---- L|       |
       | 1M- 1| 1M- 1| 1M- 1| -0.01|    10|     10|     0
       |______|______|______|______|______|_______|______
Demand |   10     15     25     20     30|
       |__________________________________|
  v[j] |   1.1   1.11   1.13   1.14     0|
```

Optimal Solution:

Quantity	Production Month	Installation Month
10	1	1
15	1	2
5	2	4
25	3	3
5	3	4
10	4	4

This schedule incurs a cost of 77.3 million dollars.

8.2-11.

(a)

(0)

	Destination 1	2	3	4	Supply	u[i]
1	500 P 10	750 P 2	300 -50	450 50	12	0
2	650 100	800 P 8	400 P 9	600 150	17	50
3	400 E -250	700 -200	500 L 1	550 B 10	11	150
Demand	10	10	10	10		
v[j]	500	750	350	400		

(b)

(1)

	Destination 1	2	3	4	Supply	u[i]
1	500 L 9	750 B 3	300 -50	450 E -200	12	0
2	650 100	800 B 7	400 B 10	600 -100	17	50
3	400 P 1	700 50	500 250	550 P 10	11	-100
Demand	10	10	10	10		
v[j]	500	750	350	650		

(2)

	Destination 1	2	3	4	Supply	u[i]
1	500 / 200	750 / L / 3	300 / E / -50	450 / B / 9	12	0
2	650 / 300	800 / P / 7	400 / P / 10	600 / 100	17	50
3	400 / B / 10	700 / -150	500 / 50	550 / B / 1	11	100
Demand	10	10	10	10		
v[j]	300	750	350	450		

(3)

	Destination 1	2	3	4	Supply	u[i]
1	500 / 200	750 / 50	300 / P / 3	450 / P / 9	12	0
2	650 / 250	800 / P / 10	400 / P / 7	600 / 50	17	100
3	400 / B / 10	700 / E / -100	500 / 100	550 / L / 1	11	100
Demand	10	10	10	10		
v[j]	300	700	300	450		

(4)

	Destination 1	2	3	4	Supply	u[i]
1	500 ---- 100	750 ---- 50	300 ---- B 2	450 ---- B 10	12	0
2	650 ---- 150	800 ---- B 9	400 ---- B 8	600 ---- 50	17	100
3	400 ---- B 10	700 ---- B 1	500 ---- 200	550 ---- 100	11	0
Demand	10	10	10	10		
v[j]	400	700	300	450		

The optimal solution is to send 2 shipments from plant 1 to center 3, 10 to center 4, 9 from plant 2 to center 2, 8 to center 3, 10 from plant 3 to center 1 and 1 to center 2. This has a total cost of $20,200.

8.2-12.

(0)

	Destination 1	2	3	Supply	u[i]
1	3 ---- P 3	2.7 ---- P 2	0 ---- 0.1	5	0
2	2.9 ---- E -0.2	2.8 ---- L 2	0 ---- B 2	4	0.1
Demand	3	4	2		
v[j]	3	2.7	-0.1		

(1)

	Destination 1	2	3	Supply	u[i]
1	3 ---- L 1	2.7 ---- B 4	0 ---- E -0.1	5	0
2	2.9 ---- P 2	2.8 ---- 0.2	0 ---- P 2	4	-0.1
Demand	3	4	2		
v[j]	3	2.7	0.1		

	Destination 1	2	3	Supply	u[i]
1	3 ---- 0.1	2.7 ---- B 4	0 ---- B 1	5	0
2	2.9 ---- B 3	2.8 ---- 0.1	0 ---- B 1	4	0
Demand	3	4	2		
v[j]	2.9	2.7	0		

The optimal solution is to purchase 4 pints from Dick tomorrow and 3 pints from Harry today, with a cost $19.50.

8.2-13.

```
         |       Destination         |
         |   1   |   2   |   3   |   4   |Supply  u[i]
_________|_______|_______|_______|_______|____________
         |  41|  |  55|  |  48|  |   0|  |       |
    1    |---- B|----   |----   |----   |       |
         |   400|      2|-1M+47|     -1|   400|     41
         |_______|_______|_______|_______|       |
         |  39|  |  51|  |  45|  |   0|  |       |
    2    |---- B|---- P|---- E|----   |       |
         |   300|    300|-1M+46|      1|   600|     39
         |_______|_______|_______|_______|       |
         |  42|  |  56|  |  50|  |   0|  |       |
    3    |----   |---- B|----   |----   |       |
         |    -2|    400|-1M+46|     -4|   400|     44
         |_______|_______|_______|_______|       |
         |  38|  |  52|  |   M|  |   0|  |       |
    4    |----   |---- P|---- L|----   |       |
         |    -2|    300|    300|      0|   600|     40
         |_______|_______|_______|_______|       |
         |  39|  |  53|  |   M|  |   0|  |       |
    5    |----   |----   |---- B|---- B|       |
         |    -1|      1|    600|    400|  1000|     40
         |_______|_______|_______|_______|_______|______
Demand   |   700   1000    900    400|
_________|___________________________|
  v[j]   |      0     12     1M    -40|
         |                 -  40      |  Z = 900M+81400

         |       Destination         |
         |   1   |   2   |   3   |   4   |Supply  u[i]
_________|_______|_______|_______|_______|____________
         |  41|  |  55|  |  48|  |   0|  |       |
    1    |---- B|----   |----   |----   |       |
         |   400|      2|      1| 1M-47|   400|      2
         |_______|_______|_______|_______|       |
         |  39|  |  51|  |  45|  |   0|  |       |
    2    |---- L|---- B|---- P|----   |       |
         |   300|      0|    300| 1M-45|   600|      0
         |_______|_______|_______|_______|       |
         |  42|  |  56|  |  50|  |   0|  |       |
    3    |----   |---- B|----   |----   |       |
         |    -2|    400|      0| 1M-50|   400|      5
         |_______|_______|_______|_______|       |
         |  38|  |  52|  |   M|  |   0|  |       |
    4    |----   |---- B|----   |----   |       |
         |    -2|    600| 1M-46| 1M-46|   600|      1
         |_______|_______|_______|_______|       |
         |  39|  |  53|  |   M|  |   0|  |       |
    5    |---- E|----   |---- P|---- B|       |
         |-1M+45|-1M+47|    600|    400|  1000|     1M
         |_______|_______|_______|_______|_______|-   45
Demand   |   700   1000    900    400|
_________|___________________________|
  v[j]   |     39     51     45    -1M|
         |                     +  45|  Z = 600M+95200
```

```
         |       Destination         |
         |  1   |  2   |  3   |  4   |Supply  u[i]
_________|______|______|______|______|____________
         | 41|  | 55|  | 48|  |  0|  |      |
    1    |---- B|----  |----  |----  |      |
         |   400|-1M+47|-1M+46|    -2|   400|      2
         |______|______|______|______|      |
         | 39|  | 51|  | 45|  |  0|  |      |
    2    |----  |---- L|---- P|----  |      |
         | 1M-45|     0|   600| 1M-45|   600|    -1M
         |______|______|______|______|      |+   45
         | 42|  | 56|  | 50|  |  0|  |      |
    3    |----  |---- B|----  |----  |      |
         | 1M-47|   400|     0| 1M-50|   400|    -1M
         |______|______|______|______|      |+   50
         | 38|  | 52|  |  M|  |  0|  |      |
    4    |----  |---- B|----  |----  |      |
         | 1M-47|   600| 1M-46| 1M-46|   600|    -1M
         |______|______|______|______|      |+   46
         | 39|  | 53|  |  M|  |  0|  |      |
    5    |---- B|---- E|---- P|---- B|      |
         |   300|-1M+47|   300|   400|  1000|      0
         |______|______|______|______|______|_____
Demand   |   700  1000    900    400|
_________|__________________________|
   v[j]  |    39     1M     1M      0|
         |       +    6             | Z = 300M+1E5
```

```
         |       Destination         |
         |  1   |  2   |  3   |  4   |Supply  u[i]
_________|______|______|______|______|____________
         | 41|  | 55|  | 48|  |  0|  |      |
    1    |---- P|----  |---- E|----  |      |
         |   400|     0|-1M+46|    -2|   400|      2
         |______|______|______|______|      |
         | 39|  | 51|  | 45|  |  0|  |      |
    2    |----  |----  |---- B|----  |      |
         | 1M-45| 1M-47|   600| 1M-45|   600|    -1M
         |______|______|______|______|      |+   45
         | 42|  | 56|  | 50|  |  0|  |      |
    3    |----  |---- B|----  |----  |      |
         |     0|   400|-1M+47|    -3|   400|      3
         |______|______|______|______|      |
         | 38|  | 52|  |  M|  |  0|  |      |
    4    |----  |---- B|----  |----  |      |
         |     0|   600|     1|     1|   600|     -1
         |______|______|______|______|      |
         | 39|  | 53|  |  M|  |  0|  |      |
    5    |---- P|---- B|---- L|---- B|      |
         |   300|     0|   300|   400|  1000|      0
         |______|______|______|______|______|_____
Demand   |   700  1000    900    400|
_________|__________________________|
   v[j]  |    39    53     1M      0|
         |                          | Z = 300M+1E5
```

```
         |          Destination          |
         |   1   |   2   |   3   |   4   |Supply  u[i]
_________|_______|_______|_______|_______|_____________
         |  41|  |  55|  |  48|  |   0|  |       |
    1    |---- B|----   |---- B|----    |       |
         |   100|      0|   300|     -2|   400|      2
         |_______|_______|_______|_______|       |
         |  39|  |  51|  |  45|  |   0|  |       |
    2    |----   |----   |---- B|----    |       |
         |      1|     -1|   600|      1|   600|     -1
         |_______|_______|_______|_______|       |
         |  42|  |  56|  |  50|  |   0|  |       |
    3    |----   |---- L|----   |---- E|       |
         |      0|   400|      1|     -3|   400|      3
         |_______|_______|_______|_______|       |
         |  38|  |  52|  |   M|  |   0|  |       |
    4    |----   |---- B|----   |----    |       |
         |      0|   600| 1M-45|      1|   600|     -1
         |_______|_______|_______|_______|       |
         |  39|  |  53|  |   M|  |   0|  |       |
    5    |---- B|---- P|----   |---- P|       |
         |   600|      0| 1M-46|    400|  1000|      0
         |_______|_______|_______|_______|_______|______
Demand   |   700   1000    900    400|
_________|_______________________________|
  v[j]   |    39     53     46      0|
         |                               | Z = 122500
```

```
         |          Destination          |
         |   1   |   2   |   3   |   4   |Supply  u[i]
_________|_______|_______|_______|_______|_____________
         |  41|  |  55|  |  48|  |   0|  |       |
    1    |---- P|----   |---- B|---- E|       |
         |   100|      0|   300|     -2|   400|      2
         |_______|_______|_______|_______|       |
         |  39|  |  51|  |  45|  |   0|  |       |
    2    |----   |----   |---- B|----    |       |
         |      1|     -1|   600|      1|   600|     -1
         |_______|_______|_______|_______|       |
         |  42|  |  56|  |  50|  |   0|  |       |
    3    |----   |----   |----   |---- B|       |
         |      3|      3|      4|    400|   400|      0
         |_______|_______|_______|_______|       |
         |  38|  |  52|  |   M|  |   0|  |       |
    4    |----   |---- B|----   |----    |       |
         |      0|   600| 1M-45|      1|   600|     -1
         |_______|_______|_______|_______|       |
         |  39|  |  53|  |   M|  |   0|  |       |
    5    |---- P|---- B|----   |---- L|       |
         |   600|   400| 1M-46|      0|  1000|      0
         |_______|_______|_______|_______|_______|______
Demand   |   700   1000    900    400|
_________|_______________________________|
  v[j]   |    39     53     46      0|
         |                               | Z = 121300
```

```
          |         Destination        |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|_____________
          |  41|  |  55|  |  48|  |   0|  |       |
        1 |---- L|----   |---- P|---- B|       |
          |   100|     0|   300|     0|   400|      0
          |_______|_______|_______|_______|       |
          |  39|  |  51|  |  45|  |   0|  |       |
        2 |----   |---- E|---- P|----   |       |
          |     1|    -1|   600|     3|   600|     -3
          |_______|_______|_______|_______|       |
          |  42|  |  56|  |  50|  |   0|  |       |
        3 |----   |----   |----   |---- B|       |
          |     1|     1|     2|   400|   400|      0
          |_______|_______|_______|_______|       |
          |  38|  |  52|  |   M|  |   0|  |       |
        4 |----   |---- B|----   |----   |       |
          |     0|   600| 1M-45|     3|   600|     -3
          |_______|_______|_______|_______|       |
          |  39|  |  53|  |   M|  |   0|  |       |
        5 |---- P|---- P|----   |----   |       |
          |   600|   400| 1M-46|     2|  1000|     -2
          |_______|_______|_______|_______|_______|______
Demand    |   700    1000     900     400|
__________|______________________________|
   v[j]   |    41      55      48       0|
          |                              |  Z = 121300
```

```
          |         Destination        |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|_____________
          |  41|  |  55|  |  48|  |   0|  |       |
        1 |----   |----   |---- B|---- B|       |
          |     1|     1|   400|     0|   400|     54
          |_______|_______|_______|_______|       |
          |  39|  |  51|  |  45|  |   0|  |       |
        2 |----   |---- B|---- B|----   |       |
          |     2|   100|   500|     3|   600|     51
          |_______|_______|_______|_______|       |
          |  42|  |  56|  |  50|  |   0|  |       |
        3 |----   |----   |----   |---- B|       |
          |     2|     2|     2|   400|   400|     54
          |_______|_______|_______|_______|       |
          |  38|  |  52|  |   M|  |   0|  |       |
        4 |----   |---- B|----   |----   |       |
          |     0|   600| 1M-46|     2|   600|     52
          |_______|_______|_______|_______|       |
          |  39|  |  53|  |   M|  |   0|  |       |
        5 |---- B|---- B|----   |----   |       |
          |   700|   300| 1M-47|     1|  1000|     53
          |_______|_______|_______|_______|_______|______
Demand    |   700    1000     900     400|
__________|______________________________|
   v[j]   |   -14       0      -6     -54|
          |                              |  Z = 121200
```

Optimal Solution:

$x_{13} = x_{34} = 400, x_{22} = 100, x_{23} = 500, x_{42} = 600, x_{51} = 700, x_{52} = 300,$

Cost: $\$121,200$

8.2-14.

Using Russell's approximation method:

(a)

	Destination 1	2	3	4	5	Supply	u[i]
1	−800 L 40	−700 P 20	−500 100	−200 600	−500 300	60	−500
2	−500 E −200	−200 B 0	−100 B 20	−300 P 60	−100 200	80	0
3	−600 −100	−400 P 40	−300 0	−500 0	−300 200	40	−200
4	M 1M+ 0	M 1M−1e	M 1M−2e	0 B 0	0 B 60	60	300
Demand	40	60	20	60	60		
v[j]	−300	−200	−100	−300	−300		

Z = −82000

(b)

	Destination 1	2	3	4	5	Supply	u[i]
1	−800 P 40	−700 P 20	−500 −100	−200 400	−500 100	60	−300
2	−500 P 0	−200 200	−100 L 20	−300 B 60	−100 200	80	0
3	−600 −100	−400 P 40	−300 E −200	−500 −200	−300 0	40	0
4	M 1M+2e	M 1M+1e	M 1M−2e	0 B 0	0 B 60	60	300
Demand	40	60	20	60	60		
v[j]	−500	−400	−100	−300	−300		

Z = −82000

(2)

	Destination 1	2	3	4	5	Supply	u[i]
1	−800 P 20	−700 P 40	−500 100	−200 400	−500 100	60	0
2	−500 P 20	−200 200	−100 200	−300 P 60	−100 200	80	300
3	−600 −100	−400 L 20	−300 B 20	−500 E −200	−300 0	40	300
4	M 1M+2e	M 1M+1e	M 1M+ 0	0 B 0	0 B 60	60	600
Demand	40	60	20	60	60		
v[j]	−800	−700	−600	−600	−600		

Z = −86000

(3)

	Destination 1	2	3	4	5	Supply	u[i]
1	−800 B 0	−700 B 60	−500 −100	−200 400	−500 100	60	−600
2	−500 B 40	−200 200	−100 0	−300 B 40	−100 200	80	−300
3	−600 100	−400 200	−300 B 20	−500 B 20	−300 200	40	−500
4	M 1M+2e	M 1M+1e	M 1M−2e	0 B 0	0 B 60	60	0
Demand	40	60	20	60	60		
v[j]	−200	−100	200	0	0		

Z = −90000

The optimal solution is to ship 60 units from plant 1 to customer 2, 40 from plant 2 to customer 1, 40 from plant 2 to customer 4, 20 from plant 3 to customer 3 and 4. This offers a profit of \$90, 000.

8.2-15.

(a) - (b) - (c)

Using northwest corner rule:

(0)

	Destination 1	2	3	4	Supply	u[i]
1	800 P 20	700 B 20	400 P 10	0 100	50	−100
2	600 E −300	800 0	500 L 10	0 B 40	50	0
Demand	20	20	20	40		
v[j]	900	800	500	0	Z = 39000	

(1)

	Destination 1	2	3	4	Supply	u[i]
1	800 L 10	700 B 20	400 B 20	0 E −200	50	0
2	600 P 10	800 300	500 300	0 P 40	50	−200
Demand	20	20	20	40		
v[j]	800	700	400	200	Z = 36000	

(2)

	Destination 1	2	3	4	Supply	u[i]
1	800 200	700 B 20	400 B 20	0 B 10	50	0
2	600 B 20	800 100	500 100	0 B 30	50	0
Demand	20	20	20	40		
v[j]	600	700	400	0	Z = 34000	

With northwest corner rule, it took 22 seconds to find the initial BF solution and its objective value is 15% above the optimal cost. The two iterations took 48 seconds.

Using Vogel's approximation method:

(0)

	Destination 1	2	3	4	Supply	u[i]
1	800 L 10	700 B 20	400 B 20	0 E −200	50	200
2	600 P 10	800 300	500 300	0 P 40	50	0
Demand	20	20	20	40		
v[j]	600	500	200	0		

Z = 36000

(1)

	Destination 1	2	3	4	Supply	u[i]
1	800 200	700 B 20	400 B 20	0 B 10	50	0
2	600 B 20	800 100	500 100	0 B 30	50	0
Demand	20	20	20	40		
v[j]	600	700	400	0		

Z = 34000

With Vogel's approximation method, it took 44 seconds to find the initial BF solution and its objective value is 6% above the optimal cost. One iteration took 28 seconds.

Using Russell's approximation method:

(0)

	Destination 1	2	3	4	Supply	u[i]
1	800 300	700 0	400 P 10	0 P 40	50	0
2	600 B 20	800 B 20	500 L 10	0 E −100	50	100
Demand	20	20	20	40		
v[j]	500	700	400	0		

Z = 37000

(1)	Destination 1	2	3	4	Supply	u[i]
1	800 / 200	700 E / −100	400 B / 20	0 P / 30	50	0
2	600 B / 20	800 L / 20	500 / 100	0 P / 10	50	0
Demand	20	20	20	40		
v[j]	600	800	400	0	Z = 36000	

(2)	Destination 1	2	3	4	Supply	u[i]
1	800 / 200	700 B / 20	400 B / 20	0 B / 10	50	0
2	600 B / 20	800 / 100	500 / 100	0 B / 30	50	0
Demand	20	20	20	40		
v[j]	600	700	400	0	Z = 34000	

With Russell's approximation method, it took 25 seconds to find the initial BF solution and its objective value is 9% above the optimal cost. The two iterations took 45 seconds.

Let x_0 denote the initial BF solution. The results are summarized in the following table.

Method	Time to Get x_0	Opt. Gap of x_0	No. Iter.'ns	Time Iter.'ns	Total Time
NW Corner	22 seconds	15%	2	48 seconds	70 seconds
Vogel's	44 seconds	6%	1	28 seconds	72 seconds
Russell's	25 seconds	9%	2	45 seconds	70 seconds

8.2-16.

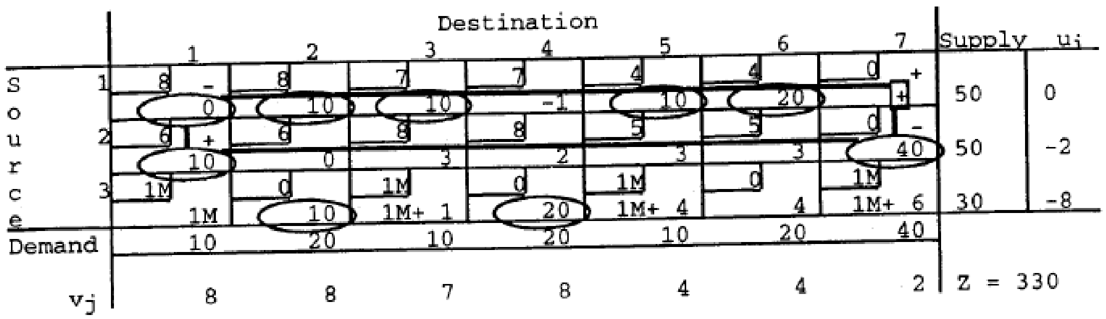
(a) - (b) - (c)

Using northwest corner rule:

Source	Destination 1	2	3	4	5	6	7	Supply	u_i
1	8 / 10	8 − / 20	7 / 10	7 + / 10	4 / 0	4 / 0	0 / 1	50	0
2	6 / −3	6 / −3	8 / 0	8 − / 10	5 / 10	5 / 20	0 + / 10	50	1
3	1M / −9	0 +	1M / −8	0 / −1M− 8	1M / −5	0 / −1M− 5	1M − / 30	30	1M +1
Demand	10	20	10	20	10	20	40		
v_j	8	8	7	7	4	4	−1	Z = 30M +610	

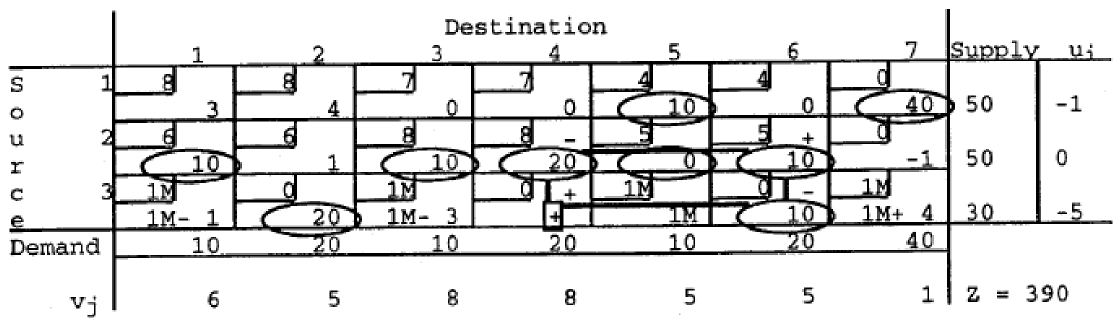
With northwest corner rule, it took 40 seconds to find the initial BF solution and its objective value is M% above the optimal cost. The seven iterations took 4 minutes.

Using Vogel's approximation method:



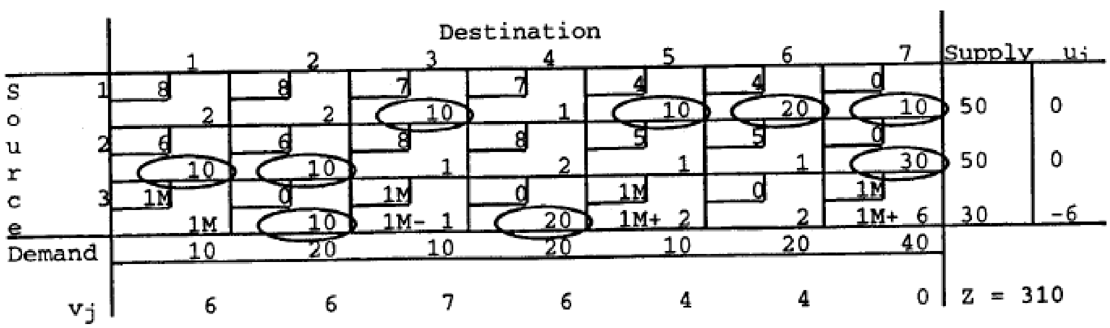
With Vogel's approximation method, it took 55 seconds to find the initial BF solution and its objective value is 6% above the optimal cost. The two iterations took 1 minute.

Using Russell's approximation method:



With Russell's approximation method, it took 63 seconds to find the initial BF solution and its objective value is 26% above the optimal cost. The five iterations took 2 minutes.

Optimal Solution: cost $31,000$



Let x_0 denote the initial BF solution. The results are summarized in the following table.

Method	Time to Get x_0	Opt. Gap of x_0	No. Iter.'ns	Time Iter.'ns	Total Time
NW Corner	40 seconds	M%	7	4 minutes	280 seconds
Vogel's	55 seconds	6%	2	1 minute	115 seconds
Russell's	63 seconds	26%	5	2 minutes	183 seconds

8.2-17.

(a) Initial solution using northwest corner rule:

			u_i
	8: 3	5: 1	0
	6: -1	4: 2	-1
v_j	8	5	

Final tableau: cost 35

			u_i
	8: 1	5: 3	0
	6: 2	4: 1	-2
v_j	8	5	

(b) minimize $8x_{11} + 5x_{12} + 6x_{21} + 4x_{22}$

subject to

$$
\begin{aligned}
x_{11} + x_{12} \qquad\qquad\qquad &= 4 \\
x_{21} + x_{22} &= 2 \\
x_{11} + x_{21} &= 3 \\
x_{12} + x_{22} &= 3 \\
x_{11}, x_{12}, x_{21}, x_{22} &\geq 0
\end{aligned}
$$

Iter.	B.V.	Eq. #	Z	X_{11}	X_{12}	X_{21}	X_{22}	W_1	W_2	W_3	W_4	RHS
0	Z	0	-1	8-2M	5-2M	6-2M	4-2M	0	0	0	0	-12M
	W_1	1	0	1	1	0	0	1	0	0	0	4
	W_2	2	0	0	0	1	1	0	1	0	0	2
	W_3	3	0	1	0	1	0	0	0	1	0	3
	W_4	4	0	0	1	0	1	0	0	0	1	3
4	Z	0	-1	0	0	0	1	2M-8	2M-6	0	3	-35
	X_{11}	1	0	1	0	0	-1	1	0	0	-1	1
	X_{21}	2	0	0	0	1	1	0	1	0	0	2
	W_3	3	0	0	0	0	0	-1	-1	1	1	0
	X_{12}	4	0	0	1	0	1	0	0	0	1	3

Hence, the transportation simplex method takes one iteration while the general simplex method takes four iterations. The computation times vary.

8.2-18.

Let $z_1 = x_1 - 10$,
$z_2 = x_1 + x_2 - 25$,
$z_3 = x_1 + x_2 + x_3 - 50$,
$z_4 = x_1 + x_2 + x_3 + x_4 - 70$.

minimize $\quad 1.08x_1 + 1.11x_2 + 1.10x_3 + 1.13x_4 + 0.15(z_1 + z_2 + z_3 + z_4)$

subject to

$$\begin{array}{lllllllll} x_1 & & & & -z_1 & & & & = 10 \\ x_1 + & x_2 & & & & -z_2 & & & = 25 \\ x_1 + & x_2 + & x_3 & & & & -z_3 & & = 50 \\ x_1 + & x_2 + & x_3 + & x_4 & & & & -z_4 & = 70 \\ \multicolumn{9}{l}{0 \le x_1 \le 25} \\ \multicolumn{9}{l}{0 \le x_2 \le 35} \\ \multicolumn{9}{l}{0 \le x_3 \le 30} \\ \multicolumn{9}{l}{0 \le x_4 \le 10} \\ \multicolumn{9}{l}{z_1, z_2, z_3, z_4 \ge 0} \end{array}$$

Initial simplex tableau:

B.V.	Eq.#	Z	x_1	x_2	x_3	x_4	z_1	z_2	z_3	z_4	w_1	w_2	w_3	w_4	y_1	y_2	y_3	y_4	RHS
Z	0	-1	$-4M+1.08$	$-3M+1.11$	$-2M+1.1$	$-M+1.13$	$M+.015$	$M+.015$	$M+.015$	$M+.015$	0	0	0	0	0	0	0	0	$-155M$
w_1	1		1				-1				1								10
w_2	2		1	1				-1				1							25
w_3	3		1	1	1				-1				1						50
w_4	4		1	1	1	1				-1				1					70
y_1	5		1												1				25
y_2	6			1												1			35
y_3	7				1												1		30
y_4	8					1												1	10

Simplex tableau: 16 variables and 8 constraints

Transportation tableau: 20 variables and 9 constraints

Even though the transportation tableau is larger, it requires less work than the simplex tableau.

8.2-19.

If we multiply the demand constraints by -1, each constraint column will have exactly two nonzero entries, one -1 and one $+1$. Summing all these constraints gives the equality:

$$0x = \sum \text{supplies} - \sum \text{demands} = 0,$$

since the total supply equals the total demand. Hence, there is a redundant constraint.

8.2-20.

In the initialization step, after selecting the next basic variable, the allocation made is equal to either the (remaining) supply or demand for that row or column. Since these quantities are known to be integer, the allocation will be integer.

Given a current BF solution that is integer, step 3 of an iteration adds and subtracts, around the chain-reaction cycle, the current value of the leaving basic variable. Since we know this is an integer, and all the other basic variables on the cycle began with integer values, the new BF solution must be all integer.

During the initialization step, we can select the next basic variable for allocation arbitrarily from among the rows and columns not already eliminated. Thus, by altering our selections, we can construct any BF solution as our initial one. Because we have shown that the initialization step gives integer solutions, all BF solutions must be integer.

8.2-21.

(a) Let x_{ij} be the number of tons hauled from pit $i = 1, 2$ (North, South) to site $j = 1, 2, 3$.

minimize $\quad 400x_{11} + 490x_{12} + 460x_{13} + 600x_{21} + 530x_{22} + 560x_{23}$

subject to
$$\begin{aligned}
&x_{11} + x_{12} + x_{13} \leq 18 \\
&x_{21} + x_{22} + x_{23} \leq 14 \\
&x_{11} + x_{21} = 10 \\
&x_{12} + x_{22} = 5 \\
&x_{13} + x_{23} = 10 \\
&x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0
\end{aligned}$$

Initial tableau:

Bas Var	Eq No	Z	X11	X12	X13	X21	X22	X23	X7	X8	X9	X10	X11	Right side
Z	0	-1	-M+400	-M+490	-M+460	-M+600	-M+530	-M+560	0	0	0	0	0	-25M
X7	1	0	1	1	1	0	0	0	1	0	0	0	0	18
X8	2	0	0	0	0	1	1	1	0	1	0	0	0	14
X9	3	0	1	0	0	1*	0	0	0	0	1	0	0	10
X1	4	0	0	1	0	0	1	0	0	0	0	1	0	5
X1	5	0	0	0	1	0	0	1	0	0	0	0	1	10

(b) This table is much smaller than the simplex tableau and it stores the same information.

Cost Per Unit Distributed

		Destination 1	2	3	4	Supply
Source	1	400	490	460	0	18
	2	600	530	560	0	14
Demand		10	5	10	7	

(c) The solution is not optimal, since $c_{13} - u_1 - v_3 = -100$.

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|______________
          | 400|  | 490|  | 460|  |   0|  |       |
    1     |---- B|---- B|----   |---- B|       |
          |    10|     5|  -100|     3|     18|      0
          |_______|_______|_______|_______|       |
          | 600|  | 530|  | 560|  |   0|  |       |
    2     |----   |----   |---- B|---- B|       |
          |   200|    40|    10|     4|     14|      0
          |_______|_______|_______|_______|_______|______
 Demand   |    10      5     10      7|
__________|_______________________________|
   v[j]   |   400    490    560      0|
          |                               |  Z = 12050
```

(d)

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|______________
          | 400|  | 490|  | 460|  |   0|  |       |
    1     |---- B|---- B|---- B|----   |       |
          |    10|     5|     3|     0|     18|      0
          |_______|_______|_______|_______|       |
          | 600|  | 530|  | 560|  |   0|  |       |
    2     |----   |----   |---- B|---- B|       |
          |     0|     0|     7|     7|     14|      0
          |_______|_______|_______|_______|_______|______
 Demand   |    10      5     10      7|
__________|_______________________________|
   v[j]   |     0      0      0      0|
          |                               |  Z = 11750
```

```
          |          Destination          |
          |   1   |   2   |   3   |   4   |Supply  u[i]
__________|_______|_______|_______|_______|______________
          | 400|  | 490|  | 460|  |   0|  |       |
    1     |---- B|---- L|---- P|----   |       |
          |    10|     5|     3|   100|     18|      0
          |_______|_______|_______|_______|       |
          | 600|  | 530|  | 560|  |   0|  |       |
    2     |----   |---- E|---- P|---- B|       |
          |   100|   -60|     7|     7|     14|    100
          |_______|_______|_______|_______|_______|______
 Demand   |    10      5     10      7|
__________|_______________________________|
   v[j]   |   400    490    460   -100|
          |                               |  Z = 11750
```

```
           |          Destination          |
           |   1   |   2   |   3   |   4   |Supply  u[i]
___________|_______|_______|_______|_______|______________
           | 400|  | 490|  | 460|  |    0| |        |
         1 |---- B|----   |---- B|----    |        |
           |     10|     60|      8|    100|     18|  -100
           |_______|_______|_______|_______|        |
           | 600|  | 530|  | 560|  |    0| |        |
         2 |----   |---- B|---- B|---- B|        |
           |    100|      5|      2|      7|     14|     0
           |_______|_______|_______|_______|_______|______
    Demand |     10       5      10       7|
___________|_______________________________|
      v[j] |    500     530     560       0|
           |                               | Z = 11450
```

The optimal solution is to haul 10 tons from the north pit to site 1 and 8 tons to site 3, 5 tons from the south pit to site 2 and 2 tons from the south pit to site 3. This incurs a cost of \$$11,450$.

(e) From the reduced costs $(c_{ij} - u_i - v_j)$ in the final tableau, we see that

$$\Delta c_{12} \geq -60 \quad \Rightarrow c_{12} \geq 430$$

$$\Delta c_{21} \geq -100 \;\; \Rightarrow c_{21} \geq 500.$$

If the contractor can negotiate a hauling cost per ton of 130 or less from the north pit to site 2, or of 80 or less from the south pit to site 1, a new solution using these options would give a cost at least as small as the current optimal cost \$$11,450$.

8.2-22.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D15	Colombo Berdoo	2.22045E-16	0	160	0	20
E15	Colombo Los Devils	5	0	130	20	1E+30
F15	Colombo San Go	0	10	220	1E+30	10
G15	Colombo Hollyglass	0	0	170	1E+30	0
D16	Sacron Berdoo	2	0	140	10	0
E16	Sacron Los Devils	0	20	130	1E+30	20
F16	Sacron San Go	2.5	0	190	10	10
G16	Sacron Hollyglass	1.5	0	150	0	1E+30
D17	Calorie Berdoo	0	10	190	1E+30	10
E17	Calorie Los Devils	0	50	200	1E+30	50
F17	Calorie San Go	1.5	0	230	10	20
G17	Calorie Hollyglass	0	-190	0	1E+30	190

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
H15	Colombo Totals	5	-20	5	1.5	0
H16	Sacron Totals	6	-40	6	1.5	2.5
H17	Calorie Totals	1.5	0.0	5	1E+30	3.5
D18	Totals Berdoo	2	180	2	2.5	1.5
E18	Totals Los Devils	5	150	5	0	1.5
F18	Totals San Go	4	230	4	3.5	1.5
G18	Totals Hollyglass	1.5	190	1.5	2.5	1.5

(a) The optimal solution would change because the decrease of $30 million is outside the allowable decrease of $20 million.

(b) The optimal solution would remain the same, since the allowable increase is ∞.

(c) By the 100% rule for simultaneous changes, the optimal solution must remain the same.

$$C_{CS}: \quad \$230 \rightarrow \$215 \;\; \text{\% of allowable decrease} = 100\left(\frac{230-215}{20}\right) = 75\%$$

$$C_{SL}: \quad \$130 \rightarrow \$145 \;\; \text{\% of allowable decrease} = 100\left(\frac{130-145}{\infty}\right) = 0\%$$

These sum up to 75%.

(d) By the 100% rule for simultaneous changes, the shadow prices may or may not remain valid.

$$C_S: \quad \$6 \rightarrow \$5.5 \qquad \text{\% of allowable decrease} = 100\left(\frac{6-5.5}{2.5}\right) = 20\%$$

$$C_H: \quad \$1.5 \rightarrow \$1 \qquad \text{\% of allowable decrease} = 100\left(\frac{1.5-1}{0}\right) = \infty\%$$

These sum up to ∞%.

8.2-23.

(a) $\Delta c_{34} = -3 \Rightarrow \Delta(c_{34} - u_3 - v_4)^* = -3$, $(c_{34} - u_3 - v_4)^* = -2$

Iteration 3		Destination 1	2	3	4	5	Supply	u_i
Source	1	16; +4	16; +4	13; 50	22; +7	17; +2	50	−7
	2	14; +2	14; +2	13; 20	19; +4	15; 40	60	−7
	3	19; 30	19; 20	20; 0	20; −2	M; M − 22	50	0
	4(D)	M; M + 3	0; +3	M; M + 2	0; 30	0; 20	50	−22
Demand		30	20	70	30	60	Z = Z+60	
v_j		19	19	20	22	22		

The current feasible solution is feasible, but not optimal.

(b) $\Delta c_{23} = 3 \Rightarrow \Delta(c_{23} - u_2 - v_3)^* = 3$

We can revise the tableau by changing u_2 from -7 to $-7 + 3 = -4$. This causes v_5 to change to $22 - 3 = 19$, u_4 to $-22 + 3 = -19$, and v_4 to $22 - 3 = 19$.

Δ(reduced cost x_{41}) $=$ Δ(reduced cost x_{42}) $=$ Δ(reduced cost x_{43}) $= -\Delta u_4 = -3$

Δ(reduced cost x_{34}) $=$ Δ(reduced cost x_{14}) $= -\Delta v_4 = 3$

Δ(reduced cost x_{35}) $=$ Δ(reduced cost x_{15}) $= -\Delta v_5 = 3$

Iteration 3	Destination 1	2	3	4	5	Supply	u_i
Source 1	16; +4	16; +4	13; (50)	22; 10	17; 5	50	−7
2	14; +2	14; +2	16; (20)	19; +4	15; (40)	60	−4
3	19; (30)	19; (20)	20; (0)	23; 4	M; M−19	50	0
4(D)	M; M	0; 0	M; M−1	0; (30)	0; (20)	50	−19
Demand	30	20	70	30	60	Z = 2460	
v_j	19	19	20	19	19		

The basic solution remains feasible and optimal.

(c) $\Delta s_2 = -10, \Delta d_5 = 10 \Rightarrow \Delta x_{25} = 10$

Iteration 3	Destination 1	2	3	4	5	Supply	u_i
Source 1	16; +4	16; +4	13; (50)	22; +7	17; +2	50	−7
2	14; +2	14; +2	13; (20)	19; +4	15; (30)	50	−7
3	19; (30)	19; (20)	20; (0)	23; +1	M; M − 22	50	0
4(D)	M; M + 3	0; +3	M; M + 2	0; (30)	0; (20)	50	−22
Demand	30	20	70	30	50	Z = 2460	
v_j	19	19	20	22	22		

The basic solution remains feasible and optimal.

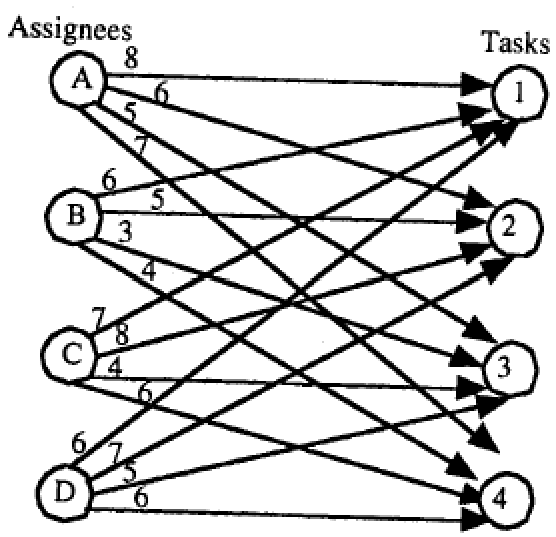
(d) $\Delta s_2 = \Delta d_2 = 20 \Rightarrow \Delta x_{23} = \Delta x_{32} = 20$ and $\Delta x_{33} = -20$

Iteration 3	Destination 1	2	3	4	5	Supply	u_i
Source 1	16 / +4	16 / +4	13 / (50)	22 / +7	17 / +2	50	−7
2	14 / +2	14 / +2	13 / (40)	19 / +4	15 / (40)	80	−7
3	19 / (30)	19 / (40)	20 / (−20)	23 / +1	M / M − 22	50	0
4(D)	M / M + 3	0 / +3	M / M + 2	0 / (30)	0 / (20)	50	−22
Demand	30	40	70	30	60	Z = 2460	
v_j	19	19	20	22	22		

This solution satisfies the optimality criterion, but it is infeasible.

8.3-1.

(a)



(b) - (c)

Unit Cost ($)

		Task 1	2	3	4	Supply
Assignee	A	8	6	5	7	1
	B	6	5	3	4	1
	C	7	8	4	6	1
	D	6	7	5	6	1
Demand		1	1	1	1	

(d)

Assignments

		Task 1	2	3	4	Totals		Supply
Assignee	A	0	1	0	0	1	=	1
	B	0	0	0	1	1	=	1
	C	0	0	1	0	1	=	1
	D	1	0	0	0	1	=	1
Totals		1	1	1	1			
		=	=	=	=	Total Cost	=	$20
Demand		1	1	1	1			

8.3-2.

(a) Ships are assignees and ports are assignments.

(b)

```
                        Task
Assignee| A       B       C       D
        |
   1    | 500     400     600     700
   2    | 600     600     700     500
   3    | 700     500     700     600
   4    | 500     400     600     600
        |
```

Optimal Solution:

```
Task A is assigned to Assignee 1
Task D is assigned to Assignee 2
Task B is assigned to Assignee 3
Task C is assigned to Assignee 4
```

This incurs a cost of $2, 100.

(c)

	Destination 1	2	3	4	Supply
1	500	400	600	700	1
2	600	600	700	500	1
3	700	500	700	600	1
4	500	400	600	600	1
Demand	1	1	1	1	

(d) - (e)

(o)	Destination 1	2	3	4	Supply	u[i]
1	500 ---- P 1	400 ---- P 0	600 ---- 100	700 ---- 300	1	0
2	600 ---- -100	600 ---- P 1	700 ---- P 0	500 ---- -100	1	200
3	700 ---- 0	500 ---- -100	700 ---- L 1	600 ---- P 0	1	200
4	500 ---- E -200	400 ---- -200	600 ---- -100	600 ---- P 1	1	200
Demand	1	1	1	1		
v[j]	500	400	500	400		

(1)

	Destination 1	2	3	4	Supply	u[i]
1	500 P 0	400 P 1	600 100	700 100	1	0
2	600 -100	600 L 0	700 B 1	500 E -300	1	200
3	700 200	500 100	700 200	600 B 1	1	0
4	500 P 1	400 0	600 100	600 P 0	1	0
Demand	1	1	1	1		
v[j]	500	400	500	600		

(2)

	Destination 1	2	3	4	Supply	u[i]
1	500 B 0	400 B 1	600 -200	700 100	1	600
2	600 200	600 300	700 P 1	500 P 0	1	500
3	700 200	500 100	700 -100	600 B 1	1	600
4	500 B 1	400 0	600 E -200	600 L 0	1	600
Demand	1	1	1	1		
v[j]	-100	-200	200	0		

(3)

	Destination 1	2	3	4	Supply	u[i]
1	500 P 0	400 P 1	600 0	700 300	1	0
2	600 0	600 100	700 L 1	500 P 0	1	100
3	700 0	500 E -100	700 -100	600 P 1	1	200
4	500 P 1	400 0	600 P 0	600 200	1	0
Demand	1	1	1	1		
v[j]	500	400	600	400		

(4)

	Destination 1	2	3	4	Supply	u[i]
1	500 B 1	400 B 0	600 0	700 200	1	0
2	600 100	600 200	700 100	500 B 1	1	0
3	700 100	500 B 1	700 0	600 B 0	1	100
4	500 B 0	400 0	600 B 1	600 100	1	0
Demand	1	1	1	1		
v[j]	500	400	600	500		

One optimal assignment is: $(1,1),(2,4),(3,2),(4,3)$, where the first entry is ship and the second port.

(f) Continuing to pivot where reduced costs are zero:

(5)	Destination 1	2	3	4	Supply	u[i]
1	500 B 1	400 B 0	600 0	700 200	1	400
2	600 100	600 200	700 100	500 B 1	1	400
3	700 100	500 P 1	700 E 0	600 B 0	1	500
4	500 0	400 P 0	600 L 1	600 100	1	400
Demand	1	1	1	1		
v[j]	100	0	200	100		

Alternative optimal matching: $(1,1),(2,4),(3,3),(4,2)$

(6)	Destination 1	2	3	4	Supply	u[i]
1	500 L 1	400 P 0	600 0	700 200	1	-100
2	600 100	600 200	700 100	500 B 1	1	-100
3	700 100	500 B 0	700 B 1	600 B 0	1	0
4	500 E 0	400 P 1	600 0	600 100	1	-100
Demand	1	1	1	1		
v[j]	600	500	700	600		

Alternative optimal matching: $(1,2),(2,4),(3,3),(4,1)$

(7)

	Destination 1	2	3	4	Supply	u[i]
1	500 / 0	400 P / 1	600 E / 0	700 / 200	1	-100
2	600 / 100	600 / 200	700 / 100	500 B / 1	1	-100
3	700 / 100	500 P / 0	700 L / 1	600 B / 0	1	0
4	500 B / 1	400 B / 0	600 / 0	600 / 100	1	-100
Demand	1	1	1	1		
v[j]	600	500	700	600		

Alternative optimal matching: $(1,3),(2,4),(3,2),(4,1)$

8.3-3.

(a) Costs are expressed in thousands of dollars.

	Task				
Assignee	A	B	C	D	E
1	11.48	22	16.8	0	0
2	10.92	20.4	15.75	0	0
3	11.76	22.4	17.5	0	0
4	10.64	20.8	M	0	0
5	10.92	21.2	M	0	0

(b) The optimal cost is 47.47 thousand dollars.

```
Task D is assigned to Assignee 1
Task C is assigned to Assignee 2
Task E is assigned to Assignee 3
Task B is assigned to Assignee 4
Task A is assigned to Assignee 5
```

(c)

Cost Per Unit Distributed

		Destination 1	2	3	4	5	Supply
Source	1	11.48	22	16.8	0	0	1
	2	10.92	20.4	15.75	0	0	1
	3	11.76	22.4	17.5	0	0	1
	4	10.64	20.8	1M	0	0	1
	5	10.92	21.2	1M	0	0	1
Demand		1	1	1	1	1	

(d)

```
         |            Destination            |
         |   1  |   2  |   3  |   4  |   5  |Supply  u[i]
_________|______|______|______|______|______|____________
         |11.5| |  22| |16.8| |   0| |   0| |       |
     1   |---- B|----  |---- B|---- B|----  |       |
         |     0|  0.24|     0|     1|     0|      1|  0.56
         |______|______|______|______|______|       |
         |10.9| |20.4| |15.8| |   0| |   0| |       |
     2   |----  |----  |---- B|----  |----  |       |
         |  0.49| -0.31|     1|  1.05|  1.05|      1| -0.49
         |______|______|______|______|______|       |
         |11.8| |22.4| |17.5| |   0| |   0| |       |
     3   |----  |----  |----  |---- B|---- B|       |
         |  0.28|  0.64|   0.7|     0|     1|      1|  0.56
         |______|______|______|______|______|       |
         |10.6| |20.8| |    M| |   0| |   0| |       |
     4   |----  |---- B|----  |----  |----  |       |
         |  0.12|     1| 1M-16|  0.96|  0.96|      1|  -0.4
         |______|______|______|______|______|       |
         |10.9| |21.2| |    M| |   0| |   0| |       |
     5   |---- B|---- B|----  |----  |----  |       |
         |     1|     0| 1M-16|  0.56|  0.56|      1|     0
         |______|______|______|______|______|_______|_____
Demand   |     1      1      1      1      1|
_________|__________________________________|
   v[j]  | 10.92   21.2  16.24  -0.56  -0.56|
         |                                  | Z = 47.47
```

The initial solution from Vogel's approximation method is optimal. Plant 2 produces product 3, plant 4 produces product 2, plant 5 produces product 1. This incurs a cost of $47,470.

8.3-4.

(a) After adding a dummy stroke, which everyone can swim in zero seconds, the problem becomes that of assigning 5 swimmers to 5 strokes. The optimal solution turns out to be the following: David swims the backstroke, Tony swims the breaststroke, Chris swims the butterfly, and Carl swims the freestyle.

		Task				
	Carl	Chris	David	Tony	Ken	
Assignee	A	B	C	D	E	Row Min
Back 1	37.7	32.9	33.8	37	35.4	32.9
Breast 2	43.4	33.1	42.2	34.7	41.8	33.1
Fly 3	33.3	28.5	38.9	30.4	33.6	28.5
Free 4	29.2	26.4	29.6	28.5	31.1	26.4
Dummy 5	0	0	0	0	0	0

(b) Cost: 126.2

```
Task C is assigned to Assignee 1
Task D is assigned to Assignee 2
Task B is assigned to Assignee 3
Task A is assigned to Assignee 4
Task E is assigned to Assignee 5
```

8.3-5.

(a)

		Product				
	1	2	3	4	5	Supply
Plant 1	820	810	840	960	0	2
Plant 2	800	870	M	920	0	2
Plant 3	740	900	810	840	0	1
Demand	1	1	1	1	1	

(b) - (c)

	Destination 1	2	3	4	5	Supply	u[i]
1	820 / 20	810 B / 1	840 B / 1	960 / 40	0 B / 0	2	0
2	800 B / 1	870 / 60	M / 1M-e3	920 B / 0	0 B / 1	2	0
3	740 / 20	900 / 170	810 / 50	840 B / 1	0 / 80	1	-80
Demand	1	1	1	1	1		
v[j]	800	810	840	920	0		

Since all the reduced costs are nonnegative, this solution is optimal.

(d)

	Product 1	2	3	4	5	Supply
Plant 1	820	810	840	960	0	1
2	820	810	840	960	0	1
3	800	870	M	920	0	1
4	800	870	M	920	0	1
5	740	900	810	840	M	1
Demand	1	1	1	1	1	

This is identical to the table in (a) except that plants 1 and 2 have been split into two plants each.

(e)

	Destination 1	2	3	4	5	Supply	u[i]
1	820 / 20	810 B / 1	840 / 0	960 / 40	0 / 0	1	0
2	820 / 20	810 B / 0	840 B / 1	960 / 40	0 B / 0	1	0
3	800 B / 1	870 / 60	M / 1M-e3	920 / 0	0 / 0	1	0
4	800 B / 0	870 / 60	M / 1M-e3	920 B / 0	0 B / 1	1	0
5	740 / 20	900 / 170	810 / 50	840 B / 1	M / 1M+80	1	-80
Demand	1	1	1	1	1		
v[j]	800	810	840	920	0		

The basic feasible solution for the transformed problem above corresponds to that given in part (c).

8.3-6.

0

Source	Dest. 1	Dest. 2	Dest. 3	Dest. 4	Supply	u[i]
1	13 / 7	16 / 8	12 / 8	11 B / 1	1	-9
2	15 B / 1	M / 1M-17	13 P / 0	20 L / 0	1	0
3	5 B / 0	7 B / 1	10 / 7	6 / -4	1	-10
4	0 / -2	0 / -4	0 P / 1	0 E / -7	1	-13
Demand	1	1	1	1		
v[j]	15	17	13	20		

(1)

Source	Dest. 1	Dest. 2	Dest. 3	Dest. 4	Supply	u[i]
1	13 / 0	16 / 1	12 / 1	11 B / 1	1	-2
2	15 L / 1	M / 1M-17	13 P / 0	20 / 7	1	0
3	5 P / 0	7 P / 1	10 / 7	6 / 3	1	-10
4	0 / -2	0 E / -4	0 P / 1	0 B / 0	1	-13
Demand	1	1	1	1		
v[j]	15	17	13	13		

(2)

	Destination 1	2	3	4	Supply	u[i]
1	13 ---- 4	16 ---- 5	12 ---- 1	11 ---- B 1	1	11
2	15 ---- 4	M ---- 1M-13	13 ---- B 1	20 ---- 7	1	13
3	5 ---- B 1	7 ---- L 0	10 ---- 3	6 ---- E -1	1	7
4	0 ---- 2	0 ---- P 1	0 ---- B 0	0 ---- P 0	1	0
Demand	1	1	1	1		
v[j]	-2	0	0	0		

(3)

	Destination 1	2	3	4	Supply	u[i]
1	13 ---- 3	16 ---- 5	12 ---- 1	11 ---- B 1	1	11
2	15 ---- 3	M ---- 1M-13	13 ---- B 1	20 ---- 7	1	13
3	5 ---- B 1	7 ---- 1	10 ---- 4	6 ---- B 0	1	6
4	0 ---- 1	0 ---- B 1	0 ---- B 0	0 ---- B 0	1	0
Demand	1	1	1	1		
v[j]	-1	0	0	0		

This solution corresponds to that given in Section 8.3; although the set of basic variables is different, the values of the variables are the same.

8.3-7.

(a) Let assignees 1 and 2 represent plant A, assignees 3 and 4 represent plant B, and the tasks be the distribution centers.

Cost Table

	Task 1	Task 2	Task 3	Task 4 (dummy)
Assignee 1	8000	14000	12000	0
Assignee 2	8000	14000	12000	0
Assignee 3	6000	16000	15000	0
Assignee 4	6000	16000	15000	0

(b) Cost: $32,000$

	Task 1	Task 2	Task 3	Task 4
Assignee 1		X		
Assignee 2			X	
Assignee 3	X			
Assignee 4				X

(c)

Cost Per Unit Distributed

	Destination 1	2	3	4	Supply
Source 1	8000	14000	12000	0	1
Source 2	8000	14000	12000	0	1
Source 3	6000	16000	15000	0	1
Source 4	6000	16000	15000	0	1
Demand	1	1	1	1	

(d)

	Destination 1	2	3	4	Supply
Source 1		1			1
Source 2			1		1
Source 3	1				1
Source 4				1	1
Demand	1	1	1	1	Cost is 32000

(e)

Cost Per Unit Distributed

	Destination 1	2	3	4	Supply
Source 1	8000	14000	12000	0	2
Source 2	6000	16000	15000	0	2
Demand	1	1	1	1	

(f)

		Destination				
		1	2	3	4	Supply
	1		1	1		2
Source	2	1			1	2
Demand		1	1	1	1	Cost is 32000

8.3-8.

(a)

		Task		
		1	2	3
	1			X
Assignee	2	X		
	3		X	

Cost 10

(b)

Cost Per Unit Distributed

		Destination			
		1	2	3	Supply
	1	5	7	4	1
Source	2	3	6	5	1
	3	2	3	4	1
Demand		1	1	1	

(c)

		Destination			
		1	2	3	Supply
	1			1	1
Source	2	1			1
	3		1		1
Demand		1	1	1	Cost is 10

(d) A transportation problem of size $m \times n$ has $m+n-1$ basic variables. Since $m = n$ for the assignment problem, there are $2(3) - 1 = 5$ basic variables, but only 3 assignments. Thus, 2 basic variables are degenerate, they equal zero. Assignment problems are always highly degenerate. This can be seen using the interactive routine in the OR Courseware.

(e) x_{A1}, x_{A2}, x_{B2} and one of (x_{B3}, x_{C3}) are nonbasic, too. x_{C1} and one of (x_{B3}, x_{C3}) are basic and equal zero.

Dual variables:

				u_i
	5 / +3	7 / +4	4 / 1	0
	3 / 1	6 / +2	5 / 0	1
	2 / 0	3 / 1	4 / 0	0
v_j	2	3	4	

Looking at $c_{ij} - u_i - v_j$, we see that the allowable ranges for this solution to stay optimal are: $c_{A1} \geq 2, c_{A2} \geq 3, c_{B1} \geq 4, c_{B2} \geq 5$.

8.3-9.

$$\text{minimize} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0 \qquad \text{for } i, j = 1, 2, \ldots, n$$

The table of constraint coefficients is identical to that for the transportation problem (Table 8.6). The assignment problem has a more special structure because $m = n$ and $s_i = d_i = 1$ for every i.

8.4-1.

Start with:

5	4	6	7
6	6	7	5
7	5	7	6
5	4	6	6

Subtract the minimum element from each element in the column and continue the algorithm.

0	0	0	2		0	0	$\boxed{0}$	3
1	2	1	0		0	1	0	$\boxed{0}$
2	1	1	1		1	$\boxed{0}$	0	1
0	0	0	1		$\boxed{0}$	0	0	2

One optimal solution is to assign ships $(1, 2, 3, 4)$ to ports $(3, 4, 2, 1)$, with cost 21.

8.4-2.

Subtract the minimum element in each row from each element in the row and continue the algorithm.

4.8	0	0.9	4.1	25		3.9	0	0	3.2	1.6
10.3	0	9.1	1.6	8.7		9.4	0	8.2	0.7	7.8
4.8	0	10.4	1.9	5.1		3.9	0	9.5	1	4.2
2.8	0	3.2	2.1	4.7		1.9	0	2.3	1.2	3.6
0	0	0	0	0		0	0.9	0	0	0

3.9	0.7	0	3.2	1.6		2.7	0.7	$\boxed{0}$	3.2	0.4
8.7	0	7.5	0	7.1		7.5	0	7.5	$\boxed{0}$	5.9
3.2	0	8.8	0.3	3.5		2.0	$\boxed{0}$	8.8	0.3	2.3
1.2	0	1.6	0.5	2.9		$\boxed{0}$	0	1.6	0.5	1.7
0	1.6	0	0	0		0	2.8	1.2	1.2	$\boxed{0}$

One optimal solution is that David swims the backstroke, Tony the breaststroke, Chris the butterfly and Carl the freestyle. The total time is 126.2.

Note: This application of the Hungarian algorithm uses the table in Problem 8.3-4 just as shown, where the strokes (including a dummy stroke) are the rows (assignees) and the swimmers are the columns (tasks). It would be more natural to first take the extra step of rewriting the table in the form shown in the back of the book for the solution for Problem 8.3-4, where the swimmers are the rows (assignees) and the strokes (including a dummy stroke) are the columns (tasks). However, the Hungarian algorithm leads to an optimal solution with either formulation.

8.4-3.

Cost: $3,260$

0	0	30	120	0
100	0	30	120	0
80	60	M	80	0
80	60	M	80	0
20	90	0	0	M

0	0	30	120	60
100	0	30	120	60
20	0	M	20	0
20	0	M	20	0
20	90	0	0	M

$\boxed{0}$	0	10	100	60
100	$\boxed{0}$	10	100	60
20	0	M	$\boxed{0}$	0
20	0	M	0	$\boxed{0}$
40	110	$\boxed{0}$	0	M

8.4-4.

Subtract the minimum element in each row from every element in the row and continue the algorithm. This gives an optimal solution with cost 12.

M	1	0
3	2	0
0	0	0

M	$\boxed{0}$	0
2	1	$\boxed{0}$
$\boxed{0}$	0	1

8.4-5.

Subtract the minimum element in each column from every element in the column and continue the algorithm.

3	0	0	1
0	2	4	0
2	1	1	3
1	1	3	0

3	0	$\boxed{0}$	2
$\boxed{0}$	2	4	1
1	$\boxed{0}$	0	3
0	0	2	$\boxed{0}$

An optimal assignment is $(A, 3)$, $(B, 1)$, $(C, 2)$, $(D, 4)$, with cost 3.

8.4-6.

Assignee	Task A	B	C	D	Row Min
1	5	8	6	7	5
2	9	5	7	8	5
3	5	9	8	4	4
4	6	3	5	9	3

Subtract the smallest number in each row from every number in the row.

Assignee	Task A	B	C	D
1	0	3	1	2
2	4	0	2	3
3	1	5	4	0
4	3	0	2	6
Col Min	0	0	1	0

Subtract the smallest number in each column of the new table from every number in the column

Assignee	Task A	B	C	D
1	0	3	0	2
2	4	0	1	3
3	1	5	3	0
4	3	0	1	6

Determine the minimum number of lines needed to cross out all zeros.

```
                         Task
 Assignee|  A       B       C       D       |
_________|__________________________________|_____
         |          |                       |
   1     |-0-------3-------0-------2-------|
         |          |                       |
   2     |  4       0       1       3       |
         |          |                       |
   3     |-1-------5-------3-------0-------|
         |          |                       |
   4     |  3       0       1       6       |
         |          |                       |
```

Select the smallest number from all the uncovered numbers.

Assignee	Task A	B	C	D
1	0	3	0	2
2	4	0	[1]	3
3	1	5	3	0
4	3	0	1	6

Subtract this number from every uncovered number and add it to every number at the intersection of covering lines.

Assignee	Task A	B	C	D
1	0	4	0	2
2	3	0	0	2
3	1	6	3	0
4	2	0	0	5

Determine the minimum number of lines needed to cross out all zeros.

```
                          Task
Assignee|  A       B       C       D       |
________|__________________________________|____
        |          |       |               |
   1    |-0--------4-------0-------2-------|
        |          |       |               |
   2    | 3        0       0       2       |
        |          |       |               |
   3    |-1--------6-------3-------0-------|
        |          |       |               |
   4    | 2        0       0       5       |
        |          |       |               |
```

Assignee	Task A	B	C	D
1	[0]	4	0	2
2	3	0	0	2
3	1	6	3	0
4	2	0	[0]	5

Task A is assigned to Assignee 1
Task B is assigned to Assignee 2
Task D is assigned to Assignee 3
Task C is assigned to Assignee 4

CASES

CASE 8.1 Shipping Wood to Market

Option 1:

		Unit Cost (1,000's) Destination (Market) 1	2	3	4	5	Supply
Source 1	1	61	72	45	55	66	15
	2	69	78	60	49	56	20
	3	59	66	63	61	47	15
Demand		11	12	9	10	8	

		Unit Cost (1,000's) Destination (Market) 1	2	3	4	5	Totals		Supply
Source 1	1	6	0	9	0	0	15	=	15
	2	2	0	0	10	8	20	=	20
	3	3	12	0	0	0	15	=	15
Totals		11	12	9	10	8			
		=	=	=	=	=	Total Cost	=	2,816.00
Demand		11	12	9	10	8			

Option 2:

		Unit Cost (1,000's) Destination (Market) 1	2	3	4	5	Supply
Source 1	1	58.5	68.3	47.8	55	63.5	15
	2	65.3	74.8	55	49	57.5	20
	3	59	61.3	63.5	58.8	50	15
Demand		11	12	9	10	8	

		Unit Cost (1,000's) Destination (Market) 1	2	3	4	5	Totals		Supply
Source 1	1	6	0	9	0	0	15	=	15
	2	5	0	0	10	5	20	=	20
	3	0	12	0	0	3	15	=	15
Totals		11	12	9	10	8			
		=	=	=	=	=	Total Cost	=	2,770.80
Demand		11	12	9	10	8			

Option 3:

		Unit Cost (1,000's) Destination (Market)					
		1	2	3	4	5	Supply
	1	58.5	68.3	45	55	63.5	15
Source 1	2	65.3	74.8	55	49	56	20
	3	59	61.3	63	58.8	47	15
Demand		11	12	9	10	8	

		Unit Cost (1,000's) Destination (Market)							
		1	2	3	4	5	Totals		Supply
	1	6	0	9	0	0	15	=	15
Source 1	2	5	0	0	10	5	20	=	20
	3	0	12	0	0	3	15	=	15
Totals		11	12	9	10	8			
		=	=	=	=	=	Total Cost	=	2,729.10
Demand		11	12	9	10	8			

The combination plan, i.e., shipping by either rail or water offers the best cost whereas shipping by rail is the most expensive. If the costs of shipping by water are expected to rise considerably more than those of shipping by rail, it is best to use option 1 and ship by rail. If the reverse is true, then it is better to use option 2. If the cost comparisons will remain roughly the same, then using option 3 is best. This option is clearly the most feasible, but it may not be chosen if it is logistically too cumbersome. Further information is needed to determine this.

Case 8.2

a) $20 million is saved in comparison with the results in Figure 6.13 by shipping 20 million fewer barrels to Charleston and 20 million more to St. Louis.

	B	C	D	E	F	G	H	I	J
3			Refineries						
4	**Unit Cost ($millions)**		New Orleans	Charleston	Seattle	St. Louis			
5		Texas	2	4	5	1			
6	Oil	California	5	5	3	4			
7	Fields	Alaska	5	7	3	7			
8		Middle East	2	3	5	4			
9									
10									
11	**Shipment Quantity**		Refineries						
12	**(millions of barrels)**		New Orleans	Charleston	Seattle	St. Louis	Total Shipped		Supply
13		Texas	0	0	0	80	80	=	80
14	Oil	California	0	0	0	60	60	=	60
15	Fields	Alaska	20	0	80	0	100	=	100
16		Middle East	80	40	0	0	120	=	120
17		Total Received	100	40	80	140			
18			≤	≤	≤	≤			Total Cost
19		Capacity	100	60	80	150			($millions)
20									940

b) $40 million is saved in comparison with the results in Figure 6.17.

	B	C	D	E	F	G	H	I	J
3			Distribution Center						
4	**Unit Cost ($millions)**		Pittsburgh	Atlanta	Kansas City	San Francisco			
5		New Orleans	6.5	5.5	6	8			
6	Refineries	Charleston	7	5	4	7			
7		Seattle	7	8	4	3			
8		St. Louis	4	3	1	5			
9									
10									
11	**Shipment Quantity**		Distribution Center						
12	**(millions of barrels)**		Pittsburgh	Atlanta	Kansas City	San Francisco	Total Shipped		Supply
13		New Orleans	100	0	0	0	100	=	100
14	Refineries	Charleston	0	20	0	20	40	=	40
15		Seattle	0	0	0	80	80	=	80
16		St. Louis	0	60	80	0	140	=	140
17		Total Received	100	80	80	100			
18			=	=	=	=			Total Cost
19		Demand	100	80	80	100			($millions)
20									1,390

The cost of shipping both crude oil and finished product under this plan is $940 million + $1,390 million = $2,330 million or $2.33 billion — a savings of $60 million compared to the original results in Table 6.20.

c) $35 million is saved in comparison with the results in part (b).
$75 million is saved in comparison with the results in Figure 6.17.

	B	C	D	E	F	G	H	I	J
3			Distribution Center						
4	**Unit Cost ($millions)**		Pittsburgh	Atlanta	Kansas City	San Francisco			
5		New Orleans	6.5	5.5	6	8			
6	Refineries	Charleston	7	5	4	7			
7		Seattle	7	8	4	3			
8		St. Louis	4	3	1	5			
9									
10									
11	**Shipment Quantity**		Distribution Center						
12	**(millions of barrels)**		Pittsburgh	Atlanta	Kansas City	San Francisco	Total Shipped		Capacity
13		New Orleans	50	20	0	0	70	≤	100
14	Refineries	Charleston	0	60	0	0	60	≤	60
15		Seattle	0	0	0	80	80	≤	80
16		St. Louis	50	0	80	20	150	≤	150
17		Total Received	100	80	80	100			
18			=	=	=	=			Total Cost
19		Demand	100	80	80	100			($millions)
20									1,355

d) This solution costs $40 million more than the solution in part (a).
This solution costs $20 million more than the solution is Figure 6.13.

	B	C	D	E	F	G	H	I	J
3			Refineries						
4	**Unit Cost ($millions)**		New Orleans	Charleston	Seattle	St. Louis			
5		Texas	2	4	5	1			
6	Oil	California	5	5	3	4			
7	Fields	Alaska	5	7	3	7			
8		Middle East	2	3	5	4			
9									
10									
11	**Shipment Quantity**		Refineries						
12	**(millions of barrels)**		New Orleans	Charleston	Seattle	St. Louis	Total Shipped		Supply
13		Texas	0	0	0	80	80	=	80
14	Oil	California	0	0	0	60	60	=	60
15	Fields	Alaska	10	0	80	10	100	=	100
16		Middle East	60	60	0	0	120	=	120
17		Total Received	70	60	80	150			
18			=	=	=	=			Total Cost
19		Demand	70	60	80	150			($millions)
20									980

The total cost of shipping both crude oil and finished product under this plan is $1,355 million + $980 million = $2,335 million or $2.335 billion. This is $5 million more than the cost of the combined total obtained in part (b), but $55 million less than the total in Table 6.20.

e) The two transportation problems (shipping to refineries and shipping to distributions centers) are combined into a single model. The amount shipped to the refineries is constrained to be no more than capacity: TotalReceived(D16:G16) = Capacity(D18:G18). The total shipped out of the refineries is constrained to equal the total amount shipped in: ShippedOut(H31:H34) = ShippedIn(J31:J34). The goal is to minimize the total combined cost (in J45) which is the sum of the two intermediate costs (in J20 and J39).

	A	B	C	D	E	F	G	H	I	J
1	**Shipping to Refineries**									
2					Refineries					
3		**Unit Cost ($millions)**		New Orleans	Charleston	Seattle	St. Louis			
4			Texas	2	4	5	1			
5		Oil	California	5	5	3	4			
6		Fields	Alaska	5	7	3	7			
7			Middle East	2	3	5	4			
8										
9										
10		**Shipment Quantity**			Refineries					
11		**(millions of barrels)**		New Orleans	Charleston	Seattle	St. Louis	Total Shipped		Supply
12			Texas	0	0	0	80	80	=	80
13		Oil	California	0	0	0	60	60	=	60
14		Fields	Alaska	20	0	80	0	100	=	100
15			Middle East	80	30	0	10	120	=	120
16			Total Received	100	30	80	150			
17				≤	≤	≤	≤			Cost
18			Capacity	100	60	80	150			(Oil Fields --> Refineries)
19										($millions)
20	**Shipping to Distribution Centers**									950
21					Distribution Center					
22		**Unit Cost ($millions)**		Pittsburgh	Atlanta	Kansas City	San Francisco			
23			New Orleans	6.5	5.5	6	8			
24		Refineries	Charleston	7	5	4	7			
25			Seattle	7	8	4	3			
26			St. Louis	4	3	1	5			
27										
28										
29		**Shipment Quantity**			Distribution Center					
30		**(millions of barrels)**		Pittsburgh	Atlanta	Kansas City	San Francisco	Shipped Out		Shipped In
31			New Orleans	100	0	0	0	100	=	100
32		Refineries	Charleston	0	10	0	20	30	=	30
33			Seattle	0	0	0	80	80	=	80
34			St. Louis	0	70	80	0	150	=	150
35			Total Received	100	80	80	100			
36				=	=	=	=			Cost
37			Demand	100	80	80	100			(Refineries --> D.C.'s)
38										($millions)
39										1,370
40										
41										Combined
42										Total
43										Cost
44										($millions)
45										2,320

The total combined cost is $2,320 million or $2.32 billion, which is $10 million less than in part (b), $15 million less than in part (d), and $70 million less than in Table 6.20.

f) If the Los Angeles refinery is chosen instead, then the combined shipping cost is $2,450 million.

	A	B	C	D	E	F	G	H	I	J
1	**Shipping to Refineries**									
2						Refineries				
3		**Unit Cost ($millions)**		New Orleans	Charleston	Seattle	Los Angeles			
4			Texas	2	4	5	3			
5		Oil	California	5	5	3	1			
6		Fields	Alaska	5	7	3	4			
7			Middle East	2	3	5	4			
8										
9										
10		**Shipment Quantity**				Refineries				
11		**(millions of barrels)**		New Orleans	Charleston	Seattle	St. Louis	Total Shipped		Supply
12			Texas	40	0	0	40	80	=	80
13		Oil	California	0	0	0	60	60	=	60
14		Fields	Alaska	0	0	80	20	100	=	100
15			Middle East	60	60	0	0	120	=	120
16			Total Received	100	60	80	120			
17				≥	≥	≥	≥			Cost
18			Capacity	100	60	80	150			(Oil Fields --> Refineries)
19										($millions)
20	**Shipping to Distribution Centers**									880
21						Distribution Center				
22		**Unit Cost ($millions)**		Pittsburgh	Atlanta	Kansas City	San Francisco			
23			New Orleans	6.5	5.5	6	8			
24		Refineries	Charleston	7	5	4	7			
25			Seattle	7	8	4	3			
26			Los Angeles	8	6	3	2			
27										
28										
29		**Shipment Quantity**				Distribution Center				
30		**(millions of barrels)**		Pittsburgh	Atlanta	Kansas City	San Francisco	Shipped Out		Shipped In
31			New Orleans	80	20	0	0	100	=	100
32		Refineries	Charleston	0	60	0	0	60	=	60
33			Seattle	20	0	60	0	80	=	80
34			St. Louis	0	0	20	100	120	=	120
35			Total Received	100	80	80	100			
36				=	=	=	=			Cost
37			Demand	100	80	80	100			(Refineries --> D.C.'s)
38										($millions)
39										1,570
40										
41										Combined
42										Total
43										Cost
44										($millions)
45										2,450

If the Galveston refinery is chosen instead, then the combined shipping cost is $2,470 million.

	A	B	C	D	E	F	G	H	I	J
1	**Shipping to Refineries**									
2					Refineries					
3		**Unit Cost ($millions)**		New Orleans	Charleston	Seattle	Galveston			
4			Texas	2	4	5	1			
5		Oil	California	5	5	3	3			
6		Fields	Alaska	5	7	3	5			
7			Middle East	2	3	5	3			
8										
9										
10		**Shipment Quantity**			Refineries					
11		**(millions of barrels)**		New Orleans	Charleston	Seattle	Galveston	Total Shipped		Supply
12			Texas	0	0	0	80	80	=	80
13		Oil	California	0	0	0	60	60	=	60
14		Fields	Alaska	10	0	80	10	100	=	100
15			Middle East	90	30	0	0	120	=	120
16			Total Received	100	30	80	150			
17				≤	≤	≤	≤			Cost
18			Capacity	100	60	80	150			(Oil Fields --> Refineries)
19										($millions)
20	**Shipping to Distribution Centers**									870
21					Distribution Center					
22		**Unit Cost ($millions)**		Pittsburgh	Atlanta	Kansas City	San Francisco			
23			New Orleans	6.5	5.5	6	8			
24		Refineries	Charleston	7	5	4	7			
25			Seattle	7	8	4	3			
26			Galveston	5	4	3	6			
27										
28										
29		**Shipment Quantity**			Distribution Center					
30		**(millions of barrels)**		Pittsburgh	Atlanta	Kansas City	San Francisco	Shipped Out		Shipped In
31			New Orleans	100	0	0	0	100	=	100
32		Refineries	Charleston	0	0	30	0	30	=	30
33			Seattle	0	0	0	80	80	=	80
34			Galveston	0	80	50	20	150	=	150
35			Total Received	100	80	80	100			
36				=	=	=	=			Cost
37			Demand	100	80	80	100			(Refineries --> D.C.'s)
38										($millions)
39										1,600
40										
41										Combined
42										Total
43										Cost
44										($millions)
45										2,470

Site	Total Cost of Shipping Crude Oil	Total Cost of Shipping Finished Product	Operating Cost for New Refinery	Total Variable Cost
Los Angeles	$880 million	$1.57 billion	$620 million	$3.07 billion
Galveston	870 million	1.60 billion	570 million	3.12 billion
St. Louis	950 million	1.37 billion	530 million	2.92 billion

g) Answers will vary.

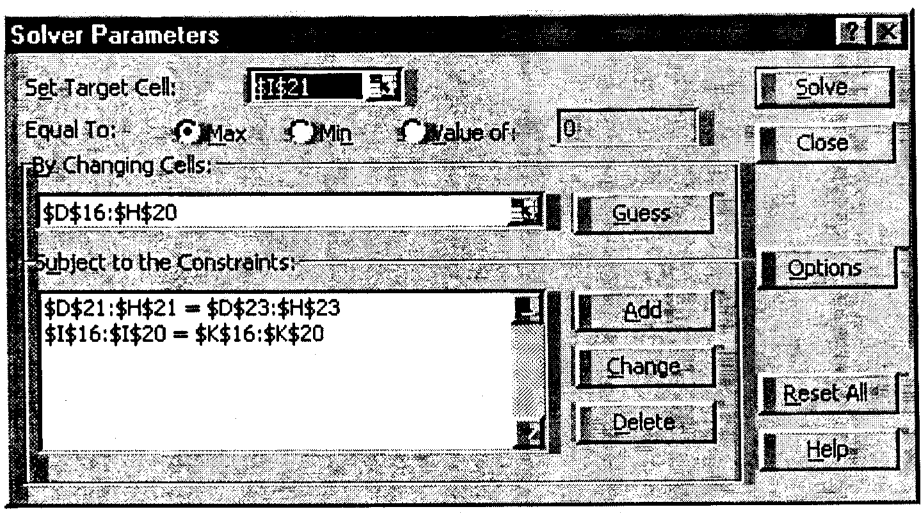
The problem in this case can be solved using assignment problem.

8.3 Throughout this case. we use the template for the assignment problem.

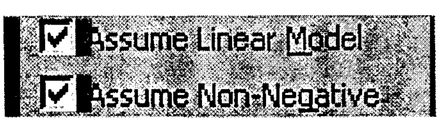
a) The projects are the tasks, and the scientists are the assignees in this assignment problem.

	A	B	C	D	E	F	G	H	I	J	K
1											
2						Points					
3						Task					
4				Up	Stable	Choice	Hope	Release	Supply		
5			Kvaal	100	400	200	200	100	1		
6		Assignee	Zuner	0	200	800	0	0	1		
7			Tsai	100	100	100	100	600	1		
8			Mickey	267	153	99	451	30	1		
9			Rollins	100	33	33	34	800	1		
10		Demand		1	1	1	1	1			
11											
12											
13						Assignments					
14						Task					
15				Up	Stable	Choice	Hope	Release	Totals		Supply
16			Kvaal	0	1	0	0	0	1	=	1
17		Assignee	Zuner	0	0	1	0	0	1	=	1
18			Tsai	1	0	0	0	0	1	=	1
19			Mickey	0	0	0	1	0	1	=	1
20			Rollins	0	0	0	0	1	1	=	1
21		Totals		1	1	1	1	1	2551	=	Total Points
22				=	=	=	=	=			
23		Demand		1	1	1	1	1			

The solver dialogue box appears as follows:



The solver options throughout this case are:



To maximize the scientists preferences you want to assign Dr. Tsai to lead project Up, Dr. Kvaal to lead project Stable, Dr. Zuner to lead project Choice, Dr. Mickey to lead project Hope, and Dr. Rollins to lead project Release.

b) Since there are only four assignees we introduce a dummy assignee with preferences of −1. The task that gets assigned the dummy assignee will not be done.

	A	B	C	D	E	F	G	H	I	J	K
1											
2						Points					
3						Task					
4				Up	Stable	Choice	Hope	Release	Supply		
5			Kvaal	100	400	200	200	100	1		
6		Assignee	Zuner	0	200	800	0	0	1		
7			Tsai	100	100	100	100	600	1		
8			Mickey	267	153	99	451	30	1		
9			dummy	-1	-1	-1	-1	-1	1		
10		Demand		1	1	1	1	1			
11											
12											
13						Assignments					
14						Task					
15				Up	Stable	Choice	Hope	Release	Totals		Supply
16			Kvaal	0	1	0	0	0	1	=	1
17		Assignee	Zuner	0	0	1	0	0	1	=	1
18			Tsai	0	0	0	0	1	1	=	1
19			Mickey	0	0	0	1	0	1	=	1
20			dummy	1	0	0	0	0	1	=	1
21		Totals		1	1	1	1	1	2250	=	Total Points
22				=	=	=	=	=			
23		Demand		1	1	1	1	1			

The solver dialogue box remains the same.

We give up on project Up.

c) Since two of the assignees can do two tasks we need to double them. We include assignees Zuner-1, Zuner-2, Mickey-1, and Mickey-2 into the problem. In order to have an equal number of assignees and tasks we also need to include one dummy task. In order to ensure that neither Dr. Kvaal nor Dr. Tsai can get assigned the dummy task and thus no project, we insert a large negative number as their point bid for the dummy project.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2				Points								
3				Task								
4				Up	Stable	Choice	Hope	Release	dummy	Supply		
5			Kvaal	100	400	200	200	100	-10000	1		
6			Zuner-1	0	200	800	0	0	-1	1		
7		Assignee	Zuner-2	0	200	800	0	0	-1	1		
8			Tsai	100	100	100	100	600	-10000	1		
9			Mickey-1	267	153	99	451	30	-1	1		
10			Mickey-2	267	153	99	451	30	-1	1		
11		Demand		1	1	1	1	1	1			
12												
13												
14				Assignments								
15				Task								
16				Up	Stable	Choice	Hope	Release	dummy	Totals		Supply
17			Kvaal	0	1	0	0	0	0	1	=	1
18			Zuner-1	0	0	0	0	0	1	1	=	1
19		Assignee	Zuner-2	0	0	1	0	0	0	1	=	1
20			Tsai	0	0	0	0	1	0	1	=	1
21			Mickey-1	0	0	0	1	0	0	1	=	1
22			Mickey-2	1	0	0	0	0	0	1	=	1
23		Totals		1	1	1	1	1	1	2517	=	Total Points
24				=	=	=	=	=	=			
25		Demand		1	1	1	1	1	1			

Dr. Kvaal leads project Stable, Dr. Zuner leads project Choice, Dr. Tsai leads project Release, and Dr. Mickey leads the projects Hope and Up.

d) Under the new bids of Dr. Zuner the assignment does not change:

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2				Points								
3				Task								
4				Up	Stable	Choice	Hope	Release	dummy	Supply		
5			Kvaal	100	400	200	200	100	-10000	1		
6			Zuner-1	20	450	451	39	40	-1	1		
7		Assignee	Zuner-2	20	450	451	39	40	-1	1		
8			Tsai	100	100	100	100	600	-10000	1		
9			Mickey-1	267	153	99	451	30	-1	1		
10			Mickey-2	267	153	99	451	30	-1	1		
11		Demand		1	1	1	1	1	1			
12												
13												
14				Assignments								
15				Task								
16				Up	Stable	Choice	Hope	Release	dummy	Totals		Supply
17			Kvaal	0	1	0	0	0	0	1	=	1
18			Zuner-1	0	0	1	0	0	0	1	=	1
19		Assignee	Zuner-2	0	0	0	0	0	1	1	=	1
20			Tsai	0	0	0	0	1	0	1	=	1
21			Mickey-1	0	0	0	1	0	0	1	=	1
22			Mickey-2	1	0	0	0	0	0	1	=	1
23		Totals		1	1	1	1	1	1	2168	=	Total Points
24				=	=	=	=	=	=			
25		Demand		1	1	1	1	1	1			

e) Certainly Dr. Zuner could be disappointed that she is not assigned to project Stable, especially when she expressed a higher preference for that project than the scientist assigned. The optimal solution maximizes the preferences overall, but individual scientists may be disappointed. We should therefore make sure to communicate the reasoning behind the assignments to the scientists.

f) Whenever a scientist cannot lead a particular project we use a large negative number as the point bid.

	A	B	C	D	E	F	G	H	I	J	K
1											
2						Points					
3						Task					
4				Up	Stable	Choice	Hope	Release	Supply		
5			Kvaal	86	343	171	-10000	-10000	1		
6		Assignee	Zuner	0	200	800	0	0	1		
7			Tsai	100	100	100	100	600	1		
8			Mickey	300	-10000	125	-10000	175	1		
9			Rollins	-10000	50	50	100	600	1		
10		Demand		1	1	1	1	1			
11											
12											
13						Assignments					
14						Task					
15				Up	Stable	Choice	Hope	Release	Totals		Supply
16			Kvaal	0	1	0	0	0	1	=	1
17		Assignee	Zuner	0	0	1	0	0	1	=	1
18			Tsai	0	0	0	1	0	1	=	1
19			Mickey	1	0	0	0	0	1	=	1
20			Rollins	0	0	0	0	1	1	=	1
21		Totals		1	1	1	1	1	2143	=	Total Points
22				=	=	=	=	=			
23		Demand		1	1	1	1	1			

Dr. Kvaal leads project Stable, Dr. Zuner leads project Choice, Dr. Tsai leads project Hope, Dr. Mickey leads project Up, and Dr. Rollins leads project Release.

g) When we want to assign two assignees to the same task we need to duplicate that task.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2							Points						
3							Task						
4				Up	Stable	Choice	Hope-A	Hope-B	Release-A	Release-B	Supply		
5			Kvaal	86	343	171	-10000	-10000	-10000	-10000	1		
6		Assignee	Zuner	0	200	800	0	0	0	0	1		
7			Tsai	100	100	100	100	100	600	600	1		
8			Mickey	300	-10000	125	-10000	-10000	175	175	1		
9			Rollins	-10000	50	50	100	100	600	600	1		
10			Arriaga	250	250	-10000	250	250	250	250	1		
11			Santos	111	1	-10000	333	333	555	555	1		
12		Demand		1	1	1	1	1	1	1			
13													
14													
15							Assignments						
16							Task						
17				Up	Stable	Choice	Hope-A	Hope-B	Release-A	Release-B	Totals		Supply
18			Kvaal	0	1	0	0	0	0	0	1	=	1
19		Assignee	Zuner	0	0	1	0	0	0	0	1	=	1
20			Tsai	0	0	0	0	0	1	0	1	=	1
21			Mickey	1	0	0	0	0	0	0	1	=	1
22			Rollins	0	0	0	0	0	0	1	1	=	1
23			Arriaga	0	0	0	0	1	0	0	1	=	1
24			Santos	0	0	0	1	0	0	0	1	=	1
25		Totals		1	1	1	1	1	1	1	3226	=	Total Points
26				=	=	=	=	=	=	=			
27		Demand		1	1	1	1	1	1	1			

Project Up is led by Dr. Mickey, Stable by Dr. Kvaal, Choice by Dr. Zuner, Hope by Dr. Arriaga and Dr. Santos, and Release by Dr. Tsai and Dr. Rollins.

h) No. Maximizing overall preferences does not maximize individual preferences. Scientists who do not get their first choice may become resentful and therefore lack the motivation to lead their assigned project. For example, in the optimal solution of part (g), Dr. Santos clearly elected project Release as his first choice, but he was assigned to lead project Hope.

In addition, maximizing preferences ignores other considerations that should be factored into the assignment decision. For example, the scientist with the highest preference for a project may not be the scientist most qualified to lead the project.

CHAPTER 9: NETWORK OPTIMIZATION MODELS

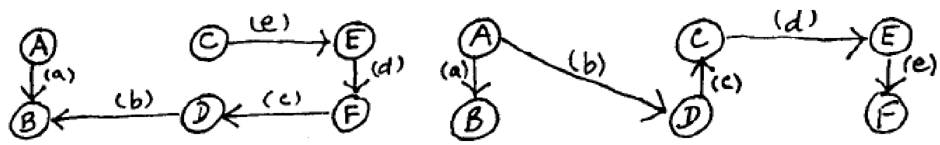
9.2-1.

(a) Directed path: AD-DC-CE-EF ($A \rightarrow D \rightarrow C \rightarrow E \rightarrow F$)
Undirected paths: AD-FD ($A \rightarrow D \rightarrow F$)
CA-CE-EF ($A \rightarrow C \rightarrow E \rightarrow F$)
AD-ED-EF ($A \rightarrow D \rightarrow E \rightarrow F$)

(b) Directed cycles: AD-DC-CA
DC-CE-ED
DC-CE-EF-FD
Undirected cycle that includes every node: CA-CE-EF-FD-DB-AB

(c) {CA, CE, DC, FD, DB} is a spanning tree.

(d)



9.3-1.

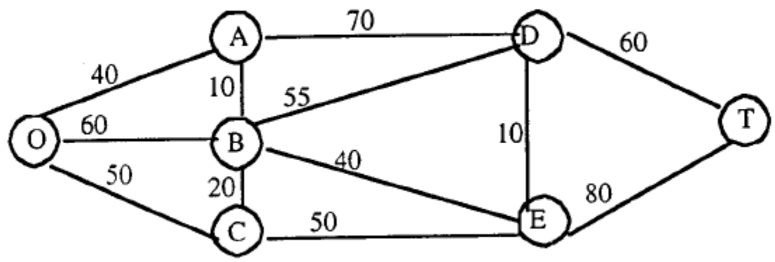
Prior to this study, Canadian Pacific Railway (CPR) used to run trains only after a sufficient level of freight was attained. This policy resulted in unreliable delivery times, so poor customer service. In order to improve customer service and utilization of available resources, CPR designed the railway operating plan called Integrated Operating Plan (IOP). "The problem of designing a railway operating plan is to satisfy a set of customer requirements expressed in terms of origin-destination traffic movements, using a blocking plan and a train plan. Thus, the primary variables are the blocks and trains. The constraints are the capacities of the lines and yards, the customer-service requirements, and the availability of various assets, such as crews and locomotives. The objective function in an abstract sense is to maximize profits" [p. 8].

Developing the blocking plan, i.e., determining the group of railcars to move together at some point during their trips, involves solving a series of shortest-path problems over a directed graph. The train plan is based on the blocking plan. It includes departure and arrival times for the trains, blocks they pick up and crew schedules. This problem is solved for each train using heuristics. Following this, simulation models and locomotive cycle plans are developed.

This study enabled CPR to save $170 million in half a year. "Total documented cost savings through the end of 2002 have exceeded half a billion dollars" [p. 12]. More savings are expected in following years. The improvements in CPR's profitability and operations can be attributed to the decrease in transit and dwelling times, lowered fuel consumption, reduction of the workforce and of the number of railcars, and balanced workloads. CPR can now schedule the trains and the crew more efficiently and provide a more reliable customer service. By allowing variability in the parameters of its plans, CPR gained flexibility and agility. It can now respond to disruptions more effectively by shifting resources quickly. These improvements earned CPR many awards and more importantly a significant competitive advantage.

9.3-2.

(a)



(b)

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	nth Nearest Node	Minimum Distance	Last Connection
1	O	A	40	A	40	OA
2, 3	O A	C B	50 40+10 = 50	C B	50 50	OC AB
4	A B C	D E E	40+70 = 110 50+40 = 90 50+50 = 100	E	90	BE
5	A B E	D D D	40+70 = 110 50+55 = 115 90+10 = 100	D	100	ED
6	D E	T T	100+60 = 160 90+80 = 170	T	160	DT

The shortest path from the origin to the destination is O → A → B → E → D → T, with a total distance of 160 miles.

(c)

From	To	On Route	Distance
Origin	A	1	40
Origin	B	0	60
Origin	C	0	50
A	B	1	10
A	D	0	70
B	C	0	20
B	D	0	55
B	E	1	40
C	E	0	50
D	E	0	10
D	Destination	1	60
E	Destination	0	80
E	D	1	10
	Total Distance =	160	

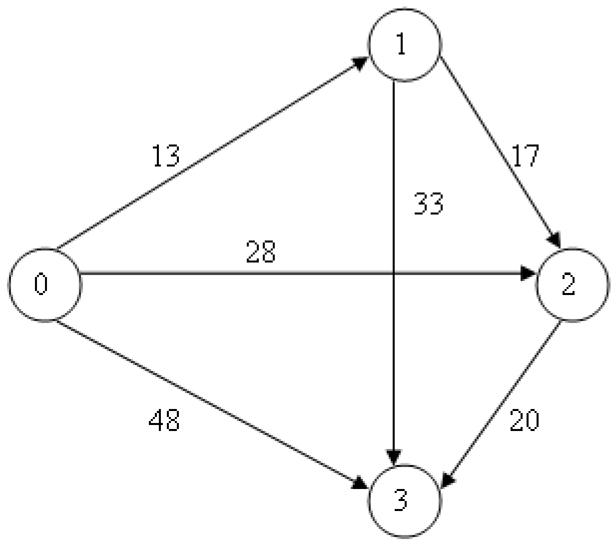
Nodes	Net Flow		Supply/Demand
Origin	1	=	1
A	0	=	0
B	0	=	0
C	0	=	0
D	0	=	0
E	0	=	0
Destination	-1	=	-1

(d) Yes.

(e) Yes.

9.3-3.

(a) The nodes represent the years. Let d_{ij} be the cost (in thousand dollars) of using the same tractor from the end of year i to the end of year j.



(b)

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	nth Nearest Node	Minimum Distance	Last Connection
1	0	1	13	1	13	01
2	0 1	2 2	28 13+17=30	2	28	02
3	0 1 2	3 3 3	48 13+33=46 28+20=48	3	46	13

The minimum-cost strategy is to replace the tractor at the end of the first year and keep the new one until the end of the third year. This incurs a total cost of 46 thousand dollars.

(c)

From	To	On Route	Cost
0	1	1	$13,000
0	2	0	$28,000
0	3	0	$48,000
1	2	0	$17,000
1	3	1	$33,000
2	3	0	$20,000

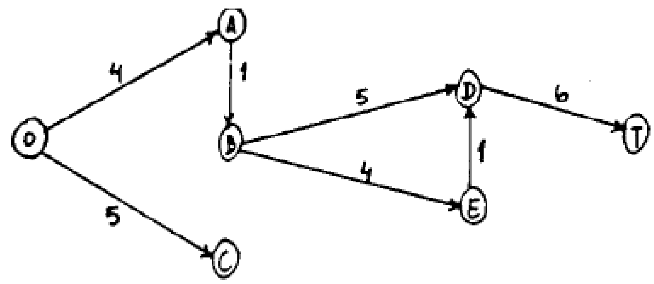
Total Cost= $46,000

Nodes	Net Flow		Supply demand
0	1	=	1
1	0	=	0
2	0	=	0
3	-1	=	-1

9.3-4.

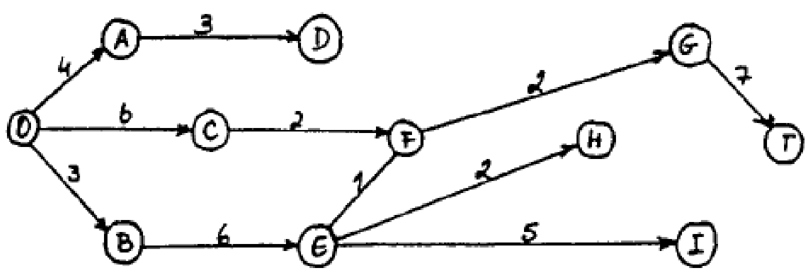
(a) Length of the shortest path: 16

0	4	5	5	10	9	16
O	A	B	C	D	E	T
OA-4 OC-5 ~~OD-6~~	AB-1 ~~AD-7~~	~~BA-1~~ ~~BD-2~~ BE-4 BD-5	~~CB-2~~ ~~CO-5~~	~~DE-1~~ ~~DB-5~~ DT-6 DA-7	ED-1 ~~EB-4~~ ~~EC-5~~ ~~ET-8~~	



(b) Length of the shortest path: 17

0	4	3	6	7	9	8	10	11	14	17
O	A	B	C	D	E	F	G	H	I	T
OB-3 OA-4 OC-6	AD-3 ~~AC-5~~	~~BC-4~~ BE-6	~~CD-2~~ CF-2 ~~CB-4~~ ~~CE-5~~ ~~CA-5~~	~~DC-2~~ ~~DF-2~~ ~~DA-3~~ ~~DG-4~~	~~EF-1~~ EH-2 EI-5 ~~EC-5~~ ~~EB-6~~	FE-1 ~~FC-2~~ ~~FD-2~~ FG-2 ~~FH-5~~	~~GF-2~~ ~~GH-2~~ ~~GD-4~~ GT-7	~~HE-2~~ ~~HG-2~~ ~~HI-3~~ ~~HF-5~~ ~~HT-8~~	~~IH-3~~ ~~IT-4~~ ~~IE-5~~	



9.3-5.

The shortest-path problem is a minimum cost flow problem with a unit supply at the origin and a unit demand at the destination. Label the origin as node 1 and the destination as node n. Then, the LP formulation is as follows:

$$
\begin{aligned}
\text{minimize} \quad & z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{1j} - \sum_{j=1}^{n} x_{j1} = 1 \\
& \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = 0, \text{ for } 2 \leq i \leq n-1 \\
& \sum_{j=1}^{n} x_{nj} - \sum_{j=1}^{n} x_{jn} = -1 \\
& 0 \leq x_{ij} \leq 1, \text{ for } 1 \leq i, j \leq n.
\end{aligned}
$$

9.3-6.

(a) The flying times play the role of "distances."

(b) Shortest path: SE → C → E → LN, with total flight time 11.3

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	nth Nearest Node	Minimum Distance	Last Connection
1	SE	C	4.2	C	4.2	SE-C
2	SE C	A F	4.6 4.2+3.4 = 7.6	A	4.6	SE-A
3	SE C A	B F E	4.7 4.2+3.4 = 7.6 4.6+3.4 = 8	B	4.7	SE-B
4	A B C	E E F	4.6+3.4 = 8 4.7+3.2 = 7.9 4.2+3.4 = 7.6	F	7.6	C-F
5	A B C F	E E E LN	4.6+3.4 = 8 4.7+3.2 = 7.9 4.2+3.5 = 7.7 7.6+3.8 = 11.4	E	7.7	C-E
6	A B F E	D D LN LN	4.6+3.5 = 8.1 4.7+3.6 = 8.3 7.6+3.8 = 11.4 7.7+3.6 = 11.3	D	8.1	A-D
7	D E F	LN LN LN	8.1+3.4 = 11.5 7.7+3.6 = 11.3 7.6+3.8 = 11.4	LN	11.3	E-LN

(c)

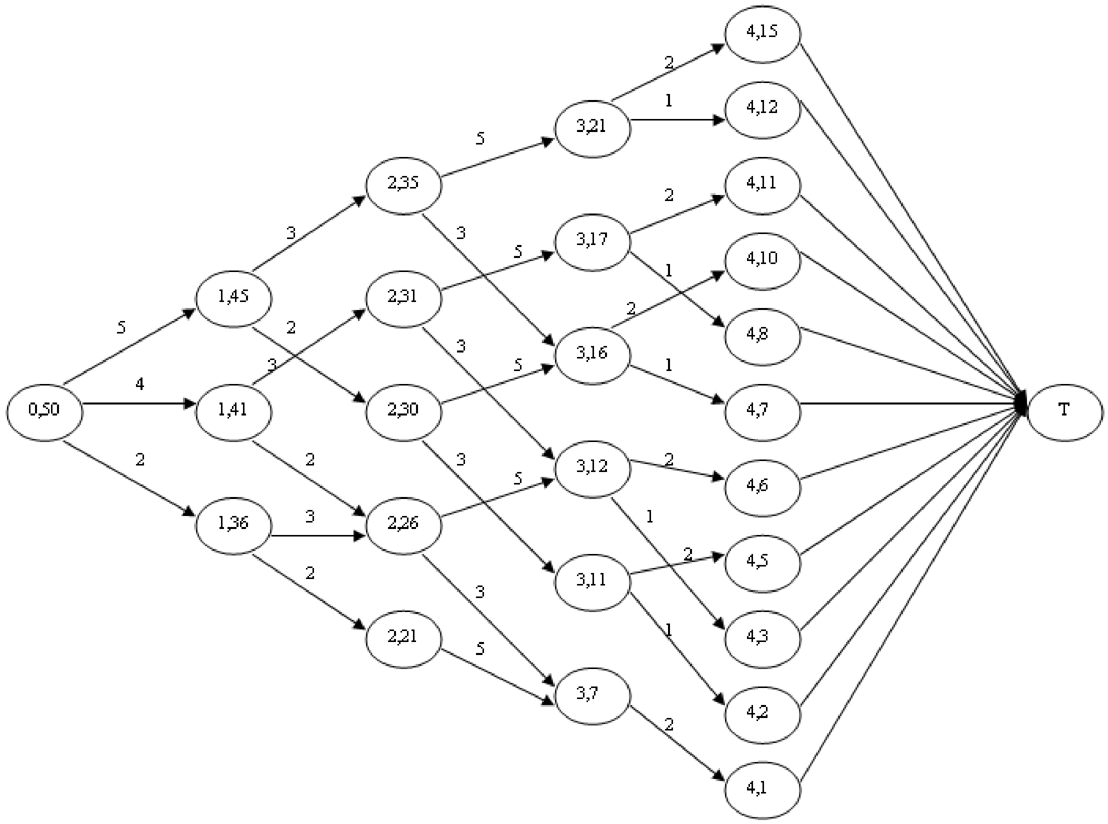
From	To	On Route	Time
SE	A	0	4.6
SE	B	0	4.7
SE	C	1	4.2
A	D	0	3.5
A	E	0	3.4
B	D	0	3.6
B	E	0	3.2
B	F	0	3.3
C	E	1	3.5
C	F	0	3.4
D	LN	0	3.4
E	LN	1	3.6
F	LN	0	3.8

Total Time = 11.3

Nodes	Net Flow		Supply/Demand
SE	1	=	1
A	0	=	0
B	0	=	0
C	0	=	0
D	0	=	0
E	0	=	0
F	0	=	0
LN	-1	=	-1

9.3-7.

(a) Let node (i, j) denote phase i being completed with j million dollars left to spent and $t_{(i,j),(i+1,k)}$ be the time to complete phase $i+1$ if a cost of $(j-k)$ million dollars is spent.



(b)

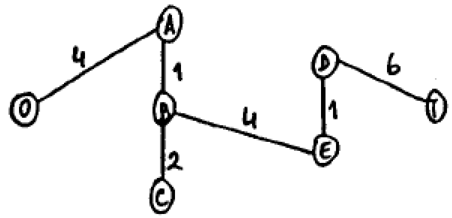
n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	nth Nearest Node	Minimum Distance	Last Connection
1	(0,50)	(1,36)	2	(1,36)	2	(0,50)-(1,36)
2	(0,50) (1,36)	(1,41) (2,21)	4 2+2=4	(1,41) (2,21)	4 4	(0,50)-(1,41) (1,36)-(2,21)
4	(0,50) (1,36) (1,41) (2,21)	(1,45) (2,26) (2,26) (3,7)	5 2+3=5 4+2=6 4+5=9	(1,45) (2,26)	5 5	(0,50)-(1,45) (1,36)-(2,26)
6	(1,41) (1,45) (2,21) (2,26)	(2,31) (2,30) (3,7) (3,7)	4+3=7 5+2=7 4+5=9 5+3=8	(2,31) (2,30)	7 7	(1,41)-(2,31) (1,45)-(2,30)
8	(1,45) (2,21) (2,26) (2,30) (2,31)	(2,35) (3,7) (3,7) (3,11) (3,12)	5+3=8 4+5=9 5+3=8 7+3=10 7+3=10	(2,35) (3,7)	8 8	(1,45)-(2,35) (2,26)-(3,7)
10	(2,26) (2,30) (2,31) (2,35) (3,7)	(3,12) (3,11) (3,12) (3,16) (4,1)	5+5=10 7+3=10 7+3=10 8+3=11 8+2=10	(3,12) (3,11) (3,12) (4,1)	10 10 10 10	(2,26)-3,12) (2,30)-(3,11) (2,31)-(3,12) (3,7)-(4,1)
13	(2,30) (2,31) (2,35) (3,11) (3,12) (4,1)	(3,16) (3,17) (3,16) (4,2) (4,3) T	7+5=12 7+5=12 8+3=11 10+1=11 10+1=11 10+0=10	 T	 10	 (4,1)-T

Shortest path: $(0,50) \xrightarrow{2} (1,36) \xrightarrow{3} (2,26) \xrightarrow{3} (3,7) \xrightarrow{2} (4,1) \xrightarrow{0}$ T, with a total time of 10 months.

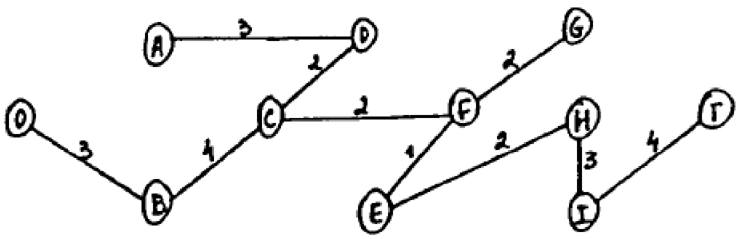
Phase	Level	Cost	Time
Research	Crash	14	2
Development	Priority	10	3
Design	Crash	19	3
Production	Priority	6	2

9.4-1.

(a) Length: 18



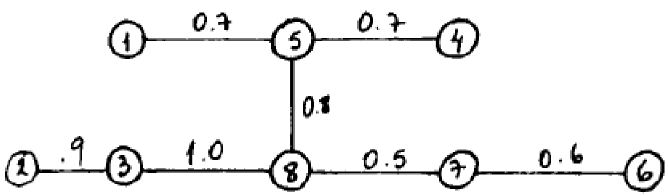
(b) Length: 26



9.4-2.

(a) The nodes represent the groves and the branches represent the roads.

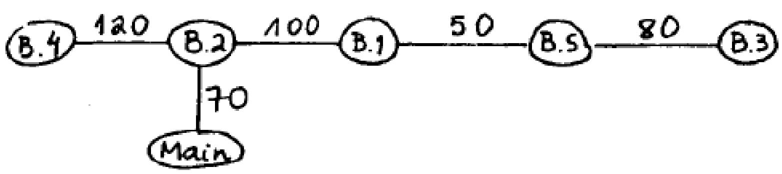
(b) Length: 5.2



9.4-3.

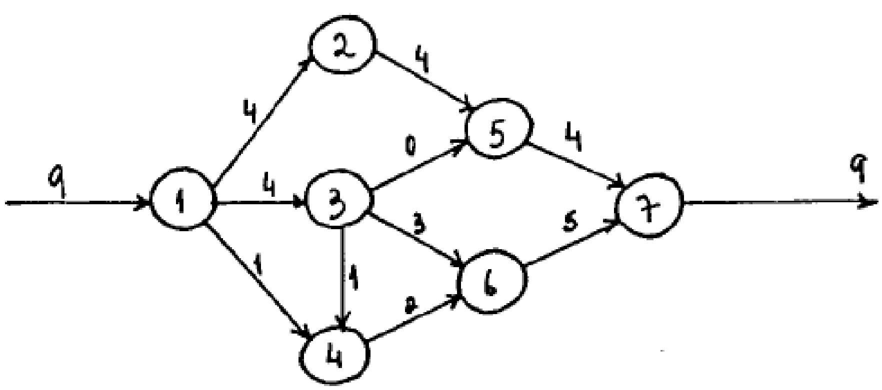
(a) The nodes are Main Office, Branch 1, Branch 2, Branch 3, Branch 4, and Branch 5. The branches are the phones lines.

(b)



9.5-1.

Maximum flow: 9



9.5-2.

Let node 1 be the source and node N be the sink.

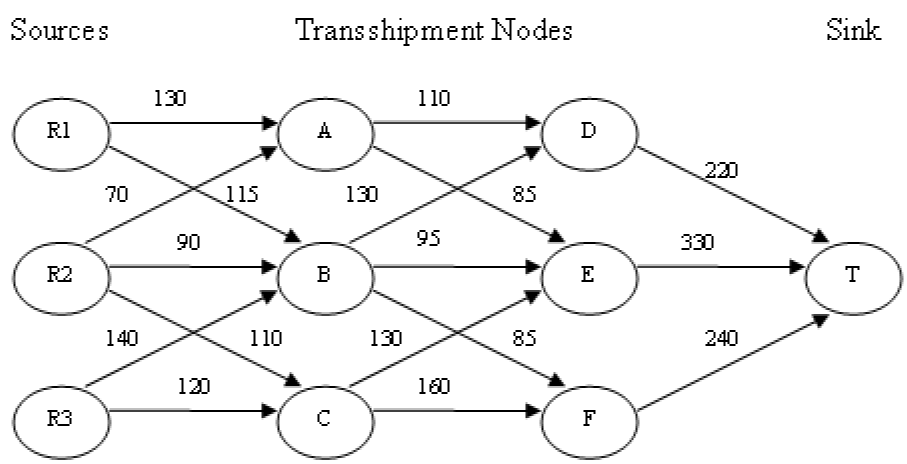
maximize $\quad z = \sum_{j=2}^{N} x_{1j}$

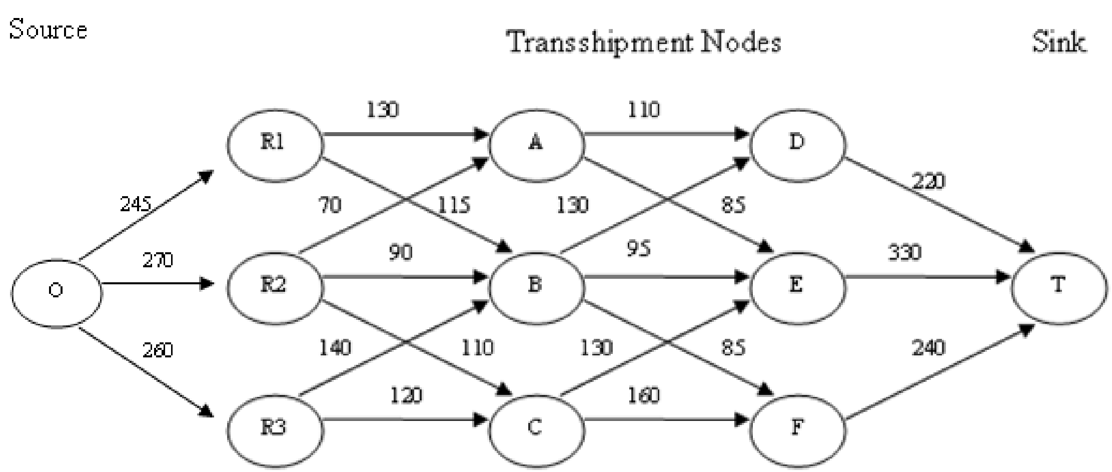
subject to $\quad \sum_{j=1, j\neq i}^{N} x_{ij} - \sum_{j=1, j\neq i}^{N} x_{ji} = 0$, for $i = 2, 3, \ldots, N-1$

$0 \leq x_{ij} \leq c_{ij}$, where $c_{ij} = 0$ if (i, j) is not a branch.

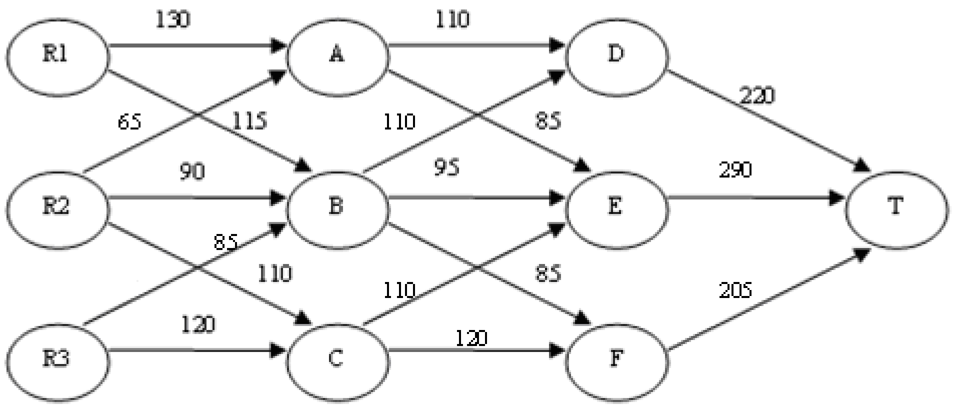
9.5-3.

(a)





(b) Maximum flow: 715



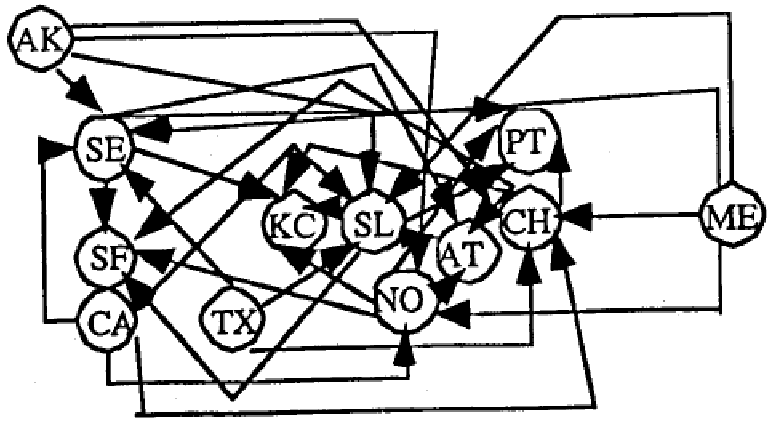
(c) Maximum flow: 715

From	To	Ship	Capacity
R1	A	130	130
R1	B	115	115
R2	A	65	70
R2	B	90	90
R2	C	110	110
R3	B	85	140
R3	C	120	120
A	D	110	110
A	E	85	85
B	D	110	130
B	E	95	95
B	F	85	85
C	E	75	130
C	F	155	160
D	T	220	220
E	T	255	330
F	T	240	240

Nodes	Net Flow		Supply/Demand
R1	245		
R2	265		
R3	205		
A	0	=	0
B	0	=	0
C	0	=	0
D	0	=	0
E	0	=	0
F	0	=	0
T	-715		

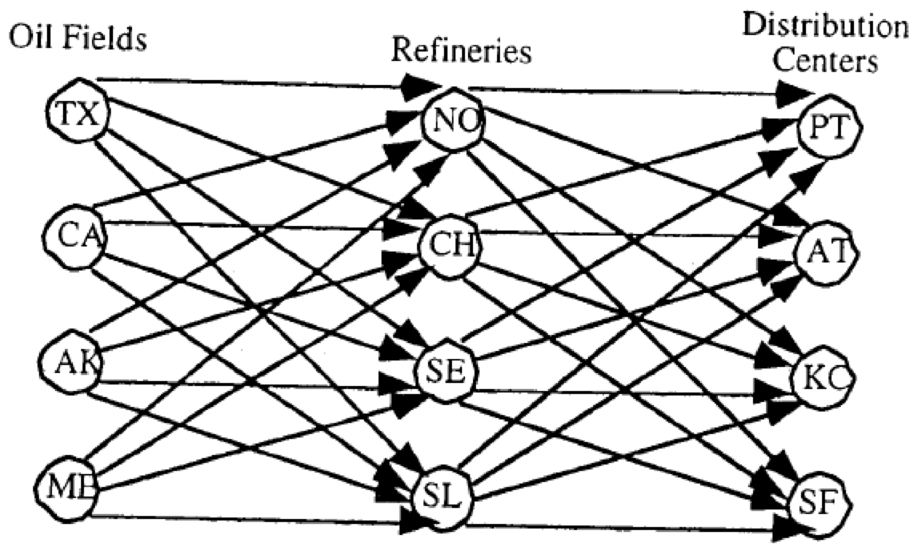
9.5-4.

(a)



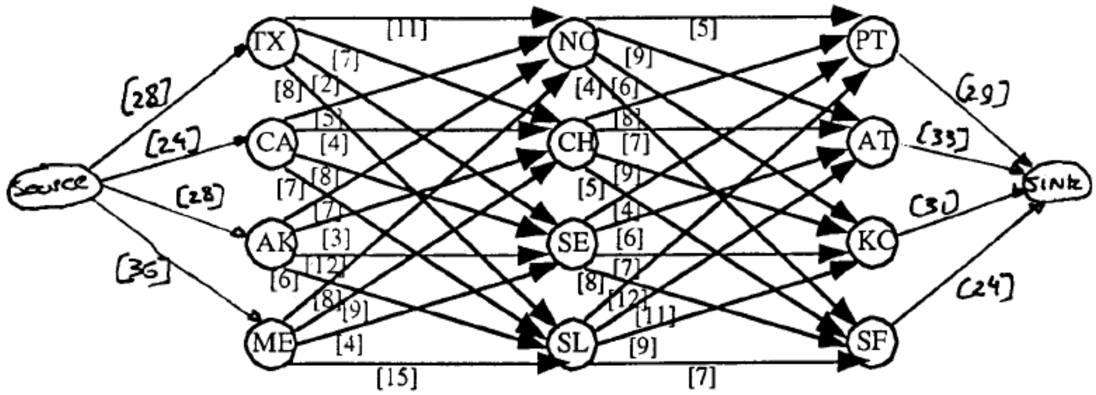
AK
SE
SF
CA
TX
KC
SL
NO
AT
PT
CH
ME

(b)



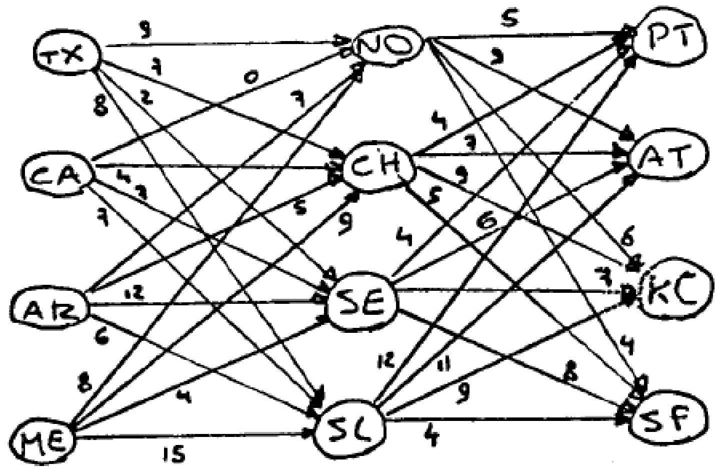
Oil Fields
Refineries
Distribution Centers
TX
CA
AK
ME
NO
CH
SE
SL
PT
AT
KC
SF

(c)



Source
[28]
[24]
[28]
[36]
TX
CA
AK
ME
[11]
[15]
NO
CH
SE
SL
[5]
[7]
PT
AT
KC
SF
[29]
[33]
[30]
[24]
Sink

(d)



(e)

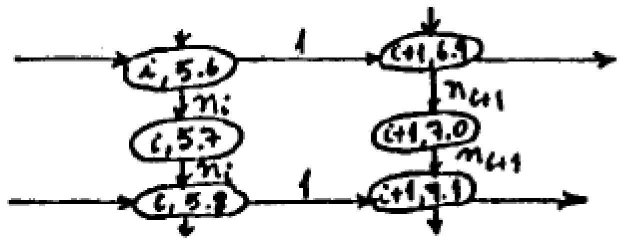
From	To	Ship	Capacity
TX	NO	9	11
TX	CH	7	7
TX	SE	2	2
TX	SL	8	8
CA	NO	0	5
CA	CH	4	4
CA	SE	7	8
CA	SL	7	7
AK	NO	7	7
AK	CH	5	5
AK	SE	12	12
AK	SL	6	6
ME	NO	8	8
ME	CH	9	9
ME	SE	4	4
ME	SL	15	15
NO	PT	5	5
NO	AT	9	9
NO	KC	6	6
NO	SF	4	4
CH	PT	4	8
CH	AT	7	7
CH	KC	9	9
CH	SF	5	5
SE	PT	4	4
SE	AT	6	6
SE	KC	7	7
SE	SF	8	8
SL	PT	12	12
SL	AT	11	11
SL	KC	9	9
SL	SF	4	7

Maximum Flow = 110

Nodes	Net Flow		Supply/Demand
TX	26		
CA	18		
AK	30		
ME	36		
NO	0	=	0
CH	0	=	0
SE	0	=	0
SL	0	=	0
PT	-25		
AT	-33		
KC	-31		
SF	-21		

9.5-5.

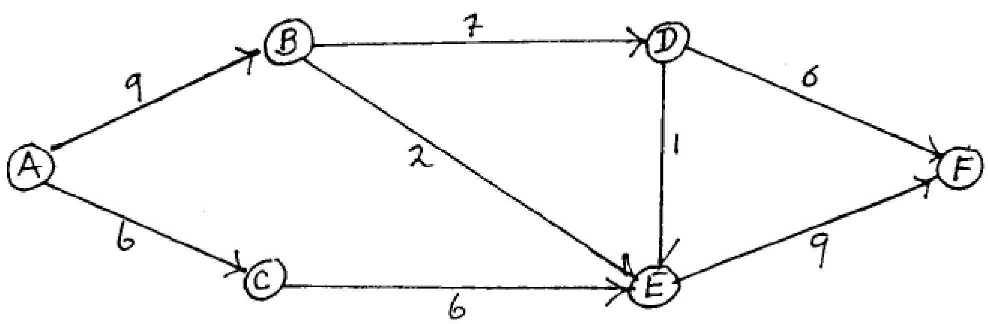
For convenience, call the Faireparc station siding 0 and the Portstown station siding $s{+}1$. Let node (i, j) represent siding i at time j for $i = 0, 1, \ldots, s, s{+}1$ and $j = 0.0, 0.1, 0.2, \ldots, 23.9$. Node $(0, 0)$ is the source and node $(s{+}1, 23.9)$ is the sink. Arcs with unit capacity exist between nodes (i, j) and $(i{+}1, j{+}t_i)$ if and only if a freight train leaving siding i at time j could not be overtaken by a scheduled passenger train before it reached siding $i{+}1$. Arcs with capacity n_i exist between nodes (i, j) and $(i, j{+}1)$ for $j = 0.0, 0.1, 0.2, \ldots, 23.8$. There are no other arcs. For example, if $t_i = 1.3$ and a scheduled passenger train could overtake a freight train leaving siding i at time 5.7 before it reached siding $i{+}1$, the following is part of the network:



The maximum flow problem in this case maximizes the number of sent freight trains.

9.5-6.

(a)



(b)

From	To	Ship	Capacity
A	B	8	9
A	C	7	7
B	D	7	7
B	E	1	2
C	D	2	4
C	E	5	6
D	E	3	3
D	F	6	6
E	F	9	9

Maximum Flow = 15

Nodes	Net Flow		Supply/Demand
A	15		
B	0	=	0
C	0	=	0
D	0	=	0
E	0	=	0
F	-15		

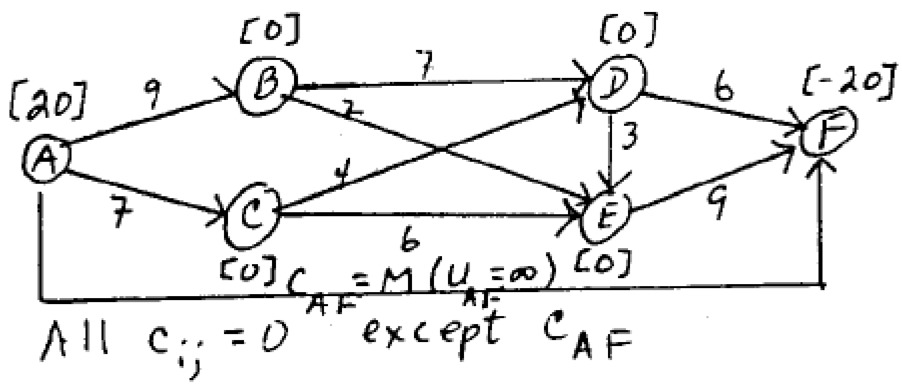
9.6-1.

In this study, flight delay and cancellation problems faced by United Airlines (UA) are modeled as minimum-cost-flow network models. The overall objective is to minimize a weighted sum of various measures related to delay. These include the total number of delay minutes for every passenger, the number of passengers affected by delays and the number of aircraft swaps. Nodes represent "arriving and departing aircraft, spare aircraft, and recovered aircraft" on a two-dimensional network, with time and airport being the two dimensions. Arcs represent "scheduled flights, connections, and aircraft substitutions" [p. 56]. Costs include the revenue loss, the costs from swapping aircraft and from delaying aircraft.

The delay problem is solved for each airport separately as a minimum-cost-flow network problem. The flow on each arc can be at most one. The solution is a set of arcs starting at a supply node and ending at a demand node, which determines flight delays due to shortage in aircraft. The cancellation model is a minimum-cost-flow network problem on the entire network. Again, the flow on each arc cannot exceed one. The solution determines which flight is canceled and what flight its aircraft is assigned to.

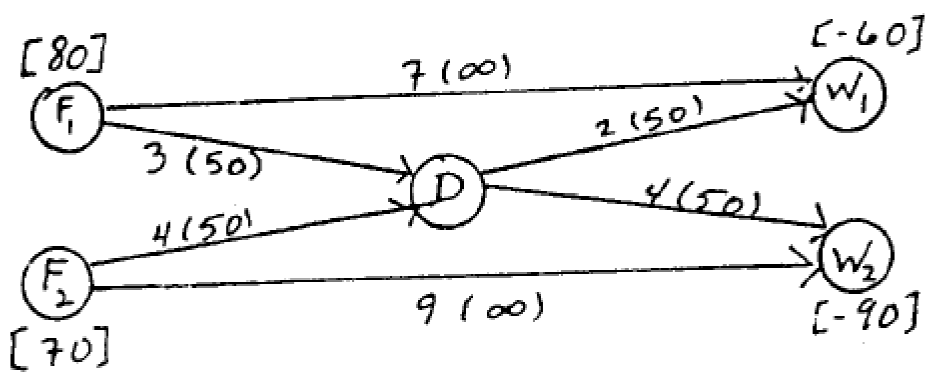
This study has saved UA over half a billion dollars in delay costs alone in less than a year. Many potential delays were prevented and hence the number of flight delays was reduced by 50%. Customer inconveniences due to delays and cancellations were reduced. Additionally, developing an efficient way of addressing these problems helped UA respond to changes in the conditions quickly.

9.6-2.



9.6-3.

(a)



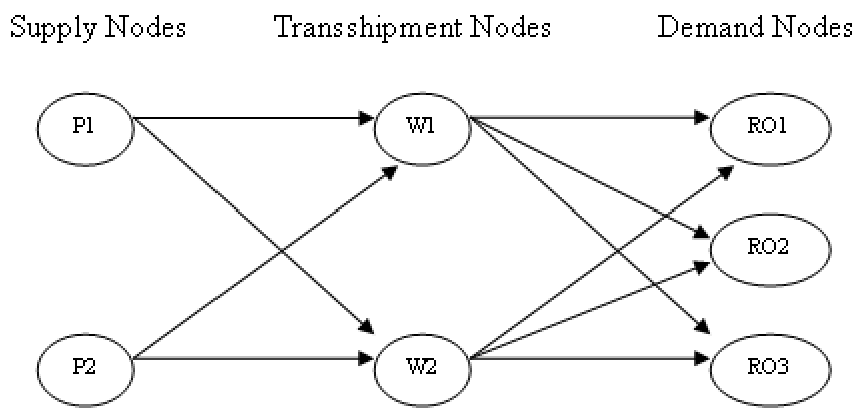
(b) minimize $7x_{F_1W_1} + 3x_{F_1D} + 2x_{DW_1} + 4x_{F_2D} + 4x_{DW_2} + 9x_{F_2W_2}$

subject to
$$x_{F_1W_1} + x_{F_1D} = 80$$
$$x_{F_2D} + x_{F_2W_2} = 70$$
$$x_{F_1W_1} + x_{DW_1} = 60$$
$$x_{DW_2} + x_{F_2W_2} = 90$$
$$x_{F_1D} - x_{DW_1} + x_{F_2D} - x_{DW_2} = 0$$
$$0 \le x_{F_1D}, x_{DW_1}, x_{F_2D}, x_{DW_2} \le 50$$

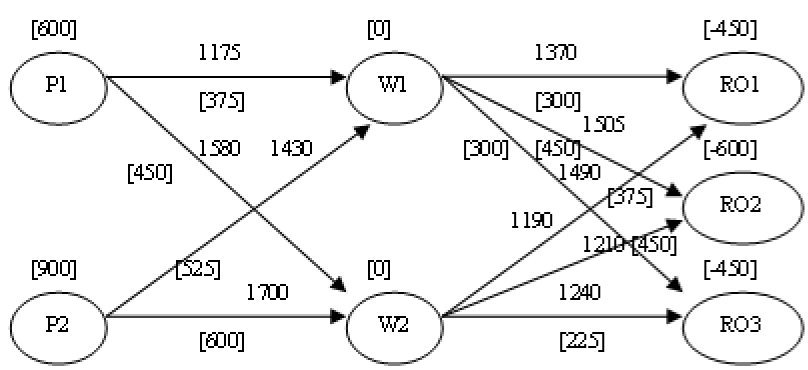
9.6-4. Please see paperclip attachment for solution:

9.6-5.

(a)



(b)



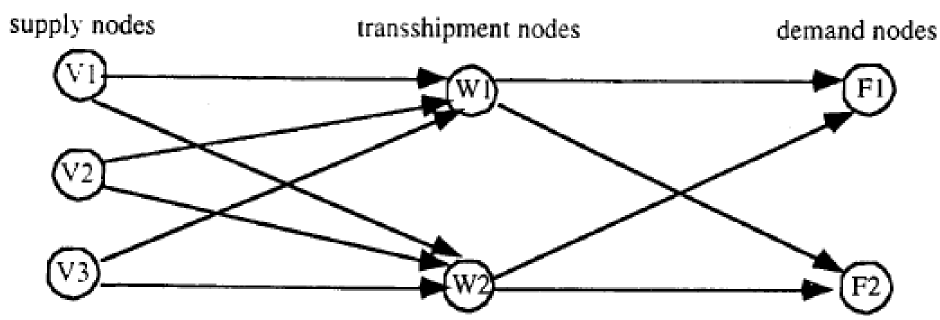
(c) Total cost: $4,217,625

From	To	Ship	Capacity	Unit Cost
P1	W1	375	375	$1,175
P1	W2	225	450	$1,580
P2	W1	375	525	$1,430
P2	W2	525	600	$1,700
W1	RO1	300	300	$1,370
W1	RO2	150	450	$1,505
W1	RO3	300	300	$1,490
W2	RO1	150	375	$1,190
W2	RO2	450	450	$1,210
W2	RO3	150	225	$1,240

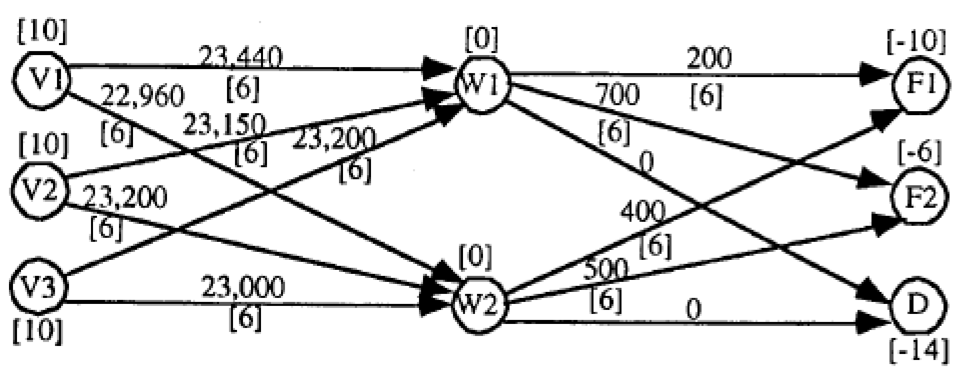
Nodes	Net Flow		Output/ Demand
P1	600	=	600
P2	900	=	900
W1	0	=	0
W2	0	=	0
RO1	-450	=	-450
RO2	-600	=	-600
RO3	-450	=	-450

9.6-6.

(a)



(b)



(c)

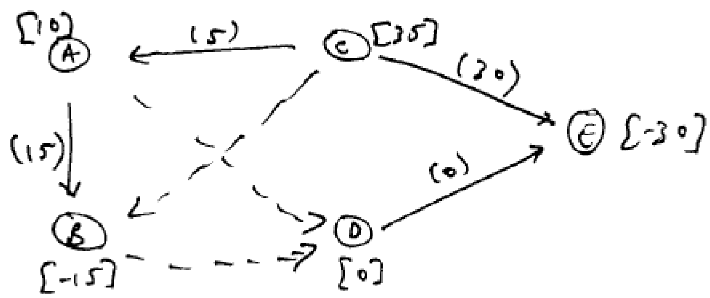
From	To	Ship		Capacity	Unit Cost
V1	W1	4	≤	6	$23,440
V1	W2	6	≤	6	$22,960
V2	W1	6	≤	6	$23,150
V2	W2	4	≤	6	$23,200
V3	W1	4	≤	6	$23,200
V3	W2	6	≤	6	$23,000
W1	F1	6	≤	6	$200
W1	F2	0	≤	6	$700
W1	D	8		-	$0
W2	F1	4	≤	6	$400
W2	F2	6	≤	6	$500
W2	D	6		-	$0

Total Cost = $699,820

Nodes	Net Flow		Output/Demand
V1	10	=	10
V2	10	=	10
V3	10	=	10
W1	0	=	0
W2	0	=	0
F1	-10	=	-10
F2	-6	=	-6
D	-14	=	-14

9.7-1.

(a)



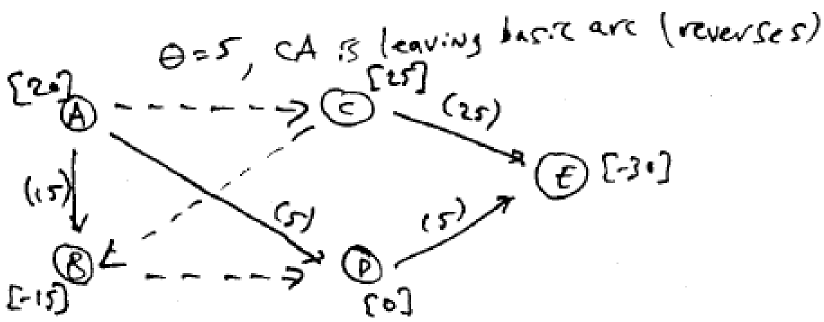
(b) Compute Δ for nonbasic arcs:

$$\Delta_{\text{BD}} = 5 + 4 - 3 + (-6) + 2 = 2$$

$$\Delta_{\text{AD}} = 5 + 4 - 3 + (-6) = 0$$

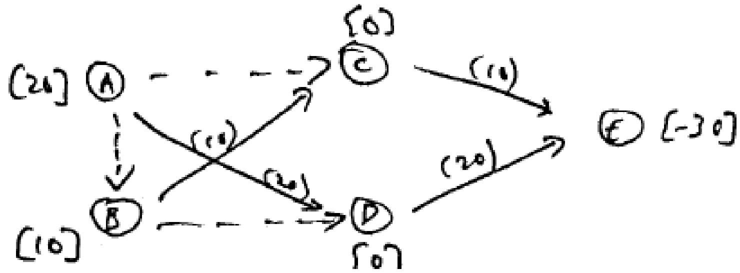
$$\Delta_{\text{CB}} = (-3) - 2 - (-6) = 1$$

All of them are nonnegative, so this solution is optimal. Since $\Delta_{\text{AD}} = 0$, multiple optima exist. Network simplex:



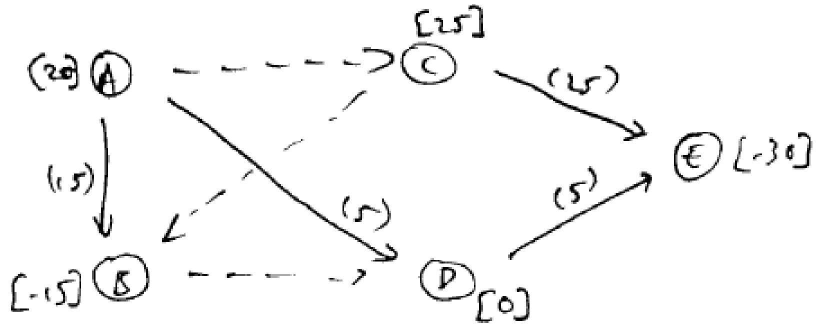
Optimal nonbasic solutions have $x_{\text{AB}} = 15$, $x_{\text{AC}} = \theta$, $x_{\text{AD}} = 5 - \theta$, $x_{\text{CE}} = 25 + \theta$, and $x_{\text{DE}} = 5 - \theta$, where $0 \le \theta \le 5$ and C → B and B → D are nonbasic arcs.

(c) Start with:



Network simplex: $\Delta_{\text{AC}} = 6 + 3 - 4 - 5 = 0$
$\Delta_{\text{AB}} = 2 + 3 + 3 - 4 - 5 = -1 < 0 \leftarrow$ entering arc
$\Delta_{\text{BD}} = 5 + 4 - 3 - 3 = 3$

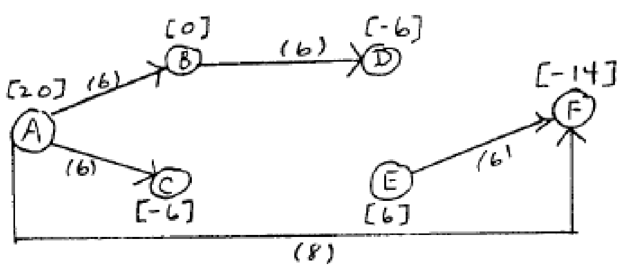
$\theta = 15$ and BC is leaving arc (reverses). The next BF solution is:



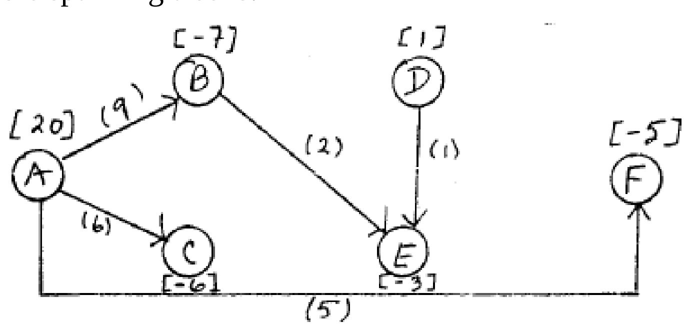
From (b), we recognize this solution as optimal.

9.7-2.

(a)



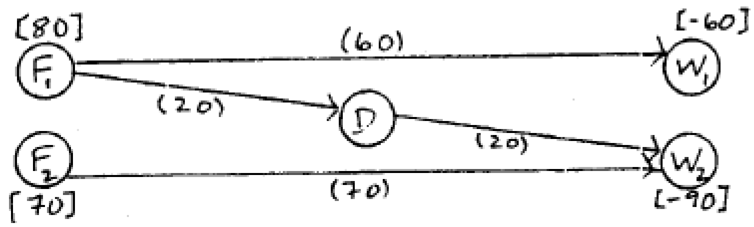
(b) The final feasible spanning tree is:



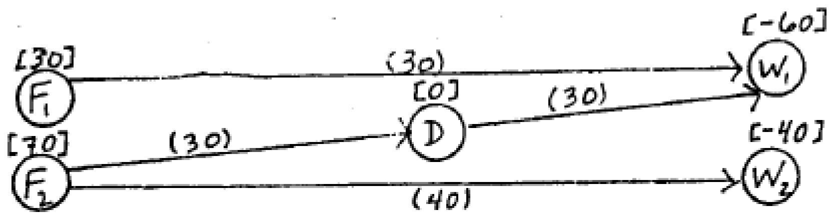
The flow to which it corresponds is the same as in Prob. 9.5-6.

9.7-3.

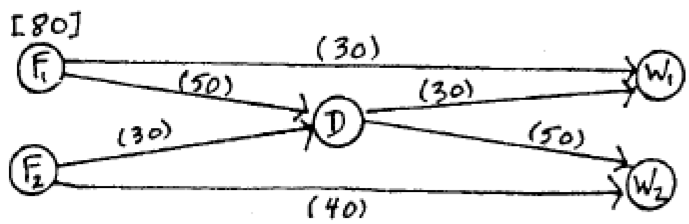
(a) There are no reverse arcs in this solution.



(b) The optimal BF spanning tree is:



which corresponds to a real flow of:

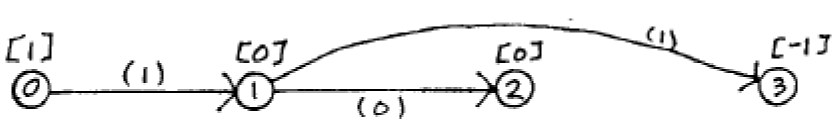


with cost $1,100$.

9.7-4.

Initial BF spanning tree:

Optimal BF spanning tree:

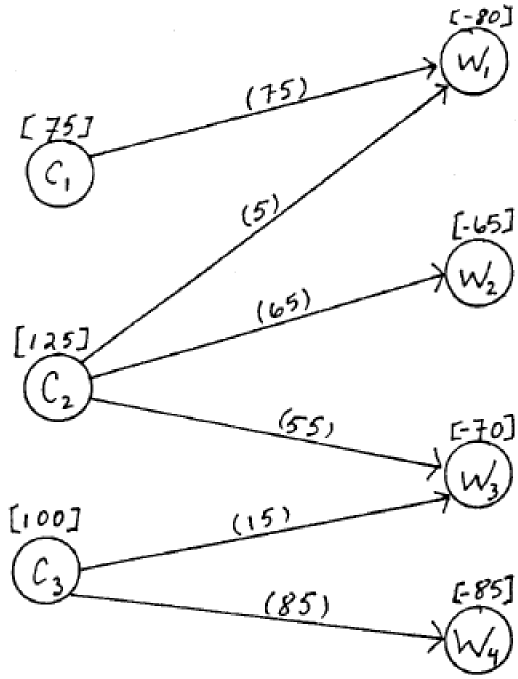


which has a real flow of:

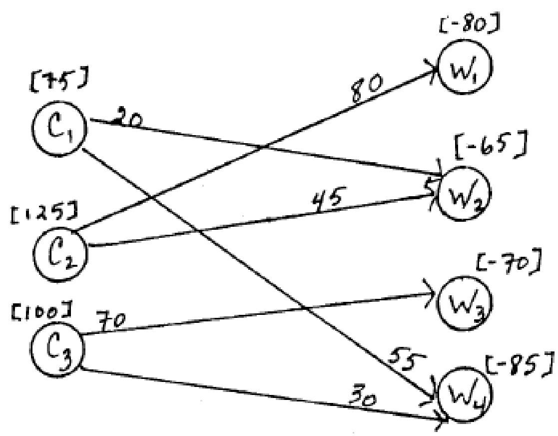
with cost 46.

9.7-5.

Initial BF spanning tree:



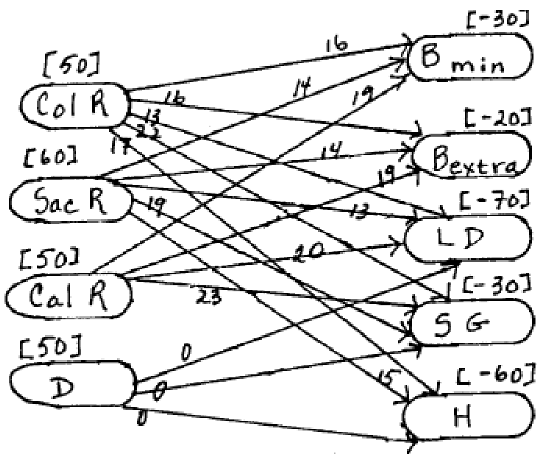
Optimal BF spanning tree:



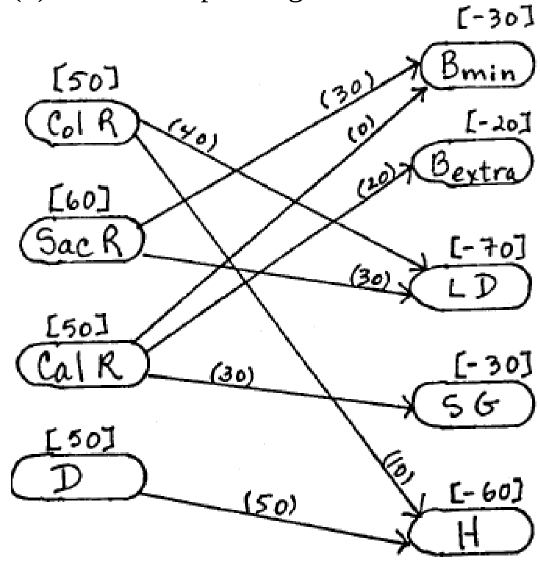
which corresponds to the optimal solution given in Sec. 8.1.

9.7-6.

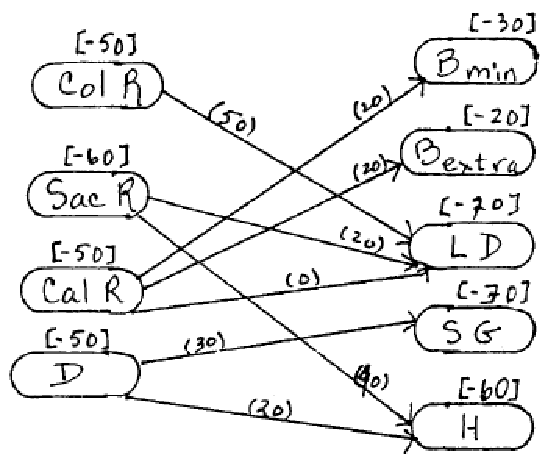
(a)



(b) Initial BF spanning tree:

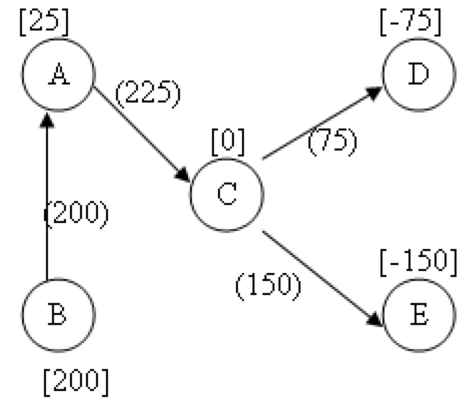


(c) Optimal BF spanning tree:

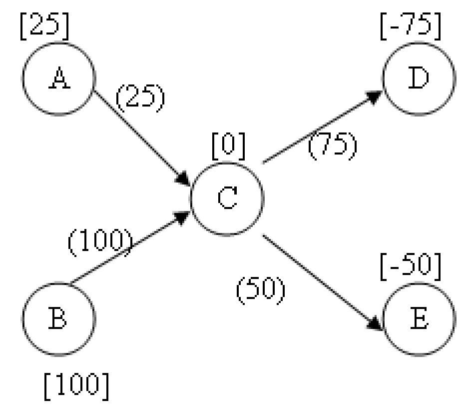


The sequence of basic feasible solutions is identical with the transportation simplex method.

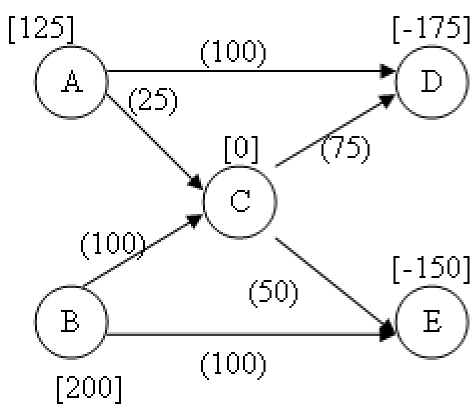
9.7-7.



Optimal BF spanning tree:



which correspond to the real flow of:



with a total cost of 2925.

9.8-1.

		Length of Path	
Activity to Crash	Crash Cost	$A-C$	$B-D$
		14	16
B	$\$5,000$	14	15
B	$\$5,000$	14	15
D	$\$6,000$	14	14
C	$\$4,000$	13	14
D	$\$6,000$	13	13
C	$\$4,000$	12	13
D	$\$6,000$	12	12

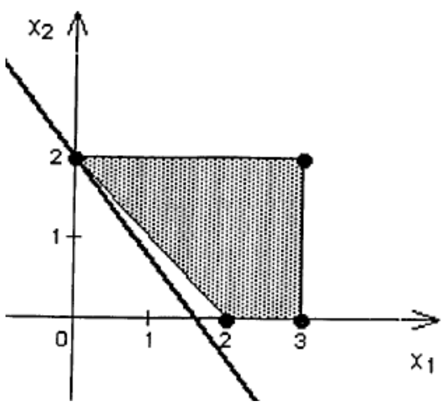
9.8-2.

(a) Let x_A and x_C be the reduction in A and C respectively, due to crashing.

minimize $\quad C = 5000x_A + 4000x_C$

subject to $\quad x_A \leq 3$
$x_C \leq 2$
$x_A + x_C \geq 2$

and $\quad x_A, x_C \geq 0$



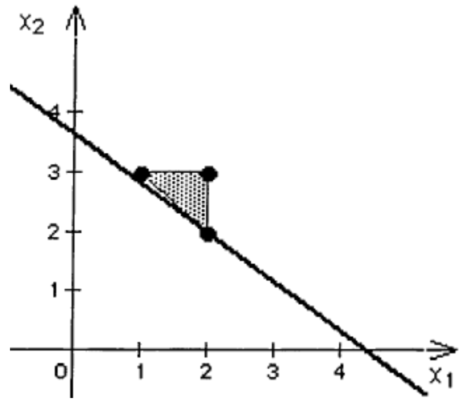
Optimal Solution: $(x_A, x_C) = (0, 2)$ and $C^* = 8,000$.

(b) Let x_B and x_D be the reduction in B and D respectively, due to crashing.

minimize $\quad C = 5000x_B + 6000x_D$

subject to $\quad x_B \leq 2$
$x_D \leq 3$
$x_B + x_D \geq 4$

and $\quad x_B, x_D \geq 0$



Optimal Solution: $(x_B, x_D) = (2, 2)$ and $C^* = 22,000$.

(c) Let x_A, x_B, x_C, and x_D be the reduction in the duration of A, B, C, and D respectively, due to crashing.

$$\begin{array}{ll} \text{minimize} & C = 5000x_A + 5000x_B + 4000x_C + 6000x_D \\ \text{subject to} & x_A \leq 3 \\ & x_B \leq 2 \\ & x_C \leq 2 \\ & x_D \leq 3 \\ & x_A + x_C \geq 2 \\ & x_B + x_D \geq 4 \\ \text{and} & x_A, x_B, x_C, x_D \geq 0 \end{array}$$

Optimal Solution: $(x_A, x_B, x_C, x_D) = (0, 2, 2, 2)$ and $C^* = 30,000$.

(d) Let x_j be the reduction in the duration of activity j due to crashing for $j = A, B, C, D$. Also let y_j denote the start time of activity j for $j = C, D$ and y_{FINISH} the project duration.

$$\begin{array}{ll} \text{minimize} & C = 5000x_A + 5000x_B + 4000x_C + 6000x_D \\ \text{subject to} & x_A \leq 3, x_B \leq 2, x_C \leq 2, x_D \leq 3 \\ & y_C \geq 0 + 8 - x_A \\ & y_D \geq 0 + 9 - x_B \\ & y_{\text{FINISH}} \geq y_C + 6 - x_C \\ & y_{\text{FINISH}} \geq y_D + 7 - x_D \\ & y_{\text{FINISH}} \leq 12 \\ \text{and} & x_A, x_B, x_C, x_D, y_C, y_D, y_{\text{FINISH}} \geq 0 \end{array}$$

(e)

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	8	5	$25000	$40000	3	$5000	0	0	8
B	9	7	$20000	$30000	2	$5000	0	2	7
C	6	4	$16000	$24000	2	$4000	8	2	12
D	7	4	$27000	$45000	3	$6000	7	2	12

Finish Time = 12
Total Cost = $118000

(f) The solution found using LINGO agrees with the solution in (e), i.e., it is optimal to reduce the duration of activities B, C, and D by two months. Then the entire project takes 12 months and costs $25 + 30 + 24 + (27 + 12) = 118$ thousand dollars.

```
Variable           Value          Reduced Cost
      XA        0.000000              0.000000
      XB        2.000000              0.000000
      XC        2.000000              0.000000
      XD        2.000000              0.000000

 Row     Slack or Surplus       Dual Price
   1          30000.00          -1.000000
   2          3.000000           0.000000
   3          0.000000           1000.000
   4          0.000000           1000.000
   5          1.000000           0.000000
   6          0.000000          -5000.000
   7          0.000000          -6000.000
```

(g) Deadline of 11 months

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	8	5	$25000	$40000	3	$5000	0	1	7
B	9	7	$20000	$30000	2	$5000	0	2	7
C	6	4	$16000	$24000	2	$4000	7	2	11
D	7	4	$27000	$45000	3	$6000	7	3	11

Finish Time = 11
Total Cost = $129000

Deadline of 13 months

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	8	5	$25000	$40000	3	$5000	0	0	8
B	9	7	$20000	$30000	2	$5000	0	2	7
C	6	4	$16000	$24000	2	$4000	8	1	13
D	7	4	$27000	$45000	3	$6000	7	1	13

Finish Time = 13
Total Cost = $108000

9.8-3.

(a) $7,834 is saved by the new plan given below.

		Length of Path		
Activity to Crash	Crash Cost	$A-B-D$	$A-B-E$	$A-C-E$
		10	11	12
C	$\$1,333$	10	11	11
E	$\$2,500$	10	10	10
D & E	$\$4,000$	9	9	9
B & C	$\$4,333$	8	8	8

Activity	Duration	Cost
A	3 weeks	$\$54,000$
B	3 weeks	$\$65,000$
C	3 weeks	$\$58,666$
D	2 weeks	$\$41,500$
E	2 weeks	$\$80,000$

(b)

	Time		Cost		Maximum	Crash Cost			
Activity	Normal	Crash	Normal	Crash	Time Reduction	per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	0	0	3
B	4	3	$62000	$65000	1	$3000	4	0	8
C	5	2	$66000	$70000	3	$1333	3	0	8
D	3	1	$40000	$43000	2	$1500	9	0	12
E	4	2	$75000	$80000	2	$2500	8	0	12

Finish Time = 12
Total Cost = $297000

	Time		Cost		Maximum	Crash Cost			
Activity	Normal	Crash	Normal	Crash	Time Reduction	per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	0	0	3
B	4	3	$62000	$65000	1	$3000	3	0	7
C	5	2	$66000	$70000	3	$1333	3	1	7
D	3	1	$40000	$43000	2	$1500	8	0	11
E	4	2	$75000	$80000	2	$2500	7	0	11

Finish Time = 11
Total Cost = $298333

	Time		Cost		Maximum	Crash Cost			
Activity	Normal	Crash	Normal	Crash	Time Reduction	per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	0	0	3
B	4	3	$62000	$65000	1	$3000	3	0	7
C	5	2	$66000	$70000	3	$1333	3	1	7
D	3	1	$40000	$43000	2	$1500	7	1.22E-15	10
E	4	2	$75000	$80000	2	$2500	7	1	10

Finish Time = 10
Total Cost = $300833

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	\$54000	\$60000	1	\$6000	0	0	3
B	4	3	\$62000	\$65000	1	\$3000	3	4.66E-12	7
C	5	2	\$66000	\$70000	3	\$1333	3	1	7
D	3	1	\$40000	\$43000	2	\$1500	7	1	9
E	4	2	\$75000	\$80000	2	\$2500	7	2	9

Finish Time = 9
Total Cost = \$304833

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	\$54000	\$60000	1	\$6000	0	3.66E-11	3
B	4	3	\$62000	\$65000	1	\$3000	3	1	6
C	5	2	\$66000	\$70000	3	\$1333	3	2	6
D	3	1	\$40000	\$43000	2	\$1500	6	1	8
E	4	2	\$75000	\$80000	2	\$2500	6	2	8

Finish Time = 8
Total Cost = \$309167

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	\$54000	\$60000	1	\$6000	0	1	2
B	4	3	\$62000	\$65000	1	\$3000	2	1	5
C	5	2	\$66000	\$70000	3	\$1333	2	2	5
D	3	1	\$40000	\$43000	2	\$1500	5	1	7
E	4	2	\$75000	\$80000	2	\$2500	5	2	7

Finish Time = 7
Total Cost = \$315167

Crash to 8 weeks.

9.8-4.

(a) Let x_j be the reduction in the duration of activity j and y_j be the start time of activity j.

minimize $\quad C = 6x_A + 12x_B + 4x_C + 6.67x_D + 10x_E + 7.33x_F + 5.75x_G + 8x_H$

subject to
$$
\begin{array}{ll}
0 \le x_A \le 2 \quad 0 \le x_B \le 1 & 0 \le x_C \le 2 \quad 0 \le x_D \le 3 \\
0 \le x_E \le 1 \quad 0 \le x_F \le 3 & 0 \le x_G \le 4 \quad 0 \le x_H \le 2 \\
y_A + 5 - x_A \le y_C & y_A + 5 - x_A \le y_D \\
y_B + 3 - x_B \le y_E & y_B + 3 - x_B \le y_F \\
y_C + 4 - x_C \le y_G & y_D + 6 - x_D \le y_H \\
y_E + 5 - x_E \le y_G & y_F + 7 - x_F \le y_H \\
y_G + 9 - x_G \le y_{\text{FINISH}} & y_H + 8 - x_H \le y_{\text{FINISH}} \\
0 \le y_{\text{FINISH}} \le 15 & \\
y_j \ge 0 &
\end{array}
$$

(b) Finish Time: 15 weeks, total crashing cost: $\$45.75$ million, total cost: $\$259.75$ million.

Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Maximum Time Reduction	Crash Cost per Week Saved	Start Time	Time Reduction	Finish Time
A	5	3	24	36	2	6.00	0	2	3
B	3	2	13	25	1	12.00	0	1	2
C	4	2	21	29	2	4.00	3	0	7
D	6	3	30	50	3	6.67	3	0	9
E	5	4	26	36	1	10.00	2	0	7
F	7	4	35	57	3	7.33	2	0	9
G	9	5	30	53	4	5.75	7	1	15
H	8	6	35	51	2	8.00	9	2	15

9.8-5.

(a) Let x_j be the reduction in the duration of activity j and y_j be the start time of activity j.

minimize $\quad C = 5x_A + 7x_B + 8x_C + 4x_D + 5x_E + 6x_F + 3x_G + 4x_H + 9x_I + 2x_J$

subject to
$$0 \le x_A \le 4 \quad 0 \le x_B \le 3 \quad 0 \le x_C \le 5 \quad 0 \le x_D \le 3 \quad 0 \le x_E \le 5$$
$$0 \le x_F \le 7 \quad 0 \le x_G \le 2 \quad 0 \le x_H \le 3 \quad 0 \le x_I \le 4 \quad 0 \le x_J \le 2$$
$$y_A + 32 - x_A \le y_C \qquad y_B + 28 - x_B \le y_D$$
$$y_B + 28 - x_B \le y_E \qquad y_B + 28 - x_B \le y_F$$
$$y_C + 36 - x_C \le y_J \qquad y_D + 16 - x_D \le y_G$$
$$y_E + 32 - x_E \le y_H \qquad y_E + 32 - x_E \le y_I$$
$$y_F + 54 - x_F \le y_J \qquad y_G + 17 - x_G \le y_H$$
$$y_G + 17 - x_G \le y_I \qquad y_H + 20 - x_H \le y_{\text{FINISH}}$$
$$y_I + 34 - x_I \le y_{\text{FINISH}} \qquad y_J + 18 - x_J \le y_{\text{FINISH}}$$
$$0 \le y_{\text{FINISH}} \le 92$$
$$y_j \ge 0$$

(b) Finish Time: 92 weeks, total crashing cost: $\$43$ million, total cost: $\$1.388$ billion.

Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Maximum Time Reduction	Crash Cost per Week Saved	Start Time	Time Reduction	Finish Time
A	32	28	160	180	4	5	8	0	40
B	28	25	125	146	3	7	0	3	25
C	36	31	170	210	5	8	40	0	76
D	16	13	60	72	3	4	25	0	41
E	32	27	135	160	5	5	26	0	58
F	54	47	215	257	7	6	25	3	76
G	17	15	90	96	2	3	41	0	58
H	20	17	120	132	3	4	58	0	78
I	34	30	190	226	4	9	58	0	92
J	18	16	80	84	2	2	76	2	92

9.9-1.

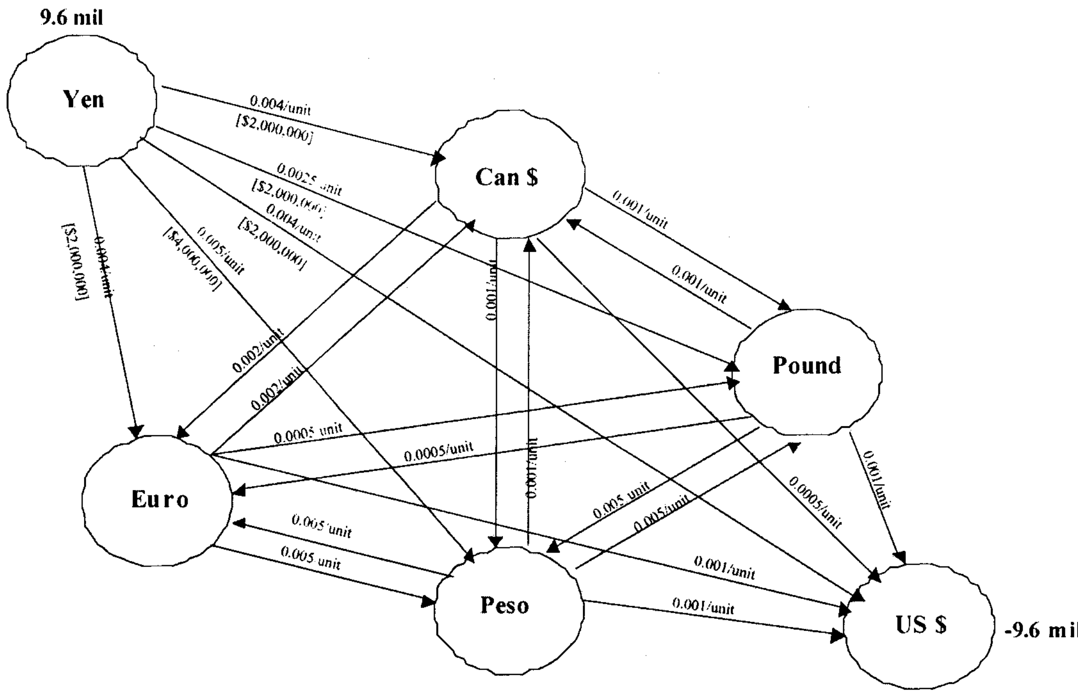
Answers will vary.

9.9-2.

Answers will vary.

Cases

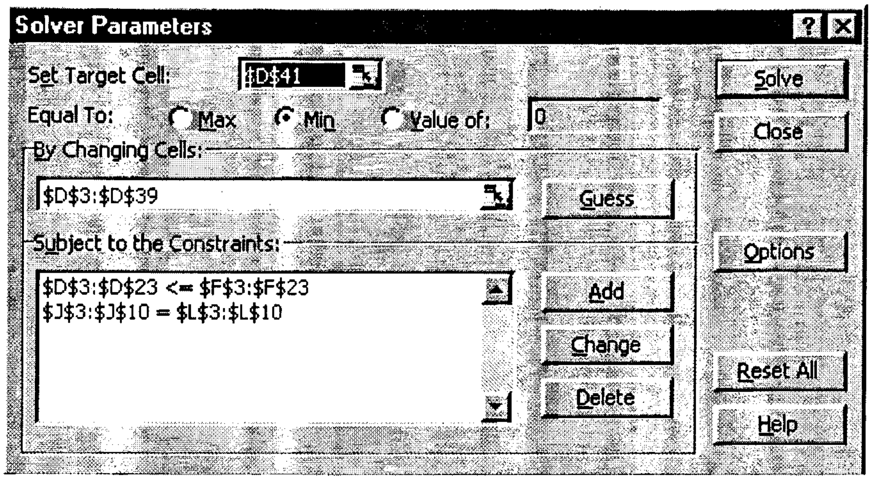
9.1 a) There are three supply nodes – the Yen node, the Rupiah node, and the Ringgit node. There is one demand node – the US$ node. Below, we draw the network originating from only the Yen supply node to illustrate the overall design of the network. In this network, we exclude both the Rupiah and Ringgit nodes for simplicity.

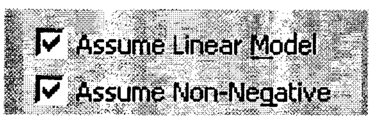


b) Since all transaction limits are given in the equivalent of 1000 dollars we define the flow variables as the amount in 1000's of dollars that Jake converts from one currency into another one. His total holdings in Yen, Rupiah, and Ringgit are equivalent to $9.6 million, $1.68 million, and $5.6 million, respectively. So, the supplies at the supply nodes Yen, Rupiah, and Ringgit are -$9.6 million, -$1.68 million, and -$5.6 million, respectively. The demand at the only demand node US$ equals $16.88 million. The transaction limits are capacity constraints for all arcs leaving from the nodes Yen, Rupiah, and Ringgit. The unit cost for every arc is given by the transaction cost for the currency conversion.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2		From	To	Ship		Capacity	Unit Cost		Nodes	Net Flow	Supply /Demand		
3		Yen	Rupiah	0	<=	5000	0.005		Yen	-9600	=	-9600	
4		Yen	Ringgit	0	<=	5000	0.005		Rupiah	-1680	=	-1680	
5		Yen	US$	2000	<=	2000	0.004		Ringgit	-5600	=	-5600	
6		Yen	Can$	2000	<=	2000	0.004		US$	16880	=	16880	
7		Yen	Euro	2000	<=	2000	0.004		Can$	0	=	0	
8		Yen	Pound	2000	<=	2000	0.0025		Euro	0	=	0	
9		Yen	Peso	1600	<=	4000	0.005		Pound	0	=	0	
10		Rupiah	Yen	0	<=	5000	0.005		Peso	0	=	0	
11		Rupiah	Ringgit	0	<=	2000	0.007						
12		Rupiah	US$	200	<=	200	0.005						
13		Rupiah	Can$	200	<=	200	0.003						
14		Rupiah	Euro	1000	<=	1000	0.003						
15		Rupiah	Pound	80	<=	500	0.0075						
16		Rupiah	Peso	200	<=	200	0.0075						
17		Ringgit	Yen	0	<=	3000	0.005						
18		Ringgit	Rupiah	0	<=	4500	0.007						
19		Ringgit	US$	1100	<=	1500	0.007						
20		Ringgit	Can$	0	<=	1500	0.007						
21		Ringgit	Euro	2500	<=	2500	0.004						
22		Ringgit	Pound	1000	<=	1000	0.0045						
23		Ringgit	Peso	1000	<=	1000	0.005						
24		Can$	US$	2200		-	0.0005						
25		Can$	Euro	0		-	0.002						
26		Can$	Pound	0		-	0.001						
27		Can$	Peso	0		-	0.001						
28		Euro	US$	5500		-	0.001						
29		Euro	Can$	0		-	0.002						
30		Euro	Pound	0		-	0.0005						
31		Euro	Peso	0		-	0.005						
32		Pound	US$	3080		-	0.001						
33		Pound	Can$	0		-	0.001						
34		Pound	Euro	0		-	0.0005						
35		Pound	Peso	0		-	0.005						
36		Peso	US$	2800		-	0.001						
37		Peso	Can$	0		-	0.001						
38		Peso	Euro	0		-	0.005						
39		Peso	Pound	0		-	0.005						
40													
41			Total Cost	$83,380.00									

	J
1	
2	Net Flow
3	=-SUM(D3:D9)+D10+D17
4	=-SUM(D10:D16)+D3+D18
5	=-SUM(D17:D23)+D4+D11
6	=D5+D12+D19+D24+D28+D32+D36
7	=D6+D13+D20-SUM(D24:D27)+D29+D33+D37
8	=D7+D14+D21+D25-SUM(D28:D31)+D34+D38
9	=D8+D15+D22+D26+D30-SUM(D32:D35)+D39
10	=D9+D16+D23+D27+D31+D35-SUM(D36:D39)
11	

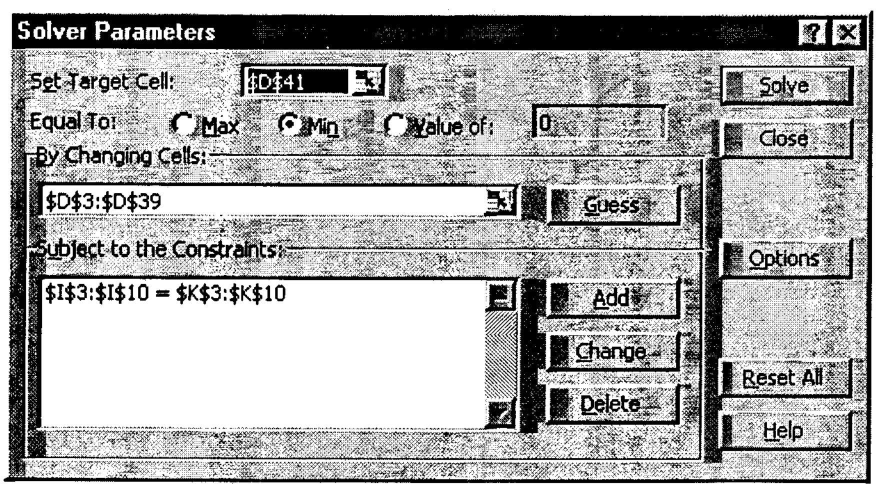




Jake should convert the equivalent of $2 million from Yen to each US$, Can$, Euro, and Pound. He should convert $1.6 million from Yen to Peso. Moreover, he should convert the equivalent of $200,000 from Rupiah to each US$, Can$, and Peso, $1 million from Rupiah to Euro, and $80,000 from Rupiah to Pound. Furthermore, Jake should convert the equivalent of $1.1 million from Ringgit to US$, $2.5 million from Ringgit to Euro, and $1 million from Ringgit to each Pound and Peso. Finally, he should convert all the money he converted into Can$, Euro, Pound, and Peso directly into US$. Specifically, he needs to convert into US$ the equivalent of $2.2 million, $5.5 million, $3.08 million, and $2.8 million Can$, Euro, Pound, and Peso, respectively. Assuming Jake pays for the total transaction costs of $83,380 directly from his American bank accounts he will have $16,880,000 dollars to invest in the US.

c) We eliminate all capacity restrictions on the arcs.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		From	To	Ship	Capacity	Unit Cost		Nodes	Net Flow		Supply /Demand	
3		Yen	Rupiah	0	-	0.005		Yen	-9600	=	-9600	
4		Yen	Ringgit	0	-	0.005		Rupiah	-1680	=	-1680	
5		Yen	US$	0	-	0.004		Ringgit	-5600	=	-5600	
6		Yen	Can$	0	-	0.004		US$	16880	=	16880	
7		Yen	Euro	0	-	0.004		Can$	0	=	0	
8		Yen	Pound	9600	-	0.0025		Euro	0	=	0	
9		Yen	Peso	0	-	0.005		Pound	0	=	0	
10		Rupiah	Yen	0	-	0.005		Peso	0	=	0	
11		Rupiah	Ringgit	0	-	0.007						
12		Rupiah	US$	0	-	0.005						
13		Rupiah	Can$	1680	-	0.003						
14		Rupiah	Euro	0	-	0.003						
15		Rupiah	Pound	0	-	0.0075						
16		Rupiah	Peso	0	-	0.0075						
17		Ringgit	Yen	0	-	0.005						
18		Ringgit	Rupiah	0	-	0.007						
19		Ringgit	US$	0	-	0.007						
20		Ringgit	Can$	0	-	0.007						
21		Ringgit	Euro	5600	-	0.004						
22		Ringgit	Pound	0	-	0.0045						
23		Ringgit	Peso	0	-	0.005						
24		Can$	US$	1680	-	0.0005						
25		Can$	Euro	0	-	0.002						
26		Can$	Pound	0	-	0.001						
27		Can$	Peso	0	-	0.001						
28		Euro	US$	5600	-	0.001						
29		Euro	Can$	0	-	0.002						
30		Euro	Pound	0	-	0.0005						
31		Euro	Peso	0	-	0.005						
32		Pound	US$	9600	-	0.001						
33		Pound	Can$	0	-	0.001						
34		Pound	Euro	0	-	0.0005						
35		Pound	Peso	0	-	0.005						
36		Peso	US$	0	-	0.001						
37		Peso	Can$	0	-	0.001						
38		Peso	Euro	0	-	0.005						
39		Peso	Pound	0	-	0.005						
40												
41			Total Cost	$67,480.0								
42												



Jake should convert the entire holdings in Japan from Yen into Pound and then into US$, the entire holdings in Indonesia from Rupiah into Can$ and then into US$, and the entire holdings in Malaysia from Ringgit into Euro and then into US$. Without the capacity limits the transaction costs are reduced to $67,480.00.

d) We multiply all unit cost for Rupiah by 6.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		From	To	Ship	Capacity	Unit Cost		Nodes	Net Flow		Supply/ Demand	
3		Yen	Rupiah	0	-	0.005		Yen	-9600	=	-9600	
4		Yen	Ringgit	0	-	0.005		Rupiah	-1680	=	-1680	
5		Yen	US$	0	-	0.004		Ringgit	-5600	=	-5600	
6		Yen	Can$	0	-	0.004		US$	16880	=	16880	
7		Yen	Euro	0	-	0.004		Can$	0	=	0	
8		Yen	Pound	9600	-	0.0025		Euro	0	=	0	
9		Yen	Peso	0	-	0.005		Pound	0	=	0	
10		Rupiah	Yen	0	-	0.03		Peso	0	=	0	
11		Rupiah	Ringgit	0	-	0.042						
12		Rupiah	US$	0	-	0.03						
13		Rupiah	Can$	1680	-	0.018						
14		Rupiah	Euro	0	-	0.018						
15		Rupiah	Pound	0	-	0.045						
16		Rupiah	Peso	0	-	0.045						
17		Ringgit	Yen	0	-	0.005						
18		Ringgit	Rupiah	0	-	0.007						
19		Ringgit	US$	0	-	0.007						
20		Ringgit	Can$	0	-	0.007						
21		Ringgit	Euro	5600	-	0.004						
22		Ringgit	Pound	0	-	0.0045						
23		Ringgit	Peso	0	-	0.005						
24		Can$	US$	1680	-	0.0005						
25		Can$	Euro	0	-	0.002						
26		Can$	Pound	0	-	0.001						
27		Can$	Peso	0	-	0.001						
28		Euro	US$	5600	-	0.001						
29		Euro	Can$	0	-	0.002						
30		Euro	Pound	0	-	0.0005						
31		Euro	Peso	0	-	0.005						
32		Pound	US$	9600	-	0.001						
33		Pound	Can$	0	-	0.001						
34		Pound	Euro	0	-	0.0005						
35		Pound	Peso	0	-	0.005						
36		Peso	US$	0	-	0.001						
37		Peso	Can$	0	-	0.001						
38		Peso	Euro	0	-	0.005						
39		Peso	Pound	0	-	0.005						
40												
41			Total Cost =	$92,680.00								
42												

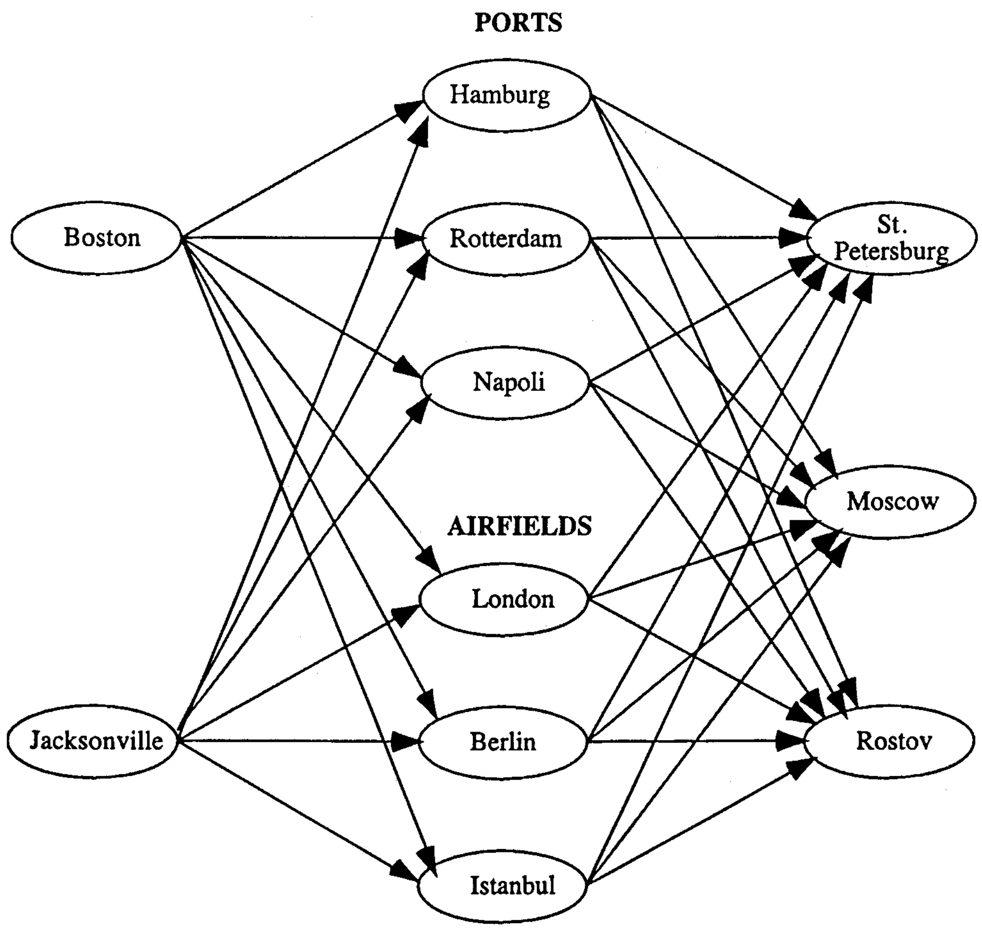
The optimal routing for the money doesn't change, but the total transaction costs are now increased to $92,680.

e) In the described crisis situation the currency exchange rates might change every minute. Jake should carefully check the exchange rates again when he performs the transactions.

The European economies might be more insulated from the Asian financial collapse than the US economy. To impress his boss Jake might want to explore other investment opportunities in safer European economies that provide higher rates of return than US bonds.

Cases

9.2 a) The network showing the different routes troops and supplies may follow to reach the Russian Federation appears below.



b) The President is only concerned about how to most quickly move troops and supplies from the United States to the three strategic Russian cities. Obviously, the best way to achieve this goal is to find the fastest connection between the US and the three cities. We therefore need to find the shortest path between the US and each of the three cities.

The President only cares about the time it takes to get the troops and supplies to Russia. It does not matter how great a distance the troops and supplies cover. Therefore we define the arc length between two nodes in the network to be the time it takes to travel between the respective cities. For example, the distance between Boston and London equals 6,200 km. The mode of transportation between the cities is a Starfighter traveling at a speed of 400 miles per hour * 1.609 km per mile = 643.6 km per hour. The time is takes to bring troops and supplies from Boston to London equals 6,200 km / 643.6 km per hour = 9.6333 hours. Using this approach we can compute the time of travel along all arcs in the network.

By simple inspection and common sense it is apparent that the fastest transportation involves using only airplanes. We therefore can restrict ourselves to only those arcs in the network where the mode of transportation is air travel. We can omit the three port cities and all arcs entering and leaving these nodes.

Finally, we define a new node ("dummy" node) in the network called "US," and we introduce two new arcs: one going from the US to Boston and the other going from the US to Jacksonville. The arc length on both new arcs equals 0. The objective is now to find the shortest path from the US to each of the three Russian cities. We define the US node to be a supply node with supply 3, and we define each of the three nodes representing Russian cities as demand nodes with a demand of -1. The nodes representing the three European airfields – London, Berlin, and Istanbul – are all transshipment nodes.

The following spreadsheet shows the entire linear programming model, which identifies the three shortest paths.

	A	B	C	D	E	F	G	H	I	J	K
1		From	To	On Route	Distance	Time (hr)		Nodes	Net Flow		Supply/Demand
2		US	Boston	3	0	0		US	3	=	3
3		US	Jacksonville	0	0	0		Boston	0	=	0
4		Boston	London	2	6200	9.63331		Jacksonville	0	=	0
5		Boston	Berlin	1	7250	11.2648		London	0	=	0
6		Boston	Istanbul	0	8300	12.8962		Berlin	0	=	0
7		Jacksonville	London	0	7900	12.2747		Istanbul	0	=	0
8		Jacksonville	Berlin	0	9200	14.2946		St. Petersburg	-1	=	-1
9		Jacksonville	Istanbul	0	10100	15.693		Moscow	-1	=	-1
10		London	St. Petersburg	1	1980	3.07644		Rostov	-1	=	-1
11		London	Moscow	1	2300	3.57365					
12		London	Rostov	0	2860	4.44375					
13		Berlin	St. Petersburg	0	1280	1.98881					
14		Berlin	Moscow	0	1600	2.48602					
15		Berlin	Rostov	1	1730	2.688					
16		Istanbul	St. Petersburg	0	2040	3.16967					
17		Istanbul	Moscow	0	1700	2.64139					
18		Istanbul	Rostov	0	990	1.53822					
19											
20			Total Time =	39.86948415							

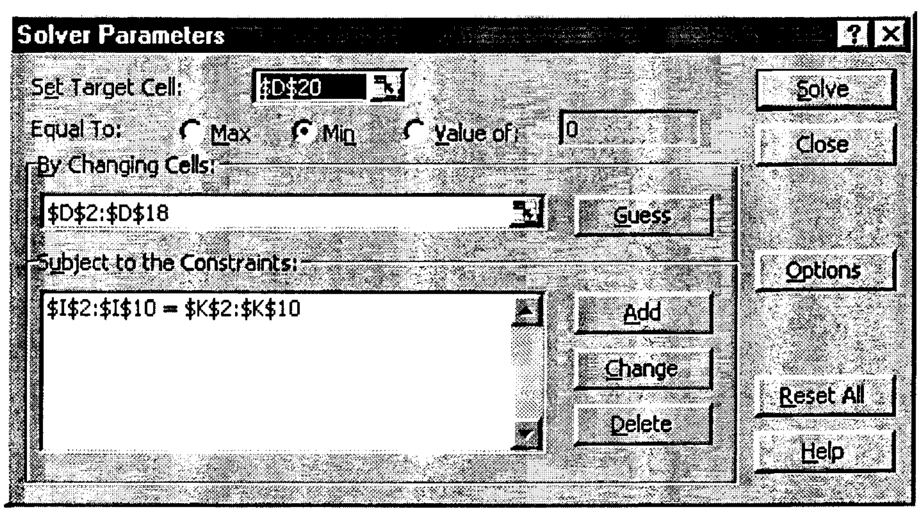
The spreadsheet contains the following formulas:

	F
1	Time (hr)
2	0
3	0
4	=E4/(400*1.609)
5	=E5/(400*1.609)
6	=E6/(400*1.609)
7	=E7/(400*1.609)
8	=E8/(400*1.609)
9	=E9/(400*1.609)
10	=E10/(400*1.609)
11	=E11/(400*1.609)
12	=E12/(400*1.609)
13	=E13/(400*1.609)
14	=E14/(400*1.609)
15	=E15/(400*1.609)
16	=E16/(400*1.609)
17	=E17/(400*1.609)
18	=E18/(400*1.609)
19	
20	

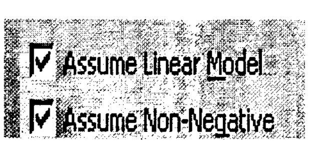
	I
1	Net Flow
2	=SUM(D2:D3)
3	=-D2+SUM(D4:D6)
4	=-D3+SUM(D7:D9)
5	=-D4-D7+D10+D11+D12
6	=-D5-D8+D13+D14+D15
7	=-D6-D9+D16+D17+D18
8	=-D10-D13-D16
9	=-D11-D14-D17
10	=-D12-D15-D18
11	

	C	D
20	Total Time =	=SUMPRODUCT(D2:D18,F2:F18)

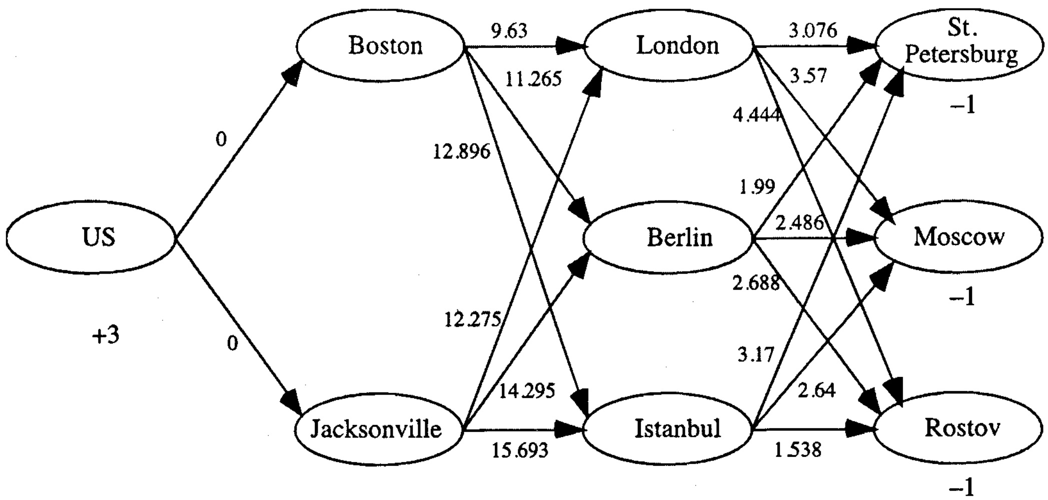
The solver dialogue box appears as follows.



Throughout the analysis of this case we use the following solver options.



From the optimal solution to the linear programming model we see that the shortest path from the US to Saint Petersburg is Boston → London → Saint Petersburg with a total travel time of 12.710 hours. The shortest path from the US to Moscow is Boston → London → Moscow with a total travel time of 13.207 hours. The shortest path from the US to Rostov is Boston → Berlin → Rostov with a total travel time of 13.953 hours. The following network diagram highlights these shortest paths.



c) The President must satisfy each Russian city's military requirements at minimum cost. Therefore, this problem can be solved as a minimum-cost network flow problem. The two nodes representing US cities are supply nodes with a supply of 500 each (we measure all weights in 1000 tons). The three nodes representing Saint Petersburg, Moscow, and Rostov are demand nodes with demands of –320. -440, and –240, respectively. All nodes representing European airfields and ports are transshipment nodes. We measure the flow along the arcs in 1000 tons. For some arcs, capacity constraints are given. All arcs from the European ports into Saint Petersburg have zero capacity. All truck routes from the European ports into Rostov have a transportation limit of 2,500*16 = 40,000 tons. Since we measure the arc flows in 1000 tons, the corresponding arc capacities equal 40. An analogous computation yields arc capacities of 30 for both the arcs connecting the nodes London and Berlin to Rostov. For all other nodes we determine natural arc capacities based on the supplies and demands at the nodes. We define the unit costs along the arcs in the network in S1000 per 1000 tons. For example, the cost of transporting 1 ton of material from Boston to Hamburg equals $30,000 / 240 = $125, so the costs of transporting 1000 tons from Boston to Hamburg equals $125,000.

The objective is to satisfy all demands in the network at minimum cost. The following spreadsheet shows the entire linear programming model.

	A	B	C	D	E	F	G	H	I	J	K
1	From	To	Ship	Capacity (in 1000 tons)	Cost of Transport	Unit Cost (in $1000 per 1000 tons)		Nodes	Net Flow		Supply/Demand
2	Boston	Hamburg	440	500	30000	125		Boston	500	=	500
3	Boston	Rotterdam	0	500	30000	125		Jacksonville	500	=	500
4	Boston	Napoli	0	500	32000	133.3333333		Hamburg	0	=	0
5	Boston	London	0	500	45000	300		Rotterdam	0	=	0
6	Boston	Berlin	0	500	50000	333.3333333		Napoli	0	=	0
7	Boston	Istanbul	60	500	55000	366.6666667		London	0	=	0
8	Jacksonville	Hamburg	0	500	48000	200		Berlin	0	=	0
9	Jacksonville	Rotterdam	0	500	44000	183.3333333		Istanbul	0	=	0
10	Jacksonville	Napoli	0	500	56000	233.3333333		St. Petersburg	-320	=	-320
11	Jacksonville	London	350	500	49000	326.6666667		Moscow	-440	=	-440
12	Jacksonville	Berlin	0	500	57000	380		Rostov	-240	=	-240
13	Jacksonville	Istanbul	150	500	61000	406.6666667					
14	Hamburg	St. Petersburg	0	0	3000	187.5					
15	Rotterdam	St. Petersburg	0	0	3000	187.5					
16	Napoli	St. Petersburg	0	0	5000	312.5					
17	London	St. Petersburg	320	320	22000	146.6666667					
18	Berlin	St. Petersburg	0	320	24000	160					
19	Istanbul	St. Petersburg	0	320	28000	186.6666667					
20	Hamburg	Moscow	440	440	4000	250					
21	Rotterdam	Moscow	0	440	5000	312.5					
22	Napoli	Moscow	0	440	5000	312.5					
23	London	Moscow	0	440	19000	126.6666667					
24	Berlin	Moscow	0	440	22000	146.6666667					
25	Istanbul	Moscow	0	440	25000	166.6666667					
26	Hamburg	Rostov	0	40	7000	437.5					
27	Rotterdam	Rostov	0	40	8000	500					
28	Napoli	Rostov	0	40	9000	562.5					
29	London	Rostov	30	30	4000	26.66666667					
30	Berlin	Rostov	0	30	23000	153.3333333					
31	Istanbul	Rostov	210	240	2000	13.33333333					
32											
33		Total Cost =	412866.6667								

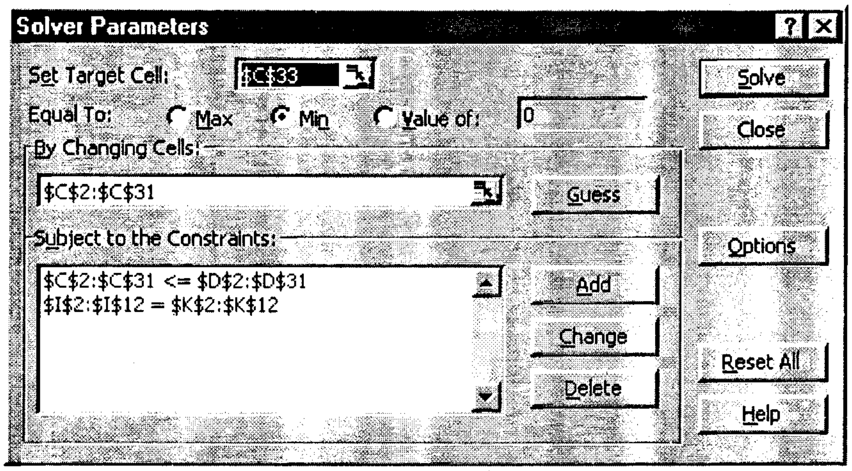
The following formulas appear in the spreadsheet.

	F
1	Unit Cost (in $1000 per 1000 tons)
2	=E2/240
3	=E3/240
4	=E4/240
5	=E5/150
6	=E6/150
7	=E7/150
8	=E8/240
9	=E9/240
10	=E10/240
11	=E11/150
12	=E12/150
13	=E13/150
14	=E14/16
15	=E15/16
16	=E16/16
17	=E17/150
18	=E18/150
19	=E19/150
20	=E20/16
21	=E21/16
22	=E22/16
23	=E23/150
24	=E24/150
25	=E25/150
26	=E26/16
27	=E27/16
28	=E28/16
29	=E29/150
30	=E30/150
31	=E31/150
32	

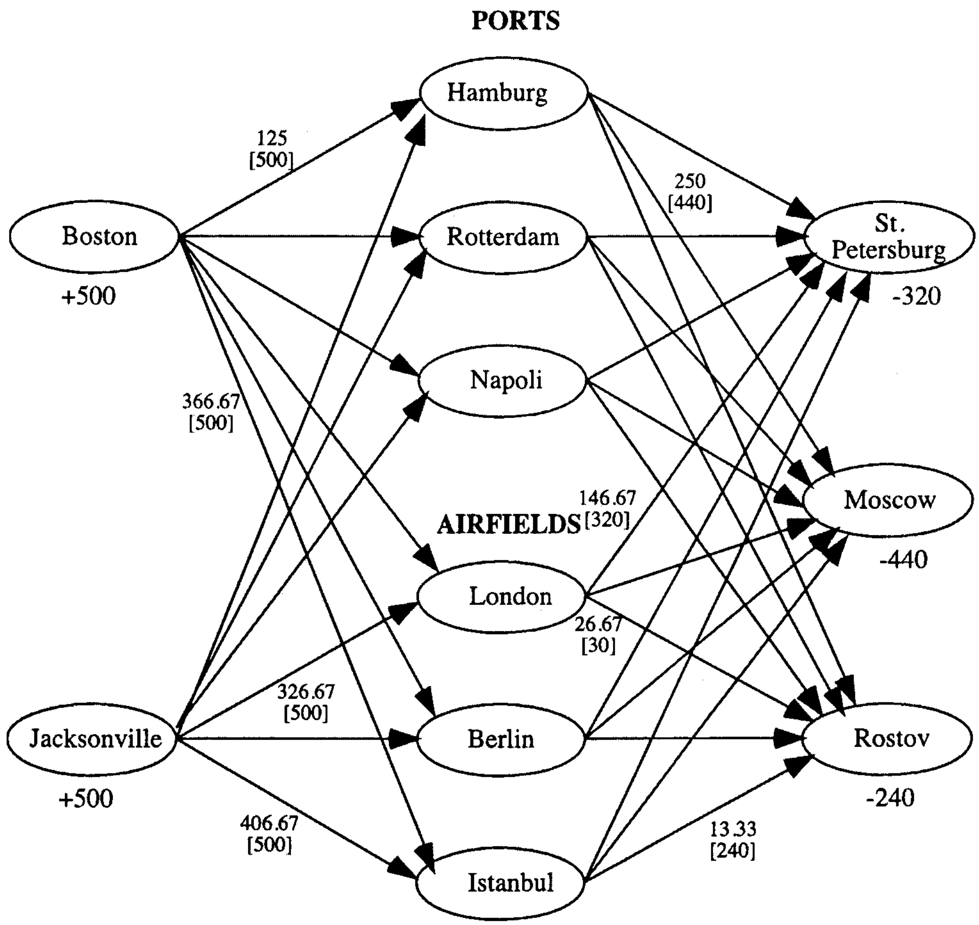
	I
1	Net Flow
2	=SUM(C2:C7)
3	=SUM(C8:C13)
4	=-C2-C8+C14+C20+C26
5	=-C3-C9+C15+C21+C27
6	=-C4-C10+C16+C22+C28
7	=-C5-C11+C17+C23+C29
8	=-C6-C12+C18+C24+C30
9	=-C7-C13+C19+C25+C31
10	=-SUM(C14:C19)
11	=-SUM(C20:C25)
12	=-SUM(C26:C31)
13	

	B	C
33	Total Cost =	=SUMPRODUCT(C2:C31,F2:F31)

We use the following solver dialogue box for this model.



The total cost of the operation equals $412,866,666.67. The entire supply for Saint Petersburg is supplied from Jacksonville via London. The entire supply for Moscow is supplied from Boston via Hamburg. Of the 240 (= 240,000 tons) demanded by Rostov, 60 are shipped from Boston via Istanbul, 150 are shipped from Jacksonville via Istanbul, and 30 are shipped from Jacksonville via London. The paths used to ship supplies to Saint Petersburg, Moscow, and Rostov are highlighted on the following network diagram.


PORTS
Hamburg
Rotterdam
Napoli
Boston
+500
125
[500]
366.67
[500]
250
[440]
St.
Petersburg
-320
AIRFIELDS
146.67
[320]
Moscow
-440
London
26.67
[30]
326.67
[500]
Jacksonville
+500
Berlin
Rostov
-240
406.67
[500]
Istanbul
13.33
[240]

d) Now the President wants to maximize the amount of cargo transported from the US to the Russian cities. In other words, the President wants to maximize the flow from the two US cities to the three Russian cities. All the nodes representing the European ports and airfields are once again transshipment nodes. The flow along an arc is again measured in thousands of tons. The new restrictions can be transformed into arc capacities using the same approach that was used in part (c). The objective is now to maximize the combined flow into the three Russian cities.

The linear programming model describing the maximum flow problem appears as follows.

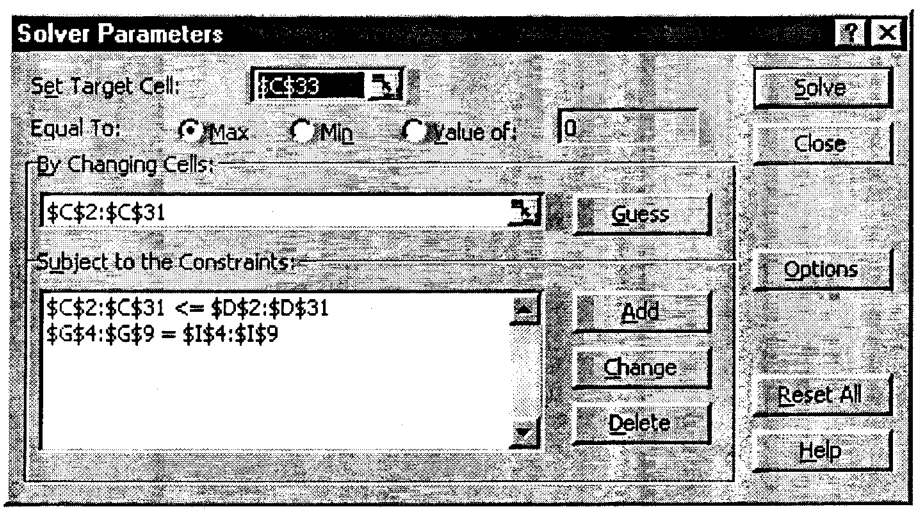
	A	B	C	D	E	F	G	H	I
1	From	To	Ship	Capacity (in 1000 tons)		Nodes	Net Flow		Supply/Demand
2	Boston	Hamburg	19.2	500		Boston	282.2		
3	Boston	Rotterdam	21.6	500		Jacksonville	240		
4	Boston	Napoli	46.4	500		Hamburg	0	=	0
5	Boston	London	75	75		Rotterdam	0	=	0
6	Boston	Berlin	45	45		Napoli	0	=	0
7	Boston	Istanbul	75	75		London	0	=	0
8	Jacksonville	Hamburg	0	500		Berlin	0	=	0
9	Jacksonville	Rotterdam	0	500		Istanbul	0	=	0
10	Jacksonville	Napoli	0	500		St. Petersburg	-225		
11	Jacksonville	London	90	90		Moscow	-104.8		
12	Jacksonville	Berlin	75	75		Rostov	-192.4		
13	Jacksonville	Istanbul	75	105					
14	Hamburg	St. Petersburg	0	0					
15	Rotterdam	St. Petersburg	0	0					
16	Napoli	St. Petersburg	0	0					
17	London	St. Petersburg	150	150					
18	Berlin	St. Petersburg	75	75					
19	Istanbul	St. Petersburg	0	0					
20	Hamburg	Moscow	11.2	11.2					
21	Rotterdam	Moscow	9.6	9.6					
22	Napoli	Moscow	24	24					
23	London	Moscow	0	30					
24	Berlin	Moscow	45	45					
25	Istanbul	Moscow	15	15					
26	Hamburg	Rostov	8	8					
27	Rotterdam	Rostov	12	12					
28	Napoli	Rostov	22.4	22.4					
29	London	Rostov	15	15					
30	Berlin	Rostov	0	0					
31	Istanbul	Rostov	135	135					
32									
33		Total Cost =	522.2						

The following formulas appear in the spreadsheet.

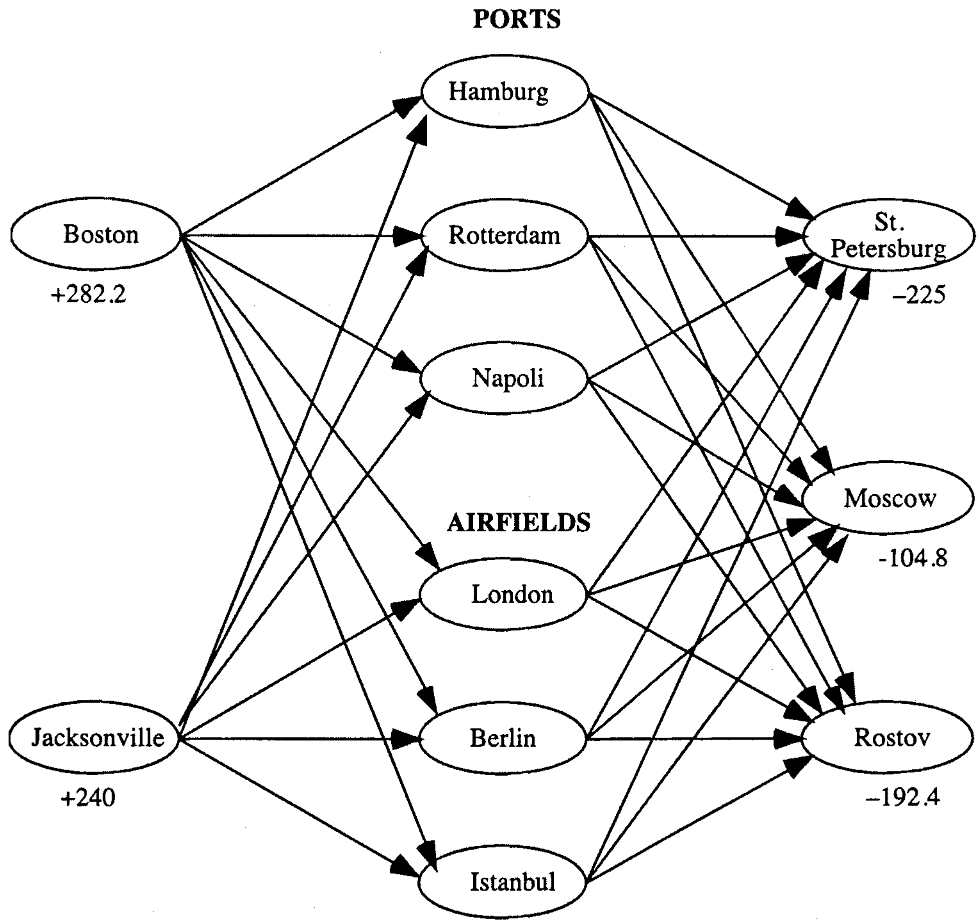
	G
1	Net Flow
2	=SUM(C2:C7)
3	=SUM(C8:C13)
4	=-C2-C8+C14+C20+C26
5	=-C3-C9+C15+C21+C27
6	=-C4-C10+C16+C22+C28
7	=-C5-C11+C17+C23+C29
8	=-C6-C12+C18+C24+C30
9	=-C7-C13+C19+C25+C31
10	=-SUM(C14:C19)
11	=-SUM(C20:C25)
12	=-SUM(C26:C31)
13	

	B	C
33	Total Cost =	=SUM(G2:G3)

We use the following solver dialogue box.



The worksheet shows all the amounts that are shipped between the various cities. The total supply for Saint Petersburg, Moscow, and Rostov equals 225,000 tons, 104,800 tons, and 192,400 tons, respectively. The following network diagram highlights the paths used to ship supplies between the US and the Russian Federation.



e) The creation of the new communications network is a minimum spanning tree problem. As usual, a greedy algorithm solves this type of problem.

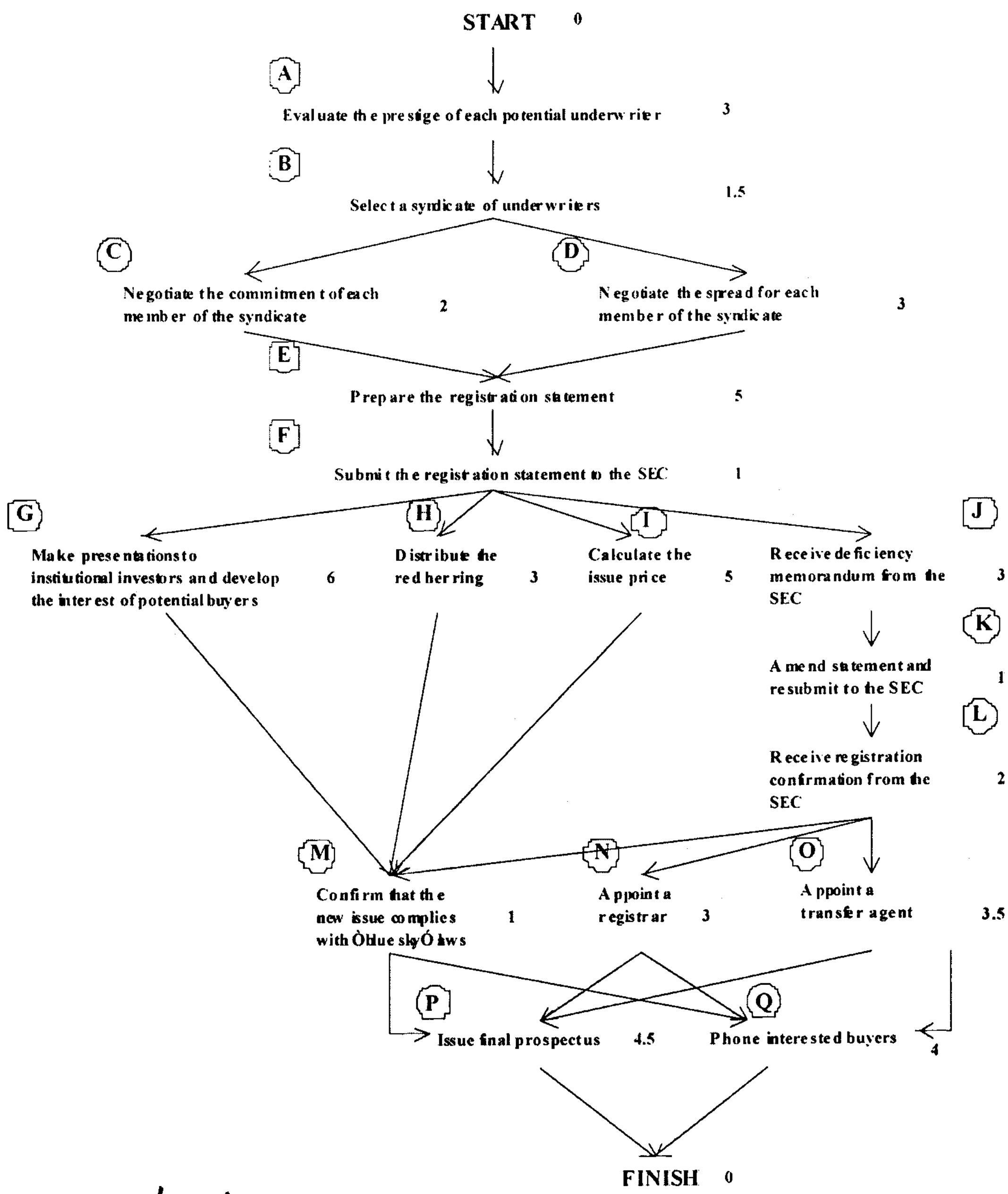
Arcs are added to the network in the following order (one of several optimal solutions):

Rostov - Orenburg	120
Ufa - Orenburg	75
Saratov - Orenburg	95
Saratov - Samara	100
Samara - Kazan	95
Ufa – Yekaterinburg	125
Perm – Yekaterinburg	85

The minimum cost of reestablishing the communication lines is $695,000.

Cases

9.3 a) A diagram of the project network appears below.



By inspection, the longest path and so the critical path is START → A → B → C → D → E → F → J → K → L → O → P → Finish. The length of this path and so the duration of the initial public offering process is 27.5 weeks.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Evaluate prestige	3	0	=E4+D4	=H4-D4	=G5	=H4-F4	=IF(I4=0,"Yes","No")
5		B	Select syndicate	1.5	=MAX(F4)	=E5+D5	=H5-D5	=MIN(G6,G7)	=H5-F5	=IF(I5=0,"Yes","No")
6		C	Negotiate commitment	2	=MAX(F5)	=E6+D6	=H6-D6	=G8	=H6-F6	=IF(I6=0,"Yes","No")
7		D	Negotiate spread	3	=MAX(F5)	=E7+D7	=H7-D7	=G8	=H7-F7	=IF(I7=0,"Yes","No")
8		E	Prepare registration	5	=MAX(F6,F7)	=E8+D8	=H8-D8	=G9	=H8-F8	=IF(I8=0,"Yes","No")
9		F	Submit registration	1	=MAX(F8)	=E9+D9	=H9-D9	=MIN(G10,G11,G12,G13)	=H9-F9	=IF(I9=0,"Yes","No")
10		G	Present	6	=MAX(F9)	=E10+D10	=H10-D10	=G16	=H10-F10	=IF(I10=0,"Yes","No"
11		H	Distribute red herring	3	=MAX(F9)	=E11+D11	=H11-D11	=G16	=H11-F11	=IF(I11=0,"Yes","No"
12		I	Calculate price	5	=MAX(F9)	=E12+D12	=H12-D12	=G16	=H12-F12	=IF(I12=0,"Yes","No"
13		J	Receive deficiency	3	=MAX(F9)	=E13+D13	=H13-D13	=G14	=H13-F13	=IF(I13=0,"Yes","No"
14		K	Amend statement	1	=MAX(F13)	=E14+D14	=H14-D14	=G15	=H14-F14	=IF(I14=0,"Yes","No"
15		L	Receive registration	2	=MAX(F14)	=E15+D15	=H15-D15	=MIN(G18,G17,G16)	=H15-F15	=IF(I15=0,"Yes","No"
16		M	Confirm blue sky	1	=MAX(F10,F11,F12,F15)	=E16+D16	=H16-D16	=MIN(G19,G20)	=H16-F16	=IF(I16=0,"Yes","No"
17		N	Appoint registrar	3	=MAX(F15)	=E17+D17	=H17-D17	=MIN(G19,G20)	=H17-F17	=IF(I17=0,"Yes","No"
18		O	Appoint transfer	3.5	=MAX(F15)	=E18+D18	=H18-D18	=MIN(G19,G20)	=H18-F18	=IF(I18=0,"Yes","No"
19		P	Issue prospectus	4.5	=MAX(F16,F17,F18)	=E19+D19	=H19-D19	=F22	=H19-F19	=IF(I19=0,"Yes","No"
20		Q	Phone buyers	4	=MAX(F16,F17,F18)	=E20+D20	=H20-D20	=F22	=H20-F20	=IF(I20=0,"Yes","No"
21										
22					Project Duration	=MAX(F19,F20)				

The values in the new spreadsheet appear below.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Evaluate prestige	3	0	3	0	3	0	Yes
5		B	Select syndicate	1.5	3	4.5	3	4.5	0	Yes
6		C	Negotiate commitment	2	4.5	6.5	5.5	7.5	1	No
7		D	Negotiate spread	3	4.5	7.5	4.5	7.5	0	Yes
8		E	Prepare registration	5	7.5	12.5	7.5	12.5	0	Yes
9		F	Submit registration	1	12.5	13.5	12.5	13.5	0	Yes
10		G	Present	6	13.5	19.5	16	22	2.5	No
11		H	Distribute red herring	3	13.5	16.5	19	22	5.5	No
12		I	Calculate price	5	13.5	18.5	17	22	3.5	No
13		J	Receive deficiency	3	13.5	16.5	13.5	16.5	0	Yes
14		K	Amend statement	1	16.5	17.5	16.5	17.5	0	Yes
15		L	Receive registration	2	17.5	19.5	17.5	19.5	0	Yes
16		M	Confirm blue sky	1	19.5	20.5	22	23	2.5	No
17		N	Appoint registrar	3	19.5	22.5	20	23	0.5	No
18		O	Appoint transfer	3.5	19.5	23	19.5	23	0	Yes
19		P	Issue prospectus	4.5	23	27.5	23	27.5	0	Yes
20		Q	Phone buyers	4	23	27	23.5	27.5	0.5	No
21										
22					Project Duration	27.5				

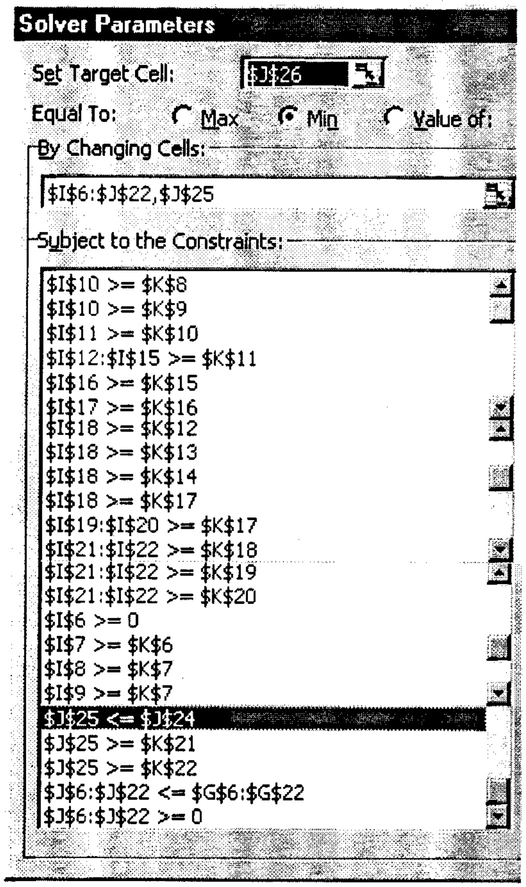
b) We formulate a linear programming problem to make the crashing decisions.

	A	B	C	D	E	F	G	H	I	J	K
3							Maximum	Crash Cost			
4			Time		Cost		Time	per Week	Start	Time	Finish
5		Activity	Normal	Crash	Normal	Crash	Reduction	saved	Time	Reduction	Time
6		A	3	1.5	8000	14000	=C6-D6	=(F6-E6)/G6	0	1.5	=I6+C6-J6
7		B	1.5	0.5	4500	8000	=C7-D7	=(F7-E7)/G7	1.5	1	=I7+C7-J7
8		C	2	2	9000	0	=C8-D8	0	2	0	=I8+C8-J8
9		D	3	3	12000	0	=C9-D9	0	2	0	=I9+C9-J9
10		E	5	4	50000	95000	=C10-D10	=(F10-E10)/G10	5	0	=I10+C10-J10
11		F	1	1	1000	0	=C11-D11	0	10	0	=I11+C11-J11
12		G	6	4	25000	60000	=C12-D12	=(F12-E12)/G12	11	0	=I12+C12-J12
13		H	3	2	15000	22000	=C13-D13	=(F13-E13)/G13	14	0	=I13+C13-J13
14		I	5	3.5	12000	31000	=C14-D14	=(F14-E14)/G14	11	0	=I14+C14-J14
15		J	3	3	0	0	=C15-D15	0	11	0	=I15+C15-J15
16		K	1	0.5	6000	9000	=C16-D16	=(F16-E16)/G16	14	0.5	=I16+C16-J16
17		L	2	2	0	0	=C17-D17	0	14.5	0	=I17+C17-J17
18		M	1	0.5	5000	8300	=C18-D18	=(F18-E18)/G18	17	0	=I18+C18-J18
19		N	3	1.5	12000	19000	=C19-D19	=(F19-E19)/G19	16.5	1.5	=I19+C19-J19
20		O	3.5	1.5	13000	21000	=C20-D20	=(F20-E20)/G20	16.5	2	=I20+C20-J20
21		P	4.5	2	40000	99000	=C21-D21	=(F21-E21)/G21	18	0.5	=I21+C21-J21
22		Q	4	1.5	9000	20000	=C22-D22	=(F22-E22)/G22	18	0	=I22+C22-J22
23											
24									Desired Finish	22	
25									Finish Time =	22	
26									Total Cost =	=SUM(E6:E22)+SUMPRODUCT(H6:H22,J6:J22)	

The values used in the spreadsheet appear below.

	A	B	C	D	E	F	G	H	I	J	K
3							Maximum	Crash Cost			
4			Time		Cost		Time	per Week	Start	Time	Finish
5		Activity	Normal	Crash	Normal	Crash	Reduction	saved	Time	Reduction	Time
6		A	3	1.5	$8000	$14000	1.5	$4000	0.0	1.5	1.5
7		B	1.5	0.5	$4500	$8000	1	$3500	1.5	1	2
8		C	2	2	$9000	$0	0	$0	2.0	0	4
9		D	3	3	$12000	$0	0	$0	2.0	0	5
10		E	5	4	$50000	$95000	1	$45000	5.0	0	10
11		F	1	1	$1000	$0	0	$0	10.0	0	11
12		G	6	4	$25000	$60000	2	$17500	11.0	0	17
13		H	3	2	$15000	$22000	1	$7000	14.0	0	17
14		I	5	3.5	$12000	$31000	1.5	$12667	11.0	0	16
15		J	3	3	$0	$0	0	$0	11.0	0	14
16		K	1	0.5	$6000	$9000	0.5	$6000	14.0	0.5	14.5
17		L	2	2	$0	$0	0	$0	14.5	0	16.5
18		M	1	0.5	$5000	$8300	0.5	$6600	17.0	0	18
19		N	3	1.5	$12000	$19000	1.5	$4667	16.5	1.5	18
20		O	3.5	1.5	$13000	$21000	2	$4000	16.5	2.0	18
21		P	4.5	2	$40000	$99000	2.5	$23600	18.0	0.5	22
22		Q	4	1.5	$9000	$20000	2.5	$4400	18.0	0	22
23											
24									Desired Finish	22	
25									Finish Time =	22	
26									Total Cost =	$260800	

The Solver settings for the linear programming appear below.



Janet and Gilbert should reduce the time for step A (evaluating the prestige of each potential underwriter) by 1.5 weeks, the time for step B (selecting a syndicate of underwriters) by one week, the time for step K (amending statement and resubmitting it to the SEC) by 0.5 weeks, the time for step N (appointing a registrar) by 1.5 weeks, the time for step O (appointing a transfer agent) by two weeks, and the time for step P (issuing final prospectus) by 0.5 weeks. Janet and Gilbert can now meet the new deadline of 22 weeks at a total cost of $260,800.

c) We use the same model formulation that was used in part (c). We change one constraint, however. The project duration now has to be greater than or equal to 24 weeks instead of 22 weeks. We obtain the following solution in Excel.

	A	B	C	D	E	F	G	H	I	J	K
3							Maximum	Crash Cost			
4			Time		Cost		Time	per Week	Start	Time	Finish
5		Activity	Normal	Crash	Normal	Crash	Reduction	saved	Time	Reduction	Time
6		A	3	1.5	$8000	$14000	1.5	$4000	0.0	1.5	1.5
7		B	1.5	0.5	$4500	$8000	1	$3500	1.5	1	2
8		C	2	2	$9000	$0	0	$0	2.0	0	4
9		D	3	3	$12000	$0	0	$0	2.0	0	5
10		E	5	4	$50000	$95000	1	$45000	5.0	0	10
11		F	1	1	$1000	$0	0	$0	10.0	0	11
12		G	6	4	$25000	$60000	2	$17500	12.5	0	18.5
13		H	3	2	$15000	$22000	1	$7000	15.5	0	18.5
14		I	5	3.5	$12000	$31000	1.5	$12667	11.0	0	16
15		J	3	3	$0	$0	0	$0	11.0	0	14
16		K	1	0.5	$6000	$9000	0.5	$6000	14.0	0.5	14.5
17		L	2	2	$0	$0	0	$0	14.5	0	16.5
18		M	1	0.5	$5000	$8300	0.5	$6600	18.5	0	19.5
19		N	3	1.5	$12000	$19000	1.5	$4667	16.5	0	19.5
20		O	3.5	1.5	$13000	$21000	2	$4000	16.5	0.5	19.5
21		P	4.5	2	$40000	$99000	2.5	$23600	19.5	0	24
22		Q	4	1.5	$9000	$20000	2.5	$4400	19.5	0	23.5
23											
24									Desired Finish	24	
25									Finish Time =	24	
26									Total Cost =	$236000	

Janet and Gilbert should reduce the time for step A (evaluating the prestige of each potential underwriter) by 1.5 weeks, the time for step B (selecting a syndicate of underwriters) by one week, the time for step K (amending statement and resubmitting it to the SEC) by 0.5 weeks, and the time for step O (appointing a transfer agent) by 0.5 weeks. Janet and Gilbert can now meet the new deadline of 24 weeks at a total cost of $236,000.

CHAPTER 10: DYNAMIC PROGRAMMING

10.2-1.

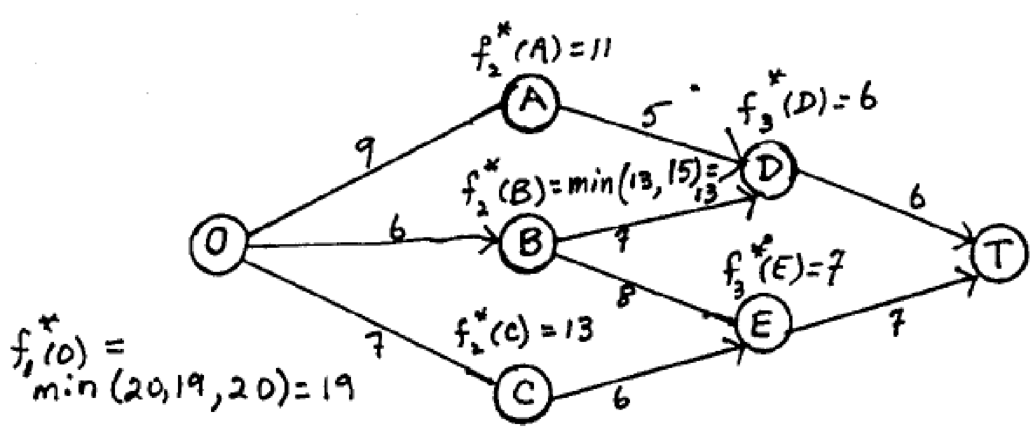
(a) The nodes of the network can be divided into "layers" such that the nodes in the nth layer are accessible from the origin only through the nodes in the $(n-1)$st layer. These layers define the stages of the problem, which can be labeled as $n = 1, 2, 3, 4$. The nodes constitute the states.

Let S_n denote the set of the nodes in the nth layer of the network, i.e., $S_1 = \{O\}$, $S_2 = \{A, B, C\}$, $S_3 = \{D, E\}$ and $S_4 = \{T\}$. The decision variable x_n is the immediate destination at stage n. Then the problem can be formulated as follows:

$f_n^*(s) = \min_{x_n \in S_{n+1}} [c_{sx_n} + f_{n+1}^*(x_n)] \equiv \min_{x_n \in S_{n+1}} f_n(s, x_n)$ for $s \in S_n$ and $n = 1, 2, 3$

$f_4^*(T) = 0$

(b) The shortest path is $O - B - D - T$.



(c) Number of stages: 3

s_3	$f_3^*(s)$	x_3^*
D	6	T
E	7	T

s_2	$f_2(s,D)$	$f_2(s,E)$	$f_2^*(s)$	x_2^*
A	11	–	11	D
B	13	15	13	D
C	–	13	13	E

s_1	$f_1(s,A)$	$f_1(s,B)$	$f_1(s,C)$	$f_1^*(s)$	x_1^*
O	20	19	20	19	B

Optimal Solution: $x_1^* = B$, $x_2^* = D$ and $x_3^* = D$.

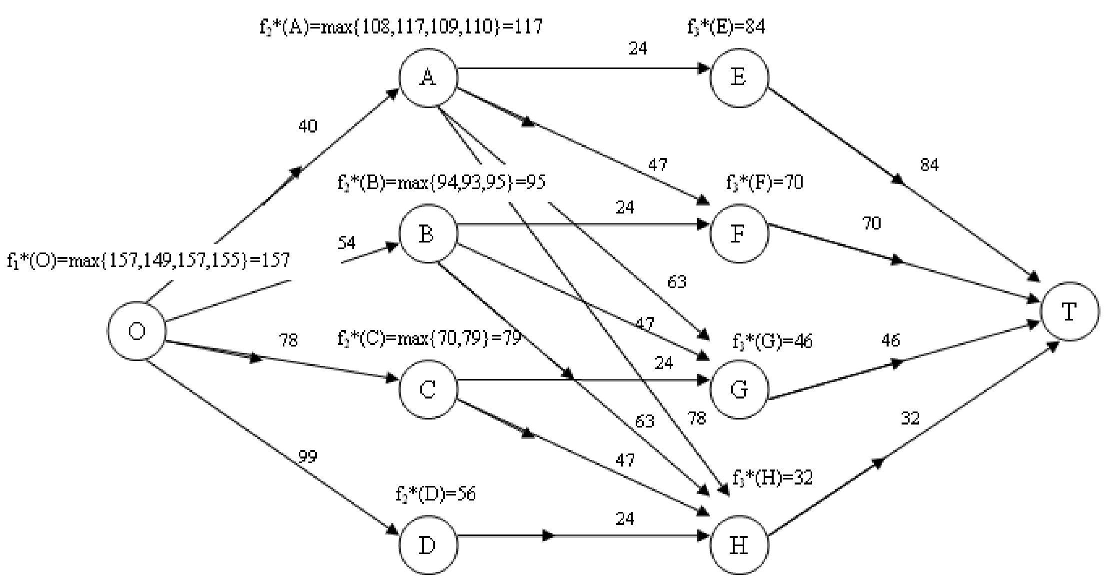
(d) Shortest-Path Algorithm:

	Solved nodes	Closest		nth	Distance to	
	directly connected	connected	total	nearest	nth nearest	Last
n	to unsolved nodes	unsolved node	distance	node	node	connection
1	O	B	6	B	6	OB
2	O	C	7	C	7	OC
	B	D	$6+7=13$			
3	O	A	9	A	9	OA
	B	D	$6+7=13$			
	C	E	$7+6=13$			
4	A	D	$9+5=14$	D	13	BD
	B	D	$6+7=13$			
	C	E	$7+6=13$	E		CE
5	D	T	$13+6=19$	T	19	DT
	E	T	$13+7=20$			

The shortest-path algorithm required 8 additions and 6 comparisons whereas dynamic programming required 7 additions and 3 comparisons. Hence, the latter seems to be more efficient for shortest-path problems with "layered" network graphs.

10.2-2.

(a)



The optimal routes are $O-A-F-T$ and $O-C-H-T$, the associated sales income is 157. The route $O-A-F-T$ corresponds to assigning 1, 2, and 3 salespeople to regions 1, 2, and 3 respectively. The route $O-C-H-T$ corresponds to assigning 3, 2, and 1 salespeople to regions 1, 2, and 3 respectively.

(b) The regions are the stages and the number of salespeople remaining to be allocated at stage n are possible states at stage n, say s_n. Let x_n be the number of salespeople assigned to region n and $c_n(x_n)$ be the increase in sales in region n if x_n salespeople are assigned to it. Number of stages: 3.

s_3	$f_3^*(s_3)$	x_3^*
1	32	1
2	46	2
3	70	3
4	84	4

	$f_2(s_2, x_2)$					
s_2	1	2	3	4	$f_2^*(s_2)$	x_2^*
2	56	–	–	–	56	1
3	70	79	–	–	79	2
4	94	93	95	–	95	3
5	108	117	109	110	117	2

	$f_1(s_1, x_1)$					
s_1	1	2	3	4	$f_1^*(s_1)$	x_1^*
6	157	149	157	155	157	$1, 3$

The optimal solutions are $(x_1^* = 1, x_2^* = 2, x_3^* = 3)$ and $(x_1^* = 3, x_2^* = 2, x_3^* = 1)$.

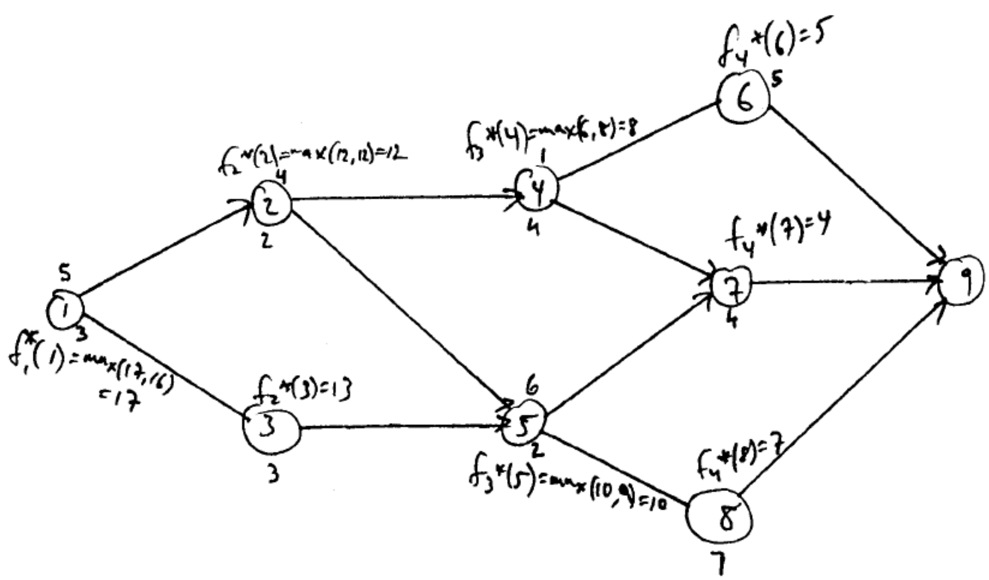
10.2-3.

(a) The five stages of the problem correspond to the five columns of the network graph. The states are the nodes of the graph. Given the activity times t_{ij}, the problem can be formulated as follows:

$$f_n^*(s) = \max_{x_n} \left[t_{sx_n} + f_{n+1}^*(x_n) \right]$$

$$f_6^*(9) = 0$$

(b) The critical paths are $1 \to 2 \to 4 \to 7 \to 9$ and $1 \to 2 \to 5 \to 7 \to 9$.



(c) Interactive Deterministic Dynamic Programming Algorithm: Number of stages: 4

s_4	$f_4^*(s_4)$	x_4^*
1	5	1
2	4	1
3	7	1

s_3 \ x_3	$f_3(s_3,x_3)$ 1	2	$f_3^*(s_3)$	x_3^*
1	6	8	8	2
2	10	9	10	1

s_2 \ x_2	$f_2(s_2,x_2)$ 1	2	$f_2^*(s_2)$	x_2^*
1	12	12	12	1,2
2	13	---	13	1

s_1 \ x_1	$f_1(s_1,x_1)$ 1	2	$f_1^*(s_1)$	x_1^*
1	17	16	17	1

Optimal Sol.	x_1^*	x_2^*	x_3^*	x_4^*
1	1	1	2	1
2	1	2	1	1

10.2-4.

(a) FALSE. It uses a recursive relationship that enables solving for the optimal policy for stage n given the optimal policy for stage $(n+1)$ [Feature 7, Section 10.2, p.446].

(b) FALSE. Given the current state, an optimal policy for remaining stages is independent of the policy decisions adopted in previous stages. Therefore, the optimal immediate decision depends on only the current state and not on how you got there. This is the Principle of Optimality for dynamic programming [Feature 5, Section 10.2, p.446].

(c) FALSE. The optimal decision at any stage depends on only the state at that stage and not on the past. This is again the Principle of Optimality [Feature 5, Section 10.2, p.446].

10.3-1.

The Military Airlift Command (MAC) employed dynamic programming in scheduling its aircraft, crew and mission support resources during Operation Desert Storm. The primary goal was to deliver cargo and passengers on time in an environment with time and space constraints. The missions are scheduled sequentially. The schedule of a mission imposes resource constraints on the schedules of following missions. A balance among various objectives is sought. In addition to maximizing timely deliveries, MAC aimed at reducing late deliveries, total flying time of each mission, ground time and frequency of crew changes. Maximizing on-time deliveries is included in the model as a lower bound on the load of the mission. The problem for any given mission is then to determine a feasible schedule that minimizes a weighted sum of the remaining objectives. The constraints are the lower bound constraints, crew and ground-support availability constraints. Stages are the airfields in the network and states are defined as airfield, departure time, and remaining duty day. The solution of the problem is made more efficient by exploiting the special structure of the objective function.

The software developed to solve the problems cost around \$2 million while the airlift operation cost over \$3 billion. Hence, even a small improvement in efficiency meant a considerable return on investment. A systematic approach to scheduling yielded better coordination, improved efficiency, and error-proof schedules. It enabled MAC not only to respond quickly to changes in the conditions, but also to be proactive by evaluating different scenarios in short periods of time.

10.3-2.

Let x_n be the number of crates allocated to store n, $p_n(x_n)$ be the expected profit from allocating x_n to store n and s_n be the number of crates remaining to be allocated to stores $k \geq n$. Then $f_n^*(s_n) = \max_{0 \leq x_n \leq s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)]$. Number of stages: 3

S3	f3*(S3)	X3*
0	0	0
1	4	1
2	9	2
3	13	3
4	18	4
5	20	5

S2 \ X2	f2(S2, X2): 0	1	2	3	4	5	f2*(S2)	X2*
0	0	---	---	---	---	---	0	0
1	4	6	---	---	---	---	6	1
2	9	10	11	---	---	---	11	2
3	13	15	15	15	---	---	15	1,2,3
4	18	19	20	19	19	---	20	2
5	20	24	24	24	23	22	24	1,2,3

S1 \ X1	f1(S1, X1): 0	1	2	3	4	5	f1*(S1)	X1*
5	24	25	24	25	23	21	25	1,3

Optimal solution	X1*	X2*	X3*
1	1	2	2
2	3	2	0

10.3-3.

Let x_n be the number of study days allocated to course n, $p_n(x_n)$ be the number of grade points expected when x_n days are allocated to course n and s_n be the number of study days remaining to be allocated to courses $k \geq n$. Then

$$f_n^*(s_n) = \max_{1 \leq x_n \leq \min(s_n, 4)} [p_n(x_n) + f_{n+1}^*(s_n - x_n)].$$

Number of stages: 4

s_4	$f_4^*(s_4)$	x_4^*
1	4	1
2	4	2
3	5	3
4	8	4

	$f_3(s_3, x_3)$					
s_3	1	2	3	4	$f_3^*(s_3)$	x_3^*
2	8	–	–	–	8	1
3	8	10	–	–	10	2
4	9	10	11	–	11	3
5	12	11	11	13	13	4

	$f_2(s_2, x_2)$					
s_2	1	2	3	4	$f_2^*(s_2)$	x_2^*
3	13	–	–	–	13	1
4	15	14	–	–	15	1
5	16	16	16	–	16	$1, 2, 3$
6	18	17	18	16	18	$1, 3$

	$f_1(s_1, x_1)$					
s_1	1	2	3	4	$f_1^*(s_1)$	x_1^*
7	19	19	21	21	21	$3, 4$

Optimal Solution	x_1^*	x_2^*	x_3^*	x_4^*
1	3	1	2	1
2	4	1	1	1

10.3-4.

Let x_n be the number of commercials run in area n, $p_n(x_n)$ be the number of votes won when x_n commercials are run in area n and s_n be the number of commercials remaining to be allocated to areas $k \geq n$. Then

$$f_n^*(s_n) = \max_{0 \leq x_n \leq s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)].$$

Number of stages: 4

S4	f4*(S4)	X4*
0	0	0
1	3	1
2	7	2
3	12	3
4	14	4
5	16	5

S3 \ X3	f3(S3, X3) 0	1	2	3	4	5	f3*(S3)	X3*
0	0	---	---	---	---	---	0	0
1	3	5	---	---	---	---	5	1
2	7	8	9	---	---	---	9	2
3	12	12	12	11	---	---	12	0,1,2
4	14	17	16	14	10	---	17	1
5	16	19	21	18	13	9	21	2

S2 \ X2	f2(S2, X2) 0	1	2	3	4	5	f2*(S2)	X2*
0	0	---	---	---	---	---	0	0
1	5	6	---	---	---	---	6	1
2	9	11	8	---	---	---	11	1
3	12	15	13	10	---	---	15	1
4	17	18	17	15	11	---	18	1
5	21	23	20	19	16	12	23	1

S1 \ X1	f1(S1, X1) 0	1	2	3	4	5	f1*(S1)	X1*
5	23	22	22	20	18	15	23	0

Optimal solution	X1*	X2*	X3*	X4*
1	0	1	1	3

10.3-5.

Let x_n be the number of workers allocated to precinct n, $p_n(x_n)$ be the increase in the number of votes if x_n workers are assigned to precinct n and s_n be the number of workers remaining at stage n. Then

$$f_n^*(s_n) = \max_{0 \le x_n \le s_n} [p_n(x_n) + f_{n+1}^*(s_n - x_n)].$$

Number of stages: 4

S4	f4*(S4)	X4*
0	0	0
1	6	1
2	11	2
3	14	3
4	15	4
5	17	5
6	18	6

S3 \ X3	f3(S3, X3) 0	1	2	3	4	5	6	f3*(S3)	X3*
0	0	---	---	---	---	---	---	0	0
1	6	5	---	---	---	---	---	6	0
2	11	11	10	---	---	---	---	11	0,1
3	14	16	16	15	---	---	---	16	1,2
4	16	19	21	21	18	---	---	21	2,3
5	17	21	24	26	24	21	---	26	3
6	18	22	26	29	29	27	22	29	3,4

S2 \ X2	f2(S2, X2) 0	1	2	3	4	5	6	f2*(S2)	X2*
0	0	---	---	---	---	---	---	0	0
1	6	7	---	---	---	---	---	7	1
2	11	13	11	---	---	---	---	13	1
3	16	18	17	16	---	---	---	18	1
4	21	23	22	22	18	---	---	23	1
5	26	28	27	27	24	20	---	28	1
6	29	33	32	32	29	26	21	33	1

S1 \ X1	f1(S1, X1) 0	1	2	3	4	5	6	f1*(S1)	X1*
6	33	32	32	33	31	29	24	33	0,3

Optimal solution	X1*	X2*	X3*	X4*
1	0	1	3	2
2	3	1	0	2
3	3	1	1	1

10.3-6.

Let $5x_n$ be the number of jet engines produced in month n and s_n be the inventory at the beginning of month n. Then $f_n^*(s_n)$ is:

$$\min_{\max(r_n - s_n, 0) \leq x_n \leq m_n} [c_n x_n + d_n \max(s_n + x_n - r_n, 0) + f_{n+1}^*(\max(s_n + x_n - r_n, 0))]$$

and $f_4^*(s_4) = c_4 \max(s_4 - r_4, 0)$.

Using the following data adjusted to reflect that x_n is one fifth of the actual production,

Month	r_n	m_n	c_n	d_n
1	2	5	5.40	0.075
2	3	7	5.55	0.075
3	5	6	5.50	0.075
4	4	2	5.65	0.075

the following tables are produced:

s_4	$f_4^*(s_4)$	x_4^*
2	11.30	2
3	5.65	1
4	0.00	0

	$f_3(s_3, x_3)$								
s_3	0	1	2	3	4	5	6	$f_3^*(s_3)$	x_3^*
1	–	–	–	–	–	–	44.45	44.45	6
2	–	–	–	–	–	38.95	38.875	38.875	6
3	–	–	–	–	33.45	33.375	33.30	33.30	6
4	–	–	–	27.95	27.875	27.80	–	27.80	5
5	–	–	22.45	22.375	22.30	–	–	22.30	4
6	–	16.95	16.875	16.80	–	–	–	16.80	3
7	11.45	11.375	11.30	–	–	–	–	11.30	2

	$f_2(s_2, x_2)$									
s_2	0	1	2	3	4	5	6	7	$f_2^*(s_2)$	x_2^*
0	–	–	–	–	66.725	66.775	66.825	66.95	66.725	4
1	–	–	–	61.175	61.225	61.275	61.40	61.525	61.175	3
2	–	–	55.625	55.675	55.725	55.85	55.975	56.10	55.625	2
3	–	50.075	50.125	50.175	50.30	50.425	50.55	50.675	50.075	1

	$f_1(s_1, x_1)$							
s_1	0	1	2	3	4	5	$f_1^*(s_1)$	x_1^*
0	–	–	77.525	77.45	77.375	77.30	77.30	5

Hence, the optimal production schedule is to produce $5 \cdot 5 = 25$ units in the first month, $1 \cdot 5 = 5$ in the second, $6 \cdot 5 = 30$ in the third and $2 \cdot 5 = 10$ in the last month.

10.3-7.

(a) Let x_n be the amount in million dollars spent in phase n, s_n be the amount in million dollars remaining, $p_1(x_1)$ be the initial share of the market attained in phase 1 when x_1 is spent in phase 1, and $p_n(x_n)$ be the fraction of this market share retained in phase n if x_n is spent in phase n, for $n = 2, 3$. Number of stages: 3

s_3	$f_3^*(s_3)$	x_3^*
0	0.3	0
1	0.5	1
2	0.6	2
3	0.7	3

	$f_2(s_2, x_2)$					
s_2	0	1	2	3	$f_2^*(s_2)$	x_2^*
0	0.06	–	–	–	0.06	0
1	0.1	0.12	–	–	0.12	1
2	0.12	0.2	0.15	–	0.2	1
3	0.14	0.24	0.25	0.18	0.25	2

	$f_1(s_1, x_1)$					
s_1	1	2	3	4	$f_1^*(s_1)$	x_1^*
4	5	6	4.8	3	6	2

The optimal solution is $x_1^* = 2$, $x_2^* = 1$, and $x_3^* = 1$. Hence, it is optimal to spend two million dollars in phase 1 and one million dollar in each one the phases 2 and 3. This will result in a final market share of 6%.

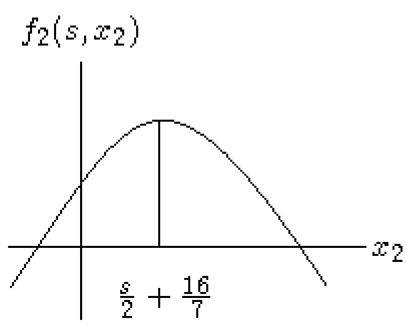
(b) Phase 3:

s	$f_3^*(s)$	x_3^*
$0 \leq s \leq 4$	$0.6 + 0.07s$	s

Phase 2: $f_2(s, x_2) = (0.4 + 0.1x_2)[0.6 + 0.07(s - x_2)]$

$$= -0.07x_2^2 + (0.07s + 0.032)x_2 + (0.24 + 0.028s)$$

$$\frac{\partial f_2(s,x_2)}{\partial x_2} = -0.014x_2 + 0.007s + 0.032 = 0 \;\Rightarrow x_2^* = \frac{s}{2} + \frac{16}{7}$$



If $s \leq \frac{s}{2} + \frac{16}{7}$: $x_2^* = s$ because $f_2(s, x_2)$ is strictly increasing on the interval $[0, \frac{s}{2} + \frac{16}{7}]$, so on $[0, s]$.

If $s > \frac{s}{2} + \frac{16}{7}$: $x_2^* = \frac{s}{2} + \frac{16}{7}$ because then the global maximizer is feasible.

We can summarize this result as:

$$x_2^*(s) = \min\left(\frac{s}{2} + \frac{16}{7}, s\right).$$

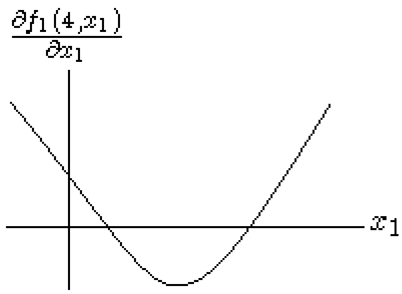
Now since $0 \leq s \leq 4 \leq \frac{32}{7}$, $s \leq \frac{s}{2} + \frac{16}{7}$, so $x_2^*(s) = s$ and $f_2^*(s) = 0.06s + 0.24$.

<u>Phase 1:</u> $f_1(4, x_1) = (10x_1 - x_1^2)[0.06(4 - x_1) + 0.24]$

$$= 0.06x_1^3 - 1.08x_1^2 + 4.8x_1$$

$$\frac{\partial f_1(4,x_1)}{\partial x_1} = 0.18x_1^2 - 2.16x_1 + 4.8 = 0$$

$$\Rightarrow x_1^* = \frac{2.16 \pm \sqrt{2.16^2 - 4(0.18)(4.8)}}{2(0.18)} = 2.945 \text{ or } 9.055.$$



The derivative of $f_1(4, x_1)$ is nonnegative for $x_1 \leq 2.945$ and $x_1 \geq 9.055$ and nonpositive otherwise, so $f_1(4, x_1)$ is nonincreasing on the interval $[2.945, 9.055]$, and nondecreasing else. Thus, $f_1(4, x_1)$ attains its maximum over the interval $[0, 4]$ at $x_1^* = 2.945$ with $f_1^*(4) = 6.302$. Accordingly, it is optimal to spend 2.945 million dollars in Phase 1, 1.055 in Phase 2 and Phase 3. This returns a market share of 6.302%.

10.3-8.

Let x_n be the number of parallel units of component n that are installed, $p_n(x_n)$ be the probability that the component will function if it contains x_n parallel units, $c_n(x_n)$ be the cost of installing x_n units of component n, s_n be the amount of money remaining in hundreds of dollars. Then

$$f_n^*(s_n) = \max_{x_n = 0, \ldots, \min(3, \alpha_{s_n})} [p_n(x_n) f_{n+1}^*(s_n - c_n(x_n))]$$

where $\alpha_{s_n} \equiv \max\{\alpha : c_n(\alpha) \leq s_n, \alpha \text{ integer}\}$.

s_4	$f_4^*(s_4)$	x_4^*
$0, 1$	0	0
2	0.5	1
3	0.7	2
$4 \leq s_4 \leq 10$	0.9	3

$f_3(s_3, x_3) = \mathrm{P}_3(x_3) f_4^*(s_3 - c_3(x_3))$

	$f_3(s_3,x_3)$					
s_3	0	1	2	3	$f_3^*(s_3)$	x_3^*
0	0	–	–	–	0	0
$1,2$	0	0	–	–	0	$0,1$
3	0	0.35	0	–	0.35	1
4	0	0.49	0	0	0.49	1
5	0	0.63	0.40	0	0.63	1
6	0	0.63	0.56	0.45	0.63	1
7	0	0.63	0.72	0.63	0.72	2
$8 \leq s_3 \leq 10$	0	0.63	0.72	0.81	0.81	3

$f_2(s_2,x_2) = \mathrm{P}_2(x_2) f_3^*(s_2 - c_2(x_2))$

	$f_2(s_2,x_2)$					
s_2	0	1	2	3	$f_2^*(s_2)$	x_2^*
$0,1$	0	–	–	–	0	0
$2,3$	0	0	–	–	0	$0,1$
4	0	0	0	–	0	$0,1,2$
5	0	0.210	0	0	0.210	1
6	0	0.294	0	0	0.294	1
7	0	0.378	0.245	0	0.378	1
8	0	0.378	0.343	0.280	0.378	1
9	0	0.432	0.441	0.392	0.441	2
10	0	0.486	0.441	0.504	0.504	3

$f_1(s_1,x_1) = \mathrm{P}_1(x_1) f_2^*(s_1 - c_1(x_1))$

	$f_1(s_1,x_1)$					
s_1	0	1	2	3	$f_1^*(s_1)$	x_1^*
10	0	0.22	0.227	0.302	0.302	3

The optimal solution is $x_1^* = 3$, $x_2^* = 1$, $x_3^* = 1$ and $x_4^* = 3$, yielding a system reliability of 0.3024.

10.3-9.

The stages are $n = 1,2$ and the state is the amount of slack remaining in the constraint, the goal is to find $f_1^*(4)$.

s_2	$f_2^*(s_2)$	x_2^*
0	0	0
1	0	0
2	4	1
3	4	1
4	12	2

	$f_1(s_1,x_1)$						
s_1	0	1	2	3	4	$f_1^*(s_1)$	x_1^*
4	12	6	8	0	-16	12	0

The optimal solution is $x_1^* = 0$ and $x_2^* = 2$.

10.3-10.

The stages are $n = 1, 2, 3$ and the state is the slack remaining in the constraint, the goal is to find $f_1^*(20)$.

s_3	$f_3^*(s_3)$	x_3^*
$0-4$	0	0
$5-9$	20	1
$10-14$	40	2
$15-19$	60	3
20	80	4

	$f_2(s_2, x_2)$				
s_2	0	1	2	$f_2^*(s_2)$	x_2^*
$0-4$	0	$-$	$-$	0	0
$5-6$	20	$-$	$-$	20	0
$7-9$	20	30	$-$	30	1
$10-11$	40	30	$-$	40	0
$12-13$	40	50	$-$	50	1
14	40	50	60	60	2
$15-16$	60	50	60	60	$0, 2$
$17-18$	60	70	60	70	1
19	60	70	80	80	2
20	80	70	80	80	$0, 2$

	$f_1(s_1, x_1)$								
s_1	0	1	2	3	4	5	6	$f_1^*(s_1)$	x_1^*
20	80	100	116	118	126	130	120	130	5

The optimal solution is $x_1^* = 5$, $x_2^* = 0$, $x_3^* = 1$ with an objective value $z^* = 130$.

10.3-11.

Let s_n denote the slack remaining in the constraint.

$f_2^*(s_2) = \max_{0 \le x_2 \le s_2} \left(36x_2 - 3x_2^3\right)$

$$\frac{\partial f_2(s, x_2)}{\partial x_2} = 36 - 9x_2^2 \begin{cases} > 0 & \text{for } 0 \le x_2 < 2 \\ = 0 & \text{for } x_2 = 2 \\ < 0 & \text{for } x_2 > 2 \end{cases} \quad \Rightarrow x_2^* = \begin{cases} s_2 & \text{for } 0 \le s_2 < 2 \\ 2 & \text{for } 2 \le s_2 \le 3 \end{cases}$$

$$f_1^*(3) = \max_{0 \le x_1 \le 3} [36x_1 + 9x_1^2 - 6x_1^3 + f_2^*(3 - x_1)]$$

$$= \max \begin{cases} \max_{0 \le x_1 \le 1} [36x_1 + 9x_1^2 - 6x_1^3 + 48] \\ \max_{1 \le x_1 \le 3} [36x_1 + 9x_1^2 - 6x_1^3 + 36(3 - x_1) - 3(3 - x_1)^3] \end{cases}$$

$$\frac{\partial f_1(3, x_1)}{\partial x_1} = \begin{cases} -18(x_1^2 - x_1 - 2) > 0 & \text{for } 0 \le x_1 \le 1 \ \Rightarrow x_1^{\text{max}} = 1 \\ -9(x_1^2 + 4x_1 - 9) \begin{cases} > 0 & \text{for } 1 \le x_1 < -2 + \sqrt{13} \\ = 0 & \text{for } x_1 = -2 + \sqrt{13} \\ < 0 & \text{for } x_1 > -2 + \sqrt{13} \end{cases} & \Rightarrow x_1^{\text{max}} = -2 + \sqrt{13} \end{cases}$$

Since $f_1(3, 1) < f_1(3, -2 + \sqrt{13})$, $x_1^* = -2 + \sqrt{13} \simeq 1.61$ and $x_2^* = 5 - \sqrt{13} \simeq 1.39$ with the optimal objective value being $f_1^*(3) \simeq 98.23$.

10.3-12.

$f_n^*(s_n) = \min_{r_n \leq x_n \leq 255} [100(x_n - s_n)^2 + 2000(x_n - r_n) + f_{n+1}^*(x_n)]$

$\underline{n=4}$:

s_4	$f_4^*(s_4)$	x_4^*
$200 \leq s_4 \leq 255$	$100(255-s_4)^2$	255

$\underline{n=3}$: $f_3(s_3, x_3) = 100(x_3 - s_3)^2 + 2000(x_3 - 200) + 100(255 - x_3)^2$

$$\frac{\partial f_3(s_3,x_3)}{\partial x_3} = 200(x_3 - s_3) + 2000 - 200(255 - x_3)$$

$$= 200[2x_3 - (s_3 + 245)] = 0 \Rightarrow x_3 = \frac{s_3+245}{2}$$

If $155 \leq s_3 \leq 265$, $200 \leq \frac{s_3+245}{2} \leq 255$, so $x_3 = \frac{s_3+245}{2}$ is feasible for $240 \leq s_3 \leq 255$ and $f_3^*(s_3) = 25(245 - s_3)^2 + 25(265 - s_3)^2 + 1000(s_3 - 155)$.

s_3	$f_3^*(s_3)$	x_3^*
$240 \leq s_3 \leq 255$	$25(245-s_3)^2 + 25(265-s_3)^2 + 1000(s_3 - 155)$	$\frac{s_3+245}{2}$

$\underline{n=2}$: $f_2(s_2, x_2) = 100(x_2 - s_2)^2 + 2000(x_2 - 240) + f_3^*(x_2)$

$$\frac{\partial f_2(s_2,x_2)}{\partial x_2} = 200(x_2 - s_2) + 2000 - 50(245 - x_2) - 50(265 - x_2) + 1000$$

$$= 100[3x_2 - (2s_2 + 225)] = 0 \Rightarrow x_2 = \frac{2s_2+225}{3}$$

If $247.5 \leq s_2 \leq 255$, $240 \leq \frac{2s_2+225}{3} \leq 255$, so $x_2^* = \frac{2s_2+225}{3}$ and

$$f_2^*(s_2) = 100\left(\frac{2s_2+225}{3} - s_2\right)^2 + 2000\left(\frac{2s_2+225}{3} - 240\right) + f_3^*\left(\frac{2s_2+225}{3}\right)$$

$$= \frac{100}{9}[(225 - s_2)^2 + (255 - s_2)^2 + (285 - s_2)^2 + 60(3s_2 - 615)].$$

If $220 \leq s_2 \leq 247.5$, $\frac{2s_2+225}{3} \leq 240 \leq x_2$, so $\frac{\partial f_2(s_2,x_2)}{\partial x_2} \geq 0$ and hence $x_2^* = 240$ and

$f_2^*(s_2) = 100(240 - s_2)^2 + 2000(240 - 240) + f_3^*(240) = 100(240 - s_2)^2 + 101,250.$

s_2	$f_2^*(s_2)$	x_2^*
$220 \leq s_2 \leq 247.5$	$100(240-s_2)^2 + 101,250$	240
$247.5 \leq s_2 \leq 255$	$\frac{100}{9}[(225-s_2)^2 + (255-s_2)^2 + (285-s_2)^2 + 60(3s_2 - 615)]$	$\frac{2s_2+225}{3}$

$\underline{n=1}$: $f_1(255, x_1) = 100(x_1 - 255)^2 + 2000(x_1 - 220) + f_2^*(x_1)$

If $220 \leq x_1 \leq 247.5$:

$$\frac{\partial f_2(255,x_1)}{\partial x_1} = 200(x_1 - 485) = 0 \Rightarrow x_1^* = 242.5.$$

If $247.5 \leq x_1 \leq 255$:

$$\frac{\partial f_2(255,x_1)}{\partial x_1} = \frac{800}{3}(x_1 - 240) > 0 \Rightarrow x_1^* = 247.5.$$

The optimal solution is $x_1^* = 242.5$ and

$f_1^*(255) = 100(242.5 - 255)^2 + 2000(242.5 - 220) + f_2^*(242.5) = 162,500.$

s_1	$f_1^*(s_1)$	x_1^*
255	$162,500$	242.5

Summer	Autumn	Winter	Spring
242.5	240	242.5	255

10.3-13.

Let s_n be the amount of the resource remaining at beginning of stage n.

<u>$n=3$:</u> $\max\limits_{0\leq x_3\leq s_3}(4x_3-x_3^2)$

$\frac{\partial}{\partial x_3}(4x_3-x_3^2)=4-2x_3=0\Rightarrow x_3^*=2$

$\frac{\partial^2}{\partial x_3^2}(4x_3-x_3^2)=-2<0\Rightarrow x_3^*=2$ is a maximum.

s_3	$f_3^*(s_3)$	x_3^*
$0\leq s_3\leq 2$	$4s_3-s_3^2$	s_3
$2\leq s_3\leq 4$	4	2

<u>$n=2$:</u> $\max\limits_{0\leq x_2\leq s_2}[2x_2+f_3^*(s_2-x_2)]$

If $0\leq s_2-x_2\leq 2$: $\max\limits_{0\leq x_2\leq s_2}[2x_2+4(s_2-x_2)-(s_2-x_2)^2]$

$\frac{\partial}{\partial x_2}[2x_2+4(s_2-x_2)-(s_2-x_2)^2]=-2+2s_2-2x_2=0\Rightarrow x_2^*=s_2-1$

$\frac{\partial^2}{\partial x_2^2}[2x_2+4(s_2-x_2)-(s_2-x_2)^2]=-2<0\Rightarrow x_2^*=s_2-1$ is a maximum.

$f_2^*(s_2)=2s_2+1.$

If $2\leq s_2-x_2\leq 4$: $\max\limits_{0\leq x_2\leq s_2}(2x_2+4)$, $x_2^*=s_2-2$ and $f_2^*(s_2)=2s_2<2s_2+1.$

s_2	$f_2^*(s_2)$	x_2^*
$0\leq s_2\leq 1$	$4s_2-s_2^2$	0
$1\leq s_2\leq 4$	$2s_2+1$	s_2-1

<u>$n=1$:</u> $\max\limits_{0\leq x_1\leq s_1}[2x_1^2+f_2^*(4-2x_1)]$

If $0\leq 4-2x_1\leq 1$: $\max\limits_{0\leq x_1\leq s_1}[2x_1^2+4(4-2x_1)-(4-2x_1)^2]=(-2x_1^2+8x_1)$

$\frac{\partial}{\partial x_1}(-2x_1^2+8x_1)=-4x_1+8=0\Rightarrow x_1^*=2$

$\frac{\partial^2}{\partial x_1^2}(-2x_1^2+8x_1)=-4<0\Rightarrow x_1^*=2$ is a maximum.

$f_1(4,2)=8.$

If $1\leq 4-2x_1\leq 4$: $\max\limits_{0\leq x_1\leq s_1}[2x_1^2+2(4-2x_1)+1]=(2x_1^2-4x_1+9)$

$\frac{\partial}{\partial x_1}(2x_1^2-4x_1+9)=4x_1-4=0\Rightarrow x_1=1$

$\frac{\partial^2}{\partial x_1^2}(2x_1^2 - 4x_1 + 9) = 4 > 0 \Rightarrow x_1 = 1$ is a minimum.

Corner points: $1 = 4 - 2x_1 \Rightarrow x_1 = 3/2$, $f_1(4, 3/2) = 7.5$

$4 = 4 - 2x_1 \Rightarrow x_1 = 0$, $f_1(4, 0) = 9$ is maximum.

Hence, $x_1^* = 0$, $x_2^* = 3$, $x_3^* = 1$ and $f_1^*(4) = 9$.

10.3-14.

<u>$n = 2$:</u> $\min_{x_2^2 \geq s_2} 2x_2^2 \Rightarrow x_2^* = \sqrt{s_2}$ and $f_2^*(s_2) = 2s_2$,

where s_2 represents the amount of 2 used by x_2^2.

<u>$n = 1$:</u> $\min_{x_1}[x_1^4 + f_2^*((2 - x_1^2)^+)] = [x_1^4 + 2(2 - x_1^2)^+]$,

where $(2 - x_1^2)^+ = \max\{0, 2 - x_1^2\}$.

If $x_1^2 \leq 2$: $\frac{\partial}{\partial x_1}(x_1^4 + 4 - 2x_1^2) = 4x_1^3 - 4x_1 = 0 \Rightarrow x_1 = 0, 1, -1$.

$\frac{\partial^2}{\partial x_1^2}(x_1^4 + 4 - 2x_1^2) = 12x_1^2 - 4$

$x_1 = 0$, $\frac{\partial^2}{\partial x_1^2}(x_1^4 + 4 - 2x_1^2) = -4 < 0$, so $x_1 = 0$ is a local maximum.

$x_1 = 1, -1$, $\frac{\partial^2}{\partial x_1^2}(x_1^4 + 4 - 2x_1^2) = 8 > 0$, so $x_1 = 1, -1$ are local minima with $z = 3$.

If $x_1^2 \geq 2$: $x_1 = 0$ and $z = 4 > 3$.

Hence, $(x_1^*, x_2^*) \in \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$, all with $z^* = 3$.

10.3-15.

(a) Let $s_n \in \{1, 2, 4\}$ be the remaining factor 4 entering stage n.

<u>$n = 3$:</u>

s_3	$f_3^*(s_3)$	x_3^*
1	16	1
2	32	2
4	64	3

<u>$n = 2$:</u>

	$f_2(s_2, x_2)$				
s_2	1	2	4	$f_2^*(s_2)$	x_2^*
1	20	–	–	20	1
2	36	32	–	36	1
4	68	48	80	80	4

<u>$n = 1$:</u>

	$f_1(s_1, x_1)$				
s_1	1	2	4	$f_1^*(s_1)$	x_1^*
4	81	44	84	84	4

The optimal solution is $(x_1^*, x_2^*, x_3^*) = (4, 1, 1)$ with $z^* = 84$.

(b) As in part (a), let s_n be the remaining factor (not necessarily integer) at stage n.

$f_3^*(s_3) = 16s_3$ and $x_3^* = s_3$

$f_2^*(s_2) = \max_{1 \le x_2 \le s_2} \{4x_2^2 + f_3^*(s_2/x_2)\} = \max_{1 \le x_2 \le s_2} \{4x_2^2 + 16s_2/x_2\}$

$\frac{\partial f_2(s_2,x_2)}{\partial x_2} = 4x_2 - 16s_2/x_2^2$ and $\frac{\partial^2 f_2(s_2,x_2)}{\partial x_2^2} = 4 + 32s_2/x_2^3 > 0$

when s_2, $x_2 \ge 0$. Thus $f_2(s_2,x_2)$ is convex in x_2 when s_2, $x_2 \ge 0$. The maximum should occur at one of the endpoints.

$x_2 = 1$, $f_2(s_2,1) = 4 + 16s_2$

$x_2 = s_2$, $f_2(s_2,s_2) = 4s_2^2 + 16$

$4 + 16s_2 \ge 4s_2^2 + 16 \Leftrightarrow (s_2 - 3)(s_2 - 1) \le 0 \Leftrightarrow 1 \le s_2 \le 3$

$x_2^* = \begin{cases} 1 & \text{if } 1 \le s_2 \le 3 \\ s_2 & \text{if } 3 \le s_2 \le 4 \end{cases}$ and $f_2^*(s_2) = \begin{cases} 4 + 16s_2 & \text{if } 1 \le s_2 \le 3 \\ 4s_2^2 + 16 & \text{if } 3 \le s_2 \le 4 \end{cases}$

$f_1^*(s_1) = \max_{1 \le x_1 \le 4} \{x_1^3 + f_2^*(4/x_1)\}$

$= \max\left\{ \max_{1 \le x_1 \le 4/3} \left\{x_1^3 + 4\left(\frac{16}{x_1^2}\right) + 16\right\}, \max_{4/3 \le x_1 \le 4} \left\{x_1^3 + 4 + 16\left(\frac{4}{x_1}\right)\right\}\right\}$

$\frac{\partial^2}{\partial x_1^2}\left\{x_1^3 + 4\left(\frac{16}{x_1^2}\right) + 16\right\} = 6x_1 + 204/x_1^4 > 0$ when $x_1 \ge 0$

$\frac{\partial^2}{\partial x_1^2}\left\{x_1^3 + 4 + 16\left(\frac{4}{x_1}\right)\right\} = 6x_1 + 128/x_1^2 > 0$ when $x_1 \ge 0$

Hence, the maximum occurs at an endpoint.

$x_1 = 1$, $f_1(s_1,1) = 81$

$x_1 = 4/3$, $f_1(s_1,4/3) \approx 54.37$

$x_1 = 4$, $f_1(s_1,4) = 84$

$f_1^*(s_1) = \max\{81, 54.37, 84\} = 84$ and $(x_1^*, x_2^*, x_3^*) = (4,1,1)$, just as when the variables are restricted to be integers.

10.3-16.

Let s_n be the slack remaining in the constraint $x_1 - x_2 + x_3 \le 1$, entering the nth stage.

$f_3^*(s_3) = \max_{0 \le x_3 \le s_3} x_3 = s_3$ and $x_3^* = s_3$

$f_2^*(s_2) = \max_{s_2^- \le x_2} \{(1 - x_2) f_3^*(s_2 + x_2)\} = \max_{s_2^- \le x_2} \{(1 - x_2)(s_2 + x_2)\}$

where $s_2^- = \max\{-s_2, 0\}$.

$\frac{\partial f_2(s_2,x_2)}{\partial x_2} = -2x_2 - (s_2 - 1) = 0 \Rightarrow x_2 = (1 - s_2)/2$

$\frac{\partial^2 f_2(s_2,x_2)}{\partial x_2^2} = -2 < 0$, so $f_2(s_2,x_2)$ is concave in x_2.

$x_2 = (s_2 - 1)/2$, $f_2(s_2,(1 - s_2)/2) = (1 + s_2)^2/4$

$x_2 = s_2^-$, $f_2(s_2,s_2^-) = \begin{cases} 0 & \text{if } s_2 \le 0 \\ s_2 & \text{if } s_2 \ge 0 \end{cases}$

$(1+s_2)^2/2 \geq \max\{0, s_2\}$

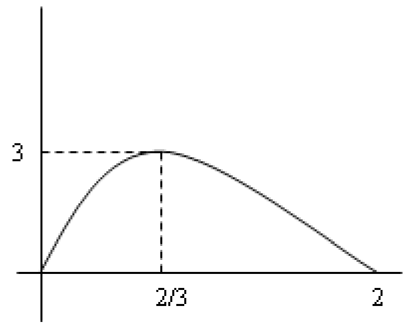
$x_2 = (1-s_2)/2$ is feasible if and only if $s_2^- \leq (1-s_2)/2$, equivalently when $s_2 \geq -1$.

$f_2^*(s_2) = \begin{cases} 0 & \text{if } s_2 \leq -1 \\ (1+s_2)^2/4 & \text{if } s_2 \geq -1 \end{cases}$ and $x_2^* = \begin{cases} s_2^- = -s_2 & \text{if } s_2 \leq -1 \\ (1-s_2)/2 & \text{if } s_2 \geq -1 \end{cases}$

$$f_1^*(s_1) = \max_{x_1 \geq 0}\{x_1 f_2^*(1-x_1)\} = \max\left\{\max_{0\leq x_1\leq 2}\left\{x_1\left(\frac{x_1^2}{4} + (1-x_1)\right)\right\}, 0\right\}$$

$$= \max_{0\leq x_1 \leq 2}\left\{\frac{x_1^3}{4} - x_1^2 + x_1\right\}$$

$$\frac{\partial}{\partial x_1}\left\{\frac{x_1^3}{4} - x_1^2 + x_1\right\} = \frac{3x_1^2}{4} - 2x_1 + 1 = 0 \Rightarrow x_1 = \frac{2\pm\sqrt{4-3}}{3/2} = \frac{4}{3} \pm \frac{2}{3}$$



Hence, $(x_1^*, x_2^*, x_3^*) = (2/3, 1/3, 2/3)$ and $z^* = 8/27$.

10.3-17.

Let $s = (R_1, R_2)$, where R_i is the slack in the ith constraint.

<u>$n=2$:</u> $f_2(R_1, R_2, x_2) = 2x_2$, $0 \leq x_2 \leq \min\{R_1/2, R_2\}$

s	$f_2^*(s)$	x_2^*
$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$	$10\min\{R_1/2, R_2\}$	$\min\{R_1/2, R_2\}$

<u>$n=1$:</u> $f_1(6,8,x_1) = 15x_1 + f_2^*(6-x_1, 8-3x_1)$

$$= 15x_1 + 10\min\{(6-x_1)/2, 8-3x_1\}, \text{ for } 0 \leq x_1 \leq 8/3$$

$$= \begin{cases} 10x_1 + 30 & \text{if } 0 \leq x_1 \leq 2 \\ 80 - 15x_1 & \text{if } 2 \leq x_1 \leq 8/3 \end{cases}$$

$$\max_{0\leq x_1\leq 8/3} f_1(6,8,x_1) = \max\left\{\max_{0\leq x_1 \leq 2} f_1(6,8,x_1), \max_{2\leq x_1\leq 8/3} f_1(6,8,x_1)\right\} = 50$$

and $x_1^* = 2$.

The optimal solution is $(x_1^*, x_2^*) = (2, 2)$ and $z^* = 50$.

10.3-18.

Let $s = (R_1, R_2)$, where R_i is the slack in the ith constraint.

$$f_3(R_1, R_2, x_3) = \begin{cases} 0 & \text{if } x_3 = 0 \\ -1 + x_3 & \text{if } x_3 > 0 \end{cases}$$

$$f_3^*(R_1, R_2) = \max\left\{0, \max_{0 \le x_3 \le R_1/2} (-1 + x_3)\right\} = \max\{0, -1 + (R_1/2)\}$$

$$= \begin{cases} -1 + (R_1/2) & \text{if } 0 \le R_1 \le 2 \\ 0 & \text{if } R_1 \ge 2 \end{cases}$$

$$x_3^* = \begin{cases} R_1/2 & \text{if } 0 \le R_1 \le 2 \\ 0 & \text{if } R_1 \ge 2 \end{cases}$$

$$f_2(R_1, R_2, x_2) = 7x_2 + f_3^*(R_1 - 3x_2, R_2)$$

$$= \begin{cases} 7x_2 - 1 + (R_1 - 3x_2)/2 & \text{if } 0 \le R_1 - 3x_2 \le 2 \\ 7x_2 & \text{if } R_1 - 3x_2 \ge 2 \end{cases}$$

$$f_2^*(R_1, R_2) = \max_{0 \le x_2 \le \min\{R_1/3, R_2\}} \left\{0, \max_{0 \le x_3 \le R_1/2} (-1 + x_3)\right\} = \max\{0, -1 + (R_1/2)\}$$

$$= \begin{cases} \frac{7R_1}{3} & \text{if } \frac{R_1}{3} \le R_2 \\ 7R_2 & \text{if } \frac{R_1 - 2}{3} \le R_2 \le \frac{R_1}{3} \\ \frac{17R_2}{2} - 1 + \frac{R_1}{2} & \text{if } R_2 \le \frac{R_1 - 2}{3} \end{cases}$$

$$x_2^* = \begin{cases} \frac{R_1}{3} & \text{if } \frac{R_1}{3} \le R_2 \\ R_2 & \text{if } \frac{R_1 - 2}{3} \le R_2 \le \frac{R_1}{3} \\ R_2 & \text{if } R_2 \le \frac{R_1 - 2}{3} \end{cases}$$

$$f_1^*(6, 5) = \max_{0 \le x_1 \le 5} [3x_1 + f_2^*(6 - x_1, 5 - x_1)]$$

$$= \max\left\{ \max_{0 \le x_1 \le 9/2} \left[3x_1 + \frac{7(6 - x_1)}{3}\right], \max_{9/2 \le x_1 \le 5} [3x_1 + 7(5 - x_1)] \right\}$$

$$= \max\left\{ \max_{0 \le x_1 \le 9/2} \left[\frac{2x_1}{3} + 14\right], \max_{9/2 \le x_1 \le 5} [35 - 2x_1] \right\} = 17$$

The optimal solution is $(x_1^*, x_2^*, x_3^*) = \left(\frac{9}{2}, \frac{1}{2}, 0\right)$ and $z^* = 17$.

10.4-1.

Let s_n be the current fortune of the player, A be the event to have \$100 at the end and X_n be the amount bet at the nth match.

$$f_3^*(s_3) = \max_{0 \le x_3 \le s_3} \{\mathrm{P}\{A | s_3\}\}$$

$0 \le s_3 < 50$, $f_3^*(s_3) = 0$.

$50 \leq s_3 < 100$, $f_3^*(s_3) = \begin{cases} 0 & \text{if } x_3^* \neq 100 - s_3 \\ 1/2 & \text{if } x_3^* = 100 - s_3 \end{cases}$

$s_3 = 100$, $f_3^*(s_3) = \begin{cases} 0 & \text{if } x_3^* > 0 \\ 1 & \text{if } x_3^* = 0 \end{cases}$

$s_3 > 100$, $f_3^*(s_3) = \begin{cases} 0 & \text{if } x_3^* \neq s_3 - 100 \\ 1/2 & \text{if } x_3^* = s_3 - 100 \end{cases}$

s_3	$f_3^*(s_3)$	x_3^*
$0 \leq s_3 < 50$	0	$0 \leq x_3^* \leq 50$
$50 \leq s_3 < 100$	1/2	$100 - s_3$
$s_3 = 100$	1	0
$100 < s_3$	1/2	$s_3 - 100$

$$f_2^*(s_2) = \max_{0 \leq x_2 \leq s_2} \left[\tfrac{1}{2} f_3^*(s_2 - x_2) + \tfrac{1}{2} f_3^*(s_2 + x_2) \right]$$

s_2	$f_2^*(s_2)$	x_2^*
$0 \leq s_2 < 25$	**0**	$0 \leq x_2 \leq s_2$
$25 \leq s_2 < 50$	0	$0 \leq x_2 \leq 50 - s_2$
	1/4	$50 - s_2 \leq x_2 \leq s_2$
$s_2 = 50$	1/4	$0 \leq x_2 < 50$
	1/2	$x_2 = 50$
$50 < s_2 < 75$	**1/2**	$0 \leq x_2 < s_2 - 50$
	1/4	$s_2 - 50 < x_2 < 100 - s_2$
	1/2	$x_2 = 100 - s_2$
	1/4	$100 - s_2 < x_2 \leq s_2$
$s_2 = 75$	1/2	$0 \leq x_2 < 25$
	3/4	$x_2 = 25$
	1/4	$25 \leq x_2 \leq 75$
$75 < s_2 < 100$	1/2	$0 \leq x_2 < 100 - s_2$
	3/4	$x_2 = 100 - s_2$
	1/2	$100 - s_2 < x_2 \leq s_2 - 50$
	1/4	$s_2 - 50 < x_2 \leq s_2$
$s_2 = 100$	**1**	$x_2 = 0$
	1/2	$0 < x_2 \leq 50$
	1/4	$50 \leq x_2 \leq 100$
$100 < s_2$	1/2	$0 \leq x_2 \leq s_2 - 100$
	3/4	$x_2 = s_2 - 100$
	1/2	$s_2 - 100 < x_2 \leq s_2 - 50$
	1/4	$s_2 - 50 < x_2 \leq s_2$

The entries in bold represent the maximum value in each case.

$$f_1^*(75) = \max_{0 \leq x_1 \leq 75} \left[\tfrac{1}{2} f_2^*(75 - x_1) + \tfrac{1}{2} f_2^*(75 + x_1) \right]$$

$$f_1(75, x_1) = \begin{cases} 3/4 & \text{if } x_1 = 0 \\ 5/8 & \text{if } 0 < x_1 < 25 \\ 3/4 & \text{if } x_1 = 25 \\ 1/2 & \text{if } 25 < x_1 \leq 50 \\ 3/8 & \text{if } 50 < x_1 \leq 75 \end{cases}$$

s_1	$f_1^*(s_1)$	x_1^*
75	$3/4$	0 or 25

Policy	x_1	won 1st bet	lost 1st bet	won 2nd bet	lost 2nd bet
1	0	25	25	0	50
2	25	0	50	0	0

10.4-2.

(a) Let $x_n \in \{0, A, B\}$ be the investment made in year n, s_n be the amount of money on hand at the beginning of year n and $f_n(s_n, x_n)$ be the maximum expected amount of money by the end of the third year given s_n and x_n.

For $0 \leq s_n < 10,000$, since one cannot invest less than \$$10,000$, $f_n(s_n, x_n) = f_{n+1}^*(s_n)$ and $x_n^* = 0$.

For $s_n \geq 10,000$,

$$f_n(s_n, x_n) = \begin{cases} f_{n+1}^*(s_n) & \text{if } x_n = 0 \\ 0.25 f_{n+1}^*(s_n - 10,000) + 0.75 f_{n+1}^*(s_n + 10,000) & \text{if } x_n = A \\ 0.9 f_{n+1}^*(s_n) + 0.1 f_{n+1}^*(s_n + 10,000) & \text{if } x_n = B \end{cases}$$

s_3	$f_3^*(s_3)$	x_3^*
$0 \leq s_3 < 10,000$	s_3	0
$s_3 \geq 10,000$	$s_3 + 5,000$	A

	$f_2(s_2, x_2)$				
s_2	0	A	B	$f_2^*(s_2)$	x_2^*
$0 \leq s_2 < 10,000$	s_2	$-$	$-$	s_2	0
$10,000 \leq s_2 < 20,000$	$s_2 + 5,000$	$s_2 + 8,750$	$s_2 + 6,000$	$s_2 + 8,750$	A
$s_2 \geq 20,000$	$s_2 + 5,000$	$s_2 + 10,000$	$s_2 + 6,000$	$s_2 + 10,000$	A

	$f_1(s_1, x_1)$				
s_1	0	A	B	$f_1^*(s_1)$	x_1^*
$10,000$	$18,750$	$22,500$	$19,875$	$22,500$	B

The optimal policy is to invest in A as long as there is enough money. The expected fortune after three years using this strategy is \$22,500.

(b) Let x_n and s_n be defined as in (a). Let $f_n(s_n, x_n)$ be the maximum probability of having at least \$20,000 after 3 years given s_n and x_n.

	$f_3(s_3, x_3)$				
s_3	0	A	B	$f_3^*(s_3)$	x_3^*
$0 \leq s_3 < 10,000$	0	–	–	0	0
$10,000 \leq s_3 < 20,000$	0	0.75	0.1	0.75	A
$20,000 \leq s_3 < 30,000$	1	0.75	1	1	$0, B$
$s_3 \geq 30,000$	1	1	1	1	$0, A, B$

	$f_2(s_2, x_2)$				
s_2	0	A	B	$f_2^*(s_2)$	x_2^*
$0 \leq s_2 < 10,000$	0	–	–	0	0
$10,000 \leq s_2 < 20,000$	0.75	0.75	0.775	0.775	B
$s_2 \geq 20,000$	1	0.75	1	1	$0, B$

	$f_1(s_1, x_1)$				
s_1	0	A	B	$f_1^*(s_1)$	x_1^*
$10,000$	0.775	0.75	0.7975	0.7975	B

With this objective, there is a number of optimal policies. The optimal action in the first period is to invest in B. If the return from it is only \$$10,000$, one is indifferent between investing in B or not investing at all in the second year. Depending on the second year's investment choice and its return, third year's starting budget can be either \$$10,000$, \$$20,000$ or \$$30,000$. If it is \$$10,000$, then it is best to invest it in A. If it is \$$20,000$, investing in B or not investing are best. Finally if it is \$$30,000$, anything is optimal, since \$$20,000$ is guaranteed. Using this policy, the probability of having at least \$$20,000$ by the end of the third year is 0.7975.

10.4-3.

$$\begin{aligned} f_n(1, x_n) &= K(x_n) + x_n + \left(\tfrac{1}{3}\right)^{x_n} f_{n+1}^*(1) + \left[1 - \left(\tfrac{1}{3}\right)^{x_n}\right] f_{n+1}^*(0) \\ &= K(x_n) + x_n + \left(\tfrac{1}{3}\right)^{x_n} f_{n+1}^*(1) \end{aligned}$$

since $f_n^*(0) = 0$ for every n. $f_3^*(1) = 16$, $f_3^*(0) = 0$ and $K(x_n) = 0$ if $x_n = 0$, $K(x_n) = 3$ if $x_n > 0$.

	$f_2(s_2, x_2)$						
s_2	0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	–	–	–	–	0	0
1	16	9.33	6.78	6.59	7.20	6.59	3

	$f_1(s_1, x_1)$						
s_1	0	1	2	3	4	$f_1^*(s_1)$	x_1^*
1	6.59	6.20	5.73	6.24	7.08	5.73	2

The optimal policy is to produce two in the first run and to produce three in the second run if none of the items produced in the first run is acceptable. The minimum expected cost is \$573.

10.4-4.

$f_n^*(s_n) = \max_{x_n \geq 0} \left\{ \frac{1}{3} f_{n+1}^*(s_n - x_n) + \frac{2}{3} f_{n+1}^*(s_n + x_n) \right\},$

with $f_6^*(s_6) = 0$ for $s_6 < 5$ and $f_6^*(s_6) = 1$ for $s_6 \geq 5$.

s_5	$f_5^*(s_5)$	x_5^*
0	0	0
1	0	0
2	0	0
3	$2/3$	$x_5^* \geq 2$
4	$2/3$	$x_5^* \geq 1$
$s_5 \geq 5$	1	$x_5^* \leq s_5 - 5$

	$f_4(s_4, x_4)$						
s_4	0	1	2	3	4	$f_4^*(s_4)$	x_4^*
0	0	–	–	–	–	0	0
1	0	0	–	–	–	0	0
2	0	$4/9$	$4/9$	–	–	$4/9$	$1, 2$
3	$2/3$	$4/9$	$2/3$	$2/3$	–	$2/3$	$0.2, 3$
4	$2/3$	$8/9$	$2/3$	$2/3$	$2/3$	$8/9$	1
$s_4 \geq 5$	1	–	–	–	–	1	$x_4^* \leq s_4 - 5$

	$f_3(s_3, x_3)$						
s_3	0	1	2	3	4	$f_3^*(s_3)$	x_3^*
0	0	–	–	–	–	0	0
1	0	$8/27$	–	–	–	$8/27$	1
2	$4/9$	$4/9$	$16/27$	–	–	$16/27$	2
3	$2/3$	$20/27$	$2/3$	$2/3$	–	$20/27$	1
4	$8/9$	$8/9$	$22/27$	$2/3$	$2/3$	$22/27$	$0, 1$
$s_3 \geq 5$	1	–	–	–	–	1	$x_3^* \leq s_3 - 5$

	$f_2(s_2, x_2)$						
s_2	0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	–	–	–	–	0	0
1	$8/27$	$32/81$	–	–	–	$32/81$	1
2	$16/27$	$48/81$	$48/81$	–	–	$48/81$	$0, 1, 2$
3	$20/27$	$64/81$	$62/81$	$2/3$	–	$64/81$	1
4	$24/27$	$74/81$	$70/81$	$62/81$	$2/3$	$74/81$	1
$s_2 \geq 5$	1	–	–	–	–	1	$x_2^* \leq s_2 - 5$

	$f_1(s_1, x_1)$				
s_1	0	1	2	$f_1^*(s_1)$	x_1^*
2	$48/81$	$160/243$	$124/243$	$160/243$	1

The probability of winning the bet using the policy given above is $160/243 = 0.658$.

10.4-5.

Let $x_n \in \{A, D\}$ denote the decision variable of quarter $n = 1, 2, 3$, where A and D represent advertising or discontinuing the product respectively. Let s_n be the level of sales (in millions) above ($s_n \geq 0$) or below ($s_n \leq 0$) the break-even point for quarter $(n-1)$. Let $f_n(s_n, x_n)$ represent the maximum expected discounted profit (in millions) from the beginning of quarter n onwards given the state s_n and decision x_n.

$f_n(s_n, x_n) = -30 + 5\left[s_n + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} t dt\right] + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} f_{n+1}^*(s_n + t) t dt,$

where a_n and b_n are given in the table that follows.

n	a_n	b_n
1	1	5
2	0	4
3	-1	3

For $1 \leq n \leq 3$,

$f_n(s_n, A) = -30 + 5\left[s_n + \frac{a_n + b_n}{2}\right] + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} f_{n+1}^*(s_n + t) dt,$

$f_n(s_n, D) = -20.$

Note that once discontinuing is chosen the process stops.

$f_n^*(s_n) = \max\{f_n(s_n, A), f_n(s_n, D)\}$

<u>$n = 4$:</u>

$f_4^*(s_4) = \begin{cases} -20 & \text{if } s_4 < 0 \\ 40 s_4 & \text{if } s_4 \geq 0 \end{cases}$

<u>$n = 3$:</u>

$f_3(s_3, D) = -20$

$f_3(s_3, A) = -30 + 5(s_3 + 1) + \frac{1}{4}\int_{-1}^{3} f_4^*(s_3 + t) dt,$

For $-3 \leq s_3 \leq 1$,

$f_3(s_3, A) = -30 + 5(s_3 + 1) + \frac{1}{4}\left[\int_{-1}^{-s_3} -20 dt + \int_{-s_3}^{3} 40(s_3 + t) dt\right] = 5(s_3 + 4)^2 - 65$

$f_3^*(s_3) = \max\{5(s_3 + 4)^2 - 65, -20\} = \begin{cases} -20 & \text{if } -3 \leq s_3 \leq -1 \text{, and } x_3^* = D, \\ 5(s_3 + 4)^2 - 65 & \text{if } -1 \leq s_3 \leq 1 \text{, and } x_3^* = A. \end{cases}$

For $1 \leq s_3 \leq 5$,

$f_3(s_3, A) = -30 + 5(s_3 + 1) + \frac{1}{4}\int_{-1}^{3} 40(s_3 + t) dt = 15 + 45 s_3$

$f_3^*(s_3) = \max\{15 + 45 s_3, -20\} = 15 + 45 s_3$ and $x_3^* = A$.

s_3	$f_3^*(s_3)$	x_3^*
$-3 \le s_3 \le -1$	-20	D
$-1 \le s_3 \le 1$	$5(s_3+4)^2-65$	A
$1 \le s_3 \le 5$	$15+45s_3$	A

<u>$n=2$:</u>

$f_2(s_2, D) = -20$

$f_2(s_2, A) = -30 + 5(s_2+1) + \frac{1}{4}\int_{-1}^{3} f_3^*(s_2+t)dt,$

For $-3 \le s_2 \le -1$,

$\int_{-1}^{3} f_3^*(s_2+t)dt = \int_{-1}^{-s_2-1} -20dt + \int_{-s_2-1}^{1-s_2}[5(s_2+t+4)^2-65]dt + \int_{1-s_2}^{4}[15+45(s_2+t)]dt$

$f_2(s_2, A) = \frac{5}{4}(\frac{9}{2}s_2^2 + 47s_2 + \frac{427}{6})$

Observe that $f_2(-3, A) = -110/3 < f_2(s_2, D) = -20 < f_2(-1, A) = 215/6$, so we need to find $-3 \le s_2 \le -1$ such that $f_2(s_2, A) = f_2(s_2, D)$.

$\frac{5}{4}(\frac{9}{2}s_2^2 + 47s_2 + \frac{427}{6}) = -20$ & $-3 \le s_2 \le -1 \Rightarrow s_2^* = \frac{-47+8\sqrt{10}}{9} = -2.411$

For $-1 \le s_2 \le 1$,

$\int_{-1}^{3} f_3^*(s_2+t)dt = \int_{0}^{1-s_2}[5(s_2+t+4)^2-65]dt + \int_{1-s_2}^{4}[15+45(s_2+t)]dt$

$f_2(s_2, A) = \frac{5}{4}\left[-\frac{1}{3}(s_2+4)^3 + \frac{9}{2}(s_2+4)^2 + 20s_2 + \frac{103}{6}\right]$

Since $f_2(-1, A) = 215/6$ and $f_2(s_2, A)$ is increasing in $-1 \le s_2 \le 1$, $x_2^* = A$ is the optimal decision in this interval.

s_2	$f_2^*(s_2)$	x_2^*
$-3 \le s_2 \le s_2^*$	-20	D
$s_2^* < s_2 \le -1$	$\frac{5}{4}(\frac{9}{2}s_2^2 + 47s_2 + \frac{427}{6})$	A
$-1 \le s_2 \le 1$	$\frac{5}{4}\left[-\frac{1}{3}(s_2+4)^3 + \frac{9}{2}(s_2+4)^2 + 20s_2 + \frac{103}{6}\right]$	A

<u>$n=1$:</u>

$f_1(-4, D) = -20$

$f_1(-4, A) = -30 + 5(-4+3) + \frac{1}{4}\int_{1}^{5} f_2^*(-4+t)dt$

$= -35 + \frac{1}{4}\left[\int_{1}^{s_2^*+4} -20dt + \frac{5}{4}\int_{s_2^*+4}^{3}(\frac{9}{2}(-4+t)^2 + 47(-4+t) + \frac{427}{6})dt\right.$

$\left. + \frac{5}{4}\int_{3}^{5}\left[-\frac{1}{3}t^3 + \frac{9}{2}t^2 + 20(-4+t) + \frac{103}{6}\right]dt\right] = 4.77$

s_1	$f_1^*(s_1)$	x_1^*
-4	4.77	A

1st Quarter	2nd Quarter	3rd Quarter
Advertise.	If $s_2 \le -2.411$, discontinue.	If $s_3 \le -1$, discontinue.
	If $s_2 > -2.411$, advertise.	If $s_3 > -1$, advertise.

11.1-1.

(a) $x_j = \begin{cases} 1 & \text{if the decision is to build a factory in city } j, \\ 0 & \text{otherwise} \end{cases}$

$y_j = \begin{cases} 1 & \text{if the decision is to build a factory in city } j, \\ 0 & \text{otherwise} \end{cases}$

for $j = $ LA, SF, SD.

maximize $\quad \text{NPV} = 9x_{\text{LA}} + 5x_{\text{SF}} + 7x_{\text{SD}} + 6y_{\text{LA}} + 4y_{\text{SF}} + 5y_{\text{SD}}$

subject to

$$6x_{\text{LA}} + 3x_{\text{SF}} + 4x_{\text{SD}} + 5y_{\text{LA}} + 2y_{\text{SF}} + 3y_{\text{SD}} \leq 10$$
$$y_{\text{LA}} + y_{\text{SF}} + y_{\text{SD}} \leq 1$$
$$-x_{\text{LA}} + y_{\text{LA}} \leq 0$$
$$-x_{\text{SF}} + y_{\text{SF}} \leq 0$$
$$-x_{\text{SD}} + y_{\text{SD}} \leq 0$$
$$x_{\text{LA}}, x_{\text{SF}}, x_{\text{SD}}, y_{\text{LA}}, y_{\text{SF}}, y_{\text{SD}} \text{ binary}$$

(b) - (c)

Constraint	Yes-or-No Question: Warehouse in LA?	Factory in LA?	Warehouse in SD?	Factory in SD?	Warehouse in SF?	Factory in SF?	Totals		Right-Hand Side
Capital ($millions)	5	6	3	4	2	3	10	≤	10
≤ 1 Warehouse	1	0	1	0	1	0	1	≤	1
NPV ($millions)	6	9	5	7	4	5	17		
Solution	0 ≤	0	1 ≤	1	0 ≤	1			

11.1-2.

(a) $M_j = \begin{cases} 1 & \text{if } j \text{ does marketing,} \\ 0 & \text{otherwise} \end{cases} \qquad C_j = \begin{cases} 1 & \text{if } j \text{ does cooking,} \\ 0 & \text{otherwise} \end{cases}$

$D_j = \begin{cases} 1 & \text{if } j \text{ does dishwashing,} \\ 0 & \text{otherwise} \end{cases} \qquad L_j = \begin{cases} 1 & \text{if } j \text{ does laundry,} \\ 0 & \text{otherwise} \end{cases}$

for $j = \text{E (Eve), S (Steven)}$.

min $\quad \text{T} = 4.5M_{\text{E}} + 7.8C_{\text{E}} + 3.6D_{\text{E}} + 2.9L_{\text{E}} + 4.9M_{\text{S}} + 7.2C_{\text{S}} + 4.3D_{\text{S}} + 3.1L_{\text{S}}$

st

$$M_{\text{E}} + C_{\text{E}} + D_{\text{E}} + L_{\text{E}} = 2$$
$$M_{\text{S}} + C_{\text{S}} + D_{\text{S}} + L_{\text{S}} = 2$$
$$M_{\text{E}} + M_{\text{S}} = 1$$
$$C_{\text{E}} + C_{\text{S}} = 1$$
$$D_{\text{E}} + D_{\text{S}} = 1$$
$$L_{\text{E}} + L_{\text{S}} = 1$$
$$M_{\text{E}}, M_{\text{S}}, C_{\text{E}}, C_{\text{S}}, D_{\text{E}}, D_{\text{S}}, L_{\text{E}}, L_{\text{S}} \text{ binary}$$

(b) - (c)

Constraint	Yes-or-No Question: Marketing by Eve	Marketing by Steven	Cooking by Eve	Cooking by Steven	Dishes by Eve	Dishes by Steven	Laundry by Eve	Laundry by Steven	Total		Right-Hand Side
Eve's Chores	1	0	1	0	1	0	1	0	2	=	2
Steven's Chores	0	1	0	1	0	1	0	1	2	=	2
Marketing	1	1	0	0	0	0	0	0	1	=	1
Cooking	0	0	1	1	0	0	0	0	1	=	1
Dishwashing	0	0	0	0	1	1	0	0	1	=	1
Laundry	0	0	0	0	0	0	1	1	1	=	1
Time Needed	4.5	4.9	7.8	7.2	3.6	4.3	2.9	3.1	18.4 hours		
Solution	1	0	0	1	1	0	0	1			

11.1-3.

(a) $$x_j = \begin{cases} 1 & \text{if the decision is to invest in project } j, \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, 2, 3, 4, 5$.

maximize $\quad \text{NPV} = x_1 + 1.8x_2 + 1.6x_3 + 0.8x_4 + 1.4x_5$

subject to $\quad 6x_1 + 12x_2 + 10x_3 + 4x_4 + 8x_5 \leq 20$

x_1, x_2, x_3, x_4, x_5 binary

(b) - (c)

Constraint	Yes-or-No Question: Project 1	Project 2	Project 3	Project 4	Project 5	Total		Right-Hand Side
Capital	6	12	10	4	8	20	≤	20
Net Present Value	1	1.8	1.6	0.8	1.4	\$3.4 million		
Solution	1	0	1	1	0			

11.1-4.

(a) $$x_j = \begin{cases} 1 & \text{if the decision is to invest in opportunity } j, \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, 2, 3, 4, 5, 6$.

Let p_j denote the estimated profit of opportunity j and c_j the capital required for opportunity j in millions of dollars.

maximize $\quad \sum_{j=1}^{6} x_j p_j$

subject to $\quad \sum_{j=1}^{6} x_j c_j \leq 100$

$x_1 + x_2 \leq 1$
$x_3 + x_4 \leq 1$
$x_3 \leq x_1 + x_2$
$x_4 \leq x_1 + x_2$
x_j binary, for $j = 1, \ldots, 6$

(b) Solution: Invest in opportunities 1, 3 and 5.

	Investment Opportunity					
	1	2	3	4	5	6
Estimated Profit	15	12	16	18	9	11
Capital Required	38	33	39	45	23	27
Invest or Not	1	0	1	0	1	0
Total Profit	40					
Total Capital Req.	100	100				
x1 +x2≤1	1	1				
x3 +x4≤1	1	1				
x3≤x1 +x2	1	1				
x4≤x1 +x2	0	1				

11.1-5.

Unit Cost (Seconds)

		Task (Swimmer)					
		Carl	Chris	David	Tony	Ken	Supply
	Back	37.7	32.9	33.8	37	35.4	1
Assignee	Breast	43.4	33.1	42.2	34.7	41.8	1
(Stroke)	Fly	33.3	28.5	38.9	30.4	33.6	1
	Free	29.2	26.4	29.6	28.5	31.1	1
Demand		1	1	1	1	1	

Assignment

		Task (Swimmer)							
		Carl	Chris	David	Tony	Ken	Totals		Supply
	Back	0	0	1	0	0	1	=	1
Assignee	Breast	0	0	0	1	0	1	=	1
(Stroke)	Fly	0	1	0	0	0	1	=	1
	Free	1	0	0	0	0	1	=	1
Totals		1	1	1	1	0			
		≤	≤	≤	≤	≤	Total Cost	=	126.20
Demand		1	1	1	1	1			

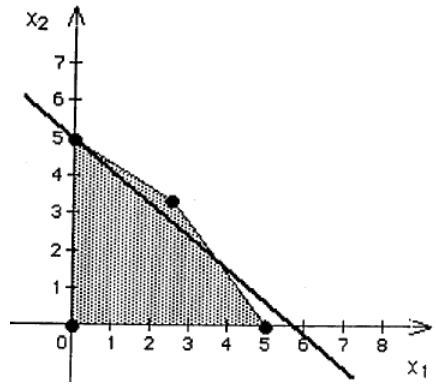
Each swimmer can swim only one stroke and each stroke can be assigned to only one swimmer.

11.1-6.

(a) Let T be the number of tow bars produced and S be the number of stabilizer bars produced.

$$\begin{aligned} \text{maximize} \quad & P = 130T + 150S \\ \text{subject to} \quad & 3.2T + 2.4S \leq 16 \\ & 2T + 3S \leq 15 \\ & T, S \geq 0 \text{ integers} \end{aligned}$$

(b) Optimal Solution: $(T, S) = (0, 5), P = \$750$



(c)

	Resource Usage Per Unit of Each Activity				Resource
Resource	Tow Bars	Stabilizer Bars	Totals		Available
Machine 1	3.2	2.4	12	≤	16
Machine 2	2	3	15	≤	15
Unit Profit	$ 130.00	$ 150.00	$750		
Solution	0	5			

11.1-7.

(a) Let x_{ij} be the number of trucks hauling from pit i to site j and y_{ij} be the number of tons of gravel hauled from pit i to site j, for $i = N, S$ and $j = 1, 2, 3$.

minimize $C = 400y_{N1} + 490y_{N2} + 460y_{N3} + 600y_{S1} + 530y_{S2} + 560y_{S3}$
$\qquad + 150(x_{N1} + x_{N2} + x_{N3} + x_{S1} + x_{S2} + x_{S3})$

subject to
$y_{N1} + y_{N2} + y_{N3} \le 18$
$y_{S1} + y_{S2} + y_{S3} \le 14$
$y_{ij} \le 5x_{ij}$, for $i = N, S$ and $j = 1, 2, 3$
$y_{N1} + y_{S1} \ge 10$
$y_{N2} + y_{S2} \ge 5$
$y_{N3} + y_{S3} \ge 10$
$y_{ij} \ge 0, x_{ij} \ge 0$ integers, for $i = N, S$ and $j = 1, 2, 3$

(b)

	Resource Usage Per Unit of Each Activity														
	Truck						Gravel								Resource
Resource	N1	N2	N3	S1	S2	S3	N1	N2	N3	S1	S2	S3	Total		Available
North	0	0	0	0	0	0	1	1	1	0	0	0	18	≤	18
South	0	0	0	0	0	0	0	0	0	1	1	1	7	≤	14
N1	-5	0	0	0	0	0	1	0	0	0	0	0	0	≤	0
N2	0	-5	0	0	0	0	0	1	0	0	0	0	0	≤	0
N3	0	0	-5	0	0	0	0	0	1	0	0	0	-2	≤	0
S1	0	0	0	-5	0	0	0	0	0	1	0	0	0	≤	0
S2	0	0	0	0	-5	0	0	0	0	0	1	0	0	≤	0
S3	0	0	0	0	0	-5	0	0	0	0	0	1	-3	≤	0
Site 1	0	0	0	0	0	0	-1	0	0	-1	0	0	-10	≤	-10
Site 2	0	0	0	0	0	0	0	-1	0	0	-1	0	-5	≤	-5
Site 3	0	0	0	0	0	0	0	0	-1	0	0	-1	-10	≤	-10
Unit Cost	150	150	150	150	150	150	400	490	460	600	530	560	12350		
Solution	2	0	2	0	1	1	10	0	8	0	5	2			

11.2-1.

Air New Zealand used integer programming to solve its aircrew-scheduling problem that consists of two subproblems: the tours-of-duty (ToD) planning and rostering. A ToD is a sequence of duty and rest periods for a single crew member. The ToD planning problem is to construct minimum-cost ToDs to crew all scheduled flights. The last duty period in a ToD should end at the crew base where the first duty period started. A ToD has to satisfy a number of rules and regulations pertaining to the length of duty and rest periods, the latest possible starting time for a flight, maximum allowable flight time, the number of crew members needed to operate a flight, etc. The second subproblem, rostering assigns planned ToDs to individual crew members. Just like ToDs, rosters should meet some rules such as minimum number of days off, total duty time, flight time limits, minimum rest time between ToDs and qualifications needed to perform a ToD. Both subproblems are instances of generalized set-partitioning problem. A set partitioning problem is of the form:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = e \\ & x \in \{0,1\}^n, \end{aligned}$$

where e is a column-vector of ones and the elements of A are zeros and ones. The generalized set-partitioning problem also includes constraints with right-hand-sides that are not one. To solve these problems, revised simplex method is used together with various pricing and constraint branching techniques.

The new scheduling approach has saved Air New Zealand over NZ$15 million annually whereas it cost only NZ$2 million. Direct savings resulted from reduced crew size and eliminated expenses of the crew that had to stay overseas because of inefficient scheduling practices. Additionally, the cost of scheduling has decreased. While the airline has expanded, the number of people needed to solve the scheduling problem has decreased. This study allowed Air New Zealand to obtain high-quality schedules that respect individual preferences and meet regulations. Furthermore, robust schedules are obtained quickly, so responding to changes promptly is now possible. The airline's dependence on a small number of highly skilled schedulers is eliminated. Schedulers can now concentrate their efforts on analyzing and evaluating solutions. Managers can review strategic decisions in the light of the information provided by optimizers. As a consequence of these improvements, Air New Zealand provides a better customer service.

11.2-2.

Answers will vary.

11.2-3.

Answers will vary.

11.3-1.

(a) Let M be a very large number, say 100 million.

$$\begin{array}{ll}
\text{max} & 70x_1 - 50,000y_1 + 60x_2 - 40,000y_2 + 90x_3 - 70,000y_3 + 80x_4 - 60,000y_4 \\
\text{st} & y_1 + y_2 + y_3 + y_4 \leq 2 \\
& y_3 \leq y_1 + y_2 \\
& y_4 \leq y_1 + y_2 \\
& 5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6000 + My_5 \\
& 4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6000 + M(1 - y_5) \\
& 0 \leq xi \leq My_i, \text{ for } i = 1, 2, 3, 4 \\
& y_i \text{ binary, for } i = 1, 2, 3, 4
\end{array}$$

(b)

Constraint	Product 1	Product 2	Product 3	Product 4	Totals		Modified Right-Hand Side	Original Right-Hand Side
First	5	3	6	4	6000	≤	6000	6000
Second	4	6	3	5	12000	≤	105999	6000
Marginal revenue	$70	$60	$90	$80	$80000			
Solution	0	2000	0	0				
	≤	≤	≤	≤				
	0	9999	0	0				
Set Up?	0	1	0	0	1	≤	2	
Start-up Cost	$50,000	$40,000	$70,000	$60,000				

Contingency Constraints:

Product 3:	0	≤	1	:Product 1 or 2
Product 4:	0	≤	1	:Product 1 or 2

Which Constraint (0 = First, 1 = Second):	0

11.3-2.

$x_1 - x_2 = 0y_1 + 3y_2 - 3y_3 + 6y_4 - 6y_5$, $y_i \in \{0, 1\}$, for $i = 1, \ldots, 5$.

11.3-3.

1.
$$\begin{array}{l}
3x_1 - x_2 - x_3 + x_4 \leq 12 + My_1 \\
x_1 + x_2 + x_3 + x_4 \leq 15 + M(1 - y_1) \\
y_1 \text{ binary}
\end{array}$$

2.
$$\begin{array}{l}
2x_1 + 5x_2 - x_3 + x_4 \leq 30 + My_2 \\
-x_1 + 3x_2 + 5x_3 + x_4 \leq 40 + My_3 \\
3x_1 - x_2 + 3x_3 - x_4 \leq 60 + My_4 \\
y_2 + y_3 + y_4 \leq 1 \\
y_i \text{ binary, for } i = 2, 3, 4
\end{array}$$

11.3-4.

(a) Let y_1 and y_2 be binary variables that indicate whether or not toys 1 and 2 are produced. Let x_1 and x_2 be the number of toys 1 and 2 that are produced. Also, let z be 0 if factory 1 is used and 1 if factory 2 is used.

$$
\begin{array}{ll}
\text{maximize} & 10x_1 + 15x_2 - 50,000y_1 - 80,000y_2 \\
\text{subject to} & x_1 \leq My_1 \\
& x_2 \leq My_2 \\
& \frac{1}{50}x_1 + \frac{1}{40}x_2 \leq 500 + Mz \\
& \frac{1}{40}x_1 + \frac{1}{25}x_2 \leq 700 + M(1-z) \\
& x_1, x_2 \geq 0 \text{ integers} \\
& y_1, y_2, z \text{ binary}
\end{array}
$$

(b)

Constraint	Toy 1	Toy 2	Totals		Modified Right-Hand Side	Original Right-Hand Side
Factory 1	0.02	0.025	560	≤	10499	500
Factory 2	0.025	0.04	700	≤	700	700
Unit Profit	$10	$15	$230000			
Solution	28000	0				
	≤	≤				
	99999	0				
Set Up?	1	0				
Setup Cost	$50,000	$80,000				

Which factory? (0=Factory 1, 1=Factory 2)	1

11.3-5.

(a) Let L, M, and S be the number of long-, medium-, and short-range jets to buy respectively.

$$
\begin{array}{ll}
\text{maximize} & P = 4.2L + 3M + 2.3S \\
\text{subject to} & 67L + 50M + 35S \leq 1500 \\
& L + M + S \leq 30 \\
& \frac{5}{3}L + \frac{4}{3}M + S \leq 40 \\
& L, M, S \geq 0 \text{ integers}
\end{array}
$$

(b)

Resource	Resource Usage Per Unit of Each Activity: Long-range	Medium-range	Short-range	Totals		Resource Available
Money	67	50	35	1498	≤	1500
Pilots	1	1	1	30	≤	30
Maintenance	1.667	1.333	1	39.338	≤	40
Profit	$ 4.20	$ 3.00	$ 2.30	$95.6		
Solution	14	0	16			

(c) $L \leq \min\left\{\frac{1500}{67}, \frac{30}{1}, \frac{40}{5/3}\right\} = 24$

$M \leq \min\left\{\frac{1500}{50}, \frac{30}{1}, \frac{40}{4/3}\right\} = 30$

$S \leq \min\left\{\frac{1500}{35}, \frac{30}{1}, \frac{40}{1}\right\} = 30$

$L = 2^0 l_0 + 2^1 l_1 + 2^2 l_2 + 2^3 l_3 + 2^4 l_4$

$M = 2^0 m_0 + 2^1 m_1 + 2^2 m_2 + 2^3 m_3 + 2^4 m_4$

$S = 2^0 s_0 + 2^1 s_1 + 2^2 s_2 + 2^3 s_3 + 2^4 s_4$

maximize $\quad P = 4.2\sum_{i=0}^{4} 2^i l_i + 3\sum_{i=0}^{4} 2^i m_i + 2.3\sum_{i=0}^{4} 2^i s_i$

subject to $\quad 67\sum_{i=0}^{4} 2^i l_i + 50\sum_{i=0}^{4} 2^i m_i + 35\sum_{i=0}^{4} 2^i s_i \leq 1500$

$\sum_{i=0}^{4} 2^i l_i + \sum_{i=0}^{4} 2^i m_i + \sum_{i=0}^{4} 2^i s_i \leq 30$

$\frac{5}{3}\sum_{i=0}^{4} 2^i l_i + \frac{4}{3}\sum_{i=0}^{4} 2^i m_i + \sum_{i=0}^{4} 2^i s_i \leq 40$

l_i, m_i, s_i binary, for $i = 0, 1, 2, 3, 4$

(d) Solution: $l_0 = l_4 = 0, l_1 = l_2 = l_3 = 1, \sum_{i=0}^{4} 2^i l_i = 14$

$m_0 = m_1 = m_2 = m_3 = m_4 = 0, \sum_{i=0}^{4} 2^i m_i = 0$

$s_0 = s_1 = s_2 = s_3 = 0, s_4 = 1, \sum_{i=0}^{4} 2^i s_i = 16$

$P =$ \$95.6 (same as in (b))

11.3-6.

(a) $x_1 = y_{11} + 2y_{12}$, $x_2 = y_{21} + 2y_{22}$

maximize $\quad Z = y_{11} + 2y_{12} + 5y_{21} + 10y_{22}$

subject to $\quad y_{11} + 2y_{12} + 10y_{21} + 20y_{22} \leq 20$

$y_{11} + 2y_{12} \leq 2$

y_{ij} binary, for $i, j = 1, 2$

(b) Solution: $y_{11} = y_{12} = 0 \Rightarrow x_1 = 0$, $y_{21} = 0$, $y_{22} = 1 \Rightarrow x_2 = 2$, $Z = 10$

11.3-7.

(a) Let x_i be the number of units to produce of product $i = 1, 2, 3$.

$$y_i = \begin{cases} 1 & \text{if product } i \text{ is produced,} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 + 0.8x_3 - 3y_1 - 2y_2 \\ \text{subject to} & 0.2x_1 + 0.4x_2 + 0.2x_3 \leq 1 \\ & x_1 \leq My_1 \\ & x_2 \leq My_2 \\ & 0 \leq x_1 \leq 3 \text{ integer} \\ & 0 \leq x_2 \leq 2 \text{ integer} \\ & 0 \leq x_3 \leq 5 \text{ integer} \\ & y_1, y_2 \text{ binary} \end{array}$$

(b)

Constraint	Amt. For Customer 1	Amt. For Customer 2	Amt. For Customer 3	Totals		Right-Hand Side
Capacity	0.2	0.4	0.2	1	≤	1
Max. Sales - Customer 1	1	0	0	0	≤	3
Max. Sales - Customer 2	0	1	0	2	≤	2
Max. Sales - Customer 3	0	0	1	1	≤	5
Marginal Net Revenue	\$2	\$3	\$1	\$4.8	million	
Solution	0	2	1			
	≤	≤	≤			
	0	99	99			
Set Up?	0	1	1			
Setup Cost(millions)	\$3	\$2	\$0			

11.4-1.

(a) $$y_{ij} = \begin{cases} 1 & \text{if } x_i = j \text{ (i.e., produce } j \text{ units of } i\text{),} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, 3$ and $j = 1, 2, 3, 4, 5$.

$$\begin{array}{ll} \text{max} & -y_{11} + 2y_{12} + 4y_{13} + y_{21} + 5y_{22} + y_{31} + 3y_{32} + 5y_{33} + 6y_{34} + 7y_{35} \\ \text{st} & y_{11} + y_{12} + y_{13} \leq 1 \\ & y_{21} + y_{22} \leq 1 \\ & y_{31} + y_{32} + y_{33} + y_{34} + y_{35} \leq 1 \\ & y_{11} + 2y_{12} + 3y_{13} + 2y_{21} + 4y_{22} + y_{31} + 2y_{32} + 3y_{33} + 4y_{34} + 5y_{35} \leq 5 \\ & y_{ij} \text{ binary} \end{array}$$

(b) Solution: $y_{ij} = 0$ except for $(i, j) = (3, 5)$, $y_{35} = 1 \Rightarrow x_3 = 5$, $Z = 7$

(c) $$y_{ij} = \begin{cases} 1 & \text{if } x_i \geq j, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, 3$ and $j = 1, 2, 3, 4, 5$.

$$
\begin{array}{ll}
\text{max} & -y_{11} + 3y_{12} + 2y_{13} + y_{21} + 4y_{22} + y_{31} + 2y_{32} + 2y_{33} + y_{34} + y_{35} \\
\text{st} & y_{13} \le y_{12} \le y_{11} \\
& y_{22} \le y_{21} \\
& y_{35} \le y_{34} \le y_{33} \le y_{32} \le y_{31} \\
& y_{11} + y_{12} + y_{13} + 2y_{21} + 2y_{22} + y_{31} + y_{32} + y_{33} + y_{34} + y_{35} \le 5 \\
& y_{ij} \text{ binary}
\end{array}
$$

(d) Solution: $y_{ij} = 0$ for $i = 1, 2$, $y_{3j} = 1$ for $j = 1, \dots, 5 \Rightarrow x_3 = 5$, $Z = 7$

11.4-2.

Introduce the binary variables y_1 and y_2 and add constraints $x_1 \le My_1$, $x_2 \le My_2$, $y_1 + y_2 = 1$.

11.4-3.

(a) Introduce the binary variables y_1, y_2, and y_3 to represent positive (nonzero) production levels.

$$
\begin{array}{ll}
\text{maximize} & Z = 50x_1 + 20x_2 + 25x_3 \\
\text{subject to} & 9x_1 + 3x_2 + 5x_3 \le 500 \\
& 5x_1 + 4x_2 \qquad\quad \le 350 \\
& 3x_1 \qquad\quad + 2x_3 \le 150 \\
& \qquad\qquad\quad x_3 \le 20 \\
& x_1 \le My_1, x_2 \le My_2, x_3 \le My_3 \\
& y_1 + y_2 + y_3 \le 2 \\
& x_1, x_2, x_3 \ge 0 \\
& y_1, y_2, y_3 \text{ binary}
\end{array}
$$

(b)

Constraint	Product 1	Product 2	Product 3	Total		Right-Hand Side
Milling	9	3	5	498	≤	500
Lathe	5	4	0	349	≤	350
Grinder	3	0	2	135	≤	150
Sales Potential	0	0	1	0	≤	20
Unit Profit	50	20	25	$2870		
Solution	45	31	0			
	≤	≤	≤			
	999	999	0			
Produce?	1	1	0	2	≤	2

11.4-4.

(a) $$y_{ij} = \begin{cases} 1 & \text{if } x_i = j, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2$ and $j = 1, 2, 3$.

Work out by hand the objective function contribution for $x_1, x_2 = 0, 1, 2, 3$.

$$\begin{array}{ll}
\text{maximize} & 3y_{11} + 8y_{12} + 9y_{13} + 9y_{21} + 24y_{22} + 9y_{23} \\
\text{subject to} & y_{11} + y_{12} + y_{13} \leq 1 \\
& y_{21} + y_{22} + y_{23} \leq 1 \\
& y_{11} + y_{23} \leq 1 \\
& y_{13} + y_{23} \leq 1 \\
& y_{12} + y_{23} \leq 1 \\
& y_{12} + y_{22} \leq 1 \\
& y_{13} + y_{22} \leq 1 \\
& y_{13} + y_{21} \leq 1 \\
& y_{ij} \text{ binary}
\end{array}$$

(b) Solution: $y_{ij} = 0$ except $y_{11} = y_{22} = 1 \Rightarrow x_1 = 1$, $x_2 = 2$, $Z = 27$

(c) $$y_{ij} = \begin{cases} 1 & \text{if } x_i \geq j, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2$ and $j = 1, 2, 3$.

Work out by hand the objective function contribution for $x_1, x_2 = 0, 1, 2, 3$.

$$\begin{array}{ll}
\text{maximize} & 3y_{11} + 5y_{12} + y_{13} + 9y_{21} + 15y_{22} - 15y_{23} \\
\text{subject to} & y_{13} \leq y_{12} \leq y_{11} \\
& y_{23} \leq y_{22} \leq y_{21} \\
& y_{11} + y_{23} \leq 1 \\
& y_{12} + y_{22} \leq 1 \\
& y_{13} + y_{21} \leq 1 \\
& y_{ij} \text{ binary}
\end{array}$$

(d) Solution: $y_{ij} = 0$ except $y_{11} = y_{21} = y_{22} = 1 \Rightarrow x_1 = 1$, $x_2 = 2$, $Z = 27$

11.4-5.

(a) $$x_{ij} = \begin{cases} 1 & \text{if arc from node } i \text{ to node } j \text{ is in the shortest path,} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{lll}
\text{min} & 3x_{12} + 6x_{13} + 6x_{24} + 5x_{25} + 4x_{34} + 3x_{35} + 3x_{46} + 2x_{56} & \\
\text{st} & x_{12} + x_{13} = 1 & (1) \\
& x_{24} + x_{25} + x_{34} + x_{35} = 1 & (2) \\
& x_{46} + x_{56} = 1 & (3) \\
& x_{24} + x_{25} \leq x_{12} & (4) \\
& x_{34} + x_{35} \leq x_{13} & (5) \\
& x_{46} \leq x_{24} + x_{34} & (6) \\
& x_{56} \leq x_{25} + x_{35} & (7) \\
& x_{ij} \text{ binary} &
\end{array}$$

(1), (2), (3) ensure that exactly one arc is used at each stage and they represent mutually exclusive alternatives. (4), (5), (6) ensure that node i is left only if it is entered and they represent contingent decisions.

(b) Solution: $x_{ij} = 0$ except $x_{12} = x_{25} = x_{56} = 1$, $Z = 10$

Shortest path: $1 \rightarrow 2 \rightarrow 5 \rightarrow 6$

11.4-6.

(a) $y_j = \begin{cases} 1 & \text{if route } j \text{ is chosen,} \\ 0 & \text{otherwise} \end{cases}$

Let x_{ij} be the ijth element of the location/route matrix, for $i = A, \ldots, I$ and $j = 1, \ldots, 10$. Let c_j denote the cost of route j, for $j = 1, \ldots, 10$.

$$\text{minimize} \quad \sum_{j=1}^{10} c_j y_j$$

$$\text{subject to} \quad \sum_{j=1}^{10} x_{ij} y_j \geq 1, \text{ for } i = A, \ldots, I$$

$$\sum_{j=1}^{10} y_j = 3$$

$$y_j \text{ binary, for } j = 1, \ldots, 10$$

(b)

Delivery Location	Yes-or-No? Possible Routes 1	2	3	4	5	6	7	8	9	10	Total		Right-Hand Side
A	1	0	0	0	1	0	0	0	1	0	1	≥	1
B	0	1	0	1	0	1	0	0	1	1	1	≥	1
C	0	0	1	1	0	0	1	0	1	0	1	≥	1
D	1	0	0	0	0	1	0	1	0	0	1	≥	1
E	0	0	1	1	0	1	0	0	0	0	1	≥	1
F	0	1	0	0	1	0	0	0	0	0	1	≥	1
G	1	0	0	0	0	0	1	1	0	1	1	≥	1
H	0	0	1	0	1	0	0	0	0	1	1	≥	1
I	0	1	0	1	0	0	1	0	0	0	1	≥	1
3 routes	1	1	1	1	1	1	1	1	1	1	3	=	3
Cost	6	4	7	5	4	6	5	3	7	6	12		
Solution	0	0	0	1	1	0	0	1	0	0			

11.4-7.

$$x_{ij} = \begin{cases} 1 & \text{if tract } j \text{ is assigned to station located in tract } i, \\ 0 & \text{otherwise} \end{cases}$$

Let a_{ij} be the response time to a fire in tract j if that tract is served by a station located in tract i.

min $\quad 2\sum_{i=1}^{5} a_{i1}x_{i1} + \sum_{i=1}^{5} a_{i2}x_{i2} + 3\sum_{i=1}^{5} a_{i3}x_{i3} + \sum_{i=1}^{5} a_{i4}x_{i4} + 3\sum_{i=1}^{5} a_{i5}x_{i5}$

st $\quad \sum_{i=1}^{5} x_{ii} = 2$ (1) Two fire stations have to be located.

$\sum_{i=1}^{5} x_{ij} = 1$, for $j = 1, \ldots, 5$ (2) Each tract needs to be assigned to a station.

$x_{ij} \leq x_{ii}$, for $i = 1, \ldots, 5$ and $j = 1, \ldots, 5$ (3) Tract j can be assigned to the station tract i only if there is a station located in tract i.

x_{ij} binary

(1) and (2) correspond to mutually exclusive alternatives and (3) represent contingent decisions.

11.4-8.

(a) $\quad x_i = \begin{cases} 1 & \text{if a station is located in tract } i, \\ 0 & \text{otherwise} \end{cases}$

minimize $\quad 300x_1 + 350x_2 + 600x_3 + 450x_4 + 700x_5$

subject to
$$\begin{aligned} & x_1 + x_5 \geq 1 \\ & x_1 + x_2 \geq 1 \\ & x_3 \geq 1 \\ & x_2 + x_4 + x_5 \geq 1 \\ & x_3 + x_4 + x_5 \geq 1 \\ & x_i \text{ binary} \end{aligned}$$

(b) Yes, this is a set covering problem. The activities are locating stations and the characteristics are the fires. S_i is the set of all locations that could cover a fire in tract i, e.g., $S_1 = \{1, 5\}$. There has to be at least one station, so $\sum_{j \in S_i} x_j \geq 1$ for all i.

(c) Solution: $x_1 = x_2 = x_3 = 1$, $Z = \$1,250$ thousand

11.4-9.

$$x_j = \begin{cases} 1 & \text{if district } j \text{ is chosen,} \\ 0 & \text{otherwise} \end{cases}$$

Let y_j be auxiliary variables that are zero for all j, except for the index of the district with largest c_j that is chosen, y_j is 1.

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{N} c_j y_j \\ \text{subject to} \quad & \sum_{j=1}^{N} y_j = 1 \\ & \sum_{j=1}^{N} c_j y_j \geq c_i x_i, \text{ for } i = 1, \ldots, N \\ & \sum_{j=1}^{N} x_j = R \\ & \sum_{j=1}^{N} a_{ij} x_j = 1, \text{ for } i = 1, \ldots, D \\ & x_j, y_j \text{ binary} \end{aligned}$$

This is a set partitioning problem with additional constraints.

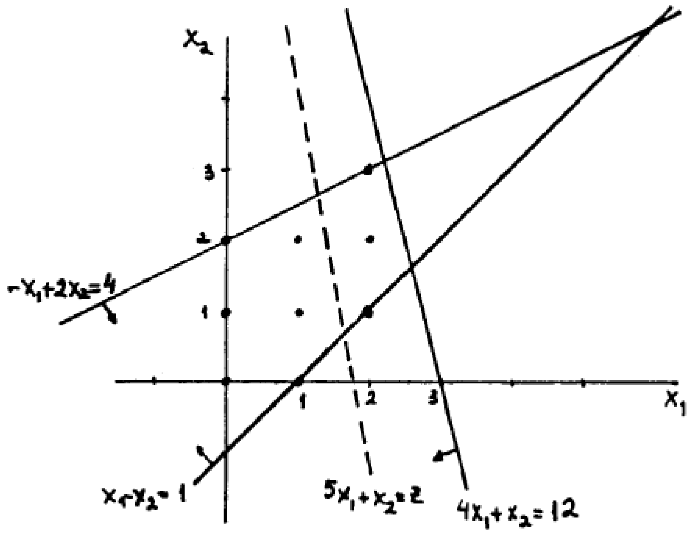
11.5-1.

This study uses integer programming to model employee scheduling problem of Taco Bell restaurants. In this integer program, the decision variables correspond to the number of employees scheduled to start working at time t and to work for s time units. The objective is to minimize the total payroll for the scheduling horizon. At any point in time, the labor requirements in each store have to be met. The total number of employees is bounded above. Without the upper bound, the problem could be solved efficiently as a network flow problem using out-of-kilter algorithms, so the upper bound is eliminated from the constraint set by using generalized Lagrange multipliers.

The new scheduling approach increased labor cost savings significantly. Additional benefits include enhanced flexibility, elimination of variability among stores, improved customer service and quality. Mathematical modeling served as a rational basis for the evaluation of new ideas, buildings, equipment and menu items. It also allowed Taco Bell to eliminate redundant tasks and to schedule balanced workloads. Consequently, productivity is improved and Taco Bell saved $13 million each year in labor costs.

11.5-2.

(a) The dots represent the feasible solutions in the graph below.



Optimal Solution: $(x_1, x_2) = (2, 3), Z = 5x_1 + x_2 = 13$

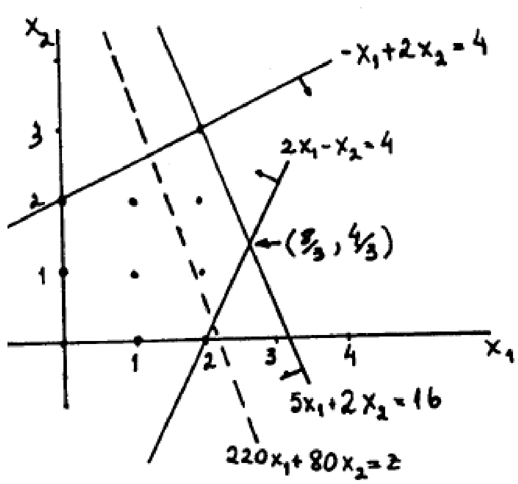
(b) The optimal solution of the LP relaxation is $(x_1, x_2) = (2.6, 1.6), Z = 14.6$. The nearest integer point is $(x_1, x_2) = (3, 2)$, which is not feasible, since $4 \cdot 3 + 2 > 12$.

Rounded Solutions	Violated Constraints	Z
$(3, 2)$	3rd	–
$(3, 1)$	2nd and 3rd	–
$(2, 2)$	none	12
$(2, 1)$	none	11

Hence, none of the feasible rounded solutions is optimal for the IP problem.

11.5-3.

(a) The dots represent the feasible solutions in the graph below.



Optimal Solution: $(x_1, x_2) = (2, 3), Z = 220x_1 + 80x_2 = 680$

(b) The optimal solution of the LP relaxation is $(x_1, x_2) = (8/3, 4/3), Z = 2080/3$. The nearest integer point is $(x_1, x_2) = (3, 1)$, which is not feasible, since $5 \cdot 3 + 2 \cdot 1 > 16$.

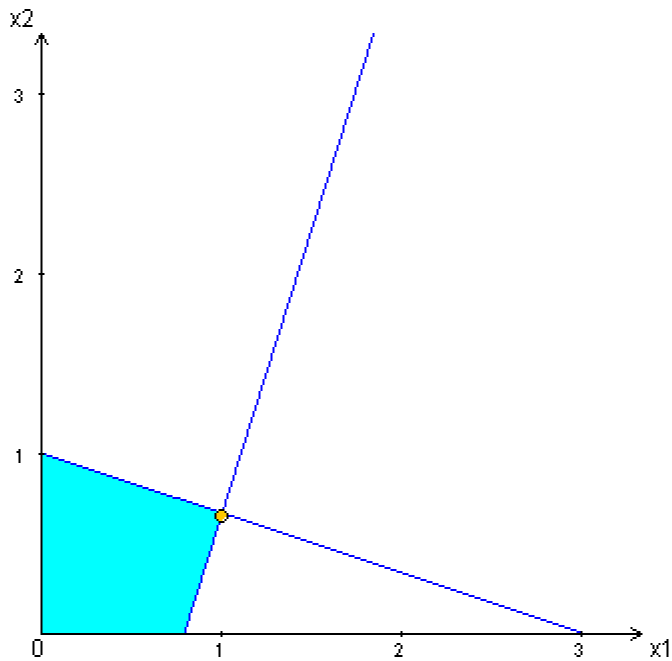
Rounded Solutions	Violated Constraints	Z
$(3, 2)$	2nd	–
$(3, 1)$	2nd and 3rd	–
$(2, 2)$	none	600
$(2, 1)$	none	520

Hence, none of the feasible rounded solutions is optimal for the IP problem.

11.5-4.

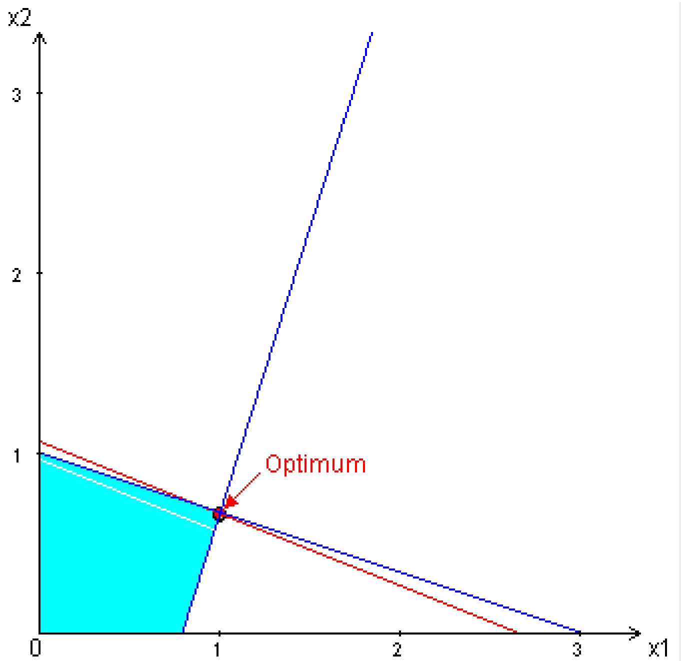
(a)

Solution	Feasible?	$P = 10x_1 + 25x_2$	Optimal?
$(0, 0)$	Yes	0	No
$(1, 0)$	No		
$(0, 1)$	Yes	25	Yes
$(1, 1)$	No		



Optimal Solution: $(x_1, x_2) = (0, 1)$

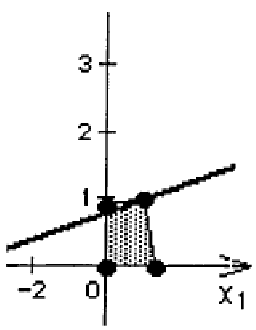
(b) The optimal solution of the LP relaxation is $(x_1, x_2) = (1, 0.667)$. The nearest integer point is $(x_1, x_2) = (1, 1)$, which is not feasible. The other rounded solution is $(1, 0)$, which is not feasible either.



11.5-5.

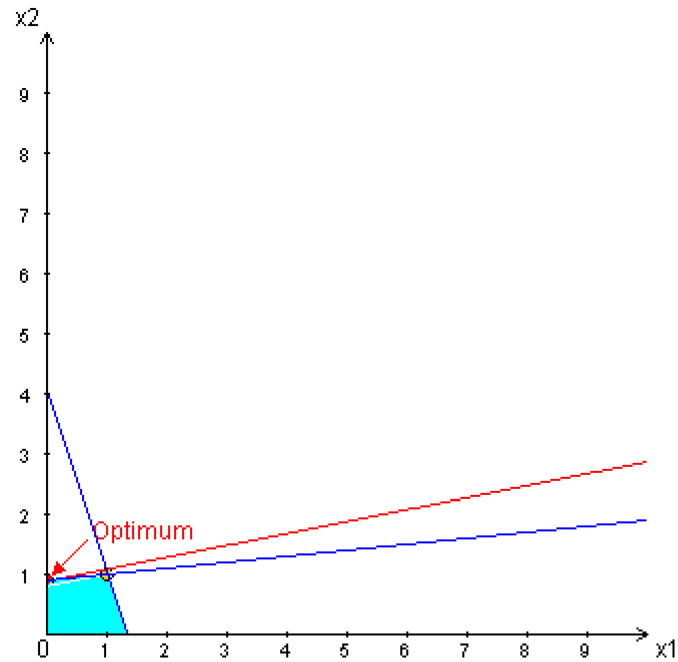
(a)

Solution	Feasible?	$P = -5x_1 + 25x_2$	Optimal?
$(0, 0)$	Yes	0	No
$(1, 0)$	Yes	-5	No
$(0, 1)$	No		
$(1, 1)$	Yes	20	Yes



Optimal Solution: $(x_1, x_2) = (1, 1)$

(b) The optimal solution of the LP relaxation is $(x_1, x_2) = (0, 0.9)$. The nearest integer point is $(x_1, x_2) = (0, 1)$, which is not feasible. The other rounded solution is $(0, 0)$, which is feasible, but not optimal.



11.5-6.

(a) TRUE, Sec. 11.5, 4th paragraph, p. 501.

(b) TRUE, Sec. 11.5, 9th paragraph, p. 502.

(c) FALSE, the result need not be feasible, see Fig. 11.2 for a counterexample, p. 503. Sec. 11.5, 11th paragraph explains this pitfall.

11.6-1.

```
                                    ,-- X4=1,X=(1,1,1,1,0.75),Z=6,F
                              --- X3=1,X=(1,1,1,1,0.75),Z=6
                             /      `-- X4=0,Inf.,F
                    --- X2=1,X=(1,1,0.86,1,1),Z=6.2857
                   /         \
                  /           --- X3=0,X=(1,1,0,0.5,1),Z=3.5,F
                 /
        --- X1=1,X=(1,1,0.86,1,1),Z=6.2857
       /         \
      /           \
     /             \
    /               --- X2=0,X=(1,0,0.67,1,0.83),Z=5.6667,F
   /
  /
 /
X=(0.67,1,1,1,1),Z=6.3333
 \
  \
   \
    \
     \
      \
       \
        --- X1=0,X=(0,0,1,1,1),Z=6
```

Optimal Solution: $(0,0,1,1,1)$, $Z = 6$

11.6-2.

```
                    --- X2=1,X=(1,1,1,0,0),Z=18,F
                   /
                  /
                 /
        --- X1=1,X=(1,0,0.67,0,0),Z=9.6667
       /         \
      /           \             --- X3=1,X=(1,0,1,0,0),Z=12
     /             \           /
    /               --- X2=0,X=(1,0,0.67,0,0),Z=9.6667
   /                           \
  /                             --- X3=0,X=(1,0,0,1,1),Z=22,F
 /
X=(0.5,0,0.5,0,0),Z=6
 \
  \
   \
    \
     \
      \
       \
        --- X1=0,Inf.,F
```

Optimal Solution: $(1,0,1,0,0)$, $Z = 12$

11.6-3.

```
                                  ,-- X4=1,X=(1,1,1,1,0.56),Z=8.4444- X5=1,X=(1,1,1,1,1),Z=8
                            --- X3=1,X=(1,1,1,0.72,0.83),Z=8.7222\ X5=0,Inf.,F
                           /      `-- X4=0,Inf.,F
                  --- X2=1,X=(1,1,1,0.72,0.83),Z=8.7222
                 /         \
                /           --- X3=0,X=(1,1,0,1,0),Z=4,F
               /
       --- X1=1,X=(1,1,1,0.72,0.83),Z=8.7222
      /        \
     /          \
    /            \
   /              --- X2=0,X=(1,0,0.71,1,1),Z=3.5714,F
  /
 /
/
X=(1,1,1,0.72,0.83),Z=8.7222
\
 \
  \
   \
    \
     \
      \
       --- X1=0,X=(0,1,1,0.39,0.83),Z=6.3889,F
```

Optimal Solution: $(1, 1, 1, 1, 1)$, $Z = 8$

11.6-4.

```
                    --- X2=1,Inf.,F
                   /
                  /
                 /
          --- X1=1,X=(1,0.5,0,0.9),Z=11
         /       \
        /         \
       /           \
      /             --- X2=0,X=(1,0,0,0.95),Z=10.5,F
     /
    /
   /
X=(0,1,0,0.9),Z=11
 \                               ,-- X4=1,Inf.,F
  \                         --- X3=1,X=(0,1,1,0.4),Z=11
   \                       /     `-- X4=0,X=(0,1,1,0),Z=7,F
    \               --- X2=1,X=(0,1,0,0.9),Z=11
     \             /       \     ,-- X4=1,Inf.,F
      \           /         --- X3=0,X=(0,1,0,0.9),Z=11
       \         /               `-- X4=0,X=(0,1,0,0),Z=2,F
          --- X1=0,X=(0,1,0,0.9),Z=11
                 \
                  \
                   \
                    --- X2=0,X=(0,0,0,1),Z=10
```

Optimal Solution: $(0, 0, 0, 1)$, $Z = 10$

11.6-5.

```
Optimal Solution:
  (X1, X2, X3, X4, X5) = (1, 1, 1, 0, 0)
  Z = 1250
```

11.6-6.

(a) FALSE. The feasible region for the IP problem is a subset of the feasible region for the LP relaxation. It is called a relaxation because it relaxes the feasible region.

(b) TRUE. If the optimal solution for the LP relaxation is integer, then it is feasible for the IP problem and since the solution for the latter cannot be better than the solution for the former, it has to be optimal.

(c) FALSE. Figure 11.2 is a counterexample for this statement.

11.6-7.

(a) Initialization: Set $Z^* = +\infty$. Apply the bounding and fathoming steps and the optimality test as described below for the whole problem. If the whole problem is not fathomed, then it becomes the initial subproblem for the first iteration below.

Iteration:

1. Branching: Choose the most recently created unfathomed subproblem (in case of a tie, select the one with the smallest bound). Among the assignees not yet assigned for the current subproblem, choose the first one in the natural ordering to be the branching variable. Subproblems correspond to each of the possible remaining assignments for the branching assignee. Form a subproblem for each remaining assignment by deleting the constraint that each of the unassigned assignees must perform exactly one assignment.

2. Bounding: For each new subproblem, obtain its bound by choosing the cheapest assignee for each remaining assignment and totaling the costs.

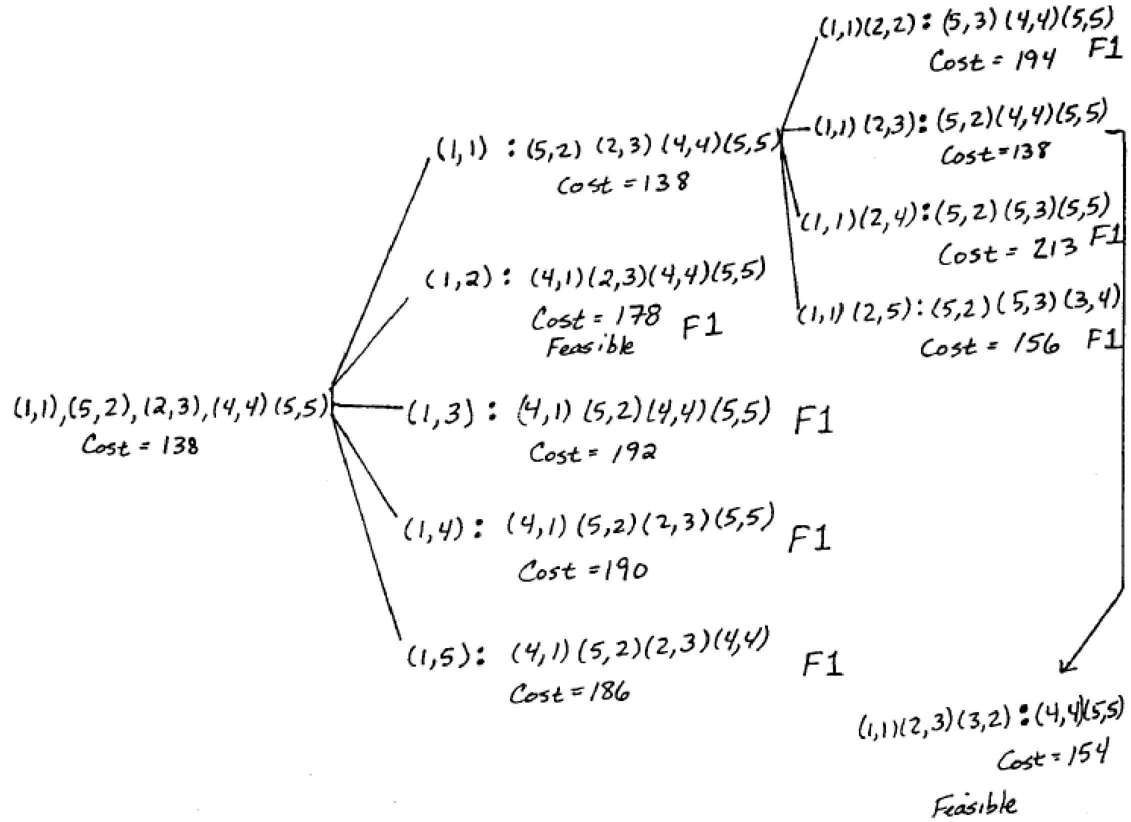
3. Fathoming: For each new subproblem, apply the two fathoming tests:

Test 1. bound $\geq Z^*$

Test 2. The optimal solution for its relaxation is a feasible assignment (If this solution is better than the incumbent, it becomes the new incumbent and Test 1 is reapplied to all unfathomed subproblems with the new smaller Z^*).

Optimality Test: Identical to the one given in the text.

(b) Matchings are indicated with the notation (assignee, assignment).



Optimal matching: $(1, 1), (2, 3), (3, 2), (4, 4), (5, 5)$, with total cost 154.

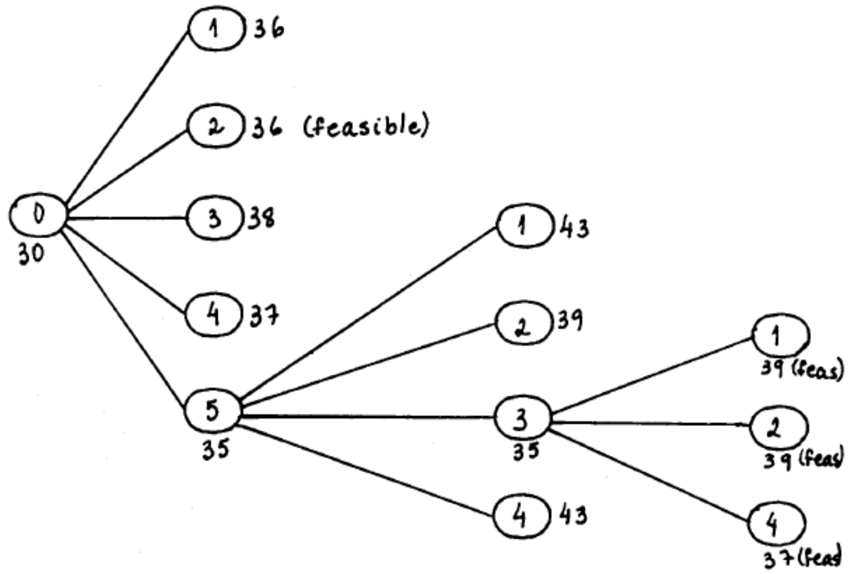
11.6-8.

(a) Branch Step: Use the best bound rule.

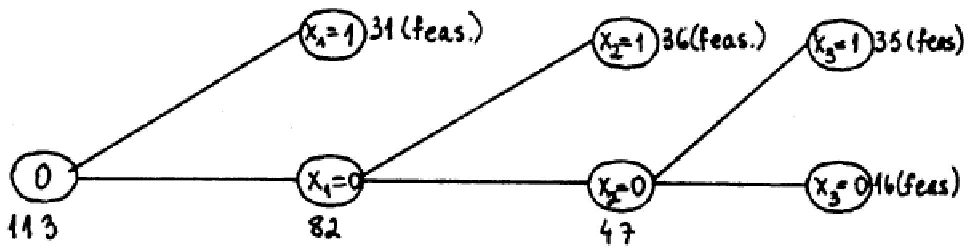
Bound Step: Given a partial sequencing $J_1, \ldots, J_k$ of the first k jobs, a lower bound on the time for the setup of the remaining $5-k$ jobs is found by adding the minimum elements of the columns corresponding to the remaining jobs, excluding those elements in rows "None", $J_1, J_2, \ldots, J_{k-1}$.

Fathoming Step: see the summary of the Branch-and-Bound technique in Sec. 11.6.

(b) The optimal sequence is $2-1-4-5-3$, with a total setup time of 36.



11.6-9.



Optimal Solution: $(x_1, x_2, x_3, x_4) = (0, 1, 1, 0)$, $Z^* = 36$

11.6-10.

(a) The only constraints of the Lagrangian relaxation are nonnegativity and integrality. Since **x** is feasible for an MIP problem, it already satisfies these constraints, so it is feasible for the corresponding Lagrangian relaxation.

(b) **x*** is feasible for an MIP problem, so from (a), it has to be feasible for its Lagrangian relaxation. Also, $A\mathbf{x}^* \leq b$ and $\lambda \geq 0$, so $Z_R^* \geq c\mathbf{x}^* - \lambda(A\mathbf{x}^* - b) \geq c\mathbf{x}^* = Z$.

11.7-1.

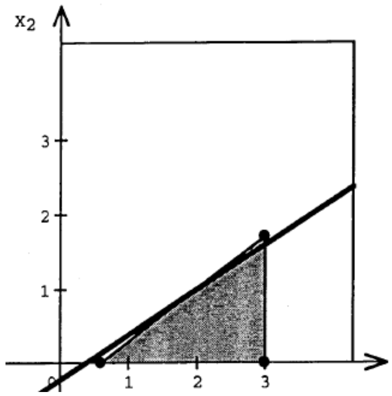
Prior to this study, Waste Management, Inc. (WM) encountered several operational inefficiencies concerning the routing of its trucks. The routes served by different trucks had overlaps and route planners or drivers determined in what order they were going to visit the stops. The result was inefficient sequences and communication gaps between customers and customer-service personnel. The problem is formulated as a mixed integer program, or more specifically as a vehicle routing problem with time windows. The goal is to obtain routes with minimum number of vehicles and travel time, maximum visual attractiveness and a balanced workload. First, a network with nodes that represent actual stops, landfills, lunch break and the depot is constructed. The binary variables x_{ijk} refer to whether arc (i,j) is included in the route of vehicle k or not. The integer variables N_k denote the number of disposal trips and the continuous variables w_{ik} correspond to the beginning time of service for node i by vehicle k. The objective function to be minimized is the total travel time. The constraints make sure that each stop is served by exactly one truck, each truck starts at the depot, the amount of garbage at the stops does not exceed the vehicle capacity and each route includes a lunch break. An iterative two-phase algorithm enhanced with metaheuristics is employed to solve the problem.

Financial benefits of this study include savings of approximately \$18 million in 2003 and estimated savings of \$44 million in 2004. WM expects to save more and to increase its cash flow by \$648 million over a five-year interval. The savings in operational costs over five years is expected to be \$498 million. By using mathematical modeling, WM now generates more efficient routes with minimal overlaps, a reduced number of vehicles and cost-effective sequences. All these contribute to the decrease in operational costs. At the same time, centralized routing made communication in the organization and with the customers easier. Customer-service personnel can now address customer problems more quickly, since they know the routes of the vehicles. As a result, WM provides a more reliable customer service. Operational efficiency also affected the environment and the employees positively. Emissions and noise are reduced. Finally, the benefits from this study led WM to exploit operations research techniques in other operational areas, too.

11.7-2.

(a)

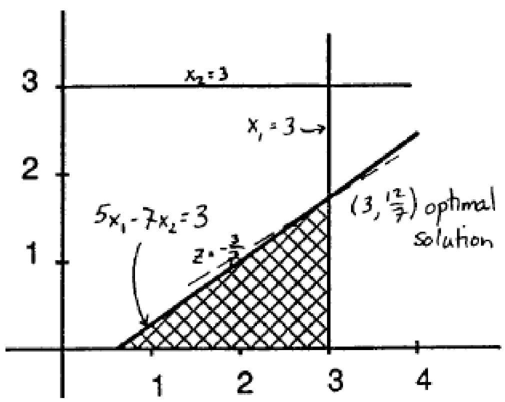
Corner Points	Z
$(3, 1.7143)$	-0.429
$(0.6, 0)$	-1.8
$(3, 0)$	-9



Optimal solution for the LP relaxation: $(3, 1.7143)$ with $Z^* = -0.429$

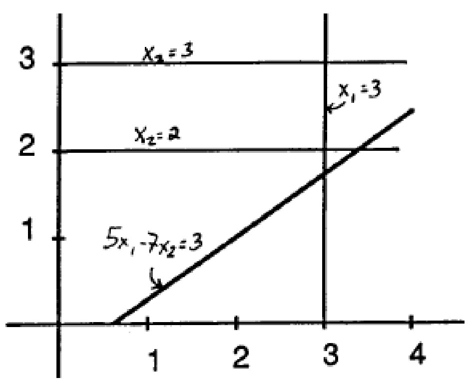
Optimal integer solution: $(2, 1)$ with $Z^* = -1$

(b) LP relaxation of the entire problem:



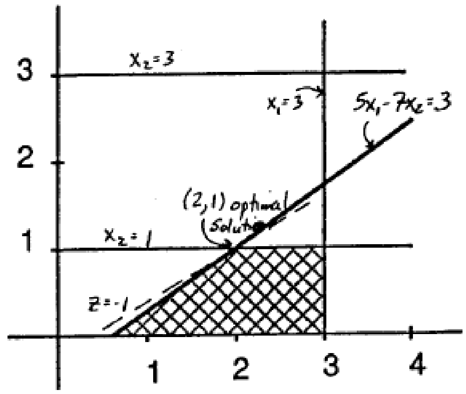
Optimal Solution: $(x_1, x_2) = (3, 12/7)$, $Z = -3/7$

Branch $x_2 \geq 2$:



This subproblem is infeasible, so the branch is fathomed.

Branch $x_2 \leq 1$:



Optimal Solution: $(x_1, x_2) = (2, 1)$, $Z = -1$, feasible for the original problem

Hence, the optimal solution for the original problem is $(x_1, x_2) = (2, 1)$ with $Z = -1$.

(c) Let $x_1 = y_{11} + 2y_{12}$ and $x_2 = y_{21} + 2y_{22}$.

maximize	$Z = -3y_{11} - 6y_{12} + 5y_{21} + 10y_{22}$
subject to	$5y_{11} + 10y_{12} - 7y_{21} - 14y_{22} \geq 3$
	$y_{11}, y_{12}, y_{21}, y_{22}$ binary

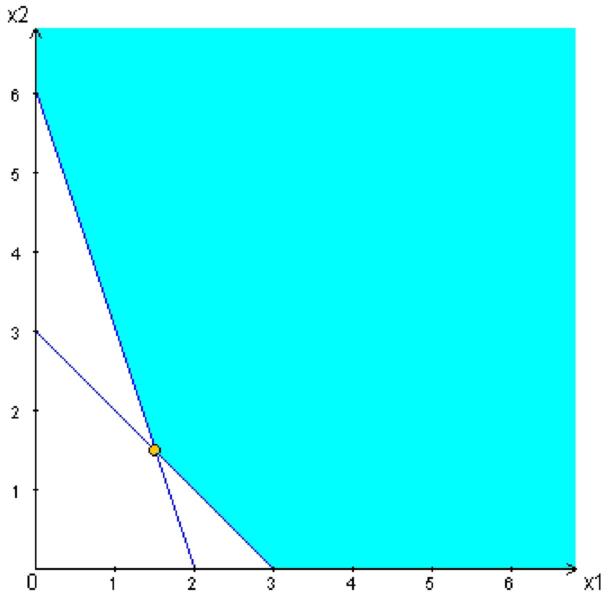
(d) Optimal Solution: $(y_{11}, y_{12}, y_{21}, y_{22}) = (0, 1, 1, 0)$, $Z = -1$, so $x_1 = 2$ and $x_2 = 1$ as in (a).

```
                                                  ,-- X4=1,Inf.,F
                                           --- X3=1,X=(1,1,1,0.36),Z=-0.429
                                          /       `-- X4=0,X=(1,1,1,0),Z=-4,F
                                  --- X2=1,X=(1,1,0,0.86),Z=-0.429
                                 /        \       ,-- X4=1,Inf.,F
                                /          --- X3=0,X=(1,1,0,0.86),Z=-0.429
                               /                  `-- X4=0,X=(1,1,0,0),Z=-9,F
                  --- X1=1,X=(1,1,0,0.86),Z=-0.429
                 /             \
                /               \
               /                 \
              /                   --- X2=0,X=(1,0,0,0.14),Z=-1.571,F
             /
            /
           /
X=(1,1,0,0.86),Z=-0.429
           \
            \                              --- X3=1,X=(0,1,1,0),Z=-1
             \                            /
              \                   --- X2=1,X=(0,1,0,0.5),Z=-1
               \                 /        \
                \               /          --- X3=0,X=(0,1,0,0.5),Z=-1,F
                 \             /
                  --- X1=0,X=(0,1,0,0.5),Z=-1
                               \
                                \
                                 \
                                  --- X2=0,Inf.,F
```

11.7-3.

(a)

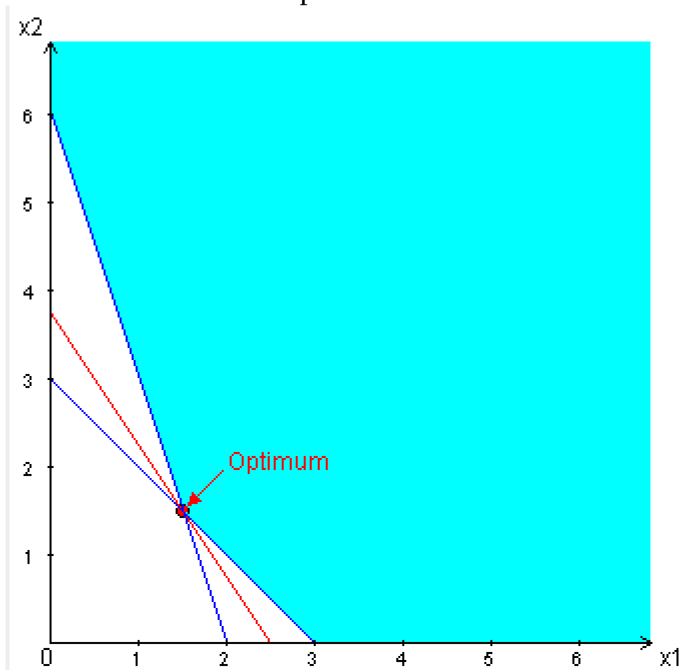
Corner Points	Z
$(1.5, 1.5)$	37.5
$(0, 6)$	60
$(3, 0)$	45



Optimal solution for the LP relaxation: $(1.5, 1.5)$ with $Z^* = 37.5$

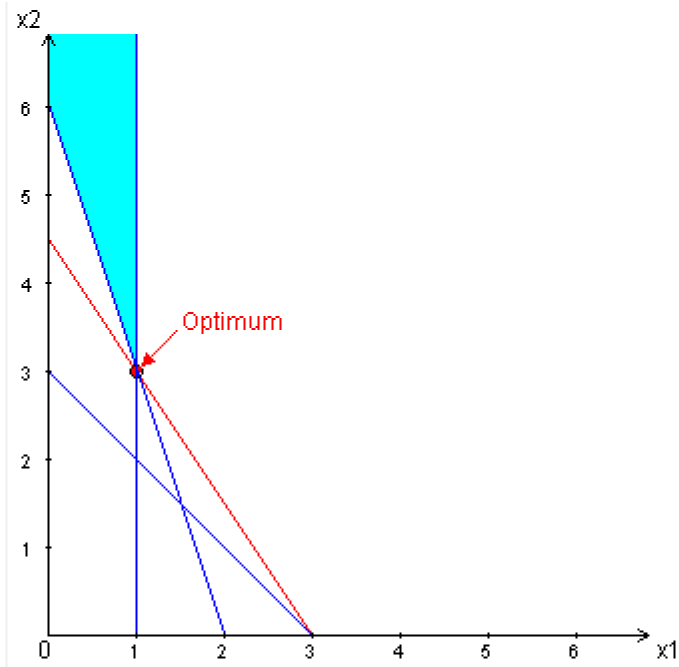
Optimal integer solution: $(2, 1)$ with $Z^* = 40$

(b) LP relaxation of the entire problem:



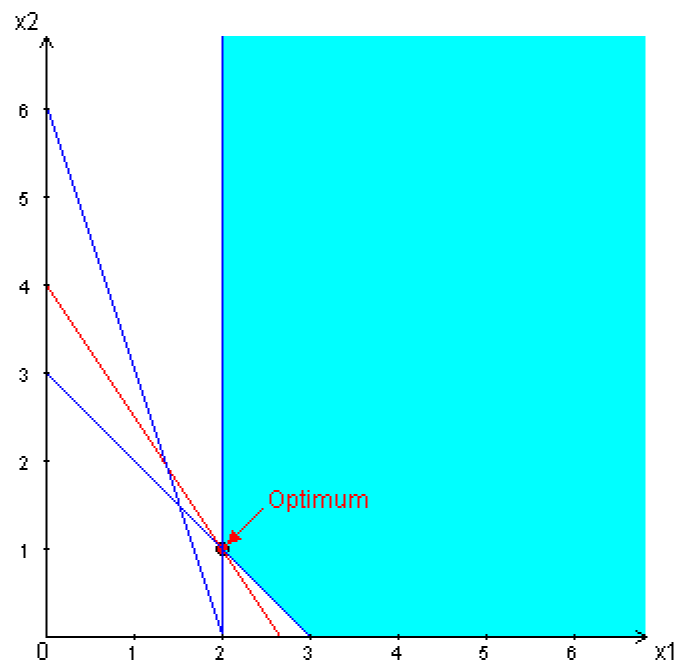
Optimal Solution: $(x_1, x_2) = (1.5, 1.5)$, $Z = 37.5$

Branch $x_1 \leq 1$:



Optimal Solution: $(x_1, x_2) = (1, 2)$, $Z = 45$, feasible for the original problem

Branch $x_1 \geq 2$:



Optimal Solution: $(x_1, x_2) = (2, 1)$, $Z = 40$

Hence, the optimal solution for the original problem is $(x_1, x_2) = (2, 1)$ with $Z = 40$.

(c) Let $x_1 = y_{11} + 2y_{12}$ and $x_2 = y_{21} + 2y_{22}$.

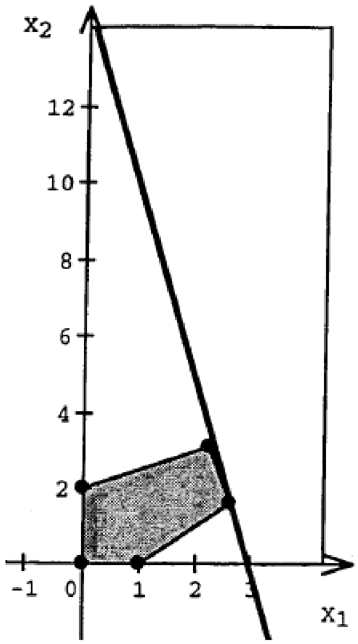
$$\begin{array}{ll} \text{minimize} & Z = 15y_{11} + 30y_{12} + 10y_{21} + 20y_{22} \\ \text{subject to} & 15y_{11} + 30y_{12} + 5y_{21} + 10y_{22} \geq 30 \\ & 10y_{11} + 20y_{12} + 10y_{21} + 20y_{22} \geq 30 \\ & y_{11}, y_{12}, y_{21}, y_{22} \text{ binary} \end{array}$$

(d) Optimal Solution: $(y_{11}, y_{12}, y_{21}, y_{22}) = (0,1,1,0)$, $Z = 40$, so $x_1 = 2$ and $x_2 = 1$ as in part (a).

```
                      --- X2=1,X=(1,1,0,0),Z=45,F
                     /
                    /
                   /
           --- X1=1,X=(1,0.25,0,0.75),Z=37.5
          /        \
         /          \
        /            \
       /              --- X2=0,X=(1,0,1,1),Z=45,F
      /
     /
    /
X=(0,0.75,0,0.75),Z=37.5
    \
     \                          --- X3=1,X=(0,1,1,0),Z=40
      \                        /
       \              --- X2=1,X=(0,1,0,0.5),Z=40
        \            /         \
         \          /           --- X3=0,X=(0,1,0,0.5),Z=40,F
          \        /
           --- X1=0,X=(0,0.75,0,0.75),Z=37.5
                   \
                    \
                     \
                      --- X2=0,Inf.,F
```

11.7-4.

(a)

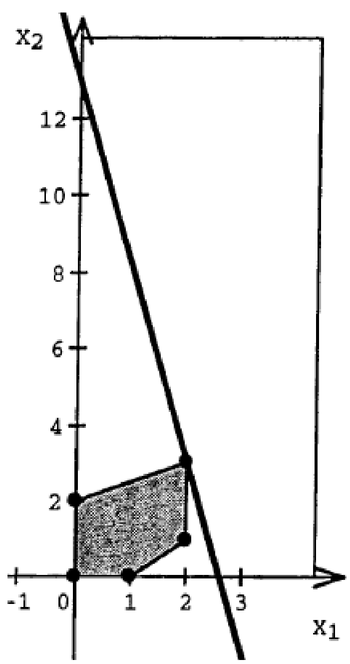


Corner Points	Z
(2.2222,3.1111)	14.222
(0, 2)	2
(2.6, 1.6)	14.6*
(1, 0)	5
(0, 0)	0

Optimal Solution: $(x_1, x_2) = (2.6, 1.6)$, $Z = 14.6$

<u>Branch $x_1 \geq 3$:</u> Infeasible

<u>Branch $x_1 \leq 2$:</u>

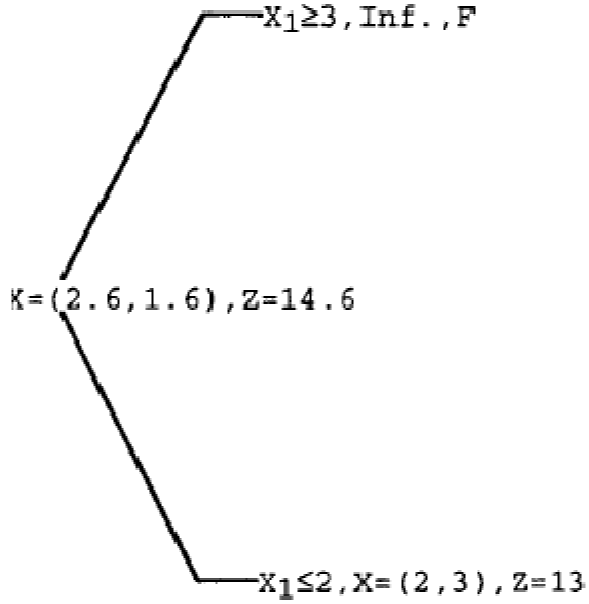


Corner Points	Z
(2, 3)	13*
(0, 2)	2
(2, 1)	11
(1, 0)	5
(0, 0)	0

Optimal Solution: $(x_1, x_2) = (2, 3)$, $Z = 13$, feasible for the original problem

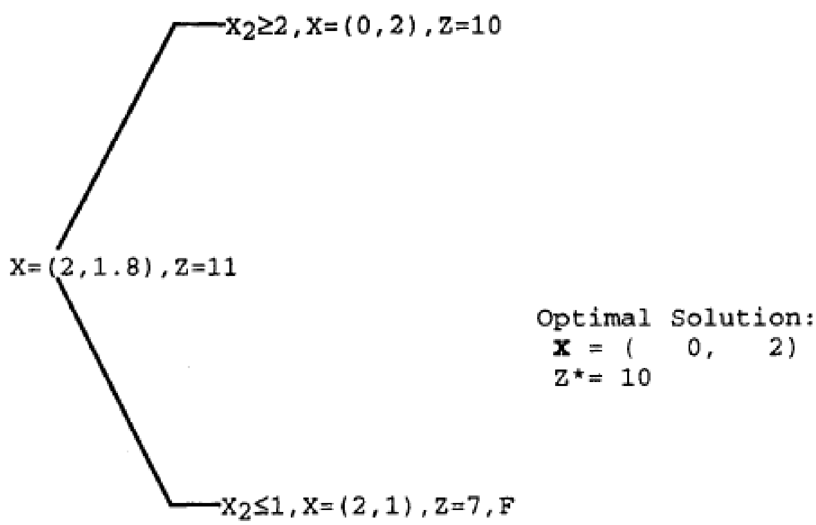
Hence, the optimal solution for the original problem is $(x_1, x_2) = (2, 3)$ with $Z = 13$.

(b) Optimal Solution: $(x_1, x_2) = (2, 3)$, $Z = 13$



(c) Solution: $(x_1, x_2) = (2, 3)$, $Z = 13$

11.7-5.



11.7-6.

```
                 --- X1>=15,X=(15,0,14.1),Z=95.529,F
                /
               /
              /
             /
            /
           /
          /
X=(14.1,0,15.9),Z=95.719
          \
           \
            \
             \              --- X2>=1,X=(13.6,1,15.4),Z=95.528,F
              \            /
               \          /
                \        /
                 --- X1<=14,X=(14,0.13,15.9),Z=95.693
                         \
                          \
                           \
                            --- X2<=0,X=(14,0,16),Z=95.6
```

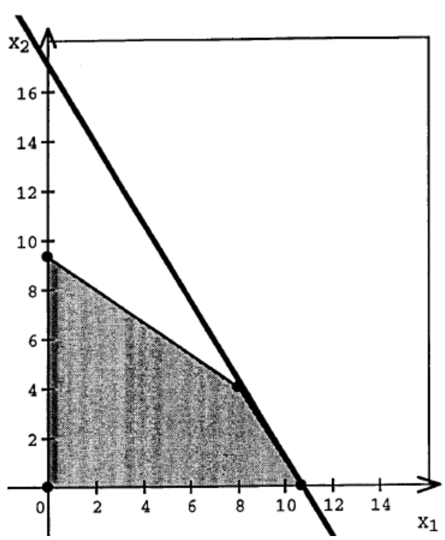
Optimal Solution: $\mathbf{x} = (14, 0, 16)$, $Z = 95.6$.

11.7-7.

(a) Let x_i be the number of ¼ units of product i to be produced, for $i = 1, 2$.

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 2.5x_2 \\ \text{subject to} \quad & \tfrac{3}{4}x_1 + \tfrac{1}{2}x_2 \leq 8 \\ & \tfrac{1}{2}x_1 + \tfrac{3}{4}x_2 \leq 7 \\ & x_1, x_2 \geq 0 \text{ integers} \end{aligned}$$

(b)

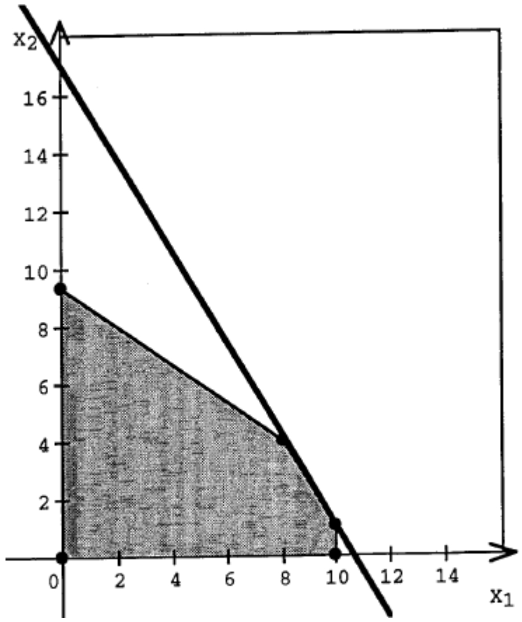


Corner Points	Z
(8, 4)	42
(10.667, 0)	42.667*
(0, 9.3333)	23.333
(0, 0)	0

Optimal Solution: $(x_1, x_2) = (10.667, 0)$, $Z = 42.667$

(c) Branch $x_1 \geq 11$: Infeasible

Branch $x_1 \leq 10$:

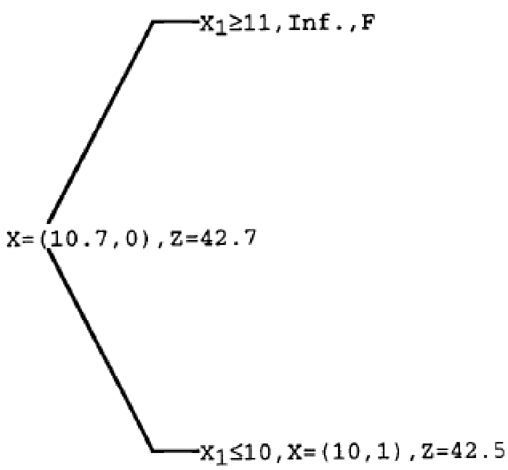


Corner Points	Z
(8, 4)	42
(10, 1)	42.5*
(0, 9.3333)	23.333
(10, 0)	40
(0, 0)	0

Optimal Solution: $(x_1, x_2) = (10, 1)$, $Z = 42.5$, feasible for the original problem

Hence, the optimal solution for the original problem is $(x_1, x_2) = (10, 1)$ with $Z = 42.5$.

(d) Optimal Solution: $(x_1, x_2) = (10, 1)$, $Z = 42.5$



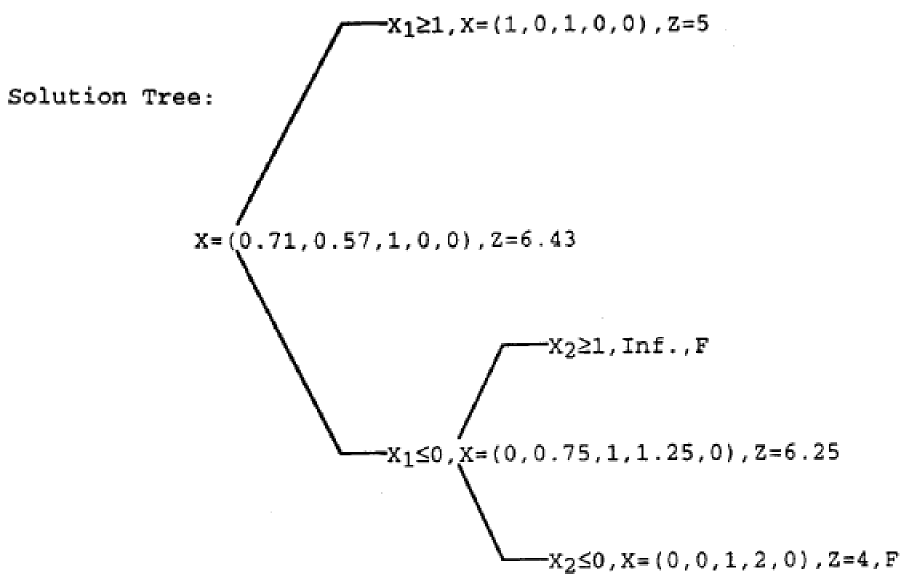
(e) Solution: $(x_1, x_2) = (10, 1)$, $Z = 42.5$

11.7-8.

```
                         --- X3>=6,Inf.,F
                        /
                       /
                      /
             --- X1>=3,X=(3,0,5.25,0.25),Z=196.25
            /         \
           /           \           --- X4>=1,X=(3,0,4.5,1),Z=192.5,F
          /             \         /
         /               --- X3<=5,X=(3,0,5,0.5),Z=195
        /                         \
       /                           --- X4<=0,X=(3,1,5,0),Z=195
      /
X=(2.13,0,3.73,3.07),Z=197.33
      \
       \                           --- X4>=3,X=(1,0,4,3),Z=180,F
        \                         /
         \               --- X3>=4,X=(2,0,4,2.67),Z=193.33
          \             /         \      ,-- X3>=5,X=(2,0,5,1),Z=185,F
           \           /           --- X4<=2,X=(2,0,4.4,2),Z=190
            \         /                  `-- X3<=4,X=(2,1,4,2),Z=190,F
             --- X1<=2,X=(2,0,3.71,3.14),Z=195.71
                      \
                       \           --- X4>=4,X=(1,0,3,4),Z=175,F
                        \         /
                         --- X3<=3,X=(2,0,3,3.5),Z=185
                                  \
                                   --- X4<=3,X=(2,1,3,3),Z=185,F
```

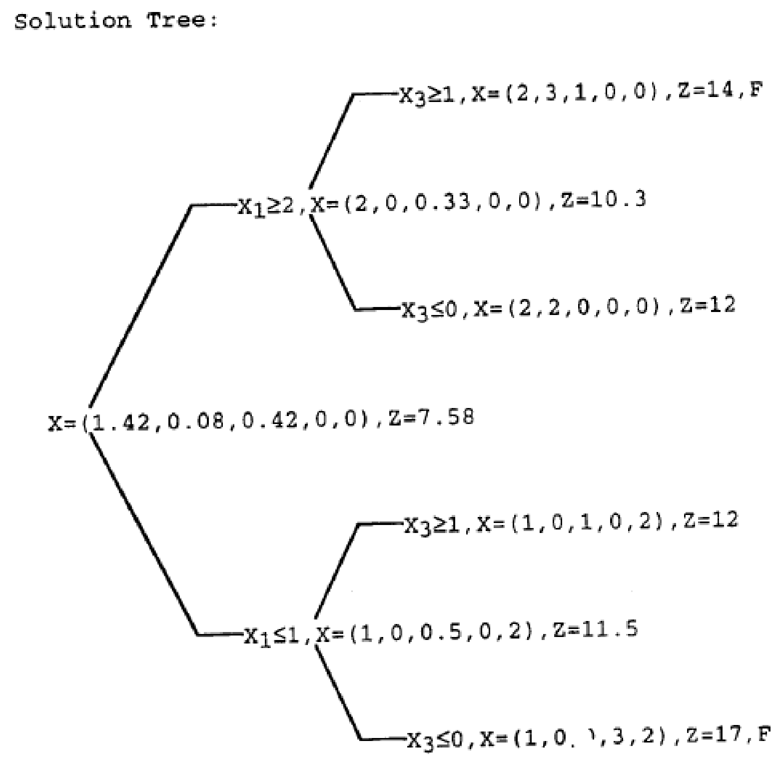
Optimal Solution: $\mathbf{x} = (3,1,5,0)$, $Z = 195$

11.7-9.



Optimal Solution: $\mathbf{x} = (1,0,1,0,0)$, $Z = 5$

11.7-10.



Optimal Solution: $\mathbf{x} = (1, 0, 1, 0, 2)$ and $\mathbf{x} = (2, 2, 0, 0, 0)$, $Z = 12$

11.8-1.

(a) $x_1 = 0, x_3 = 0$

(b) $x_1 = 0$

(c) $x_1 = 1, x_3 = 1$

11.8-2.

(a) $x_1 = 0$

(b) $x_1 = 1, x_2 = 0$

(c) $x_1 = 0, x_2 = 1$

11.8-3.

From the first equation, , $x_3 = 0$. Then, this equation becomes redundant. From the third equation, $x_5 = 0$ and $x_6 = 1$. Now, this equation is redundant, too. Since $x_6 = 1$, from the second equation, $x_2 = x_4 = 0$ and this equation becomes redundant. Finally, the fourth equation reduces to $x_1 = 0$. Consequently, all equations become redundant. The solution is then fixed to $(0, 0, 0, 0, 0, 1, x_7)$.

11.8-4.

(a) Redundant. Even if all the variables are set to their upper bounds, $x_i = 1$, $2 + 1 + 2 \leq 5$.

(b) Not redundant. For example, $(1, 0, 1)$ violates this constraint.

(c) Not redundant. For example $(0.0.0)$ violates this constraint.

(d) Redundant. The least value of $3x_1 - x_2 - 2x_3$ is attained by $(0, 1, 1)$ and it is -3, so the constraint is still satisfied.

11.8-5.

$$
\begin{aligned}
& \quad 4x_1 - 3x_2 + x_3 + 2x_4 \leq 5 \\
& b = 5, S = 7, S < b + |a_1| \Rightarrow \overline{a}_1 = S - b = 2, \overline{b} = S - a_1 = 3 \\
& \Rightarrow \quad 2x_1 - 3x_2 + x_3 + 2x_4 \leq 3 \\
& b = 3, S = 5, S < b + |a_2| \Rightarrow \overline{a}_2 = b - S = -2 \\
& \Rightarrow \quad 2x_1 - 2x_2 + x_3 + 2x_4 \leq 3 \\
& b = 3, S = 5, S \geq b + |a_j| \text{ for } j = 1, 2, 3, 4
\end{aligned}
$$

11.8-6.

$$
\begin{aligned}
& \quad 5x_1 - 10x_2 + 15x_3 \leq 15 \\
& b = 15, S = 20, S < b + |a_2| \Rightarrow \overline{a}_2 = b - S = -5 \\
& \Rightarrow \quad 5x_1 - 5x_2 + 15x_3 \leq 15 \\
& b = 15, S = 20, S < b + |a_3| \Rightarrow \overline{a}_3 = S - b = 5, \overline{b} = S - a_3 = 5 \\
& \Rightarrow \quad 5x_1 - 5x_2 + 5x_3 \leq 5 \\
& b = 5, S = 10, S \geq b + |a_j| \text{ for } j = 1, 2, 3
\end{aligned}
$$

11.8-7.

$$
\begin{aligned}
& \quad x_1 - x_2 + 3x_3 + 4x_4 \geq 1 \\
& \Leftrightarrow \quad -x_1 + x_2 - 3x_3 - 4x_4 \leq -1 \\
& b = -1, S = 1, S < b + |a_3| \Rightarrow \overline{a}_3 = b - S = -2 \\
& \Rightarrow \quad -x_1 + x_2 - 2x_3 - 4x_4 \leq -1 \\
& b = -1, S = 1, S < b + |a_4| \Rightarrow \overline{a}_4 = b - S = -2 \\
& \Rightarrow \quad -x_1 + x_2 - 2x_3 - 2x_4 \leq -1 \\
& b = -1, S = 1, S \geq b + |a_j| \text{ for } j = 1, 2, 3, 4
\end{aligned}
$$

11.8-8.

(a) $\quad x_1 + 3x_2 - 4x_3 \leq 2$

$$
\begin{aligned}
& b = 2, S = 4, S < b + |a_2| \Rightarrow \overline{a}_2 = S - b = 2, \overline{b} = S - a_2 = 1 \\
& \Rightarrow \quad x_1 + 2x_2 - 4x_3 \leq 1 \\
& b = 1, S = 3, S < b + |a_3| \Rightarrow \overline{a}_3 = b - S = -2 \\
& \Rightarrow \quad x_1 + 2x_2 - 2x_3 \leq 1 \\
& b = 1, S = 3, S \geq b + |a_j| \text{ for } j = 1, 2, 3
\end{aligned}
$$

(b) $\quad 3x_1 - x_2 + 4x_3 \geq 1$

$$
\begin{aligned}
& \Leftrightarrow \quad -3x_1 + x_2 - 4x_3 \leq -1 \\
& b = -1, S = 1, S < b + |a_1| \Rightarrow \overline{a}_1 = b - S = -2 \\
& \Rightarrow \quad -2x_1 + x_2 - 4x_3 \leq -1
\end{aligned}
$$

$b = -1, S = 1, S < b + |a_3| \Rightarrow \overline{a}_3 = b - S = -2$
$\Rightarrow \quad -2x_1 + x_2 - 2x_3 \leq -1$
$b = -1, S = 1, S \geq b + |a_j|$ for $j = 1, 2, 3$

11.8-9.

The minimum cover for the constraint $2x_1 + 3x_2 \leq 4$ is $\{x_1, x_2\}$, so the resulting cutting plane is $x_1 + x_2 \leq 1$, which is the same constraint obtained using the tightening procedure.

11.8-10.

$\{x_2, x_4\} \rightarrow x_2 + x_4 \leq 1$

$\{x_3, x_4\} \rightarrow x_3 + x_4 \leq 1$

$\{x_1, x_2, x_3\} \rightarrow x_1 + x_2 + x_3 \leq 2$

11.8-11.

$\{x_1, x_2\} \rightarrow x_1 + x_2 \leq 1$

$\{x_1, x_3\} \rightarrow x_1 + x_3 \leq 1$

$\{x_2, x_3, x_4\} \rightarrow x_2 + x_3 + x_4 \leq 2$

11.8-12.

$\{x_1, x_4\} \rightarrow x_1 + x_4 \leq 1$

$\{x_2, x_4\} \rightarrow x_2 + x_4 \leq 1$

$\{x_3, x_4\} \rightarrow x_3 + x_4 \leq 1$

$\{x_1, x_2, x_3\} \rightarrow x_1 + x_2 + x_3 \leq 2$

11.8-13.

$\{x_1, x_3\} \rightarrow x_1 + x_3 \leq 1$

$\{x_1, x_5\} \rightarrow x_1 + x_5 \leq 1$

$\{x_2, x_3\} \rightarrow x_2 + x_3 \leq 1$

$\{x_3, x_4\} \rightarrow x_3 + x_4 \leq 1$

$\{x_3, x_5\} \rightarrow x_3 + x_5 \leq 1$

$\{x_4, x_5\} \rightarrow x_4 + x_5 \leq 1$

$\{x_1, x_2, x_4\} \rightarrow x_1 + x_2 + x_4 \leq 2$

11.8-14.

(1) $3x_2 + x_4 + x_5 \geq 3 \Rightarrow x_2 = 1$

(2) $x_1 + x_2 \leq 1$ and $x_2 = 1 \Rightarrow x_1 = 0$

(3) $x_2 + x_4 - x_5 - x_6 \leq -1$ and $x_2 = 1 \Rightarrow x_4 = 0, x_5 = x_6 = 1$

(4) $x_2 + 2x_6 + 3x_7 + x_8 + 2x_9 \geq 4$ and $x_2 = x_6 = 1$

$\Rightarrow 3x_7 + x_8 + 2x_9 \geq 1 \Rightarrow x_7 + x_8 + x_9 \geq 1$

(5) $-x_3 + 2x_5 + x_6 + 2x_7 - 2x_8 + x_9 \leq 5$ and $x_5 = x_6 = 1$

$\Rightarrow -x_3 + 2x_7 - 2x_8 + x_9 \leq 2 \Rightarrow -x_3 + x_7 - x_8 + x_9 \leq 1$

Hence, the problem is reduced to finding binary variables x_3, x_7, x_8, x_9 that

$$\begin{array}{ll} \text{maximize} & x_3 + 2x_7 + x_8 + 3x_9 \\ \text{subject to} & \quad\;\; x_7 + x_8 + \;\; x_9 \geq 1 \\ & -x_3 + x_7 - x_8 + \;\; x_9 \leq 1. \end{array}$$

The objective is maximized when all variables with positive coefficients are set to their upper bounds, so when $x_3 = x_7 = x_8 = x_9 = 1$. This solution also satisfies the constraints, so it is optimal.
Optimal Solution: **x** $= (0,1,1,0,1,1,1,1,1)$, $Z = 15$

11.9-1.

Since the variables x_1, x_2, x_3 take values from the set $\{2,3,4\}$ and all the variables must have different values, $x_4 = 1$. There are two feasible solutions, $(2,4,3,1)$ and $(3,2,4,1)$. Their objective function values are $Z = 290$ and $Z = 280$ respectively, so $(2,4,3,1)$ is optimal.

11.9-2.

$\underline{x_1 = 12}$: $x_4 = 6$, $x_2 = 3$, and $x_3 = 9$, but $12 + 9 + 6 > 25$, so this is not feasible.

$\underline{x_1 = 6}$: $x_2 = 3$, $x_4 = 12$, and $x_3 = 9$, $6 + 9 + 12 > 25$, so this is not feasible.

$\underline{x_1 = 3}$: $x_2 = 6$, $x_4 = 12$, and $x_3 = 9$, $3 + 9 + 12 \leq 25$, so this is feasible. There are two feasible solutions, $(3,6,9,12,15)$ with $Z = 138$ and $(3,6,9,12,18)$ with $Z = 99$. Hence, $(3,6,9,12,15)$ is optimal.

11.9-3.

$\underline{x_1 = 25}$: $x_4 = 20$ and $x_3 = 30$, but $25 + 30 > 55$, so this is not feasible.

$\underline{x_1 = 30}$: $x_3 \in \{20, 25\}$, but $30 + 25 > 55$, so $x_3 = 20$ and $x_4 = 25$. There are two feasible solutions, $(30,35,20,25)$ with $Z = 11825$ and $(30,40,20,25)$ with $Z = 11950$, so $(30,40,20,25)$ is optimal.

11.9-4.

Let y_i denote the task to which the assignee i is assigned.

$$\begin{array}{ll} \text{minimize} & z_1 + z_2 + z_3 + z_4 \\ \text{subject to} & \text{element}(y_1, [13,16,12,11], z_1) \\ & \text{element}(y_2, [15,\text{M},13,20], z_2) \\ & \text{element}(y_3, [5,7,10,6], z_3) \\ & \text{element}(y_4, [0,0,0,0], z_4) \\ & \text{all-different}(y_1, y_2, y_3, y_4) \\ & y_i \in \{1,2,3,4\}, \text{ for } i = 1,2,3,4 \end{array}$$

11.9-5.

Relabel Carl, Chris, David, Tony and Ken as assignee $1, 2, 3, 4, 5$ respectively. Relabel Backstroke, Breaststroke, Butterfly, Freestyle and Dummy as tasks $1, 2, 3, 4, 5$ respectively. Let y_i be the task to which assignee i is assigned.

minimize	$z_1 + z_2 + z_3 + z_4 + z_5$
subject to	element$(y_1, [37.7, 43.4, 33.3, 29.2, 0], z_1)$
	element$(y_2, [32.9, 33.1, 28.5, 26.4, 0], z_2)$
	element$(y_3, [33.8, 42.2, 38.9, 29.6, 0], z_3)$
	element$(y_4, [37.0, 34.7, 30.4, 28.5, 0], z_4)$
	element$(y_5, [35.4, 41.8, 33.6, 31.1, 0], z_5)$
	all-different$(y_1, y_2, y_3, y_4, y_5)$
	$y_i \in \{1, 2, 3, 4, 5\}$, for $i = 1, 2, 3, 4, 5$

11.9-6.

Let y_i be the number of study days allocated to course i for $i = 1, 2, 3, 4$.

minimize	$z_1 + z_2 + z_3 + z_4$
subject to	element$(y_1, [1, 3, 6, 8], z_1)$
	element$(y_2, [5, 6, 8, 8], z_2)$
	element$(y_3, [4, 6, 7, 9], z_3)$
	element$(y_4, [4, 4, 5, 8], z_4)$
	$y_1 + y_2 + y_3 + y_4 \leq 7$
	$y_i \in \{1, 2, 3, 4\}$, for $i = 1, 2, 3, 4$

11.9-7.

Let y_i be the number of crates allocated to store i for $i = 1, 2, 3$.

minimize	$z_1 + z_2 + z_3$
subject to	element$(y_1 + 1, [0, 5, 9, 14, 17, 21], z_1)$
	element$(y_2 + 1, [0, 6, 11, 15, 19, 22], z_2)$
	element$(y_3 + 1, [0, 4, 9, 13, 18, 20], z_3)$
	$y_1 + y_2 + y_3 \leq 5$
	$y_i \in \{0, 1, 2, 3, 4, 5\}$, for $i = 1, 2, 3$

11.9-8.

minimize	$Z = \sum_{j=1}^{n} c_{x_j x_{j+1}}$
subject to	$x_j \in \{2, 3, \ldots, n\}$, for $j = 2, 3, \ldots, n$
	$x_1 = 1$
	$x_{n+1} = 1$
	all-different$(x_2, \ldots, x_n)$

11.10-1.

Answers will vary.

11.10-2.

Answers will vary.

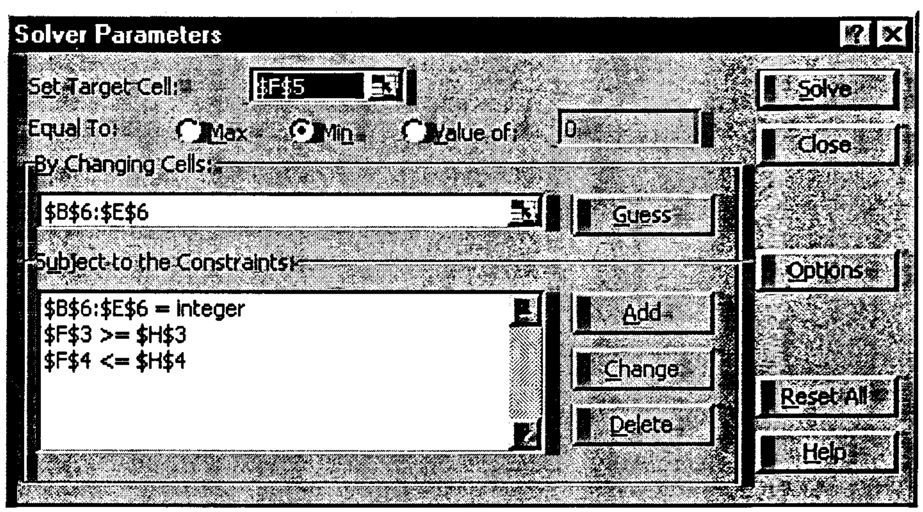
Cases

11-1 a) With this approach, we need to formulate an integer program for each month and optimize each month individually.

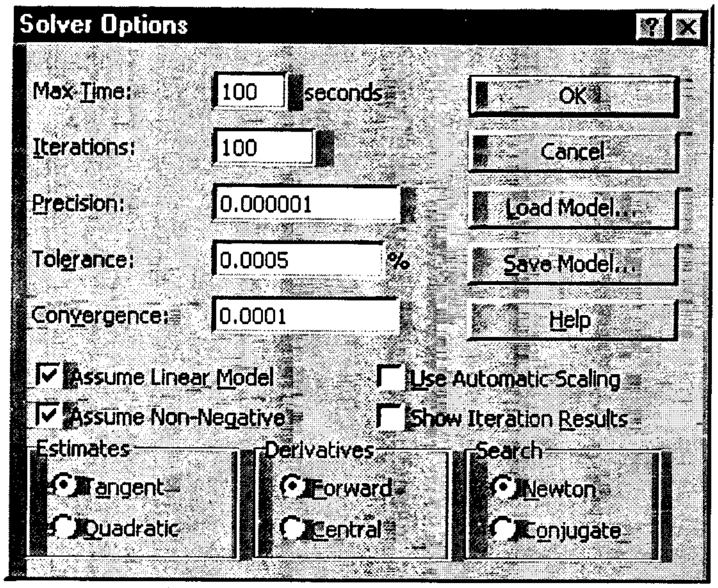
In the first month, Emily does not buy any servers since none of the departments implement the intranet in the first month. In the second month she must buy computers to ensure that the Sales Department can start the intranet. Emily can formulate her decision problem as an integer problem. Her objective is to minimize the purchase cost. She has to satisfy to constraints. She cannot spend more than $9500 (she still has her entire budget for the first two months since she didn't buy any computers in the first month) and the computer(s) must support at least 60 employees. She solves her integer programming problem using the Excel solver.

	A	B	C	D	E	F	G	H
1								
2	Server	Std. PC	Enh. PC	SGI	Sun	Totals		
3	Support	30	80	200	2000	80	>=	60
4	Capital m2	$2500	$5000	$9000	$18750	$5000	<=	$9500
5	Capital	$2500	$5000	$9000	$18750	$5000		
6	Solution	0	1	0	0			
7								
8		Formula in cell F3:		"=SUMPRODUCT(B3:E3,B6:E6)"				
9		Formula in cell F4:		"=SUMPRODUCT(B4:E4,B6:E6)"				
10		Formula in cell F5:		"=SUMPRODUCT(B5:E5,B6:E6)"				

The solver dialogue box looks as follows:



Emily uses the following options throughout her entire analysis of this case:

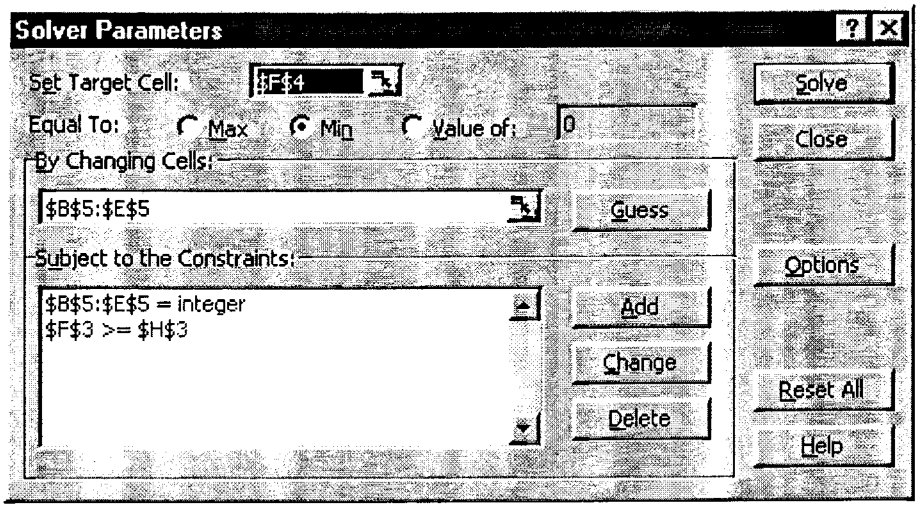


Notice that the price of the SGI server is 10 percent less than the actual price and the price of the Sun server is 25 percent less than the actual price because these two manufacturers offer discounts in the second month. Emily decides to buy one enhanced PC in the second month.

Note, that there is a second optimal solution to this integer programming problem. For the same amount of money Emily could buy two standard PC's that would also support 60 employees. However, since Emily knows that she needs to support more employees in the near future, she decides to buy the enhanced PC since it supports more users.

For the third month Emily needs to support 260 users. Since she has already computing power to support 80 users, she now needs to figure out how to support additional 180 users at minimum cost. She can disregard the constraint that the Manufacturing Department needs one of the three larger servers, since she already bought such a server in the previous month. Her task leads her to the following integer programming problem:

	A	B	C	D	E	F	G	H
1								Support
2	Server	Std. PC	Enh. PC	SGI	Sun	Totals		Needed
3	Support	30	80	200	2000	200	>=	180
4	Capital	$2,500	$5,000	$10,000	$25,000	$10,000		
5	Solution	0	0	1	0			
6								
7								
8		Formula in cell F3:		"=SUMPRODUCT(B3:E3,B6:E6)"				
9		Formula in cell F4:		"=SUMPRODUCT(B4:E4,B6:E6)"				



Emily decides to buy one SGI Workstation in month 3. The network is now able to support 280 users.

In the fourth month Emily needs to support a total of 290 users. Since she has already computing power to support 280 users, she now needs to figure out how to support additional 10 users at minimum cost. This task leads her to the following integer programming problem:

	A	B	C	D	E	F	G	H
1								Support
2	Server	Std. PC	Enh. PC	SGI	Sun	Totals		Needed
3	Support	30	80	200	2000	30	>=	10
4	Capital	$2,500	$5,000	$10,000	$25,000	$2,500		
5	Solution	1	0	0	0			
6								
7								
8		Formula in cell F3:		"=SUMPRODUCT(B3:E3,B6:E6)"				
9		Formula in cell F4:		"=SUMPRODUCT(B4:E4,B6:E6)"				

The solver dialogue box appears just as in her previous problem. Emily decides to buy a standard PC in the fourth month. The network is now able to support 310 users.

Finally, in the fifth and last month Emily needs to support the entire company with a total of 365 users. Since she has already computing power to support 310 users, she now needs to figure out how to support additional 55 users at minimum cost. This task leads her to the following integer programming problem:

	A	B	C	D	E	F	G	H
1								Support
2	Server	Std. PC	Enh. PC	SGI	Sun	Totals		Needed
3	Support	30	80	200	2000	80	>=	55
4	Capital	$2,500	$5,000	$10,000	$25,000	$5,000		
5	Solution	0	1	0	0			
6								
7								
8		Formula in cell F3:		"=SUMPRODUCT(B3:E3,B6:E6)"				
9		Formula in cell F4:		"=SUMPRODUCT(B4:E4,B6:E6)"				

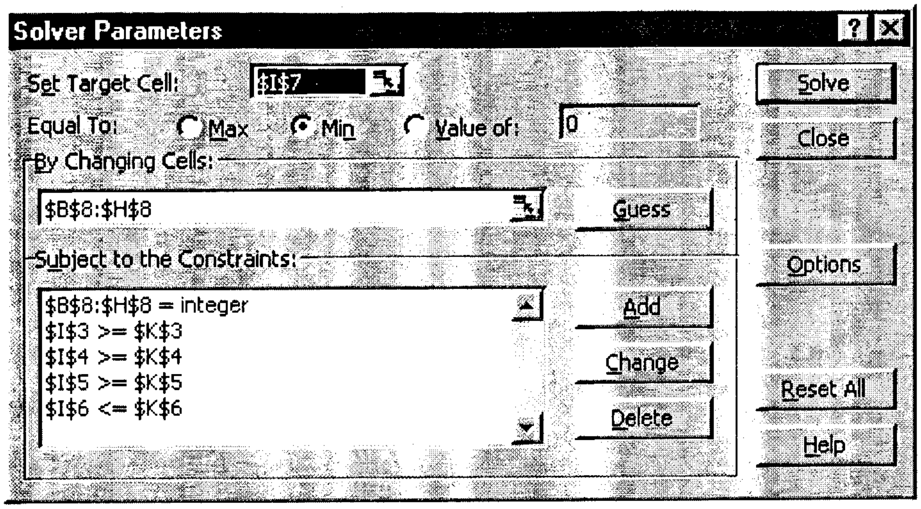
Again, the solver dialogue box has not changed. Emily decides to buy another enhanced PC in the fifth month. (Note that again she could have also bought two standard PC's, but clearly the enhanced PC provides more room for the workload of the system to grow.) The entire network of CommuniCorp consists now of 1 standard PC, 2 enhanced PC's and 1 SGI workstation and it is able to support 390 users. The total purchase cost for this network is $22,500.

b) Emily realizes that she will not be able to buy a Sun workstation during the first and second month, since the cost of such a server exceeds her budget even after the 25% discount. However, she could buy any one of the other three servers during the first two months. Due to the budget restriction she faces in the first two months she needs to distinguish between the computers she buys in those early months and in the later months. Therefore, Emily introduces two variables for each of the first three servers but only one for the Sun workstation:

Std. PC m2 = number of standard PCs bought during the first two months
Std. PC = number of standard PCs bought during the later three months
Enh. PC m2 = number of enhanced PCs bought during the first two months
Enh. PC = number of enhanced PCs bought during the later three months
SGI m2 = number of SGI workstations bought during the first two months
SGI = number of SGI workstations bought during the later three months
Sun = number of Sun workstations bought during the later three months

Emily essentially faces four constraints. First, she must support the 60 users in the sales department in the second month. She realizes that, since she no longer buys the computers sequentially after the second month, that it suffices to include only the constraint on the network to support the all users in the entire company. This second constraint requires her to support a total of 365 users. The third constraint requires her to buy at least one of the three large servers. Finally, Emily has to make sure that she stays within her budget during the second month.

	A	B	C	D	E	F	G	H	I	J	K
1											
2	Server	Stdd. PC m2	Stdd. PC	Enh. PC m2	Enh. PC	SGI m2	SGI	Sun	Totals		
3	Support m2	30	0	80	0	200	0	0	200	>=	60
4	Support	30	30	80	80	200	200	2000	400	>=	365
5	Large Comp.	0	0	1	1	1	1	1	2	>=	1
6	Capital m2	$2500	$0	$5,000	$0	$9,000	$0	$0	$9000	<=	$9500
7	Capital	$2,500	$2,500	$5,000	$5,000	$9,000	$10,000	$25,000	$19,000		
8	Solution	0	0	0	0	1	1	0			
9											
10											
11		Formula for cell I3:		"=SUMPRODUCT(B3:H3,B8:H8)"							
12		Formula for cell I4:		"=SUMPRODUCT(B4:H4,B8:H8)"							
13		Formula for cell I5:		"=SUMPRODUCT(B5:H5,B8:H8)"							
14		Formula for cell I6:		"=SUMPRODUCT(B6:H6,B8:H8)"							
15		Formula for cell I7:		"=SUMPRODUCT(B7:H7,B8:H8)"							



Emily should purchase a discounted SGI workstation in the second month, and another regular priced one in the third month. The total purchase cost is $19,000.

c) Emily's second method in part (b) finds the cost for the best overall purchase policy. The method in part (a) only finds the best purchase policy for the given month, ignoring the fact that the decision in a particular month has an impact on later decisions. The method in (a) is very short-sighted and thus yields a worse result that the method in part (b).

d) Installing the intranet will incur a number of other costs. These costs include:

Training cost,
Labor cost for network installation,
Additional hardware cost for cabling, network interface cards, necessary hubs, etc.,
Salary and benefits for a network administrator and web master,
Cost for establishing or outsourcing help desk support.

e) The intranet and the local area network are complete departures from the way business has been done in the past. The departments may therefore be concerned that the new technology will eliminate jobs. For example, in the past the manufacturing department has produced a greater number of pagers than customers have ordered. Fewer employees may be needed when the manufacturing department begins producing only enough pagers to meet orders. The departments may also become territorial about data and procedures, fearing that another department will encroach on their business. Finally, the departments may be concerned about the security of their data when sending it over the network.

11-2 a) We want to maximize the number of pieces displayed in the exhibit. For each piece, we therefore need to decide whether or not we should display the piece. Each piece becomes a binary decision variable. The decision variable is assigned 1 if we want to display the piece and assigned 0 if we do not want to display the piece.

We group our constraints into four categories – the artistic constraints imposed by Ash, the personal constraints imposed by Ash, the constraints imposed by Celeste, and the cost constraint. We now step through each of these constraint categories.

Artistic Constraints Imposed by Ash

Ash imposes the following constraints that depend upon the type of art that is displayed. The constraints are as follows:

1. Ash wants to include only one collage. We have four collages available: "Wasted Resources" by Norm Marson, "Consumerism" by Angie Oldman, "My Namesake" by Ziggy Lite, and "Narcissism" by Ziggy Lite. This constraint forces us to include only one of these four pieces.

2. Ash wants at least one wire-mesh sculpture displayed if a computer-generated drawing is displayed. We have three wire-mesh sculptures available and two computer-generated drawings available. Thus, if we include either one or two computer-generated drawings, we have to include at least one wire-mesh sculpture.

3. Ash wants at least one computer-generated drawing displayed if a wire-mesh sculpture is displayed. We have two computer-generated drawings available and three wire-mesh sculptures available. Thus, if we include one, two, or three wire-mesh sculptures, we have to include either one or two computer-generated drawings.

4. Ash wants at least one photo-realistic painting displayed. We have three photo-realistic paintings available: "Storefront Window" by David Lyman, "Harley" by David Lyman, and "Rick" by Rick Rawls. At least one of these three paintings has to be displayed.

5. Ash wants at least one cubist painting displayed. We have three cubist paintings available: "Rick II" by Rick Rawls, "Study of a Violin" by Helen Row, and "Study of a Fruit Bowl" by Helen Row. At least one of these three paintings has to be displayed.

6. Ash wants at least one expressionist painting displayed. We have only one expressionist painting available: "Rick III" by Rick Rawls. This painting has to be displayed.

7. Ash wants at least one watercolor painting displayed. We have six watercolor paintings available: "Serenity" by Candy Tate, "Calm Before the Storm" by Candy Tate, "All That Glitters" by Ash Briggs, "The Rock" by Ash Briggs, "Winding Road" by Ash Briggs, and "Dreams Come True" by Ash Briggs. At least one of these six paintings has to be displayed.

8. Ash wants at least one oil painting displayed. We have five oil paintings available: "Void" by Robert Bayer, "Sun" by Robert Bayer, "Beyond" by Bill Reynolds, "Pioneers" by Bill Reynolds, and "Living Land" by Bear Canton. At least one of these five paintings has to be displayed.

9. Finally, Ash wants the number of paintings to be no greater than twice the number of other art forms. We have 18 paintings available and 16 other art forms available. We classify the following pieces as paintings: "Serenity," "Calm Before the Storm," "Void," "Sun," "Storefront Window," "Harley," "Rick," "Rick II," "Rick III," "Beyond," "Pioneers," "Living Land," "Study of a Violin," "Study of a Fruit Bowl," "All That Glitters," "The Rock," "Winding Road," and "Dreams Come True." The maximum number of these paintings that we display has to be less than or equal to twice the number of other art forms we display.

Personal Constraints Imposed by Ash

1. Ash wants all of his own paintings included in the exhibit, so we must include "All That Glitters," "The Rock," "Winding Road," and "Dreams Come True."

2. Ash wants all of Candy Tate's work included in the exhibit, so we must include "Serenity" and "Calm Before the Storm."

3. Ash wants to include at least one piece from David Lyman, so we have to include one or more of the following pieces: "Storefront Window" and "Harley."

4. Ash wants to include at least one piece from Rick Rawls, so we have to include one or more of the following pieces: "Rick," "Rick II," and "Rick III."

5. Ash wants to display as many pieces from David Lyman as from Rick Rawls. Because the number of displayed pieces from David Lyman has to equal the number of displayed pieces from Rick Rawls and because David Lyman only has two pieces available, we can only display a maximum of two pieces from each of these artists.

6. Finally, Ash wants at most one piece from Ziggy Lite displayed. We can therefore include either none or one of the following pieces: "My Namesake" and "Narcissism."

Constraints Imposed by Celeste

1. Celeste wants to include at least one piece from a female artist for every two pieces included from a male artist. We have 11 pieces by female artists available: "Chaos Reigns" by Rita Losky, "Who Has Control?" by Rita Losky, "Domestication" by Rita Losky, "Innocence" by Rita Losky, "Serenity" by Candy Tate, "Calm Before the Storm" by Candy Tate, "Consumerism" by Angie Oldman, "Reflection" by Angie Oldman, "Trojan Victory" by Angie Oldman, "Study of a Violin" by Helen Row, and "Study of a Fruit Bowl" by Helen Row. One or more of these pieces has to be displayed for every two pieces by male artists displayed.

2. Celeste wants either one or both of the pieces "Aging Earth" and "Wasted Resources" displayed.

3. Celeste wants to include at least one piece by Bear Canton, so we must include one or more of the following pieces: "Wisdom," "Superior Powers," and "Living Land."

4. Celeste wants to include one or more of the following pieces: "Chaos Reigns," "Who Has Control," "Beyond," and "Pioneers."

5. Celeste knows that the museum only has enough floor space for four sculptures. We have six sculptures available: "Perfection" by Colin Zweibell, "Burden" by Colin Zweibell, "The Great Equalizer" by Colin Zweibell, "Aging Earth" by Norm Marson, "Reflection" by Angie Oldman, and "Trojan Victory" by Angie Oldman. We can only include a maximum of four of these six sculptures.

6. Celeste also knows that the museum only has enough wall space for 20 paintings, collages, and drawings. We have 28 paintings, collages, and drawings available: "Chaos Reigns," "Who Has Control," "Domestication," "Innocence," "Wasted Resources," "Serenity," "Calm Before the Storm," "Void," "Sun," "Storefront Window," "Harley," "Consumerism," "Rick," "Rick II," "Rick III," "Beyond," "Pioneers," "Wisdom," "Superior Powers," "Living Land," "Study of a Violin," "Study of a Fruit Bowl," "My Namesake," "Narcissism," "All That Glitters," "The Rock," "Winding Road," and "Dreams Come True." We can only include a maximum of 20 of these 28 wall pieces.

7. Finally, Celeste wants "Narcissism" displayed if "Reflection" is displayed. So if the decision variable for "Reflection" is 1, the decision variable for "Narcissism" must also be 1. However, the decision variable for "Narcissism" can still be 1 even if the decision variable for "Reflection" is 0.

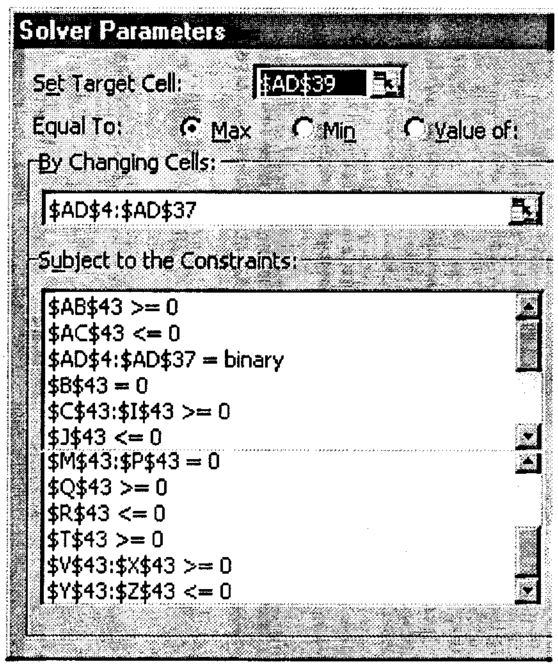
Cost Constraint

The cost of all of the pieces displayed has to be less than or equal to $4 million.

The problem formulation in an Excel spreadsheet follows.

	A	B	C	D	E	F	G	H	I	J	K	M	N	O	P	Q	R	T	U	V	W	X	Y	Z	AA	AB	AC	AD
1		Ash: Artistic Constraints										Ash: Personal Constraints						Celeste: Constraints										
2		Clg?	PC?	Wire?	Photo?	Cube?	Exprs?	Water?	Oil?	Paint?	Other?	Ash	Tate	Lym	Rawl	Frnd	Lite	Fem?	Male?	Enviro?	NtvAm?	Sci?	Flr	Wall	Narc	Rflct	Cost	Sol
3		(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)							(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)					000	
4	CZ: Prf			1							1								1				1				$300	0
5	CZ: Brd			1							1								1				1				$250	0
6	CZ: Gun			1							1								1				1				$125	1
7	RL: Cha		1								1							1				1		1			$400	1
8	RL: Who		1								1							1				1		1			$500	0
9	RL: Dom										1							1						1			$400	0
10	RL: Inn										1							1						1			$550	0
11	NM: Age										1								1	1			1				$700	0
12	NM: Wst	1									1								1	1				1			$575	1
13	CT: Srn							1		1			1					1						1			$200	1
14	CT: Clm							1		1			1					1						1			$225	1
15	RB: Void								1	1									1					1			$150	0
16	RB: Sun								1	1									1					1			$150	1
17	DL: Str				1					1				1		1			1					1			$850	0
18	DL: Har				1					1				1		1			1					1			$750	1
19	AO: Con	1									1							1						1			$400	0
20	AO: Ref										1							1					1			1	$175	1
21	AO: Tro										1							1					1				$450	0
22	RR: R1				1					1					1				1					1			$500	0
23	RR: R2					1				1					1				1					1			$500	0
24	RR: R3						1			1					1				1					1			$500	1
25	BR: Bnd								1	1									1			1		1			$650	0
26	BR: Pnr								1	1									1			1		1			$650	0
27	BC: Wis										1								1		1			1			$250	1
28	BC: Pwr										1								1		1			1			$350	0
29	BC: Lnd								1	1									1		1			1			$450	0
30	HR: Vln					1				1								1						1			$400	1
31	HR: Frt					1				1								1						1			$400	0
32	ZL: Nam	1									1						1		1					1			$300	0
33	ZL: Nrc	1									1						1		1					1	1		$300	0
34	AB: All							1		1		1							1					1			$50	1
35	AB: Rck							1		1		1							1					1			$50	1
36	AB: Rd							1		1		1							1					1			$50	1
37	AB: Drm							1		1		1							1					1			$50	1
38																												Max
39	Total	1	1	1	1	1	1	6	1	10	5	4	2	1	1	1	0	5	10	1	1	1	2	13	0	1	3950	15
40		=	>	>	>	>	>	>	>	<		=	=	=	=	>	<	>		>	>	>	<	<		>	<	
41			=	=	=	=	=	=	=	=						=	=	=		=	=	=	=	=		=	=	
42		1	0.500	0.333	1	1	1	1	1	10		4	2	1	1	1	1	5		1	1	1	4	20		0	4000	
43	Diff	0	1	1	0	0	0	5	0	0	5	0	0	0	0	0	-1	0	10	0	0	0	-2	-7	0	1	-50	
44		=	>	>	>	>	>	>	>	<		=	=	=	=	>	<	>		>	>	>	<	<		>	<	
45			=	=	=	=	=	=	=	=						=	=	=		=	=	=	=	=		=	=	
46		0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0		0	0	0	0	0		0	0	

The Solver settings used in the problem are shown below.



In the optimal solution, 15 pieces are displayed at a cost of $3.95 million. The following pieces are displayed:

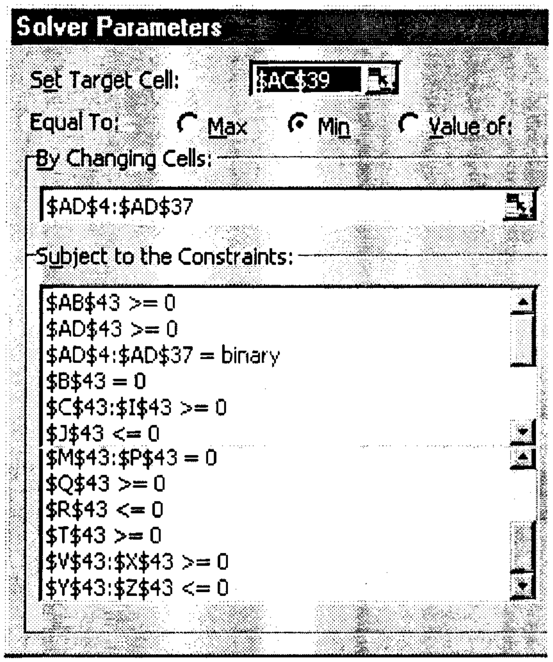
1. "The Great Equalizer" by Colin Zweibell
2. "Chaos Reigns" by Rita Losky
3. "Wasted Resources" by Norm Marson
4. "Serenity" by Candy Tate
5. "Calm Before the Storm" by Candy Tate
6. "Sun" by Robert Bayer
7. "Harley" by David Lyman
8. "Reflection" by Angie Oldman
9. "Rick III" by Rick Rawls
10. "Wisdom" by Bear Canton
11. "Study of a Violin" by Helen Row
12. "All That Glitters" by Ash Briggs
13. "The Rock" by Ash Briggs
14. "Winding Road" by Ash Briggs
15. "Dreams Come True" by Ash Briggs

b) The formulation of this problem is the same as the formulation in part (a) except that the objective function from part (a) now becomes a constraint and the cost constraint from part (a) now becomes the objective function. Thus, we have the new constraint that we need to select 20 or more pieces to display in the exhibit. We also have the new objective to minimize the cost of the exhibit.

The new formulation of the problem in an Excel spreadsheet appears below.

	A	B	C	D	E	F	G	H	I	J	K	M	N	O	P	Q	R	T	U	V	W	X	Y	Z	AA	AB	AC	AD
1		Ash: Artistic Constraints										Ash: Personal Constraints						Celeste: Constraints										
2		Cig?	PC?	Wire?	Photo?	Cube?	Expis?	Water?	Oil?	Paint?	Other?	Ash	Tate	Lymn	Rawl	Pind	Lite	Fem?	Male?	Enviro?	NtvAm?	Sci?	Flr	Wall	Natc	Rflct	Cost	Sol
3		(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)							(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)					000	
4	CZ: Prf			1							1								1				1				$300	0
5	CZ: Brd			1							1								1				1				$250	1
6	CZ: Gun			1							1								1				1				$125	1
7	RL: Cha		1								1							1				1		1			$400	1
8	RL: Who		1								1							1				1		1			$500	0
9	RL: Dom										1							1						1			$400	1
10	RL: Inn										1							1						1			$550	0
11	NM: Age										1								1	1			1				$700	0
12	NM: Wst	1									1								1	1				1			$575	1
13	CT: Srn							1		1			1					1						1			$200	1
14	CT: Clm							1		1			1					1						1			$225	1
15	RB: Void								1	1									1					1			$150	1
16	RB: Sun								1	1									1					1			$150	1
17	DL: Str				1					1				1		1			1					1			$850	0
18	DL: Har				1					1				1		1			1					1			$750	1
19	AO: Con	1									1							1						1			$400	0
20	AO: Ref										1							1					1			1	$175	1
21	AO: Tro										1							1					1				$450	0
22	RR: R1				1					1					1				1					1			$500	0
23	RR: R2					1				1					1				1					1			$500	0
24	RR: R3						1			1					1				1					1			$500	1
25	BR: Bnd								1	1									1			1		1			$650	0
26	BR: Pnr								1	1									1			1		1			$650	0
27	BC: Wis										1								1		1			1			$250	1
28	BC: Pwr										1								1		1			1			$350	1
29	BC: Lnd								1	1									1		1			1			$450	0
30	HR: Vln					1				1								1						1			$400	1
31	HR: Frt					1				1								1						1			$400	1
32	ZL: Nam	1									1						1		1					1			$300	-0
33	ZL: Nrc	1									1						1		1					1	1		$300	0
34	AB: All							1		1		1							1					1			$50	1
35	AB: Rck							1		1		1							1					1			$50	1
36	AB: Rd							1		1		1							1					1			$50	1
37	AB: Drm							1		1		1							1					1			$50	1
38																											Min	
39	Total	1	1	2	1	2	1	6	2	12	8	4	2	1	1	1	-0	7	13	1	2	1	3	17	0	1	5500	20
40		=	>	>	>	>	>	>	>	<		=	=	=	=	>	<	>		>	>	>	<	<		>		>
41			=	=	=	=	=	=	=	=						=	=	=		=	=	=	=	=		=		=
42		1	1.000	0.333	1	1	1	1	1	16		4	2	1	1	1	1	6.5		1	1	1	4	20		0		20
43	Diff	0	0	2	0	1	0	5	1	-4	0	0	0	0	0	0	-1	0	10	0	1	0	-1	-0	0	1		-0
44		=	>	>	>	>	>	>	>	<		=	=	=	=	>	<	>		>	>	>	<	<		>		>
45			=	=	=	=	=	=	=	=						=	=	=		=	=	=	=	=		=		=
46		0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0		0	0	0	0	0		0		0

The Solver settings used in the problem appear below.



In the optimal solution, exactly 20 pieces are displayed at a cost of $5.5 million – $1.5 million more than Ash decided to allocate in part (a). All pieces from part (a) are displayed in addition to the following five new pieces:

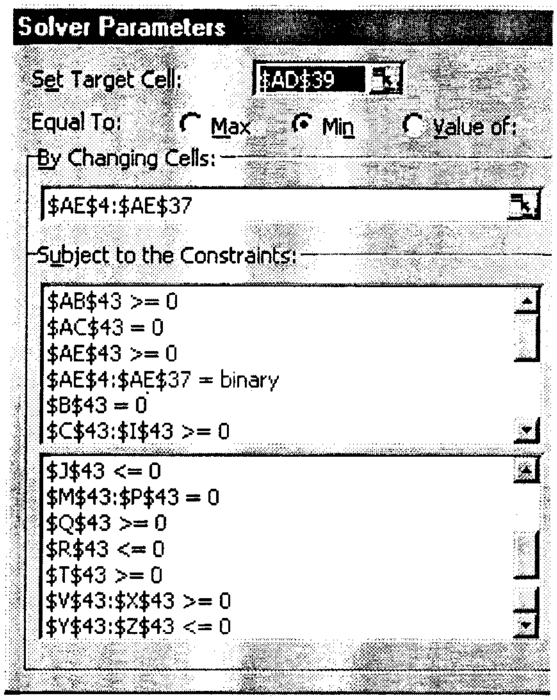
1. "Burden" by Colin Zweibell
2. "Domestication" by Rita Losky
3. "Void" by Robert Bayer
4. "Superior Powers" by Bear Canton
5. "Study of a Fruit Bowl" by Helen Row

c) This problem is also a cost minimization problem. The problem formulation is the same as that used in part (b). A new constraint is added, however. The patron wants all of Rita's pieces displayed. Rita has four pieces: "Chaos Reigns," "Who Has Control?," "Domestication," and "Innocence." All of these four pieces must be displayed.

The problem formulation in Excel appears below.

	A	B	C	D	E	F	G	H	I	J	K	M	N	O	P	Q	R	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE
1		Ash: Artistic Constraints										Ash: Personal Constraints						Celeste: Constraints											
2		Cg?	PC?	Wire?	Photo?	Cube?	Exprs?	H2O?	Oil?	Pant?	Other?	Ash	Tate	Lym	Rawl	Frnd	Lrte	Fem?	Male?	Envr?	NtvA?	Sci?	Flr	Wal	Narc	Rflct	Rita	Cost	Sol
3		(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)							(Y/N)	(Y/N)	(Y/N)	(Y/N)	(Y/N)						000	
4	CZ: Prf			1							1								1				1					$300	0
5	CZ: Brd			1							1								1				1					$250	1
6	CZ: Gun			1							1								1				1					$125	1
7	RL: Cha		1								1							1				1		1			1	$400	1
8	RL: Who		1								1							1				1		1			1	$500	1
9	RL: Dom										1							1						1			1	$400	1
10	RL: Inn										1							1						1			1	$550	1
11	NM: Age										1								1	1			1					$700	0
12	NM: Wst	1									1								1	1				1				$575	1
13	CT: Srn							1		1			1					1						1				$200	1
14	CT: Clm							1		1			1					1						1				$225	1
15	RB: Void								1	1									1					1				$150	1
16	RB: Sun								1	1									1					1				$150	1
17	DL: Str				1					1				1		1			1					1				$850	0
18	DL: Har				1					1				1		1			1					1				$750	1
19	AO: Con	1									1							1						1				$400	0
20	AO: Ref										1							1					1			1		$175	1
21	AO: Tro										1							1					1					$450	0
22	RR: R1				1					1					1				1					1				$500	0
23	RR: R2					1				1					1				1					1				$500	0
24	RR: R3						1			1					1				1					1				$500	1
25	BR: Bnd								1	1									1			1		1				$650	0
26	BR: Prr								1	1									1			1		1				$650	0
27	BC: Ws										1								1		1			1				$250	1
28	BC: Pwr										1								1		1			1				$350	0
29	BC: Lrd								1	1									1		1			1				$450	0
30	HR: Vh					1				1								1						1				$400	0
31	HR: Frt					1				1								1						1				$400	1
32	ZL: Nam	1									1						1		1					1				$300	0
33	ZL: Nrc	1									1						1		1					1	1			$300	0
34	AB: All							1		1		1							1					1				$50	1
35	AB: Rck							1		1		1							1					1				$50	1
36	AB: Rd							1		1		1							1					1				$50	1
37	AB: Drm							1		1		1							1					1				$50	1
38																												Min	
39	Total	1	2	2	1	1	1	6	2	11	9	4	2	1	1	1	0	8	12	1	1	2	3	17	0	1	4	5800	20
40		=	>	>	>	>	>	>	>	<		=	=	=	=	>	<	>		>	>	>	<	<		>	=		>
41			=	=	=	=	=	=	=	=						=	=	=		=	=	=	=	=		=			=
42		1	1.000	0.667	1	1	1	1	1	18		4	2	1	1	1	1	6		1	1	1	4	20		0	4		20
43	Diff	0	1	1	0	0	0	5	1	-7	9	0	0	0	0	0	-1	2	12	0	0	1	-1	-3	0	1	0		0
44		=	>	>	>	>	>	>	>	<		=	=	=	=	>	<	>		>	>	>	<	<		>	=		>
45			=	=	=	=	=	=	=	=						=	=	=		=	=	=	=	=		=			=
46		0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0		0	0	0	0	0		0	0		0

The Solver settings used in this problem appear below.



In the optimal solution, exactly 20 pieces are displayed at a total cost of $5.8 million. The patron has to pay $1.8 million. The following pieces are displayed:

1. "Burden" by Colin Zweibell
2. "The Great Equalizer" by Colin Zweibell
3. "Chaos Reigns" by Rita Losky
4. "Who Has Control?" by Rita Losky
5. "Domestication" by Rita Losky
6. "Innocence" by Rita Losky
7. "Wasted Resources" by Norm Marson
8. "Serenity" by Candy Tate
9. "Calm Before the Storm" by Candy Tate
10. "Void" by Robert Bayer
11. "Sun" by Robert Bayer
12. "Harley" by David Lyman
13. "Reflection" by Angie Oldman
14. "Rick III" by Rick Rawls
15. "Wisdom" by Bear Canton
16. "Study of a Fruit Bowl" by Helen Row
17. "All That Glitters" by Ash Briggs
18. "The Rock" by Ash Briggs
19. "Winding Road" by Ash Briggs
20. "Dreams Come True" by Ash Briggs

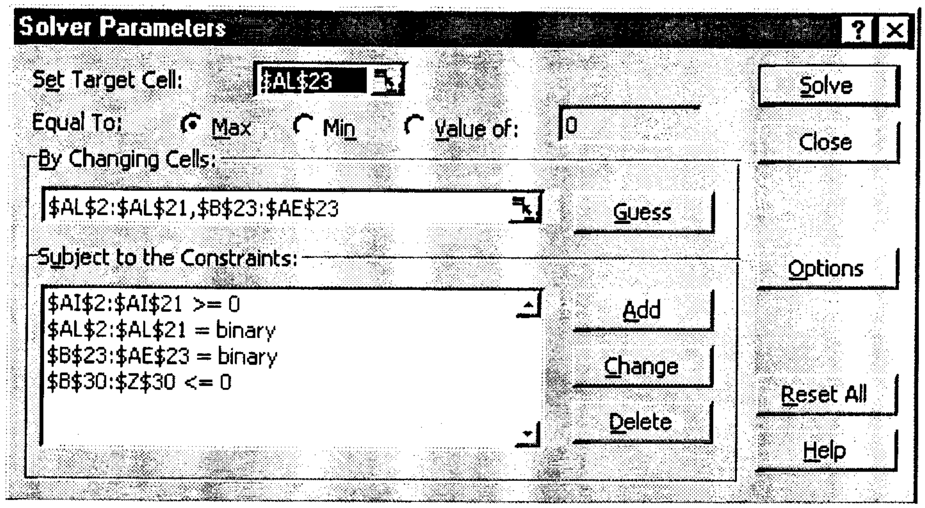
11-3 a) We want to maximize the total number of kitchen sets, so each of the 20 kitchen sets becomes a decision variable. But the kitchen sets are not our only decision variables. Because we assume that any particular item composing a kitchen set is replenished immediately, we only need to stock one of each item. A particular item may compose multiple kitchen sets. For example, tile T1 is part of kitchen sets 3, 7, 10, and 17. So a kitchen set exists when all of the items composing that kitchen set are in stock. Therefore, each of 30 items also becomes a decision variable. These decision variables are binary decision variables. If a kitchen set or item is in stock, the decision variable is 1. If a kitchen set or item is not in stock, the decision variable is 0.

A handful of constraints exist in this problem.
1. We cannot indicate that a kitchen set is in stock unless all the items composing that kitchen set are also in stock. Thus, a kitchen set decision variable is 1 only if all the decision variables for the items composing that kitchen set are also 1. For example, for set 1 this constraint equals 8*(Set 1) <= T2+W2+L4+C2+O4+S2+D2+R2
2. Each kitchen set requires 20 square feet of tile. Thus, if a particular tile is in stock, 20 square feet of that tile are in stock. The warehouse can only hold 50 square feet of tile, so only a maximum of two different styles of tile can be in stock.
3. Each kitchen set requires five rolls of wallpaper. Thus, if a particular style of wallpaper is in stock, five rolls of that wallpaper are in stock. The warehouse can only hold 12 rolls of wallpaper, so only a maximum of two different styles of wallpaper can be in stock.
4. A maximum of two different styles of light fixtures can be in stock.
5. A maximum of two different styles of cabinets can be in stock.
6. A maximum of three different styles of countertops can be in stock.
7. A maximum of two different sinks can be in stock.
8. A combination of four different styles of dishwashers and ranges can be held in stock.

The problem formulated in an Excel spreadsheet follows.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH	AI	AJ	AK	AL
1		T1	T2	T3	T4	W1	W2	W3	W4	L1	L2	L3	L4	C1	C2	C3	C4	O1	O2	O3	O4	S1	S2	S3	S4	D1	D2	R1	R2	R3	R4	Total		Set	Diff			Sol
2	Set 1		1				1						1		1						1		1				1		1			2	>=	0	2	>=	0	0
3	Set 2		1			1				1							1				1				1		1		1			4	>=	0	4	>=	0	0
4	Set 3	1						1			1			1				1						1		1				1		6	>=	0	6	>=	0	0
5	Set 4			1				1				1				1				1		1				1		1				6	>=	0	6	>=	0	0
6	Set 5				1				1	1				1					1				1			1		1				4	>=	0	4	>=	0	0
7	Set 6		1				1				1						1				1			1			1				1	4	>=	0	4	>=	0	0
8	Set 7	1						1					1			1			1			1				1		1				5	>=	0	5	>=	0	0
9	Set 8		1			1						1		1				1						1			1				1	8	>=	8	0	>=	0	1
10	Set 9		1			1					1					1			1				1				1		1			5	>=	0	5	>=	0	0
11	Set 10	1				1				1				1						1					1	1				1		5	>=	0	5	>=	0	0
12	Set 11			1		1						1				1		1				1				1				1		8	>=	8	0	>=	0	1
13	Set 12		1				1			1					1				1						1		1		1			4	>=	0	4	>=	0	0
14	Set 13				1				1			1				1		1					1			1				1		5	>=	0	5	>=	0	0
15	Set 14				1				1				1	1						1		1						1				2	>=	0	2	>=	0	0
16	Set 15			1				1		1				1				1						1						1		7	>=	7	0	>=	0	1
17	Set 16			1				1					1	1						1			1					1				3	>=	0	3	>=	0	0
18	Set 17	1							1		1					1				1					1					1		2	>=	0	2	>=	0	0
19	Set 18		1					1				1			1						1	1							1			4	>=	0	4	>=	0	0
20	Set 19		1						1				1				1				1		1								1	2	>=	0	2	>=	0	0
21	Set 20		1					1		1				1					1					1							1	7	>=	7	0	>=	0	1
22																																						Max
23	Sol	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0	1	1	0	0	1	1							4
24																																						
25																																						
26	Capacity	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0	0	4												
27		<				<				<				<				<				<				<												
28		=				=				=				=				=				=				=												
29		2				2				2				2				3				2				4												
30	Diff	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0												
31		<				<				<				<				<				<				<												
32		=				=				=				=				=				=				=												
33		0				0				0				0				0				0				0												

The solver settings are the following:



b) Four different kitchen sets are in stock. We should stock the following number of items:

Item	Quantity
T1	0
T2	1
T3	1
T4	0
W1	1
W2	0
W3	1
W4	0
L1	1
L2	0
L3	1
L4	0
C1	1
C2	0
C3	1
C4	0
O1	1
O2	1
O3	0
O4	0
S1	1
S2	0
S3	1
S4	0
D1	1
D2	1
R1	0
R2	0
R3	1
R4	1

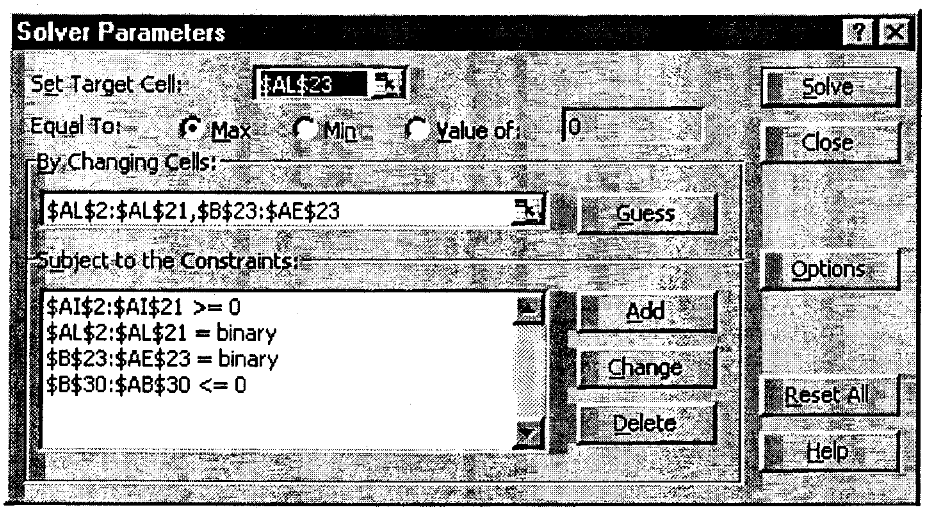
Note that the above optimal solution is not a unique solution. The value of the objective function is always four complete kitchen sets, but the specific items and kitchen sets stocked may be different. Throughout this solution, we will refer to the optimal solution shown above, but because other optimal solutions exist, student answers may differ from the solution somewhat.

c) We model this new problem by changing the capacity constraint for the dishwashers and ranges. Now, instead of being able to stock a combination of only four different styles of dishwashers and ranges, we can stock a maximum of two different styles of dishwashers and a maximum of three different styles of ranges. Because we only have two different styles of dishwashers available, we now effectively do not have a constraint on the number of dishwashers we can carry.

The formulation of the problem in Excel follows:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH	AI	AJ	AK	AL
1		T1	T2	T3	T4	W1	W2	W3	W4	L1	L2	L3	L4	C1	C2	C3	C4	O1	O2	O3	O4	S1	S2	S3	S4	D1	D2	R1	R2	R3	R4	Total		Set	Diff			Sol
2	Set 1		1				1						1		1						1		1				1		1			2	>=	0	2	>=	0	0
3	Set 2		1			1				1							1				1				1		1		1			4	>=	0	4	>=	0	0
4	Set 3	1						1			1			1				1						1		1				1		6	>=	0	6	>=	0	0
5	Set 4			1				1				1				1				1		1				1		1				8	>=	8	0	>=	0	1
6	Set 5				1				1	1				1					1				1			1		1				5	>=	0	5	>=	0	0
7	Set 6		1				1				1						1				1			1			1				1	4	>=	0	4	>=	0	0
8	Set 7	1						1					1			1			1			1				1		1				6	>=	0	6	>=	0	0
9	Set 8		1			1						1		1				1						1			1				1	8	>=	8	0	>=	0	1
10	Set 9		1			1					1					1			1				1				1		1			5	>=	0	5	>=	0	0
11	Set 10	1				1				1				1						1					1	1				1		6	>=	0	6	>=	0	0
12	Set 11			1		1						1				1		1				1				1				1		8	>=	8	0	>=	0	1
13	Set 12		1				1			1					1				1						1		1		1			4	>=	0	4	>=	0	0
14	Set 13				1				1			1				1		1					1			1				1		5	>=	0	5	>=	0	0
15	Set 14				1				1				1	1						1		1						1				4	>=	0	4	>=	0	0
16	Set 15			1				1		1				1				1						1						1		7	>=	7	0	>=	0	1
17	Set 16			1				1					1	1						1			1					1				5	>=	0	5	>=	0	0
18	Set 17	1							1		1					1				1					1					1		3	>=	0	3	>=	0	0
19	Set 18		1					1				1			1						1	1							1			4	>=	0	4	>=	0	0
20	Set 19		1						1				1				1				1		1								1	2	>=	0	2	>=	0	0
21	Set 20		1					1		1				1					1					1							1	7	>=	7	0	>=	0	1
22																																						Max
23	Sol	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	0	1	0	1	0	1	1	1	0	1	1							5
24																																						
25																																						
26	Capacity	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0	0	3	0	0	0	2	0	0	0	0	0	3										
27		<				<				<				<				<				<						<										
28		=				=				=				=				=				=						=										
29		2				2				2				2				3				2						3										
30	Diff	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
31		<				<				<				<				<				<						<										
32		=				=				=				=				=				=						=										
33		0				0				0				0				0				0						0										

The Solver settings for this problem follow.



With the extra space, the number of kitchen sets we can stock increases from four to five. In part (a), the constraint on the number of different styles of dishwashers and ranges was binding, and we could only stock four different kitchen sets – sets 8, 11, 15, and 20. With the extra space, we can add set 4 to our stock. Set 4 requires two items that are not required by sets 8, 11, 15, and 20 – a different countertop O3 and a different range R1. In part (a), the constraint limiting the maximum number of different styles of countertops was non-binding, so we can add a new countertop style to our stock. The new space vacated by the nursery department provides us with the space to stock the new range.

d) With the additional space, our constraints change. We eliminate the constraints limiting the maximum number of different styles of sinks and countertops we can stock. Instead of stocking two of the four styles of light fixtures, we can now stock three of the four styles of light fixtures. Finally, instead of stocking only two of the four cabinet styles, we can now stock three of the four cabinet styles.

The problem formulated in Excel follows.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH	AI	AJ	AK	AL
1		T1	T2	T3	T4	W1	W2	W3	W4	L1	L2	L3	L4	C1	C2	C3	C4	O1	O2	O3	O4	S1	S2	S3	S4	D1	D2	R1	R2	R3	R4	Total		Set	Diff			Sol
2	Set 1		1				1						1		1						1		1				1		1			4	>=	0	4	>=	0	0
3	Set 2		1			1				1							1				1				1		1		1			4	>=	0	4	>=	0	0
4	Set 3	1						1			1			1				1						1		1				1		6	>=	0	6	>=	0	0
5	Set 4			1				1				1				1				1		1				1		1				8	>=	8	0	>=	0	1
6	Set 5				1				1	1				1					1				1			1		1				6	>=	0	6	>=	0	0
7	Set 6		1				1				1						1				1			1			1				1	4	>=	0	4	>=	0	0
8	Set 7	1						1					1			1			1			1				1		1				7	>=	0	7	>=	0	0
9	Set 8		1			1						1		1				1						1			1				1	8	>=	8	0	>=	0	1
10	Set 9		1			1					1					1			1				1				1		1			6	>=	0	6	>=	0	0
11	Set 10	1				1				1				1						1					1	1				1		6	>=	0	6	>=	0	0
12	Set 11			1		1						1				1		1				1				1				1		8	>=	8	0	>=	0	1
13	Set 12		1				1			1					1				1						1		1		1			4	>=	0	4	>=	0	0
14	Set 13				1				1			1				1		1					1			1				1		6	>=	0	6	>=	0	0
15	Set 14				1				1				1	1						1		1						1				5	>=	0	5	>=	0	0
16	Set 15			1				1		1				1				1						1						1		7	>=	7	0	>=	0	1
17	Set 16			1				1					1	1						1			1					1				7	>=	7	-0	>=	0	1
18	Set 17	1							1		1					1				1					1					1		3	>=	0	3	>=	0	0
19	Set 18		1					1				1			1						1	1							1			4	>=	0	4	>=	0	0
20	Set 19		1						1				1				1				1		1								1	4	>=	0	4	>=	0	0
21	Set 20		1					1		1				1					1					1							1	7	>=	7	0	>=	0	1
22																																						Max
23	Sol	0	1	1	0	1	0	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1							6
24																																						
25																																						
26	Capacity	2	0	0	0	2	0	0	0	3	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	3										
27		<				<				<				<														<										
28		=				=				=				=														=										
29		2				2				3				3														3										
30	Diff	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0										
31		<				<				<				<														<										
32		=				=				=				=														=										
33		0				0				0				0														0										

The Solver settings are the same settings used in part (c).

With the extra space, we are now able to stock six complete kitchen kits. In part (c), we stocked five sets – sets 4, 8, 11, 15, and 20. Now we add set 16 to the inventory. All of the extra space does not increase the number of complete sets we can stock significantly because the capacity constraints on the countertops, and cabinets were not binding in part (c). Only the capacity constraints on the sinks and light fixtures were binding. Set 16 requires both a sink and a light fixture that sets 4, 8, 11, 15, and 20 do not require. We therefore add an additional style of sink and light fixture to our stock. We still have space for one additional sink, so the constraint is not binding in this problem.

e) If the items composing a kitchen set could not be replenished immediately, we could not formulate this problem as a binary integer program. We would have to formulate the problem as an integer program since we may have to store more than one kitchen component or kitchen set to ensure that we meet demand.

The assumption of immediate replenishment is justified if the average time to replenish the component is less than the average time between demands for that component.

12-4 a)

Let $x_{ij} = 1$ if students from area i are assigned to school j; 0 if not

C_{ij} = bussing cost

S_i = student population of area i

K_j = capacity of school j

P_{ik} = % of students in area i in grade k

(for $i = 1,2,3,4,5,6 \quad j = 1,2,3$ and $k = 6,7,8$).

Minimize Cost $= \sum_{i=1}^{6}\sum_{j=1}^{3}(C_{ij})(S_i)(x_{ij})$

subject to $\sum_i S_i x_{ij} \le K_j$

$$\sum_j x_{ij} = 1$$

$$0.30\sum_i S_i x_{ij} \le \sum_i P_{ik} S_i x_{ij} \le 0.36 \sum_i S_i x_{ij}, \qquad (k = 6,7,8)$$

and x_{ij} are binary variables (for $i = 1,2,3,4,5,6$ and $j = 1,2,3$).

Note $x_{21} = x_{43} = x_{52} = 0$ due to infeasibility.

b) The models really aren't too different. x_{ij} are binary here, which amounts to forcing their value in the LP of Case 4-3 to be either 0 or S_i. We can leave out the three variables known to be 0, and also 9 redundant constraints. The LP-relaxation of this model, with $0 \le x_{ij} \le 1$ would allow us to interpret x_{ij} as the fraction of students from area i to be assigned to school j. This obviously would be a more general model, equivalent to that in Case 4-3.

c)

Area	Number of Students	Percentage in 6th grade	Percentage in 7th grade	Percentage in 8th grade	Bussing Cost ($/student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	9999	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	9999
5	500	0.39	0.34	0.27	0	9999	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	950	1150	1050

Solution:

Area	Area Assignments School 1	School 2	School 3	Total		
1	0	1	0	1	=	1
2	0	1	0	1	=	1
3	0	0	1	1	=	1
4	1	0	0	1	=	1
5	0	0	1	1	=	1
6	1	0	0	1	=	1

Total Bussing Cost = 845000

Area	Number of Students School 1	School 2	School 3
1	0	450	0
2	0	600	0
3	0	0	550
4	350	0	0
5	0	0	500
6	450	0	0
Total	800	1050	1050
	<=	<=	<=
Capacity	950	1150	1050

Grade Constraints:

	School 1	School 2	School 3
6th graders	251	366	360
7th graders	266	339	346
8th graders	283	345	344
30% of total	240	315	315
36% of total	288	378	378

d)
Without prohibiting the splitting of residential areas, the total cost was 555,555. Thus, adding this restriction increases the cost by $845000 - $555,555 = $289,445

e)

Area	Number of Students	Percentage in 6th grade	Percentage in 7th grade	Percentage in 8th grade	Bussing Cost ($/student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	9999	400	500
3	550	0.3	0.32	0.38	600	300	0
4	350	0.28	0.4	0.32	0	500	9999
5	500	0.39	0.34	0.27	0	9999	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	950	1150	1050

Solution:

Area	Area Assignments School 1	School 2	School 3	Total		
1	0	1	0	1	=	1
2	0	1	0	1	=	1
3	0	0	1	1	=	1
4	1	0	0	1	=	1
5	0	0	1	1	=	1
6	1	0	0	1	=	1

Total Bussing Cost = 665000

Area	Number of Students School 1	School 2	School 3
1	0	450	0
2	0	600	0
3	0	0	550
4	350	0	0
5	0	0	500
6	450	0	0
Total	800	1050	1050
	<=	<=	<=
Capacity	950	1150	1050

Grade Constraints:

	School 1	School 2	School 3
6th graders	251	366	360
7th graders	266	339	346
8th graders	283	345	344
30% of total	240	315	315
36% of total	288	378	378

As shown in the spreadsheet, the solution remains the same, but the busing costs are reduced to $665000.

f)

Area	Number of Students	Percentage in 6th grade	Percentage in 7th grade	Percentage in 8th grade	Bussing Cost ($/student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	0	0	700
2	600	0.37	0.28	0.35	9999	400	500
3	550	0.3	0.32	0.38	600	0	0
4	350	0.28	0.4	0.32	0	500	9999
5	500	0.39	0.34	0.27	0	9999	400
6	450	0.34	0.28	0.38	500	0	0
				Capacity:	950	1150	1050

Solution:

Area	Area Assignments School 1	School 2	School 3	Total		
1	0	1	0	1	=	1
2	0	1	0	1	=	1
3	0	0	1	1	=	1
4	1	0	0	1	=	1
5	0	0	1	1	=	1
6	1	0	0	1	=	1

Total Bussing Cost = 665000

Area	Number of Students School 1	School 2	School 3
1	0	450	0
2	0	600	0
3	0	0	550
4	350	0	0
5	0	0	500
6	450	0	0
Total	800	1050	1050
	<=	<=	<=
Capacity	950	1150	1050

Grade Constraints:

	School 1	School 2	School 3
6th graders	251	366	360
7th graders	266	339	346
8th graders	283	345	344
30% of total	240	315	315
36% of total	288	378	378

As shown in the spreadsheet, the solution and the bussing cost remains the same as for Option 1.

g)

For all three options, the assignments are identical. For the current alternative, the bussing costs are $845000. For option 1, the bussing costs are $665000 (a reduction of $180000). This savings results from the fact that students from area 3 would no longer be bussed to school 3 and the same happens for area 4 and school 1.

For option 2, the bussing costs are the same as those for option 1, the reason being that under the optimal assignment policy there are no students bused 1.5 to 2 miles.

h) Arguments can be made for the current situation and option 1. Option 2 can be discarded since it provides no improvement over option 1.

CHAPTER 12: NONLINEAR PROGRAMMING

12.1-1.

In 1995, a number of factors including increased competition, the lack of quantitative tools to support financial advices and the introduction of new regulations compelled Bank Hapoalim to review its investment advisory process. Consequently, the Opti-Money system was developed as a tool to offer systematic financial advice. The underlying mathematical model is a constrained nonlinear program with continuous or discontinuous derivatives depending on the selected risk measure. The variables x_i denote the fraction of asset i. The goal is to choose a portfolio that minimizes "risk" among all portfolios with a fixed expected return. Opti-Money allows the investor to choose among four risk measures, viz., symmetric return variability, asymmetric downside risk, asymmetric return variability around more than one benchmark, and classical Markowitz risk of a portfolio. Once the risk measure and the benchmark(s) are specified, the objective function is formulated as a weighted sum of this risk measure and a market-portfolio tracking term. Then the efficient frontier is constructed.

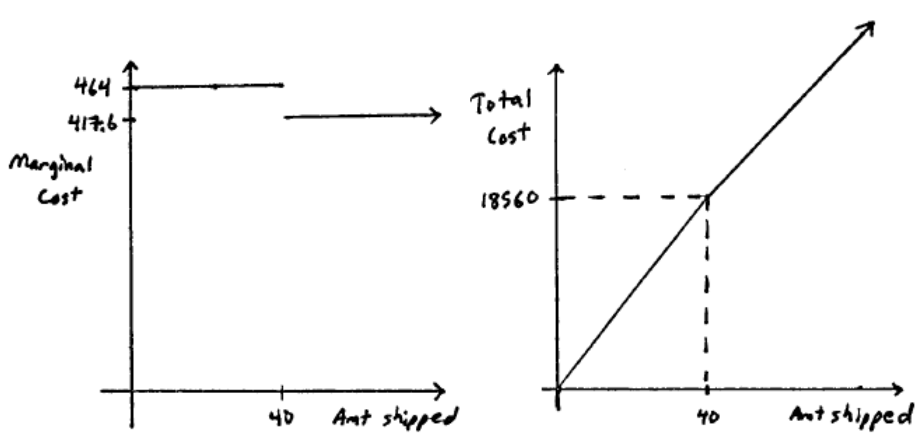
The Opti-Money system increased average monthly profit of Bank Hapoalim significantly. The average annual return for customers has also increased. The excess earnings using Opti-Money exceeds $200 million per year. The subsidiaries of the bank like Continental Mutual Fund benefit from Opti-Money, too. The new system resulted in "an organizational revolution in the investment advisory process at Bank Hapoalim" [p. 46]. As a result of this study, additional consultation-support systems are developed to help the customer relations managers.

12.1-2.

maximize $\quad f(\boldsymbol{x}) = 100x_1^{2/3} + 10x_1 + 40x_2^{3/4} + 5x_2 + 50x_3^{1/2} + 5x_3$

subject to
$$\begin{aligned} 9x_1 + 3x_2 + 5x_3 &\leq 500 \\ 5x_1 + 4x_2 \quad\quad\quad &\leq 350 \\ 3x_1 \quad\quad + 2x_3 &\leq 150 \\ x_3 &\leq 20 \\ x_1, x_2, x_3 \quad &\geq 0 \end{aligned}$$

12.1-3.



Each term in the objective function changes (as above) from $a_{ij}x_{ij}$ to

$$a_{ij}x_{ij} - 0.1a_{ij}(x_{ij} - 40)S(x_{ij} - 40)$$

where a_{ij} is the shipping cost from cannery i to warehouse j and

$$S(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

The rest of the formulation is the same.

12.1-4.

Let S_1 and S_2 be the number of blocks of stock 1 and 2 to purchase respectively.

$$\begin{array}{ll} \text{minimize} & f(S_1, S_2) = 4S_1^2 + 100S_2^2 + 5S_1S_2 \\ \text{subject to} & 20S_1 + 30S_2 \leq 50 \\ & \quad 5S_1 + 10S_2 \geq \text{minimum acceptable expected return} \\ & \quad\; S_1, S_2 \qquad \geq 0 \end{array}$$

12.2-1.

$f(\boldsymbol{x}) = f_1(x_1) + f_2(x_2) + f_3(x_3)$

with $f_1(x_1) = 100x_1^{2/3} + 10x_1$, $f_2(x_2) = 40x_2^{3/4} + 5x_2$, $f_3(x_3) = 50x_3^{1/2} + 5x_3$.

$\frac{d^2 f_1(x_1)}{dx_1^2} = -\frac{200}{9}x_1^{-4/3} \leq 0$ for $x_1 \geq 0$

$\frac{d^2 f_2(x_2)}{dx_2^2} = -\frac{120}{16}x_2^{-5/4} \leq 0$ for $x_2 \geq 0$

$\frac{d^2 f_3(x_3)}{dx_3^2} = -\frac{50}{4}x_3^{-3/2} \leq 0$ for $x_3 \geq 0$

f_1, f_2 and f_3 are concave on the nonnegative orthant so f is concave in the same region. The constraints are linear. Hence, the problem is a convex programming problem.

12.2-2.

$\frac{d^2 f(S_1,S_2)}{dS_1^2} = 8 \geq 0$, $\frac{d^2 f(S_1,S_2)}{dS_2^2} = 200 \geq 0$, $\frac{d^2 f(S_1,S_2)}{dS_1 dS_2} = 5 \geq 0$

$\frac{d^2 f(S_1,S_2)}{dS_1^2}\frac{d^2 f(S_1,S_2)}{dS_2^2} - \left[\frac{d^2 f(S_1,S_2)}{dS_1 dS_2}\right]^2 = 1575 \geq 0$

Hence, f is convex everywhere.

12.2-3.

Objective function: $Z = 3x_1 + 5x_2 \Rightarrow x_2 = -(3/5)x_1 + (1/5)Z \Rightarrow$ slope: $-(3/5)$

Constraint boundary: $9x_1^2 + 5x_2^2 = 216 \Rightarrow x_2 = \sqrt{(1/5)(216 - 9x_1^2)}$

$\Rightarrow \frac{\partial x_2}{\partial x_1} = -\frac{1}{5}\frac{9x_1}{\sqrt{(1/5)(216 - 9x_1^2)}} = -\frac{3}{5}$ for $x_1 = 2$

Hence, the objective function is tangent to this constraint at $(x_1, x_2) = (2, 6)$.

12.2-4.

Constraint boundary: $3x_1 + 2x_2 = 18 \Rightarrow g(x_1) = x_2 = -\frac{3}{2}x_1 + 9 \Rightarrow \frac{dg(x_1)}{dx_1} = -\frac{3}{2}$

Objective function at $(8/3, 5)$: $(9x_1^2 - 126x_1 + 857) - 182x_2 + 13x_2^2 = 0$

$\Rightarrow f(x_1) = x_2 = \frac{182 - 2\sqrt{-2860 + 1638x_1 - 117x_1^2}}{26} \Rightarrow \frac{df(x_1)}{dx_1} = -\frac{3}{2}$

$f(8/3) = g(8/3) = 5$

Hence, the objective function is tangent to this constraint at $(x_1, x_2) = (8/3, 5)$.

12.2-5.

(a) $\frac{df(x)}{dx} = 240 - 600x + 30x^2 = 0$

$\Rightarrow x^* = \frac{600 \pm \sqrt{600^2 - 4 \cdot 30 \cdot 240}}{60} = 0.408$ or 19.592

$\frac{d^2 f(x)}{dx^2} = -600 + 60x$

$\frac{d^2 f(0.408)}{dx^2} = -575.5 \Rightarrow f(0.408) = 48.66$ is a local maximum.

$\frac{d^2 f(19.592)}{dx^2} = 575.5 \Rightarrow f(19.592) = -35,248.7$ is a local minimum.

(b) For $x > 19.592$, $\frac{df(x)}{dx} > 0$ and $\frac{d^2 f(x)}{dx^2} = 60x - 600 > 0 \Rightarrow f$ is unbounded above.

For $x < 0.408$, $\frac{df(x)}{dx} < 0$ and $\frac{d^2 f(x)}{dx^2} = 60x - 600 < 0 \Rightarrow f$ is unbounded below.

12.2-6.

(a) $\frac{d^2 f(x)}{dx^2} = -2 < 0$ for all $x \Rightarrow f$ is concave.

(b) $\frac{d^2 f(x)}{dx^2} = 12x^2 + 12 > 0$ for all $x \Rightarrow f$ is convex.

(c) $\frac{d^2 f(x)}{dx^2} = 12x - 6 \begin{cases} > 0 & \text{for } x > 1/2 \\ < 0 & \text{for } x < 1/2 \end{cases} \Rightarrow f$ is neither convex nor concave.

(d) $\frac{d^2 f(x)}{dx^2} = 12x^2 + 2 > 0$ for all $x \Rightarrow f$ is convex.

(e) $\frac{d^2 f(x)}{dx^2} = 6x + 12x^2 \begin{cases} > 0 & \text{for } x < -1/2 \text{ or } x > 0 \\ < 0 & \text{for } -1/2 < x < 0 \end{cases} \Rightarrow f$ is neither convex nor concave.

12.2-7.

(a) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = -2 < 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 4 - 1^2 = 3 > 0$ for all (x_1, x_2)

$\Rightarrow f$ is concave.

(b) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = 4 > 0$, $\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = 2 > 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 8 - 2^2 = 4 > 0$ for all (x_1, x_2)

$\Rightarrow f$ is convex.

(c) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = 2 > 0$, $\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = 4 > 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 8 - 3^2 = -1 < 0$ for all (x_1, x_2)

$\Rightarrow f$ is neither convex nor concave.

(d) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} = 0$

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 0$

$\Rightarrow f$ is both convex and concave.

(e) $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 0 - 1^2 = -1 < 0$ for all (x_1, x_2)

$\Rightarrow f$ is neither convex nor concave.

12.2-8.

$f(x) = f_1(x_1) + f_2(x_2) + f_{34}(x_3, x_4) + f_{56}(x_5, x_6) + f_{67}(x_6, x_7)$

with $f_1(x_1) = 5x_1$, $f_2(x_2) = 2x_2^2$, $f_{34}(x_3, x_4) = x_3^2 - 3x_3x_4 + 4x_4^2$,

$f_{56}(x_5, x_6) = x_5^2 + 3x_5x_6 + 3x_6^2$, $f_{67}(x_6, x_7) = 3x_6^2 + 3x_6x_7 + x_7^2$.

$\frac{d^2 f_1(x_1)}{dx_1^2} = 0$ for all $x_1 \Rightarrow f_1$ is convex (and concave).

$\frac{d^2 f_2(x_2)}{dx_2^2} = 4 > 0$ for all $x_2 \Rightarrow f_2$ is convex.

$\frac{d^2 f_{34}(x_3,x_4)}{dx_3^2} = 2 > 0$, $\frac{d^2 f_{34}(x_3,x_4)}{dx_4^2} = 8 > 0$ for all (x_3, x_4)

$\frac{d^2 f_{34}(x_3,x_4)}{dx_3^2} \frac{d^2 f_{34}(x_3,x_4)}{dx_4^2} - \left[\frac{d^2 f_{34}(x_3,x_4)}{dx_3 dx_4}\right]^2 = 16 - 3^2 = 7 > 0$ for all (x_3, x_4)

$\Rightarrow f_{34}$ is convex.

$\frac{d^2 f_{56}(x_5,x_6)}{dx_5^2} = 2 > 0$, $\frac{d^2 f_{56}(x_5,x_6)}{dx_6^2} = 6 > 0$ for all (x_5, x_6)

$\frac{d^2 f_{56}(x_5,x_6)}{dx_5^2} \frac{d^2 f_{56}(x_5,x_6)}{dx_6^2} - \left[\frac{d^2 f_{56}(x_5,x_6)}{dx_5 dx_6}\right]^2 = 12 - 3^2 = 3 > 0$ for all (x_5, x_6)

$\Rightarrow f_{56}$ is convex.

$f_{67}(x_6, x_7) = f_{56}(x_7, x_6) \Rightarrow f_{67}$ is convex.

Hence, f is convex.

12.2-9.

(a) maximize $f(\boldsymbol{x}) = x_1 + x_2$

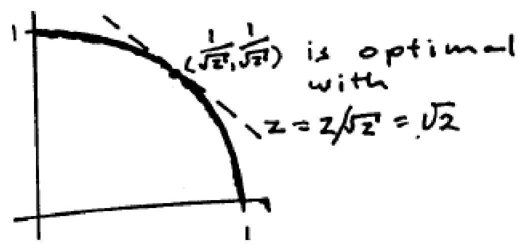
subject to $g(\boldsymbol{x}) = x_1^2 + x_2^2 \leq 1, \boldsymbol{x} \geq 0$

$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} = \frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} = \frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2} = 0, \frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = 0 \Rightarrow f$ is concave (convex).

$\frac{\partial^2 g(\boldsymbol{x})}{\partial x_1^2} = \frac{\partial^2 g(\boldsymbol{x})}{\partial x_2^2} = 2 > 0, \frac{\partial^2 g(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 g(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 g(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = 4 - 0^2 = 4 > 0 \Rightarrow g$ is convex.

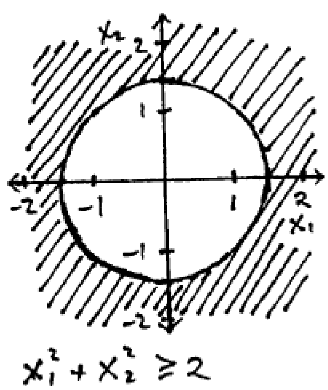
The problem is a convex programming problem.

(b)



12.2-10.

(a)



Clearly, this is not a convex feasible region. For example, take the points $(0, \sqrt{2})$ and $(0, -\sqrt{2})$, $(0, 0) = \frac{1}{2}(0, \sqrt{2}) + \frac{1}{2}(0, -\sqrt{2})$ is not feasible.

(b) Feasible region: $-x_1^2 - x_2^2 \leq -2$

Both $g_1(x_1) = -x_1^2$ and $g_2(x_2) = -x_2^2$ are concave functions, so the feasible region need not be convex.

$$\frac{d^2 g_1(x_1)}{dx_1^2} = \frac{d^2 g_2(x_2)}{dx_2^2} = -1 < 0$$

To prove that the feasible region is not convex, one needs to find two feasible points y and z, a scalar $\alpha \in [0, 1]$ such that $\alpha y + (1 - \alpha) z$ is not feasible. Such points are given in part (a).

12.3-1.

Since the objective is to minimize a concave function, as shown in Problem 12.1-3, this is a nonconvex programming problem.

12.3-2.

$\frac{df(x)}{dx} = -120 + 30x - 30x^2 = 0 \Rightarrow x = \frac{-30 \pm \sqrt{30^2 - 4 \cdot 30 \cdot 120}}{-60}$ no real solution

$\frac{d^2 f(x)}{dx^2} = 30 - 60x \begin{cases} > 0 & \text{for } x < 1/2 \\ < 0 & \text{for } x > 1/2 \end{cases}$

The slope of f increases from -120 at $x = 0$ to -112.5 at $x = 1/2$ and decreases for all x thereafter. It is always negative, so $x^* = 0$ is optimal.

12.3-3.

(a) Linearly Constrained Convex Programming:

$g_1(x_1, x_2) = 2x_1 + x_2$ and $g_2(x_1, x_2) = x_1 + 2x_2$ are linear.

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = -12x_1^2 - 4 < 0$, $\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = -8 < 0$ for all (x_1, x_2)

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 = 96x_1^2 + 32 - 2^2 > 0$ for all (x_1, x_2)

$\Rightarrow f$ is concave.

Geometric Programming:

$f(\boldsymbol{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}} + c_3 x_1^{a_{31}} x_2^{a_{32}} + c_4 x_1^{a_{41}} x_2^{a_{42}}$

where $c_1 = 1, a_{11} = 4, a_{12} = 0$

$c_2 = 2, a_{21} = 2, a_{22} = 0$

$c_3 = 2, a_{31} = 1, a_{32} = 1$

$c_4 = 4, a_{41} = 0, a_{42} = 2$

$g_1(\boldsymbol{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}}$

where $c_1 = -2, a_{11} = 1, a_{12} = 0$

$c_2 = -1, a_{21} = 0, a_{22} = 1$

$g_2(\boldsymbol{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}}$

where $c_1 = -1, a_{11} = 1, a_{12} = 0$

$c_2 = -2, a_{21} = 0, a_{22} = 1$

Fractional Programming:

$f' = f_1/f_2$ where $f_1 = f$ and $f_2 = 1$

(b) Let $y_1 = x_1 - 1$ and $y_2 = x_2 - 1$.

minimize $\quad y_1^4 + 4y_1^3 + 8y_1^2 + 10y_1 + 2y_1 y_2 + 4y_2^2 + 10y_2$

subject to
$$\begin{aligned} 2y_1 + \ \ y_2 &\geq 7 \\ y_1 + 2y_2 &\geq 7 \\ y_1, y_2 &\geq 0 \end{aligned}$$

12.3-4.

(a) Let $x_1 = e^{y_1}$ and $x_2 = e^{y_2}$.

minimize $\quad f(\boldsymbol{y}) = 2e^{-2y_1-y_2} + e^{-y_1-2y_2}$

subject to $\quad g(\boldsymbol{y}) = 4e^{y_1+y_2} + e^{2y_1+2y_2} - 12 \leq 0$

$e^{y_1}, e^{y_2} \geq 0$ (true for any (y_1, y_2))

(b) $\frac{\partial^2 f(\boldsymbol{y})}{\partial y_1^2} = 8e^{-2y_1-y_2} + e^{-y_1-2y_2} \geq 0$ for all (y_1, y_2)

$\frac{\partial^2 f(\boldsymbol{y})}{\partial y_2^2} = 2e^{-2y_1-y_2} + 4e^{-y_1-2y_2} \geq 0$ for all (y_1, y_2)

$\frac{\partial^2 f(\boldsymbol{y})}{\partial y_1^2}\frac{\partial^2 f(\boldsymbol{y})}{\partial y_2^2} - \left[\frac{\partial^2 f(\boldsymbol{y})}{\partial y_1 \partial y_2}\right]^2 = 18e^{-3y_1-3y_2} \geq 0$ for all (y_1, y_2)

$\Rightarrow f$ is convex.

$\frac{\partial^2 g(\boldsymbol{y})}{\partial y_1^2} = \frac{\partial^2 g(\boldsymbol{y})}{\partial y_2^2} = \frac{\partial^2 g(\boldsymbol{y})}{\partial y_1 \partial y_2} = 4e^{y_1+y_2} + 4e^{2y_1+2y_2} \geq 0$ for all (y_1, y_2)

$\frac{\partial^2 g(\boldsymbol{y})}{\partial y_1^2}\frac{\partial^2 g(\boldsymbol{y})}{\partial y_2^2} - \left[\frac{\partial^2 g(\boldsymbol{y})}{\partial y_1 \partial y_2}\right]^2 = 0$ for all (y_1, y_2)

$\Rightarrow g$ is convex.

Hence, this is a convex programming problem.

12.3-5.

(a) maximize $\quad 10y_1 + 20y_2 + 10t$

subject to

$$\begin{aligned} y_1 + \ 3y_2 - 50t &\leq 0 \\ 3y_1 + \ 4y_2 - 80t &\leq 0 \\ 3y_1 + \ 4y_2 + 20t &= 1 \\ y_1, y_2, t &\geq 0 \end{aligned}$$

(b)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
								1M	
Z	0	1	3.269	0	0	1.385	0	3.962	3.962
X2	1	0	0.654	1	0	0.077	0	0.192	0.192
X5	2	0	3.231	0	0	-1.38	1	0.538	0.538
X3	3	0	0.019	0	1	-0.02	0	0.012	0.012

The variables $(X1, X2, X3)$ in this courseware solution correspond to the variables (y_1, y_2, t) in (a), so the optimal solution is $(y_1, y_2, t) = (0, 0.192, 0.012)$ with the objective function value $Z = 3.962$. Then, the optimal solution of the original problem is $(x_1, x_2) = (0, 16.67)$ with the optimal objective function value $f(\boldsymbol{x}) = 3.962$.

12.3-6.

KKT conditions:
$$\begin{aligned} &Qx + A^T u - c = y \\ &-Ax + b = v \\ &x, u, y, v \geq 0 \\ &x^T(Qx + A^T u - c) + u^T(-Ax + b) = 0 \end{aligned}$$

This is the linear complementarity problem with:

$$Z = \begin{pmatrix} x \\ u \end{pmatrix}, M = \begin{pmatrix} Q & A^T \\ -A & 0 \end{pmatrix}, q = \begin{pmatrix} -c \\ b \end{pmatrix}, w = \begin{pmatrix} Qx + A^T u - c \\ -Ax + b \end{pmatrix}.$$

12.4-1.

(a)

Interactive One-Dimensional Search Procedure:

Max f(X) = 1 X^3 + 2 X - 2 X^2 -0.25 X^4

df(X)/dX = 3 X^2 + 2 - 4 X - 1 X^3

Lower Bound: 0 Upper Bound: 2.4

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	2.4	1.2	0.7296
1	-0.208	0	1.2	0.6	0.6636
2	+0.464	0.6	1.2	0.9	0.745
3	+0.101	0.9	1.2	1.05	0.7487
4	- 0.05	0.9	1.05	0.975	0.7497
5	+0.025	0.975	1.05	1.0125	0.7499
Stop					

Solution: X = 1.0125

(b)

Newton's method

Max f(x) = x^3 + 2x - 2 x^2 - 0.25 x^4

f'(x) = 3x^2 + 2 - 4 x - x^3

f''(x) = 6x - 4 - 3 x^2

error 0.001

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	1.2	0.7296	-0.208	-1.12	1.014286	0.185714
2	1.01428571	0.74989795	-0.014289	-1.000612	1.000006	0.01428
3	1.00000583	0.75	-5.83E-06	-1	1	5.83E-06

12.4-2.

(a)

Iteration	df(X)/dX	X(L)	X(U)	New X′	f(X′)
0		0	4.8	2.4	8.64
1	+ 1.2	2.4	4.8	3.6	8.64
2	- 1.2	2.4	3.6	3	9
3	+ 0	3	3.6	3.3	8.91
4	- 0.6	3	3.3	3.15	8.9775
5	- 0.3	3	3.15	3.075	8.9944
6	- 0.15	3	3.075	3.0375	8.9986
Stop					

(b)

Iteration	df(X)/dX	X(L)	X(U)	New X′	f(X′)
0		-4	1	-1.5	-1.688
1	- 1.5	-1.5	1	-0.25	-1.121
2	+3.188	-1.5	-0.25	-0.875	-1.984
3	+0.258	-1.5	-0.875	-1.188	-1.964
4	-0.401	-1.188	-0.875	-1.031	-1.999
5	-0.063	-1.031	-0.875	-0.953	-1.998
6	+0.094	-1.031	-0.953	-0.992	-2
Stop					

12.4-3.

(a)

Iteration	df(X)/dX	X(L)	X(U)	New X′	f(X′)
0		-1	4	1.5	-16.69
1	- 100	-1	1.5	0.25	0.3047
2	+0.156	0.25	1.5	0.875	0.2482
3	-0.923	0.25	0.875	0.5625	0.3125
4	-0.001	0.25	0.5625	0.4063	0.3124
5	+0.004	0.4063	0.5625	0.4844	0.3125
Stop					

(b)

Newton's method

Max f(x) = 48 x^5 + 42x^3 + 3.5x -16 x^6 - 61 x^4 -16.5x^2

f'(x) = 240x^4 + 126 x^2 + 3.5 - 96 x^5 - 264 x^3 - 33 x

f''(x) = 960 x^3 + 252 x - 480 x^4 - 792 x^2 - 33

error 0.001

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	1	0	-23.5	-93	0.747312	0.252688
2	0.74731183	0.30509816	-8.496421	-36.0381	0.51155	0.235762
3	0.51154965	0.31249998	-2.677284	-15.70259	0.34105	0.170499
4	0.34105018	0.31160364	-0.767583	-7.588489	0.239899	0.101151
5	0.23989924	0.302969	-0.091464	-6.461815	0.225745	0.014154
6	0.22574474	0.30003409	0.001383	-6.675803	0.225952	0.000207

12.4-4.

(a) $f(x) = 10x^3 + 60x - 2x^6 - 3x^4 - 12x^2$

$= -x[2x^2(x^3 - 5) + 3(x^3 + 4x - 20)]$

$= -x[2x^2(x^3 - 8) + 3(x^3 + 6x^2 + 4x - 20)]$

The expression in brackets is positive for all $x \geq 2$ and negative for all $x \leq 0$, so $f(x)$ is negative for $x \geq 2$ and $x < 0$. Hence, choose $\underline{x} = 0$ and $\overline{x} = 2$.

```
Interactive One-Dimensional Search Procedure:

Max f(X) =      10 X^3 +    60 X -     2 X^6 -     3 X^4 -    12 X^2

df(X)/dX =      30 X^2 +    60  -    12 X^5 -    12 X^3 -    24 X

Lower Bound: 0          Upper Bound: 2


Iteration | df(X)/dX |    X(L)    |    X(U)    |  New X'    |  f(X')
__________|__________|___________|___________|___________|__________
    0     |          |         0 |         2 |         1  |      53
    1     |  +   42  |         1 |         2 |       1.5  |  58.781
    2     |  -40.12  |         1 |       1.5 |      1.25  |  60.828
    3     |  +16.82  |      1.25 |       1.5 |     1.375  |  61.569
    4     |  -6.455  |      1.25 |     1.375 |    1.3125  |  61.561
  Stop    |          |           |           |            |

Solution: X = 1.3125
```

(b) $f(x) = 10x^3 + 60x - 2x^6 - 3x^4 - 12x^2$

$f'(x) = 30x^2 + 60 - 12x^5 - 12x^3 - 24x$

$f''(x) = 60x - 60x^4 - 36x^2 - 24$

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	1	53	42	-60	1.7	0.7
2	1.7	43.119	-123.44	-527.17	1.46584	0.23416
3	1.46584	59.971	-29.73	-290.41	1.36348	0.10236
4	1.36348	61.629	-3.919	-216.49	1.34538	0.0181
5	1.34538	61.665	-0.105	-205.02	1.34487	0.00051

12.4-5.

(a) $f'(x) = 4x^3 + 2x - 4 \Rightarrow f'(0) = -4, f'(1) = 2, f'(2) = 32$

Since $f'(x)$ is continuous, there must be a point $0 \leq x^* \leq 1$ such that $f'(x^*) = 0$ and since f is a convex function (given that this is a convex programming problem), x^* must be the optimal solution. Hence, the optimal solution lies in the interval $0 \leq x \leq 1$.

(b)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	2	1	-2
1	+ 2	0	1	0.5	-1.688
2	- 2.5	0.5	1	0.75	-2.121
3	-0.813	0.75	1	0.875	-2.148
4	+ 0.43	0.75	0.875	0.8125	-2.154
5	-0.229	0.8125	0.875	0.8438	-2.156
6	+ 0.09	0.8125	0.8438	0.8281	-2.156
Stop					

(c)

Newton's method

Max f(x) = x^4 + x^2 - 4 x **s.t. x>= 0, x<=2**

f'(x) = 4x^3 + 2x - 4

f''(x) = 12x^2 - 2

error 0.0001

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	1	-2	2	10	0.8	0.2
2	0.8	-2.1504	-0.352	5.68	0.861972	0.061972
3	0.86197183	-2.1528497	0.285708	6.915945	0.82066	0.041312
4	0.82066031	-2.1555781	-0.147875	6.0818	0.844975	0.024314
5	0.84497469	-2.1561459	0.103137	6.567787	0.829271	0.015703
6	0.82927123	-2.1564755	-0.060329	6.252289	0.83892	0.009649
7	0.83892031	-2.1565774	0.039526	6.445448	0.832788	0.006132
8	0.83278785	-2.1566242	-0.024152	6.322427	0.836608	0.00382
9	0.83660793	-2.1566409	0.015426	6.398954	0.834197	0.002411
10	0.83419717	-2.1566479	-0.009585	6.350619	0.835706	0.001509
11	0.83570643	-2.1566506	0.00606	6.380863	0.834757	0.00095
12	0.83475674	-2.1566517	-0.00379	6.361826	0.835352	0.000596
13	0.83535244	-2.1566521	0.002386	6.373764	0.834978	0.000374
14	0.83497803	-2.1566523	-0.001496	6.36626	0.835213	0.000235
15	0.83521306	-2.1566523	0.000941	6.37097	0.835065	0.000148
16	0.83506541	-2.1566524	-0.00059	6.368011	0.835158	9.27E-05

12.4-6.

(a) Consider the two cases:

Case 1:$\overline{x}_{n+1} = \overline{x}_n$ and $\underline{x}_{n+1} = x'_n$

$$\Rightarrow \overline{x}_{n+1} - \underline{x}_{n+1} = \overline{x}_n - x'_n = \overline{x}_n - \tfrac{1}{2}(\overline{x}_n + \underline{x}_n) = \tfrac{1}{2}(\overline{x}_n - \underline{x}_n)$$

Case 2:$\overline{x}_{n+1} = x'_n$ and $\underline{x}_{n+1} = \underline{x}_n$

$$\Rightarrow \overline{x}_{n+1} - \underline{x}_{n+1} = x'_n - \underline{x}_n = \tfrac{1}{2}(\overline{x}_n + \underline{x}_n) - \underline{x}_n = \tfrac{1}{2}(\overline{x}_n - \underline{x}_n)$$

In both cases: $\overline{x}_{n+1} - \underline{x}_{n+1} = \frac{1}{2}(\overline{x}_n - \underline{x}_n) = \cdots = \frac{1}{2^{n+1}}(\overline{x}_0 - \underline{x}_0)$

$$\Rightarrow \lim_{n\to\infty}(\overline{x}_{n+1} - \underline{x}_{n+1}) = \lim_{n\to\infty}\tfrac{1}{2^{n+1}}(\overline{x}_0 - \underline{x}_0) = 0$$

If the sequence of trial solutions selected by the midpoint rule did not converge to a limiting solution, then there must be an $\epsilon > 0$ such that regardless of what N is, there are $n \geq N$ and $m \geq N$ with $|x'_n - x'_m| > \epsilon$. In that case, choose N that satisfies $|\overline{x}_N - \underline{x}_N| = 2^{-N}(\overline{x}_0 - \underline{x}_0) < \epsilon$. Then for every $n \geq N$, since $x'_n \in [\overline{x}_N, \underline{x}_N]$:

$$|x'_n - x'_m| \leq |\overline{x}_N - \underline{x}_N| = 2^{-N}(\overline{x}_0 - \underline{x}_0) < \epsilon,$$

which contradicts that $|x'_n - x'_m| > \epsilon$. Hence, the sequence must converge.

(b) Let $\overline{x}$ be the limiting solution. Then, $f'(x) \geq 0$ for $x < \overline{x}$ and $f'(x) \leq 0$ for $x > \overline{x}$. Suppose now that there exists an $\widehat{x}$ with $f(\widehat{x}) > f(\overline{x})$ so that $\overline{x}$ is not a global maximum.

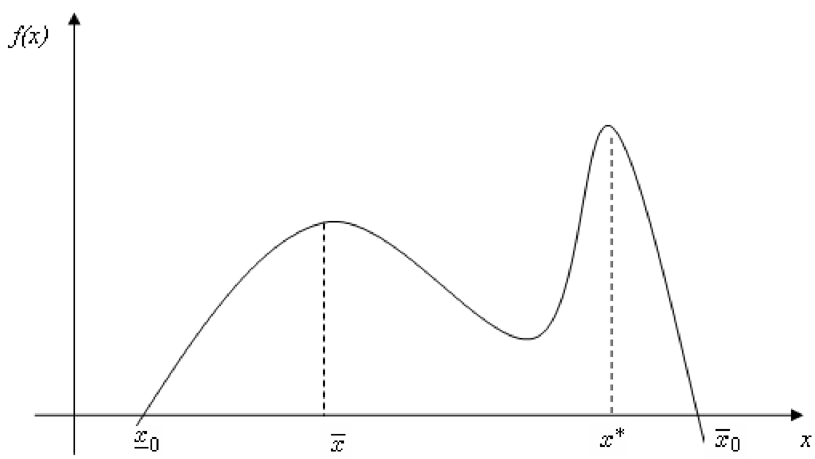
Case 1: $\widehat{x} > \overline{x}$. By the Mean Value Theorem, there exists a z such that $\widehat{x} > z > \overline{x}$ and $f(\widehat{x}) - f(\overline{x}) = (\widehat{x} - \overline{x})f'(z) \leq 0$, so $f(\widehat{x}) \leq f(\overline{x})$.

Case 2: $\widehat{x} < \overline{x}$. By the Mean Value Theorem, there exists a z such that $\widehat{x} < z < \overline{x}$ and $f(\overline{x}) - f(\widehat{x}) = (\overline{x} - \widehat{x})f'(z) \geq 0$, so $f(\widehat{x}) \leq f(\overline{x})$.

Both cases give rise to a contradiction, so $\overline{x}$ must be a global maximum.

(c) The argument is the same as the one in part (b). Observe that z that is chosen between $\widehat{x}$ and $\overline{x}$ remains in the region where f is concave and the values $\overline{x}_0$ and $\underline{x}_0$ are given as lower and upper bounds on the same global maximum.

(d) In the example illustrated in the graph below, the bisection method converges to $\overline{x}$ rather than to x^*, which is the global maximum.



(e) Suppose $f'(x) < 0$ for all x and $\widehat{x}$ is a global maximum. Then, by the Mean Value Theorem, there exists a z such that $\widehat{x} > z > x$ and $f(\widehat{x}) - f(x) = (\widehat{x} - x)f'(z) < 0$, so $f(x) = f(\widehat{x}) - (\widehat{x} - x)f'(z) > f(\widehat{x})$. The objective function value can be strictly increased by choosing smaller x values at any given point, so there exists no lower bound $\underline{x}_0$ on the global maximum, there is no global maximum indeed.

Suppose $f'(x) > 0$ for all x and $\widehat{x}$ is a global maximum. Then, by the Mean Value Theorem, there exists a z such that $x > z > \widehat{x}$ and $f(x) - f(\widehat{x}) = (x - \widehat{x})f'(z) > 0$, so $f(x) = f(\widehat{x}) + (x - \widehat{x})f'(z) > f(\widehat{x})$. The objective function value can be strictly increased by choosing larger x values at any given point, so there exists no upper bound $\overline{x}_0$ on the global maximum, there is no global maximum indeed.

(f) Suppose $f(x)$ is concave and there exists a lower bound $\underline{x}_0$ on the global maximum. In this case, $f'(\underline{x}_0) \geq 0$, but $f'(x)$ is monotone decreasing, so for $x < \underline{x}_0$, $f'(x) \geq 0$. Hence, $\lim_{x \to -\infty} f'(x) \geq 0$, so if $\lim_{x \to -\infty} f'(x) < 0$, there cannot be an $\underline{x}_0$.

Suppose $f(x)$ is concave and there exists an upper bound $\overline{x}_0$ on the global maximum. In this case, $f'(\overline{x}_0) \leq 0$, but $f'(x)$ is monotone decreasing, so for $x > \overline{x}_0$, $f'(x) \leq 0$. Hence, $\lim_{x \to \infty} f'(x) \leq 0$, so if $\lim_{x \to -\infty} f'(x) > 0$, there cannot be an $\overline{x}_0$.

In either case, there is no global maximum, since one of the bounds does not exist.

12.4-7.

$f(\boldsymbol{x}) = f_1(x_1) + f_2(x_2)$

where $f_1(x_1) = 32x_1 - x_1^4$ and $f_2(x_2) = 50x_2 - 10x_2^2 + x_2^3 - x_2^4$.

$\frac{df_1(x_1)}{dx_1} = 32 - 4x_1^3 = 0 \Leftrightarrow x_1 = 2, f_1(2) = 48$

Bisection method with $\epsilon = 0.001$ and initial bounds 0 and 4 applied to $f_2(x_2)$ gives $x_2 = 1.8076$ and $f_2(1.8076) = 52.936$, so $f(2, 1.8076) = 100.936$.

$3x_1 + x_2 = 7.8076 < 11$ and $2x_1 + 5x_2 = 13.038 < 16$

Since the optimal solution for the unconstrained problem is in the interior of the feasible region for the constrained problem, it is also optimal for the constrained problem.

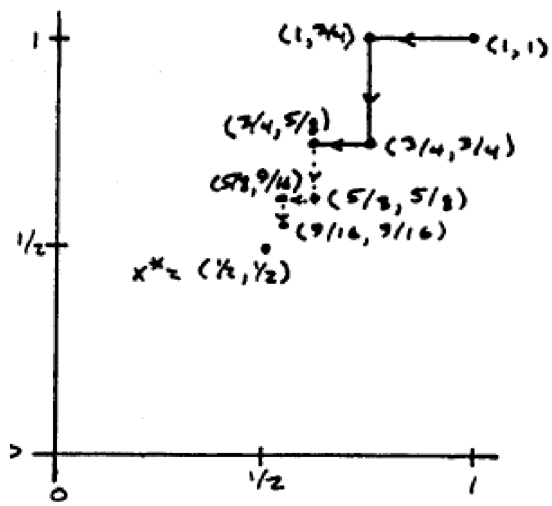
12.5-1.

(a)

It.	X'	grad f(X')	x' + t[grad f(X')]	t*	X'+t[grad f]
1	(1, 1)	(0, -1)	(1+ 0t, 1- 1t)	0.25	(1, 0.75)
2	(1, 0.75)	(-0.5, 0)	(1- 0.5t, 0.75+ 0t)	0.5	(0.75, 0.75)
3	(0.75, 0.75)	(0, -0.5)	(0.75+ 0t, 0.75- 0.5t)	0.25	(0.75,0.625)
4	(0.75,0.625)	(-0.25, 0)			

(b) $-2x_1 + 2x_2 = 0$ and $-2x_1 + 4x_2 = 1 \Rightarrow x_1 = x_2 = 0.5$ is optimal.

(c)



(d) Solution: $(x_1, x_2) = (0.508, 0.504)$, grad $f(x_1, x_2) = (-8e{-}3, 6e{-}8)$

12.5-2.

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(1, 1)	(30, -100)	(1+ 30t, 1- 100t)	0.005	(1.165,0.451)
2	(1.165,0.451)	(-7.85,-2.36)	(1.165-7.85t,0.451-2.36t)	0.13	(0.144,0.144)
3	(0.144,0.144)	(4.344,-14.5)	(0.144+4.34t,0.144-14.5t)	0.005	

```
Automatic Gradient Search Procedure:

Initial trial solution: (X1, X2) = (1, 1).

Max f(X1,X2) =      60 X1X2 -    15 X1^2 -    80 X2^2

Error Tolerance:

   abs(df/dXj) <= 0.01


Final solution: (X1, X2) = (0.0006,0.0002)

Final gradient = (-0.004,-0.001)
```

Solution: $(x_1, x_2) = (0.005, 0.003)$, grad $f(x_1, x_2) = (-7e{-}3, 3e{-}8)$

$\nabla f = (-30x_1 + 60x_2, 60x_1 - 160x_2) = 0 \Leftrightarrow (x_1, x_2) = (0, 0)$ is optimal.

12.5-3.

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(0, 0)	(8, -12)	(0+ 8t, 0- 12t)	0.191	(1.529,-2.29)
2	(1.529,-2.29)	(0.361,0.219)	(1.529+0.36t,-2.29+0.22t)	1.31	(2.002, -2)
3	(2.002, -2)	(-0,0.003)			

Solution: $(x_1, x_2) = (1.997, -2)$, grad $f(x_1, x_2) = (0.002, 0.001)$

$\nabla f = (-2x_1 + 2x_2 + 8, 2x_1 - 4x_2 - 12) = 0 \Leftrightarrow (x_1, x_2) = (2, -2)$ is optimal.

12.5-4.

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(0, 0)	(6, -2)	(0+ 6t, 0- 2t)	0.2	(1.2, -0.4)
2	(1.2, -0.4)	(0.4, 1.2)	(1.2+ 0.4t, -0.4+ 1.2t)	1	(1.6, 0.8)
3	(1.6, 0.8)	(1.2, -0.4)			

Solution: $(x_1, x_2) = (1.994, 0.989)$, grad $f(x_1, x_2) = (0.003, 0.01)$

$\nabla f = (-4x_1 + 2x_2 + 6, 2x_1 - 2x_2 - 2) = 0 \Leftrightarrow (x_1, x_2) = (2, 1)$ is optimal.

12.5-5.

Iter.	$\underline{x}_n$	$\nabla f(\underline{x}_n)$	$f(\underline{x}_n + \nabla f(\underline{x}_n))$	Iter.	t'	$f(t)$
1	$(0,0)$	$(4,2)$	$20t - 26t^2 - 256t^4$	1	0.5	-144
				2	0.25	-14
				$t^* = 0.125$		
$\Rightarrow x + t^*\nabla f(x) = (0.5, 0.25)$ is the approximate solution.						

12.5-6.

(a) $f(\boldsymbol{x}) = f_1(x_1, x_2) + f_2(x_2, x_3)$

where $f_1(x_1, x_2) = 3x_1x_2 - x_1^2 - 3x_2^2$ and $f_2(x_2, x_3) = 3x_2x_3 - x_3^2 - 3x_2^2$.

Note that $f_1(x_3, x_2) = f_2(x_2, x_3)$, so for any given x_2, the maximizers of f_1 and f_2 are the same, i.e., $x_1 = x_3$. Hence, first maximize f_1 (or f_2) and obtain (x_1, x_2). Then, set $x_3 = x_1$ and $f(\boldsymbol{x}) = 2f_1(x_1, x_2)$.

(b)

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t*[grad f]
1	(1, 1)	(1, -3)	(1+ 1t, 1- 3t)	0.135	(1.135,0.595)
2	(1.135,0.595)	(-0.49,-0.16)	(1.14-0.49t,0.59-0.16t)	1.616	(0.343,0.336)
3	(0.343,0.336)	(0.323,-0.99)	(0.34+0.32t,0.34-0.99t)	0.135	(0.387,0.202)
4	(0.387,0.202)	(-0.17,-0.05)	(0.39-0.17t, 0.2-0.05t)	1.427	(0.144,0.131)
5	(0.144,0.131)	(0.103,-0.35)	(0.14+ 0.1t,0.13-0.35t)	0.139	(0.158,0.083)
6	(0.158,0.083)	(-0.07,-0.02)	(0.16-0.07t,0.08-0.02t)	1.361	(0.063,0.056)
7	(0.063,0.056)	(0.042,-0.15)	(0.06+0.04t,0.06-0.15t)	0.135	(0.069,0.036)

Final Solution: $(x_1, x_2) = (0.069, 0.036) \Rightarrow (x_1, x_2, x_3) = (0.069, 0.036, 0.069)$ is an approximate solution.

(c) Solution: $(x_1, x_2) = (0.004, 0.002)$, grad $f(x_1, x_2) = (-2e{-}3, 6e{-}4)$

12.5-7.

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(0, 0)	(0, 3)	(0+ 0t, 0+ 3t)	0.5	(0, 1.5)
2	(0, 1.5)	(1.5, 0)	(0+ 1.5t, 1.5+ 0t)	0.5	(0.75, 1.5)
3	(0.75, 1.5)	(0, 0.75)			

Solution: $(x_1, x_2) = (0.996, 1.998)$, grad $f(x_1, x_2) = = (0.006, -2e{-}8)$

12.6-1.

KKT conditions:
(1) $-4x^3 - 2x + 4 - u \leq 0$
(2) $x(-4x^3 - 2x + 4 - u) = 0$
(3) $x - 2 \leq 0$
(4) $u(x - 2) = 0$
(5) $x \geq 0$
(6) $u \geq 0$

If $x = 2$, from (2), $-4x^3 - 2x + 4 - u = 0$, so $u = -32$, which violates (6). Hence, $x \neq 2$, then from (4), $u = 0$. From (2), either $x = 0$ or $-4x^3 - 2x + 4 = 0$. In the former case, (1) is violated, so the latter equality must hold. This gives

$$x = \sqrt[3]{\tfrac{1}{2} + \sqrt{\tfrac{55}{216}}} + \sqrt[3]{\tfrac{1}{2} - \sqrt{\tfrac{55}{216}}} = 0.83512.$$

12.6-2.

KKT conditions:
(1a) $1 - 2ux_1 \leq 0$ (1b) $1 - 2ux_2 \leq 0$
(2a) $x_1(1 - 2ux_1) = 0$ (2b) $x_2(1 - 2ux_2) = 0$
(3) $x_1^2 + x_2^2 - 1 \leq 0$
(4) $u(x_1^2 + x_2^2 - 1) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

If $x = (1/\sqrt{2}, 1/\sqrt{2})$, from (2a), $u = 1/\sqrt{2}$. This solution satisfies all KKT conditions, so it is optimal.

12.6-3.

KKT conditions:
(1a) $-4x_1^3 - 4x_1 - 2x_2 + 2u_1 + u_2 \leq 0$
(2a) $x_1(-4x_1^3 - 4x_1 - 2x_2 + 2u_1 + u_2) = 0$
(1b) $-2x_1 - 8x_2 + u_1 + 2u_2 \leq 0$
(2b) $x_2(-2x_1 - 8x_2 + u_1 + 2u_2) = 0$
(3a) $2x_1 + x_2 \geq 10$
(4a) $u_1(-2x_1 - x_2 + 10) = 0$
(3b) $x_1 + 2x_2 \geq 10$
(4b) $u_2(-x_1 - 2x_2 + 10) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

If $x = (0, 10)$, from (2b), $u_1 + 2u_2 = 80$ and from (4b), $u_2 = 0$, so $u_1 = 80$. This solution violates (1a), so it is not optimal.

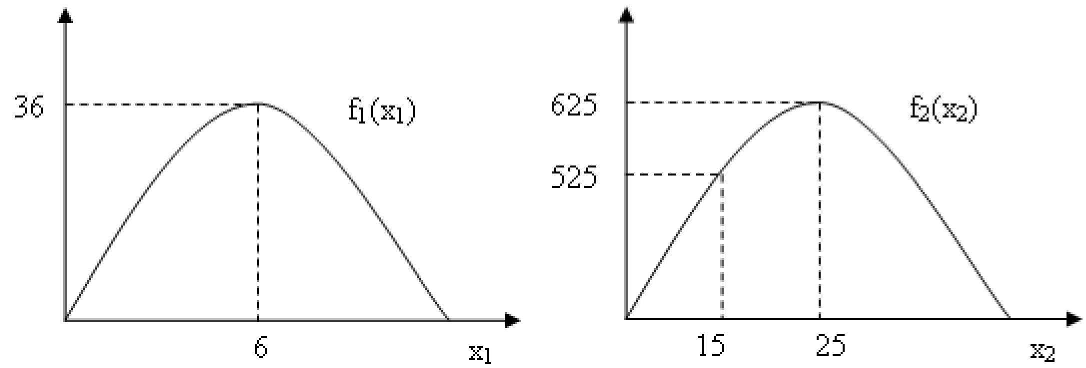
12.6-4.

(a) KKT conditions:
(1a) $12 - 2x_1 - u_1 \leq 0$ (1b) $50 - 2x_2 - u_2 \leq 0$
(2a) $x_1(12 - 2x_1 - u_1) = 0$ (2b) $x_2(50 - 2x_2 - u_2) = 0$
(3a) $x_1 \leq 10$ (3b) $x_2 \leq 15$
(4a) $u_1(x_1 - 10) = 0$ (4b) $u_2(x_2 - 15) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

Consider $x_1 = 10$. From (2a), $u_1 = -8$, which violates (6). Hence, $u_1 = 0$. The constraint (1a) together with (2a) implies $x_1 = 6$. Also, let $x_2 = 15$. From (2b), $u_2 = 20$. This solution satisfies all the conditions and since this is a convex programming problem, $(x_1, x_2) = (6, 15)$ is optimal.

(b) Subproblem 1: maximize $f_1(x_1) = 12x_1 - x_1^2$ subject to $0 \leq x_1 \leq 10$

Subproblem 2: maximize $f_2(x_2) = 50x_2 - x_2^2$ subject to $0 \leq x_1 \leq 15$



$\frac{\partial f_1(\boldsymbol{x_1})}{\partial x_1} = 12 - 2x_1 = 0$ at $x_1 = 6$ and $\frac{\partial^2 f_1(\boldsymbol{x_1})}{\partial x_1^2} = -2 < 0 \Rightarrow x_1 = 6$ is a global maximizer.

$\frac{\partial f_2(\boldsymbol{x_2})}{\partial x_2} = 50 - 2x_2 > 0$ for all $0 \leq x_2 \leq 15 \Rightarrow x_2 = 15$ is the maximizer over the feasible region.

12.6-5.

(a) $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} = -\frac{1}{(x_1+1)^2} \leq 0$ for all (x_1, x_2) such that $x_1 \neq -1$

$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} = -2 \leq 0$ for all (x_1, x_2)

$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} \frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = \frac{2}{(x_1+1)^2} \geq 0$ for all (x_1, x_2) such that $x_1 \neq -1$

$\Rightarrow f$ is concave.

Since also $g(\boldsymbol{x}) = x_1 + 2x_2 - 3$ is linear, this is a convex programming problem.

(b) KKT conditions:

(1a) $\frac{1}{(x_1+1)} - u \leq 0$ (1b) $-2x_2 - 2u \leq 0$

(2a) $x_1\left(\frac{1}{(x_1+1)} - u\right) = 0$ (2b) $x_2(-2x_2 - 2u) = 0$

(3) $x_1 + 2x_2 \leq 3$

(4) $u(x_1 + 2x_2 - 3) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u \geq 0$

Consider $u \neq 0$. From (4), $x_1 + 2x_2 = 3$. Let $x_2 = 0$. Then, $x_1 = 3$ and from (2a), $u = 0.25$. This satisfies all the conditions, so $(x_1, x_2) = (3, 0)$ is optimal.

(c) Since $-x_2^2$ is monotonically strictly decreasing in $x_2 \geq 0$ and $\ln(x_1 + 1)$ is monotonically strictly increasing in $x_1 \geq 0$, it is intuitively clear that one would like to increase x_1

and decrease x_2 towards 0 as much as possible, in order to maximize the objective function. Let $\mathcal{F}$ denote the set of feasible points. Then,

$$\max_{x_1}\left[\min_{x_2} \mathcal{F}\right] = \min_{x_2}\left[\max_{x_1} \mathcal{F}\right] = \{(3,0)\}.$$

Hence, the solution $(3,0)$ makes intuitive sense.

12.6-6.

KKT conditions:
(1a) $36 + 18x_1 - 18x_1^2 - u \leq 0$ (1b) $36 - 9x_2^2 - u \leq 0$
(2a) $x_1(36 + 18x_1 - 18x_1^2 - u) = 0$ (2b) $x_2(36 - 9x_2^2 - u) = 0$
(3) $x_1 + x_2 \leq 3$
(4) $u(x_1 + x_2 - 3) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

For $(x_1, x_2) = (1,2)$, from (2b), $u = 0$ and this violates (2a), so $(1,2)$ is not optimal.

12.6-7.

(a) KKT conditions:
(1a) $\frac{1}{(x_2+1)} - u \leq 0$ (1b) $-\frac{x_1}{(x_2+1)^2} + u \leq 0$
(2a) $x_1\left(\frac{1}{(x_2+1)} - u\right) = 0$ (2b) $x_2\left(-\frac{x_1}{(x_2+1)^2} + u\right) = 0$
(3) $x_1 - x_2 \leq 2$
(4) $u(x_1 - x_2 - 2) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

For $(x_1, x_2) = (4,2)$, from (2a), $u = 1/3$ and this violates (2b), so $(4,2)$ is not optimal.

(b) Try $x_2 = 0$ and $u \neq 0$. From (4), $x_1 = 2$ and from (2a), $u = 1$. This solution satisfies all the conditions, so $(x_1, x_2) = (2,0)$ is optimal.

(c) $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} = 0$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} = \frac{2x_1}{(x_2+1)^2} \geq 0$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2} = -\frac{1}{(x_2+1)^2} \leq 0$ for all $x_1 \geq 0, x_2 \geq 0$

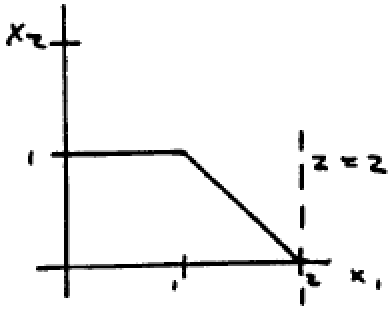
Thus, f is not concave and this is not a convex programming problem.

(d) The function $f(\boldsymbol{x})$ is monotonically strictly increasing in x_1 and monotonically strictly decreasing in x_2 if $x_2 > -1$. Any optimal solution in a bounded feasible region with $x_2 > -1$ will have x_1 increased as much as possible and x_2 decreased toward -1 as much as possible. The feasible region of the problem allows x_1 to be increased without bound. However, then x_2 can only be decreased to the line $x_1 - x_2 = 2$.

$$f(x_2 + 2, x_2) = \frac{x_2+2}{x_2+1} \to 1 \text{ as } x_2 \to \infty \text{ and } f(x_2+2, x_2) = 2 \text{ at } x_2 = 0$$

Conversely, if x_2 is decreased to 0, x_1 can be increased to $x_1 = 2$. Hence, the optimal solution is $(x_1, x_2) = (2,0)$.

(e)
$$\begin{array}{ll} \text{maximize} & x_1 \\ \text{subject to} & x_1 - x_2 - 2t \leq 0 \\ & \quad\;\; x_2 + \;\; t = 1 \\ & x_1, x_2, t \geq 0 \end{array} \quad\Leftrightarrow\quad \begin{array}{ll} \text{maximize} & x_1 \\ \text{subject to} & x_1 + x_2 \leq 2 \\ & \qquad\; x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$



$(x_1, x_2) = (2, 0)$ is optimal.

12.6-8.

(a) KKT conditions:

(1a) $1 - u \leq 0$ (1b) $2 - 3x_2^2 - u \leq 0$
(2a) $x_1(1 - u) = 0$ (2b) $x_2(2 - 3x_2^2 - u) = 0$
(3) $x_1 + x_2 \leq 1$
(4) $u(x_1 + x_2 - 1) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

The solution $(x_1, x_2, u) = (1 - 1/\sqrt{3}, 1/\sqrt{3}, 1)$ satisfies all the conditions. Since this is a convex programming problem, $(1 - 1/\sqrt{3}, 1/\sqrt{3})$ is optimal.

(a) KKT conditions:

(1a) $20 - 2u_1x_1 - u_2 \leq 0$ (1b) $10 - 2u_1x_1 - 2u_2 \leq 0$
(2a) $x_1(20 - 2u_1x_1 - u_2) = 0$ (2b) $x_2(10 - 2u_1x_1 - 2u_2) = 0$
(3a) $x_1^2 + x_2^2 \leq 1$ (3b) $x_1 + 2x_2 \leq 2$
(4a) $u_1(x_1^2 + x_2^2 - 1) = 0$ (4b) $u_2(x_1 + 2x_2 - 2) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

The solution $(x_1, x_2, u) = (2/\sqrt{5}, 1/\sqrt{5}, 5\sqrt{5}, 0)$ satisfies all the conditions. Since this is a convex programming problem, $(2/\sqrt{5}, 1/\sqrt{5})$ is optimal.

12.6-9.

minimize $f(\boldsymbol{x})$
subject to $g_i(\boldsymbol{x}) \geq b_i$ for $i = 1, 2, \ldots, m$
$\boldsymbol{x} \geq \mathbf{0}$

$\Leftrightarrow$ maximize $-f(\boldsymbol{x})$
subject to $-g_i(\boldsymbol{x}) \leq -b_i$ for $i = 1, 2, \ldots, m$
$\boldsymbol{x} \geq \mathbf{0}$

KKT conditions:

(1) $\sum_{i=1}^{m} u_i \frac{\partial g_i(\boldsymbol{x})}{\partial x_j} - \frac{\partial f(\boldsymbol{x})}{\partial x_j} \leq 0$ for $j = 1, 2, \ldots, n$

(2) $x_j \left(\sum_{i=1}^{m} u_i \frac{\partial g_i(\boldsymbol{x})}{\partial x_j} - \frac{\partial f(\boldsymbol{x})}{\partial x_j} \right) = 0$ for $j = 1, 2, \ldots, n$

(3) $g_i(\boldsymbol{x}) \geq b_i$ for $i = 1, 2, \ldots, m$
(4) $u_i(b_i - g_i(\boldsymbol{x})) = 0$ for $i = 1, 2, \ldots, m$
(5) $x_j \geq 0$ for $j = 1, 2, \ldots, n$
(6) $u_i \geq 0$ for $i = 1, 2, \ldots, m$

12.6-10.

(a) An equivalent nonlinear programming problem is:

$$\begin{array}{ll} \text{maximize} & Z = -2x_1^2 - x_2^2 \\ \text{subject to} & \begin{aligned} x_1 + x_2 &\leq 10 \\ -x_1 - x_2 &\leq -10 \\ x_1, x_2 &\geq 0. \end{aligned} \end{array}$$

This problem can be fitted to the following problems.

- Linearly Constrained Optimization Problem: All constraints are linear.

- Quadratic Programming Problem: All constraints are linear and the objective function involves only the squares of the variables.

- Convex Programming Problem: The objective function is concave and all constraints are linear.

$$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = (-4)(-2) - 0 = 8 \geq 0 \Rightarrow f \text{ is concave.}$$

- Geometric Programming Problem:

$$f(x_1, x_2) = c_1 P_1(x_1, x_2) + c_2 P_2(x_1, x_2)$$

with $c_1 = -2$, $c_2 = -1$, $P_1(x_1, x_2) = x_1^2$ and $P_2(x_1, x_2) = x_2^2$

$$g_1(x_1, x_2) = c_1 P_1(x_1, x_2) + c_2 P_2(x_1, x_2)$$

with $c_1 = c_2 = 1$, $P_1(x_1, x_2) = x_1$ and $P_2(x_1, x_2) = x_2$

$$g_2(x_1, x_2) = c_1 P_1(x_1, x_2) + c_2 P_2(x_1, x_2)$$

with $c_1 = c_2 = -1$, $P_1(x_1, x_2) = x_1$ and $P_2(x_1, x_2) = x_2$

- Fractional Programming Problem:

$$f(x_1, x_2) = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} \text{ with } f_1(x_1, x_2) = -2x_1^2 - x_2^2 \text{ and } f_2(x_1, x_2) = 1$$

(b) KKT conditions:
(1a) $-4x_1 - u_1 + u_2 \leq 0$
(2a) $x_1(-4x_1 - u_1 + u_2) = 0$
(1b) $-2x_2 - u_1 + u_2 \leq 0$
(2b) $x_2(-2x_2 - u_1 + u_2) = 0$
(3a) $x_1 + x_2 - 10 \leq 0$
(4a) $u_1(x_1 + x_2 - 10) = 0$
(3b) $-x_1 - x_2 + 10 \leq 0$
(4b) $u_2(-x_1 - x_2 + 10) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

(c) From (3a) and (3b), $x_1 + x_2 = 10$, so (4a) and (4b) are automatically satisfied. Try $x_1, x_2 \neq 0$. Then, (2a) and (2b) give $-4x_1 - u_1 + u_2 = -2x_2 - u_1 + u_2 = 0$, so $x_2 = 2x_1$. Since $x_1 + x_2 = 10$, $x_1 = 10/3$ and $x_2 = 20/3$. From (2a), $-u_1 + u_2 = 40/3$. Let $u_1 = 0$ and $u_2 = 40/3$. Indeed, any $(u_1, u_2) = (c, c + 40/3)$ with $c \geq 0$ works. This solution satisfies all the conditions, so $(x_1, x_2) = (10/3, 20/3)$ is optimal.

12.6-11.

(a) An equivalent nonlinear programming problem is:

$$\begin{array}{ll} \text{maximize} & f(\boldsymbol{y}) = -(y_1+1)^3 - 4(y_2+1)^2 - 16(y_3+1) \\ \text{subject to} & \;\; y_1+y_2+y_3 \leq \;\; 2 \\ & -y_1-y_2-y_3 \leq -2 \\ & \;\; y_1, y_2, y_3 \qquad \geq 0. \end{array}$$

(b) KKT conditions:
(1a) $-3(y_1+1)^2 - u_1 + u_2 \leq 0$
(2a) $y_1(-3(y_1+1)^2 - u_1 + u_2) = 0$
(1b) $-8(y_2+1) - u_1 + u_2 \leq 0$
(2b) $y_2(-8(y_2+1) - u_1 + u_2) = 0$
(1c) $-16 - u_1 + u_2 \leq 0$
(2c) $y_3(-16 - u_1 + u_2) = 0$
(3a) $y_1 + y_2 + y_3 \leq 2$
(4a) $u_1(y_1 + y_2 + y_3 - 2) = 0$
(3b) $-y_1 - y_2 - y_3 \leq -2$
(4b) $u_2(-y_1 - y_2 - y_3 + 2) = 0$
(5) $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

(c) If $\boldsymbol{x} = (2,1,2)$, $\boldsymbol{y} = (1,0,1)$. From (2a), $-u_1 + u_2 = 12$, which contradicts (2c), so $\boldsymbol{x} = (2,1,2)$ is not optimal.

12.6-12.

(a) KKT conditions:
(1a) $6 - 2x_1 - u \leq 0$ (1b) $3 - 3x_2^2 - u \leq 0$
(2a) $x_1(6 - 2x_1 - u) = 0$ (2b) $x_2(3 - 3x_2^2 - u) = 0$
(3) $x_1 + x_2 \leq 1$
(4) $u(x_1 + x_2 - 1) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

(b) For $\boldsymbol{x} = (1/2, 1/2)$, (2a) gives $u = 5$, which violates (2b), so this point is not optimal.

(c) $(x_1, x_2, u) = (1, 0, 4)$ satisfies all the conditions and since this is a convex programming problem, $(1, 0)$ is optimal.

12.6-13.

(a) KKT conditions:

(1a) $8 - 2x_1 - u \leq 0$ (1b) $2 - 3u \leq 0$ (1c) $1 - 2u \leq 0$
(2a) $x_1(8 - 2x_1 - u) = 0$ (2b) $x_2(2 - 3u) = 0$ (2c) $x_3(1 - 2u) = 0$
(3) $x_1 + 3x_2 + 2x_3 \leq 12$
(4) $u(x_1 + 3x_2 + 2x_3 - 12) = 0$
(5) $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$
(6) $u \geq 0$

For $\boldsymbol{x} = (2, 2, 2)$, (2a) gives $u = 4$, which violates (2b) and (2c), so it is not optimal.

(b) $(x_1, x_2, x_3, u) = (11/3, 25/9, 0, 2/3)$ satisfies all the conditions and since this is a convex programming problem, $(11/3, 25/9, 0)$ is optimal.

12.6-14.

KKT conditions:

(1a) $-2+2x_1u \leq 0$ (1b) $-3x_2^2+4x_2u \leq 0$ (1c) $-2x_3+2x_3u \leq 0$
(2a) $x_1(-2+2x_1u)=0$ (2b) $x_2(-3x_2^2+4x_2u)=0$ (2c) $x_3(-2x_3+2x_3u)=0$
(3) $x_1^2+2x_2^2+x_3^2 \geq 4$
(4) $u(4-x_1^2-2x_2^2-x_3^2)=0$
(5) $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$
(6) $u \geq 0$

For $\boldsymbol{x}=(1,1,1)$, (2a) gives $u=1$, which violates (2b), so it is not optimal.

12.6-15.

KKT conditions:

(1a) $-4x_1^3+2x_1u \leq 0$ (1b) $-4x_2+2x_2u \leq 0$
(2a) $x_1(-4x_1^3+2x_1u)=0$ (2b) $x_2(-4x_2+2x_2u)=0$
(3) $-x_1^2-x_2^2+2 \leq 0$
(4) $u(-x_1^2-x_2^2+2)=0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

For $\boldsymbol{x}=(1,1)$, (2a) gives $u=2$, and this satisfies all the conditions, so $(1,1)$ is optimal.

12.6-16.

KKT conditions:

(1a) $32-4x_1^3-3u_1-2u_2 \leq 0$
(2a) $x_1(32-4x_1^3-3u_1-2u_2)=0$
(1b) $50-20x_2+3x_2^2-4x_2^3-u_1-5u_2 \leq 0$
(2b) $x_2(50-20x_2+3x_2^2-4x_2^3-u_1-5u_2)=0$
(3a) $3x_1+x_2 \leq 11$
(4a) $u_1(3x_1+x_2-11)=0$
(3b) $2x_1+5x_2 \leq 16$
(4b) $u_2(2x_1+5x_2-16)=0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

For $\boldsymbol{x}=(2,2)$, (4a) and (4b) give $u_1=u_2=0$, and this violates (2b), so $(2,2)$ is not optimal.

12.7-1.

(a) $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2}=-4<0$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2}=-8<0$, $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2}-\left[\frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2=16>0$

$\Rightarrow f$ is strictly concave.

(b) $x^TQx=4x_1^2-8x_1x_2+8x_2^2=4(x_1-x_2)^2+4x_2^2>0$ for all $(x_1,x_2)\neq(0,0)$

$\Rightarrow Q$ is positive definite.

(c) KKT conditions:

(1a) $15+4x_2-4x_1-u \leq 0$ (1b) $30+4x_1-8x_2-2u \leq 0$
(2a) $x_1(15+4x_2-4x_1-u)=0$ (2b) $x_2(30+4x_1-8x_2-2u)=0$
(3) $x_1+2x_2 \leq 30$

(4) $u(x_1 + 2x_2 - 30) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

$\boldsymbol{x} = (12, 9)$ with $u = 3$ satisfies all these conditions.

12.7-2.

(a) KKT conditions:
(1a) $8 - 2x_1 - u \leq 0$ (1b) $4 - 2x_2 - u \leq 0$
(2a) $x_1(8 - 2x_1 - u) = 0$ (2b) $x_2(4 - 2x_2 - u) = 0$
(3) $x_1 + x_2 \leq 2$
(4) $u(x_1 + x_2 - 2) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

$\boldsymbol{x} = (2, 0)$ with $u = 4$ satisfies all these conditions. Since this is a convex programming problem, $(2, 0)$ is optimal.

(b) Objective function in vector notation:

$$\text{maximize} \quad \begin{pmatrix} 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Equivalent problem: minimize $z_1 + z_2$
subject to
$2x_1 + u - y_1 + z_1 = 8$
$2x_2 + u - y_2 + z_2 = 4$
$x_1 + x_2 + v = 2$
$x_1 \geq 0, x_2 \geq 0$
$y_1 \geq 0, y_2 \geq 0$
$u \geq 0, v \geq 0$
$z_1 \geq 0, z_2 \geq 0$

Complementarity constraint: $x_1 y_1 + x_2 y_2 + uv = 0$

(c)

```
Linear Programming Model:

Number of Decision Variables:      5

Number of Functional Constraints:  3

Max Z =       0 X1 +    0 X2 +    0 X3 -    1 X4 -    1 X5

subject to

 1)           2 X1 +    0 X2 +    1 X3 +    1 X4 +    0 X5 >=       8

 2)           0 X1 +    2 X2 +    1 X3 +    0 X4 +    1 X5 >=       4

 3)           1 X1 +    1 X2 +    0 X3 +    0 X4 +    0 X5 <=       2

and

         X1 >= 0, X2 >= 0, X3 >= 0, X4 >= 0, X5 >= 0.
```

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	-2	-2	-2	0	0	1	1	0	-12
X4	1	0	2	0	1	1	0	-1	0	0	8
X5	2	0	0	2	1	0	1	0	-1	0	4
X8	3	0	1*	1	0	0	0	0	0	1	2

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	-2	0	0	1	1	2	-8
X4	1	0	0	-2	1*	1	0	-1	0	-2	4
X5	2	0	0	2	1	0	1	0	-1	0	4
X1	3	0	1	1	0	0	0	0	0	1	2

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	-4	0	2	0	-1	1	-2	0
X3	1	0	0	-2	1	1	0	-1	0	-2	4
X5	2	0	0	4*	0	-1	1	1	-1	2	0
X1	3	0	1	1	0	0	0	0	0	1	2

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	0	1	1	0	0	0	0
X3	1	0	0	0	1	0.5	0.5	-0.5	-0.5	-1	4
X2	2	0	0	1	0	-0.25	0.25	0.25	-0.25	0.5	0
X1	3	0	1	0	0	0.25	-0.25	-0.25	0.25	0.5	2

Optimal Solution: $(x_1, x_2) = (2, 0)$ with $u = 4$

(d) Excel Solver Solution: $(x_1, x_2) = (2, 0)$

12.7-3.

(a) Objective function in vector notation:

$$\text{maximize} \quad \begin{pmatrix} 250 & 100 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 50 & -90 \\ -90 & 200 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Equivalent problem:

$$\begin{array}{ll} \text{minimize} & z_1 + z_2 \\ \text{subject to} & 50x_1 - 90x_2 + 20u_1 + 10u_2 - y_1 + z_1 = 250 \\ & -90x_1 + 200x_2 + 5u_1 + 10u_2 - y_2 + z_2 = 100 \\ & 20x_1 + 5x_2 + v_1 = 90 \\ & 10x_1 + 10x_2 + v_2 = 60 \\ & x_1 \geq 0, x_2 \geq 0 \\ & u_1 \geq 0, u_2 \geq 0 \\ & y_1 \geq 0, y_2 \geq 0 \\ & z_1 \geq 0, z_2 \geq 0 \end{array}$$

Enforced complementarity constraint: $x_1y_1 + x_2y_2 + u_1v_1 + u_2v_2 = 0$

(b)

```
Linear Programming Model:

Number of Decision Variables:      6

Number of Functional Constraints:  4

Max Z =      0 X1 +    0 X2 +    0 X3 +    0 X4 -    1 X5 -    1 X6

subject to

 1)         50 X1 -   90 X2 +   20 X3 +   10 X4 +    1 X5 +    0 X6 >=     250

 2)        -90 X1 +  200 X2 +    5 X3 +   10 X4 +    0 X5 +    1 X6 >=     100

 3)         20 X1 +    5 X2 +    0 X3 +    0 X4 +    0 X5 +    0 X6 <=      90

 4)         10 X1 +   10 X2 +    0 X3 +    0 X4 +    0 X5 +    0 X6 <=      60

and

        X1 >= 0, X2 >= 0, X3 >= 0, X4 >= 0, X5 >= 0, X6 >= 0.
```

Bas	Eq		Coefficient of										Right
Var	No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	side
Z	0	1	40	-110	-25	-20	0	0	1	1	0	0	-350
X5	1	0	50	-90	20	10	1	0	-1	0	0	0	250
X6	2	0	-90	200*	5	10	0	1	0	-1	0	0	100
X9	3	0	20	5	0	0	0	0	0	0	1	0	90
X10	4	0	10	10	0	0	0	0	0	0	0	1	60

Bas	Eq		Coefficient of										Right
Var	No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	side
Z	0	1	-9.5	0	-22.2	-14.5	0	0.55	1	0.45	0	0	-295
X5	1	0	9.5	0	22.25	14.5	1	0.45	-1	-0.45	0	0	295
X2	2	0	-0.45	1	0.025	0.05	0	0.005	0	-5e-3	0	0	0.5
X9	3	0	22.25	0	-0.12	-0.25	0	-0.02	0	0.025	1	0	87.5
X10	4	0	14.5*	0	-0.25	-0.5	0	-0.05	0	0.05	0	1	55

Bas	Eq		Coefficient of										Right
Var	No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	side
Z	0	1	0	0	-22.4	-14.8	0	0.517	1	0.483	0	0.655	-259
X5	1	0	0	0	22.41	14.83	1	0.483	-1	-0.48	0	-0.66	259
X2	2	0	0	1	0.017	0.034	0	0.003	0	-3e-3	0	0.031	2.207
X9	3	0	0	0	0.259	0.517*	0	0.052	0	-0.05	1	-1.53	3.103
X1	4	0	1	0	-0.02	-0.03	0	-3e-3	0	0.003	0	0.069	3.793

Bas	Eq		Coefficient of										Right
Var	No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	side
Z	0	1	0	0	-15	0	0	2	1	-1	28.67	-43.3	-170
X5	1	0	0	0	15*	0	1	-1	-1	1	-28.7	43.33	170
X2	2	0	0	1	0	0	0	-0	0	0	-0.07	0.133	2
X4	3	0	0	0	0.5	1	0	0.1	0	-0.1	1.933	-2.97	6
X1	4	0	1	0	0	0	0	0	0	-0	0.067	-0.03	4

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	0	0	1	1	0	-0	0	0	0
X3	1	0	0	0	1	0	0.067	-0.07	-0.07	0.067	-1.91	2.889	11.33
X2	2	0	0	1	0	0	0	-0	0	0	-0.07	0.133	2
X4	3	0	0	0	0	1	-0.03	0.133	0.033	-0.13	2.889	-4.41	0.333
X1	4	0	1	0	0	0	0	0	0	-0	0.067	-0.03	4

Optimal Solution: $(x_1, x_2) = (4, 2)$ with $(u_1, u_2) = (11.33, 0.33)$

12.7-4.

(a) KKT conditions:

(1a) $2 - 2x_1 - u \leq 0$

(1b) $3 - 2x_2 - u \leq 0$

(2a) $x_1(2 - 2x_1 - u) = 0$

(2b) $x_2(3 - 2x_2 - u) = 0$

(3) $x_1 + x_2 \leq 2$

(4) $u(x_1 + x_2 - 2) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u \geq 0$

By plotting the points obtained, one observes that one optimal solution is on the boundary , so $x_1 \neq 0$, $x_2 \neq 0$ and $u \neq 0$. The point $(x_1, x_2) = (0.75, 1.25)$ with $u = 0.5$ satisfies all the conditions, so it is optimal.

(b) minimize $z_1 + z_2$

subject to

$$\begin{aligned}
&2x_1 + u - y_1 + z_1 = 2 \\
&2x_2 + u - y_2 + z_2 = 3 \\
&x_1 + x_2 + v = 2 \\
&x_1 \geq 0, x_2 \geq 0 \\
&u \geq 0, v \geq 0 \\
&y_1 \geq 0, y_2 \geq 0 \\
&z_1 \geq 0, z_2 \geq 0
\end{aligned}$$

Enforced complementarity constraint: $x_1 y_1 + x_2 y_2 + uv = 0$

(c) Substitute $(x_1, x_2) = (0.75, 1.25)$ and $u = 0.5$ in the constraints.

$$\begin{aligned}
-y_1 + z_1 &= 0 \\
-y_2 + z_2 &= 0 \\
v &= 0
\end{aligned}$$

Enforced complementarity constraint: $0.75y_1 + 1.25y_2 = 0$

Since $y_1 \geq 0$ and $y_2 \geq 0$, the unique solution of the complementarity constraint is $y_1 = y_2 = 0$, so $z_1 = z_2 = 0$. Hence, $(x_1, x_2) = (0.75, 1.25)$ is optimal.

(d)

```
Linear Programming Model:

Number of Decision Variables:      5

Number of Functional Constraints:  3

Max Z =      0 X1 +    0 X2 +    0 X3 -    1 X4 -    1 X5

subject to

 1)          2 X1 +    0 X2 +    1 X3 +    1 X4 +    0 X5 >=       2

 2)          0 X1 +    2 X2 +    1 X3 +    0 X4 +    1 X5 >=       3

 3)          1 X1 +    1 X2 +    0 X3 +    0 X4 +    0 X5 <=       2

and

        X1 >= 0, X2 >= 0, X3 >= 0, X4 >= 0, X5 >= 0.
```

```
Bas|Eq|                    Coefficient of                        | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6     X7     X8  |  side
___|__|__|_______________________________________________________|______
   |  |  |                                                       |
 Z | 0| 1|    0     -2     -1      1      0      0      1      0  |    -3
 X1| 1| 0|    1      0    0.5    0.5      0   -0.5      0      0  |     1
 X5| 2| 0|    0      2      1      0      1      0     -1      0  |     3
 X8| 3| 0|    0      1* -0.5   -0.5      0    0.5      0      1  |     1
```

```
Bas|Eq|                    Coefficient of                        | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6     X7     X8  |  side
___|__|__|_______________________________________________________|______
   |  |  |                                                       |
 Z | 0| 1|    0      0     -2      0      0      1      1      2  |    -1
 X1| 1| 0|    1      0    0.5    0.5      0   -0.5      0      0  |     1
 X5| 2| 0|    0      0      2*     1      1     -1     -1     -2  |     1
 X2| 3| 0|    0      1   -0.5   -0.5      0    0.5      0      1  |     1
```

```
Bas|Eq|                    Coefficient of                        | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6     X7     X8  |  side
___|__|__|_______________________________________________________|______
   |  |  |                                                       |
 Z | 0| 1|    0      0      0      1      1      0      0      0  |     0
 X1| 1| 0|    1      0      0   0.25  -0.25  -0.25   0.25    0.5  |  0.75
 X3| 2| 0|    0      0      1    0.5    0.5   -0.5   -0.5     -1  |   0.5
 X2| 3| 0|    0      1      0  -0.25   0.25   0.25  -0.25    0.5  |  1.25
```

Optimal Solution: $(x_1, x_2) = (0.75, 1.25)$ with $u = 0.5$

(e) Excel Solver Solution: $(x_1, x_2) = (0.75, 1.25)$

12.7-5.

(a) KKT conditions: (1a) $126 - 18x_1 - u_1 - 3u_3 \leq 0$

(2a) $x_1(126 - 18x_1 - u_1 - 3u_3) = 0$

(1b) $182 - 26x_2 - 2u_2 - 2u_3 \leq 0$

(2b) $x_2(182 - 26x_2 - 2u_2 - 2u_3) = 0$

(3a) $x_1 \leq 4$

(4a) $u_1(x_1 - 4) = 0$

(3b) $2x_2 \leq 12$

(4b) $u_2(2x_2 - 12) = 0$

(3c) $3x_1 + 2x_2 \leq 18$

(4c) $u_3(3x_1 + 2x_2 - 18) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u_1 \geq 0, u_2 \geq 0, u_3 \geq 0$

$(x_1, x_2) = (8/3, 5)$ with $\boldsymbol{u} = (0, 0, 26)$ satisfies these conditions, so it is optimal.

(b) minimize $z_1 + z_2$

subject to

$$\begin{aligned}
&18x_1 - y_1 + y_3 + 3y_5 + z_1 = 126 \\
&26x_2 - y_2 + 2y_4 + 2y_5 + z_2 = 182 \\
&x_1 + x_3 = 4 \\
&2x_2 + x_4 = 12 \\
&3x_1 + 2x_2 + x_5 = 18 \\
&x_1, x_2, x_3, x_4, x_5 \geq 0 \\
&y_1, y_2, y_3, y_4, y_5 \geq 0 \\
&z_1 \geq 0, z_2 \geq 0
\end{aligned}$$

Enforced complementarity constraint: $x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 = 0$

(c) Substitute $(x_1, x_2) = (8/3, 5)$ and $u_3 = y_5 = 26$ in the constraints.

$$\begin{aligned}
-y_1 + y_3 + z_1 &= 0 \\
-y_2 + 2y_4 + z_2 &= 0 \\
x_3 &= 4/3 \\
x_4 &= 2 \\
x_5 &= 0
\end{aligned}$$

Enforced complementarity constraint: $(8/3)y_1 + 5y_2 + (4/3)y_3 + 2y_4 = 0$

Since $y_i \geq 0$ for $i = 1, 2, \ldots, 5$, the complementarity constraint has the unique solutions $y_1 = y_2 = y_3 = y_4 = 0$, so $z_1 = z_2 = 0$. Hence, $(x_1, x_2) = (8/3, 5)$ is optimal.

12.7-6.

(a) - (b)

Minimum acceptable expected return = 13

	Amount Per Block			Right-Hand
Factor	Stock 1	Stock 2	Totals	Side
Budget	20	30	50	50
Expected Return	5	10	13.00	13
Risk	4	100	25.56	
Solution	2.2	0.2		

Joint Risk	Stock 1	Stock 2
Stock 1		5
Stock 2		

Minimum acceptable expected return = 14

	Amount Per Block			Right-Hand
Factor	Stock 1	Stock 2	Totals	Side
Budget	20	30	50	50
Expected Return	5	10	14.00	14
Risk	4	100	51.04	
Solution	1.6	0.6		

Joint Risk	Stock 1	Stock 2
Stock 1		5
Stock 2		

Minimum acceptable expected return = 15

	Amount Per Block			Right-Hand
Factor	Stock 1	Stock 2	Totals	Side
Budget	20	30	50	50
Expected Return	5	10	15.00	15
Risk	4	100	109.00	
Solution	1.0	1.0		

Joint Risk	Stock 1	Stock 2
Stock 1		5
Stock 2		

Minimum acceptable expected return = 16

	Amount Per Block			Right-Hand
Factor	Stock 1	Stock 2	Totals	Side
Budget	20	30	50	50
Expected Return	5	10	16.00	16
Risk	4	100	199.44	
Solution	0.4	1.4		

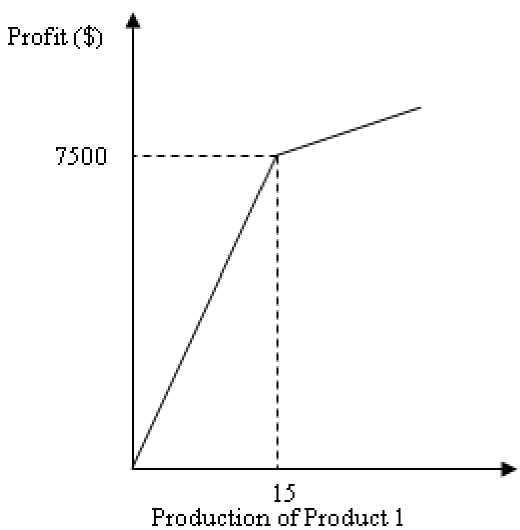
Joint Risk	Stock 1	Stock 2
Stock 1		5
Stock 2		

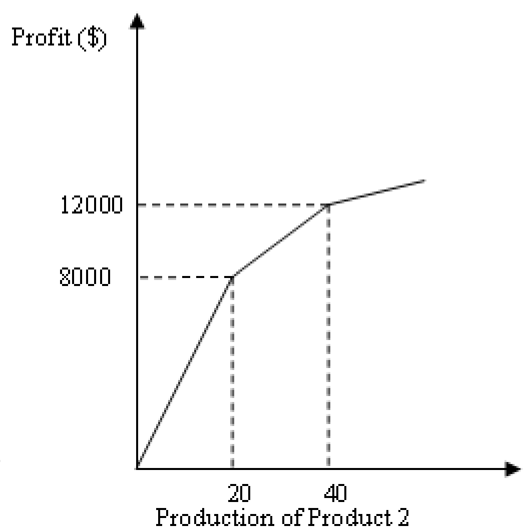
(c)

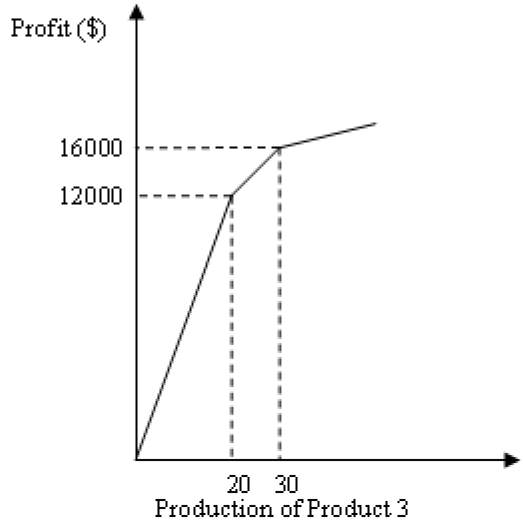
μ	σ	$\mu - \sigma$	$\mu - 3\sigma$
13	5.06	7.94	$-$ 2.18
14	7.14	6.86	$-$ 7.42
15	10.44	4.56	-16.32
16	14.12	1.88	-26.36

12.8-1.

(a)







(b) maximize $500x_{11} + 60x_{12} + 400x_{21} + 200x_{22} + 100x_{23} + 600x_{31} + 400x_{32} + 200x_{33}$

subject to
$$\begin{aligned}
&2x_{11} + 2x_{12} + 3x_{21} + 3x_{22} + 3x_{23} + 4x_{31} + 4x_{32} + 4x_{33} \leq 180 \\
&3x_{11} + 3x_{12} + x_{21} + x_{22} + x_{23} \leq 150 \\
&x_{11} + x_{12} + 3x_{31} + 3x_{32} + 3x_{33} \leq 100 \\
&0 \leq x_{11} \leq 15, 0 \leq x_{12} \\
&0 \leq x_{21} \leq 20, 0 \leq x_{22} \leq 20, 0 \leq x_{23} \\
&0 \leq x_{31} \leq 20, 0 \leq x_{32} \leq 10, 0 \leq x_{33}
\end{aligned}$$

where $x_1 = x_{11} + x_{12}, x_2 = x_{21} + x_{22} + x_{23}, x_3 = x_{31} + x_{32} + x_{33}$.

(c) Optimal solution with the simplex method:

	Coefficient Of										
	X11	X12	X21	X22	X23	X31	X32	X33	Total		RHS
	2	2	3	3	3	4	4	4	180	≤	180
	3	3	1	1	1	0	0	0	65	≤	150
	1	1	0	0	0	3	3	3	82.5	≤	100
	1	0	0	0	0	0	0	0	15	≤	15
	0	0	1	0	0	0	0	0	20	≤	20
	0	0	0	1	0	0	0	0	0	≤	20
	0	0	0	0	0	1	0	0	20	≤	20
	0	0	0	0	0	0	1	0	2.5	≤	10
Unit Profit	500	60	400	200	100	600	400	200	28500		
Solution	15	0	20	0	0	20	2.5	0			

Original variables: $x_1 = 15, x_2 = 20, x_3 = 22.5$

(d) The restriction on profit from products 1 and 2 can be modeled by introducing the constraint: $500x_{11} + 60x_{12} + 400x_{21} + 200x_{22} + 100x_{23} \geq 20,000$.

(e) Optimal solution with the simplex method:

	Coefficient Of										
	X11	X12	X21	X22	X23	X31	X32	X33	Total		RHS
	2	2	3	3	3	4	4	4	180	≤	180
	3	3	1	1	1	0	0	0	90	≤	150
	1	1	0	0	0	3	3	3	26.25	≤	100
	1	0	0	0	0	0	0	0	15	≤	15
	0	0	1	0	0	0	0	0	20	≤	20
	0	0	0	1	0	0	0	0	20	≤	20
	0	0	0	0	0	1	0	0	3.75	≤	20
	0	0	0	0	0	0	1	0	0	≤	10
Unit Profit	500	60	400	200	100	600	400	200	22250		
Solution	15	0	20	20	5	3.75	0	0			

Profit from Products 1 &2 =	20000	≥	20000

Original variables: $x_1 = 15, x_2 = 40, x_3 = 8.75$

12.8-2.

(a) KKT conditions:
(1a) $4 - 3x_1^2 - u_1 - 5u_2 \leq 0$
(2a) $x_1(4 - 3x_1^2 - u_1 - 5u_2) = 0$
(1b) $6 - 4x_2 - 3u_1 - 2u_2 \leq 0$
(2b) $x_2(6 - 4x_2 - 3u_1 - 2u_2) = 0$
(3a) $x_1 + 3x_2 \leq 8$
(4a) $u_1(x_1 + 3x_2 - 8) = 0$
(3b) $5x_1 + 2x_2 \leq 14$
(4b) $u_2(5x_1 + 2x_2 - 14) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u_1 \geq 0, u_2 \geq 0$

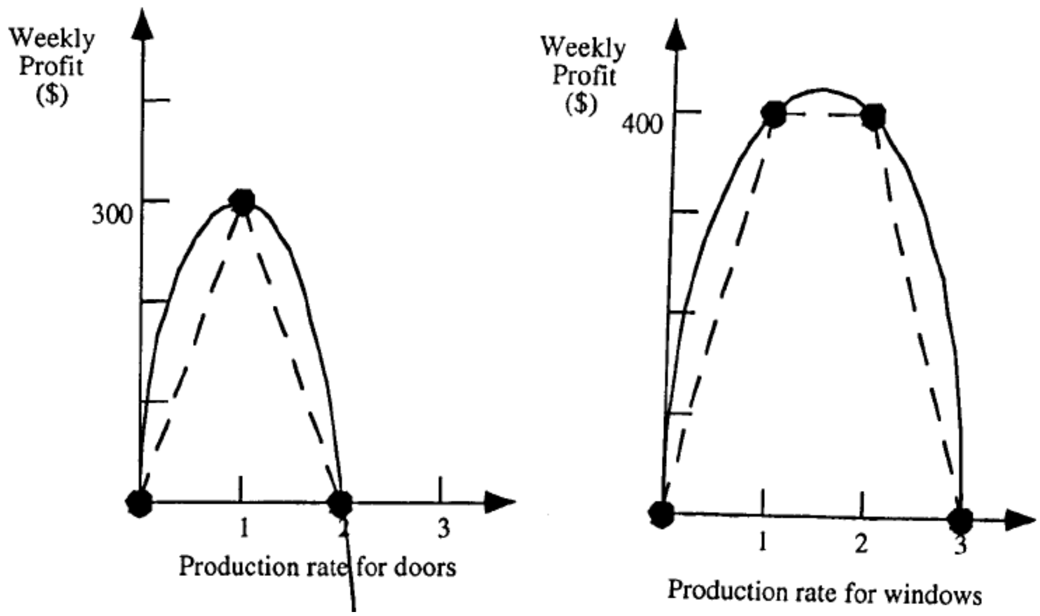
$(x_1, x_2) = (2/\sqrt{5}, 3/2)$ with $\boldsymbol{u} = (0, 0)$ satisfies these conditions, so it is optimal with $Z = 7.58$.

(b)

Profit data for doors when marketing costs are considered:				
Production Rate	Gross Profit	Marketing Cost	Net Profit	Incremental Net Profit
0	\$0	\$0	\$0	–
1	\$400	\$100	\$300	\$300
2	\$800	\$800	\$0	–\$300
3	\$1200	\$2700	–\$1900	–\$1900
D	\$$4D$	\$$D^3$	\$$4D - D^3$	

Profit data for windows when marketing costs are considered:				
Production Rate	Gross Profit	Marketing Cost	Net Profit	Incremental Net Profit
0	\$0	\$0	\$0	–
1	\$600	\$200	\$400	\$400
2	\$1200	\$800	\$400	\$0
3	\$1800	\$1800	\$0	–\$400
W	\$$6W$	\$$2W^2$	\$$6W - 2W^2$	

(c)



(d) Let $x_1 = x_{11} + x_{12} + x_{13}, x_2 = x_{21} + x_{22} + x_{23}, f_1(x_1) = 4x_1 - x_1^3$ and $f_2(x_2) = 6x_2 - 2x_2^2$.

$$f_1(0) = 0, f_1(1) = 3, f_1(2) = 0, f_1(3) = -15$$

$$f_2(0) = 0, f_2(1) = 4, f_2(2) = 4, f_2(3) = 0$$

$$s_{11} = \tfrac{3-0}{1-0} = 3, s_{12} = \tfrac{0-3}{2-1} = -3, s_{13} = \tfrac{-15-0}{3-2} = -15$$

$$s_{21} = \tfrac{4-0}{1-0} = 4, s_{12} = \tfrac{4-4}{2-1} = 0, s_{13} = \tfrac{0-4}{3-2} = -4$$

Approximate linear programming model:

maximize $\quad 3x_{11} - 3x_{12} - 15x_{13} + 4x_{21} - 4x_{23}$

subject to $\quad x_{11} + x_{12} + x_{13} + 3x_{21} + 3x_{22} + 3x_{23} \leq 8$

$\quad 5x_{11} + 5x_{12} + 5x_{13} + 2x_{21} + 2x_{22} + 2x_{23} \leq 14$

$\quad 0 \leq x_{ij} \leq 1$ for $i = 1, 2$ and $j = 1, 2, 3$

(e) Optimal solution with the simplex method:

```
Value of the
Objective Function: Z = 7
```

Variable		Value
X1	(x_{11})	1
X2	(x_{12})	0
X3	(x_{13})	0
X4	(x_{21})	1
X5	(x_{22})	0
X6	(x_{23})	0

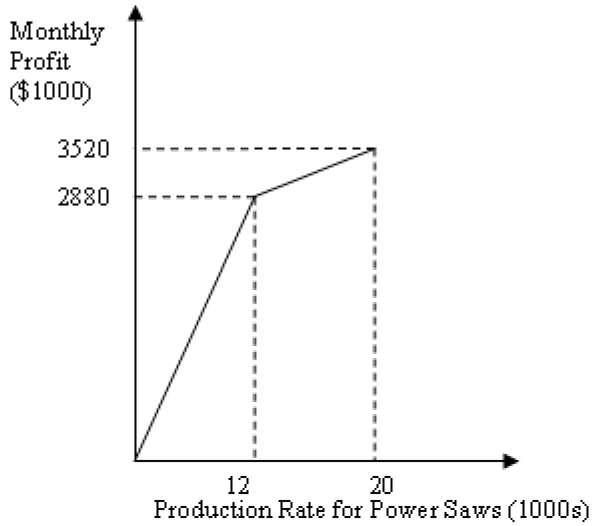
Original variables: $x_1 = 1, x_2 = 1$ (or $x_2 = 2$)

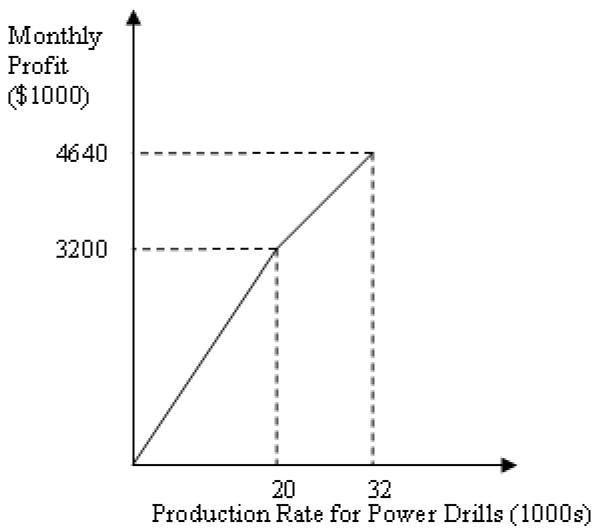
$$x_{11} = 0 \Rightarrow x_{12} = 0 \Rightarrow x_{13} = 0 \text{ and } x_{21} = 0 \Rightarrow x_{22} = 0 \Rightarrow x_{23} = 0$$

Hence, the special restriction for the model is satisfied. The approximate solutios $(1, 1)$ and $(1, 2)$ are pretty close to the optimal solution $(1.155, 1.5)$.

12.8-3.

(a)





(b) maximize $240x_{11} + 80x_{12} + 160x_{21} + 120x_{22}$

subject to

$$\begin{aligned} x_{11} + \ \ x_{12} + x_{21} + x_{22} &\leq 40,000 \\ 2x_{11} + 2x_{12} + x_{21} + x_{22} &\leq 60,000 \\ 0 \leq x_{11} \leq 12000, 0 \leq x_{12} &\leq 8000 \\ 0 \leq x_{21} \leq 20000, 0 \leq x_{22} &\leq 12000 \end{aligned}$$

(c) 12,000 power saws and 28,000 power drills should be produced in November.

Resource Usage Per Unit of Each Activity

	Power Saws		Power Drills				Resource
Resource	Regular	Overtime	Regular	Overtime	Total		Available
Power Supply Units	1	1	1	1	40000	≤	40000
Gear Assemblies	2	2	1	1	52000	≤	60000
Unit Profit	240	80	160	120	7040000		
Solution	12000	0	20000	8000			
Maximum	12000	8000	20000	12000			

12.8-4.

(a) Let $x_1 = x_{11} + x_{12} + x_{13}, x_2 = x_{21} + x_{22} + x_{23}, f_1(x_1) = 32x_1 - x_1^4$ and $f_2(x_2) = 50x_2 - 10x_2^2 + x_2^3 - x_2^4$.

$$f_1(0) = 0, f_1(1) = 31, f_1(2) = 48, f_1(3) = 15$$

$$f_2(0) = 0, f_2(1) = 40, f_2(2) = 52, f_2(3) = 6$$

$$s_{11} = 31, s_{12} = 17, s_{13} = -33$$

$$s_{21} = 40, s_{12} = 12, s_{13} = -46$$

Approximate linear programming model:

$$\begin{array}{ll} \text{maximize} & 31x_{11} + 17x_{12} - 33x_{13} + 40x_{21} + 12x_{22} - 46x_{23} \\ \text{subject to} & 3x_{11} + 3x_{12} + 3x_{13} + x_{21} + x_{22} + x_{23} \leq 11 \\ & 2x_{11} + 2x_{12} + 2x_{13} + 5x_{21} + 5x_{22} + 5x_{23} \leq 16 \\ & 0 \leq x_{ij} \leq 1 \text{ for } i = 1,2 \text{ and } j = 1,2,3 \end{array}$$

(b) Optimal solution with the simplex method:

Value of the
Objective Function: Z = 100

Variable		Value
X1	(X_{11})	1
X2	(X_{12})	1
X3	(X_{13})	0
X4	(X_{21})	1
X5	(X_{22})	1
X6	(X_{23})	0

Original variables: $x_1 = 2, x_2 = 2$

12.8-5.

Let $f_1(x_1) = \begin{cases} 5x_1 & \text{if } 0 \leq x_1 \leq 2 \\ 2 + 4x_1 & \text{if } 2 \leq x_1 \leq 5 \\ 12 + 2x_1 & \text{if } 5 \leq x_1 \end{cases}$ and $f_2(x_2) = \begin{cases} 4x_2 & \text{if } 0 \leq x_2 \leq 3 \\ 9 + x_2 & \text{if } 3 \leq x_2 \leq 4 \end{cases}$.

$$\begin{array}{ll} \text{maximize} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & 3x_1 + 2x_2 \leq 25 \\ & 2x_1 - x_2 \leq 10 \\ & x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

Possibly, the $f_i(x_i)$'s are piecewise-linear approximations of the original objective function.

12.8-6.

(a) Assume that in the optimal solution of the linear program, there exists and x_{ij} such that $x_{ij} < u_{ij}$ and $x_{i(j+1)} > 0$. Create a new solution with $x'_{ij} = \min\{u_{ij}, x_{ij} + x_{i(j+1)}\}$ and $x'_{i(j+1)} = \max\{0, x_{ij} + x_{i(j+1)} - u_{ij}\}$. This solution is feasible, since all the g_i's are linear and $x_{ij} + x_{i(j+1)} = x'_{ij} + x'_{i(j+1)}$, but

$$s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} = \begin{cases} s_{ij}(x_{ij} + x_{i(j+1)}) & \text{if } x_{ij} + x_{i(j+1)} \leq u_{ij} \\ s_{ij}u_{ij} + s_{i(j+1)}(x_{ij} + x_{i(j+1)} - u_{ij}) & \text{else.} \end{cases}$$

Clearly, $s_{ij}(x_{ij} + x_{i(j+1)}) > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$, since $s_{ij} > s_{i(j+1)}$.

Furthermore, $(s_{ij} - s_{i(j+1)})u_{ij} > (s_{ij} - s_{i(j+1)})x_{ij}$, since $x_{ij} < u_{ij}$.

$\Rightarrow s_{ij}u_{ij} + s_{i(j+1)}(x_{ij} - u_{ij}) > s_{ij}x_{ij}$

$\Rightarrow s_{ij}u_{ij} + s_{i(j+1)}(x_{ij} + x_{i(j+1)} - u_{ij}) > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$

$\Rightarrow s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$

Thus, the original solution was not optimal.

(b) Make the same assumptions as in (a) and construct x' from x in the same way. The linear approximation of g_i is of the form $\cdots + a_{ij}x_{ij} + a_{i(j+1)}x_{i(j+1)} + \cdots \leq b_i$ with $a_{ij} \leq a_{i(j+1)}$, since g_i is convex. By the same analysis as the one in (a), it can be shown that if the inequalities are reversed at appropriate places:

$$a_{ij}x'_{ij} + a_{i(j+1)}x'_{i(j+1)} < a_{ij}x_{ij} + a_{i(j+1)}x_{i(j+1)},$$

so x' is feasible. Furthermore, $s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$, so x was not optimal.

12.8-7.

$$f_1(x_1) = \begin{cases} 23x_1 & \text{if } 0 \leq x_1 \leq 6000 \\ 38x_1 - 90,000 & \text{if } 6000 \leq x_1 \end{cases}$$

$$f_2(x_2) = \begin{cases} 24x_2 & \text{if } 0 \leq x_2 \leq 3000 \\ 36x_2 - 36,000 & \text{if } 3000 \leq x_2 \end{cases}$$

$$\begin{array}{ll} \text{maximize} & z = x_1 + x_2 \\ \text{subject to} & f_1(x_1) + f_2(x_2) \leq 270,000 \\ & 0 \leq x_1 \leq 9000 \\ & 0 \leq x_2 \leq 4500 \end{array}$$

(a) Let x_i^R and x_i^O denote the regular and overtime production at plant i.

$$\begin{array}{ll} \text{maximize} & z = x_1^R + x_1^O + x_2^R + x_2^O \\ \text{subject to} & 23x_1^R + 38x_1^O + 24x_2^R + 36x_2^O \leq 270,000 \\ & 0 \leq x_1^R \leq 6000, 0 \leq x_1^O \leq 3000 \\ & 0 \leq x_2^R \leq 3000, 0 \leq x_2^O \leq 1500 \end{array}$$

(b) Since overtime production is more expensive than regular time production, the objective of maximizing the total production time will force the regular time to be used first.

12.8-8.

(a) The objective function is linear, so concave.

$$\frac{\partial^2 g_1(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 g_1(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 g_1(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = 4 \cdot 0 - 0^2 = 0$$

$$\frac{\partial^2 g_2(\boldsymbol{x})}{\partial x_1^2}\frac{\partial^2 g_2(\boldsymbol{x})}{\partial x_2^2} - \left[\frac{\partial^2 g_2(\boldsymbol{x})}{\partial x_1 \partial x_2}\right]^2 = 2 \cdot 0 - 0^2 = 0$$

$\Rightarrow g_1$ and g_2 are convex.

(b) Let $x_1 = x_{11} + x_{12} + x_{13}$. From the first constraint and $x_2 \geq 0$,

$$x_1 \leq \sqrt{13/2} \approx 2.55,$$

so using an integer breakpoint requires 3 linear pieces.

$$g_{11}(x_1) = 2x_1^2, g_{12}(x_2) = x_2, g_{21}(x_1) = x_1^2, g_{22}(x_2) = x_2$$
$$g_{11}(0) = 0, g_{11}(1) = 2, g_{11}(2) = 8, g_{11}(3) = 18$$
$$g_{21}(0) = 0, g_{21}(1) = 1, g_{21}(2) = 4, g_{21}(3) = 9$$
$$s_{11,1} = 2, s_{11,2} = 6, s_{11,3} = 10$$
$$s_{21,1} = 1, s_{21,2} = 3, s_{21,3} = 5$$

Approximate linear programming model:

maximize $\quad 5x_{11} + 5x_{12} + 5x_{13} + x_2$

subject to $\quad 2x_{11} + 6x_{12} + 10x_{13} + x_2 \leq 13$

$\quad x_{11} + 3x_{12} + 5x_{13} + x_2 \leq 9$

$\quad 0 \leq x_{11} \leq 1, 0 \leq x_{12} \leq 1, 0 \leq x_{13}, 0 \leq x_2$

We could have $0 \leq x_{13} \leq 1$, but the constraints will enforce the upper bound.

(c)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	0	0	0	1	4	2	15
X3	1	0	0	0	1	0	0.2	-0.2	-0.2	-0.6	0
X4	2	0	0	0	0	1	-1	2	0	0	5
X1	3	0	1	0	0	0	0	0	1	0	1
X2	4	0	0	1	0	0	0	0	0	1	1

Original variables: $x_1 = 1 + 1 + 0 = 2, x_2 = 5$

12.8-9.

(a) Let $x_1 = x_{11} + x_{12} + x_{13}$ and $x_2 = x_{21} + x_{22} + x_{23}$.

$f_1(x_1) = 32x_1 - x_1^4$, $\frac{d^2 f_1(x_1)}{dx_1^2} = -12x_1^2 \leq 0 \Rightarrow f_1$ concave

$f_2(x_2) = 4x_2 - x_2^2$, $\frac{d^2 f_2(x_2)}{dx_2^2} = -2 < 0 \Rightarrow f_2$ concave

$f_1(0) = 0, f_1(1) = 31, f_1(2) = 48, f_1(3) = 15$

$f_2(0) = 0, f_2(1) = 3, f_2(2) = 4, f_2(3) = 3$

$s_{11} = 31, s_{12} = 15, s_{13} = -33$

$s_{21} = 3, s_{22} = 1, s_{23} = -1$

$g_{11}(x_1) = x_1^2$, $\frac{d^2 g_{11}(x_1)}{dx_1^2} = 2 > 0 \Rightarrow g_{11}$ convex

$g_{12}(x_2) = x_2^2$, $\frac{d^2 g_{12}(x_2)}{dx_2^2} = 2 > 0 \Rightarrow g_{12}$ convex

$g_{11}(0) = 0, g_{11}(1) = 1, g_{11}(2) = 4, g_{11}(3) = 9$

$g_{21}(0) = 0, g_{21}(1) = 1, g_{21}(2) = 4, g_{21}(3) = 9$

$t_{11,1} = 1, t_{11,2} = 3, t_{11,3} = 5$

$t_{21,1} = 1, t_{21,2} = 3, t_{21,3} = 5$

Approximate linear programming model:

maximize $\quad 31x_{11} + 17x_{12} - 33x_{13} + 3x_{21} + x_{22} - x_{23}$

subject to $\quad x_{11} + 3x_{12} + 5x_{13} + x_{21} + 3x_{22} + 5x_{23} \leq 9$

$\quad 0 \leq x_{11} \leq 1, 0 \leq x_{12} \leq 1, 0 \leq x_{13} (\leq 1)$

$\quad 0 \leq x_{21} \leq 1, 0 \leq x_{22} \leq 1, 0 \leq x_{23} (\leq 1)$

(b) Solution with the simplex method:

Value of the
Objective Function: Z = 52

Variable		Value
X1	(x_{11})	1
X2	(x_{12})	1
X3	(x_{13})	0
X4	(x_{21})	1
X5	(x_{22})	1
X6	(x_{23})	0

Original variables: $x_1 = x_2 = 2$

(c) KKT conditions:

(1a) $32 - 4x_1^3 - 2x_1 u \leq 0$ (1b) $4 - 2x_2 - 2x_2 u \leq 0$

(2a) $x_1(32 - 4x_1^3 - 2x_1 u) = 0$ (2b) $x_2(4 - 2x_2 - 2x_2 u) = 0$

(3) $x_1^2 + x_2^2 - 9 \leq 0$

(4) $u(x_1^2 + x_2^2 - 9) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u \geq 0$

For $(x_1, x_2) = (2, 2)$, from (4), $u = 0$. This satisfies all the conditions, so is optimal to the original problem.

12.8-10.

(a) $f(x) = f_1(x_1) + f_2(x_2), f_1(x_1) = 3x_1^2 - x_1^3, f_2(x_2) = 5x_2^2 - x_2^3$

$\frac{d^2 f_1(x_1)}{dx_1^2} = 6 - 6x_1 > 0$ if $0 \leq x_1 < 1$

$\frac{d^2 f_2(x_2)}{dx_2^2} = 10 - 6x_2 > 0$ if $0 \leq x_2 < 5/3$

Neither f_1 nor f_2 is concave, so f is not concave. It is indeed enough to show one is not concave.

(b) Let $x_1 = x_{11} + x_{12} + x_{13} + x_{14}, x_2 = x_{21} + x_{22}$.

$f_1(0) = 0, f_1(1) = 2, f_1(2) = 4, f_1(3) = 0, f_1(4) = -16$

$f_2(0) = 0, f_2(1) = 4, f_2(2) = 12$

$s_{11} = 2, s_{12} = 2, s_{13} = -4, s_{14} = -16$

$s_{21} = 4, s_{22} = 8$

Special restrictions are needed:

(i) $x_{12} = 0$ if $x_{11} < 1$

(ii) $x_{13} = 0$ if $x_{12} < 1$

(iii) $x_{14} = 0$ if $x_{13} < 1$

(iv) $x_{22} = 0$ if $x_{21} < 1$.

Since $s_{12} > s_{13} > s_{14}$, (ii) and (iii) are automatically satisfied upon optimization.

Approximate binary integer programming model:

maximize $2x_{11} + 2x_{12} - 4x_{13} - 16x_{14} + 4x_{21} + 8x_{22}$

subject to $x_{11} + x_{12} + x_{13} + x_{14} + 2x_{21} + 2x_{22} \leq 4$
$-x_{11} + x_{12} \leq 0$
$-x_{21} + x_{22} \leq 0$
$x_{ij} \in \{0, 1\}$ for all i, j

(c) Solution with BIP automatic routine:

$x_{11} = x_{12} = x_{13} = x_{14} = 0, x_{21} = x_{22} = 1, z = 12$

Original variables: $x_1 = 0, x_2 = 2, z = 12$

Alternate solution: $x_1 = 2, x_2 = 1, z = 12$

12.9-1.

$\nabla f(x_1, x_2) = \left(\frac{1}{x_1+1}, -2x_2\right)$

Iteration 1: $\nabla f(0,0) = (1,0)$

maximize x_1

subject to $x_1 + 2x_2 \leq 3$
$x_1, x_2 \geq 0$

$\Rightarrow x_1 = 3, x_2 = 0 \Rightarrow x^{(1)} = (0,0) + t(3,0)$

$t^* = 1$ ($f(\boldsymbol{x})$ increases with t) $\Rightarrow x^{(1)} = (3,0)$ [solution found in Problem 12.6-5]

Iteration 2: $\nabla f(3,0) = (1/4, 0)$

$$\begin{array}{ll} \text{maximize} & 0.25x_1 \\ \text{subject to} & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

$\Rightarrow x_1 = 3, x_2 = 0 \Rightarrow x^{(1)} = (3,0) + t(0,0)$

Hence $x = (3,0)$ is optimal.

12.9-2.

k	x(k-1)	c1	c2	xLP(k)	t*	x(k)
1	(0, 0)	-6	-3	(1, 0)	1	(1, 0)
2	(1, 0)	-4	-3	(1, 0)	1e-8	(1, 0)

Final solution: (1, 0).

$\nabla f(x_1, x_2) = (2x_1 - 6, 3x_2^2 - 3)$

$x_1 + x_2 \leq 1, x_1, x_2 \geq 0 \Rightarrow x_1, x_2 \leq 1 \Rightarrow 2x_1 - 6 \leq -4 < -3 \leq 3x_2^2 - 3$

$$\text{Resulting LP:} \quad \begin{array}{ll} \text{maximize} & c_1x_1 + c_2x_2 \\ \text{subject to} & x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

where $c_1 < c_2$, so $(1,0)$ is always optimal.

$\Rightarrow x^{(1)} = (x_1^{(0)}, x_2^{(0)}) + t(1 - x_1^{(0)}, -x_2^{(0)})$

At $t^* = 1$, $x^{(1)} = (1,0)$ is optimal.

12.9-3.

k	x(k-1)	c1	c2	c3	xLP(k)	t*	x(k)
1	(0, 0, 0)	8	2	1	(12, 0, 0)	0.33	(4, 0, 0)
2	(4, 0, 0)	0	2	1	(0, 4, 0)	0.25	(3, 1, 0)

Final solution: (3, 1, 0).

12.9-4.

$$\begin{array}{ll} \text{maximize} & 15x_1 + 30x_2 + 4x_1x_2 - 2x_1^2 - 4x_2^2 \\ \text{subject to} & x_1 + 2x_2 \leq 30 \\ & x_1, x_2 \geq 0 \end{array}$$

k	X^(k-1)	c1	c2	X[LP]^k	t*	X^k
1	(5, 5)	15	10	(30, 0)	0.088	(7.196,4.561)
2	(7.196,4.561)	6.216	22.3	(0, 15)	0.119	(6.336,5.808)
3	(6.336,5.808)	7.898	8.884	(30, 0)	0.07	(7.998, 5.4)
4	(7.998, 5.4)	6.24	18.79	(0, 15)	0.089	(7.284,6.257)
5	(7.284,6.257)	7.466	9.076	(30, 0)	0.054	(8.516,5.918)
6	(8.516,5.918)	5.967	16.72	(0, 15)	0.072	(7.904, 6.57)
7	(7.904, 6.57)	7.054	9.058	(30, 0)	0.045	(8.888,6.277)
8	(8.888,6.277)	5.727	15.33	(0, 15)	0.06	(8.351,6.804)

Final solution: (8.3515, 6.804).

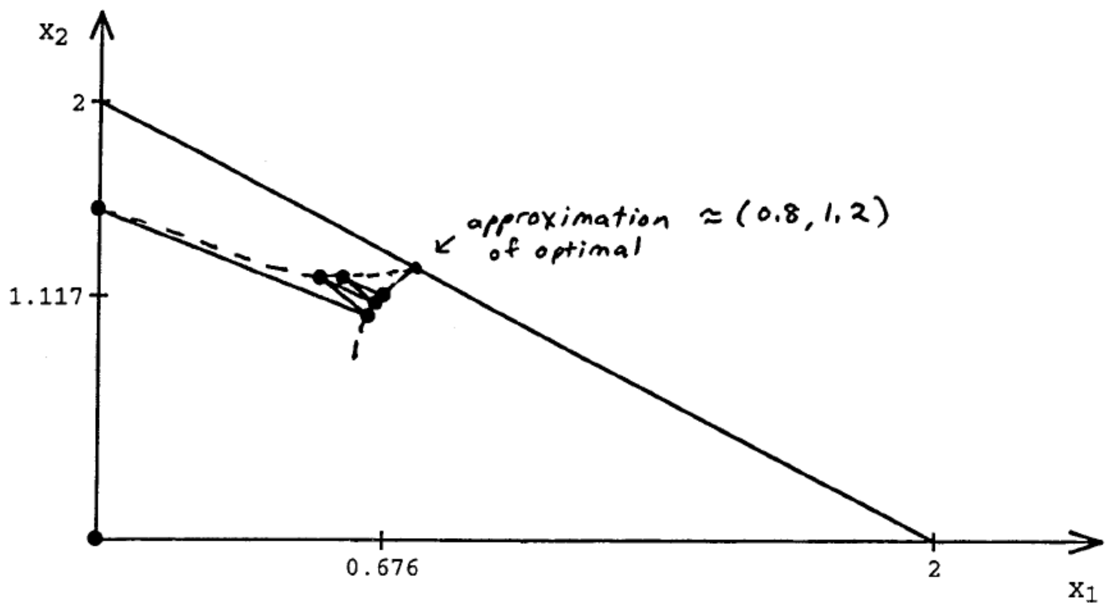
12.9-5.

(a)

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t^*	$\mathbf{x}^{(k)}$
1	(0, 0)	2	3	(0, 2)	0.75	(0, 1.5)
2	(0, 1.5)	2	0	(2, 0)	0.32	(0.64, 1.02)
3	(0.64, 1.02)	0.72	0.96	(0, 2)	0.175	(0.528,1.192)
4	(0.528,1.192)	0.944	0.617	(2, 0)	0.092	(0.663,1.082)
5	(0.663,1.082)	0.674	0.835	(0, 2)	0.126	(0.579,1.198)
6	(0.579,1.198)	0.842	0.603	(2, 0)	0.068	(0.676,1.117)

Final solution: (0.676,1.1166).

(b)



12.9-6.

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t^*	$\mathbf{x}^{(k)}$
1	(0, 0)	32	50	(3, 2)	0.729	(2.188,1.458)
2	(2.188,1.458)	-9.87	14.81	(0, 3.2)	0.131	(1.902,1.686)
3	(1.902,1.686)	4.499	5.634	(3, 2)	0.111	(2.024,1.721)
4	(2.024,1.721)	-1.15	4.078	(0, 3.2)	0.028	(1.966,1.763)

Final solution: (1.9662,1.7629).

12.9-7.

k	X^(k-1)	c1	c2	X[LP]^k	t*	X^k
1	(0, 0)	40	30	(3, 0)	0.616	(1.847, 0)
2	(1.847, 0)	0.001	35.54	(0, 2)	0.406	(1.097,0.812)

12.9-8.

(a)

k	X^(k-1)	c1	c2	X[LP]^k	t*	X^k
1	(0.25, 0.25)	2.813	3.5	(0, 1)	1	(0, 1)
2	(0, 1)	3	2	(1, 0)	0.333	(0.333,0.667)
3	(0.333,0.667)	2.667	2.667	(1, 0)	0.001	(0.334,0.666)

(b) KKT conditions: (1a) $3-3x_1^2-u \le 0$ (1b) $4-2x_2-u \le 0$
(2a) $x_1(3-3x_1^2-u)=0$ (2b) $x_2(4-2x_2-u)=0$
(3) $x_1+x_2 \le 1$
(4) $u(x_1+x_2-1)=0$
(5) $x_1 \ge 0, x_2 \ge 0$
(6) $u \ge 0$

$(x_1,x_2)=(1/3,2/3)$ with $u=8/3$ satisfies these conditions, so the estimated solution in part (a) is optimal.

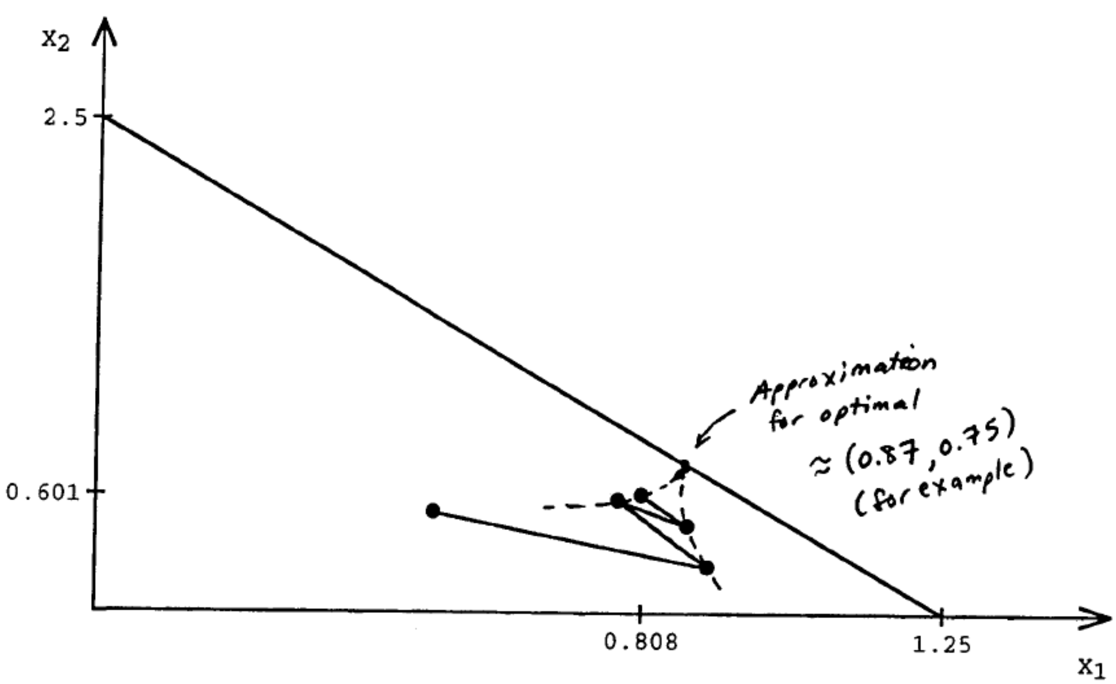
12.9-9.

(a)

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t*	$\mathbf{x}^{(k)}$
1	(0.5, 0.5)	3.5	1	(1.25, 0)	0.541	(0.906,0.229)
2	(0.906,0.229)	1.027	1.541	(0, 2.5)	0.148	(0.771,0.566)
3	(0.771,0.566)	2.164	0.867	(1.25, 0)	0.216	(0.875,0.444)
4	(0.875,0.444)	1.323	1.112	(0, 2.5)	0.076	(0.808,0.601)

Final solution: (0.8079,0.6011).

(b)



(c) KKT conditions:

(1a) $4 - 4x_1^3 - 4u \leq 0$ (1b) $2 - 2x_2 - 2u \leq 0$

(2a) $x_1(4 - 4x_1^3 - 4u) = 0$ (2b) $x_2(2 - 2x_2 - 2u) = 0$

(3) $4x_1 + 2x_2 \leq 5$

(4) $u(4x_1 + 2x_2 - 5) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u \geq 0$

$(x_1, x_2) = (0.8934, 0.7131)$ with $u = 0.5737$ satisfies these conditions, so is optimal.

12.9-10.

(a) $P(\boldsymbol{x}; r) = 3x_1 + 4x_2 - x_1^3 - x_2^2 - r\left[\frac{1}{1-x_1-x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(b)

$$\nabla P(\boldsymbol{x}; r) = \begin{pmatrix} 3 - 3x_1^2 + r\left[\frac{-1}{(1-x_1-x_2)^2} + \frac{1}{x_1^2}\right] \\ 4 - 2x_2 + r\left[\frac{-1}{(1-x_1-x_2)^2} + \frac{1}{x_2^2}\right] \end{pmatrix}$$

$$\Rightarrow \nabla P(\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \end{pmatrix}; 1) = \begin{pmatrix} 14\frac{13}{16} \\ 15\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \end{pmatrix} + t\nabla P(\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \end{pmatrix}; 1) = \begin{pmatrix} \frac{1}{4} + 14\frac{13}{16}t & \frac{1}{4} + 15\frac{1}{2}t \end{pmatrix}$$

$$t^* = 0.006606 \Rightarrow x' = \begin{pmatrix} 0.3479 & 0.3524 \end{pmatrix}$$

(c)

k	r	x_1	x_2	f(**x**)
0		0.25	0.25	1.672
1	1	0.343	0.357	2.29
2	0.01	0.322	0.619	3.023
3	0.0001	0.331	0.663	3.169

(d) True Solution: $(1/3, 2/3)$

Percentage error in x_1: $\frac{|1/3-0.331|}{1/3} = 0.70\%$

Percentage error in x_2: $\frac{|2/3-0.663|}{2/3} = 0.55\%$

Percentage error in $f(x)$: $\frac{|86/27-3.169|}{86/27} = 0.51\%$

12.9-11.

(a) $P(\boldsymbol{x};r) = 4x_1 - x_1^4 + 2x_2 - x_2^2 - r\left[\frac{1}{5-4x_1-2x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(b) $\nabla P(\boldsymbol{x};r) = \begin{pmatrix} 4 - 4x_1^3 + r\left[\frac{-4}{(5-4x_1-2x_2)^2} + \frac{1}{x_1^2}\right] \\ 2 - 2x_2 + r\left[\frac{-2}{(5-4x_1-2x_2)^2} + \frac{1}{x_2^2}\right] \end{pmatrix}$

$\Rightarrow \nabla P((\,\tfrac{1}{2} \quad \tfrac{1}{2}\,);1) = \begin{pmatrix} 6\frac{1}{2} \\ 4\frac{1}{2} \end{pmatrix}$

$(\,\tfrac{1}{2} \quad \tfrac{1}{2}\,) + t\nabla P((\,\tfrac{1}{2} \quad \tfrac{1}{2}\,);1) = (\,\tfrac{1}{2} + 6\tfrac{1}{2}t \quad \tfrac{1}{4} + 4\tfrac{1}{2}t\,)$

$t^* = 0.03167 \Rightarrow x' = (\,0.7058 \quad 0.6425\,)$

(c)

k	r	x_1	x_2	f(x)
0		0.5	0.5	2.688
1	1	0.669	0.716	3.395
2	0.01	0.871	0.671	3.801
3	0.0001	0.891	0.708	3.849
4	0.000001	0.894	0.712	3.854

12.9-12.

(a) $P(\boldsymbol{x};r) = -x_1^4 - 2x_1^2 - 2x_1x_2 - 4x_2^2 - r\left[\frac{1}{2x_1+x_2-10} + \frac{1}{x_1+2x_2-10} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(b) $\nabla P(\boldsymbol{x};r) = \begin{pmatrix} -4x_1^3 - 4x_1 - 2x_2 + r\left[\frac{2}{(2x_1+x_2-10)^2} + \frac{1}{(x_1+2x_2-10)^2} + \frac{1}{x_1^2}\right] \\ -2x_1 - 8x_2 + r\left[\frac{1}{(2x_1+x_2-10)^2} + \frac{2}{(x_1+2x_2-10)^2} + \frac{1}{x_2^2}\right] \end{pmatrix}$

$\Rightarrow \nabla P((\,5 \quad 5\,);100) = \begin{pmatrix} -514 \\ -34 \end{pmatrix}$

$(\,5 \quad 5\,) + t\nabla P((\,5 \quad 5\,);1) = (\,5 - 514t \quad 5 - 34t\,)$

$t^* = 0.003529 \Rightarrow x' = (\,3.1862 \quad 4.8802\,)$

(c)

k	r	x_1	x_2	f(x)
0		5	5	-825
1	100	2.725	6.072	-251
2	1	2.587	4.976	-183
3	0.01	2.562	4.891	-177
4	0.0001	2.557	4.888	-176

minimize $f(x) \rightarrow$ maximize $-f(x)$

$g(x) \geq b \rightarrow -g(x) \leq -b$

12.9-13.

(a) KKT conditions:

(1a) $x_2 - 4ux_1 \leq 0$ (1b) $x_1 - u \leq 0$

(2a) $x_1(x_2 - 4ux_1) = 0$ (2b) $x_2(x_1 - u) = 0$

(3) $x_1^2 + x_2 \leq 3$

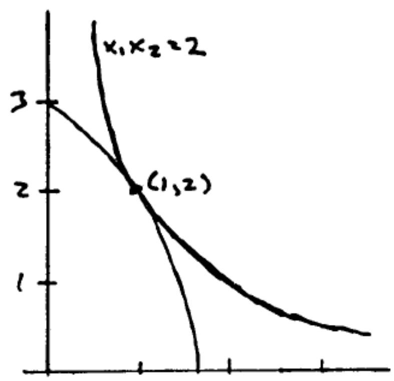
(4) $u(x_1^2 + x_2 - 3) = 0$

(5) $x_1 \geq 0, x_2 \geq 0$

(6) $u \geq 0$

$(x_1, x_2) = (1, 2)$ with $u = 1$ satisfies these conditions.

(b)



12.9-14.

(a) $P(\boldsymbol{x}; r) = -2x_1 - (x_2 - 3)^2 - r\left[\frac{1}{x_1 - 3} + \frac{1}{x_2 - 3}\right]$

(b) $\nabla P(\boldsymbol{x}; r) = \begin{pmatrix} -2 + r\left[\frac{1}{(x_1-3)^2}\right] \\ -2x_2 + 6 + r\left[\frac{1}{(x_2-3)^2}\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 = \sqrt{r/2} + 3, x_2 = \sqrt[3]{r/2} + 3$

r	x_1	x_2
1	3.7071	3.7937
10^{-2}	3.0707	3.1710
10^{-4}	3.0071	3.0368
10^{-6}	3.0007	3.0079

Note that $(x_1, x_2) \to (3, 3)$ as $r \to 0$, so $(3, 3)$ is optimal.

(c)

k	r	X1	X2	f(X)
0		4	4	-9
1	1	3.707	3.794	-8.044
2	0.01	3.07	3.179	-6.172
3	0.0001	3.007	3.056	-6.017
4	0.000001	3.001	3.011	-6.002

12.9-15.

$P(\boldsymbol{x};r) = -x_1^2 - x_2^2 - x_1 - x_2 + x_1x_2 - r/x_2$

k	r	X_1	X_2	$f(\mathbf{X})$
0		1	1	-3
1	1	-0.18	0.638	-1.01
2	0.01	-0.46	0.079	0.127
3	0.0001	-0.5	0.008	0.238

12.9-16.

$P(\boldsymbol{x};r) = 2x_1 + 3x_2 - x_1^2 - x_2^2 - r\left[\frac{1}{2-x_1-x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

k	r	X_1	X_2	$f(\mathbf{X})$
0		0.5	0.5	2
1	1	0.649	0.781	2.61
2	0.01	0.691	1.184	3.055
3	0.0001	0.743	1.243	3.118
4	0.000001	0.749	1.249	3.124

12.9-17.

$P(\boldsymbol{x};r) = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2 - r\left[\frac{1}{4-x_1} + \frac{1}{12-2x_2} + \frac{1}{18-3x_1-2x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

k	r	X_1	X_2	$f(\mathbf{X})$
0		2	3	645
1	100	2.292	4.523	798.8
2	1	2.62	4.972	851.9
3	0.01	2.661	4.999	856.5
4	0.0001	2.665	5.002	856.9

12.9-18.

(a) $P(\boldsymbol{x};r) = x_1^3 + 4x_2^2 + 16x_3 - r\left[\frac{1}{x_1-1} + \frac{1}{x_2-1} + \frac{1}{x_3-1}\right] - \frac{(5-x_1-x_2-x_3)^2}{\sqrt{r}}$

(b)

k	r	X1	X2	X3	f(X)
0		1.5	1.5	2	-44.38
1	0.01	1.95	1.434	1.047	-32.38
2	0.0001	2.179	1.743	1.007	-38.62
3	0.000001	2.208	1.784	1.001	-39.51
4	0.00000001	2.21	1.786	1.002	-39.6

(c) Standard Excel Solver

Min	f(x)	39.608		
s.t.	g1(x)	5.000	=	5
	x1	2.194	>=	1
	x2	1.806	>=	1
	x3	1.000	>=	1

(d) Evolutionary Solver

Min	f(x)	38.796		
s.t.	g1(x)	4.941	=	5
	x1	2.146	>=	1
	x2	1.783	>=	1
	x3	1.012	>=	1

(e) LINGO

```
MIN =     X1^3 + 4 * X2^2 + 16*X3;
          X1   + X2 + X3           = 5;
          X1                      >= 1;
                X2                >= 1;
                     X3           >= 1;
```

```
Local optimal solution found at iteration:              34
Objective value:                                  39.60766

Variable              Value         Reduced Cost
      X1           2.194335             0.000000
      X2           1.805665             0.000000
      X3           1.000000             0.000000
Row      Slack or Surplus        Dual Price
  1           39.60766           -1.000000
  2           0.000000          -14.44532
  3           1.194335            0.000000
  4          0.8056651            0.000000
  5           0.000000           -1.554676
```

12.10-1.

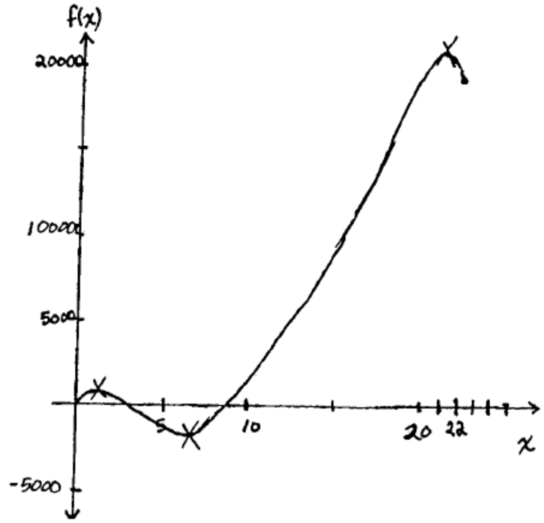
(a) Solving for the roots of $x^2 + x - 500 = 0$, one observes that x is feasible in the range $\left[0, \frac{-1+\sqrt{2001}}{2}\right] = [0, 21.866]$.

$$f'(x) = 1000 - 800x + 120x^2 - 4x^3$$

$$f''(x) = -800 + 240x - 12x^2$$

$$f'''(x) = 240 - 24x$$

A rough sketch of $f(x)$:



The points that are marked as X correspond to a local minimum or maximum.

(b)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	5	2.5	585.94
1	-312.5	0	2.5	1.25	700.68
2	+179.7	1.25	2.5	1.875	720.06
3	-104.5	1.25	1.875	1.5625	732.56
4	+27.71	1.5625	1.875	1.7188	731.48
5	-40.82	1.5625	1.7188	1.6406	733.36
6	-7.166	1.5625	1.6406	1.6016	733.3
Stop					

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		18	21.866	19.933	19931
1	+ 1053	19.933	21.866	20.899	20546
2	+180.4	20.899	21.866	21.383	20509
3	-346.2	20.899	21.383	21.141	20559
4	- 75.1	20.899	21.141	21.02	20560
5	+54.58	21.02	21.141	21.081	20562
6	-9.778	21.02	21.081	21.051	20561
Stop					

There is a local maximum near 1.6016 and a global maximum near 21.051.

(c)

Newton's method

Max f(x) = 1000 x - 400 x^2 +40 x^3 - x^4 s.t. x^2 + x <= 500, x>= 0

f'(x) = 1000 - 800 x + 120 x^2 - 4 x^3

f''(x) = -800 + 240 x - 12 x^2

error 0.001

Starting with x = 3

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	3	399	-428	-188	0.723404	2.276596
2	0.72340426	528.947606	482.56	-632.6627	1.486149	0.762744
3	1.48614867	729.11001	62.98815	-469.828	1.620215	0.134066
4	1.62021508	733.414048	1.826677	-442.6495	1.624342	0.004127
5	1.62434177	733.417819	0.001712	-441.8198	1.624346	3.88E-06

Local maximum: $x = 1.6243$

Starting with x = 15

Iteration i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}	$\lvert x_i - x_{i+1}\rvert$
1	20	20000	1000	-800	21.25	1.25
2	21.25	20544.4336	-195.3125	-1118.75	21.07542	0.174581
3	21.075419	20561.721	-4.093317	-1071.979	21.0716	0.003818
4	21.0716005	20561.7289	-0.001938	-1070.964	21.0716	1.81E-06
5	21.0715987	20561.7289	-4.51E-10	-1070.964	21.0716	4.23E-13
6	21.0715987	20561.7289	0	-1070.964	21.0716	0
7	21.0715987	20561.7289	0	-1070.964	21.0716	0
8	21.0715987	20561.7289	0	-1070.964	21.0716	0
9	21.0715987	20561.7289	0	-1070.964	21.0716	0
10	21.0715987	20561.7289	0	-1070.964	21.0716	0
11	21.0715987	20561.7289	0	-1070.964	21.0716	0
12	21.0715987	20561.7289	0	-1070.964	21.0716	0
13	21.0715987	20561.7289	0	-1070.964	21.0716	0

Local maximum: $x = 21.0716$

(d)

k	r	x_1	f(x)
0		3	399
1	1000	2.171	672.8
2	100	1.704	732
3	10	1.633	733.4
4	1	1.625	733.4
0		15	9375
1	1000	21.04	20561
2	100	21.07	20562
3	10	21.07	20562
4	1	21.07	20562

The first four iterations with initial trial solution $x = 3$, return $x = 1.625$ with $f(x) = 733.4$ as maximum. The next four iterations with initial trial solution $x = 15$, return $x = 21.07$ with $f(x) = 20562$ as maximum. The global maximum is $x = 21.07$.

(e)

Solver Table
Max f(x) = 1000 x - 400 x^2 +40 x^3 - x^4 s.t. x^2 + x <= 500, x>= 0

Max	f(x)	733.4178		
s.t.	x	1.624346	>=	0
	g(x)	4.262844	<=	500

Starting poir	optimal x*	profit f(x*)
	1.624	733.418
0	1.624	733.418
5	1.624	733.418
10	21.072	20561.729
15	21.072	20561.729
20	21.072	20561.729
25	21.072	20561.729

(f) $x = 21.0716$

Evolutionary Solver
Max f(x) = 1000 x - 400 x^2 +40 x^3 - x^4 s.t. x^2 + x <= 500, x>= 0

Max	f(x)	20561.73		
s.t.	x	21.0716	>=	0
	g(x)	465.0839	<=	500

(g)

```
                    Lingo with Global Solver
! Nonlinear constraint;
MAX = 1000 * X  -400*X^2 + 40*X^3 - X^4                ;
             X                                  >=   0;
             X^2 + X                            <= 500;

Global optimal solution found at iteration:                33
Objective value:                                     20561.73

Variable           Value        Reduced Cost
       X        21.07159            0.000000

Row      Slack or Surplus       Dual Price
  1          20561.73           1.000000
  2          21.07159           0.000000
  3          34.91636           0.000000
```

(h)

```
TITLE
    "12.10-1";

OPTIONS
    ModelType=Nonlinear

VARIABLES
    x;

MODEL

    MAX f = 1000x-400x^2+40x^3-x^4;

SUBJECT TO

    x^2+x <= 500;

END
```

```
SOLUTION RESULT


  Global solution found (1,1)


    MAX f        =        733.4178

DECISION VARIABLES


PLAIN VARIABLES

    Variable Name              Activity     Reduced Cost
  -------------------------------------------------------
    x                            1.6243           0.0000
  -------------------------------------------------------
```

12.10-2.

(a) $P(\boldsymbol{x};r) = 3x_1x_2 - 2x_1^2 - x_2^2 - r\left[\frac{1}{4-x_1^2-2x_2^2} + \frac{1}{x_2-2x_1} + \frac{1}{x_1} + \frac{1}{x_2}\right] - \frac{(2-x_1x_2^2-x_1^2x_2)^2}{\sqrt{r}}$

(b)

k	r	X1	X2	f(X)
0		1	1	0
1	1	0.915	1.007	0.0758
2	0.01	0.848	1.169	0.1692
3	0.0001	0.843	1.175	0.1697

(c) Evolutionary Solver

Max	f(x)	0.171564		
s.t.	g1(x)	3.519555	<=	4
	g2(x)	0.504927	<=	3
	g3(x)	2.030303	=	2
	x1	0.844707	>=	0
	x2	1.184488	>=	0

(d) Use global optimizer feature of LINGO.

```
! Nonlinear constraint;
MAX = 3*x1*x2-2*x1^2-x2^2;
x1^2+2*x2^2<=4;
2*x1-x2<=3;
x1*x2^2+x1^2*x2=2;
x1>=0;
x2>=0;
```

```
Global optimal solution found at iteration:          879
Objective value:                               0.1698892


                Variable           Value        Reduced Cost
                      X1       0.8382396       -0.8808309E-07
                      X2        1.181385         0.000000

                     Row    Slack or Surplus      Dual Price
                       1       0.1698892          1.000000
                       2       0.5060155          0.000000
                       3        2.504905          0.000000
                       4        0.000000         0.5662949E-01
                       5       0.8382396          0.000000
                       6        1.181385          0.000000
```

(e)

```
TITLE
    "12.10-2";

OPTIONS
    ModelType=Nonlinear

VARIABLES
    x1 x2;

MODEL

    MAX f = 3x1*x2-2x1^2-x2^2;

SUBJECT TO

    x1^2+2x2^2      <= 4;
    2x1-x2          <= 3;
    x1*x2^2+x1^2*x2  = 2;

END
```

```
SOLUTION RESULT


  Infeasible solution (4,1)


    MAX f        =          0.0000

DECISION VARIABLES


PLAIN VARIABLES

    Variable Name              Activity     Reduced Cost
  ----------------------------------------------------
    x1                           0.0000           0.0000
    x2                           0.0000           0.0000
  ----------------------------------------------------
```

12.10-3.

(a) $P(\boldsymbol{x};r) = \sin 3x_1 + \cos 3x_2 + \sin(x_1 + x_2) + r\left[\frac{1}{1+x_1^2-10x_2} + \frac{1}{100-10x_1-x_2^2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(b) SUMT can be used to obtain the global minimum if it is run with "enough" different starting points. If a lattice of points over the feasible region is chosen so that the adjacent points do not differ by more than $2\pi/3$, then this set of points works for $f(x)$. Since sin and cos have period 2π, choosing lattice points with grid size not exceeding $2\pi/3$ ensures that the arguments of the sin and cos terms in f do not differ by more than 2π between adjacent lattice points. Since the second constraint ensures $x_1 \leq 10$ and $x_2 \leq 10$, at most $[10/(2\pi/3)]^2 \approx 23$ starting points are required if chosen correctly.

(c)

```
          Use LINGO Global Solver;
MIN =    @SIN(3*X1) + @COS(3*X2) + @SIN(X1+X2);
         X1^2 - 10 * X2       >= -1;
         10*X1  + X2^2        <= 100;
           X1                 >= 0;
                X2            >= 0;

Global optimal solution found at iteration:              6
Objective value:                                 -2.999999

Variable           Value          Reduced Cost
      X1        3.665418              0.000000
      X2        1.046684              0.000000

Row     Slack or Surplus        Dual Price
  1        -2.999999            -1.000000
  2         3.968452             0.000000
  3         62.25027             0.000000
  4         3.665418             0.000000
  5         1.046684             0.000000
```

(d)

```
TITLE
    "12.10-3";

OPTIONS
    ModelType=Nonlinear

VARIABLES
    x1 x2 x3 x4 x5;

MODEL

    MIN f = SIN(x3) + COS(x4) +SIN(x5);

SUBJECT TO

    x3 = 3x1;
    x4 = 3x2;
    x5 = x1+x2;

    x1^2-10x2 >= -1;
    10x1+x2^2 <= 100;

END
```

```
SOLUTION RESULT


  Global solution found (1,1)


    MIN f        =         -3.0000

DECISION VARIABLES


PLAIN VARIABLES

    Variable Name              Activity      Reduced Cost
  -----------------------------------------------------
    x1                          -0.5236            0.0000
    x2                          -1.0472            0.0000
    x3                          -1.5708            0.0000
    x4                          -3.1416            0.0000
    x5                          -1.5708            0.0000
  -----------------------------------------------------
```

12.10-4.

(a)

	A	B	C	D	E	F
1		0		Starting		
2		<=		Point	x*	Profit*
3	x =	3.537			3.537	6.801
4		<=		0	0.405	10.735
5		5		1	0.405	10.735
6				2	3.537	6.801
7	Profit =	$x^5 - 13x^4 + 59x^3 - 107x^2 + 61x$		3	3.537	6.801
8	=	6.801		4	3.537	6.801
9				5	5	5

(b)

	A	B
1		0
2		<=
3	x =	0.405
4		<=
5		5
6		
7	Profit =	$x^5 - 13x^4 + 59x^3 - 107x^2 + 61x$
8	=	10.735

12.10-5.

(a)

	A	B	C	D	E	F
1		0		Starting		
2		<=		Point	x*	Profit*
3	x =	1.187			1.187	753.451
4		<=		0	0	0
5		5		1	1.187	753.451
6				2	1.187	753.451
7	Profit =	$100x^6 - 1{,}359x^5 + 6{,}836x^4 - 15{,}670x^3 + 15{,}870x^2 - 5{,}095x$		3	3.184	906.902
8	=	753.451		4	3.184	906.902
9				5	5	650

(b)

	A	B
1		0
2		<=
3	x =	3.184
4		<=
5		5
6		
7	Profit =	$100x^6 - 1{,}359x^5 + 6{,}836x^4 - 15{,}670x^3 + 15{,}870x^2 - 5{,}095x$
8	=	906.902

12.10-6.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	City	Democrat	Republican	Total				District								
2	1	152	62	214		1	<=	3	<=	10			Min District Population		150	
3	2	81	59	140		1	<=	4	<=	10			Max District Population		350	
4	3	75	83	158		1	<=	8	<=	10			Number of Districts		10	
5	4	34	52	86		1	<=	6	<=	10						
6	5	62	87	149		1	<=	5	<=	10						
7	6	38	87	125		1	<=	5	<=	10						
8	7	48	69	117		1	<=	7	<=	10		District	Democrat	Republican	Total	Winner
9	8	74	49	123		1	<=	1	<=	10		1	119	131	250	Republican
10	9	98	62	160		1	<=	7	<=	10		2	140	151	291	Republican
11	10	66	72	138		1	<=	9	<=	10		3	152	62	214	Democrat
12	11	83	75	158		1	<=	6	<=	10		4	174	127	301	Democrat
13	12	86	82	168		1	<=	9	<=	10		5	100	174	274	Republican
14	13	72	83	155		1	<=	10	<=	10		6	117	127	244	Republican
15	14	28	53	81		1	<=	2	<=	10		7	146	131	277	Democrat
16	15	112	98	210		1	<=	2	<=	10		8	75	83	158	Republican
17	16	45	82	127		1	<=	1	<=	10		9	152	154	306	Republican
18	17	93	68	161		1	<=	4	<=	10		10	144	181	325	Republican
19	18	72	98	170		1	<=	10	<=	10				Total Republican Districts		7
20	Total	1,319	1,321													

12.10-7.

(a)

	B	C	D	E	F	G
3		Doors	Windows			
4	Unit Profit	$300	$500			
5				Hours		Hours
6		Hours Used Per Unit Produced		Used		Available
7	Plant 1	1	0	2	Š	4
8	Plant 2	0	2	12	Š	12
9	Plant 3	3	2	18	Š	18
10						
11		Doors	Windows			Total Profit
12	Units Produced	2	6			$3,600

(b)

	B	C	D	E	F	G
3		Doors	Windows			
4	Unit Profit	$300	$500			
5				Hours		Hours
6		Hours Used Per Unit Produced		Used		Available
7	Plant 1	1	0	1.96	Š	4
8	Plant 2	0	2	11.80	Š	12
9	Plant 3	3	2	17.68	Š	18
10						
11		Doors	Windows			Total Profit
12	Units Produced	1.959	5.902			$3,538
13		<=	<=			
14		10	10			

(c) The Standard Solver gives a better solution and finds it much more quickly. It is much better suited to linear programs than the Evolutionary Solver.

12.11-1.

(a) Yes, this is a convex programming problem.

$f(\boldsymbol{x}) = f_1(x_1) + f_2(x_2), f_1(x_1) = 4x_1 - x_1^2, f_2(x_2) = 10x_2 - x_2^2$

$\frac{d^2 f_1(x_1)}{dx_1^2} = \frac{d^2 f_2(x_2)}{dx_2^2} = -2 < 0 \Rightarrow f$ is concave.

$g(\boldsymbol{x}) = g_1(x_1) + g_2(x_2), g_1(x_1) = x_1^2, g_2(x_2) = 4x_2^2$

$\frac{d^2 g_1(x_1)}{dx_1^2} = 2 > 0, \frac{d^2 g_2(x_2)}{dx_2^2} = 8 > 0 \Rightarrow g$ is convex.

(b) No, this is not a quadratic programming problem because the constraints are nonlinear.

(c) No, the Frank-Wolfe algorithm in Section 12.9 requires linear constraints, so it cannot be applied to this problem.

(d) KKT conditions:

(1a) $4 - 2x_1 - 2x_1 u \leq 0$ (1b) $10 - 2x_2 - 8x_2 u \leq 0$
(2a) $x_1(4 - 2x_1 - 2x_1 u) = 0$ (2b) $x_2(10 - 2x_2 - 8x_2 u) = 0$
(3) $x_1^2 + 4x_2^2 - 16 \leq 0$
(4) $u(x_1^2 + 4x_2^2 - 16) = 0$
(5) $x_1 \geq 0, x_2 \geq 0$
(6) $u \geq 0$

Let $x_1 = x_2 = 1$. Then from (2a), $u = 1$ and this violates (4), so it cannot be optimal.

(e) Let $x_1 = x_{11} + x_{12} + x_{13} + x_{14}$ and $x_2 = x_{21} + x_{22}$.

$f_1(x_1) = 4x_1 - x_1^2, f_2(x_2) = 10x_2 - x_2^2$

$f_1(0) = 0, f_1(1) = 3, f_1(2) = 4, f_1(3) = 3, f_1(4) = 0$

$f_2(0) = 0, f_2(1) = 9, f_2(2) = 16$

$s_{11} = 3, s_{12} = 1, s_{13} = -1, s_{14} = -3$

$s_{21} = 9, s_{22} = 7$

$g_1(x_1) = x_1^2, g_2(x_2) = 4x_2^2$

$g_1(0) = 0, g_1(1) = 1, g_1(2) = 4, g_1(3) = 9, g_1(4) = 16$

$g_2(0) = 0, g_2(1) = 4, g_2(2) = 16$

$t_{11} = 1, t_{12} = 3, t_{13} = 5, t_{14} = 7$

$t_{21} = 4, t_{22} = 12$

Approximate linear programming model:

maximize $3x_{11} + x_{12} - x_{13} + 3x_{14} + 9x_{21} + 7x_{22}$

subject to $x_{11} + 3x_{12} + 5x_{13} + 7x_{14} + 4x_{21} + 12x_{22} \leq 16$
$5x_{11} + 5x_{12} + 5x_{13} + 2x_{21} + 2x_{22} + 2x_{23} \leq 14$
$0 \leq x_{ij} \leq 1$ for all i, j

(f) Solution with the simplex method:

Value of the
Objective Function: Z = 18.4166667

Variable		Value
X1	(Y_{11})	1
X2	(Y_{12})	0
X3	(Y_{13})	0
X4	(Y_{14})	0
X5	(Y_{21})	1
X6	(Y_{22})	0.91667

Original variables: $x_1 = 1, x_2 = 1.91667$

(g) $P(\boldsymbol{x}; r) = 4x_1 - x_1^2 + 10x_2 - x_2^2 - r\left[\frac{1}{16-x_1^2-4x_2^2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

(h)

k	r	X1	X2	f(X)
0		2	1	13
1	1	1.504	1.754	18.22
2	0.01	1.409	1.862	18.8
3	0.0001	1.41	1.871	18.86
4	0.000001	1.411	1.871	18.86

(i) Standard Solver

Max	f(x) =	4x1-x1^2+10x2-x2^2		
	=	18.865		
s.t.	g1(x) =	x1^2 + 4*x2^2		
	=	16.000	<=	16
	x1 =	1.411	>=	0
	x2 =	1.871	>=	0

(j) Evolutionary Solver

Max	f(x) =	4x1-x1^2+10x2-x2^2		
	=	18.865		
s.t.	g1(x) =	x1^2 + 4*x2^2		
	=	16.000	<=	16
	x1 =	1.407	>=	0
	x2 =	1.872	>=	0

(k) LINGO Solver

```
MAX =    4*X1 - X1^2 + 10*X2 - X2^2;
         X1^2 + 4 * X2^2       <= 16;
         X1                    >= 0;
                    X2         >= 0;

Local optimal solution found at iteration:            63
Objective value:                                18.86516

Variable          Value          Reduced Cost
      X1       1.410531              0.000000
      X2       1.871524          0.1287136E-07

Row     Slack or Surplus       Dual Price
  1         18.86516            1.000000
  2         0.000000           0.4179049
  3         1.410531            0.000000
  4         1.871524            0.000000
```

CASES

Case 12.1 Savvy Stock Selection

(a) If Lydia wants to ignore the risk of her investment she should invest all her money into the stock that promises the highest expected return. According to the predictions of the investment advisors, the expected returns equal 20% for BB, 42% for LOP, 100% for ILI, 50% for HEAL, 46% for QUI, and 30% for AUA. Therefore, she should invest 100% of her money into ILI. The risk (variance) of this portfolio equals 0.333.

(b) Lydia should invest 40% of her money into the stock with the highest expected return, 40% into the stock with the second highest expected return, and 20% into the stock with the third highest expected return. This intuitive solution can be found also by solving the linear programming problem to

maximize MaxExpectedReturn = SUMPRODUCT(Portfolio, StockExpectedReturn)
subject to Total = OneHundredPercent
Portfolio ≤ MaxInSingleStock.

	A	B	C	D	E	F	G	H	I	J
1		BB	LOP	ILI	HEAL	QUI	AUA			
2	Expected Return	20%	42%	100%	50%	46%	30%			
3										
4	Covariance Matrix									
5	(Variance on Diagonal)	BB	LOP	ILI	HEAL	QUI	AUA			
6	BB	0.032	0.005	0.030	-0.031	-0.027	0.010			
7	LOP	0.005	0.1	0.085	-0.07	-0.05	0.020			
8	ILI	0.030	0.085	0.333	-0.11	-0.02	0.042			
9	HEAL	-0.031	-0.07	-0.11	0.125	0.05	-0.060			
10	QUI	-0.027	-0.05	-0.02	0.05	0.065	-0.020			
11	AUA	0.010	0.020	0.042	-0.060	-0.020	0.08			
12										
13		BB	LOP	ILI	HEAL	QUI	AUA	Total		
14	Portfolio	0%	0%	40%	40%	20%	0%	100%	=	100%
15		≤	≤	≤	≤	≤	≤			
16	Max in Single Stock	40%	40%	40%	40%	40%	40%			
17										
18		Portfolio								
19	Expected Return =	69.2%								
20										
21	Risk (Variance) =	0.04548								

Range Name	Cells
CovarianceMatrix	B6:G11
MaxInSingleStock	B16:G16
OneHundredPercent	J14
Portfolio	B14:G14
PortfolioExpectedReturn	B19
StockExpectedReturn	B2:G2
Total	H14
Variance	B21

	H
13	Total
14	=SUM(Portfolio)

	A	B
18		Portfolio
19	Expected Return =	=SUMPRODUCT(StockExpectedReturn,Portfolio)

	A	B
21	Risk (Variance) =	=SUMPRODUCT(MMULT(Portfolio,CovarianceMatrix),Portfolio)

The total expected return of her new portfolio is 69.2% with a total variance of 0.04548.

(c) The risk of Lydia's portfolio is a quadratic function of her decision variables. We apply quadratic programming to her decision problem.

(d) The expected return of Lydia's portfolio is no longer the objective function. It now becomes part of a constraint:

PortfolioExpectedReturn(C21) $\geq$ 35%(MinimumExpectedReturn).

The objective is now to minimize the risk.

	A	B	C	D	E	F	G	H	I	J
1		BB	LOP	ILI	HEAL	QUI	AUA			
2	Expected Return	20%	42%	100%	50%	46%	30%			
3										
4	Covariance Matrix									
5	(Variance on Diagonal)	BB	LOP	ILI	HEAL	QUI	AUA			
6	BB	0.032	0.005	0.030	-0.031	-0.027	0.010			
7	LOP	0.005	0.1	0.085	-0.07	-0.05	0.020			
8	ILI	0.030	0.085	0.333	-0.11	-0.02	0.042			
9	HEAL	-0.031	-0.07	-0.11	0.125	0.05	-0.060			
10	QUI	-0.027	-0.05	-0.02	0.05	0.065	-0.020			
11	AUA	0.010	0.020	0.042	-0.060	-0.020	0.08			
12										
13		BB	LOP	ILI	HEAL	QUI	AUA	Total		
14	Portfolio	31.8%	19.9%	0.0%	16.8%	20.9%	10.6%	100%	=	100%
15		≥	≥	≥	≥	≥	≥			
16	Max in Single Stock	40%	40%	40%	40%	40%	40%			
17										
18				Minimum						
19				Expected						
20		Portfolio		Return						
21	Expected Return =	35.9%	≥	35%						
22										
23	Risk (Variance) =	0.00136								

Lydia's optimal portfolio consists of 31.8% BB, 19.9% LOP, 16.8% HEAL, 20.9% QUI, and 10.6% AUA. Her expected return equals 35.9% with a risk of 0.00136.

(e) Since the return constraint is not binding in the solution of part (d), decreasing the right-hand-side will not affect the optimal solution. The minimum risk for a minimum expected return of 25% is the same as the minimum risk for a minimum expected return of 35%, which is 0.00136. However, for a minimum expected return of 40%, a new portfolio is obtained.

	A	B	C	D	E	F	G	H	I	J
1		BB	LOP	ILI	HEAL	QUI	AUA			
2	**Expected Return**	20%	42%	100%	50%	46%	30%			
3										
4	**Covariance Matrix**									
5	**(Variance on Diagonal)**	BB	LOP	ILI	HEAL	QUI	AUA			
6	BB	0.032	0.005	0.030	-0.031	-0.027	0.010			
7	LOP	0.005	0.1	0.085	-0.07	-0.05	0.020			
8	ILI	0.030	0.085	0.333	-0.11	-0.02	0.042			
9	HEAL	-0.031	-0.07	-0.11	0.125	0.05	-0.060			
10	QUI	-0.027	-0.05	-0.02	0.05	0.065	-0.020			
11	AUA	0.010	0.020	0.042	-0.060	-0.020	0.08			
12										
13		BB	LOP	ILI	HEAL	QUI	AUA	**Total**		
14	**Portfolio**	22.9%	21.0%	3.4%	22.0%	18.8%	11.9%	100%	=	100%
15		≤	≤	≤	≤	≤	≤			
16	Max in Single Stock	40%	40%	40%	40%	40%	40%			
17										
18				Minimum						
19				Expected						
20		Portfolio		Return						
21	**Expected Return =**	40.0%	≥	40%						
22										
23	**Risk (Variance) =**	0.00233								

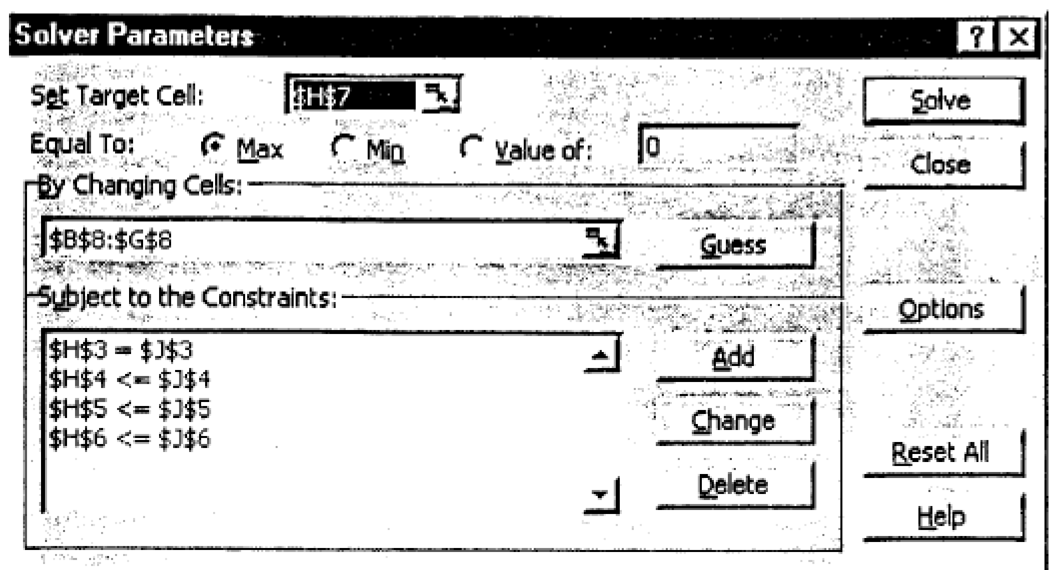
Lydia's new optimal portfolio consists of 22.9% BB, 21% LOP, 3.4% ILI, 22% HEAL, 18.8% QUI, and 11.9% AUA. Her expected return equals 40% with a risk of 0.00233.

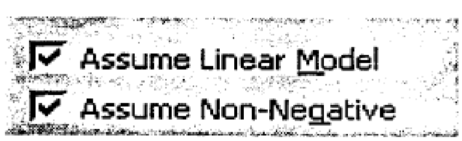
(f) Lydia's approach is very risky. She puts a lot of confidence in the advice of the two investment experts. She cannot expect to find an optimal investment strategy with her model if the estimates she uses for the input parameters are not accurate.

Case 12.2 International Investments

(a) When Charles sells a portion of his B-Bonds in a given year, the first DM 6100 of interest are tax-free, but the interest earnings exceeding DM 6100 are levied a 30% tax. Therefore, Charles encounters decreasing marginal returns and we can use separable programming to solve this problem. Let NoTax5 and Tax5 be the base amount of B-Bonds Charles sells in the fifth year that yield untaxed interest and taxed interest respectively. The variables NoTax6, Tax6, NoTax7, and Tax7 are defined in the same way. The sum of the six variables must equal the total of DM 30,000 that Charles invested at the beginning of the first year. When Charles sells B-Bonds with the base amount NoTax5, he earns 50.01% of this amount as interest. In order for him not to pay any taxes on this amount, the interest must not exceed DM 6100. This is included in the model as a constraint. Any additional base amount of B-Bonds sold in year 5 yields Charles only $0.7 \times 0.5001 = 0.35007$. A similar reasoning applies to other years. The objective is to maximize Charles' interest income.

	A	B	C	D	E	F	G	H	I	J
1										
2	Server	NoTax5	Tax5	NoTax6	Tax6	NoTax7	Tax7	Totals		
3	Selling	1	1	1	1	1	1	30000	=	30000
4	Untaxed5	0.5001	0	0	0	0	0	0	<=	6100
5	Untaxed6	0	0	0.6351	0	0	0	6100	<=	6100
6	Untaxed7	0	0	0	0	0.7823	0	6100	<=	6100
7	Interest	0.5001	0.35007	0.6351	0.44457	0.7823	0.54761	19098.62		
8	Solution	0	0	9604.79	0	7797.52	12597.69			
9										
10		Formula in cell H3:			"=SUMPRODUCT(B3:G3,B8:G8)"					
11		Formula in cell H4:			"=SUMPRODUCT(B4:G4,B8:G8)"					
12		Formula in cell H5:			"=SUMPRODUCT(B5:G5,B8:G8)"					
13		Formula in cell H6:			"=SUMPRODUCT(B6:G6,B8:G8)"					
14		Formula in cell H7:			"=SUMPRODUCT(B7:G7,B8:G8)"					





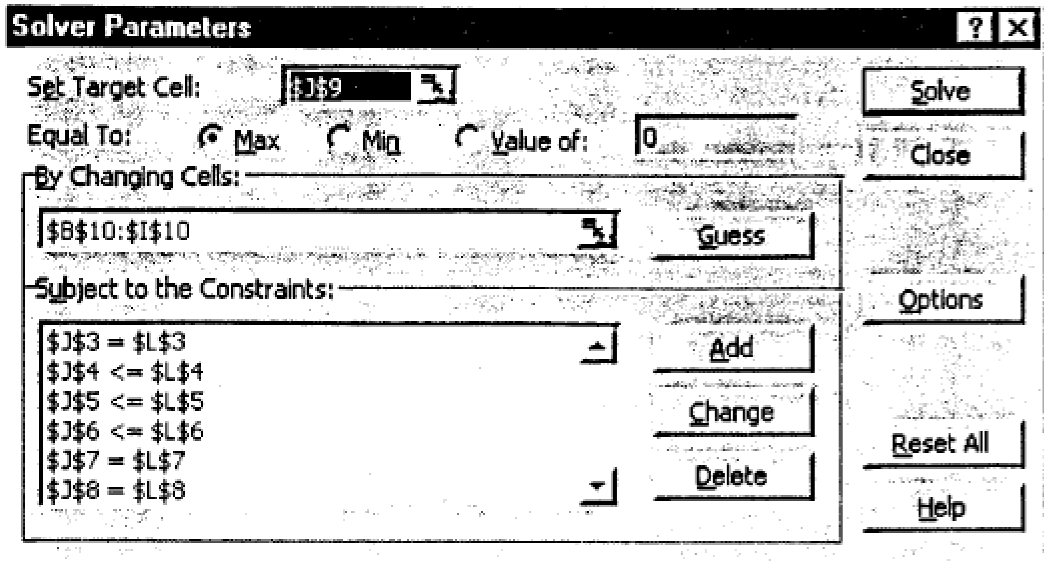
(b) The optimal investment strategy for Charles is to sell a base amount of DM 9604.79 at the end of year 6 and the remaining DM 20395.21 at the end of year 7. His total after-tax interest income equals DM 19098.62.

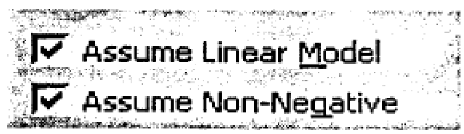
(c) When Charles sells all B-Bonds in year 7, he must pay 30% of tax on the amount of interest income exceeding DM 6100. This amount is earned interest not only from the last year, but it also includes interest from all the previous years. Hence, Charles does not pay 30% tax on the 9% interest he earned last year, but he effectively pays tax on the total interest of all the years. This tax payment decreases his after-tax interest so much that it pays for him to sell some of his bonds in year 6 in order to take advantage of the yearly tax-free income of DM 6100. Comparing the total amount of interest Charles earns if he sells tax-free after year 6 and taxed after year 7, we see that in the former case his total interest equals 63.51% while in the latter case it is only 54.761%. Therefore, it is better to sell some bonds at the end of year 6 rather than to keep them until the end of the last year.

(d) The following observation greatly simplifies the analysis of this problem: The interest rate on the CD is much lower than the yearly interest rates on the B-Bonds. Therefore, it can never be optimal for Charles to sell B-Bonds in year 5 in order to buy a CD for year 6 if he does not take advantage of the maximal tax-free amount of selling B-Bonds in year 6. In other words, Charles will only buy a CD for year 6 if he already plans to sell B-Bonds in year 6 to obtain at least the maximal tax-free amount of interest. The same argument applies to year 7. Consequently, Charles will never earn untaxed interest on a CD. Therefore, his yearly interest on the CD will always be $0.7 \times 0.04 = 0.028 = 2.8\%$.

To formulate the problem in Excel, let CD6 and CD7 be the amount invested in a CD in year 6 and 7 respectively. The amount of money Charles can invest in a CD in year 6 equals the base amount of B-Bonds sold in year 5 plus the total after-tax interest earned on the base amount. This gives the constraint CD6 = 1.5001*NoTax5 + 1.35007*Tax5. Similarly, for year 7, CD7 = 1.6351*NoTax6 + 1.44457*Tax6 + 1.028*CD6.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	Server	NoTax5	Tax5	CD6	NoTax6	Tax6	CD7	NoTax7	Tax7	Totals		
3	Selling	1	1	0	1	1	0	1	1	30000	=	30000
4	Untaxed5	0.5001	0	0	0	0	0	0	0	6100	<=	6100
5	Untaxed6	0	0	0	0.6351	0	0	0	0	6100	<=	6100
6	Untaxed7	0	0	0	0	0	0	0.7823	0	6100	<=	6100
7	CDInvest5	1.5001	1.35007	-1	0	0	0	0	0	0	=	0
8	CDInvest6	0	0	1.028	1.6351	1.44457	-1	0	0	0	=	0
9	Interest	0.5001	0.35007	0.028	0.6351	0.44457	0.028	0.7823	0.5476	19997.86		
10	Solution	12197.56	0	18297.569604.79		0	34514.687797.52400.133					
11												
12		Formula in cell J3:			"=SUMPRODUCT(B3:I3,B10:I10)"							
13		Formula in cell J4:			"=SUMPRODUCT(B4:I4,B10:I10)"							
14		Formula in cell J5:			"=SUMPRODUCT(B5:I5,B10:I10)"							
15		Formula in cell J6:			"=SUMPRODUCT(B6:I6,B10:I10)"							
16		Formula in cell J7:			"=SUMPRODUCT(B7:I7,B10:I10)"							
17		Formula in cell J8:			"=SUMPRODUCT(B8:I8,B10:I10)"							
18		Formula in cell J9:			"=SUMPRODUCT(B9:I9,B10:I10)"							





Charles should sell the maximal base amount of B-Bonds in year 5 that yields tax-free interest and then invest this money (base amount & interest) into a one-year CD for year 6. In year 6, he should sell again the maximal base amount of B-Bonds that yields tax-free interest and then invest this money (base amount & interest) and the money from his CD into a one-year CD for year 7. In year 7, he should sell the remainder of the base amount of B-Bonds. He again takes advantage of the maximum tax-free amount, but he also sells a base amount of DM 400.13 for which he must pay taxes on the interest earnings.

(e) The right-hand-side of the selling constraint should be changed.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	Server	NoTax5	Tax5	CD6	NoTax6	Tax6	CD7	NoTax7	Tax7	Totals		
3	Selling	1	1	0	1	1	0	1	1	50000	=	50000
4	Untaxed5	0.5001	0	0	0	0	0	0	0	6100	<=	6100
5	Untaxed6	0	0	0	0.6351	0	0	0	0	6100	<=	6100
6	Untaxed7	0	0	0	0	0	0	0.7823	0	6100	<=	6100
7	CDInvest5	1.5001	1.35007	-1	0	0	0	0	0	0	=	0
8	CDInvest6	0	0	1.028	1.6351	1.44457	-1	0	0	0	=	0
9	Interest	0.5001	0.35007	0.028	0.6351	0.44457	0.028	0.7823	0.54761	30950.06		
10	Solution	12197.56	0	18297.56	9604.79	0	34514.68	7797.52	20400.1			
11												
12		Formula in cell J3:			"=SUMPRODUCT(B3:I3,B10:I10)"							
13		Formula in cell J4:			"=SUMPRODUCT(B4:I4,B10:I10)"							
14		Formula in cell J5:			"=SUMPRODUCT(B5:I5,B10:I10)"							
15		Formula in cell J6:			"=SUMPRODUCT(B6:I6,B10:I10)"							
16		Formula in cell J7:			"=SUMPRODUCT(B7:I7,B10:I10)"							
17		Formula in cell J8:			"=SUMPRODUCT(B8:I8,B10:I10)"							
18		Formula in cell J9:			"=SUMPRODUCT(B9:I9,B10:I10)"							

The optimal investment strategy is similar to the previous one except that Charles must now pay taxes on the interest earned from selling a base amount of DM 20400.13 in year 7.

(f) The right-hand-sides of the Untaxed5, Untaxed6, and Untaxed7 constraints should be changed.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	Server	NoTax5	Tax5	CD6	NoTax6	Tax6	CD7	NoTax7	Tax7	Totals		
3	Selling	1	1	0	1	1	0	1	1	30000	=	30000
4	Untaxed5	0.5001	0	0	0	0	0	0	0	0	<=	12200
5	Untaxed6	0	0	0	0.6351	0	0	0	0	9148.59	<=	12200
6	Untaxed7	0	0	0	0	0	0	0.7823	0	12200	<=	12200
7	CDInvest 5	1.5001	1.35007	-1	0	0	0	0	0	0	=	0
8	CDInvest 6	0	0	1.028	1.6351	1.44457	-1	0	0	0	=	0
9	Interest	0.5001	0.35007	0.028	0.6351	0.44457	0.028	0.7823	0.54761	22008.09		
10	Solution	0	0	0	14404.96	0	23553.55	15595.04	0			
11												
12		Formula in cell J3:			"=SUMPRODUCT(B3:I3,B10:I10)"							
13		Formula in cell J4:			"=SUMPRODUCT(B4:I4,B10:I10)"							
14		Formula in cell J5:			"=SUMPRODUCT(B5:I5,B10:I10)"							
15		Formula in cell J6:			"=SUMPRODUCT(B6:I6,B10:I10)"							
16		Formula in cell J7:			"=SUMPRODUCT(B7:I7,B10:I10)"							
17		Formula in cell J8:			"=SUMPRODUCT(B8:I8,B10:I10)"							
18		Formula in cell J9:			"=SUMPRODUCT(B9:I9,B10:I10)"							

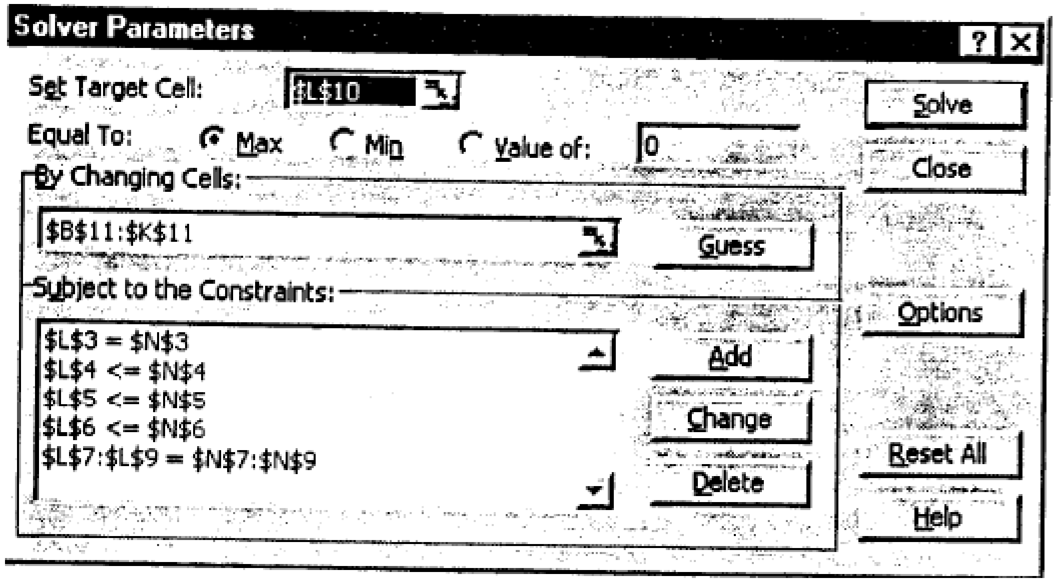
By getting married in year 5, Charles can increase his interest income by

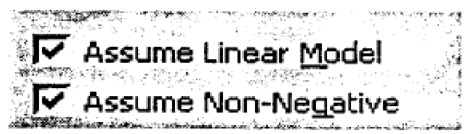
$$22008.09 - 19997.86 = \text{DM } 2010.23.$$

He should sell the maximal base amount of B-Bonds earning tax-free interest in year 7 (DM 15595.04). The remainder of DM 14404.96 should be sold at the end of year 6. His entire interest income on this base amount will be tax-free. He then should invest the total amount (base amount & interest) in a CD for year 7.

(g) Instead of maximizing his interest income, Charles now wants to maximize the expected dollar amount he will have at the end of year 7. He considers exchanging marks for dollars either at the end of year 5 or 7. Let CD-US be the amount of money in dollars that Charles invests in a two-year CD at the end of year 5 and US be the amount of money in dollars that Charles converts at the end of year 7. The total amount of money in dollars Charles has at the end of year 7 equals $(1.036)^2$*CD-US + US; this is the new objective function. At the end of year 5, \$1 is assumed to be equal to DM 1.50, so Charles can exchange marks for dollars at this rate in year 5. This is included as a constraint. Similarly, we include a constraint for the currency conversion at the end of the last year.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2	Server	NoTax5	Tax5	CD6	CD-US	NoTax6	Tax6	CD7	NoTax7	Tax7	US	Totals		
3	Selling	1	1	0	0	1	1	0	1	1	0	30000	=	30000
4	Untaxed5	0.5001	0	0	0	0	0	0	0	0	0	6100	<=	6100
5	Untaxed6	0	0	0	0	0.6351	0	0	0	0	0	0	<=	6100
6	Untaxed7	0	0	0	0	0	0	0	0.7823	0	0	6100	<=	6100
7	CDInvest5	1.5001	1.35007	-1	-1.5	0	0	0	0	0	0	0	=	0
8	CDInvest6	0	0	1.028	0	1.6351	1.44457	-1	0	0	0	0	=	0
9	Conversion	0	0	0	0	0	0	1.028	1.7823	1.5461	-1.8	0	=	0
10	Dollars	0	0	0	1.073296	0	0	0	0	0	1	30478.23		
11	Solution	12197.56	10004.92	0	21203.27	0	0	0	7797.52	0	7720.84			
12														
13														
14			Formula in cell L3:		"=SUMPRODUCT(B3:K3,B11:K11)"									
15			Formula in cell L4:		"=SUMPRODUCT(B4:K4,B11:K11)"									
16			Formula in cell L5:		"=SUMPRODUCT(B5:K5,B11:K11)"									
17			Formula in cell L6:		"=SUMPRODUCT(B6:K6,B11:K11)"									
18			Formula in cell L7:		"=SUMPRODUCT(B7:K7,B11:K11)"									
19			Formula in cell L8:		"=SUMPRODUCT(B8:K8,B11:K11)"									
20			Formula in cell L9:		"=SUMPRODUCT(B9:K9,B11:K11)"									
21			Formula in cell L10:		"=SUMPRODUCT(B10:K10,B11:K11)"									



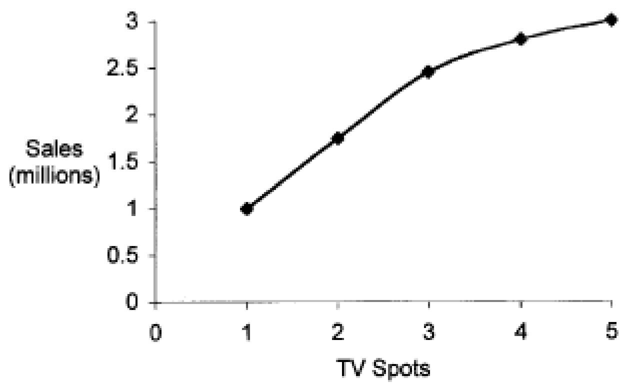


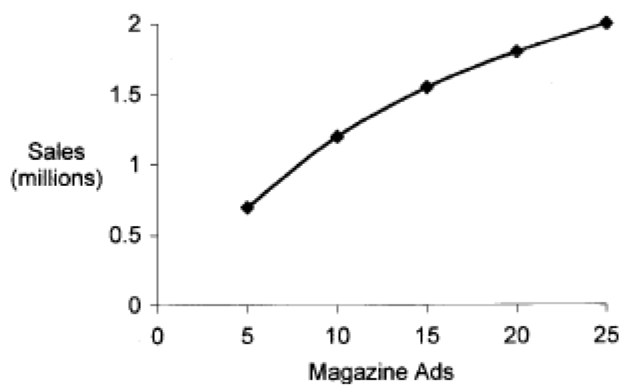
Charles converts DM (1.5001*12197.56 + 1.35007*10004.92 to dollars at the end of year 5. With the exchange rate of DM 1.50 for $1, he is able to invest $21203.27 in the American CD. At the end year 7, he converts the remaining DM 1.7823*7797.52 to dollars. The total amount of his investments at the end of year 7 is then $30478.23.

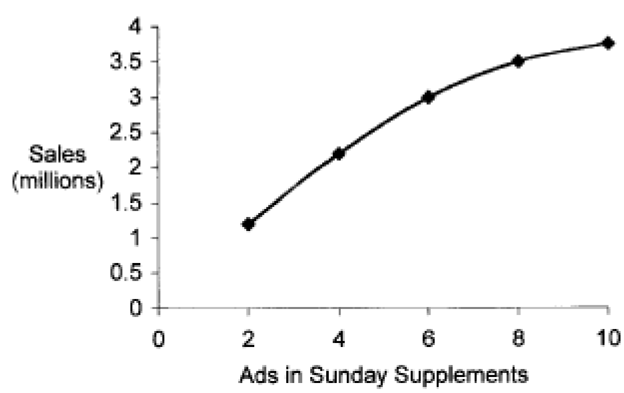
Case 12.3 Promoting a Breakfast Cereal, Revisited

(a)

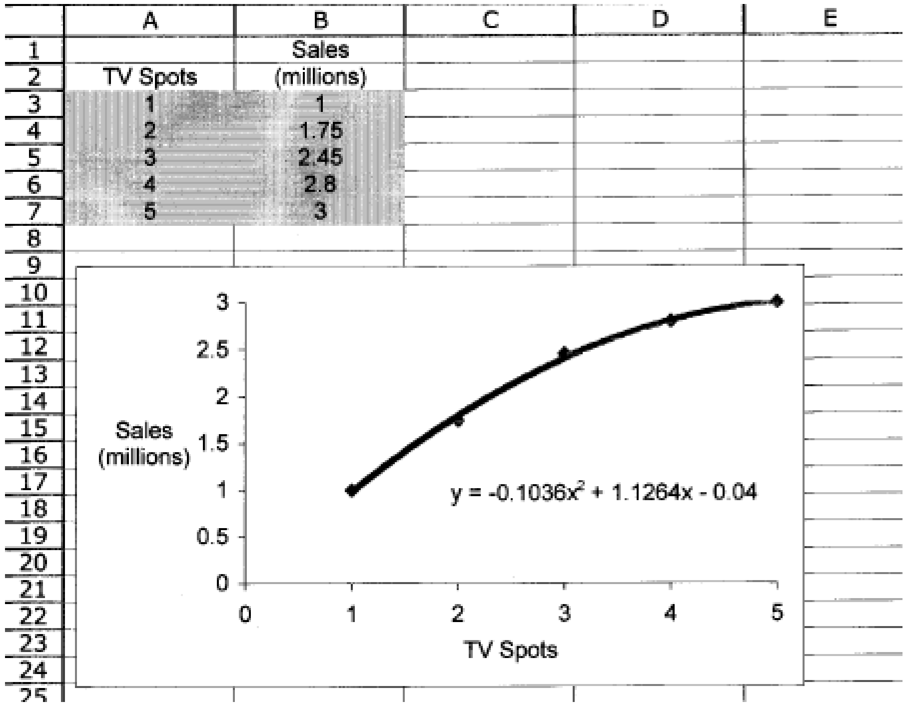
TV Spots	Sales (millions)
1	1
2	1.75
3	2.45
4	2.8
5	3



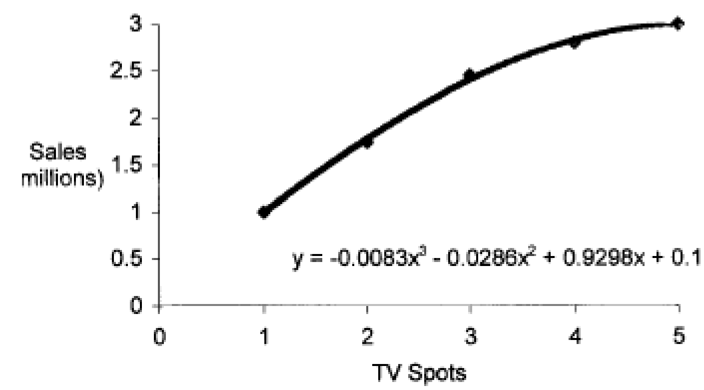




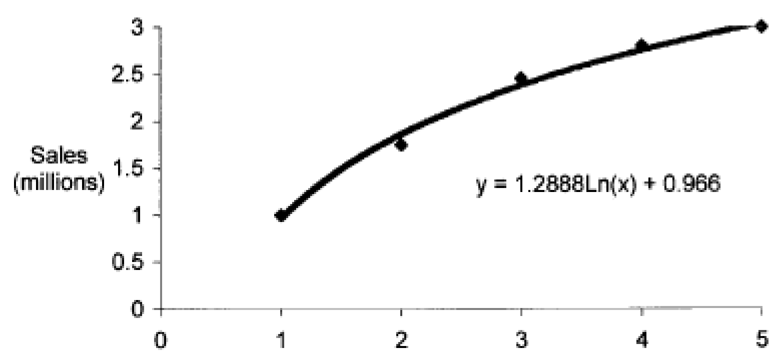
(b) TV Spots (polynomial of order 2)



TV Spots (polynomial of order 3)

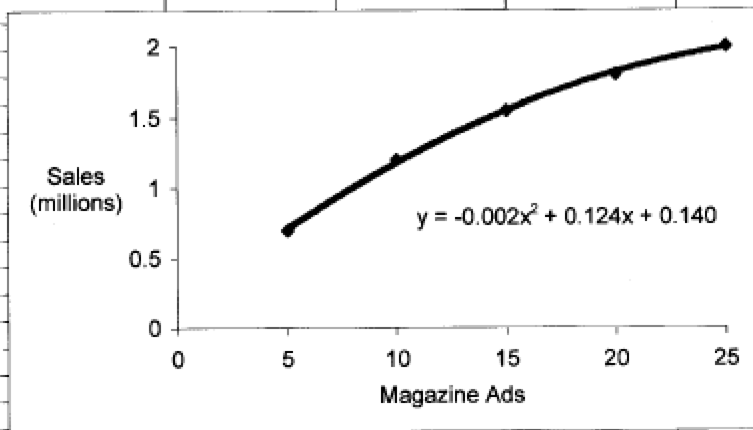


TV Spots (logarithmic form)

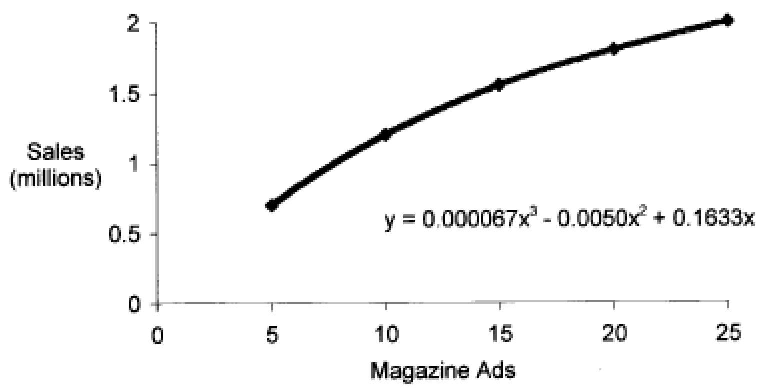


Magazine Ads (polynomial of order 2)

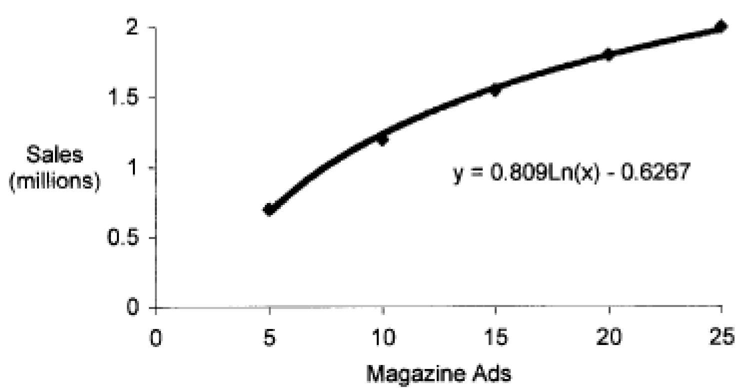
	A	B	C	D	E
1		Sales			
2	Magazine Ads	(millions)			
3	5	0.7			
4	10	1.2			
5	15	1.55			
6	20	1.8			
7	25	2			



Magazine Ads (polynomial of order 3)

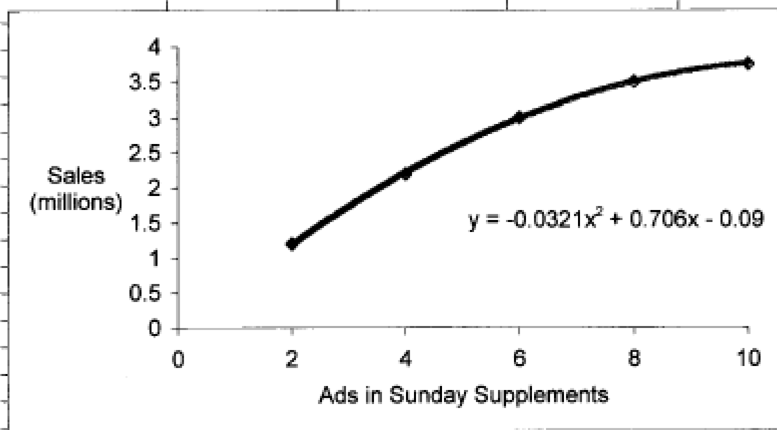


Magazine Ads (logarithmic form)

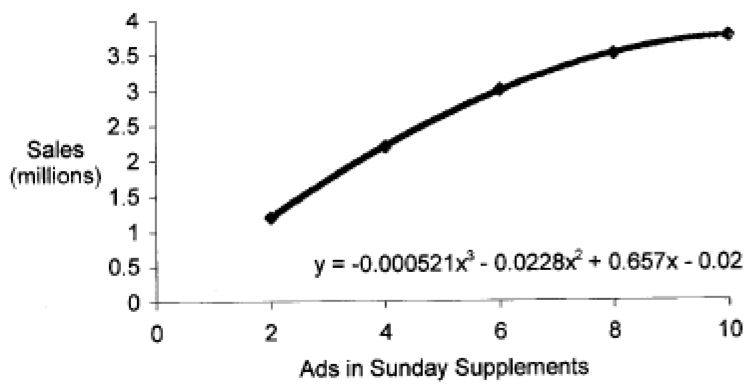


Ads in Sunday Supplements (polynomial of order 2)

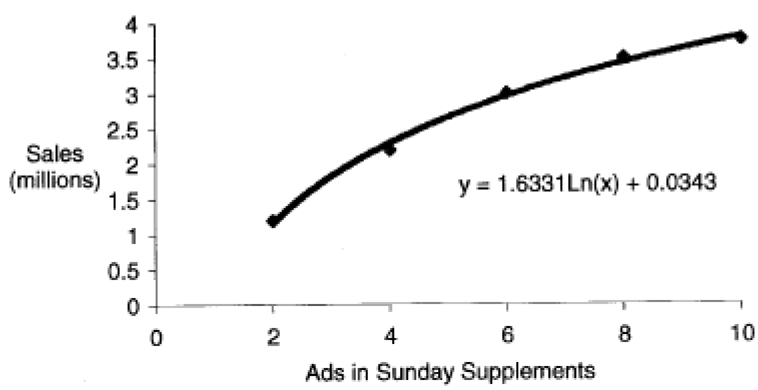
	A	B	C	D	E
1	Ads in Sunday	Sales			
2	Supplements	(millions)			
3	2	1.2			
4	4	2.2			
5	6	3			
6	8	3.5			
7	10	3.75			



Ads in Sunday Supplements (polynomial of order 3)



Ads in Sunday Supplements (logarithmic form)



In all three cases, the quadratic form is a close fit. The polynomial of order 3 is also a good fit. The logarithmic form is not a bad fit, but not as closes as the polynomial forms. We will use the quadratic form in the sequel.

(c) Let TV, M, and SS be the number of TV spots, magazine ads, and ads in Sunday supplements respectively. Based on the results of part (b), using the quadratic form gives:

$$\text{Sales} = -0.1036\text{TV}^2 + 1.1264\text{TV} - 0.04 - 0.002\text{M}^2 + 0.124\text{M} + 0.14 - 0.0321\text{SS}^2 + 0.706\text{SS} - 0.09$$

$$\text{Cost of Ads} = 0.3\text{TV} + 0.15\text{M} + 0.1\text{SS}$$

$$\text{Planning Cost} = 0.09\text{TV} + 0.03\text{M} + 0.04\text{SS}$$

$\Rightarrow$ Profit = \$0.75 $\times$ (Sales) $-$ Cost of Ads $-$ Planning Cost.

(d) The total sales generated are calculated in row 7 using the nonlinear equations from part (b). Then, the gross profit from sales are calculated in H20. The TotalProfit (H23) is the gross profit minus the cost of ads and of planning. The objective is to maximize this.

	B	C	D	E	F	G	H
3	Sales per Ad = ax^2 + bx + k, where	TV Spots	Magazine Ads	SS Ads			
4	a=	-0.1036	-0.002	-0.0321			
5	b=	1.1264	0.124	0.706			
6	k =	-0.0400	0.14	-0.09	Total		Gross Profit per Sale
7	Sales Generated (millions)	2.8296	0.5600	3.7903	7.1799		$0.75
8							
9			Cost per Ad ($thousands)		Budget Spent		Budget Available
10	Ad Budget	300	150	100	2,884	<=	4,000
11	Planning Budget	90	30	40	923	<=	1,000
12							
13			Number Reached per Ad (millions)		Total Reached		Minimum Acceptable
14	Young Children	1.2	0.1	0	5.25	>=	5
15	Parents of Young Children	0.5	0.2	0.2	5.00	>=	5
16							
17		TV Spots	Magazine Ads	SS Ads	Total Redeemed		Required Amount
18	Coupon Redemption per Ad	0	40	120	1,490	=	1,490
19	($thousands)						
20						Gross Profit	5.385
21		TV Spots	Magazine Ads	SS Ads		Cost of Ads	2.884
22	Number of Ads	4.075	3.596	11.218		Planning Cost	0.923
23		<=				Total Profit	1.578
24	Maximum TV Spots	5					($million)

	B	C	D
7	Sales Generated (millions)	=a*(NumberOfAds)^2+b*NumberOfAds+k	=a*(NumberOfAds)^2+b*NumberOfAds+k

Range Name	Cells
a	C4:E4
b	C5:E5
BudgetAvailable	H9:H10
BudgetSpent	F9:F10
CostPerAd	C9:E10
CouponRedemptionPerAd	C17:E17
GrossProfitPerSale	H7
k	C6:E6
MaxTVSpots	C23
MinimumAcceptable	H13:H14
NumberOfAds	C21:E21
NumberReachedPerAd	C13:E14
RequiredAmount	H17
SalesGenerated	C7:E7
Total Profit	H21
TotalReached	F13:F14
TotalRedeemed	F17
TotalSales	F7
TVSpots	C21

	G	H
20	Gross Profit	=GrossProfitPerSale*TotalSales
21	Cost of Ads	=F10/1000
22	Planning Cost	=F11/1000
23	Total Profit	=H20-H21-H22
24		($million)

(e) Separable programming formulation

	B	C	D	E	F	G	H	I
3	Sales per Ad	TV Spots	Magazine Ads	SS Ads				
4	Group 1	1	0.14	0.6				
5	Group 2	0.75	0.1	0.5				
6	Group 3	0.7	0.07	0.4				
7	Group 4	0.35	0.05	0.25				
8	Group 5	0.2	0.04	0.125				
9								
10							Budget	
11			Cost per Ad ($thousands)		Budget Spent		Available	
12	Ad Budget	300	150	100	3,156	<=	4,000	
13	Planning Budget	90	30	40	938	<=	1,000	
14								
15			Number Reached per Ad (millions)		Total Reached		Min. Acceptable	
16	Young Children	1.2	0.1	0	5.00	>=	5	
17	Parents of Young Children	0.5	0.2	0.2	5.23	>=	5	
18								
19		TV Spots	Magazine Ads	SS Ads	Total Redeemed		Req. Amount	
20	Coupon Redemption per Ad	0	40	120	1,490	=	1,490	
21	($thousands)							
22							Maximum	
23	Number of Ads	TV Spots	Magazine Ads	SS Ads		TV Spots	Magazine Ads	SS Ads
24	Group 1	1.000	5.000	2.000	<=	1	5	2
25	Group 2	1.000	2.250	2.000	<=	1	5	2
26	Group 3	1.000	0.000	2.000	<=	1	5	2
27	Group 4	0.563	0.000	2.000	<=	1	5	2
28	Group 5	0.000	0.000	2.000	<=	1	5	2
29	Total	3.563	7.250	10.000				
30		<=				Total Sales	7.3219	
31	Maximum TV Spots	5				Gross Profit per Sale	$0.75	
32								
33						Gross Profit	5.491	
34						Cost of Ads	3.156	
35						Planning Cost	0.938	
36						Total Profit	1.397	
37							($million)	

Range Name	Cells
BudgetAvailable	H12:H13
BudgetSpent	F12:F13
CostPerAd	C12:E13
CouponRedemptionPerAd	C20:E20
GrossProfitPerSale	H31
Maximum	G24:I28
MaxTVSpots	C31
MinimumAcceptable	H16:H17
NumberOfAds	C24:E28
RequiredAmount	H20
SalesPerAd	C4:E8
TotalAds	C29:E29
TotalProfit	H36
TotalReached	F16:F17
TotalRedeemed	F20
TotalSales	H30
TVSpots	C29

	G	H
30	Total Sales	=SUMPRODUCT(SalesPerAd,NumberOfAds)
31	Gross Profit per Sale	0.75
32		
33	Gross Profit	=GrossProfitPerSale*TotalSales
34	Cost of Ads	=F12/1000
35	Planning Cost	=F13/1000
36	Total Profit	=H33-H34-H35
37		($million)

(f) In part (d), 4.075 TV ads, 3.596 magazine ads, and 11.218 ads in Sunday supplements are placed. In part (e), 3.563 TV ads, 7.25 magazine ads, and 10 ads in Sunday supplements are placed. In Case 3.4, 3 TV ads, 14 magazine ads, and 7.75 ads in Sunday supplements are placed. Unlike linear programming, nonlinear and separable programming take into account the diminishing returns from repeated advertisements. Since the solution is fairly different, it certainly appears that it was worthwhile to refine the linear programming model used in Case 3.4.

CHAPTER 13: METAHEURISTICS

13.1-1.

(a)

Tours	Distance	Tours	Distance
1-2-3-4-5-1	34	1-3-2-4-5-1	32
1-2-3-5-4-1	34	1-3-2-5-4-1	26
1-2-4-3-5-1	36	1-3-4-2-5-1	28
1-2-4-5-3-1	31	1-3-5-2-4-1	28
1-2-5-3-4-1	30	1-4-2-3-5-1	37
1-2-5-4-3-1	25	1-4-3-2-5-1	31

Optimal Solution: 1-2-5-4-3-1 (or the reverse 1-3-4-5-2-1)

(b) Start with the initial trial solution 1-2-3-4-5-1. There are three possible sub-tour reversals that improve upon this solution.

	1-2-3-4-5-1	Distance = 34
Reverse 2-3	1-3-2-4-5-1	Distance = 32
Reverse 2-3-4	1-4-3-2-5-1	Distance = 31
Reverse 3-4-5	1-2-5-4-3-1	Distance = 25

Choose 1-2-5-4-3-1 as the next trial solution. There is no sub-tour reversal that improves upon this solution. The tour 1-2-5-4-3-1 is optimal.

(c) Start with the initial trial solution 1-2-4-3-5-1. There are four possible sub-tour reversals that improve upon this solution.

	1-2-4-3-5-1	Distance = 36
Reverse 4-3	1-2-3-4-5-1	Distance = 34
Reverse 3-5	1-2-4-5-3-1	Distance = 31
Reverse 2-4-3	1-3-4-2-5-1	Distance = 28
Reverse 4-3-5	1-2-5-3-4-1	Distance = 30

Choose 1-3-4-2-5-1 as the next trial solution. There is only one possible sub-tour reversal that improves upon this solution.

	1-3-4-2-5-1	Distance = 28
Reverse 2-5	1-3-4-5-2-1	Distance = 25

Choose 1-3-4-5-2-1 as the next trial solution. There is no sub-tour reversal that improves upon this. The solution 1-3-4-5-2-1 is optimal.

(d) Start with the initial trial solution 1-4-2-3-5-1. There are five possible sub-tour reversals that improve upon this solution.

	1-4-2-3-5-1	Distance = 37
Reverse 2-4	1-2-4-3-5-1	Distance = 36
Reverse 2-3	1-4-3-2-5-1	Distance = 31
Reverse 3-5	1-4-2-5-3-1	Distance = 28
Reverse 4-2-3	1-3-2-4-5-1	Distance = 32
reverse 2-3-5	1-4-5-3-2-1	Distance = 34

Choose 1-4-2-5-3-1 as the next trial solution. There is only one possible sub-tour reversal that improves upon this solution.

	1-4-2-5-3-1	Distance = 28
Reverse 2-5	1-4-5-2-3-1	Distance = 26

Choose 1-4-5-2-3-1 as the next trial solution. There is one possible sub-tour reversal that improves upon this.

	1-4-5-2-3-1	Distance = 26
Reverse 4-5-2	1-2-5-4-3-1	Distance = 25

Choose 1-2-5-4-3-1 as the next trial solution. There is no sub-tour reversal that improves upon this. The solution 1-2-5-4-3-1 is optimal.

13.1-2.

(a) If the second reversal were chosen, the next trial solution would be 1-2-3-5-4-6-7-1 and there is no sub-tour reversal that gives an improvement.

(b) Start with the initial trial solution 1-2-4-5-6-7-3-1. There are two possible sub-tour reversals that improve upon this solution.

	1-2-4-5-6-7-3-1	Distance = 69
Reverse 5-6	1-2-4-6-5-7-3-1	Distance = 66
Reverse 2-4-5-6-7	1-7-6-5-4-2-3-1	Distance = 68

Choose 1-2-4-6-5-7-3-1 as the next trial solution. There is only one possible sub-tour reversal that improves upon this.

	1-2-4-6-5-7-3-1	Distance = 66
Reverse 5-7	1-2-4-6-7-5-3-1	Distance = 63

Choose 1-2-4-6-7-5-3-1 as the next trial solution. This is an optimal solution.

13.1-3.

(a)

Tours	Distance	Tours	Distance
1-2-3-4-5-6-1	48	1-2-6-3-4-5-1	52
1-2-3-4-6-5-1	44	1-5-2-3-4-6-1	42
1-2-3-6-4-5-1	50	1-5-2-4-3-6-1	46
1-2-4-3-6-5-1	48	1-6-2-3-4-5-1	48
1-2-5-4-3-6-1	50	1-6-3-2-4-5-1	50

Optimal Solution: 1-5-2-3-4-6-1 (or the reverse 1-6-4-3-5-2-1)

(b) Start with the initial trial solution 1-2-3-4-5-6-1. There are two possible sub-tour reversals that improve upon this solution.

	1-2-3-4-5-6-1	Distance = 48
Reverse 5-6	1-2-3-4-6-5-1	Distance = 44
Reverse 2-3-4-5	1-5-4-3-2-6-1	Distance = 48

Choose 1-2-3-4-6-5-1 as the next trial solution. There is no sub-tour reversal that improves upon this solution.

(c) Start with the initial trial solution 1-2-5-4-3-6-1. There are two possible sub-tour reversals that improve upon this solution.

	1-2-5-4-3-6-1	Distance = 50
Reverse 2-5	1-5-2-4-3-6-1	Distance = 46
Reverse 5-4-3	1-2-3-4-5-6-1	Distance = 48

Choose 1-5-2-4-3-6-1 as the next trial solution. There is no sub-tour reversal that improves upon this solution.

13.2-1.

Sears logistics services (SLS) provides delivery with its fleet of over 1,000 vehicles. Sears product services (SPS) offers home service with its fleet of 12,500 vehicles and technicians. A customer who asks for delivery or home service is given a day and a time window based on customer preferences and working schedule in the region where the customer is located. In either case, the goal is to generate efficient routes for the vehicles and to provide customers with accurate and convenient time windows while minimizing the operational costs. Both problems are instances of vehicle routing problem with time windows (VRPTW). A basic VRPTW determines routes for M vehicles, each starting at the depot and returning to the depot after visiting a subset of customers in some order. Every customer is visited by exactly one vehicle. The capacity constraints of the vehicles and the time windows imposed by customers should be met. The objective is to minimize the total cost. The problems faced by SLS and SPS differ from the basic VRPTW in that they include additional constraints. For instance, in the case of SPS, technicians' skills need to be considered in assigning service orders to them. In both cases, there may be restrictions on total route times and travel times between any two locations. Hence, the problem is a complex one and necessitates the use of a solution procedure that can provide good solutions in acceptable time.

To solve the problem, first an initial route is found for each vehicle, then unassigned stops are inserted into a route. This solution is improved using various local heuristic techniques. In order not to be stuck at local optima, the procedure is enhanced with tabu search technique. Once a stop in a route is relocated, the move is included in a tabu list and remains prohibited for a number of iterations unless the objective function value it offers exceeds the best value obtained up to that iteration.

Financial benefits of this study include $9 million in one-time savings and over $42 million in annual savings. The savings result from the reduction in travel times, mileage and routing times. Sears now offers more timely delivery of merchandise and home service, so more reliable customer service. The utilization of the fleets is improved. The routing process became faster and the facility, equipment and personnel costs related to routing decreased. Since the problem can be solved quickly, Sears can respond to disruptions and adjust its schedules more efficiently.

13.2-2.

Start with the initial trial solution with links AB, AC, AE, CD, which costs 232.

Iteration 1:

Add	Delete	Cost
BC	AB	138
	AC	246
BD	AB	56
	AC	164
	CD	268
DE	AC	152
	AE	240
	CD	256

Adding BD and deleting AB results in the lowest cost, so choose inserting links AC, AE, BD. CD. In fact, this is the optimal solution.

13.2-3.

Start with the initial trial solution with links AB, AD, BE, CD, which costs 390.

Iteration 1: Minimum local search

Add	Delete	Cost
AC	AD	185
	CD	275
CE	AB	275
	AD	180
	CD	270
	BE	365

Current solution: AB, BE, CD, CE.
Tabu list: CE

Iteration 2: Minimum local search

Add	Delete	Cost
DE	CD	95

Current solution: AB, BE, CE, DE
Tabu list: CE, DE

Iteration 3: Minimum local search

Add	Delete	Cost
AC	BE	75

The solution AB, AC, CE, DE is optimal.

13.2-4.

Start with the initial trial solution with links OA, AB, BC, BE, ED, DT, which costs 314.

Iteration 1: Minimum local search

Add	Delete	Cost
ET	DE	122

Current solution: OA, AB, BC, BE, ET, DT
Tabu list: ET

Iteration 2: Minimum local search

Add	Delete	Cost
CE	BC	23

The solution OA, AB, CE, BE, ET, DT is optimal.

13.2-5.

Initial trial solution: 1-5-3-2-4-1 Distance $= 37$

Iteration 1: Choose to reverse 3-5.
Deleted links: 1-5 and 3-2
Added links: 1-3 and 5-2
Tabu list: Links 1-3 and 5-2
New trial solution: 1-3-5-2-4-1 Distance $= 28$

Iteration 2: Choose to reverse 5-2.
Deleted links: 3-5 and 2-4
Added links: 3-2 and 5-4
Tabu list: Links 1-3, 5-2, 3-2 and 5-4
New trial solution: 1-3-2-5-4-1 Distance $= 26$

Iteration 3: Choose to reverse 2-5-4.
Deleted links: 3-2 and 4-1
Added links: 3-4 and 2-1
Tabu list: Links 3-2, 5-4, 3-4 and 2-1
New trial solution: 1-3-4-5-2-1 Distance $= 25$

Iteration 4: Choose to reverse 3-4.
Deleted links: 1-3 and 4-5
Added links: 1-4 and 3-5
Tabu list: Links 3-4, 2-1, 1-4 and 3-5
New trial solution: 1-4-3-5-2-1 Distance $= 30$

Iteration 5: Choose to reverse 5-3.
Deleted links: 4-3 and 5-2
Added links: 4-5 and 3-2
Tabu list: Links 1-4, 3-5, 4-5 and 3-2
New trial solution: 1-4-5-3-2-1 Distance $= 34$

Iteration 6: Choose to reverse 3-2.
Deleted links: 5-3 and 2-1
Added links: 5-2 and 3-1
Tabu list: Links 4-5, 3-2, 5-2 and 3-1
New trial solution: 1-4-5-2-3-1 Distance $= 26$

The solution 1-3-4-5-2-1 is optimal.

13.2-6.

Traveling Salesman Problems:
Number of Cities: 8

City	1	2	3	4	5	6	7	8
1	0	14	15	--	--	--	--	17
2	14	0	13	14	20	--	--	21
3	15	13	0	11	21	17	9	9
4	--	14	11	0	11	10	8	20
5	--	20	21	11	0	15	18	--
6	--	--	17	10	15	0	9	--
7	--	--	9	8	18	9	0	13
8	17	21	9	20	--	--	13	0

(a) Initial trial solution: 1-2-3-4-5-6-7-8-1

Iteration	Trial Solution	Distance	Tabu List
0	1-2-3-4-5-6-7-8-1	103.0	
1	1-3-2-4-5-6-7-8-1	107.0	1-3,2-4
2	1-3-8-7-6-5-4-2-1	100.0	1-3,2-4,3-8,2-1
3	1-8-3-7-6-5-4-2-1	98.0	3-8,2-1,1-8,3-7
4	1-8-3-7-6-4-5-2-1	99.0	1-8,3-7,6-4,5-2
5	1-8-3-7-4-6-5-2-1	102.0	6-4,5-2,7-4,6-5
6	1-8-3-4-7-6-5-2-1	103.0	7-4,6-5,3-4,7-6

Best Distance = 98.0 Best Solution = 1-8-3-7-6-5-4-2-1

(b) Initial trial solution: 1-2-5-6-7-4-8-3-1

Iteration	Trial Solution	Distance	Tabu List
0	1-2-5-6-7-4-8-3-1	110.0	
1	1-2-5-6-7-4-3-8-1	103.0	4-3,8-1
2	1-2-5-6-4-7-3-8-1	102.0	4-3,8-1,6-4,7-3
3	1-2-5-4-6-7-3-8-1	99.0	6-4,7-3,5-4,6-7
4	1-2-4-5-6-7-3-8-1	98.0	5-4,6-7,2-4,5-6
5	1-2-4-5-6-7-8-3-1	100.0	2-4,5-6,7-8,3-1
6	1-8-7-6-5-4-2-3-1	107.0	7-8,3-1,1-8,2-3
7	1-8-7-6-5-4-3-2-1	103.0	1-8,2-3,4-3,2-1

Best Distance = 98.0 Best Solution = 1-2-4-5-6-7-3-8-1

(c) Initial trial solution: 1-3-2-5-6-4-7-8-1

Iteration	Trial Solution	Distance	Tabu List
0	1-3-2-5-6-4-7-8-1	111.0	
1	1-3-8-7-4-6-5-2-1	104.0	3-8,2-1
2	1-3-8-7-6-4-5-2-1	101.0	3-8,2-1,7-6,4-5
3	1-8-3-7-6-4-5-2-1	99.0	7-6,4-5,1-8,3-7
4	1-8-3-7-6-5-4-2-1	98.0	1-8,3-7,6-5,4-2
5	1-8-3-7-5-6-4-2-1	106.0	6-5,4-2,7-5,6-4
6	1-3-8-7-5-6-4-2-1	108.0	7-5,6-4,1-3,8-7
7	1-3-8-7-4-6-5-2-1	104.0	1-3,8-7,7-4,5-2

Best Distance = 98.0 Best Solution = 1-8-3-7-6-5-4-2-1

13.2-7.

Traveling Salesman Problems:
Number of Cities: 10

City	1	2	3	4	5	6	7	8	9	10
1	0	13	25	15	21	9	19	18	8	15
2	13	0	26	21	29	21	31	23	16	10
3	25	26	0	11	18	23	28	44	34	35
4	15	21	11	0	10	13	19	34	24	29
5	21	29	18	10	0	12	11	37	27	36
6	9	21	23	13	12	0	10	25	14	25
7	19	31	28	19	11	10	0	32	23	35
8	18	23	44	34	37	25	32	0	10	16
9	8	16	34	24	27	14	23	10	0	14
10	15	10	35	29	36	25	35	16	14	0

(a)

Iteration	Trial Solution	Distance	Tabu List
0	1-2-3-4-5-6-7-8-9-10-1	153.0	
1	1-9-8-7-6-5-4-3-2-10-1	144.0	1-9,2-10
2	1-9-8-10-2-3-4-5-6-7-1	132.0	1-9,2-10,8-10,7-1
3	1-9-8-10-2-3-4-5-7-6-1	121.0	8-10,7-1,5-7,6-1
4	1-9-8-10-2-4-3-5-7-6-1	124.0	5-7,6-1,2-4,3-5
5	1-8-9-10-2-4-3-5-7-6-1	132.0	2-4,3-5,1-8,9-10
6	1-8-9-6-7-5-3-4-2-10-1	138.0	1-8,9-10,9-6,10-1

Best Distance = 121.0 Best Solution = 1-9-8-10-2-3-4-5-7-6-1

(b)

Iteration	Trial Solution	Distance	Tabu List
0	1-3-4-5-7-6-9-8-10-2-1	130.0	
1	1-4-3-5-7-6-9-8-10-2-1	128.0	1-4,3-5
2	1-4-3-5-7-6-2-10-8-9-1	130.0	1-4,3-5,6-2,9-1
3	1-6-7-5-3-4-2-10-8-9-1	124.0	6-2,9-1,1-6,4-2
4	1-6-7-5-4-3-2-10-8-9-1	121.0	1-6,4-2,5-4,3-2
5	1-6-7-5-4-3-2-10-9-8-1	129.0	5-4,3-2,10-9,8-1
6	1-10-2-3-4-5-7-6-9-8-1	135.0	10-9,8-1,1-10,6-9
7	1-9-6-7-5-4-3-2-10-8-1	134.0	1-10,6-9,1-9,10-8

Best Distance = 121.0 Best Solution = 1-6-7-5-4-3-2-10-8-9-1

(c)

Iteration	Trial Solution	Distance	Tabu List
0	1-9-8-10-2-4-3-6-7-5-1	141.0	
1	1-9-8-10-2-4-3-5-7-6-1	124.0	3-5,6-1
2	1-9-8-10-2-3-4-5-7-6-1	121.0	3-5,6-1,2-3,4-5
3	1-8-9-10-2-3-4-5-7-6-1	129.0	2-3,4-5,1-8,9-10
4	1-8-9-6-7-5-4-3-2-10-1	135.0	1-8,9-10,9-6,10-1
5	1-8-10-2-3-4-5-7-6-9-1	134.0	9-6,10-1,8-10,9-1

Best Distance = 121.0 Best Solution = 1-9-8-10-2-3-4-5-7-6-1

13.3-1.

$Z_c = 30, T = 2$

(a) Maximization problem:

$Z_n = 29$, $x = (Z_n - Z_c)/T = -0.5$, P$\{$acceptance$\} = e^x = 0.607$
$Z_n = 34$, $Z_n > Z_c$, P$\{$acceptance$\} = 1$
$Z_n = 31$, $Z_n > Z_c$, P$\{$acceptance$\} = 1$
$Z_n = 24$, $x = (Z_n - Z_c)/T = -3$, P$\{$acceptance$\} = e^x = 0.05$

(b) Minimization problem:

$Z_n = 29$, $Z_n < Z_c$, P$\{$acceptance$\} = 1$
$Z_n = 34$, $x = (Z_c - Z_n)/T = -2$, P$\{$acceptance$\} = e^x = 0.135$
$Z_n = 31$, $x = (Z_c - Z_n)/T = -0.5$, P$\{$acceptance$\} = e^x = 0.607$
$Z_n = 24$, $Z_n < Z_c$, P$\{$acceptance$\} = 1$

13.3-2.

Because of the randomness in the algorithm, the output will vary.

13.3-3.

(a) Initial trial solution: 1-4-2-3-5-1, $Z_c = 37$, $T_1 = 0.2Z_c = 7.4$

0.0000 - 0.3332	Sub-tour begins in slot 2.
0.3333 - 0.6666	Sub-tour begins in slot 3.
0.6667 - 0.9999	Sub-tour begins in slot 4.

The random number is 0.09656: choose a sub-tour that begins in slot 2. The sub-tour needs to end either in slot 3 or slot 5.

0.0000 - 0.4999	Sub-tour ends in slot 3.
0.5000 - 0.9999	Sub-tour ends in slot 5.

The random number is 0.96657: choose a sub-tour that ends in slot 5.
Reverse 2-3-5 to obtain the new solution 1-4-5-3-2-1, $Z_n = 34$. Since $Z_n < Z_c$, accept this solution as the next trial solution.

(b) Because of the randomness in the algorithm, the output will vary.

13.3-4.

Because of the randomness in the algorithm, the output will vary.

13.3-5.

Because of the randomness in the algorithm, the output will vary.

13.3-6.

Because of the randomness in the algorithm, the output will vary.

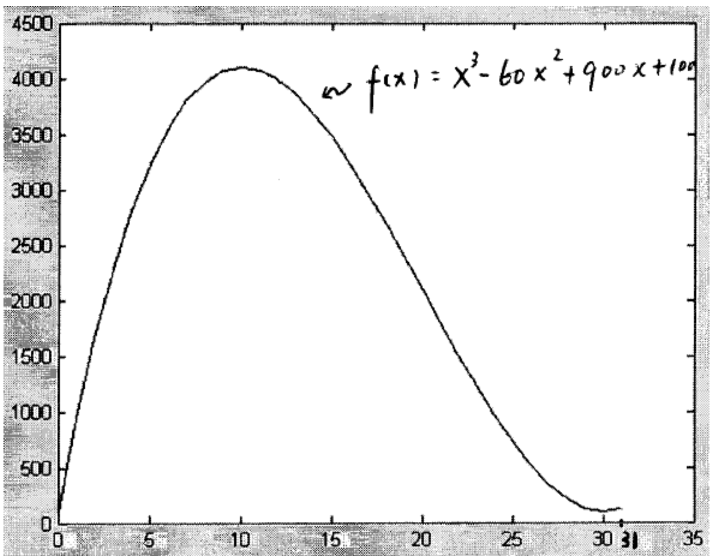
13.3-7.

(a) $f(x) = x^3 - 60x^2 + 900x + 100$
$f'(x) = 3x^2 - 120x + 900$ and $f''(x) = 6x - 120$

Stationary Points: $f'(x^*) = 0 \Rightarrow x^*$is either 10 or 30 (stationary points of f).
$f''(10) = -60 < 0 \Rightarrow x^* = 10$ is a local maximum.
$f''(30) = 60 > 0 \Rightarrow x^* = 30$ is a local minimum.

End Points: $f'(0) = 900 > 0 \Rightarrow x = 0$ is a local minimum.
$f'(31) = 63 > 0 \Rightarrow x = 31$ is a local minimum.

(b)



(c) $x = 15.5$, $f(x) = Z_c = 3558.9$, $T = 0.2Z_c = 671.775$
$L = 0$, $U = 31$, $\sigma = (U - L)/6 = 5.167$

The random number obtained from Table 20.3 is 0.09656. From Appendix 5,

P$\{Y \leq -1.315\} \simeq 0.09656$,

with Y a standard Normal random variable, $N(0, 5.167) = -1.315 \cdot 5.167 = -6.79$.

$x = 15.5 + N(0, 5.167) = 8.71$, $Z_n = f(x) = 4047.6$

Since $Z_n > Z_c$, accept $x = 8.71$ as the next trial solution.

(d) Because of the randomness in the algorithm, the output will vary.

13.3-8.

The nonconvex problem is to:

$$\begin{array}{ll} \text{maximize} & 0.5x^5 - 6x^4 + 24.5x^3 - 39x^2 + 20x \\ \text{subject to} & 0 \leq x \leq 5. \end{array}$$

(a) $x = 2.5$, $f(x) = Z_c = 3.5156$, $T = 0.2Z_c = 0.7031$

$$L = 0, U = 5, \sigma = (U - L)/6 = 0.8333$$

The random number obtained from Table 20.3 is 0.09656. From Appendix 5,

$$\text{P}\{Y \leq -1.315\} \simeq 0.09656,$$

with Y a standard Normal random variable, $N(0, 0.8333) = -1.315 \cdot 0.8333 = -1.0958$.

$$x = 2.5 + N(0, 0.8333) = 1.4042, Z_n = f(x) = -1.5782$$

Since $(Z_n - Z_c)/T = -7.2488$, the probability of accepting $x = 1.4042$ as the next trial solution is $\text{P}\{\text{acceptance}\} = e^{-7.2488} = 0.00071$. From Table 20.3, the next random number is $0.96657 > 0.00071$, so we reject $x = 1.4042$ as the next trial solution.

(b) Because of the randomness in the algorithm, the output will vary.

13.3-9.

(a) $x = 25$, $f(x) = Z_c = -13,671,875$, $T = 0.2Z_c = -2,734,375$

$$L = 0, U = 50, \sigma = (U - L)/6 = 8.333$$

The random number obtained from Table 20.3 is 0.09656. From Appendix 5,

$$\text{P}\{Y \leq -1.315\} \simeq 0.09656,$$

with Y a standard Normal random variable, $N(0, 8.333) = -1.315 \cdot 8.333 = -10.958$.

$$x = 25 + N(0, 8.333) = 14.042, Z_n = f(x) = 5,659,191.646$$

Since $Z_n > Z_c$, accept the new solution.

(b) Because of the randomness in the algorithm, the output will vary.

13.3-10.

(a) $(x_1, x_2) = (18, 25)$, $f(x_1, x_2) = Z_c = 133,509.5$, $T = 0.2Z_c = 26,701.9$

$$L = (0, 0), U = (36, 50)$$

$$\sigma_1 = (36 - 0)/6 = 6$$

$$\sigma_2 = (50 - 0)/6 = 8.333$$

The random number obtained from Table 20.3 is 0.09656. From Appendix 5,

$$\text{P}\{Y \leq -1.315\} \simeq 0.09656,$$

with Y a standard Normal random variable,

$$N(0, 6) = -1.315 \cdot 6 = -7.89$$

$x_1 = 18 + N(0,6) = 10.11$

$N(0, 8.333) = -1.315 \cdot 8.333 = -10.958$

$x_2 = 25 + N(0, 8.333) = 14.042$

This solution is feasible.

$Z_n = f(x) = -107,467$

Since $(Z_n - Z_c)/T = -9.0247$, the probability of accepting this solution as the next trial solution is P$\{\text{acceptance}\} = e^{-9.0247} = 0.00012$. From Table 20.3, the next random number is $0.96657 > 0.00012$, so we reject $(10.11, 14.042)$ as the next trial solution.

(b) Because of the randomness in the algorithm, the output will vary.

13.4-1.

(a) P1: 010011 and
P2: 100101

Only the last digits agree, the children then become:

C1: xxxxx1 and
C2: xxxxx1,

where x represents the unknown digits. Random numbers are used to identify these unknown digits and let random numbers:

0.00000 - 0.49999 correspond to x = 0,
0.50000 - 0.99999 correspond to x = 1.

Starting from the front of the top row of Table 20.3, the first 10 random numbers are: 0.09656, 0.96657, 0.64842, 0.49222, 0.49506, 0.10145, 0.48455, 0.23505, 0.90430, 0.04180. The corresponding digits are: 0,1,1,0,0,0,0,0,1,0. The children then become:

C1: 011001 and
C2: 000101.

Next, we consider the possibility of mutations. The probability of a mutation in any generation is set at 0.1, and let random numbers

0.00000 - 0.09999 correspond to a mutation,
0.10000 - 0.99999 correspond to no mutation.

Starting from the second row of Table 20.3, we obtain the next 12 random numbers. Accordingly, the 8^{th} and 11^{th} ones correspond to a mutation, so the final conclusion is that the two children are

C1: 011001 and
C2: 010111.

(b) P1: 000010 and
P2: 001101

The first and second digits agree, the children then become:

C1: 00xxxx and

C2: 00xxxx,

where x represents the unknown digits. Random numbers are used to identify these unknown digits and let random numbers:

0.00000 - 0.49999 correspond to x = 0,
0.50000 - 0.99999 correspond to x = 1.

Starting from the front of the top row of Table 20.3, the first 8 random numbers correspond to digits: 0,1,1,0,0,0,0,0. The children then become:

C1: 000110 and
C2: 000000.

Next, we consider the possibility of mutations. The probability of a mutation in any generation is set at 0.1, and let random numbers

0.00000 - 0.09999 correspond to a mutation,
0.10000 - 0.99999 correspond to no mutation.

Use Table 20.3 to obtain the next 12 random numbers. Accordingly, the 2nd and 10th ones correspond to a mutation, so the final conclusion is that the two children are

C1: 010110 and
C2: 000100.

(c) P1: 100000 and
P2: 101000

All but the third digits agree, the children then become:

C1: 10x000 and
C2: 10x000,

where x represents the unknown digits. Random numbers are used to identify these unknown digits and let random numbers:

0.00000 - 0.49999 correspond to x = 0,
0.50000 - 0.99999 correspond to x = 1.

Starting from the front of the top row of Table 20.3, the first 2 random numbers correspond to digits: 0,1. The children then become:

C1: 100000 and
C2: 101000.

Next, we consider the possibility of mutations. The probability of a mutation in any generation is set at 0.1, and let random numbers

0.00000 - 0.09999 correspond to a mutation,
0.10000 - 0.99999 correspond to no mutation.

Use Table 20.3 to obtain the next 12 random numbers. Accordingly, only the 8th one corresponds to a mutation, so the final conclusion is that the two children are

C1: 100000 and
C2: 111000.

13.4-2.

(a) P1: 1-2-3-4-7-6-5-8-1 and
P2: 1-5-3-6-7-8-2-4-1

Start from city 1.
Possible links: 1-2, 1-8, 1-5, 1-4

Random numbers:	0.09656	choose 1-2
	0.96657	no mutation

Start from city 2. Current tour: 1-2
Possible links: 2-3, 2-8, 2-4

Random numbers:	0.64842	choose 2-8
	0.49222	no mutation

Start from city 8. Current tour: 1-2-8
Possible links: 8-5, 8-7

Random numbers:	0.49506	choose 8-5
	0.10145	no mutation

Start from city 5. Current tour: 1-2-8-5
Possible links: 5-6, 5-3

Random numbers:	0.48455	choose 5-6
	0.23505	no mutation

Start from city 6. Current tour: 1-2-8-5-6
Possible links: 6-7, 6-7, 6-3

Random numbers:	0.90430	choose 6-3
	0.04180	mutation

Reject 6-3 and consider all other possible links: 6-4, 6-7

Random numbers:	0.24712	choose 6-4

Start from city 4. Current tour: 1-2-8-5-6-4
Possible links: 4-3, 4-7

Random numbers:	0.55799	choose 4-7
	0.60857	no mutation

The only remaining city is 3. Hence, C1 = 1-2-8-5-6-4-7-3-1.

(b) P1: 1-6-4-7-3-8-2-5-1 and
P2: 1-2-5-3-6-8-4-7-1

Start from city 1.
Possible links: 1-6, 1-5, 1-2, 1-7

Random numbers:	0.09656	choose 1-6
	0.96657	no mutation

Start from city 6. Current tour: 1-6
Possible links: 6-4, 6-3, 6-8

Random numbers:	0.64842	choose 6-3
	0.49222	no mutation

Start from city 3. Current tour: 1-6-3
Possible links: 3-7, 3-8, 3-5

Random numbers:	0.49506	choose 3-8
	0.10145	no mutation

Start from city 8. Current tour: 1-6-3-8
Possible links: 8-2, 8-4

Random numbers:	0.48455	choose 8-2
	0.23505	no mutation

Start from city 2. Current tour: 1-6-3-8-2
Possible links: 2-5

Random numbers:	0.04180	mutation

Reject 2-5 and consider all other possible links: 2-4, 2-7

Random numbers:	0.24712	choose 2-4

Start from city 4. Current tour: 1-6-3-8-2-4
Possible links: 4-7

Random numbers:	0.60857	no mutation

The only remaining city is 5. Hence, C1 = 1-6-3-8-2-4-7-5-1.

(c) P1: 1-5-7-4-6-2-3-8-1 and
P2: 1-3-7-2-5-6-8-4-1

Start from city 1.
Possible links: 1-5, 1-8, 1-3, 1-4

Random numbers:	0.09656	choose 1-5
	0.96657	no mutation

Start from city 5. Current tour: 1-5
Possible links: 5-7, 5-2, 5-6

Random numbers:	0.64842	choose 5-2
	0.49222	no mutation

Start from city 2. Current tour: 1-5-2
Possible links: 2-6, 2-3, 2-7

Random numbers:	0.49506	choose 2-3
	0.10145	no mutation

Start from city 3. Current tour: 1-5-2-3
Possible links: 3-8, 3-7

Random numbers:	0.48455	choose 3-8
	0.23505	no mutation

Start from city 8. Current tour: 1-5-2-3-8
Possible links: 8-6, 8-4

Random numbers:	0.90430	choose 8-4
	0.04189	mutation

Reject 8-4 and consider all other possible links: 8-6, 8-7

Random numbers:	0.24712	choose 8-6

Start from city 6. Current tour: 1-5-2-3-8-6
Possible links: 6-4

Random numbers:	0.55799	choose 6-4
	0.60857	no mutation

The only remaining city is 7. Hence, C1 = 1-5-2-3-8-6-4-7-1.

13.4-3.

(a) Because of the randomness in the algorithm, the output will vary.

(b) Because of the randomness in the algorithm, the output will vary.

13.4-4.

Integer nonlinear programming: maximize $f(x) = x^3 - 60x^2 + 900$

subject to $0 \leq x \leq 31$

(a)

```
Iter.     |     Best Solution      |     Fitness      |
1         |     (0)                |     900.0        |

Iteration 1
Population:
Member    |     Population     |     Solution        |     Fitness
1         |(00000)             |(0)                  |900.0
2         |(00001)             |(1)                  |841.0
3         |(00100)             |(4)                  |4.0
4         |(00110)             |(6)                  |-1044.0
5         |(01010)             |(10)                 |-4100.0
6         |(01110)             |(14)                 |-8116.0
7         |(10111)             |(23)                 |-18673.0
8         |(11010)             |(26)                 |-22084.0
9         |(11100)             |(28)                 |-24188.0
10        |(11101)             |(29)                 |-25171.0

Children:
Member    |  Parents   |     Children      |  Solution   |     Fitness
5         |(01010)     |( [1][1]010 )      |(26)         |-22084.0
3         |(00100)     |( 01100 )          |(12)         |-6012.0
4         |(00110)     |( 0011[1] )        |(7)          |-1697.0
6         |(01110)     |( 00110 )          |(6)          |-1044.0
2         |(00001)     |( [1]00[0]0 )      |(16)         |-10364.0
8         |(11010)     |( 11000 )          |(24)         |-19836.0
```

(b) Because of the randomness in the algorithm, the output will vary.

13.4-5.

Because of the randomness in the algorithm, the output will vary.

13.4-6.

Because of the randomness in the algorithm, the output will vary.

13.4-7.

(a) Because of the randomness in the algorithm, the output will vary.

(b) Because of the randomness in the algorithm, the output will vary.

13.4-8.

(a) Genetic Algorithm

Iteration	Best Solution	Fitness
1	1-2-5-4-3-1	24.0

Iteration 1:

Member	Population	Fitness
1	1-4-5-3-2-1	33.0
2	1-2-5-4-3-1	24.0
3	1-3-2-4-5-1	31.0
4	1-4-5-3-2-1	33.0
5	1-2-3-4-5-1	33.0
6	1-3-4-2-5-1	28.0
7	1-5-2-3-4-1	31.0
8	1-4-2-3-5-1	37.0
9	1-5-2-3-4-1	31.0
10	1-5-2-4-3-1	28.0

Member	Children	Fitness
10	1-3-4-5-2-1	24.0
3	1-3-2-4-5-1	31.0
5	1-2-3-4-5-1	33.0
1	1-2-3-5-4-1	33.0
2	1-5-2-4-3-1	28.0
8	1-4-3-2-5-1	31.0

(b) Because of the randomness in the algorithm, the output will vary.

13.4-9.

Because of the randomness in the algorithm, the output will vary.

13.4-10.

Because of the randomness in the algorithm, the output will vary.

13.5-1.

See the solution for Problem 13.2-6(a) for the output from the basic tabu search algorithm. Because of the randomness in the basic simulated annealing and genetic algorithms, their outputs will vary.

13.5-2.

See the solution for Problem 13.2-7(a) for the output from the basic tabu search algorithm. Because of the randomness in the basic simulated annealing and genetic algorithms, their outputs will vary.

CHAPTER 14: GAME THEORY

14.1-1.

Let player 1 be the labor union with strategy i being to decrease the wage demand by $10(i-1)$¢ and player 2 be the management with strategy i being to increase the offer by $10(i-1)$¢. The payoff matrix is:

	1	2	3	4	5	6
1	1.35	1.2	1.3	1.4	1.5	1.6
2	1.5	1.35	1.3	1.4	1.5	1.6
3	1.4	1.4	1.35	1.4	1.5	1.6
4	1.3	1.3	1.3	1.35	1.5	1.6
5	1.2	1.2	1.2	1.2	1.35	1.6
6	1.1	1.1	1.1	1.1	1.1	1.35

where the rows represent the strategy of player 1 and the columns the strategy of player 2.

14.1-2.

Label the products as A and B respectively. The strategies for each manufacturer are:

1- Normal development of both products
2- Crash development of product A
3- Crash development of product B.

Let $p_{ij} = \frac{1}{2}$[(% increase to manufacturer 1 from A) $+$ (% increase to manufacturer 1 from B)] when manufacturer 1 uses strategy i and manufacturer 2 uses strategy j. The payoff matrix is:

	1	2	3	row min
1	8	10	10	**8**
2	4	-4	13	-4
3	4	13	-4	-4
col max	**8**	13	13	8

The rows correspond to the strategy of manufacturer 1 and the columns to the strategy of manufacturer 2. The minimum of the column maxima and the maximum of the row minima is 8, so both manufacturers should use strategy 1, namely choose normal development of both products. Consequently, manufacturer 1 will increase its share by 8%.

14.1-3.

Each player has the same strategy set. A strategy must specify the first chip chosen, the second and third chips chosen for every choice first chip by the opponent. Denote the white, red and blue chips by W, R and B respectively. Then a strategy is of the form: Choose $i \in \{\text{W, R, B}\}$ as first chip, if the opponent chooses $j \in \{\text{W, R, B}\}$, then choose $k_j \in \{\text{W, R, B}\}\backslash\{i\}$, and let $l_j \in \{\text{W, R, B}\}\backslash\{i, k\}$. There are three choices of i and for each i, eight choices of second and third chips, so 24 strategies in total. Player 1 can either win all three games, or win one and get a draw in another one, or lose all three.

Hence, the payoff to player 1 can be either 210, 0, or -210. The payoff to player 1 in each possible scenario is given in the table below, where the rows and the columns represent the order of chips played by player 1 and 2 respectively.

	WRB	WBR	RWB	RBW	BWR	BRW
WRB	0	0	0	-210	210	0
WBR	0	0	-210	0	0	210
RWB	0	210	0	0	0	-210
RBW	210	0	0	0	-210	0
BWR	-210	0	0	210	0	0
BRW	0	-210	210	0	0	0

14.2-1.

(a) Strategies 4, 5, and 6 of each player are dominated by their strategy 3. Then strategy 1 can be eliminated, since it is dominated by strategy 3 for each player. Once these are eliminated, strategy 2 of each is dominated by strategy 3. Thus, the best strategy of the labor union is to decrease its demand by 20¢ and the best for the management if to increase its offer by 20¢. The resulting wage is $1.35.

(b)

	1	2	3	4	5	6	row min
1	1.35	1.2	1.3	1.4	1.5	1.6	1.2
2	1.5	1.35	1.3	1.4	1.5	1.6	1.3
3	1.4	1.4	1.35	1.4	1.5	1.6	**1.35**
4	1.3	1.3	1.3	1.35	1.5	1.6	1.3
5	1.2	1.2	1.2	1.2	1.35	1.6	1.2
6	1.1	1.1	1.1	1.1	1.1	1.35	1.1
col max	1.5	1.4	**1.35**	1.4	1.5	1.6	1.35

14.2-2.

Strategy 3 of player 1 is dominated by strategy 2.
Strategy 3 of player 2 is dominated by strategy 1.
Strategy 1 of player 1 is dominated by strategy 2.
Strategy 2 of player 2 is dominated by strategy 1.

Therefore, the optimal strategy is strategy 2 for player 1 and strategy 1 for player 2 and the resulting payoff is 1 to player 1.

14.2-3.

Strategies 1 and 4 of player 2 is dominated by strategy 3.
Strategies 1 and 2 of player 1 are dominated by strategy 3.
Strategy 2 of player 2 is dominated by strategy 3.

Therefore, the optimal strategy is strategy 3 for each player and the resulting payoff is 2 to player 2.

14.2-4.

	1	2	3	row min
1	3	-1	3	-1
2	-3	1	7	-3
3	7	3	5	$\mathbf{3}$
col max	7	$\mathbf{3}$	7	3

The best strategy is strategy 3 for player 1 and strategy 2 for player 2, the resulting payoff is 3 to player 1. The game is stable with a saddle point $(3, 2)$, since the minimax value equals the maximin value.

14.2-5.

	1	2	3	4	row min
1	3	-3	-2	-4	-4
2	-4	-2	-1	1	-4
3	1	-1	2	0	$-\mathbf{1}$
col max	1	$-\mathbf{1}$	2	1	-1

The best strategy is strategy 3 for player 1 and strategy 2 for player 2, the resulting payoff is 1 to player 2. The game is stable with a saddle point $(3, 2)$.

14.2-6.

(a)

	1	2	3	row min
1	2	3	1	$\mathbf{1}$
2	1	4	0	0
3	3	-2	-1	-2
col max	3	4	$\mathbf{1}$	1

The best strategy is strategy 1 for player 1 and strategy 3 for player 2, the resulting payoff is 1 to player 1. The game is stable with a saddle point $(1, 3)$.

(b) Strategy 1 of player 2 is dominated by strategy 3.
Strategy 3 of player 1 is dominated by strategies 1 and 2.
Strategy 2 of player 2 is dominated by strategy 3.
Strategy 2 of player 1 is dominated by strategy 1.

The optimal strategy is strategy 1 for player 1 and strategy 3 for player 2, with a payoff of 1 to player 1.

14.2-7.

(a)

	1	2	3	row min
1	7	-1	3	-1
2	1	0	2	$\mathbf{0}$
3	-5	-3	1	-5
col max	7	$\mathbf{0}$	$\mathbf{1}$	0

The best strategy is to use issue 2 for each politician, with zero payoff to each.

(b) Let p_{ij} be the probability that politician 1 wins the election or the election results in a tie when politician 1 chooses issue i and politician 2 issue j. Then the new payoff matrix is:

	1	2	3
1	1	0	$3/5$
2	$1/5$	0	$2/5$
3	0	0	$1/5$

Strategies 2 and 3 of politician 1 are dominated by strategy 2.
Strategies 1 and 3 of politician 2 are dominated by strategy 2.

Hence, by eliminating dominated strategies, one gets issue 1 as the best strategy for politician 1 and issue 2 for politician 2, the payoff is zero. Thus, politician 2 can prevent politician 1 from winning or getting a tie.

(c) Let $p_{ij} = \begin{cases} 1 & \text{if politician 1 will win or tie} \\ 0 & \text{if politician 2 will win} \end{cases}$

Then the payoff matrix becomes:

	1	2	3
1	1	0	0
2	0	0	0
3	0	0	0

where the minimax of the columns and the maximin of the rows both equal zero, i.e., politician 1 cannot win. Politician 1 can use any use, politician 2 can choose issue 2 or 3; however, since issue 1 offers politician 1 his only chance of winning, he should use that one and hope that politician 2 chooses issue 1 by mistake.

14.2-8.

Advantages: It provides the best possible guarantee on what the worst outcome can be, regardless of how skillfully the opponent plays the game and hence, reduces the possibility of undesirable outcomes to a minimum.

Disadvantages: Since it aims at eliminating worst cases, it is conservative and may yield payoffs that are far from the best ones.

14.3-1.

(a)

	1	2	row min
1	1	-1	-1
2	-1	1	-1
col max	1	$\mathbf{1}$	

The minimax payoff is not the same as the maximin payoff, so the game does not have a saddle point.

(b) Expected payoff for player 1: $(x_1y_1 + x_2y_2) - (x_1y_2 + x_2y_1)$

$x_1 + x_2 = y_1 + y_2 = 1$

(i) $y_1 = 1, y_2 = 0$: $x_1 - x_2 = x_1 - (1 - x_1) = 2x_1 - 1$

(ii) $y_1 = 0, y_2 = 1$: $x_2 - x_1 = (1 - x_1) - x_1 = 1 - 2x_1$

(ii) $y_1 = \frac{1}{2}, y_2 = \frac{1}{2}$: 0

(c) Expected payoff for player 1: $(x_1y_2 + x_2y_1) - (x_1y_1 + x_2y_2)$

$x_1 + x_2 = y_1 + y_2 = 1$

(i) $y_1 = 1, y_2 = 0$: $x_2 - x_1 = (1 - x_1) - x_1 = 1 - 2x_1$

(ii) $y_1 = 0, y_2 = 1$: $x_1 - x_2 = x_1 - (1 - x_1) = 2x_1 - 1$

(ii) $y_1 = \frac{1}{2}, y_2 = \frac{1}{2}$: 0

14.3-2.

(a) <u>Strategies for player1:</u> 1- Pass on heads or tails
2- Bet on heads or tails
3- Pass on heads, bet on tails
4- Bet on heads, pass on tails

<u>Strategies for player 2:</u> 1- If player 1 bets, call.
2- If player 1 bets, pass.

(b)

	1	2
1	-5	-5
2	0	5
3	-7.5	0
4	2.5	0

Strategies 1 and 3 of player 1 are dominated bye strategy 2. Upon eliminating them, the table is reduced to:

	1	2
2	0	5
4	2.5	0

(c)

	1	2	row min
1	-5	-5	-5
2	0	5	$\mathbf{0}$
3	-7.5	0	-7.5
4	2.5	0	$\mathbf{0}$
col max	$\mathbf{2.5}$	5	

The minimum of the column maxima is not equal to the maximum of the row minima, there is no saddle point. If either player chooses a pure strategy, the other one can choose a strategy to cause the first player to change his strategy. One needs mixed strategies to find an equilibrium.

(d) The dominated strategies will not be chosen. Let x_2 and x_4 be the probabilities that player 1 uses strategy 2 and 4 respectively, y_1 and y_2 be the probabilities that player 2 uses strategy 1 and 2 respectively. Hence, $x_2 + x_4 = 1$ and $y_1 + y_2 = 1$ and the expected payoff can be expressed as $p_{21}x_2y_1 + p_{22}x_2y_2 + p_{41}x_4y_1 + p_{42}x_4y_2$.

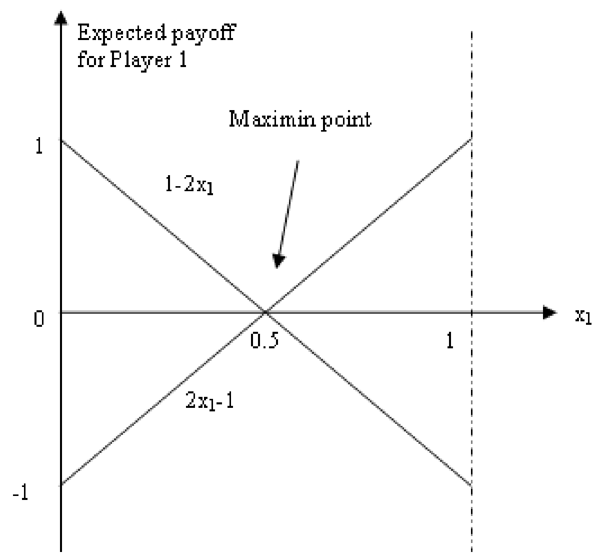
Case (i): $y_1 = 1$, $y_2 = 0$ $\Rightarrow 2.5x_4 = 2.5(1 - x_2)$

Case (ii): $y_1 = 0$, $y_2 = 1$ $\Rightarrow 5x_2 = 5(1 - x_4)$

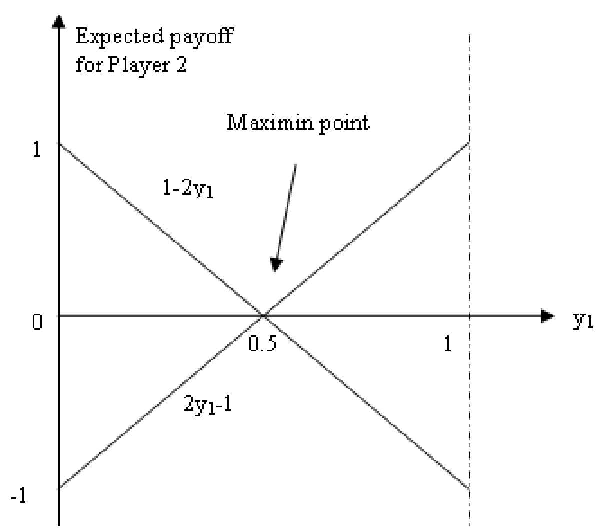
Case (iii): $y_1 = y_2 = 0.5$ $\Rightarrow 5x_2\left(\frac{1}{2}\right) + 2.5x_4\left(\frac{1}{2}\right) = 0.25x_2 + 1.25$

14.4-1.

Expected payoff for player 1: (i) $y_1 = 1, y_2 = 0$: $2x_1 - 1$
(ii) $y_1 = 0, y_2 = 1$: $1 - 2x_1$



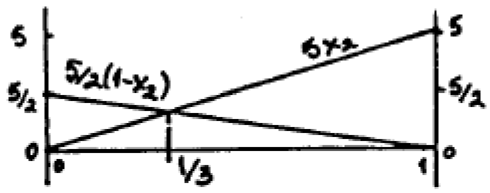
Expected payoff for player 2: (i) $x_1 = 1, x_2 = 0$: $1 - 2y_1$
(ii) $x_1 = 0, x_2 = 1$: $2y_1 - 1$



The corresponding value of the game is zero.

14.4-2.

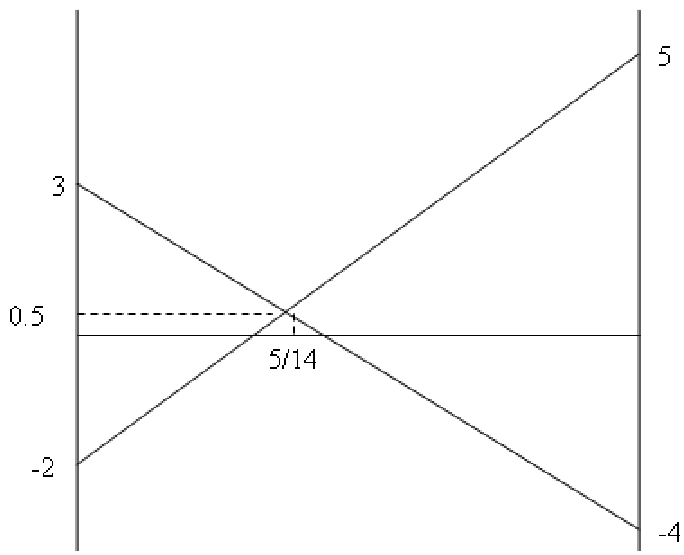
(y_1, y_2)	Expected Payoff
$(1, 0)$	$2.5(1 - x_2)$
$(0, 1)$	$5x_2$



$2.5(1 - x_2) = 5x_2 \;\Rightarrow\; (x_1^*, x_2^*, x_3^*, x_4^*) = (0, 1/3, 0, 2/3)$ and $v = 5/3$.
$2.5y_1^*(1 - x_2) + 5y_2^* x_2 = 5/3$ for $0 \leq x_2 \leq 1 \;\Rightarrow\; 2.5y_1^* = 5/3$ and $5y_2^* = 5/3$
$\Rightarrow (y_1^*, y_2^*) = (2/3, 1/3)$.

14.4-3.

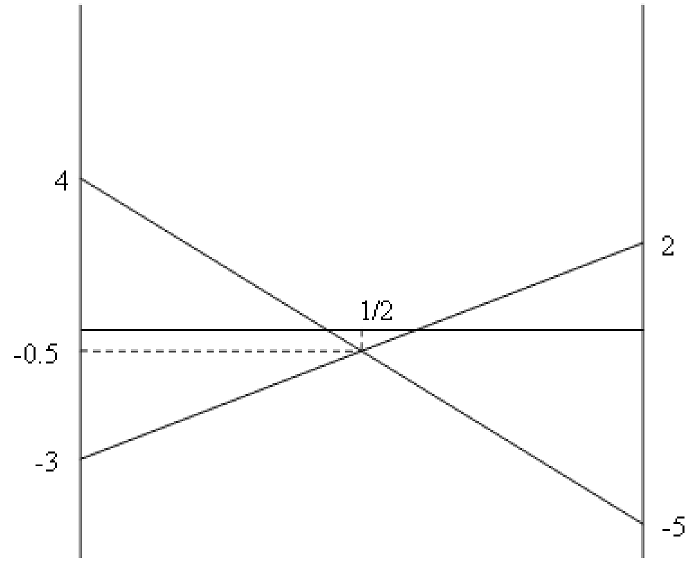
(y_1, y_2)	Expected Payoff
$(1, 0)$	$5x_1 - 2(1 - x_1) = \quad 7x_1 - 2$
$(0, 1)$	$-4x_1 + 3(1 - x_1) = -7x_1 + 3$



$7x_1 - 2 = -7x_1 + 3 \;\Rightarrow\; (x_1^*, x_2^*) = (5/14, 9/14)$ and $v = 7(5/14) - 2 = 0.5$.
$5y_1^* - 4y_2^* = 0.5$ and $-2y_1^* + 3y_2^* = 0.5 \;\Rightarrow\; (y_1^*, y_2^*) = (0.5, 0.5)$.

The payoff matrix for player 2 is:

	1	2
1	-5	4
2	2	-3

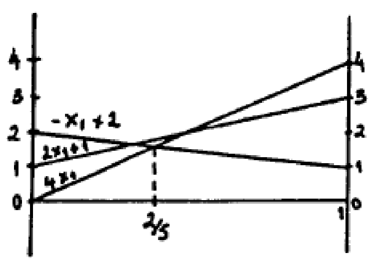


(x_1, x_2)	Expected Payoff
$(1, 0)$	$-5y_1 + 4(1 - y_1) = -9y_1 + 4$
$(0, 1)$	$2y_1 - 3(1 - y_1) = \quad 5y_1 - 3$

$-9y_1 + 4 = 5y_1 - 3 \ \Rightarrow \ y_1^* = y_2^* = 0.5$

14.4-4.

(y_1, y_2, y_3)	Expected Payoff
$(1, 0, 0)$	$4x_1$
$(0, 1, 0)$	$3x_1 + (1 - x_1) = 2x_1 + 1$
$(0, 0, 1)$	$x_1 + 2(1 - x_1) = -x_1 + 2$



$4x_1 = -x_1 + 2 \ \Rightarrow \ (x_1^*, x_2^*) = (2/5, 3/5)$ and $v = 8/5$.
$y_1^*(4x_1) + y_3^*(-x_1 + 2) = 8/5$ for $0 \leq x_1 \leq 1 \ \Rightarrow \ 2y_3^* = 8/5$ and $4y_1^* + y_3^* = 3/5$
$\Rightarrow (y_1^*, y_2^*, y_3^*) = (1/5, 0, 4/5)$.

14.4-5.

(a) Strategies for A.J. Team:
1- John does not swim butterfly.
2- John does not swim backstroke.
3- John does not swim breaststroke.

Strategies for G.N. Team:
1- Mark does not swim butterfly.
2- Mark does not swim backstroke.
3- Mark does not swim breaststroke.

Let the payoff entries be the total points earned in all three events by A.J. Team when a given pair of strategies are chosen by the teams. Then the payoff matrix becomes:

	1	2	3
1	14	13	12
2	13	12	12
3	12	12	13

Strategy 2 of A.J. Team is dominated by strategy 1 and strategy 1 of G.N. Team is dominated by strategy 2. When we eliminate these strategies we obtain the table:

	2	3
1	13	12
3	12	13

(y_1, y_2)	Expected Payoff
$(1, 0)$	$13x_1 + 12(1 - x_1) = x_1 + 12$
$(0, 1)$	$12x_1 + 13(1 - x_1) = -x_1 + 13$

$x_1 + 12 = -x_1 + 13 \Rightarrow (x_1^*, x_2^*, x_3^*) = (0.5, 0, 0.5)$ and $v = 12.5$.
$y_2^*(x_1 + 12) + y_3^*(-x_1 + 13) = 12.5$ for $0 \leq x_1 \leq 1 \Rightarrow 12y_2^* + 13y_3^* = 12.5$ and $13y_2^* + 12y_3^* = 12.5 \Rightarrow (y_1^*, y_2^*, y_3^*) = (0, 0.5, 0.5)$.

Hence, John should always swim backstroke and should swim butterfly and breaststroke each with probability $1/2$. Also, Mark should always swim butterfly and should swim backstroke and breaststroke each with probability $1/2$. Consequently, A.J. Team can expect to get 12.5 points on average in three events.

(b) The strategies for the two teams are the same as in (a). If p_{ij} denotes the total points earned by A.J. Team, let p'_{ij} be the new payoff that is defined as:

$$p'_{ij} = \begin{cases} 1/2 & \text{if } p_{ij} \geq 13 \text{, i.e., if A.J. Team wins} \\ -1/2 & \text{if } p_{ij} < 13 \text{, i.e., if A.J. Team loses} \end{cases}.$$

Then, the new payoff matrix becomes:

	1	2	3
1	$1/2$	$1/2$	$-1/2$
2	$1/2$	$-1/2$	$-1/2$
3	$-1/2$	$-1/2$	$1/2$

where strategy 2 of A.J. Team is dominated by strategy 1 and strategy 1 of G.N. Team is dominated by strategy 2. After eliminating these, the reduced payoff matrix is:

	2	3
1	$1/2$	$-1/2$
3	$-1/2$	$1/2$

Adding the constant 12.5 to every entry does not change the optimal strategies. Furthermore, the payoff matrix in (a) is obtained by doing so. Hence, the best strategies found in (a) are still optimal, the new payoff is $v' = 12.5 - 12.5 = 0$.

(c) Since John and Mark are the best swimmers of their teams, they will always swim in two events. Their teams cannot do better if they do not swim or if they swim in only one event. Hence, if either one of them does not swim in the first event, namely butterfly, he will surely swim the last two events. Accordingly, the strategies for A.J. Team are:

1- John swims butterfly and then backstroke regardless of whether Mark swims butterfly.
2- John swims butterfly and then backstroke if Mark swims butterfly, breaststroke else.
3- John swims butterfly and then breaststroke if Mark swims butterfly, backstroke else.
4- John swims butterfly and then breaststroke regardless of whether Mark swims butterfly.
5- John does not swim butterfly, swims both backstroke and breaststroke.

The strategies for G.N. Team are the same but with the roles of John and Mark are reversed. The associated payoff matrix is:

	1	2	3	4	5
1	$1/2$	$1/2$	$-1/2$	$-1/2$	$-1/2$
2	$1/2$	$1/2$	$-1/2$	$-1/2$	$1/2$
3	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
4	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$1/2$
5	$-1/2$	$1/2$	$-1/2$	$1/2$	$1/2$

	3
1	$-1/2$
2	$-1/2$
3	$-1/2$
4	$-1/2$
5	$-1/2$

Strategy 3 of G.N. Team dominates all others, by eliminating them, we obtain the payoff matrix on the right. It shows that if G.N. Team uses strategy 3, it will win regardless of what strategy is employed by A.J. Team.

(d) Strategy 2 of A.J. Team dominates strategies 1, 3, and 4. Thus, if the coach of G.N. Team may choose any of their strategies at random, the coach of A.J. Team should choose either strategy 2 or 5. After eliminating the dominated strategies, the payoff matrix becomes:

	1	2	3	4	5
2	$1/2$	$1/2$	$-1/2$	$-1/2$	$1/2$
5	$-1/2$	$1/2$	$-1/2$	$1/2$	$1/2$

The two rows are identical except for columns 1 and 4. Thus, if the coach of A.J. team knows that the other coach has a tendency to enter Mark in butterfly and backstroke more often than breaststroke, that means column 1 is more likely to be chosen than column 4, so the coach of A.J. team should choose strategy 2.

14.5-1.

(a) Player 1:

$$\begin{array}{ll} \text{maximize} & x_3 \\ \text{subject to} & x_1 - x_2 - x_3 \geq 0 \\ & -x_1 + x_2 - x_3 \geq 0 \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{array}$$

Player 2:

$$\begin{array}{ll} \text{minimize} & y_3 \\ \text{subject to} & y_1 - y_2 - y_3 \leq 0 \\ & -y_1 + y_2 - y_3 \leq 0 \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{array}$$

(b) Optimal Solution: $x_1 = x_2 = y_1 = y_2 = 0.5, x_3 = y_3 = 0$

14.5-2.

After adding 3 to the entries of Table 14.6, the payoff table becomes:

	1	2	3
1	3	1	5
2	8	7	0

The new linear programming problem for player 1 is:

$$\begin{array}{ll} \text{maximize} & x_3 \\ \text{subject to} & 3x_1 + 8x_2 - x_3 \geq 0 \\ & x_1 + 7x_2 - x_3 \geq 0 \\ & 5x_1 - x_3 \geq 0 \\ & x_1 + x_2 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

The new linear programming problem for player 2 is:

$$\begin{array}{ll} \text{maximize} & y_4 \\ \text{subject to} & 3y_1 + y_2 + 5y_3 - y_4 \leq 0 \\ & 8y_1 + 7y_2 - y_4 \leq 0 \\ & y_1 + \ y_2 + y_3 = 1 \\ & y_1, y_2, y_3, y_4 \geq \ 0 \end{array}$$

Based on the information given in Section 14.5, the optimal solutions for these new models are:

$$\begin{array}{ll} (x_1^*, x_2^*, x_3^*) & = (7/11, 4/11, 35/11) \\ (y_1^*, y_2^*, y_3^*, y_4^*) & = (0, 5/11, 6/11, 35/11). \end{array}$$

Note that $x_3^* = y_4^* = v + 3$ where v is the value for the original version of the game.

14.5-3.

(a)

$$\begin{array}{ll} \text{maximize} & x_4 \\ \text{subject to} & 5x_1 + 2x_2 + 3x_3 - x_4 \geq 0 \\ & 4x_2 + 2x_3 - x_4 \geq 0 \\ & 3x_1 + 3x_2 - x_4 \geq 0 \\ & x_1 + 2x_2 + 4x_3 - x_4 \geq 0 \\ & x_1 + \ \ x_2 + \ \ x_3 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

(b)

```
Optimal Solution
Objective Function: 2.368
 Variable    |   Value
_____________|____________
X1           |0.053
X2           |0.737
X3           |0.211
X4           |2.368
```

14.5-4.

(a) To insure $x_4 \geq 0$, add 5 to each entry of the payoff table.

$$\begin{aligned} &\text{maximize} && x_4 \\ &\text{subject to} && 12x_1 + 4x_2 + \ 8x_3 - x_4 \geq 0 \\ &&& 8x_1 + 5x_2 + 10x_3 - x_4 \geq 0 \\ &&& 10x_2 + \ \ 2x_3 - x_4 \geq 0 \\ &&& x_1 + x_2 + x_3 = \ 1 \\ &&& x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(b)

```
Optimal Solution
Objective Function: 6.462
 Variable    |    Value
_____________|____________
X1           |0.231
X2           |0.615
X3           |0.154
X4           |6.462
```

14.5-5.

(a) To insure $x_5 \geq 0$, add 4 to each entry of the payoff table.

$$\begin{aligned} &\text{maximize} && x_5 \\ &\text{subject to} && 5x_1 + 6x_2 + 4x_3 - x_5 \geq 0 \\ &&& x_1 + 7x_2 + 8x_3 + 4x_4 - x_5 \geq 0 \\ &&& 6x_1 + 4x_2 + 3x_3 + 2x_4 - x_5 \geq 0 \\ &&& 2x_1 + 7x_2 + x_3 + 6x_4 - x_5 \geq 0 \\ &&& 5x_1 + 2x_2 + 6x_3 + 3x_4 - x_5 \geq 0 \\ &&& x_1 + x_2 + x_3 + x_4 = 1 \\ &&& x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

(b)

```
Optimal Solution
Objective Function: 3.981
 Variable    |    Value
_____________|____________
X1           |0.31
X2           |0.266
X3           |0.209
X4           |0.215
X5           |3.981
```

14.5-6.

Following Table 6.14, the dual of player 1's problem is:

$$\begin{array}{lrcl}
\text{minimize} & y_{n+1} & & \\
\text{subject to} & p_{11}y_1' + p_{12}y_2' + \cdots + p_{1n}y_n' + y_{n+1} & \geq & 0 \\
& p_{21}y_1' + p_{22}y_2' + \cdots + p_{2n}y_n' + y_{n+1} & \geq & 0 \\
& \vdots & & \\
& p_{m1}y_1' + p_{m2}y_2' + \cdots + p_{mn}y_n' + y_{n+1} & \geq & 0 \\
& -y_1' - y_2' - \cdots - y_n' & = & 1 \\
& y_i' \leq 0,\ i = 1, 2, \ldots, n;\ (y_{n+1} \text{ free}). & &
\end{array}$$

Now, let $y_i = -y_i'$ for $i = 1, 2, \ldots, n$ to get the linear program for player 2.

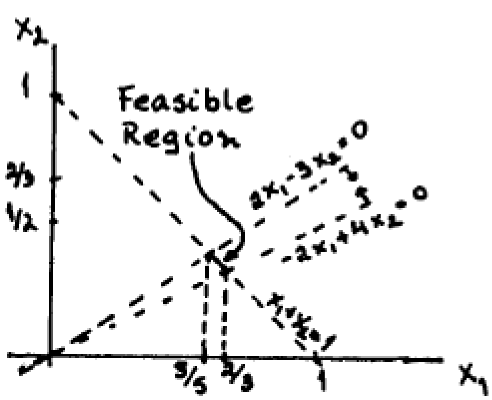
14.5-7.

Taking the dual of player 1's problem gives:

$$\begin{array}{lrcl}
\text{minimize} & y_4 & & \\
\text{subject to} & -2y_2' + 2y_3' + y_4 & \geq & 0 \\
& 5y_1' + 4y_2' - 3y_3' + y_4 & \geq & 0 \\
& -y_1' - y_2' - y_3' & = & 1 \\
& y_1', y_2', y_3 \leq 0;\ (y_4 \text{ free}). & &
\end{array}$$

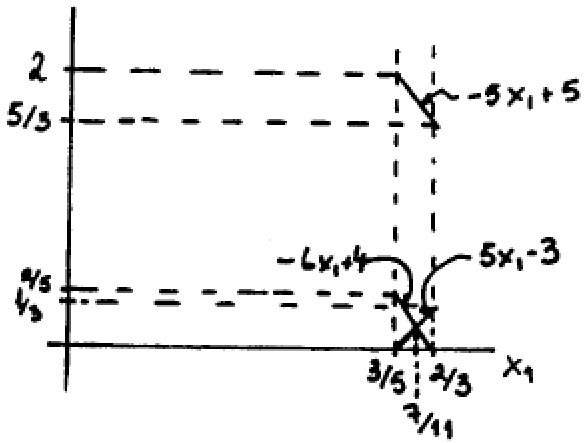
Now, let $y_i = -y_i'$ for $i = 1, 2, 3$ to get the linear program for player 2.

14.5-8.



The feasible region may be algebraically described by: $x_2 = 1 - x_1$ and $3/5 \leq x_1 \leq 2/3$. The restrictions may be rewritten as:

$$\begin{array}{ll}
x_3 \leq -5x_1 + 5 & 3/5 \leq x_1 \leq 2/3 \\
x_3 \leq -6x_1 + 4 & 3/5 \leq x_1 \leq 2/3 \\
x_3 \leq 5x_1 - 3 & 3/5 \leq x_1 \leq 2/3
\end{array}$$



$-6x_1 + 4 = 5x_1 - 3 \;\Rightarrow\; x_1 = 7/11.$

Therefore, the algebraic expression for the maximizing value of x_3 for any point in the feasible region is:

$$x_3 = \begin{cases} 5x_1 - 3 & \text{for } 3/5 \le x_1 \le 7/11 \\ -6x_1 + 4 & \text{for } 7/11 \le x_1 \le 2/3 \end{cases}$$

Hence, the optimal solution is:

$(x_1^*, x_2^*, x_3^*) = (7/11, 1 - 7/11, 5(7/11) - 3) = (7/11, 4/11, 2/11).$

14.5-9.

AUTOMATIC SIMPLEX METHOD: FINAL TABLEAU

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
									1M	1M	1M	1M	
Z	0	1	0	0	0	0	0.455	0.545	0	-0.45	-0.55	0.182	0.182
X2	1	0	0	1	0	0	-0.09	0.091	0	0.091	-0.09	0.364	0.364
X4	2	0	0	0	0	1	-0.91	-0.09	-1	0.909	0.091	1.636	1.636
X1	3	0	1	0	0	0	0.091	-0.09	0	-0.09	0.091	0.636	0.636
X3	4	0	0	0	1	0	0.455	0.545	0	-0.45	-0.55	0.182	0.182

Optimal primal solution: $(x_1, x_2) = (0.636, 0.364)$ with a payoff of 0.182
Optimal dual solution: $(y_1, y_2, y_3) = (0, 0.455, 0, 545)$

14.5-10.

(a) Since the saddle points can be found by linear programming, (a) follows from (b).

(b) Consider the linear programming formulation of the problem for player 2. The ith and kth constraints are:

$$\begin{aligned} p_{i1}y_1 + p_{i2}y_2 + \cdots + p_{in}y_n &\leq y_{n+1} \\ p_{k1}y_1 + p_{k2}y_2 + \cdots + p_{kn}y_n &\leq y_{n+1} \end{aligned}$$

If row k weakly dominates row i, then

$$p_{i1}y_1 + p_{i2}y_2 + \cdots + p_{in}y_n \leq p_{k1}y_1 + p_{k2}y_2 + \cdots + p_{kn}y_n$$

for every $y_1, \ldots, y_n$. In that case, the ith constraint is redundant, as it is implied by the kth constraint. Hence, eliminating dominated pure strategies for player 1 corresponds to eliminating redundant constraints from the linear program for player 2. Similarly, eliminating dominated strategies of player 2 is equivalent to eliminating redundant constraints of player 1's linear program. Since this process cannot eliminate any feasible solutions or create new ones, all optimal strategies are preserved and no new ones are added.

CHAPTER 15: DECISION ANALYSIS

15.2-1.

Phillips Petroleum Company developed a decision analysis tool named DISCOVERY to evaluate available investment opportunities and decide on the participation levels. The need for a systematic decision analysis tool arose from the uncertainty associated with various alternatives, the lack of a consistent risk measure across the organization and the scarcity of capital resources. The notion of risk is incorporated in the model with the use of risk-averse exponential utility function. The objective is to maximize expected utility rather than expected return. DISCOVERY provides a decision-tree display of available alternatives at various participation levels. A simple version of the problem is one where Phillips needs to decide first on the participation level and second on whether to drill or not. The exploration of petroleum when drilled is uncertain. The analysis is performed for different levels of risk-aversion and the sensitivity of the decisions to the risk-aversion level is observed. When additional seismic information is available at a cost, the value of information is computed.

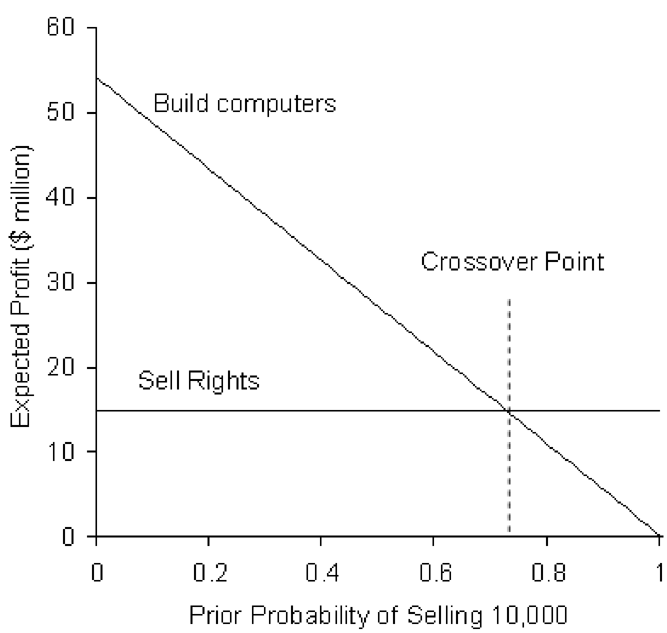
This study "has increased management's awareness of risk and risk tolerance, provided insight into the financial risks associated with its set of investment opportunities, and provided the company a formalized decision model for allocating scarce capital" [p. 55]. The software package developed has been a valuable aid in decision making. It provided a systematic treatment of risk and uncertainty. Other petroleum exploration firms started to use DISCOVERY in analyzing decisions, too.

15.2-2.

(a)

	State of Nature	
Alternative	Sell $10,000$	Sell $100,000$
Build Computers	0	54
Sell Rights	15	15

(b)



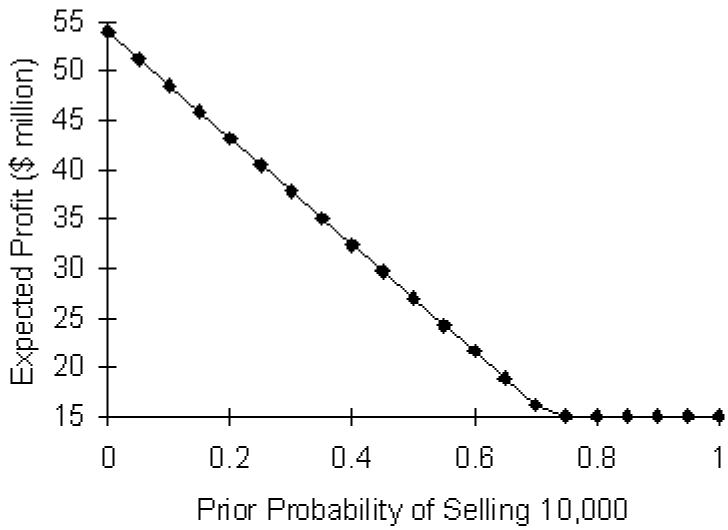
(c) Let p be the prior probability of selling 10,000 computers.

Build: EP $= p(0) + (1-p)(54) = -54p + 54$

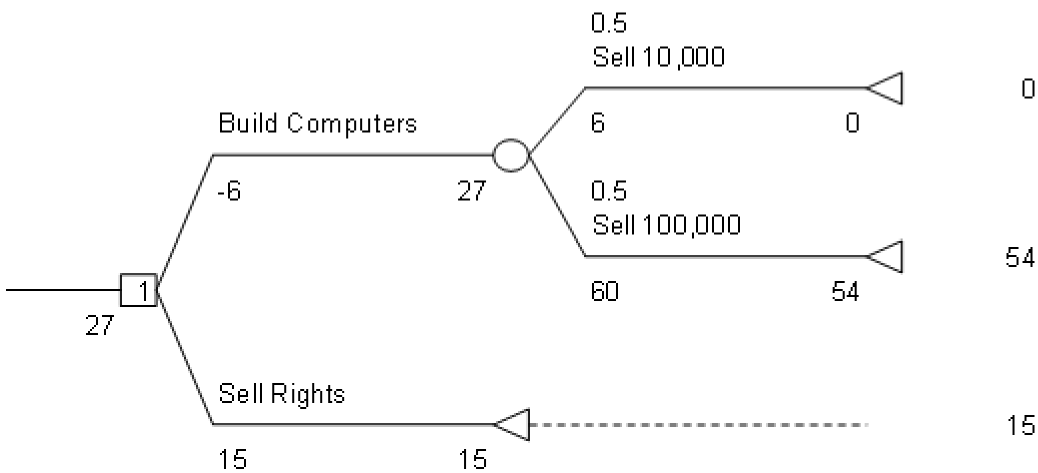
Sell: EP $= p(15) + (1-p)(15) = 15$

The expected profit for Build and Sell is the same when $-54p + 54 = 15 \Rightarrow p = 0.722$. They should build when $p \leq 0.722$ and sell if $p \geq 0.722$.

(d)



(e)



Building computers should be chosen, since it has an expected payoff of $27 million.

15.2-3.

(a)

	State of Nature			
Alternative	Sell 12 Cases	Sell 13 Cases	Sell 14 Cases	Sell 15 Cases
Buy 12 Cases	132	132	132	132
Buy 13 Cases	125	143	143	143
Buy 14 Cases	118	136	154	154
Buy 15 Cases	111	129	147	165
Prior Probability	0.1	0.3	0.4	0.2

(b) According to the maximin payoff criterion, Jean should purchase 12 cases.

	State of Nature				
Alternative	Sell 12 Cases	Sell 13 Cases	Sell 14 Cases	Sell 15 Cases	Min
Buy 12 Cases	132	132	132	132	132
Buy 13 Cases	125	143	143	143	125
Buy 14 Cases	118	136	154	154	118
Buy 15 Cases	111	129	147	165	111
Prior Probability	0.1	0.3	0.4	0.2	

(c) She will be able to sell 14 cases with highest probability and the maximum possible profit from selling 14 cases is earned when she buys 14 cases. Hence, according to the maximum likelihood criterion, Jean should purchase 14 cases.

(d) According to Bayes' decision rule, Jean should purchase 14 cases.

	State of Nature				Exp.
Alternative	Sell 12 Cases	Sell 13 Cases	Sell 14 Cases	Sell 15 Cases	Profit
Buy 12 Cases	132	132	132	132	132
Buy 13 Cases	125	143	143	143	141.2
Buy 14 Cases	118	136	154	154	145
Buy 15 Cases	111	129	147	165	141.6
Prior Probability	0.1	0.3	0.4	0.2	

(e) <u>0.2 and 0.5:</u> Jean should purchase 14 cases.

	State of Nature				Exp.
Alternative	Sell 12 Cases	Sell 13 Cases	Sell 14 Cases	Sell 15 Cases	Profit
Buy 12 Cases	132	132	132	132	132
Buy 13 Cases	125	143	143	143	141.2
Buy 14 Cases	118	136	154	154	146.8
Buy 15 Cases	111	129	147	165	143.4
Prior Probability	0.1	0.2	0.5	0.2	

<u>0.4 and 0.3:</u> Jean should purchase 14 cases.

	State of Nature				Exp.
Alternative	Sell 12 Cases	Sell 13 Cases	Sell 14 Cases	Sell 15 Cases	Profit
Buy 12 Cases	132	132	132	132	132
Buy 13 Cases	125	143	143	143	141.2
Buy 14 Cases	118	136	154	154	143.2
Buy 15 Cases	111	129	147	165	139.8
Prior Probability	0.1	0.4	0.3	0.2	

<u>0.5 and 0.2:</u> Jean should purchase 14 cases.

	State of Nature				Exp.
Alternative	Sell 12 Cases	Sell 13 Cases	Sell 14 Cases	Sell 15 Cases	Profit
Buy 12 Cases	132	132	132	132	132
Buy 13 Cases	125	143	143	143	141.2
Buy 14 Cases	118	136	154	154	141.4
Buy 15 Cases	111	129	147	165	138
Prior Probability	0.1	0.5	0.2	0.2	

15.2-4.

(a) The optimal (maximin) actions are conservative and countercyclical investments, both incur a loss of $10 million in the worst case.

(b) The economy is most likely to be stable and the alternative with the highest profit in this state of nature is to make a speculative investment. According to the maximum likelihood criterion, Warren should choose speculative investment.

(c) To maximize his expected payoff, Warren should make a countercyclical investment.

	State of Nature			Exp.
Alternative	Improving	Stable	Worsening	Profit
Conservative	30	5	−10	1.5
Speculative	40	10	−30	−3
Countercyclical	−10	0	15	5
Prior Probability	0.1	0.5	0.4	

15.2-5.

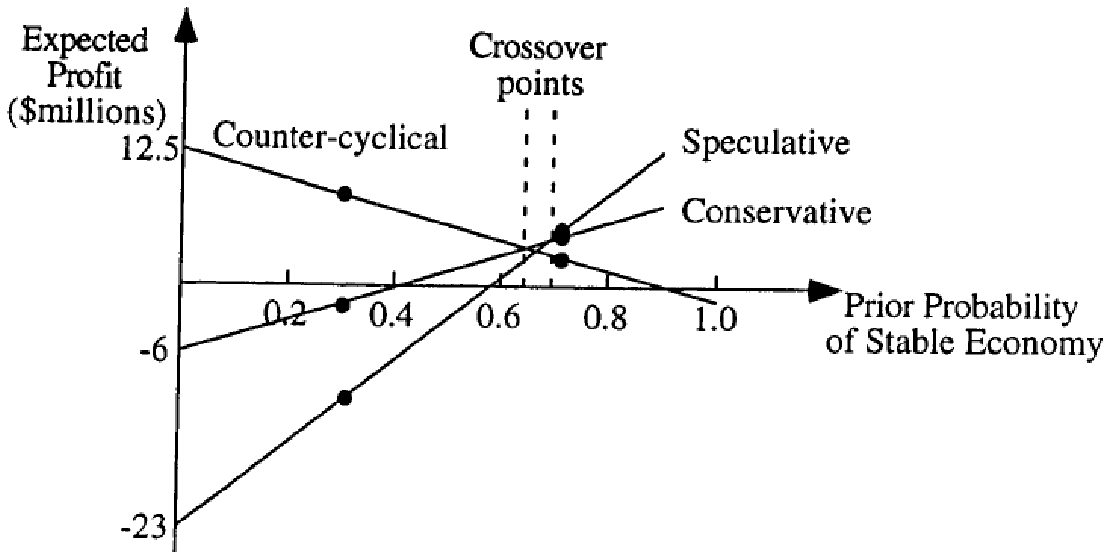
(a) Warren should make a countercyclical investment.

	State of Nature			Exp.
Alternative	Improving	Stable	Worsening	Profit
Conservative	30	5	−10	−1.5
Speculative	40	10	−30	−11
Countercyclical	−10	0	15	8
Prior Probability	0.1	0.3	0.6	

(b) Warren should make a speculative investment.

	State of Nature			Exp.
Alternative	Improving	Stable	Worsening	Profit
Conservative	30	5	−10	4.5
Speculative	40	10	−30	5
Countercyclical	−10	0	15	2
Prior Probability	0.1	0.7	0.2	

(c) The expected profit from countercyclical and conservative investments is the same when $p \approx 0.62$. The expected profit lines for conservative and speculative investments cross at $p \approx 0.68$. Those for countercyclical and speculative investments cross at $p \approx 0.65$; however, this crossover point does not result in a decision shift.



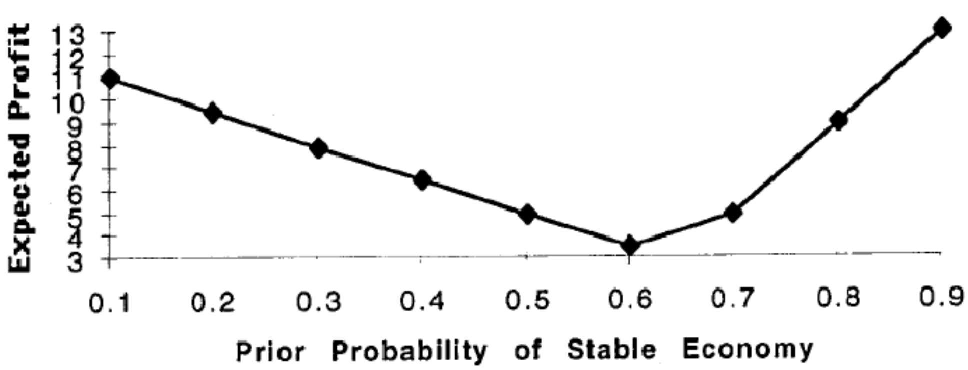
(d) Let p be the prior probability of stable economy.

Conservative: EP $= (0.1)(30) + p(5) + (1 - 0.1 - p)(-10) = 15p - 6$
Speculative: EP $= (0.1)(40) + p(10) + (1 - 0.1 - p)(-30) = 40p - 23$
Countercyclical: EP $= (0.1)(-10) + p(0) + (1 - 0.1 - p)(15) = -15p + 12.5$

Countercyclical and conservative cross when $-15p + 12.5 = 15p - 6 \Rightarrow p = 0.617$.
Conservative and speculative cross when $15p - 6 = 40p - 23 \Rightarrow p = 0.68$.

Accordingly, Warren should choose countercyclical investment when $p < 0.617$, conservative investment when $0.617 \leq p < 0.68$ and speculative investment when $p \geq 0.68$.

(e)



15.2-6.

(a) A_2 should be chosen.

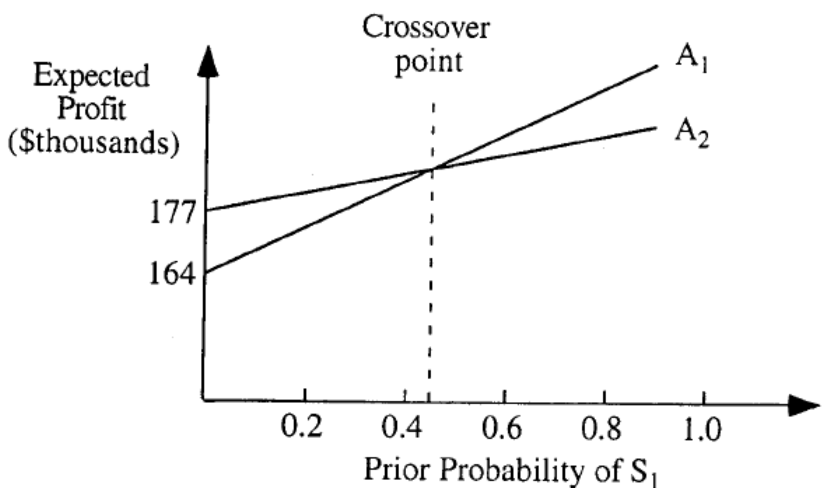
	State of Nature			
Alternative	S_1	S_2	S_3	Min
A_1	220	170	110	110
A_2	200	180	150	150
Prior Probability	0.6	0.3	0.1	

(b) The most likely state of nature is S_1 and the alternative with highest profit in this state is A_1.

(c) A_1 should be chosen.

	State of Nature			Exp.
Alternative	S_1	S_2	S_3	Payoff
A_1	220	170	110	194
A_2	200	180	150	189
Prior Probability	0.6	0.3	0.1	

(d)

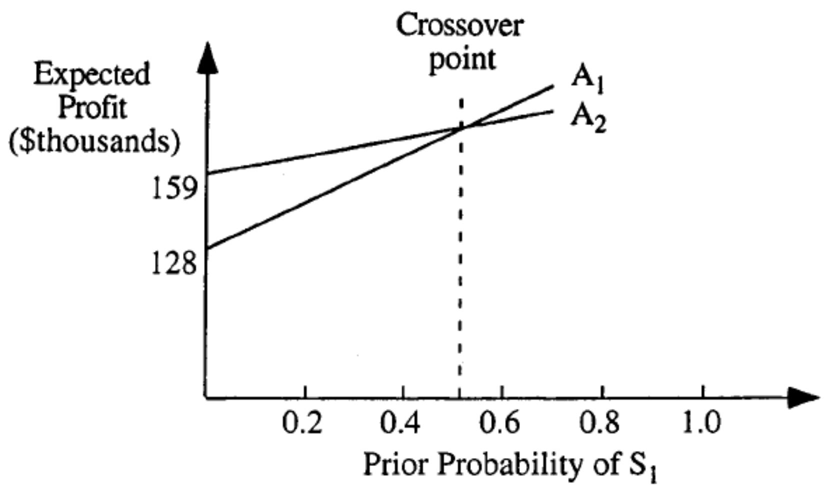


Let p be the prior probability of S_1.

A_1: EP $= p(220) + (1 - 0.1 - p)(170) + (0.1)(110) = 50p + 164$
A_2: EP $= p(200) + (1 - 0.1 - p)(180) + (0.1)(150) = 20p + 177$

A_1 and A_2 cross when $50p + 164 = 20p + 177 \Rightarrow p = 0.433$. They should choose A_2 when $p \leq 0.433$ and A_1 if $p > 0.433$.

(e)

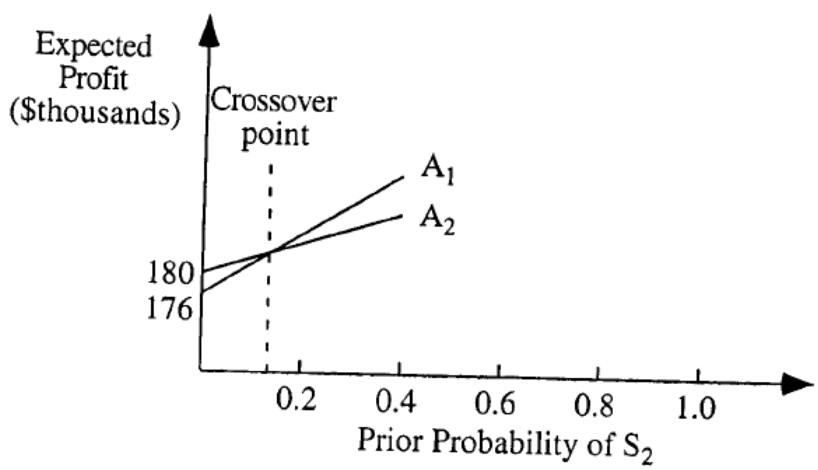


Let p be the prior probability of S_1.

A$_1$: EP $= p(220) + (0.3)(170) + (1 - 0.3 - p)(110) = 110p + 128$
A$_2$: EP $= p(200) + (0.3)(180) + (1 - 0.3 - p)(150) = 50p + 159$

A$_1$ and A$_2$ cross when $110p + 128 = 50p + 159 \Rightarrow p = 0.517$. They should choose A$_2$ when $p \leq 0.517$ and A$_1$ if $p > 0.517$.

(f)



Let p be the prior probability of S_2.

A$_1$: EP $= (0.6)(220) + p(170) + (1 - 0.6 - p)(110) = 60p + 176$
A$_2$: EP $= (0.6)(200) + p(180) + (1 - 0.6 - p)(150) = 30p + 180$

A$_1$ and A$_2$ cross when $60p + 176 = 30p + 180 \Rightarrow p = 0.133$. They should choose A$_2$ when $p \leq 0.133$ and A$_1$ if $p > 0.133$.

(g) A$_1$ should be chosen.

15.2-7.

(a)

	State of Nature		
Alternative	Dry	Moderate	Damp
Crop 1	90	150	180
Crop 2	112.5	135	180
Crop 3	120	105	105
Crop 4	90	90	90
Prior Probability	0.2	0.5	0.3

(b) Grow Crop 1.

	State of Nature			Exp.
Alternative	Dry	Moderate	Damp	Payoff
Crop 1	90	150	180	147
Crop 2	112.5	135	180	144
Crop 3	120	105	105	108
Crop 4	90	90	90	90
Prior Probability	0.2	0.5	0.3	

(c) Prior probability of moderate weather $= 0.2$: Grow Crop 2.

	State of Nature			Exp.
Alternative	Dry	Moderate	Damp	Payoff
Crop 1	90	150	180	156
Crop 2	112.5	135	180	157.5
Crop 3	120	105	105	108
Crop 4	90	90	90	90
Prior Probability	0.2	0.2	0.6	

Prior probability of moderate weather $= 0.3$: Grow Crop 1 or 2.

	State of Nature			Exp.
Alternative	Dry	Moderate	Damp	Payoff
Crop 1	90	150	180	153
Crop 2	112.5	135	180	153
Crop 3	120	105	105	108
Crop 4	90	90	90	90
Prior Probability	0.2	0.3	0.5	

Prior probability of moderate weather $= 0.4$: Grow Crop 1.

	State of Nature			Exp.
Alternative	Dry	Moderate	Damp	Payoff
Crop 1	90	150	180	150
Crop 2	112.5	135	180	148.5
Crop 3	120	105	105	108
Crop 4	90	90	90	90
Prior Probability	0.2	0.4	0.4	

Prior probability of moderate weather $= 0.6$: Grow Crop 1.

	State of Nature			Exp.
Alternative	Dry	Moderate	Damp	Payoff
Crop 1	90	150	180	144
Crop 2	112.5	135	180	139.5
Crop 3	120	105	105	108
Crop 4	90	90	90	90
Prior Probability	0.2	0.6	0.2	

15.2-8.

The prior distribution is $\mathrm{P}\{\theta = \theta_1\} = 2/3$, $\mathrm{P}\{\theta = \theta_2\} = 1/3$.

Order 15: $-\mathrm{EP} = 2/3(1.155 \cdot 10^7) + 1/3(1.414 \cdot 10^7) = 1.241 \cdot 10^7$
Order 20: $-\mathrm{EP} = 2/3(1.012 \cdot 10^7) + 1/3(1.207 \cdot 10^7) = 1.077 \cdot 10^7$
Order 25: $-\mathrm{EP} = 2/3(1.047 \cdot 10^7) + 1/3(1.135 \cdot 10^7) = 1.076 \cdot 10^7$

The maximum expected profit, or equivalently the minimum expected cost is that of ordering 25, so the optimal decision under Bayes' decision rule is to order 25.

15.3-1.

This article describes the use of decision analysis at the Workers' Compensation Board of British Columbia (WCB), which is "responsible for the occupational health and safety, rehabilitation, and compensation interests of British Columbia's workers and employers" [p. 15]. The focus of the study is on the short-term disability claims that can later turn into long-term disability claims and can be very costly for the WCB. First, logistic regression is employed to estimate the probability of conversion for each claim. Then using decision analysis, a threshold is determined to classify the claims as high- and low-risk claims. For any fixed conversion probability, the problem consists of a simple decision tree. First the WCB chooses between classifying the claim as high risk or low risk and then whether the claim converts or not determines the actual cost. If the claim is identified as a high-risk claim, the WCB intervenes. The early intervention lowers the costs and ensures faster rehabilitation. The expected total cost is computed for various cutoff points and the point with minimum expected cost is identified as the optimal threshold.

The new policy offers accurate predictions of high-risk claims. As a result, future costs are reduced and injured workers start working sooner. This study is expected to save the WCB $4.7 per year. The scorecard system developed to implement the new policy improved the efficiency of claim management and the productivity of staff. Overall, the benefits accrued from this study paved the way for the WCB's adoption of operations research in other aspects of the organization.

15.3-2.

(a)

	State of Nature	
Alternative	Sell $10,000$	Sell $100,000$
Build Computers	0	54
Sell Rights	15	15
Prior Probability	0.5	0.5
Maximum Payoff	15	54

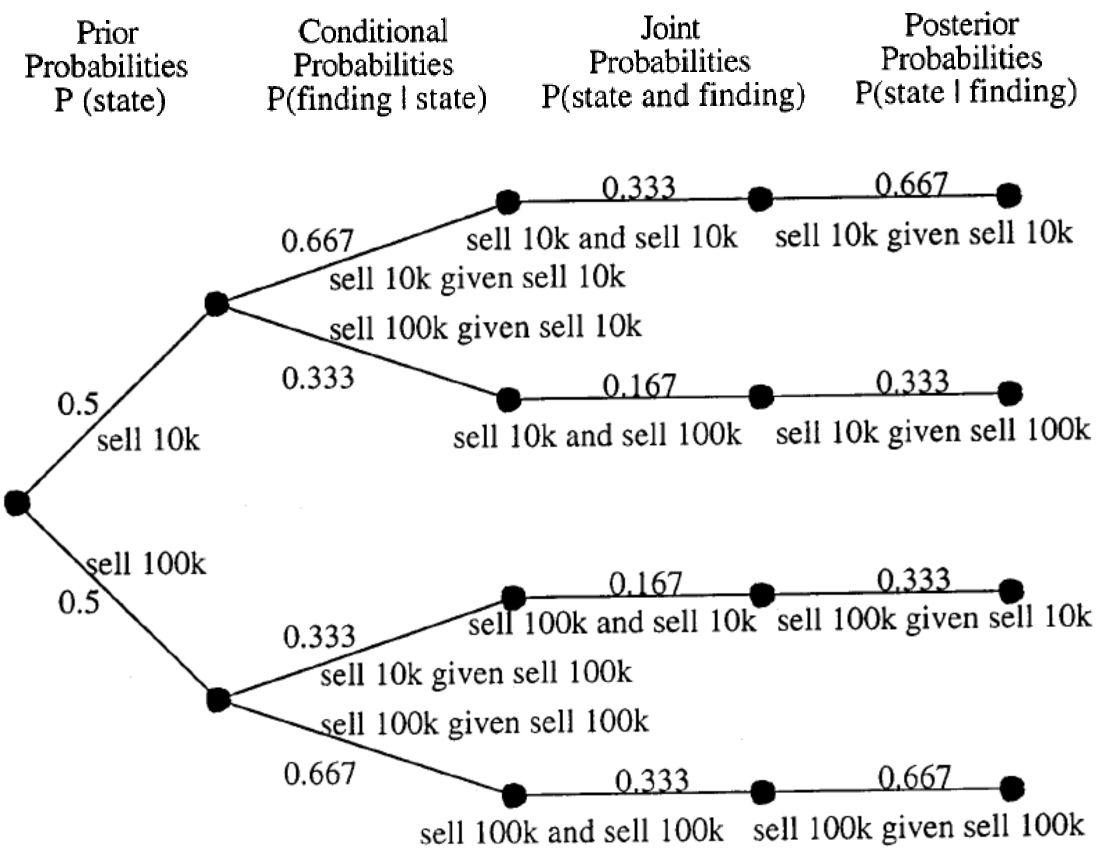
Expected Payoff with Perfect Information: $0.5(15) + 0.5(54) = 34.5$

Expected Payoff without Information: $0.5(0) + 0.5(54) = 27$

EVPI $= 34.5 - 27 =$ \$$7.5$ million

(b) Since the market research will cost \$1 million, it might be worthwhile to perform it.

(c)



(d)

Data:		P(Finding | State)	
State of	Prior	Finding	
Nature	Probability	Sell 10,000	Sell 100,000
Sell 10,000	0.5	0.6666667	0.3333333
Sell 100,000	0.5	0.3333333	0.6666667

Posterior		P(State | Finding)	
Probabilities:		State of Nature	
Finding	P(Finding)	Sell 10,000	Sell 100,000
Sell 10,000	0.5	0.6666667	0.3333333
Sell 100,000	0.5	0.3333333	0.6666667

(e) EVE $= [0.5(1800) + 0.5(3600)] - 2700 = 0$, so performing the market research is not worthwhile.

15.3-3.

(a) Choose A_1 with expected payoff \$$2,500$.

	State of Nature			Exp.
Alternative	S_1	S_2	S_3	Payoff
A_1	6	1	1	2.5
A_2	1	3	0	1.5
A_3	4	1	2	2.2
Prior Probability	0.3	0.4	0.3	

(b)

	State of Nature		
Alternative	S_1	S_2	S_3
A_1	6	1	1
A_2	1	3	0
A_3	4	1	2
Prior Probability	0.3	0.4	0.3
Maximum Payoff	6	3	2

Expected Payoff with Perfect Information: $0.3(6) + 0.4(3) + 0.3(2) = 3.6$

Expected Payoff without Information: 2.5

EVPI $= 3.6 - 2.5 =$ \$1.1 thousand

(c) Since the information will cost \$$1,00$ and the value is \$$1,100$, it might be worthwhile to spend the money.

15.3-4.

(a) Choose A_1 with expected payoff $\$35$.

	State of Nature			Exp.
Alternative	S_1	S_2	S_3	Payoff
A_1	50	100	-100	35
A_2	0	10	-10	1
A_3	20	40	-40	14
Prior Probability	0.5	0.3	0.2	

(b)

	State of Nature		
Alternative	S_1	S_2	S_3
A_1	50	100	-100
A_2	0	10	-10
A_3	20	40	-40
Prior Probability	0.5	0.3	0.2
Maximum Payoff	50	100	-10

Expected Payoff with Perfect Information: $0.5(50) + 0.3(100) + 0.2(-10) = 53$

Expected Payoff without Information: 35

EVPI $= 53 - 35 = \$18$

(c) Betsy should consider spending up to $\$18$ to obtain more information.

15.3-5.

(a) Choose A_3 with expected payoff $\$7,800$.

	State of Nature			Exp.
Alternative	S_1	S_2	S_3	Payoff
A_1	-20	3	25	4.9
A_2	-3	5	10	4.6
A_3	4	2	15	7.8
Prior Probability	0.3	0.3	0.4	

(b) If S_1 occurs for certain, then choose A_3 with expected payoff $\$4,000$. If S_1 does not occur for certain, then the probability that S_2 will occur is $3/7$ and the probability that S_3 will occur is $4/7$.

$$\begin{aligned}
\text{A}_1&: \quad (3/7)(3) + (4/7)(25) = 15.57\\
\text{A}_2&: \quad (3/7)(5) + (4/7)(10) = 7.86\\
\text{A}_3&: \quad (3/7)(2) + (4/7)(15) = 9.43
\end{aligned}$$

Hence, choose A_1 which offers an expected payoff of $\$15,570$.

Expected Payoff with Information: $0.3(4) + 0.7(15.57) = 12.01$

Expected Payoff without Information: 7.8

EVI $= 12.01 - 7.8 = \$4.21$ thousand

The maximum amount that should be paid for the information is $\$4,210$. The decision with this information will be to choose A_3 if the state of nature is S_1 and A_1 otherwise.

(c) If S_2 occurs for certain, then choose A_2 with expected payoff $\$5,000$. If S_2 does not occur for certain, then the probability that S_1 will occur is $3/7$ and the probability that S_3 will occur is $4/7$.

$$\begin{array}{ll} \mathrm{A}_1: & (3/7)(-20) + (4/7)(25) = \ \ 5.71 \\ \mathrm{A}_2: & (3/7)(-3) + (4/7)(10) \ = \ \ 4.43 \\ \mathrm{A}_3: & (3/7)(4) + (4/7)(15) \quad = 10.29 \end{array}$$

Hence, choose A_3 which offers an expected payoff of $\$10,290$.

Expected Payoff with Information: $0.3(5) + 0.7(10.29) = 9.91$

Expected Payoff without Information: 7.8

EVI $= 9.91 - 7.8 =$ $2.11 thousand

The maximum amount that should be paid for the information is $\$2,110$. The decision with this information will be to choose A_2 if the state of nature is S_2 and A_3 otherwise.

(d) If S_3 occurs for certain, then choose A_1 with expected payoff $\$25,000$. If S_3 does not occur for certain, then S_1 and S_2 occur with equal probability.

$$\begin{array}{ll} \mathrm{A}_1: & (1/2)(-20) + (1/2)(3) = -8.5 \\ \mathrm{A}_2: & (1/2)(-3) + (1/2)(5) \ = \ \ 1 \\ \mathrm{A}_3: & (1/2)(4) + (1/2)(2) \quad = \ \ 3 \end{array}$$

Hence, choose A_3 which offers an expected payoff of $\$3,000$.

Expected Payoff with Information: $0.6(3) + 0.4(25) = 11.8$

Expected Payoff without Information: 7.8

EVI $= 11.8 - 7.8 =$ $4 thousand

The maximum amount that should be paid for the information is $\$4,000$. The decision with this information will be to choose A_1 if the state of nature is S_3 and A_3 otherwise.

(e) Expected Payoff with Perfect Information: $0.3(4) + 0.3(5) + 0.4(25) = 12.7$

Expected Payoff without Information: 7.8

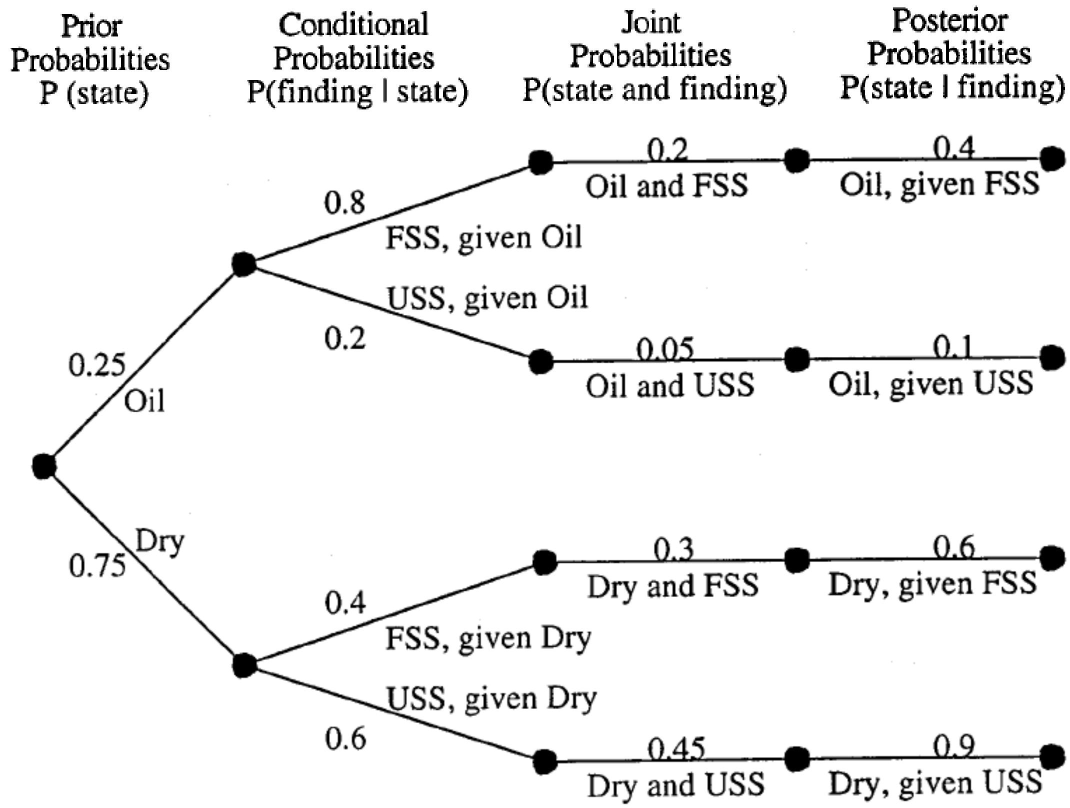
EVPI $= 12.7 - 7.8 =$ $4.9 thousand

The maximum amount that should be paid for the information is $\$4,900$. The decision with this information will be to choose A_3 if the state of nature is S_1, A_2 if the state of nature is S_2 and A_1 otherwise.

(f) The maximum amount that should be paid for testing is $\$4,900$, since any additional information cannot add more value than perfect information.

15.3-6.

(a)



(b)

Data:		P(Finding | State)			
State of	Prior	Finding			
Nature	Probability	FSS	USS		
Oil	0.25	0.8	0.2		
Dry	0.75	0.4	0.6		

Posterior		P(State | Finding)			
Probabilities:		State of Nature			
Finding	P(Finding)	Oil	Dry		
FSS	0.5	0.4	0.6		
USS	0.5	0.1	0.9		

(c) The optimal policy is to do a seismic survey, to drill if favorable seismic surroundings are obtained, and to sell if unfavorable surroundings are obtained.

15.3-7.

(a) Choose A_1 with expected payoff $100.

	State of Nature		Exp.
Alternative	S_1	S_2	Payoff
A_1	400	−100	100
A_2	0	100	60
Prior Probability	0.4	0.6	

(b)

	State of Nature	
Alternative	S_1	S_2
A_1	400	-100
A_2	0	100
Prior Probability	0.4	0.6
Maximum Payoff	400	100

Expected Payoff with Perfect Information: $0.4(400) + 0.6(100) = 220$

Expected Payoff without Information: 100

EVPI $= 220 - 100 =$ \$$120$, so it might be worthwhile to do the research.

(c) Let X denote the state of nature and Y denote the prediction. From Bayes' Rule,

$$\text{P}(X = x \text{ and } Y = y) = \text{P}(X = x)\text{P}(Y = y | X = x).$$

(i) $\text{P}(X = \text{S}_1 \text{ and } Y = \text{S}_1) = (0.4)(0.6) = 0.24$
(ii) $\text{P}(X = \text{S}_1 \text{ and } Y = \text{S}_2) = (0.4)(0.4) = 0.16$
(iii) $\text{P}(X = \text{S}_2 \text{ and } Y = \text{S}_1) = (0.6)(0.2) = 0.12$
(iv) $\text{P}(X = \text{S}_2 \text{ and } Y = \text{S}_2) = (0.6)(0.8) = 0.48$

(d) $\text{P}(\text{S}_1) = 0.24 + 0.12 = 0.36$, $\text{P}(\text{S}_2) = 0.16 + 0.48 = 0.64$

(e) Bayes' Rule: $\text{P}(X = x | Y = y) = \frac{\text{P}(X=x \text{ and } Y=y)}{\text{P}(X=x)}$

$\text{P}(\text{S}_1|\text{S}_1) = 0.24/0.36 = 0.667$
$\text{P}(\text{S}_1|\text{S}_2) = 0.16/0.64 = 0.25$
$\text{P}(\text{S}_2|\text{S}_1) = 0.12/0.36 = 0.333$
$\text{P}(\text{S}_2|\text{S}_2) = 0.48/0.64 = 0.75$

(f)

Data:		P(Finding | State)	
State of	Prior	Finding	
Nature	Probability	S1	S2
S1	0.4	0.6	0.4
S2	0.6	0.2	0.8

Posterior		P(State | Finding)	
Probabilities:		State of Nature	
Finding	P(Finding)	S1	S2
S1	0.36	0.66667	0.33333
S2	0.64	0.25	0.75

(g) If S_1 is predicted, then choose A_1 with expected payoff \$$233.33$.

	State of Nature		Exp.
Alternative	S_1	S_2	Payoff
A_1	400	-100	233.5
A_2	0	100	33.3
Prior Probability	0.667	0.333	

(h) If S_2 is predicted, then choose A_2 with expected payoff \$75.

	State of Nature		Exp.
Alternative	S_1	S_2	Payoff
A_1	400	-100	25
A_2	0	100	75
Prior Probability	0.25	0.75	

(i) Given that the research is done, the expected payoff is

$$(0.36)(233.33) + (0.64)(75) - 100 = \$32.$$

(j) The optimal policy is to not do research and to choose A_1.

15.3-8.

(a) EVPI $= [(2/3)(-1.012 \cdot 10^7) + (1/3)(-1.135 \cdot 10^7)] - (-1.076 \cdot 10^7)$

$$= \$230,000.$$

(b)

$$P(\theta = 21 | \text{ 30 spares required}) = \frac{P(\text{30 spares required } |\theta{=}21)P(\theta{=}21)}{P(\text{30 spares required } |\theta{=}21)P(\theta{=}21)+P(\text{30 spares required } |\theta{=}24)P(\theta{=}24)}$$

$$= \frac{(0.013)(2/3)}{(0.013)(2/3)+(0.036)(1/3)} = 0.419$$

$P(\theta = 24 |$ 30 spares required$) = 1 - 0.419 = 0.581$

Order 15: EP $= 0.419(-1.155 \cdot 10^7) + 0.581(-1.414 \cdot 10^7) = -1.305 \cdot 10^7$
Order 20: EP $= 0.419(-1.012 \cdot 10^7) + 0.581(-1.207 \cdot 10^7) = -1.125 \cdot 10^7$
Order 25: EP $= 0.419(-1.047 \cdot 10^7) + 0.581(-1.135 \cdot 10^7) = -1.098 \cdot 10^7$

The optimal alternative is to order 25.

15.3-9.

(a)

	State of Nature		
Alternative	Poor Risk	Average Risk	Good Risk
Extend Credit	$-15,000$	$10,000$	$20,000$
Not Extend Credit	0	0	0
Prior Probability	0.2	0.5	0.3

(b) Choose to extend credit with expected payoff \$8,000.

	State of Nature			Exp.
Alternative	Poor Risk	Average Risk	Good Risk	Payoff
Extend Credit	$-15,000$	$10,000$	$20,000$	$8,000$
Not Extend Credit	0	0	0	0
Prior Probability	0.2	0.5	0.3	

(c)

	State of Nature		
Alternative	Poor Risk	Average Risk	Good Risk
Extend Credit	$-15,000$	$10,000$	$20,000$
Not Extend Credit	0	0	0
Prior Probability	0.2	0.5	0.3
Maximum Payoff	0	$10,000$	$20,000$

Expected Payoff with Perfect Information:

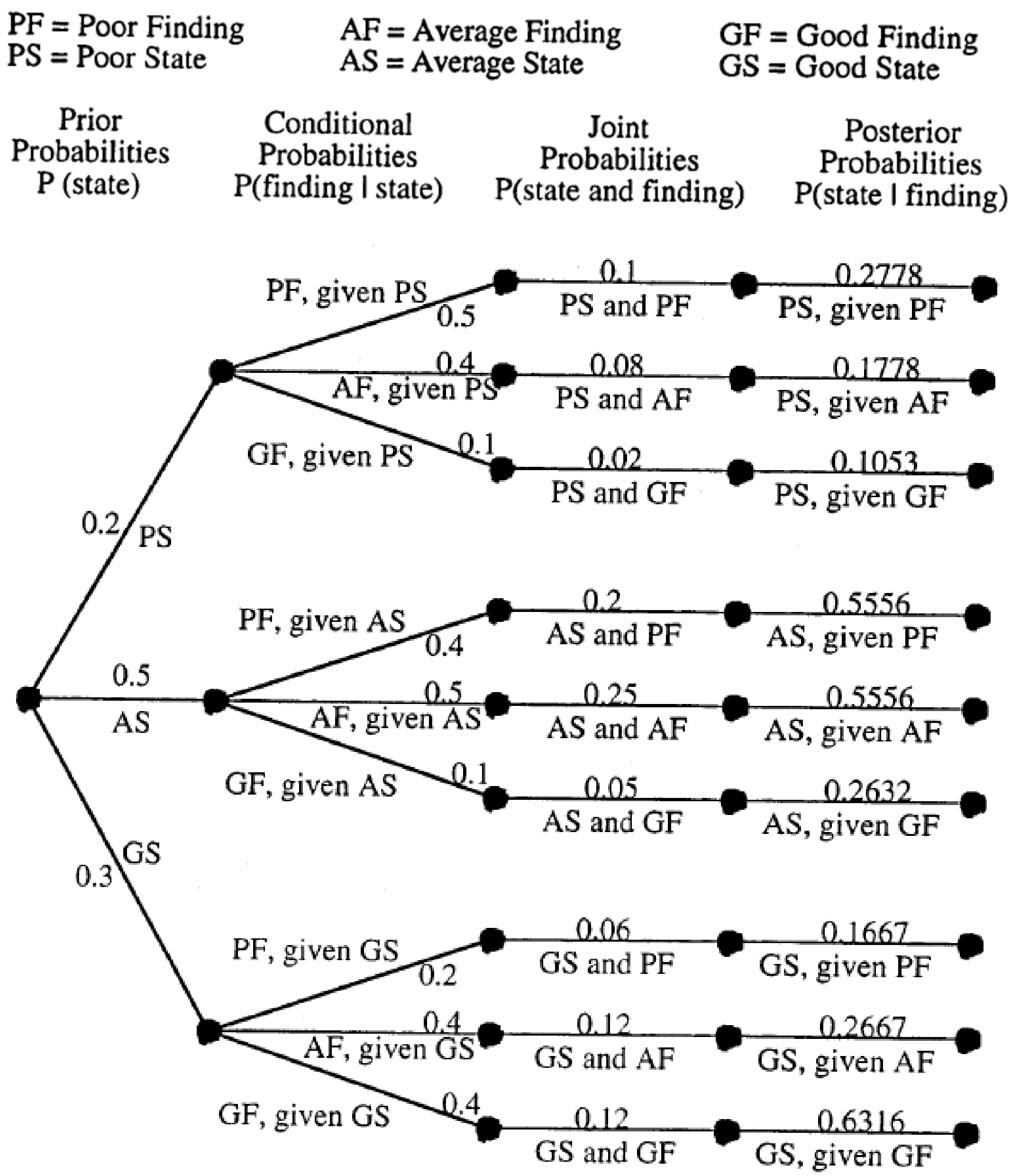
$$0.2(0) + 0.3(10,000) + 0.4(20,000) = 11,000$$

Expected Payoff without Information: $8,000$

EVPI $= 11,000 - 8,000 = \$3,000$

Hence, the credit-rating organization should not be used.

(d)



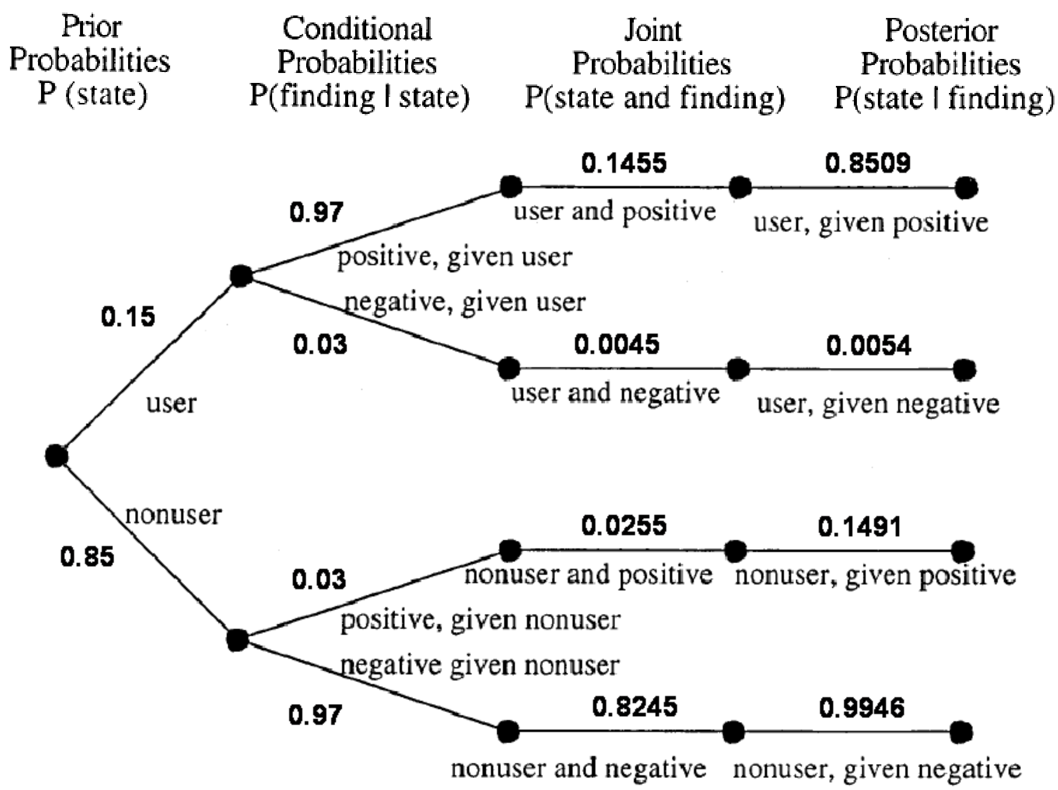
(e)

Data:		P(Finding | State)		
State of	Prior	Finding		
Nature	Probability	Poor	Average	Good
Poor	0.2	0.5	0.4	0.1
Average	0.5	0.4	0.5	0.1
Good	0.3	0.2	0.4	0.4

Posterior		P(State | Finding)		
Probabilities:		State of Nature		
Finding	P(Finding)	Poor	Average	Good
Poor	0.36	0.27778	0.55556	0.16667
Average	0.45	0.17778	0.55556	0.26667
Good	0.19	0.10526	0.26316	0.63158

(f) Vincent should not get the credit rating and extend credit.

15.3-10.



(a) Given that the test is positive, the athlete is a drug user with probability 0.8509.

(b) Given that the test is positive, the athlete is not a drug user with probability 0.1491.

(c) Given that the test is negative, the athlete is a drug user with probability 0.0054.

(d) Given that the test is negative, the athlete is not a drug user with probability 0.9946.

(e) The answers in Excel agree with those found in parts (a), (b), (c), and (d).

Data:		P(Finding | State)	
State of	Prior	Finding	
Nature	Probability	Positive	Negative
Positive	0.15	0.97	0.03
Negative	0.85	0.03	0.97

Posterior		P(State | Finding)	
Probabilities:		State of Nature	
Finding	P(Finding)	Positive	Negative
Positive	0.171	0.85088	0.14912
Negative	0.829	0.00543	0.99457

15-3.11.

(a)

	State of Nature	
Alternative	Successful	Unsuccessful
Develop New Product	$1,500,000$	$-1,800,000$
Not Develop New Product	0	0
Prior Probability	0.667	0.333

(b) Choose to develop new product with expected payoff $\$400,000$.

	State of Nature		Exp.
Alternative	Successful	Unsuccessful	Payoff
Develop New Product	$1,500,000$	$-1,800,000$	$400,000$
Not Develop New Product	0	0	0
Prior Probability	0.667	0.333	

(c)

	State of Nature	
Alternative	Successful	Unsuccessful
Develop New Product	$1,500,000$	$-1,800,000$
Not Develop New Product	0	0
Prior Probability	0.667	0.333
Maximum Payoff	$1,500,000$	0

Expected Payoff with Perfect Information: $0.667(1,500,000) + 0.333(0) = 1,000,000$

Expected Payoff without Information: $400,000$

EVPI $= 1,000,000 - 400,000 = \$600,000$

This indicates that consideration should be given to conducting the market survey.

(d)

Data:		P(Finding | State)	
State of	Prior	Finding	
Nature	Probability	Successful	Unsuccessful
Successful	0.66666667	0.8	0.2
Unsuccessful	0.33333333	0.3	0.7

Posterior		P(State | Finding)	
Probabilities:		State of Nature	
Finding	P(Finding)	Successful	Unsuccessful
Successful	0.63333333	0.8421053	0.15789474
Unsuccessful	0.36666667	0.3636364	0.63636364

(e)

Action	Prediction	Expected Payoff
Develop product	Successful	$[0.8421(1.5) + 0.1579(-1.8)] \cdot 10^6 = \$979,000$
Not develop product	Successful	0
Develop product	Unsuccessful	$[0.3636(1.5) + 0.6364(-1.8)] \cdot 10^6 = -\$600,000$
Not develop product	Unsuccessful	0

It is optimal to develop the product if it is predicted to be successful and to not develop otherwise. Let S be the event that the product is predicted to be successful. Then,

$$\mathrm{P(S)} = \mathrm{P(S}|\theta_1)\mathrm{P}(\theta_1) + \mathrm{P(S}|\theta_2)\mathrm{P}(\theta_2) = 0.8(2/3) + 0.2(1/3) = 0.6.$$

The expected payoff given the information is $0.6(979,000) + 0.4(0) = \$587,000$, so

$$\text{EVE} = 587,000 - 400,000 = \$187,000 < \$300,000 = \text{Cost of survey.}$$

Hence, the optimal strategy is to not conduct the market survey, and to market the product.

15.3-12.

(a)

	State of Nature	
Alternative	$p = 0.05$	$p = 0.25$
Screen	$-1,500$	$-1,500$
Not Screen	-750	$-3,750$
Prior Probability	0.8	0.2

(b) Choose to not screen with expected loss $\$1,350$.

	State of Nature		Exp.
Alternative	$p = 0.05$	$p = 0.25$	Payoff
Screen	$-1,500$	$-1,500$	$-1,500$
Not Screen	-750	$-3,750$	$-1,350$
Prior Probability	0.8	0.2	

(c)

	State of Nature	
Alternative	$p = 0.05$	$p = 0.25$
Screen	$-1,500$	$-1,500$
Not Screen	-750	$-3,750$
Prior Probability	0.8	0.2
Maximum Payoff	-750	$-1,500$

Expected Payoff with Perfect Information: $0.8(-750) + 0.2(-1,500) = -900$

Expected Payoff without Information: $-1,350$

EVPI $= -900 - (-1,350) = \$450$

This indicates that consideration should be given to inspecting the single item.

(d)

Data:		P(Finding | State)	
		Finding	
State of Nature	Prior Probability	Defective	Nondefective
p=0.05	0.8	0.05	0.95
p=0.25	0.2	0.25	0.75

Posterior Probabilities:		P(State | Finding)	
		State of Nature	
Finding	P(Finding)	p=0.05	p=0.25
Defective	0.09	0.4444444	0.55555556
Nondefective	0.91	0.8351648	0.16483516

(e) P(defective) $= (0.05)(0.8) + (0.25)(0;2) = 0.09$ and P(nondefective) $= 0.91$

EVE $= [(0.09)(-1500) + (0.91)(-1245)] - (-1350) = 82.05$

Since the cost of the inspection is $\$125 > \$82.05$, inspecting the single item is not worthwhile.

(f) If defective:

EP(screen, $\theta|$defective) $= 0.444(-1500) + 0.556(-1500) = -1500$
EP(no screen, $\theta|$defective) $= 0.444(-750) + 0.556(-3750) = -2418$

If nondefective:

EP(screen, $\theta|$defective) $= -1500$
EP(no screen, $\theta|$defective) $= 0.835(-750) + 0.165(-3750) = -1245$

Hence, the optimal policy with experimentation is to screen if defective is found and not screen if nondefective is found. On the other hand, from part (e), inspecting a single item, in other words experimenting is not worthwhile. Using part (b), the overall optimal policy is to not inspect the single items, to not screen each item in the lot, instead, rework each item that is ultimately found to be defective.

15.3-13.

(a) Say coin 1 tossed: EP $= 0.6(0) + 0.4(-1) = -0.4$
Say coin 2 tossed: EP $= 0.6(-1) + 0.4(0) = -0.6$

The optimal alternative is to say coin 1 is tossed.

(b) If the outcome is heads (H):

$$\text{P(coin 1|H)} = \frac{\text{P(H|coin 1)P(coin 1)}}{\text{P(H|coin 1)P(coin 1)+P(H|coin 2)P(coin 2)}} = \frac{0.3(0.6)}{0.3(0.6)+0.6(0.4)} = \frac{3}{7}$$

$$\text{P(coin 2|H)} = \frac{4}{7}$$

Say coin 1: EP $= \frac{3}{7}(0) + \frac{4}{7}(-1) = -\frac{4}{7}$

Say coin 2: EP $= \frac{3}{7}(-1) + \frac{4}{7}(0) = -\frac{3}{7}$

The optimal alternative is to say coin 2.

If the outcome is tails (T):

$$\text{P(coin 1|T)} = \frac{\text{P(T|coin 1)P(coin 1)}}{\text{P(T|coin 1)P(coin 1)+P(T|coin 2)P(coin 2)}} = \frac{0.7(0.6)}{0.7(0.6)+0.4(0.4)} = 0.7241$$

$$\text{P(coin 2|T)} = 0.2759$$

Say coin 1: EP $= 0.7241(0) + 0.2759(-1) = -0.2759$

Say coin 2: EP $= 0.7241(-1) + 0.2759(0) = -0.7241$

The optimal alternative is to say coin 1.

15.3-14.

(a)

	State of Nature	
Alternative	Coin 1	Coin 2
Predict 0 H	22.5	122.5
Predict 1 H	105	105
Predict 2 H	122.5	22.5
Prior probabilities	0.5	0.5

Predict 0 H: EP $= 0.5(22.5) + 0.5(122.5) = 72.5$
Predict 1 H: EP $= 0.5(105) + 0.5(105) = 105$
Predict 2 H: EP $= 0.5(122.5) + 0.5(22.5) = 72.5$

The optimal alternative is to predict one heads with expected payoff $105.

(b)

Data:		P(Finding | State)		
State of	Prior	Finding		
Nature	Probability	Heads	Tails	
Coin 1	0.5	0.7	0.3	
Coin 2	0.5	0.3	0.7	

Posterior		P(State | Finding)		
Probabilities:		State of Nature		
Finding	P(Finding)	Coin 1	Coin 2	
Heads	0.5	0.7	0.3	
Tails	0.5	0.3	0.7	

(c) If the outcome is heads (H):

$$\begin{aligned}
&\text{Predict 0 H:} \quad \text{EP} = 0.7(22.5) + 0.3(122.5) = 52.5 \\
&\text{Predict 1 H:} \quad \text{EP} = 0.7(105) + 0.3(105) = 105 \\
&\text{Predict 2 H:} \quad \text{EP} = 0.7(122.5) + 0.3(22.5) = 92.5
\end{aligned}$$

The optimal alternative is to predict one heads.

If the outcome is tails (T):

$$\begin{aligned}
&\text{Predict 0 H:} \quad \text{EP} = 0.3(22.5) + 0.7(122.5) = 92.5 \\
&\text{Predict 1 H:} \quad \text{EP} = 0.3(105) + 0.7(105) = 105 \\
&\text{Predict 2 H:} \quad \text{EP} = 0.3(122.5) + 0.7(22.5) = 52.5
\end{aligned}$$

The optimal alternative is to predict one heads.

Since EP(H) $=$ EP(T) $= 105$, the expected payoff is \$$105$.

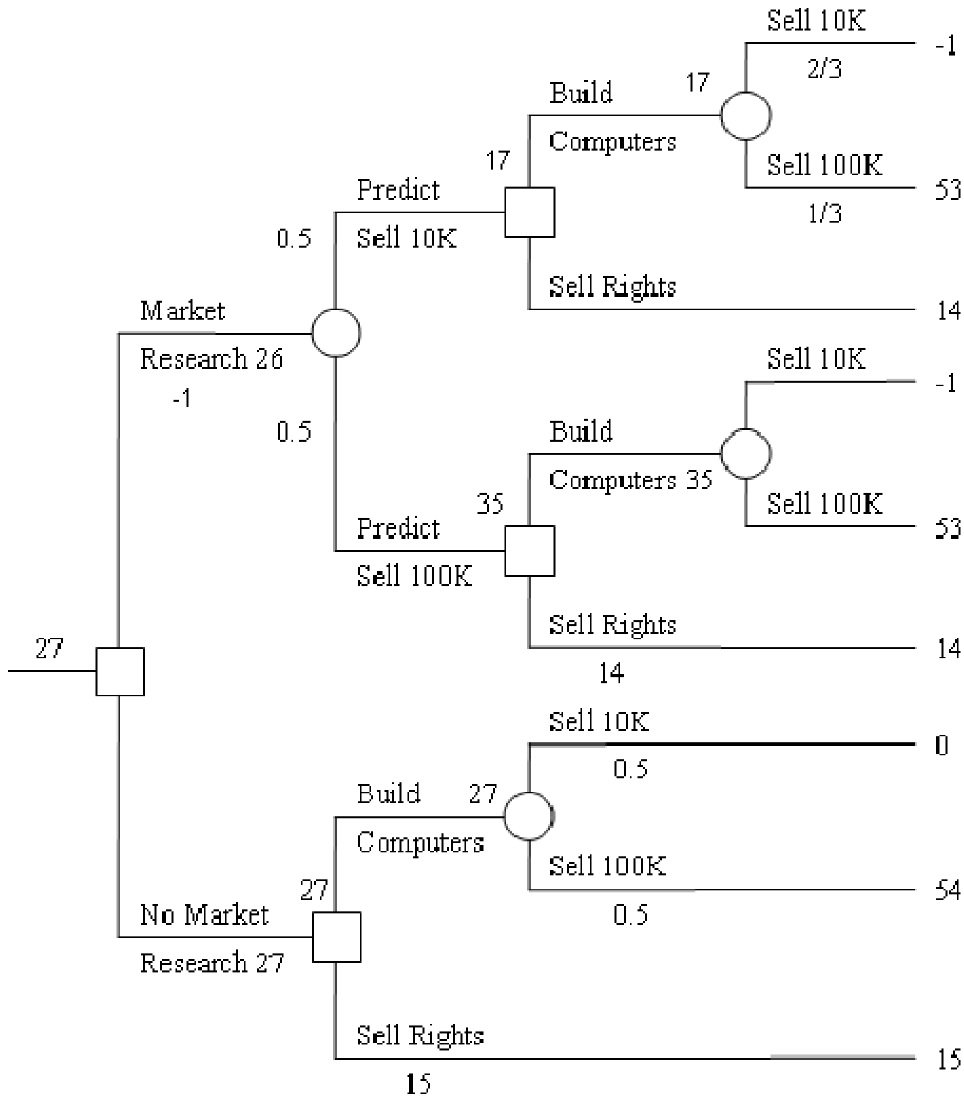
(d) EVE $= 105 - 105 = \$0 < \$30$, so it is better to not pay for the experiment and choose to predict either one or two heads.

15.4-1.

Driven by "the pressure to reduce costs and deliver high-impact technology quickly while justifying investments" [p. 57], Westinghouse initiated this study to evaluate R and D efforts effectively. At any point in time, the firm chooses between launching, delaying and abandoning an innovation. When the launch is delayed, there is a chance of losing the opportunity. R and D is hence treated as a call option with flexibility. The value of the innovation and the optimal decision rule in subsequent stages are found by using dynamic programming. This value is then used in the analysis of the decision tree constructed to find the present value of the project. In this tree, decisions consist of whether to fund or not at different stages and each decision node is followed by a chance node that represents either a technical milestone or strategic fit. Sensitivity analysis is performed to ensure robustness of the model.

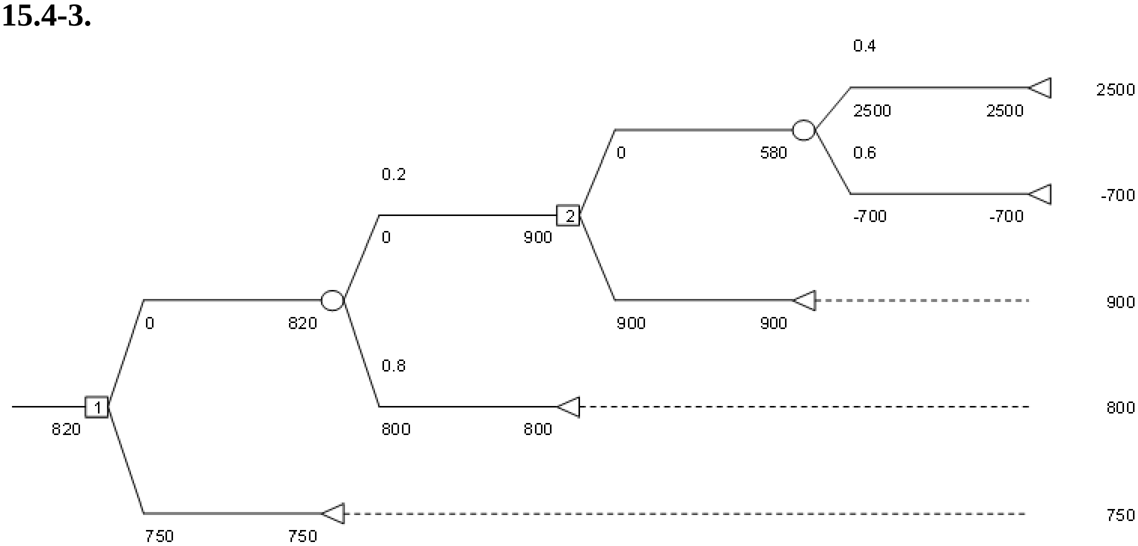
As a result of this study, explicit decision rules for funding R and D projects are obtained. Including flexibility in the model yields a more realistic model. The new system helps identifying cost-effective research portfolios with simplified data acquisition and easy implementation.

15.4-2.



The optimal policy is to build the computers without doing market research.

15.4-3.



15.4-4.

(a)

	State of Nature	
Alternative	W	L
Hold Campaign	3	−2
Not Hold Campaign	0	0
Prior Probability	0.6	0.4

(b) Choose to hold the campaign with expected payoff $1 million.

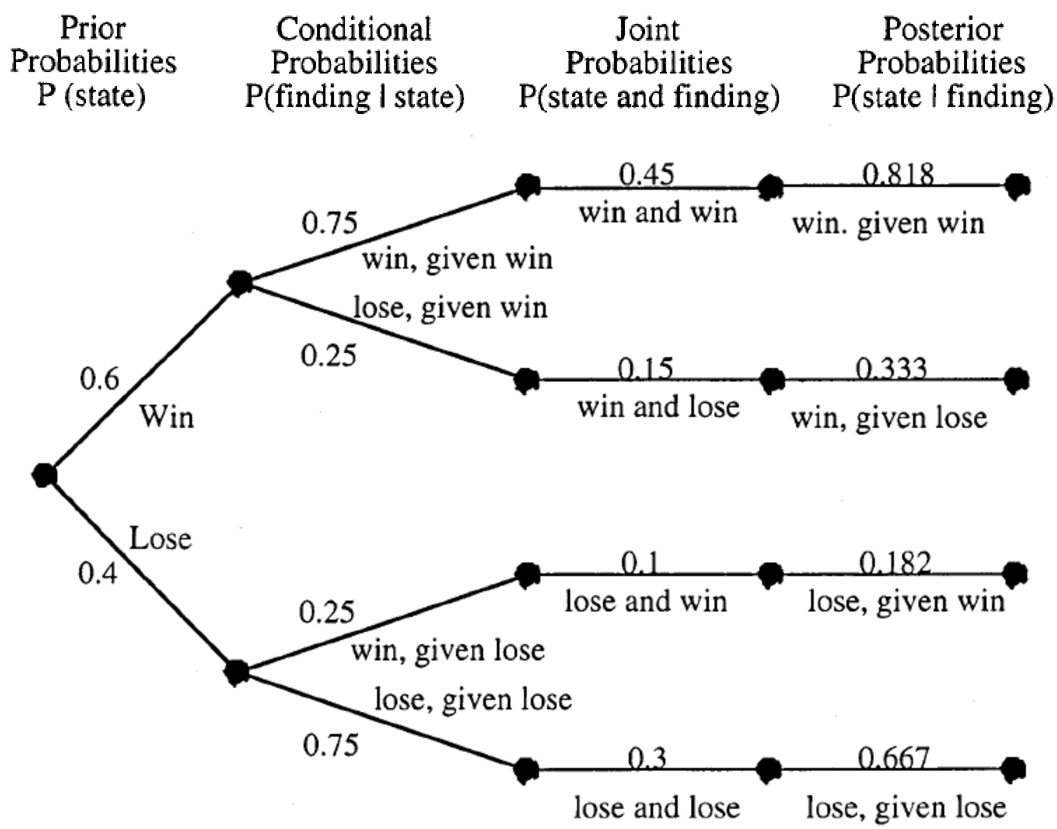
	State of Nature		Exp.
Alternative	W	L	Payoff
Hold Campaign	3	−2	1
Not Hold Campaign	0	0	0
Prior Probability	0.6	0.4	

(c)

	State of Nature	
Alternative	W	L
Hold Campaign	3	−2
Not Hold Campaign	0	0
Prior Probability	0.6	0.4
Maximum Payoff	3	0

Expected Payoff with Perfect Information: $0.6(3) + 0.4(0) = 1.8$
Expected Payoff without Information: 1
EVPI $= 1.8 - 1 =$ $0.8 million

(d)

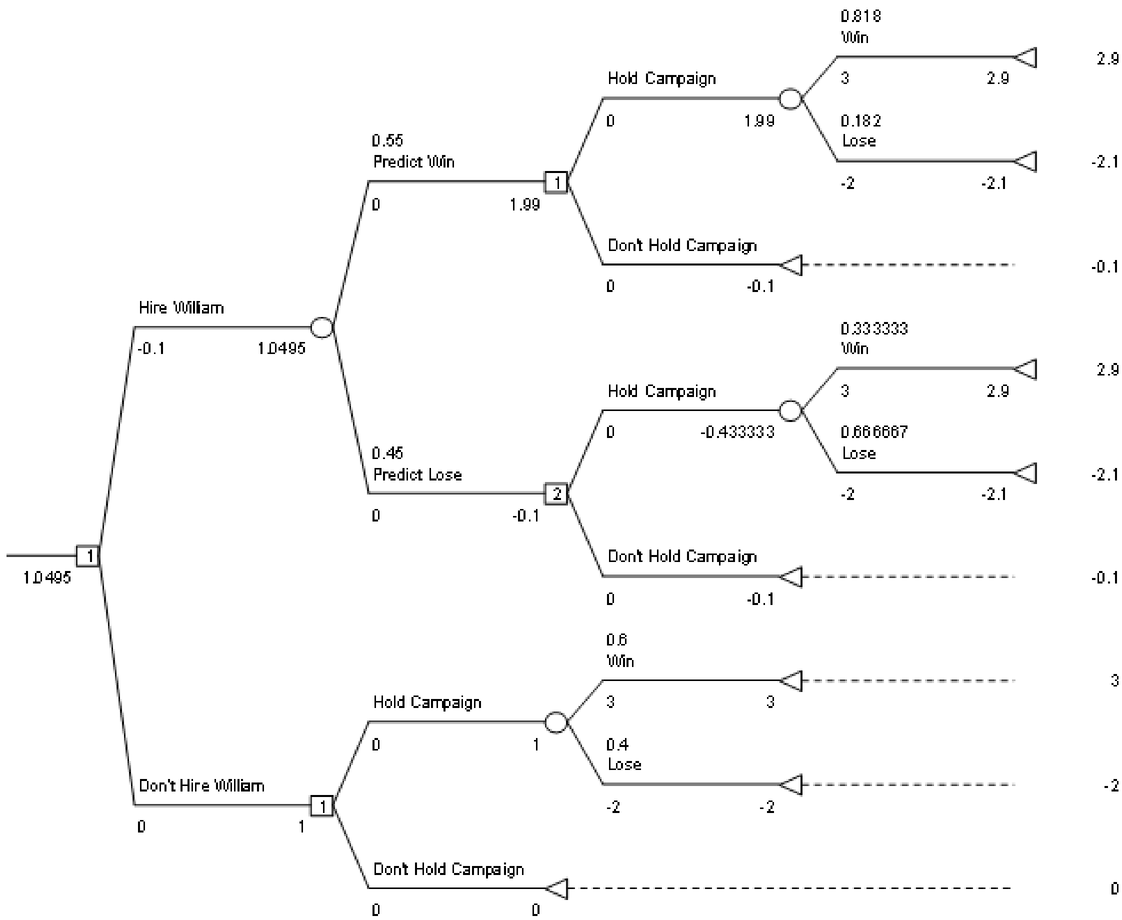


(e)

<table>
<tr><td>Data:</td><td></td><td colspan="2">P(Finding | State)</td></tr>
<tr><td>State of</td><td>Prior</td><td colspan="2">Finding</td></tr>
<tr><td>Nature</td><td>Probability</td><td>Win</td><td>Lose</td></tr>
<tr><td>Win</td><td>0.6</td><td>0.75</td><td>0.25</td></tr>
<tr><td>Lose</td><td>0.4</td><td>0.25</td><td>0.75</td></tr>
</table>

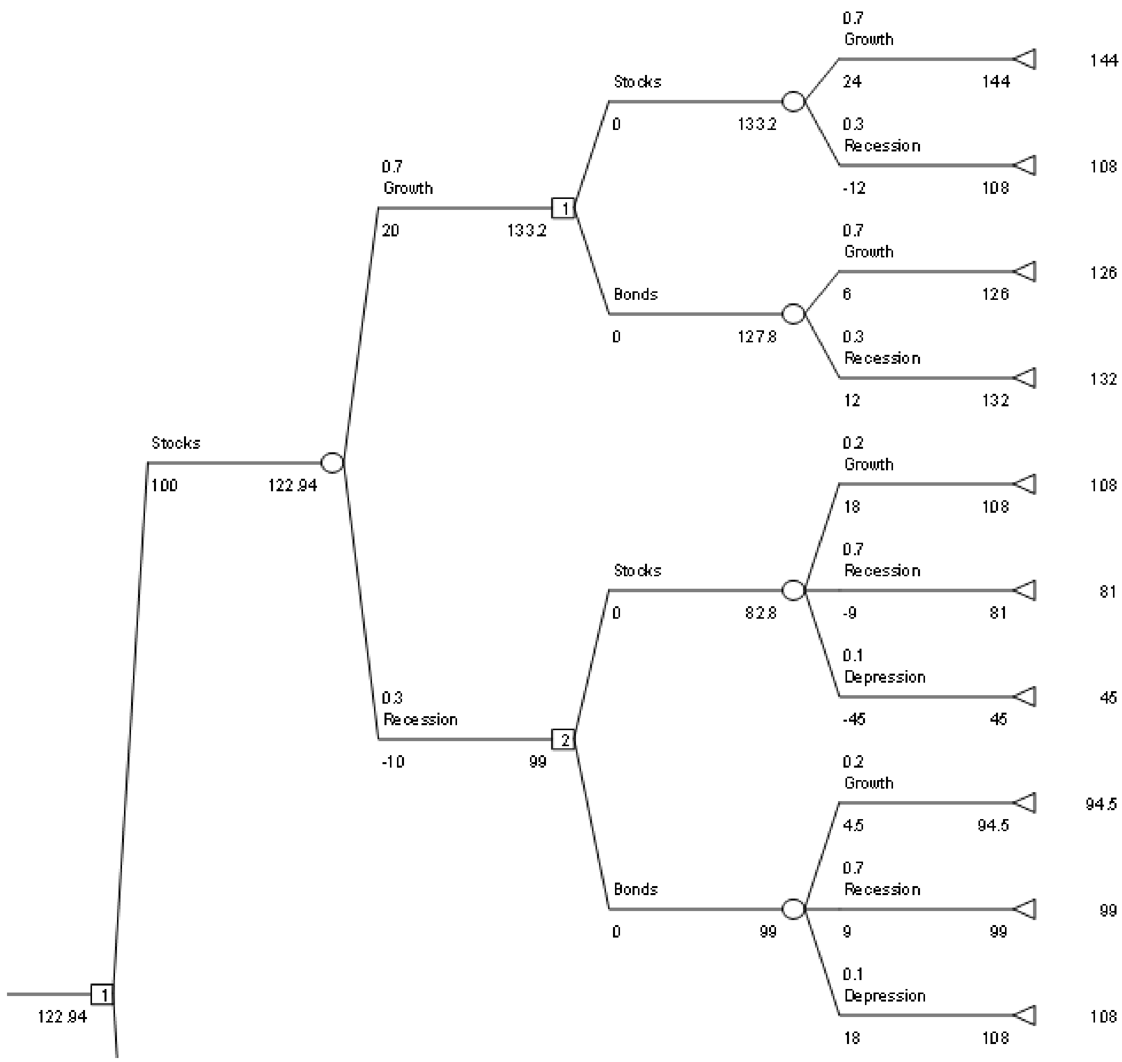
<table>
<tr><td>Posterior</td><td></td><td colspan="2">P(State | Finding)</td></tr>
<tr><td>Probabilities:</td><td></td><td colspan="2">State of Nature</td></tr>
<tr><td>Finding</td><td>P(Finding)</td><td>Win</td><td>Lose</td></tr>
<tr><td>Win</td><td>0.55</td><td>0.81818</td><td>0.1818</td></tr>
<tr><td>Lose</td><td>0.45</td><td>0.33333</td><td>0.6667</td></tr>
</table>

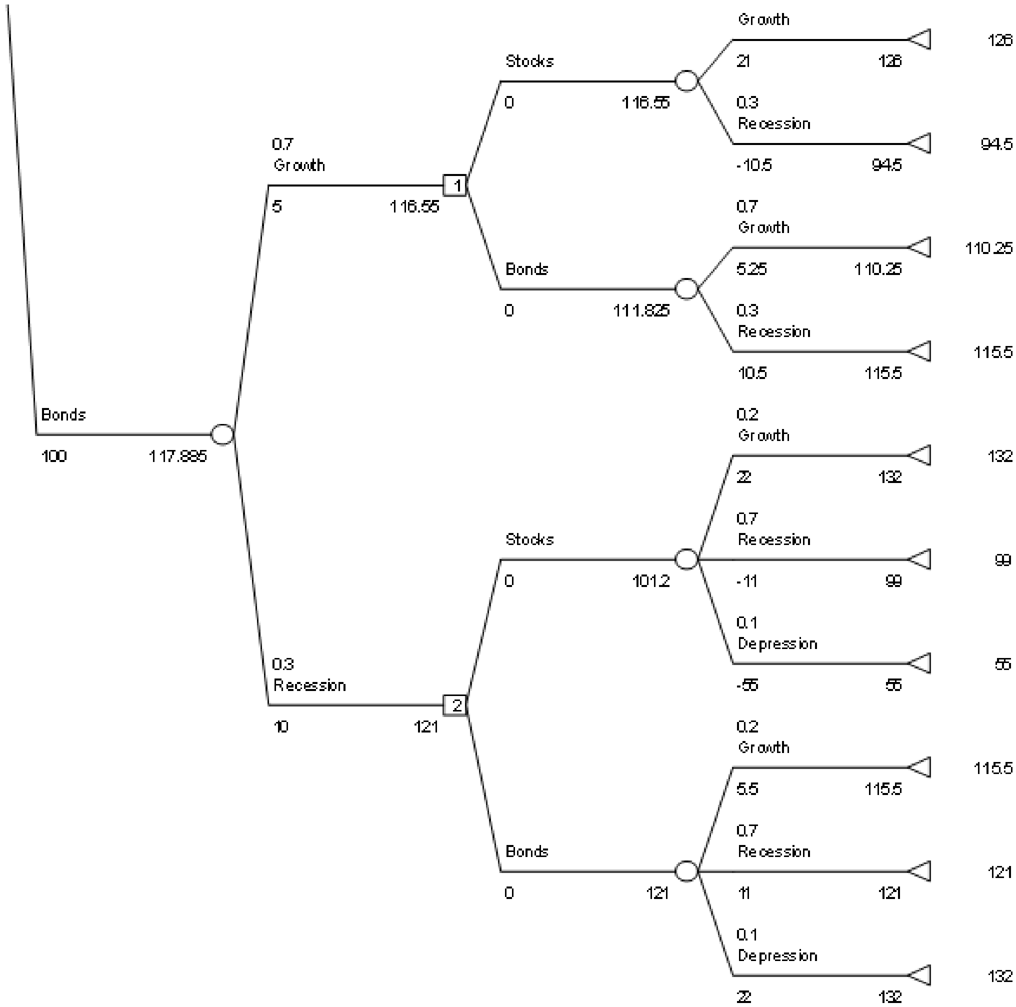
(f) Leland University should hire William. If he predicts a winning season, then they should hold the campaign and if he predicts a losing season, then they should not hold the campaign.



15.4-5.

(a)

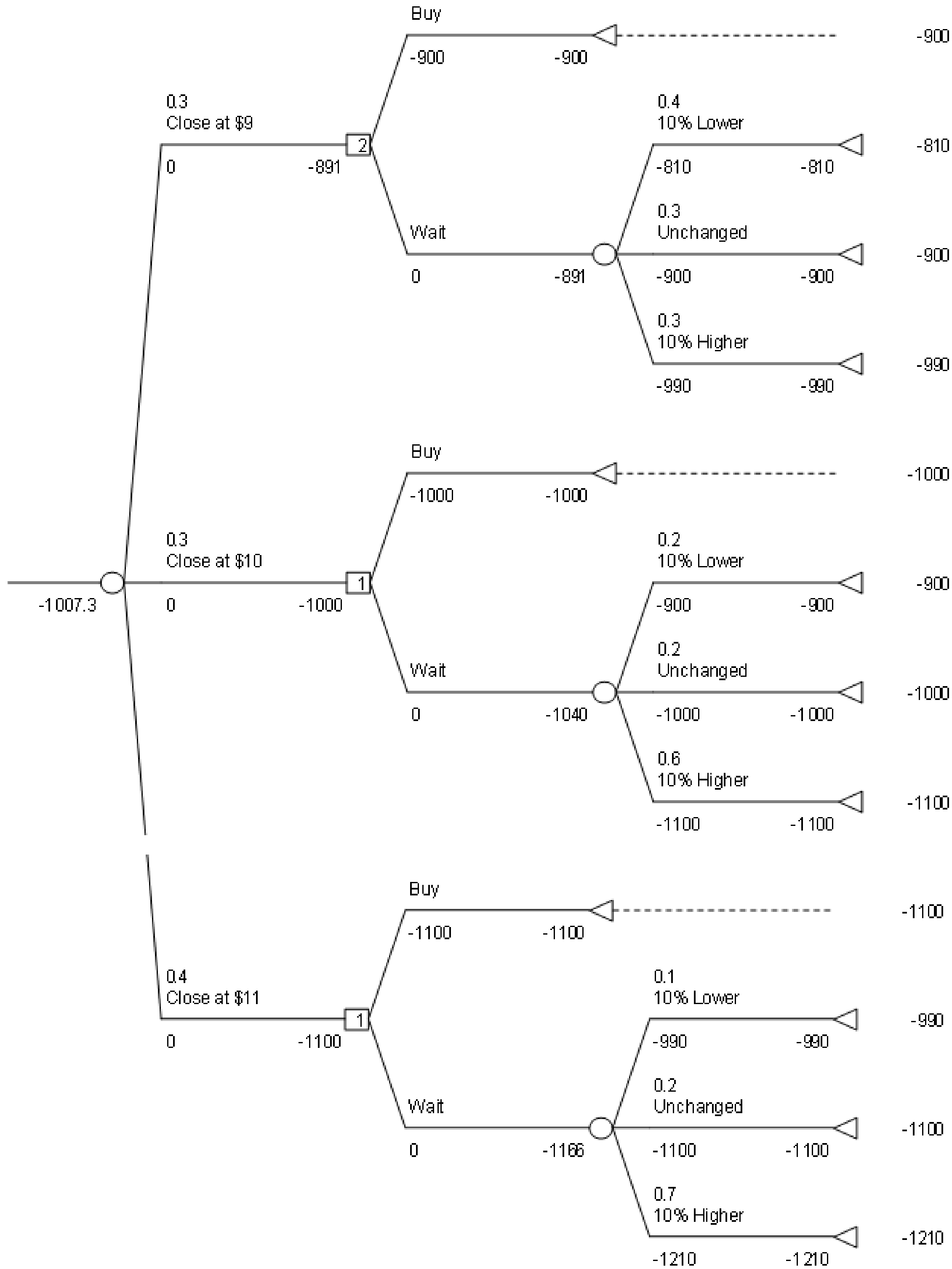

0.7
Growth
24
144
144
Stocks
0
133.2
0.3
Recession
-12
108
108
0.7
Growth
20
133.2
0.7
Growth
6
126
126
Bonds
0
127.8
0.3
Recession
12
132
132
Stocks
100
122.94
0.2
Growth
18
108
108
Stocks
0
82.8
0.7
Recession
-9
81
81
0.1
Depression
-45
45
45
0.3
Recession
-10
99
0.2
Growth
4.5
94.5
94.5
Bonds
0
99
0.7
Recession
9
99
99
0.1
Depression
18
108
108
122.94



(b) The comptroller should invest in stocks the first year. If growth occurs in the first year, then she should invest in stocks again the second year. If recession occurs in the first year, then she should invest in bonds the second year.

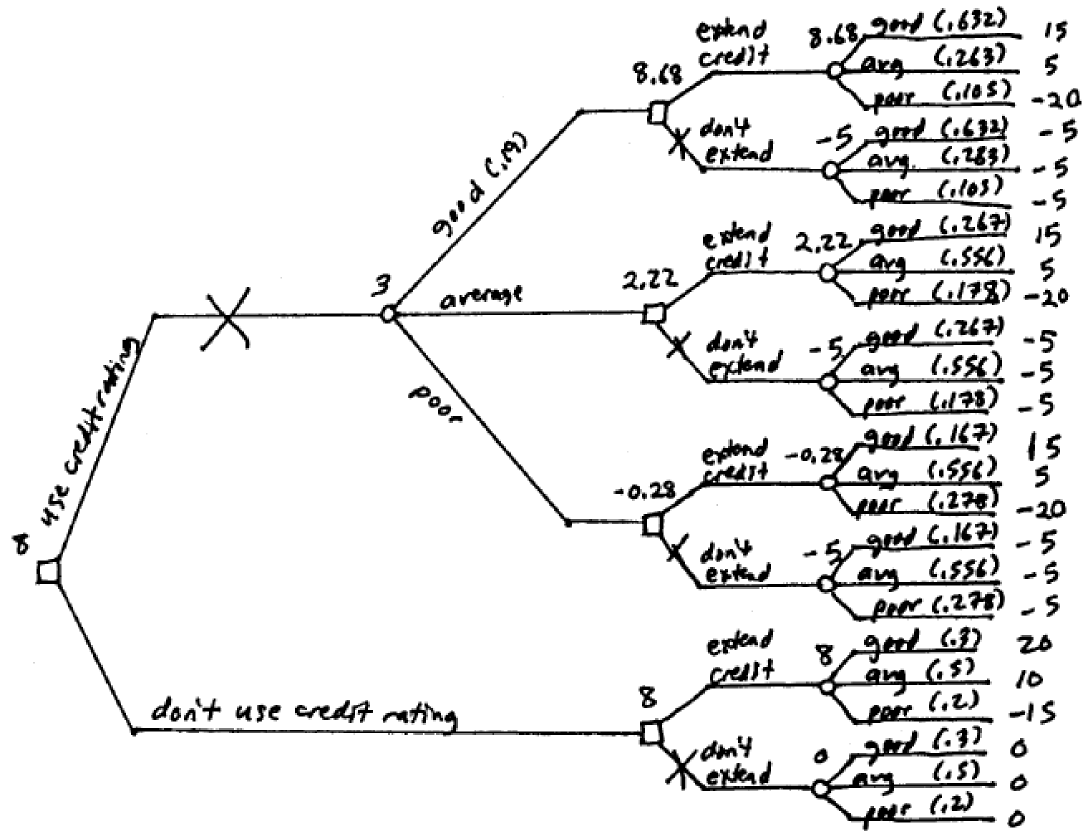
15.4-6.

The optimal policy is to wait until Wednesday to buy if the price is $9 on Tuesday. If the price is $10 or $11 on Tuesday, then buying on Tuesday is optimal.



15.4-7.

(a)



(b) Prior Distribution:

	θ_1	θ_2	θ_3
$P_\theta(k)$	0.2	0.5	0.3

	$Q_{X\|\theta=k}(x)$		
x	θ_1	θ_2	θ_3
X_1	0.5	0.4	0.2
X_2	0.4	0.5	0.4
X_3	0.1	0.1	0.4

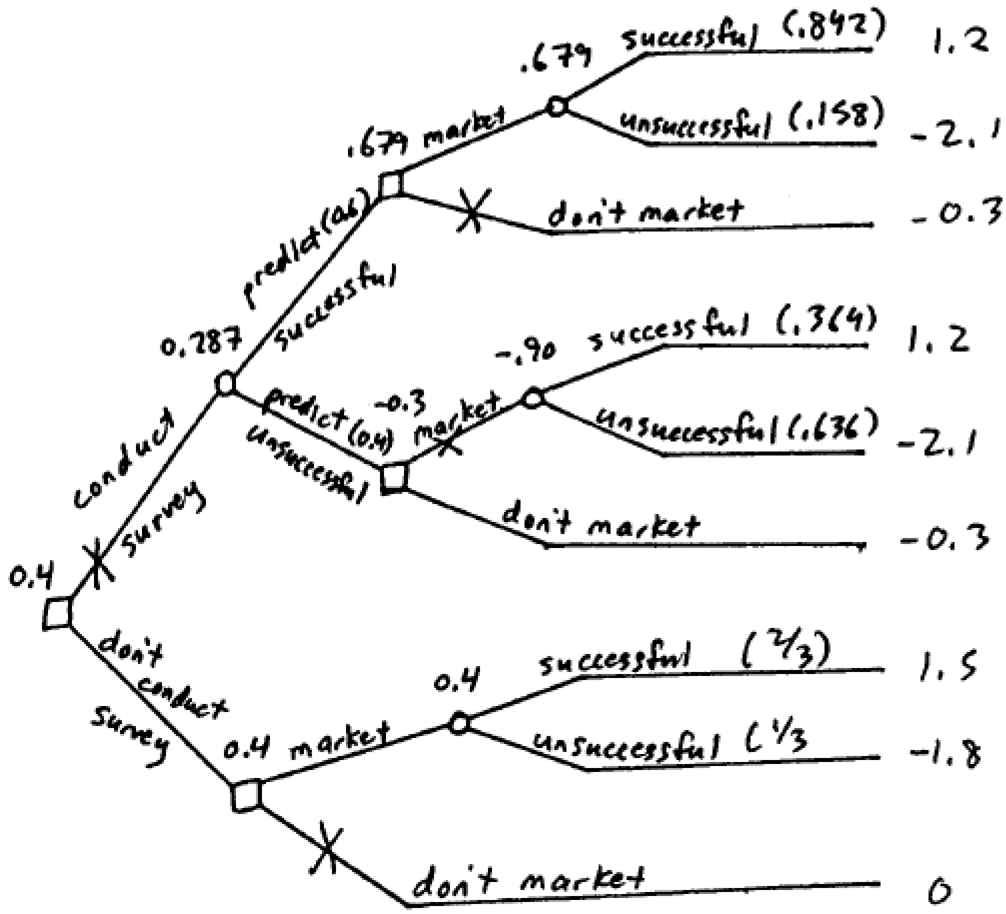
Posterior Distribution:

	$h_{\theta\|X=x}(k)$		
x	θ_1	θ_2	θ_3
X_1	0.278	0.556	0.167
X_2	0.178	0.556	0.267
X_3	0.105	0.263	0.632

(c) It is optimal to not use credit rating, but to extend credit, see part (a).

15.4-8.

(a)



(b) Prior Distribution:

	θ_1	θ_2
$P_\theta(k)$	0.667	0.333

	$Q_{X\|\theta=k}(x)$	
x	θ_1	θ_2
X_1	0.8	0.3
X_2	0.2	0.7

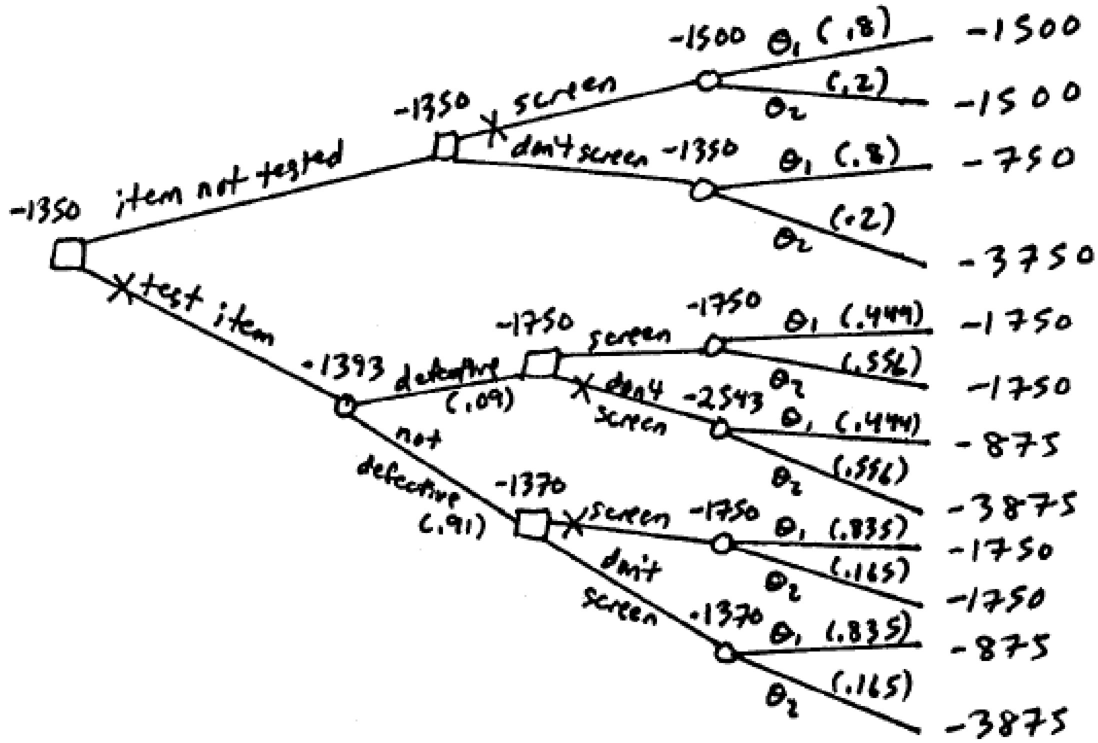
Posterior Distribution:

	$h_{\theta\|X=x}(k)$	
x	θ_1	θ_2
X_1	0.842	0.158
X_2	0.364	0.636

(c) It is optimal to not conduct a survey, but to market the new product, see part (a).

15.4-9.

(a)



(b) Prior Distribution:

	θ_1	θ_2
$P_\theta(k)$	0.8	0.2

	$Q_{X\|\theta=k}(x)$	
x	θ_1	θ_2
X_1	0.95	0.75
X_2	0.05	0.25

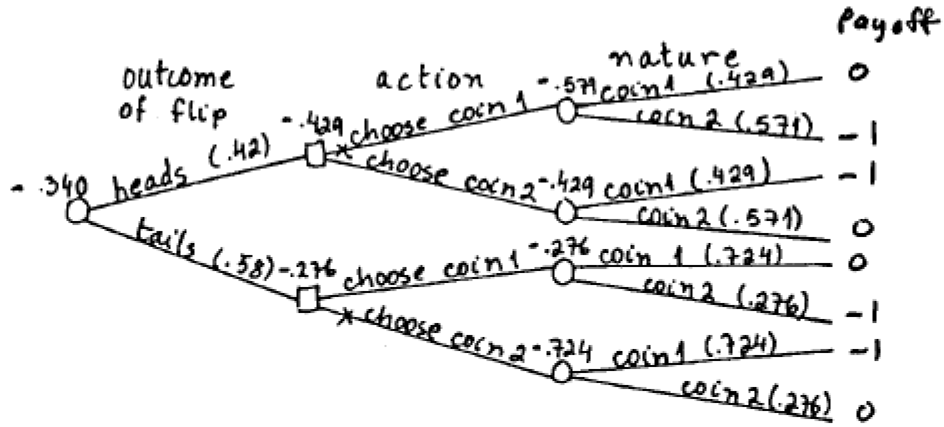
Posterior Distribution:

	$h_{\theta\|X=x}(k)$	
x	θ_1	θ_2
X_1	0.835	0.165
X_2	0.444	0.556

(c) It is optimal to not test and to not screen, see part (a).

15.4-10.

(a)



(b) Prior Distribution:

	θ_1	θ_2
$\mathrm{P}_\theta(k)$	0.6	0.4

	$Q_{X\mid\theta=k}(x)$	
x	θ_1	θ_2
X_1	0.3	0.6
X_2	0.7	0.4

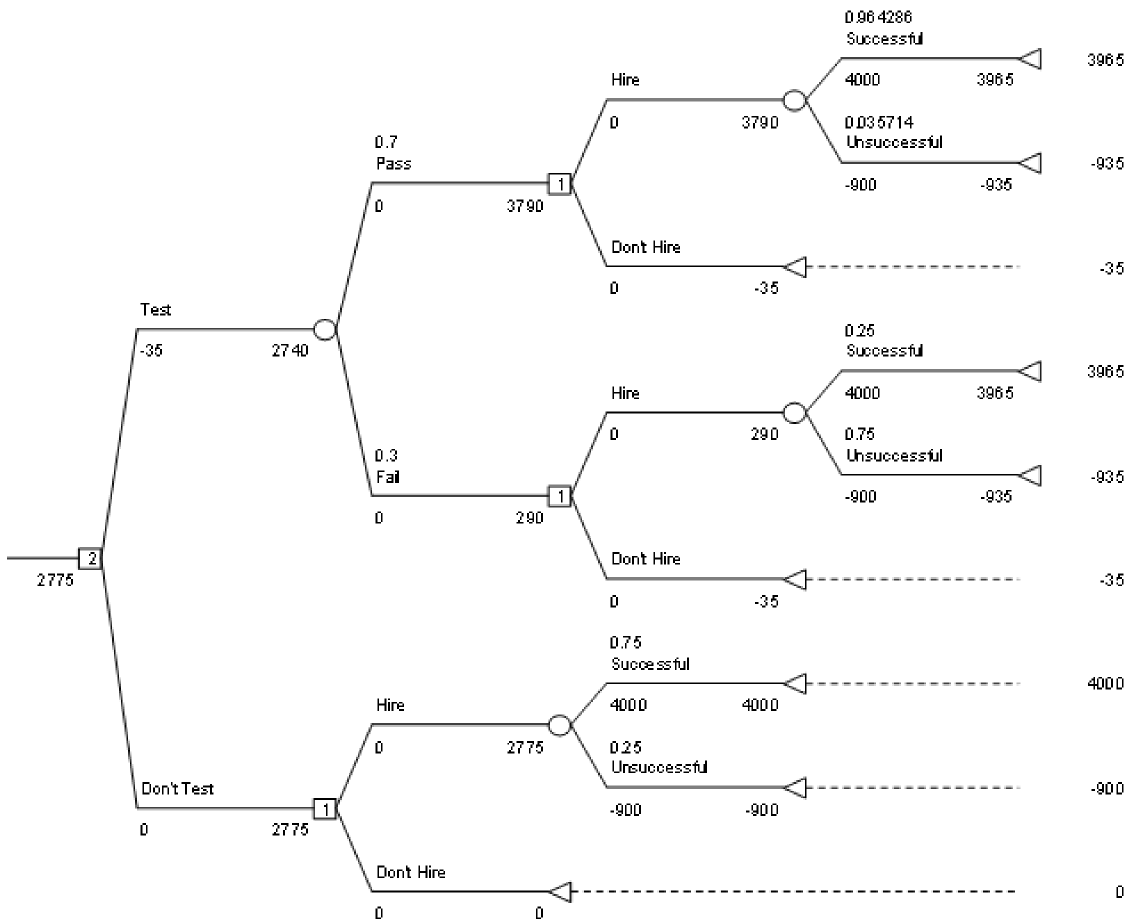
Posterior Distribution:

	$h_{\theta\mid X=x}(k)$	
x	θ_1	θ_2
X_1	0.429	0.571
X_2	0.724	0.276

(c) It is optimal to choose coin 1 if the outcome is tails and coin 2 if the outcome is heads, see part (a).

15.4-11.

(a)



(b)

Data:		P(Finding | State)		
State of	Prior	Finding		
Nature	Probability	Pass	Fail	
Successful	0.75	0.9	0.1	
Unsuccessful	0.25	0.1	0.9	

Posterior		P(State | Finding)		
Probabilities:		State of Nature		
Finding	P(Finding)	Successful	Unsuccessful	
Pass	0.7	0.9642857	0.03571429	
Fail	0.3	0.25	0.75	

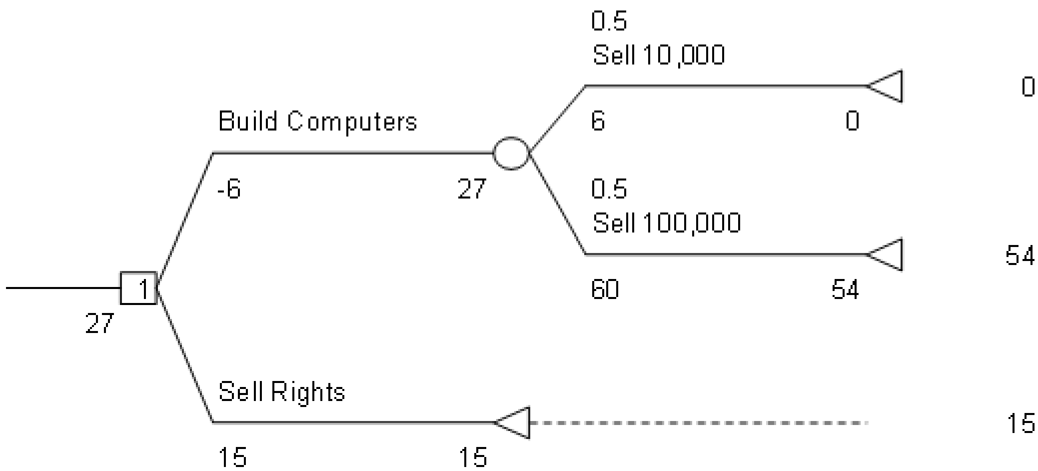
(c) The optimal policy is to not pay for testing and to hire Matthew.

(d) Even if the fee is zero, hiring Matthew without any further investigation is optimal, so Western Bank should not pay anything for the detailed report.

15.5-1.

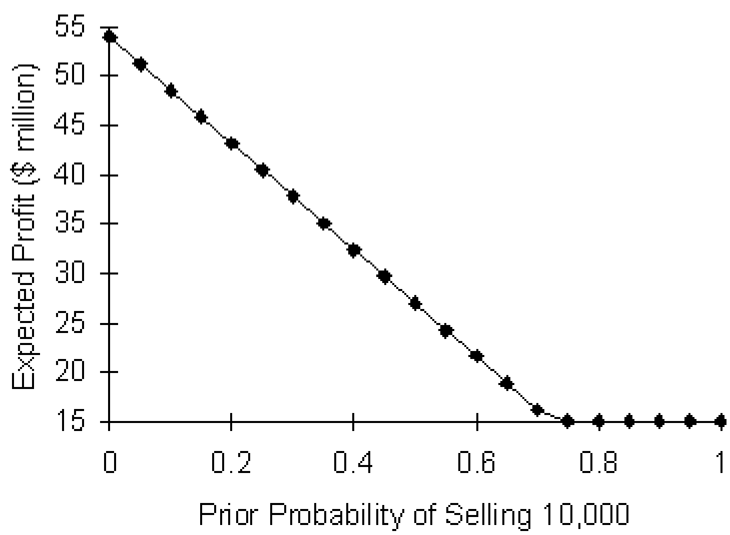
(a)

	State of Nature	
Alternative	Sell 10,000	Sell 100,000
Build Computers	0	54
Sell Rights	15	15



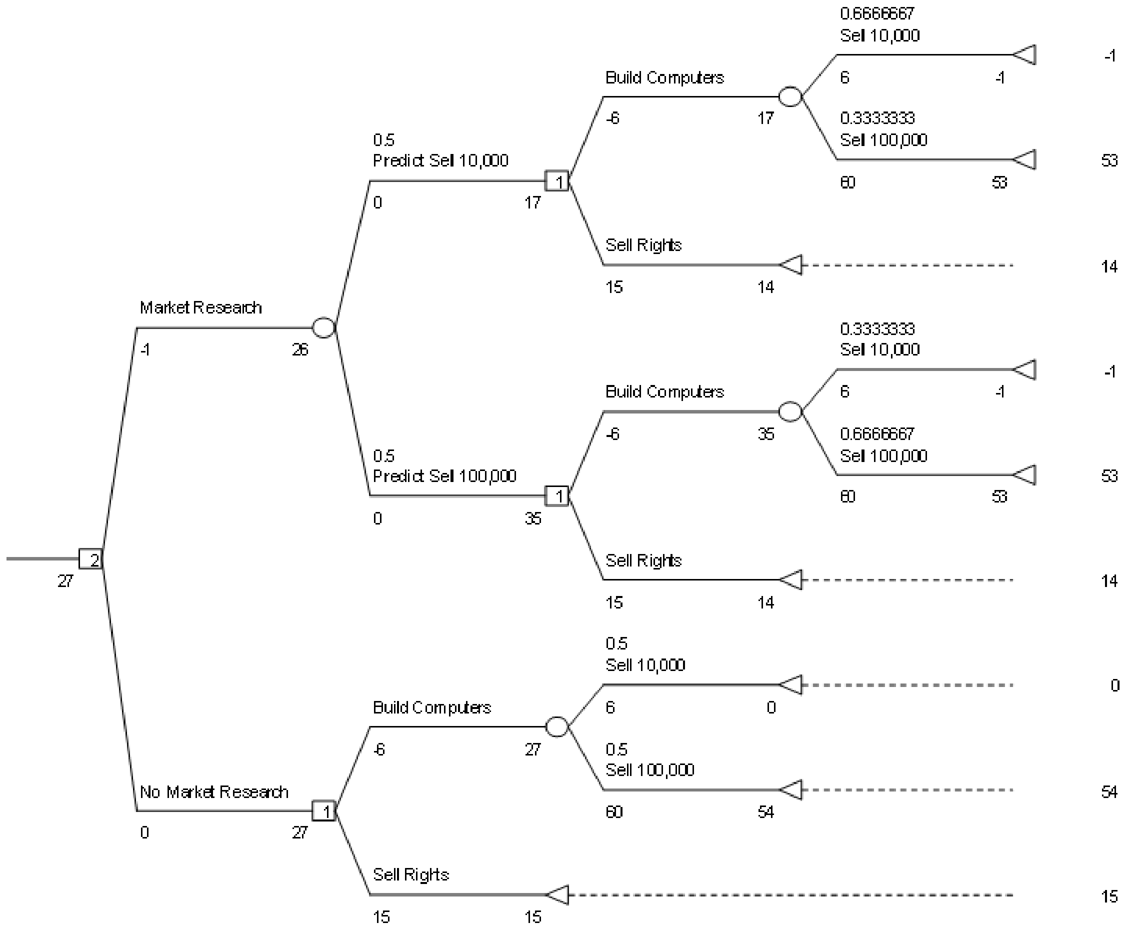
They should build computers with an expected payoff of $27 million.

(b)



15.5-2.

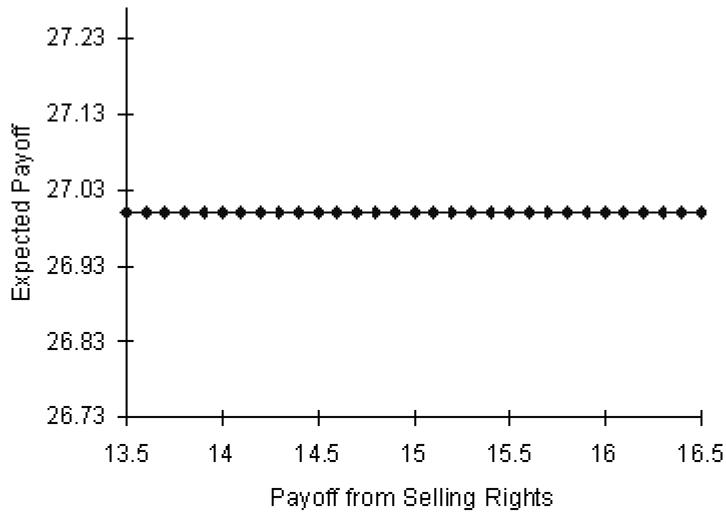
(a) The optimal policy is to not do market research and build the computers. The expected payoff is $27 million.



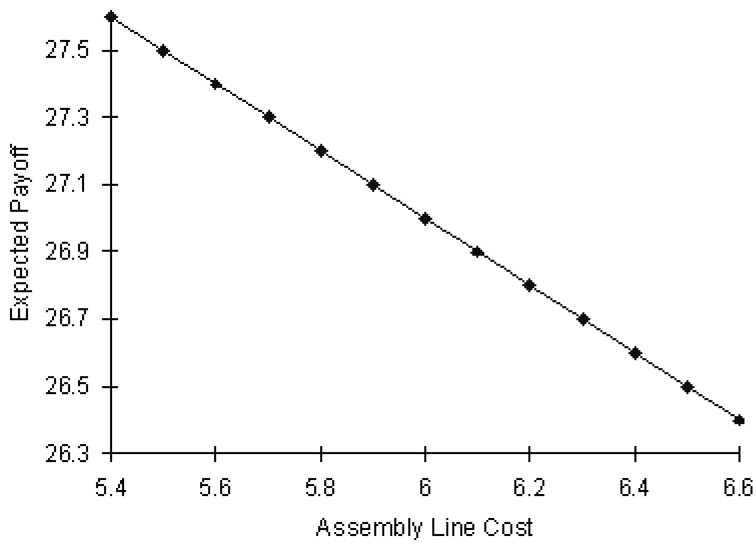
(b) If the rights can be sold for $16.5 or $13.5 million, the optimal policy is still to build the computers with an expected payoff of $27 million. If the cost of setting up the assembly line is $5.4 million or $6.6 million, the optimal policy is still to build the computers with an expected payoff of $27.6 or $26.4 million respectively. If the difference between the selling price and the variable cost of each computer is $540 or $660, the optimal policy is still to build the computers with an expected payoff of $23.7 or $33.3 million respectively. For each combination of financial data, the expected payoff is as shown in the following table. In all cases, the optimal policy is to build the computers without doing market research.

Sell Rights	Cost of Assembly Line	Selling Price – Variable Cost	Expected Payoff
$13.5 million	$5.4 million	$540	$23.4 million
$13.5 million	$5.4 million	$660	$30.9 million
$13.5 million	$6.6 million	$540	$23.1 million
$13.5 million	$6.6 million	$660	$29.7 million
$16.5 million	$5.4 million	$540	$24.3 million
$16.5 million	$5.4 million	$660	$30.9 million
$16.5 million	$6.6 million	$540	$23.1 million
$16.5 million	$6.6 million	$660	$29.7 million

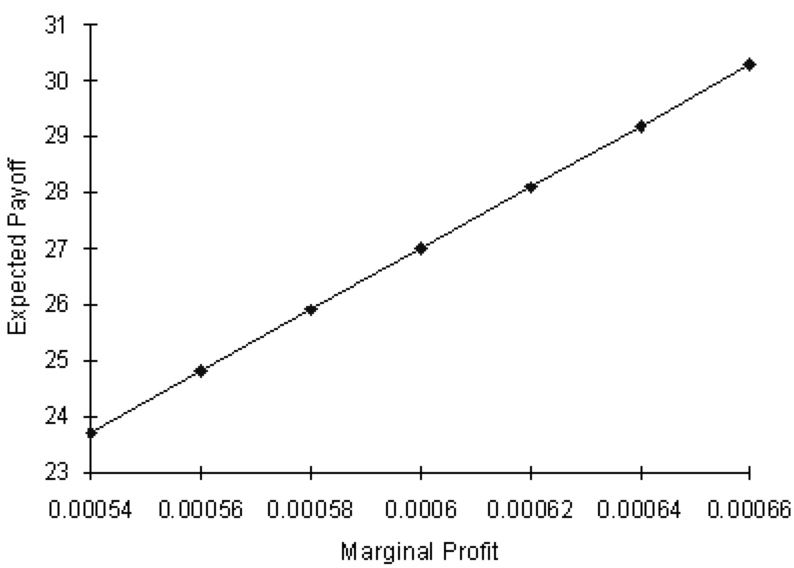
(c)



Expected Payoff
27.23
27.13
27.03
26.93
26.83
26.73
13.5
14
14.5
15
15.5
16
16.5
Payoff from Selling Rights

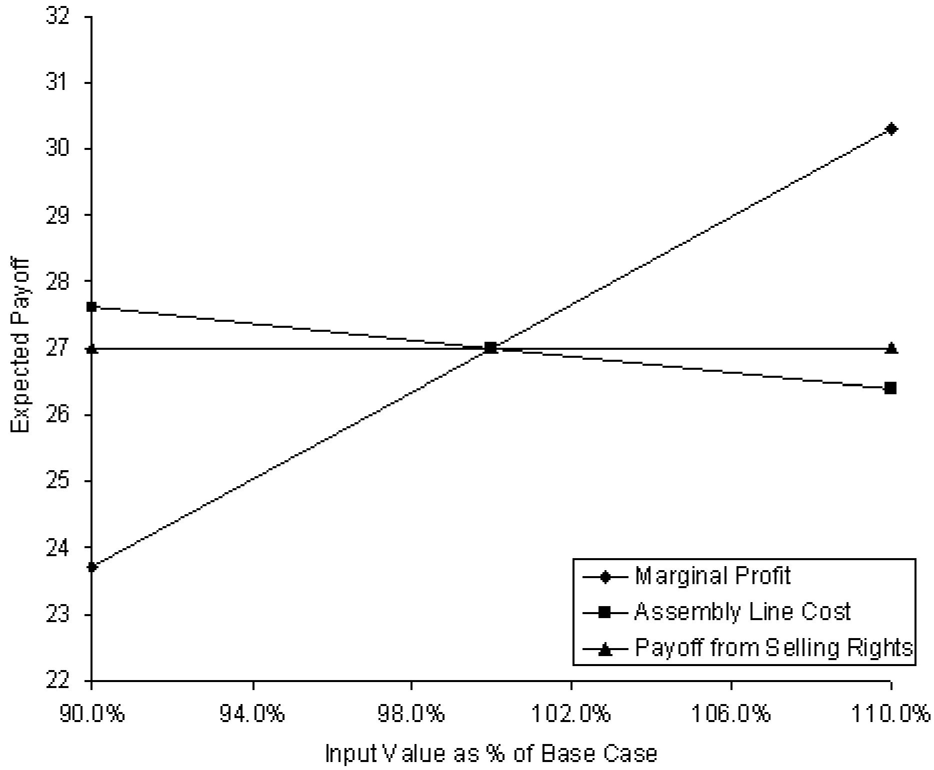


Expected Payoff
27.5
27.3
27.1
26.9
26.7
26.5
26.3
5.4
5.6
5.8
6
6.2
6.4
6.6
Assembly Line Cost

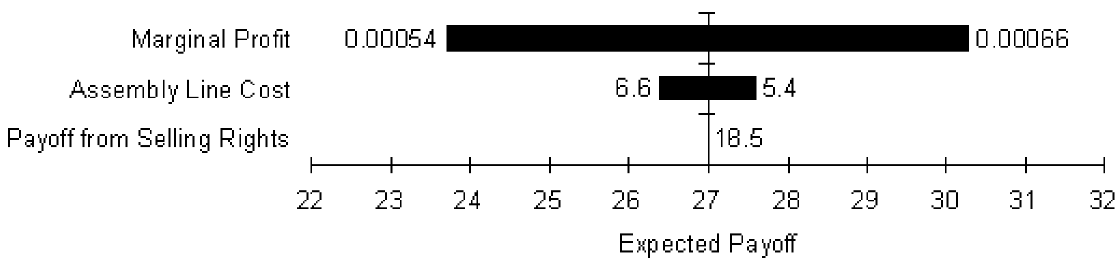


Expected Payoff
31
30
29
28
27
26
25
24
23
0.00054
0.00056
0.00058
0.0006
0.00062
0.00064
0.00066
Marginal Profit

(d)

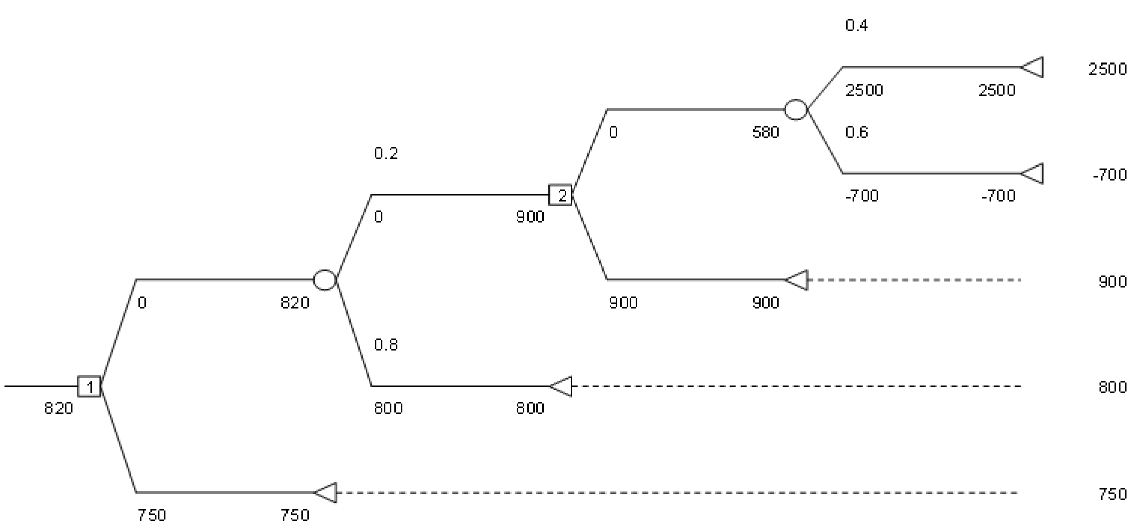


32
31
30
29
28
27
26
25
24
23
22
Expected Payoff
90.0%
94.0%
98.0%
102.0%
106.0%
110.0%
Input Value as % of Base Case
Marginal Profit
Assembly Line Cost
Payoff from Selling Rights



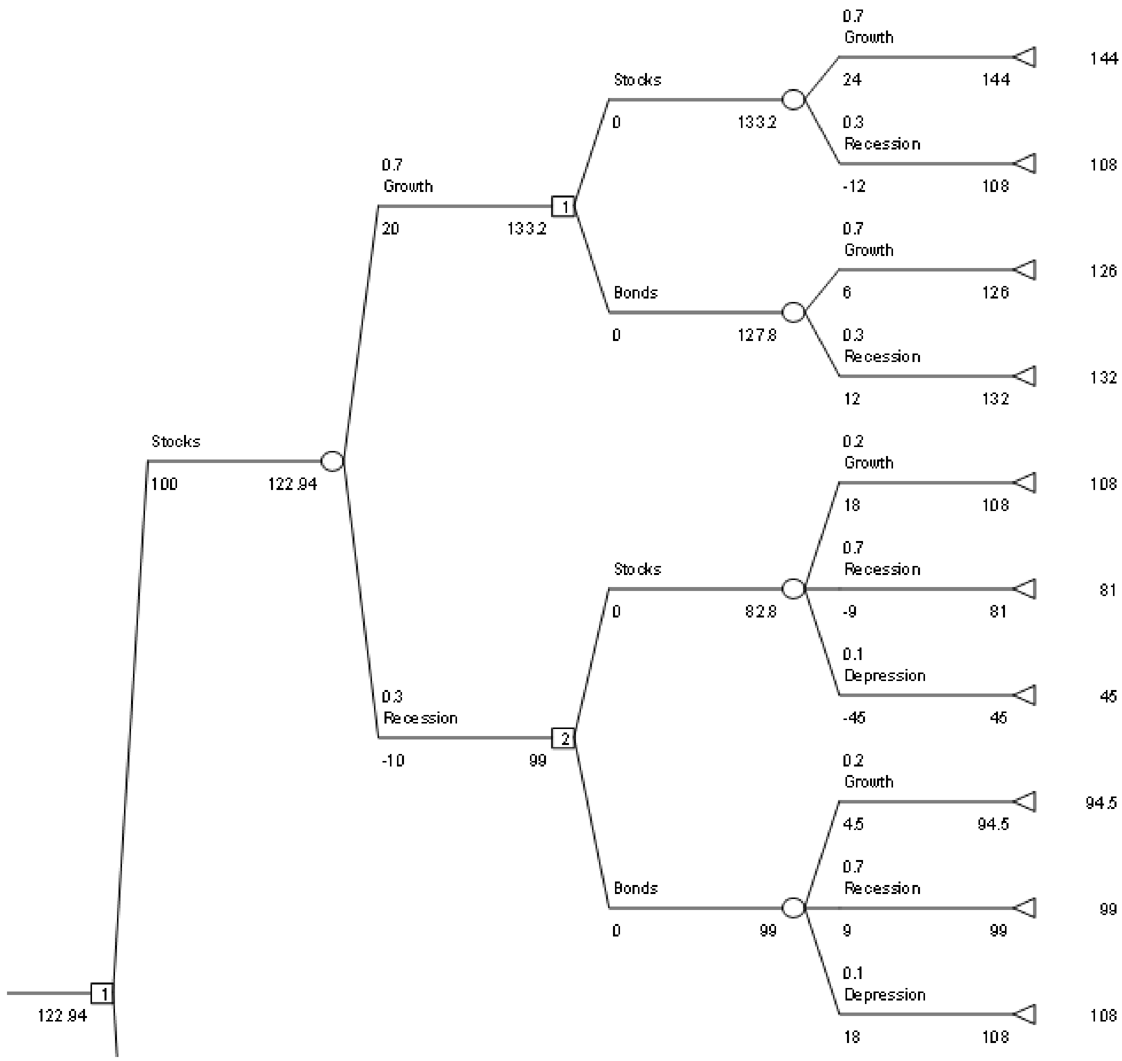
Marginal Profit
0.00054
0.00066
Assembly Line Cost
6.6
5.4
Payoff from Selling Rights
18.5
22
23
24
25
26
27
28
29
30
31
32
Expected Payoff

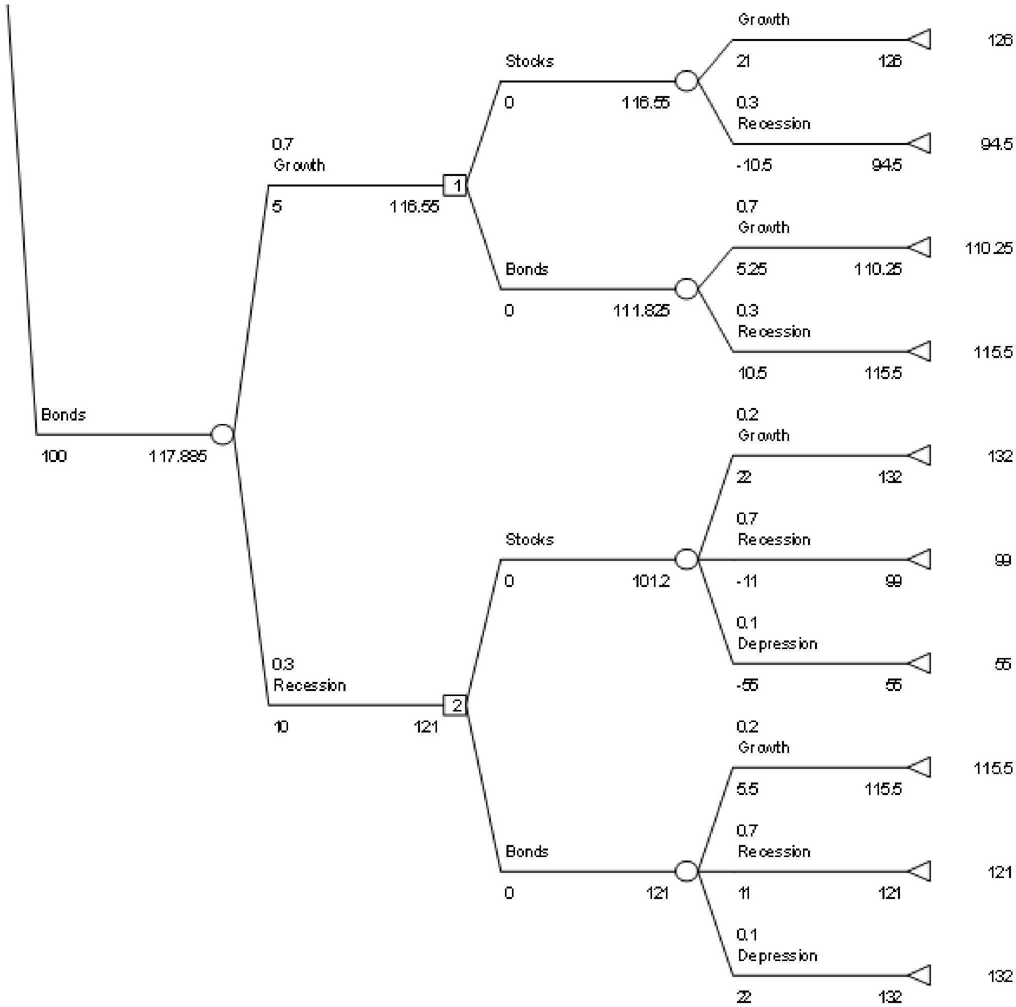
15.5-3.



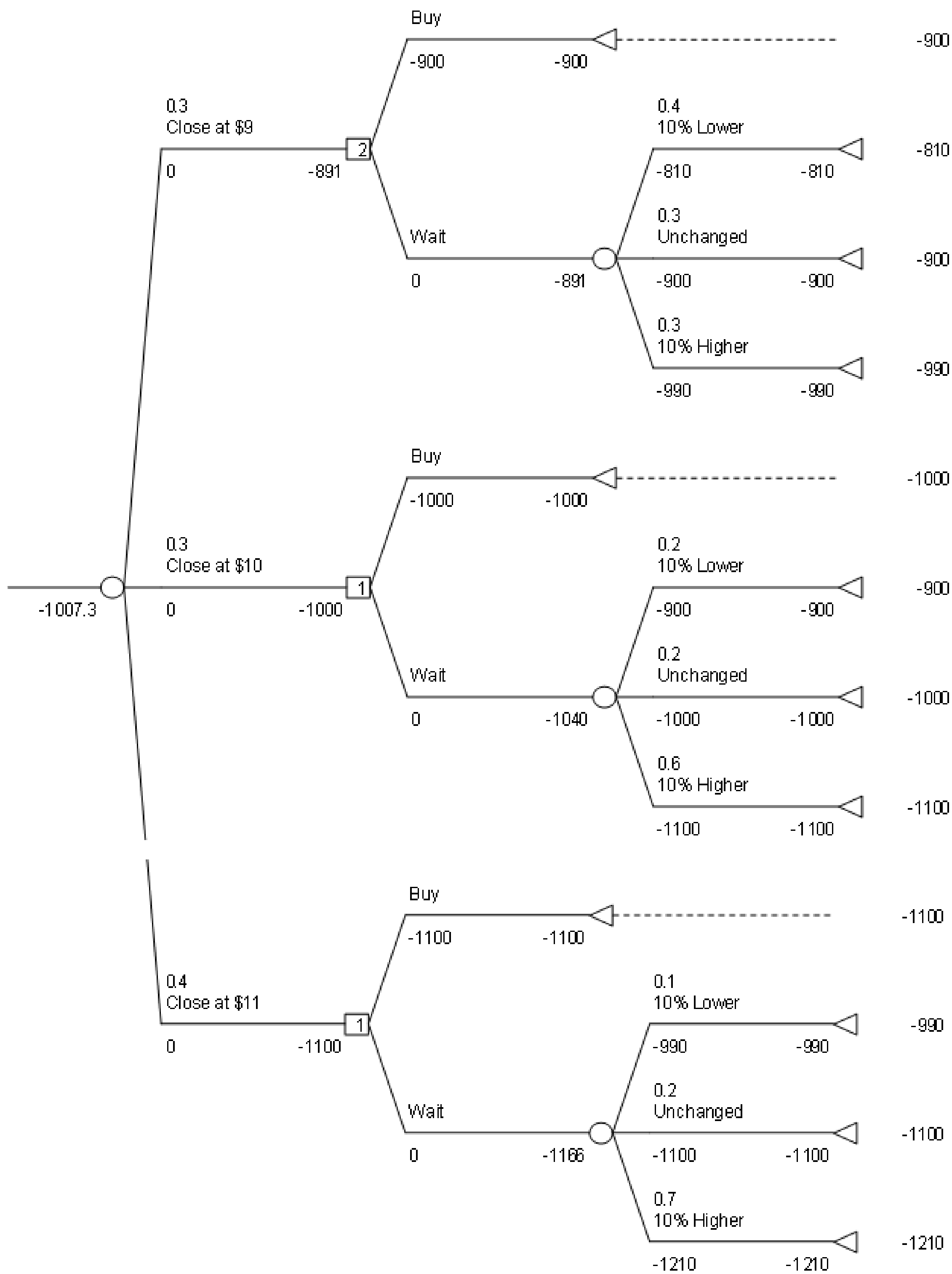
0.4
2500
2500
2500
0
580
0.6
0.2
-700
-700
-700
0
900
900
900
900
0
820
0.8
800
800
800
820
750
750
750

15.5-4.


0.7
Growth
24
144
144
Stocks
0
133.2
0.3
Recession
-12
108
108
0.7
Growth
20
133.2
0.7
Growth
6
126
126
Bonds
0
127.8
0.3
Recession
12
132
132
Stocks
100
122.94
0.2
Growth
18
108
108
Stocks
0
82.8
0.7
Recession
-9
81
81
0.1
Depression
-45
45
45
0.3
Recession
-10
99
0.2
Growth
4.5
94.5
94.5
Bonds
0
99
0.7
Recession
9
99
99
0.1
Depression
18
108
108
122.94


Bonds
100
117.885
0.7
Growth
5
116.55
1
Stocks
0
116.55
Growth
21
126
126
0.3
Recession
-10.5
94.5
94.5
Bonds
0
111.825
0.7
Growth
5.25
110.25
110.25
0.3
Recession
10.5
115.5
115.5
0.3
Recession
10
121
2
Stocks
0
101.2
0.2
Growth
22
132
132
0.7
Recession
-11
99
99
0.1
Depression
-55
55
55
Bonds
0
121
0.2
Growth
5.5
115.5
115.5
0.7
Recession
11
121
121
0.1
Depression
22
132
132

15.5-5.

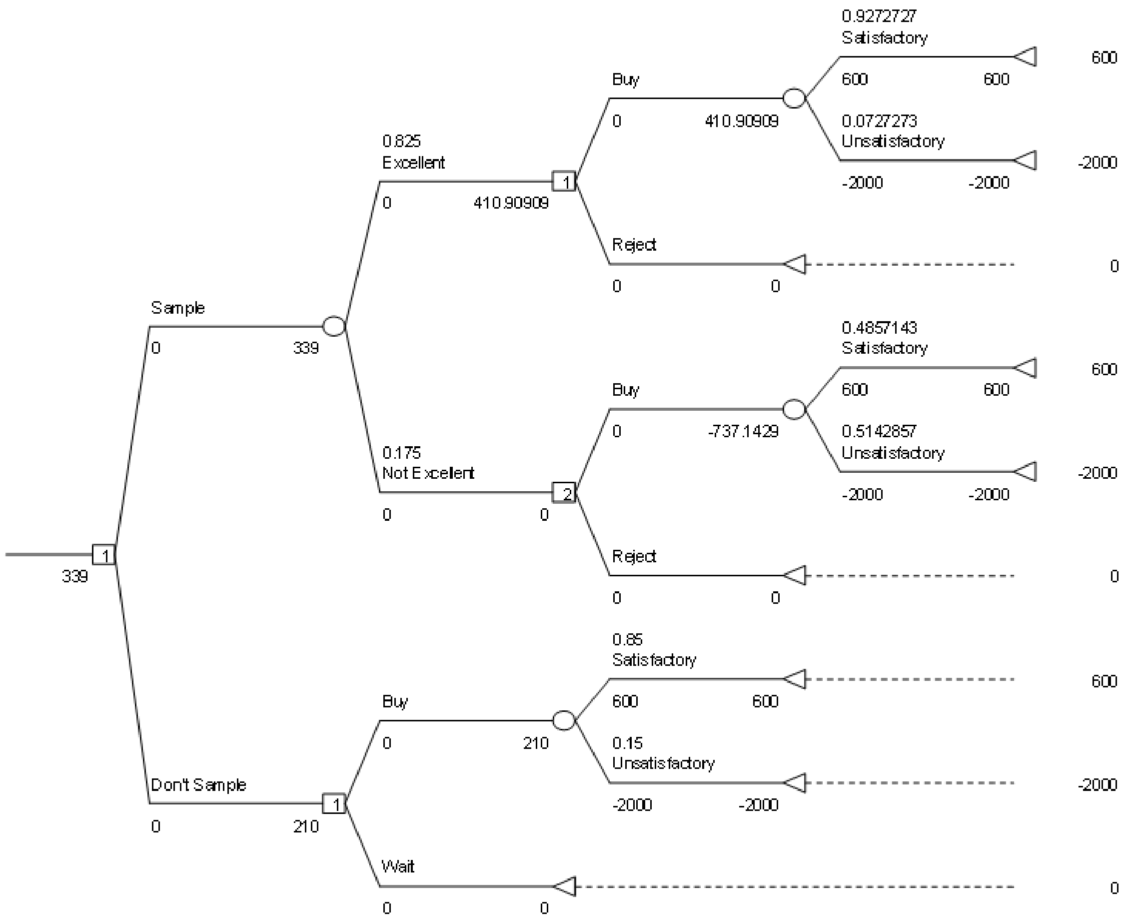


The optimal policy is to wait until Wednesday to buy if the price is $9 on Tuesday. If the price is $10 or $11 on Tuesday, then buy on Tuesday.

15.5-6.

Data:			P(Finding | State)
State of	Prior		Finding
Nature	Probability	Excellent	Not Excellent
Satisfactory Box	0.85	0.9	0.1
Unsatisfactory Box	0.15	0.4	0.6

Posterior			P(State | Finding)
Probabilities:			State of Nature
Finding	P(Finding)	Satisfactory Box	Unsatisfactory Box
Excellent	0.825	0.927272727	0.072727273
Not Excellent	0.175	0.485714286	0.514285714



The optimal policy is to sample the fruit and buy if it is excellent and reject if it is unsatisfactory.

15.5-7.

(a) Choose to introduce the new product with expected payoff of $12.5 million.

	State of Nature		Exp.
Alternative	Successful	Unsuccessful	Payoff
Introduce New Product	$40 million	$-$$15 million	$12.5 million
Don't Introduce New Product	0	0	0
Prior Probabilities	0.5	0.5	

(b) With perfect information, Morton Ward should introduce the product if it will be successful and not introduce it if it will not be successful.

Expected Payoff with Perfect Information: $0.5(40) + 0.5(0) = 20$

Expected Payoff without Information: 12.5

EVPI $= 20 - 12.5 =$ $7.5 million

(c) The optimal policy is to not test but to introduce the new product, with expected payoff $12.5 million.

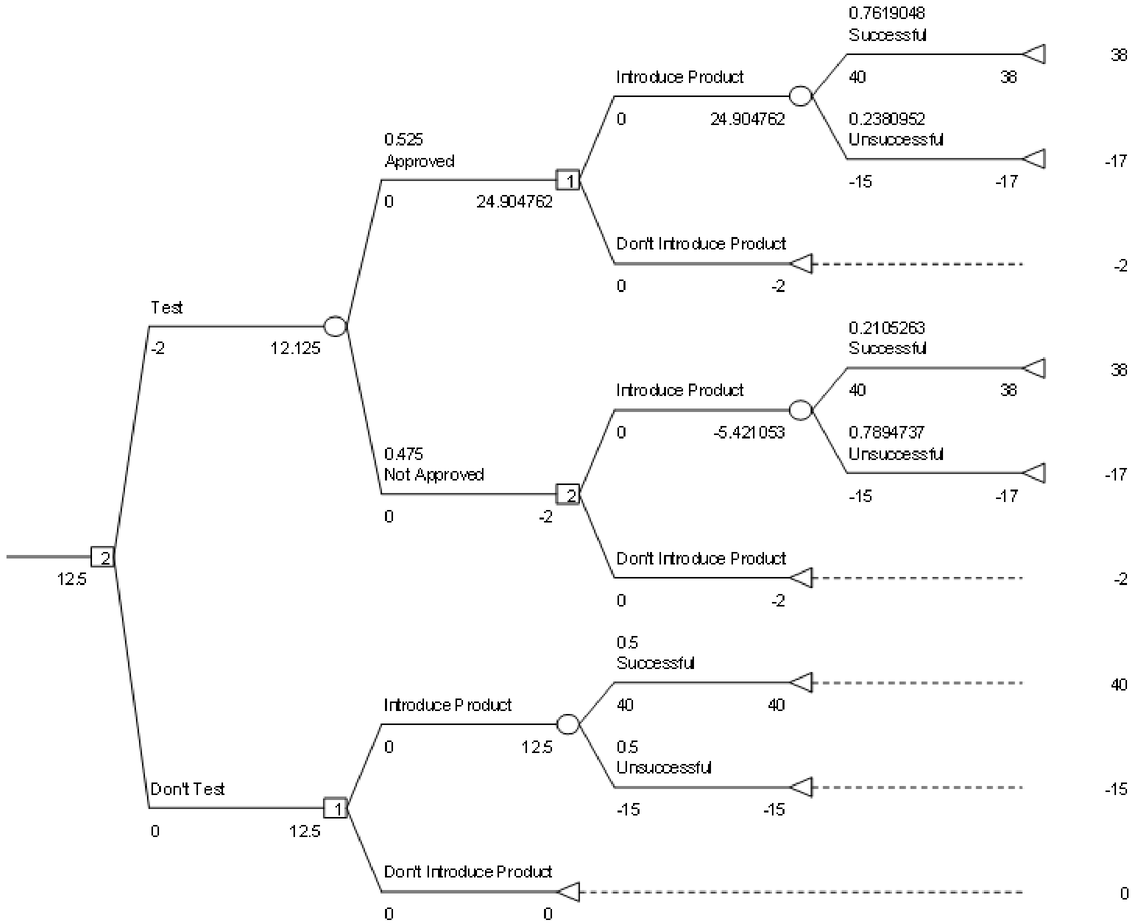
Data:		P(Finding | State)	
State of	Prior	Finding	
Nature	Probability	Approved	Not Approved
Successful	0.5	0.8	0.2
Unsuccessful	0.5	0.25	0.75

Posterior		P(State | Finding)	
Probabilities:		State of Nature	
Finding	P(Finding)	Successful	Unsuccessful
Approved	0.525	0.761904762	0.238095238
Not Approved	0.475	0.210526316	0.789473684

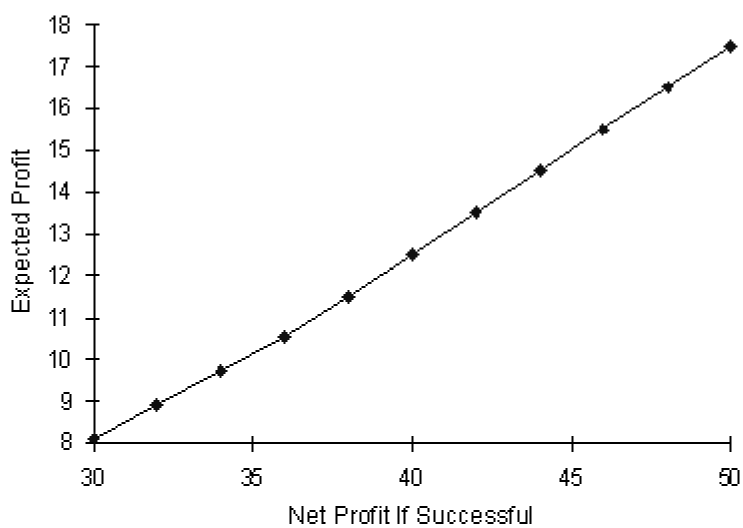
The associated decision tree is on the next page.

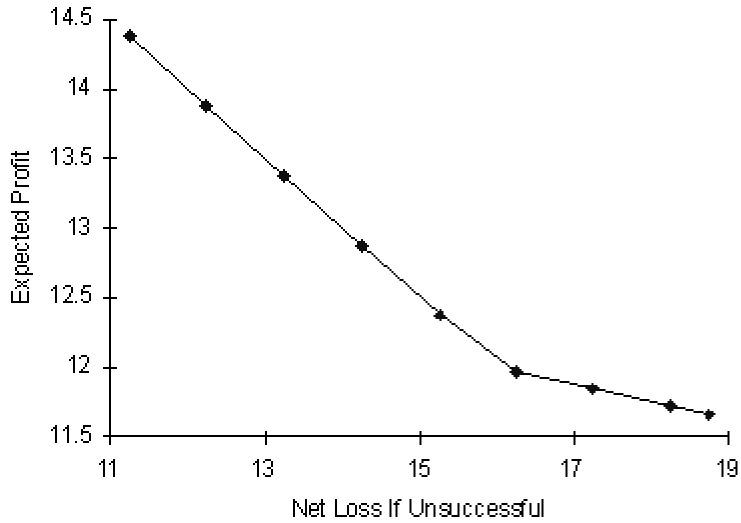
(d) If the net profit if successful is only $30 million, then the optimal policy is to test and to introduce the product only if the test market approves. The expected payoff is $8.125 million. If the net profit if successful is $50 million, then the optimal policy is to skip the test and to introduce the product, with an expected payoff of $17.5 million. If the net loss if unsuccessful is only $11.25 million, then the optimal policy is to skip the test and to introduce the product, with an expected payoff of $14.375 million. If the net loss if unsuccessful is $18.75 million, then the optimal policy is to conduct the test and to introduce the product only if the test market approves. The expected payoff is $11.656 million. For each combination of financial data, the expected payoff and the optimal policy are as shown below.

Successful	Unsuccessful	Optimal Policy	Expected Profit
$30 million	-$11.25 million	Skip Test, Introduce Product	$9.375 million
$30 million	-$18.75 million	Test, Introduce Product if Approved	$7.656 million
$50 million	-$11.25 million	Skip Test, Introduce Product	$19.375 million
$50 million	-$18.75 million	Test, Introduce if Approved	$15.656 million

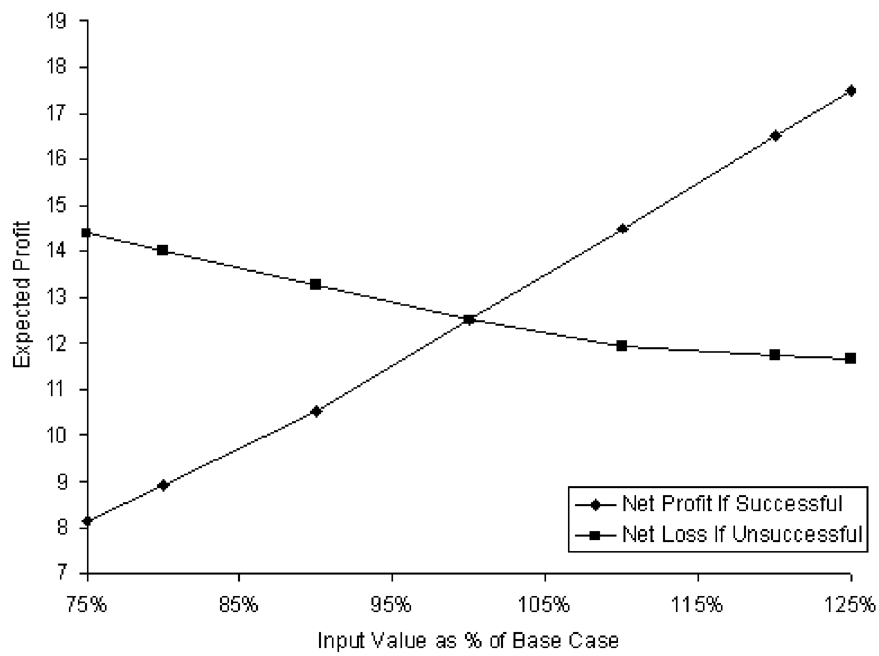

0.7619048
Successful
40
38
38
Introduce Product
0
24.904762
0.2380952
Unsuccessful
-15
-17
-17
0.525
Approved
0
24.904762
Don't Introduce Product
0
-2
-2
Test
-2
12.125
0.2105263
Successful
40
38
38
Introduce Product
0
-5.421053
0.7894737
Unsuccessful
-15
-17
-17
0.475
Not Approved
0
-2
12.5
Don't Introduce Product
0
-2
-2
0.5
Successful
40
40
40
Introduce Product
0
12.5
0.5
Unsuccessful
-15
-15
-15
Don't Test
0
12.5
Don't Introduce Product
0
0
0

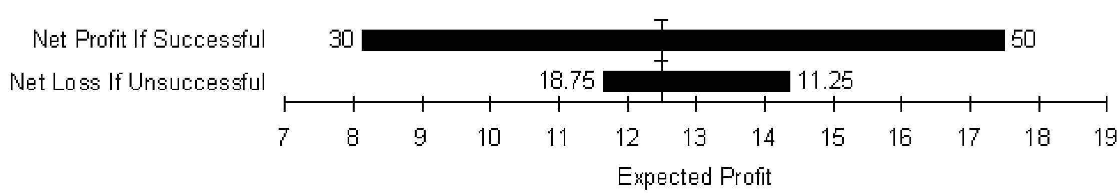
(e)


Expected Profit
Net Profit If Successful
18
17
16
15
14
13
12
11
10
9
8
30
35
40
45
50



(f)

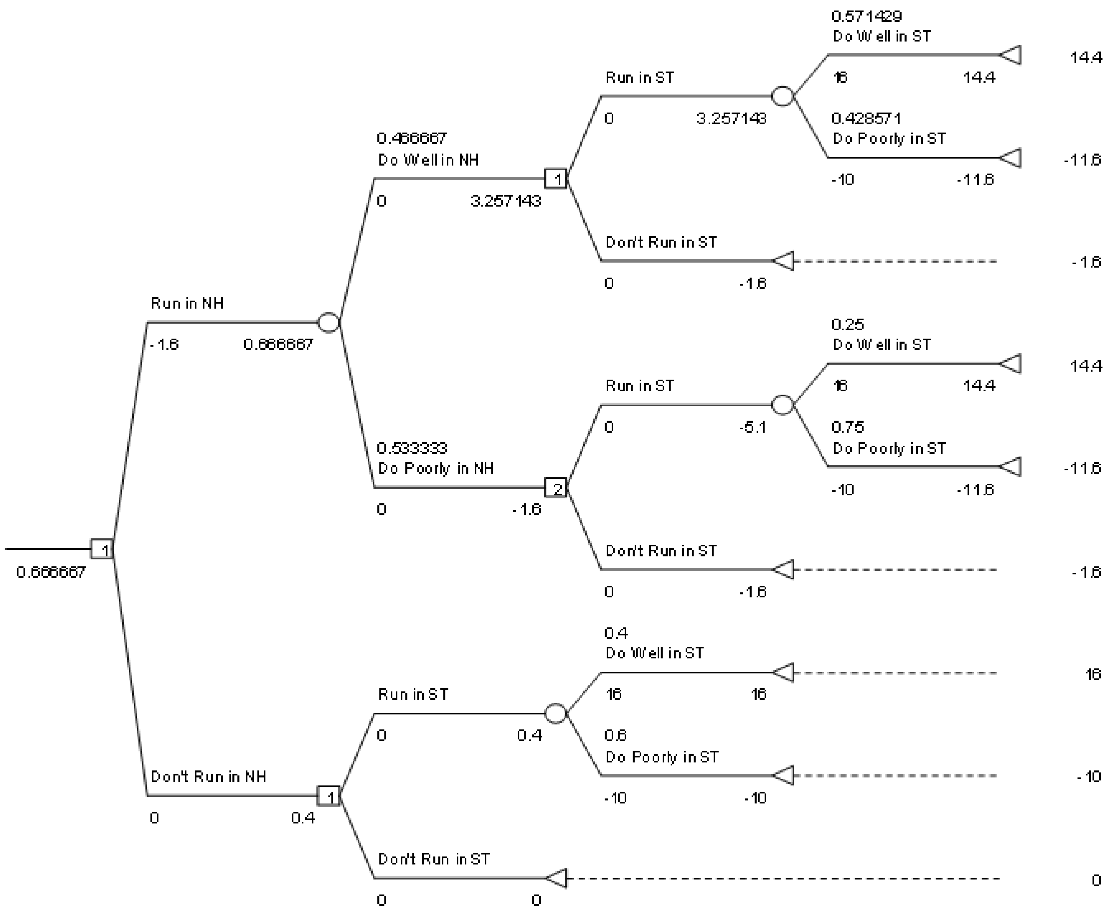




Both charts indicate that the expected profit is sensitive to both parameters, but is somewhat more sensitive to changes in the profit if successful than to changes in the loss if unsuccessful.

15.5-8.

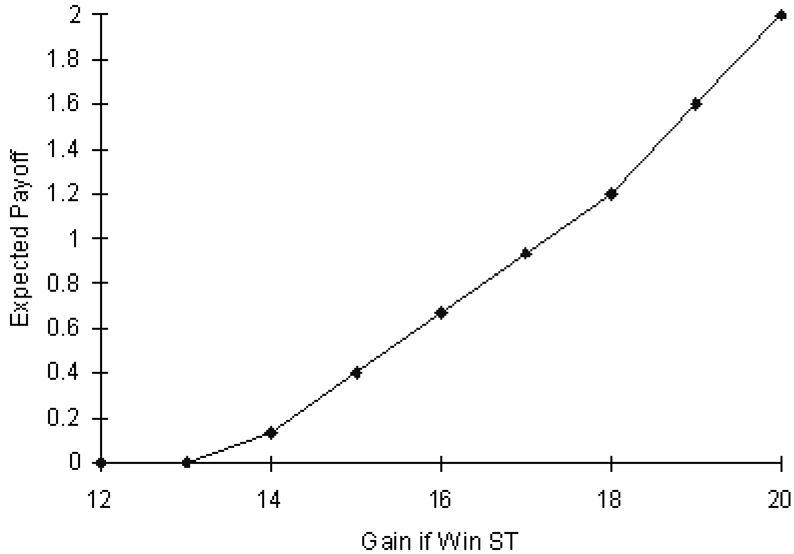
Chelsea should run in the NH primary. If she does well, then she should run in the ST primaries. If she does poorly in the NH primary, then should not run the ST primaries. The expected payoff is $\$666,667$.



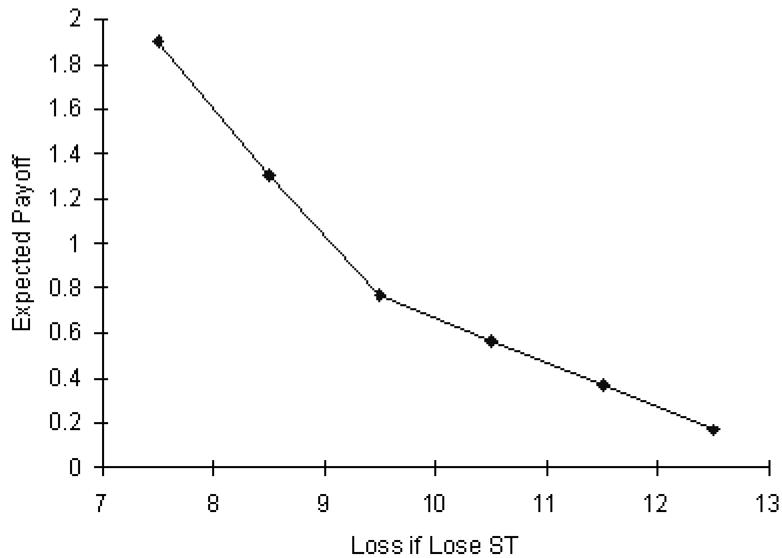
(b) If the payoff for doing well in ST is only \$12 million, Chelsea should not run in either NH or ST, with expected payoff of $\$0$. If the payoff for doing well in ST is $\$20$ million, Chelsea should not run in NH, but run in ST, with expected payoff of \$2 million. If the loss for doing poorly in ST is $\$7.5$ million, Chelsea should not run in NH, but run in ST, with expected payoff of $\$1.9$ million. If the loss for doing poorly in ST is only $\$12.5$ million, Chelsea should run in NH and run in ST if she does well in NH, with expected payoff of $\$166,667$. For each combination of financial data, the expected payoff and the optimal policy is as shown below.

Well in ST	Poorly in ST	Optimal Policy	Expected Funds
\$12 million	-\$7.5 million	Run in ST Only	\$300,000
\$12 million	-\$12.5 million	Don't Run in Either	\$0
\$20 million	-\$7.5 million	Run in ST Only	\$3.5 million
\$20 million	-\$12.5 million	Run in NH, Run in ST if Well	\$1.233 million

(c)

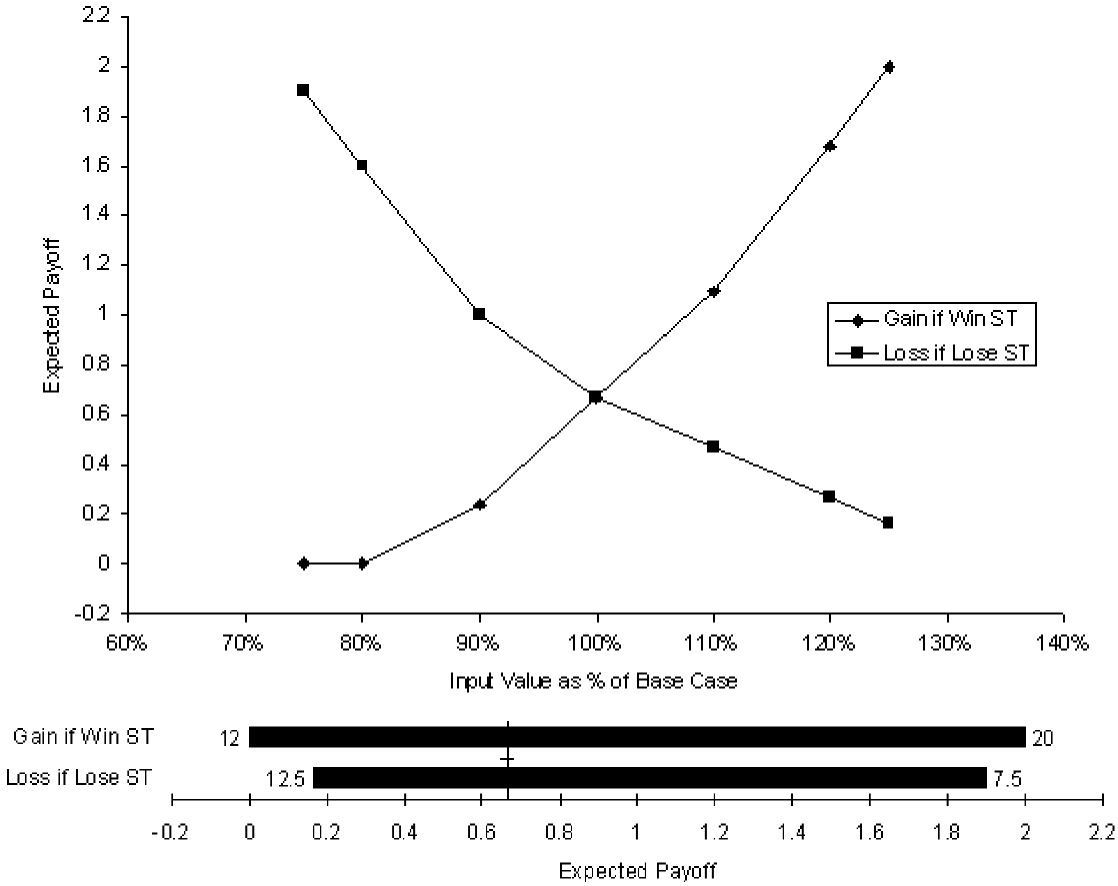


Expected Payoff
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
12
14
16
18
20
Gain if Win ST



Expected Payoff
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
7
8
9
10
11
12
13
Loss if Lose ST

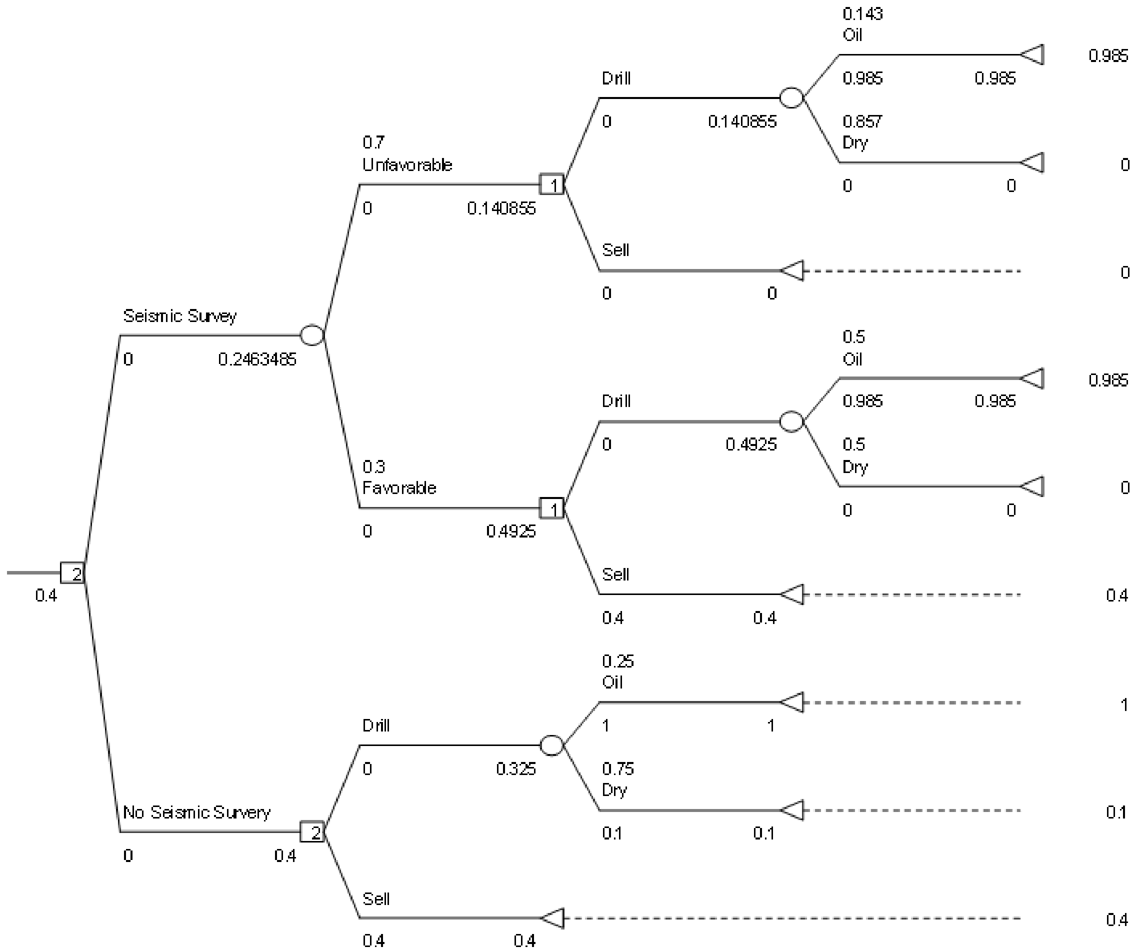
(d)



Both charts indicate that the expected payoff is sensitive to both parameters, although it is slightly more sensitive to changes in the profit if she does well than to changes in the loss if she does poorly.

15.6-1.

(a) - (b) The optimal policy is to not conduct a survey and to sell the land.



15.6-2.

(a) Choose to not buy insurance with expected payoff $\$249,840$.

	State of Nature		Exp.
Alternative	Earthquake	No Earthquake	Payoff
Buy Insurance	$249,820$	$249,820$	$249,820$
Not Buy Insurance	$90,000$	$250,000$	$249,840$
Prior Probability	0.001	0.999	

(b) $U(\text{insurance}) = U(250,000-180) = \sqrt{249,820} = 499.82$

$U(\text{no insurance}) = 0.999U(250,000) + 0.001U(90,000) = 499.8$

The optimal policy is to buy insurance.

15.6-3.

Expected utility of $\$19,000$: $U(19) = \sqrt{25} = 5$

Expected utility of investment: $0.3U(10) + 0.7U(30) = 0.3\sqrt{16} + 0.7\sqrt{36} = 5.4$

Choose the investment to maximize expected utility.

15.6-4.

Expected utility of A_1 $=$ Expected utility of A_2

$$pU(10) + (1-p)U(30) = U(19)$$

$$0.3U(10) + 0.7(20) = 16.7 \Rightarrow U(10) = 9$$

15.6-5.

(a) Expected utility of A_1 $=$ Expected utility of A_2

$$pU(10) + (1-p)U(0) = U(1)$$

$$0.125U(10) + 0.875(0) = 1 \Rightarrow U(10) = 8$$

(b) Expected utility of A_1 $=$ Expected utility of A_2

$$pU(10) + (1-p)U(0) = U(5)$$

$$0.5625(8) + 0.4375(0) = U(5) \Rightarrow U(5) = 4.5$$

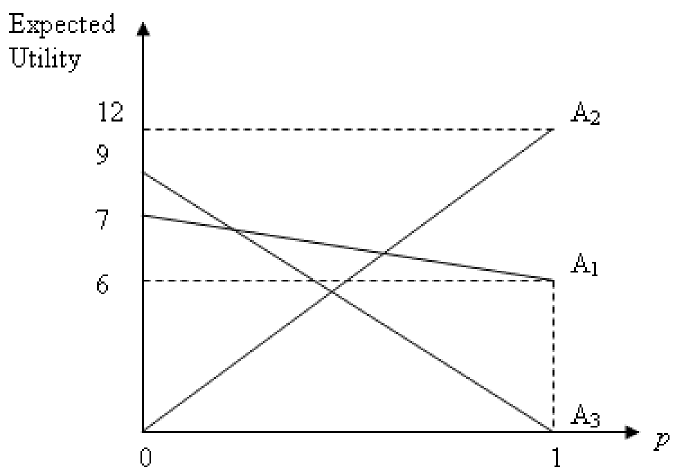
(c) Answers will vary.

15.6-6.

(a) Expected utility of $A_1 = pU(36) + (1-p)U(49) = 6p + 7(1-p) = 7 - p$

Expected utility of $A_2 = pU(144) + (1-p)U(0) = 12p + 0 = 12p$

Expected utility of $A_3 = pU(0) + (1-p)U(81) = 0 + 9(1-p) = 9 - 9p$



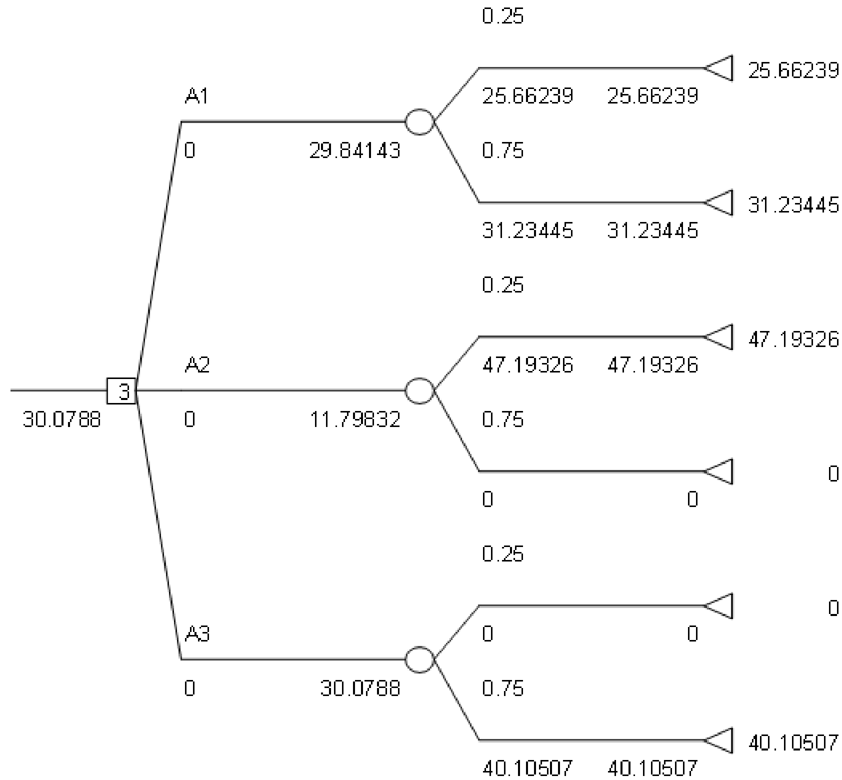
A_1 and A_2 cross when $7 - p = 12p \Rightarrow p = 7/13$.

A_1 and A_3 cross when $7 - p = 9 - 9p \Rightarrow p = 1/4$.

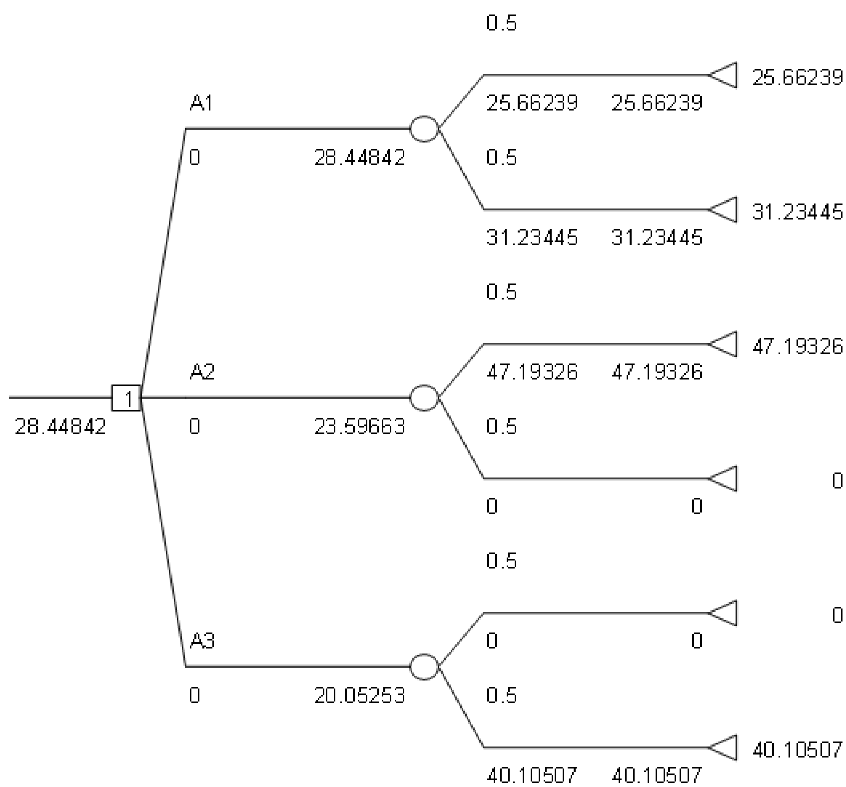
Thus, A_3 is best when $p \leq 1/4$, A_1 is best when $1/4 \leq p \leq 7/13$, and A_2 is best when $p \geq 7/13$.

(b) $U(M) = 50(1 - e^{-M/50})$

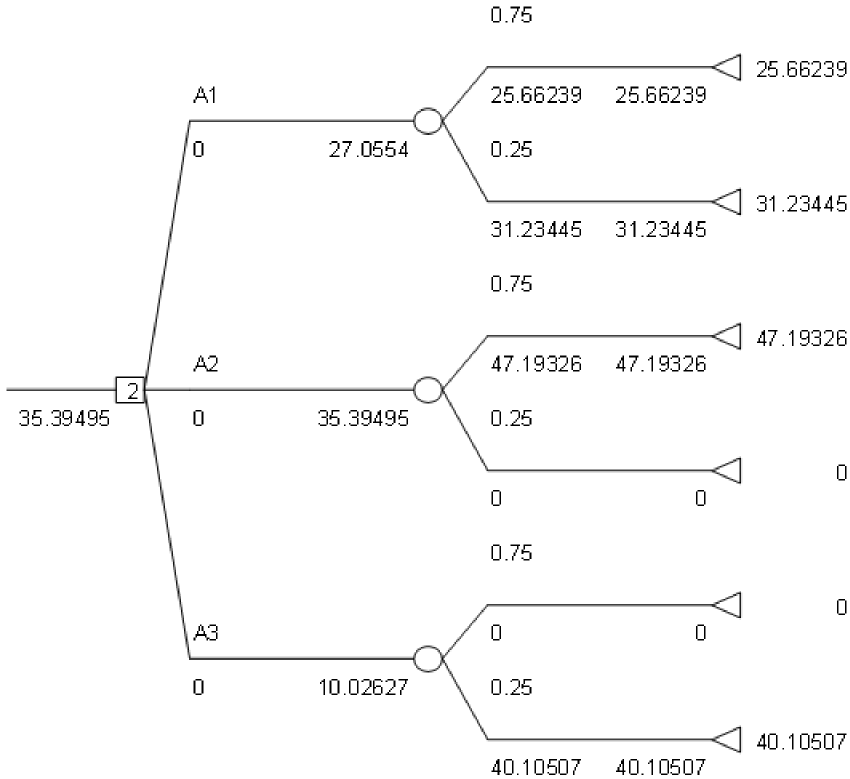
$p = 25\%$



$p = 50\%$



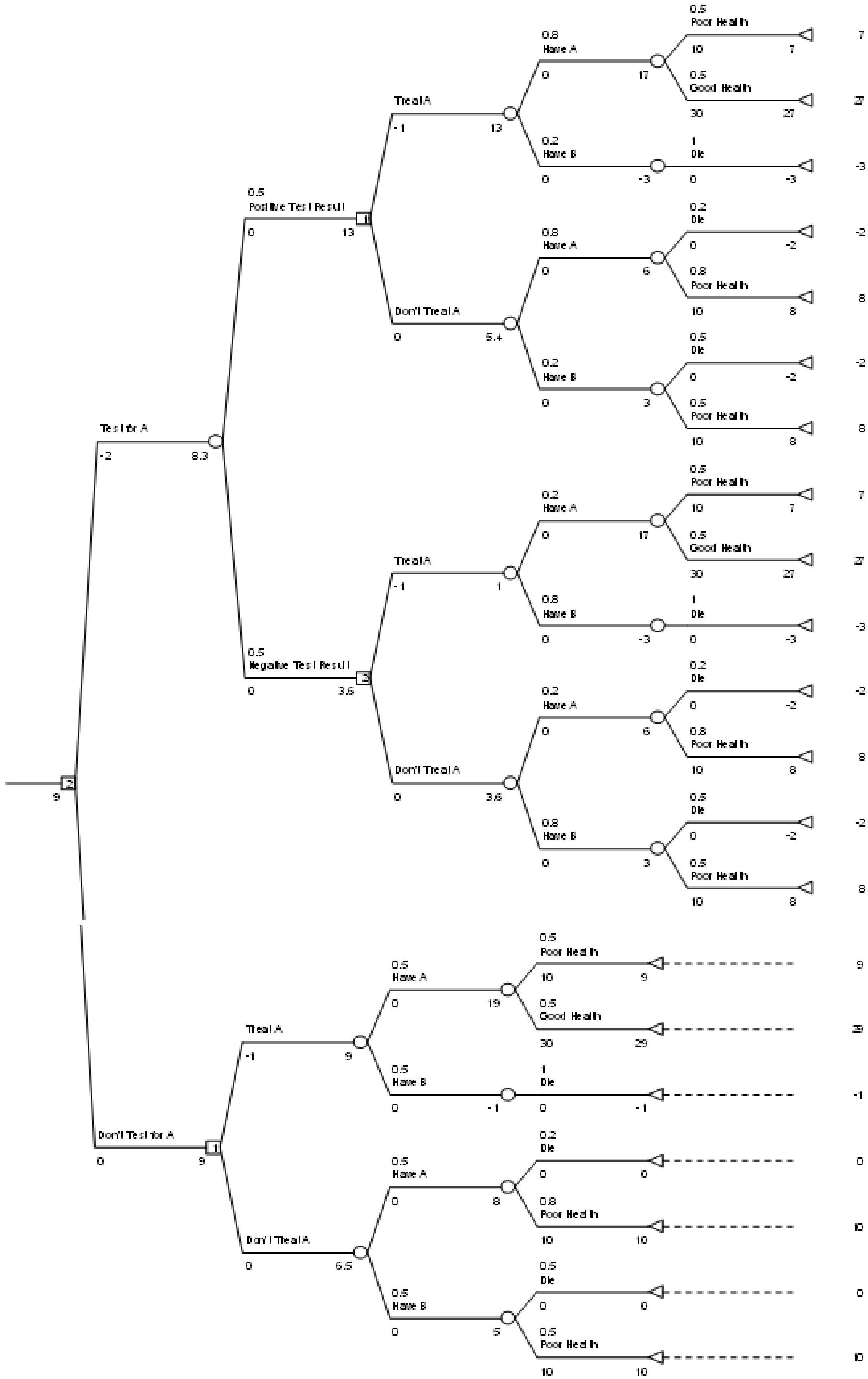
$p = 75\%$



0.75
25.66239
A1
25.66239 25.66239
0 27.0554
0.25
31.23445
31.23445 31.23445
0.75
47.19326
A2
47.19326 47.19326
2
35.39495
0 35.39495
0.25
0
0 0
0.75
0
A3
0 0
0 10.02627
0.25
40.10507
40.10507 40.10507

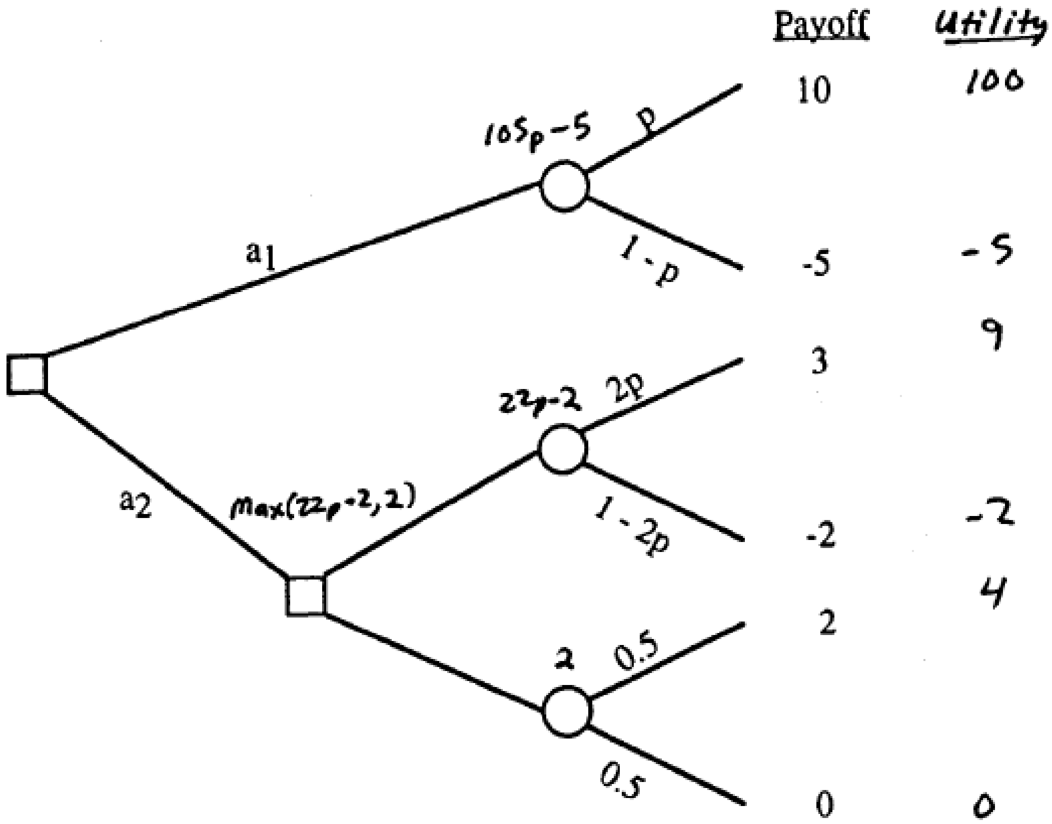
15.6-7.

The optimal policy is to not test for disease A, but to treat disease A.



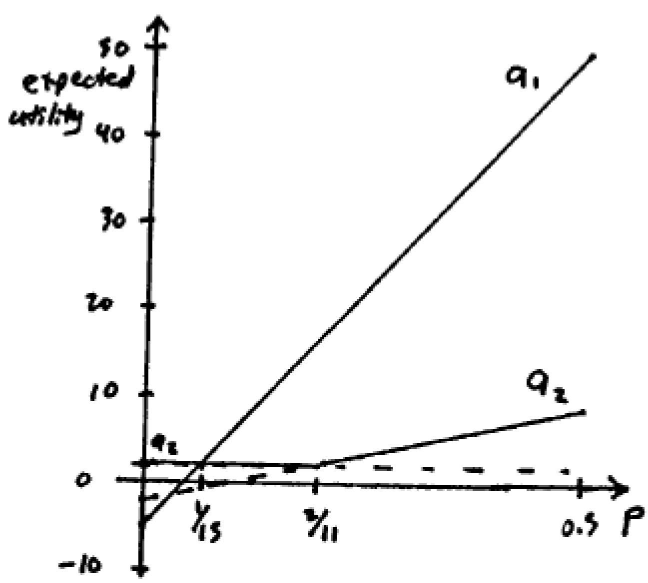
15.6-8.

(a)



At $p = 0.25$, $105p - 5 = 21.25$ and max $(22p - 2, 2) =$ max $(3.5, 2) = 3.5$, so A_1 is optimal.

(b)



As can be seen on the graph, A_1 stays optimal for $1/15 \leq p \leq 0.5$.

CASES

CASE 15.1 Brainy Business

(a) The relevant data are summarized in the following spreadsheet.

Data:

Price:

High	Medium	Low
50.00	40.00	30.00

Sales:

High	Medium	Low
50,000	30,000	20,000

Probability tables:

Prior probabilities:

Competition:	Severe	Moderate	Weak
p(competition)	0.20	0.70	0.10

Conditional Probabilities:

Price:	50.00		
Competition:	Severe	Moderate	Weak
50k Units	0.20	0.25	0.30
30k Units	0.25	0.30	0.35
20k Units	0.55	0.45	0.35

Price:	40.00		
Competition:	Severe	Moderate	Weak
50k Units	0.25	0.30	0.40
30k Units	0.35	0.40	0.50
20k Units	0.40	0.30	0.10

Price:	30.00		
Competition:	Severe	Moderate	Weak
50k Units	0.35	0.40	0.50
30k Units	0.40	0.50	0.45
20k Units	0.25	0.10	0.05

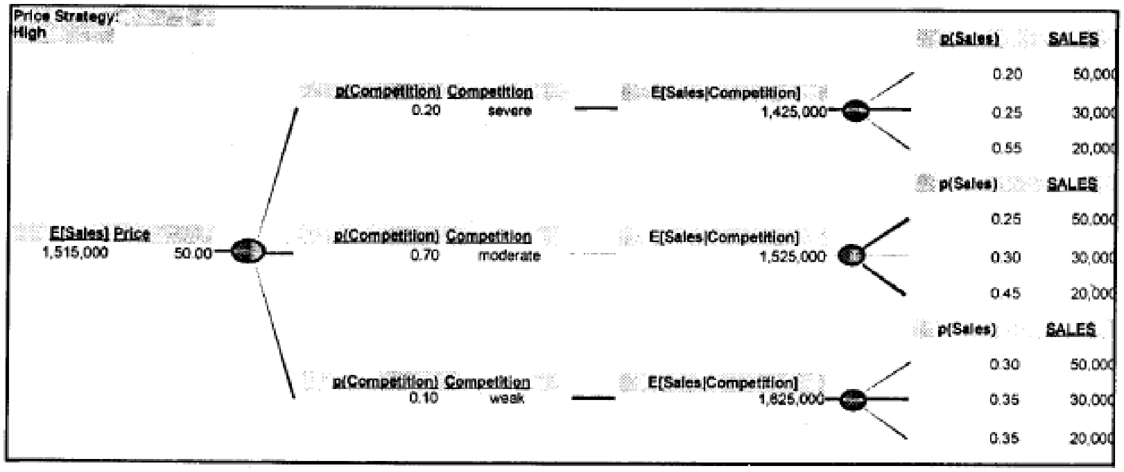
(b) The scenario "moderate competition, sales of 30,000 units at a unit price of $30" has the largest total probability. Therefore, under the maximum likelihood criterion, Charlotte should price the product at $30.

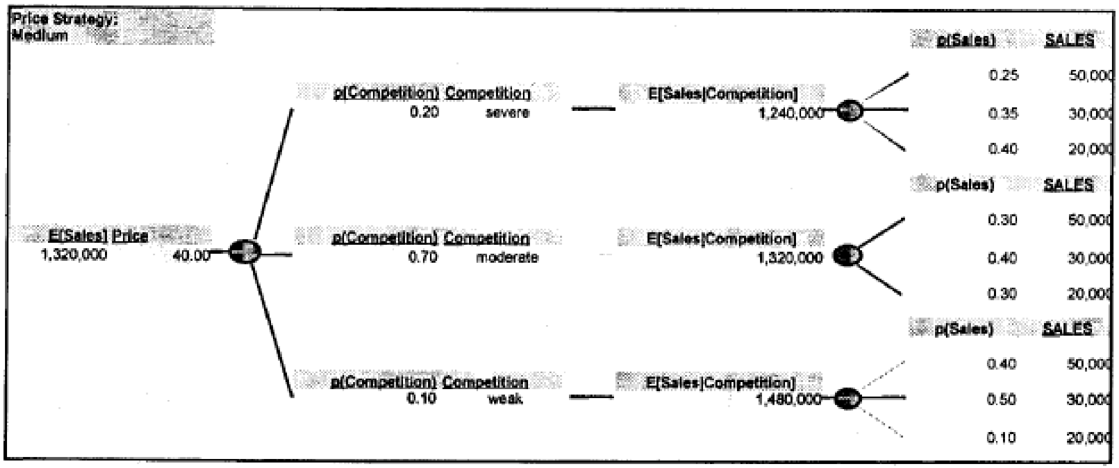
To find out best maximin alternative, note that for a price of

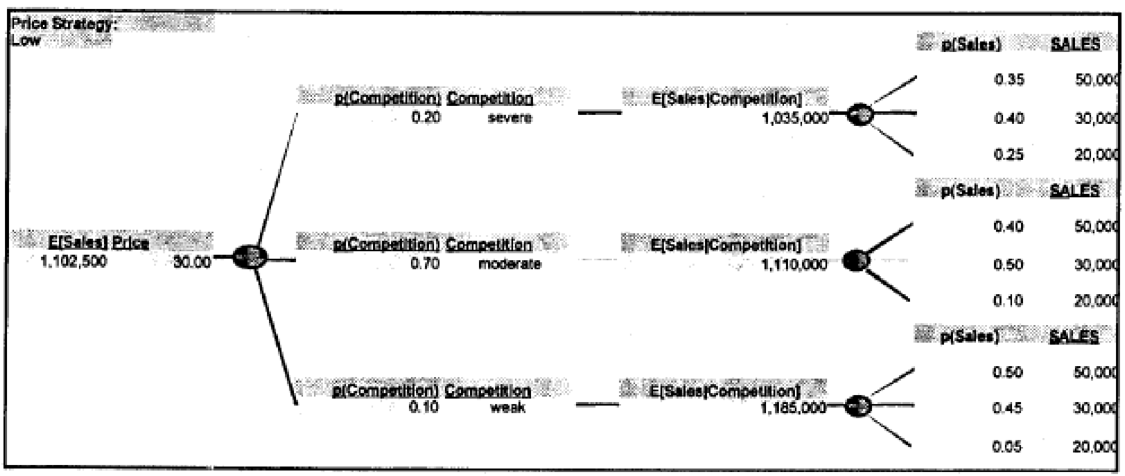
$30: 20,000 units at a unit price $30 is the worst case,
$40: 20,000 units at a unit price $40 is the worst case,
$50: 20,000 units at a unit price $50 is the worst case.

The maximum of these three is for the price of $50, so it is optimal under the maximin criterion.

(c) The three branches of the decision tree for the decision problem without additional information follow.



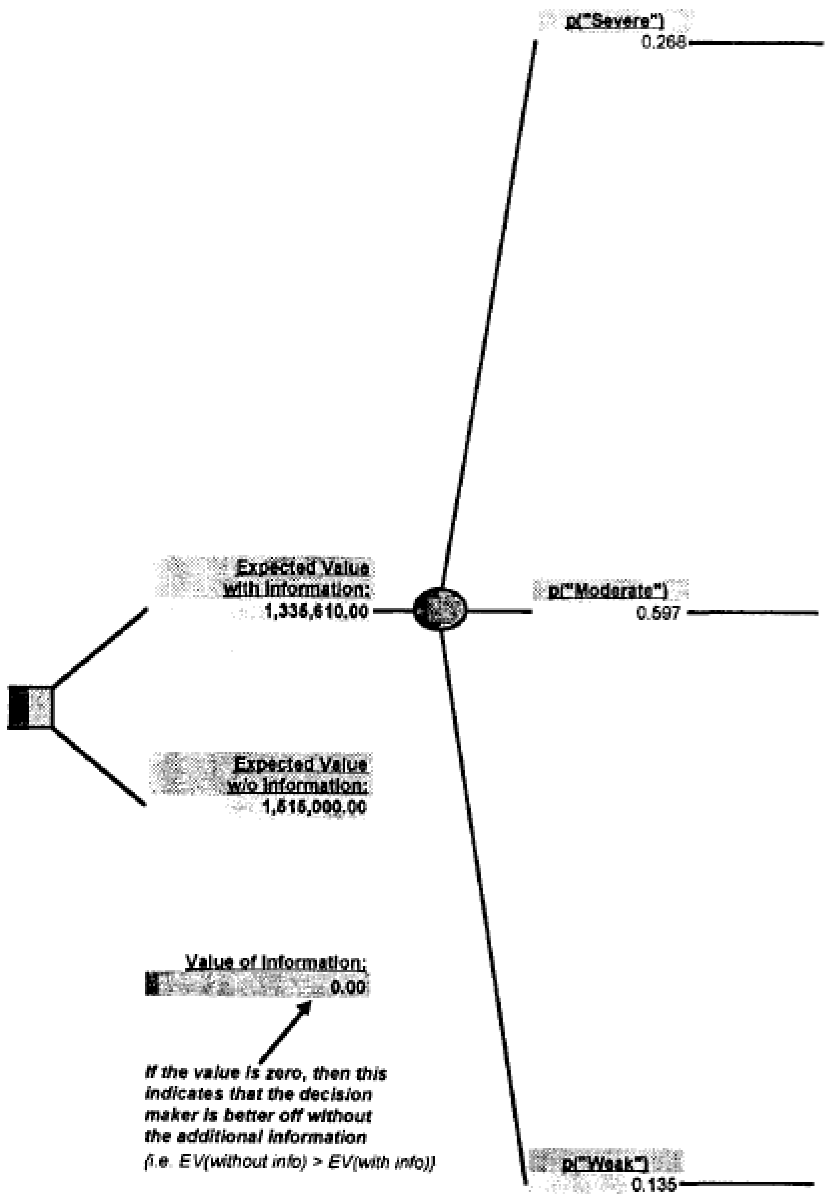


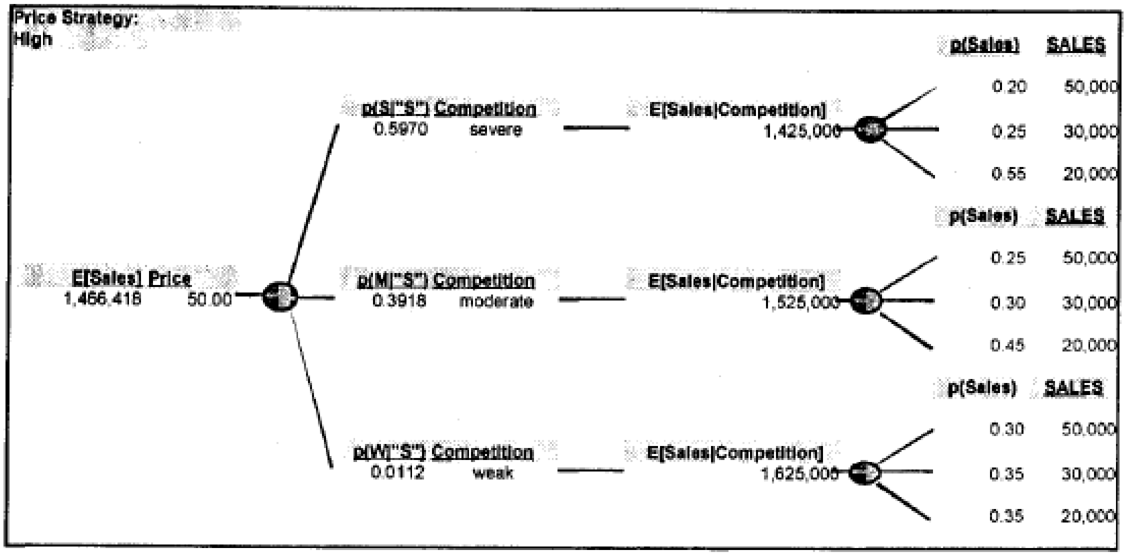


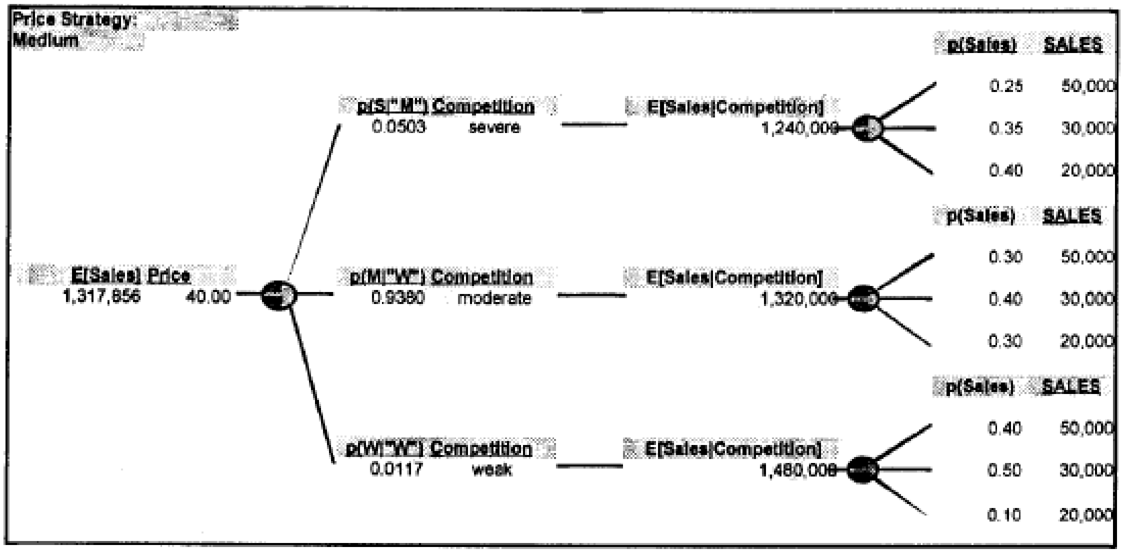
To find the expected revenue for a high price strategy, multiply each final outcome ("Sales") by its probability. For example, the expected sales given severe competition equals: $0.2(50,000) + 0.25(30,000) + 0.55(20,000) = 1,425,000$. The remaining expected values are computed similarly. The decision tree indicates that the alternative with the maximum expected value is the high price strategy. Hence, Charlotte should price the product at $50.

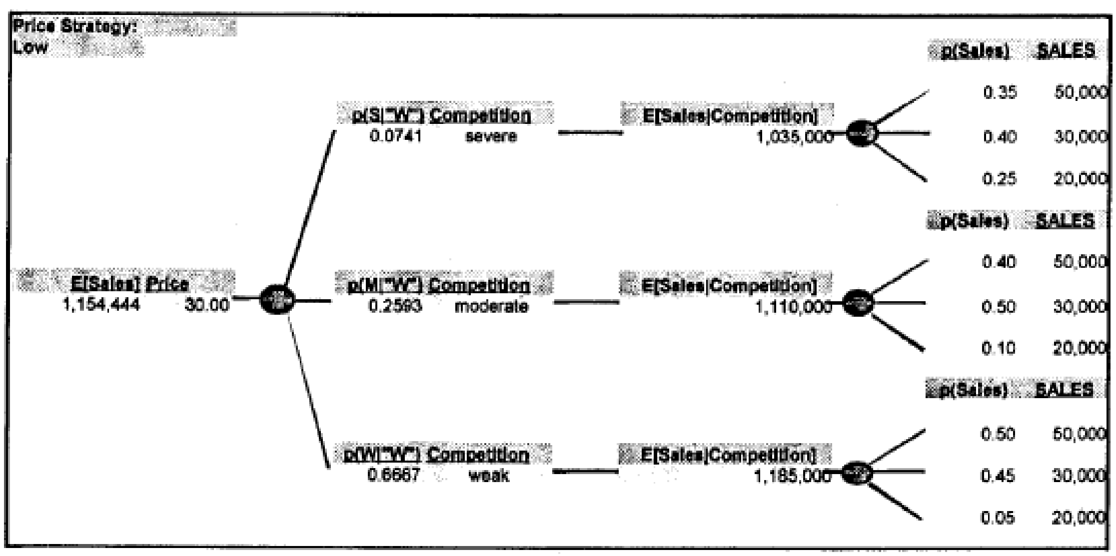
(d) The decision tree for the decision problem with additional information follows in two sections. The branches of the decisions follow the first half of the tree. Note that the decision alternative "Expected Value w/o Information" represents the entire decision tree of part (c). The computations in the tree are performed in the same manner as in part (c).

Decision tree (with additional info)





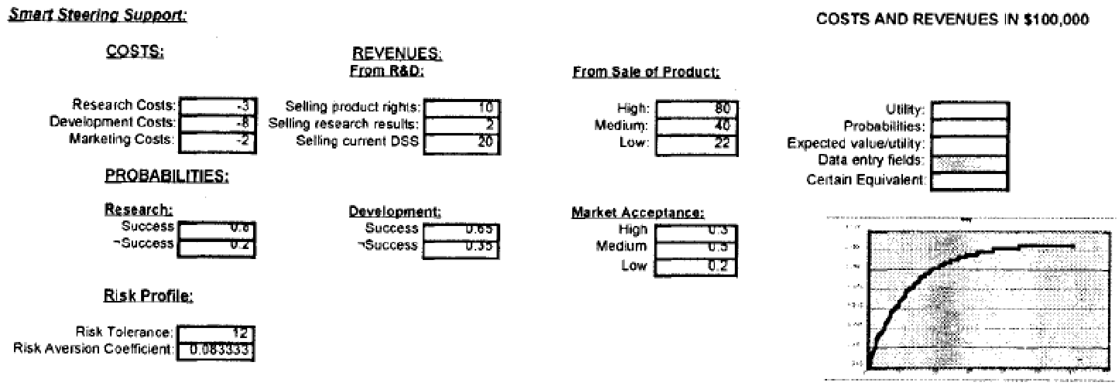




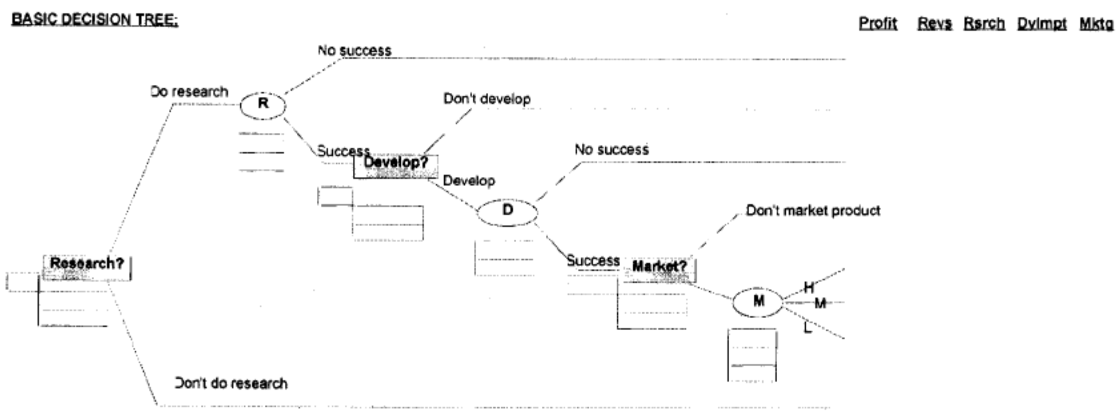
Since the expected value for the decision tree with additional (imperfect) information is less than that without information, Charlotte should not purchase the services of the market research company.

CASE 15.2 Smart Steering Support

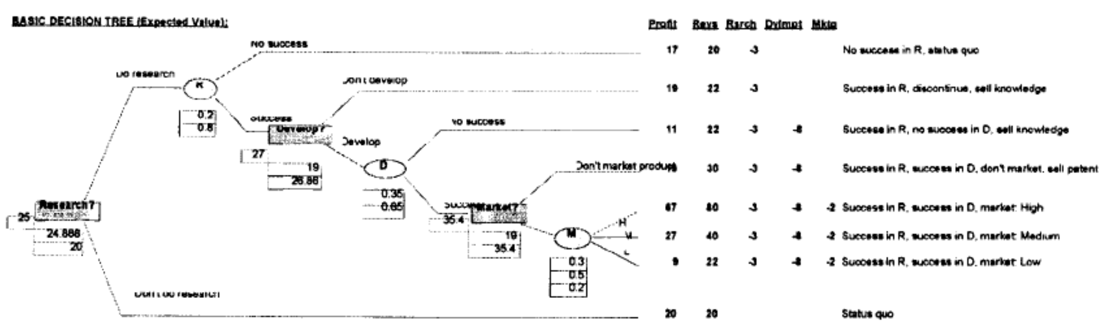
(a) The available data are summarized in the table.



(b) The basic decision tree is shown. Rectangular nodes represent decision forks and oval nodes represent chance forks.

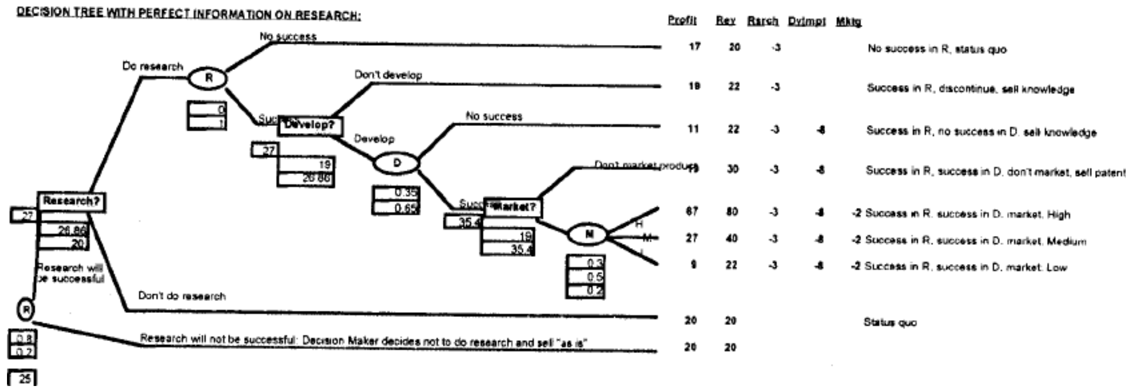


(c) The decision tree displays all the expected payoffs.

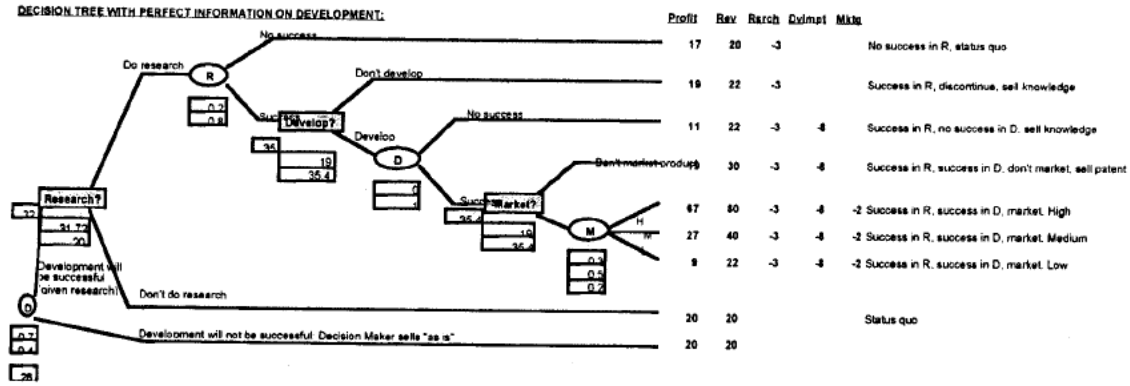


(d) The best course of action is to do the research project. The expected payoff is $2.4888 million.

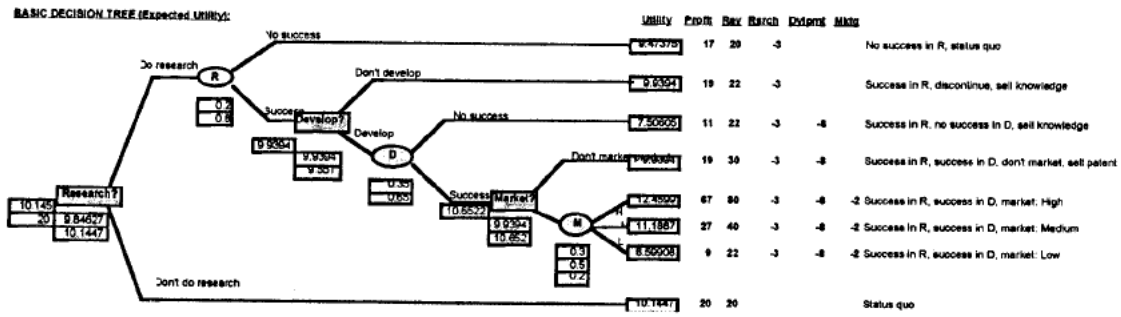
(e) The decision tree with perfect information on research is displayed. The expected value in this case equals $2.5488 million. The difference between the expected values with and without information is $60,000, which is the value of perfect information on research.



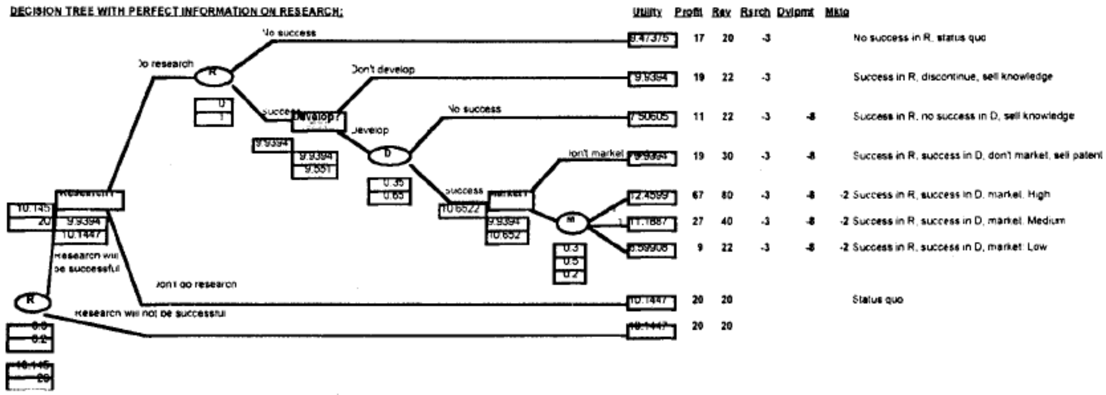
(f) The decision tree with perfect information on development is displayed. The expected value in this case equals $\$2.7618$ million. The difference between the expected values with and without information is $\$273,000$, which is the value of perfect information on development



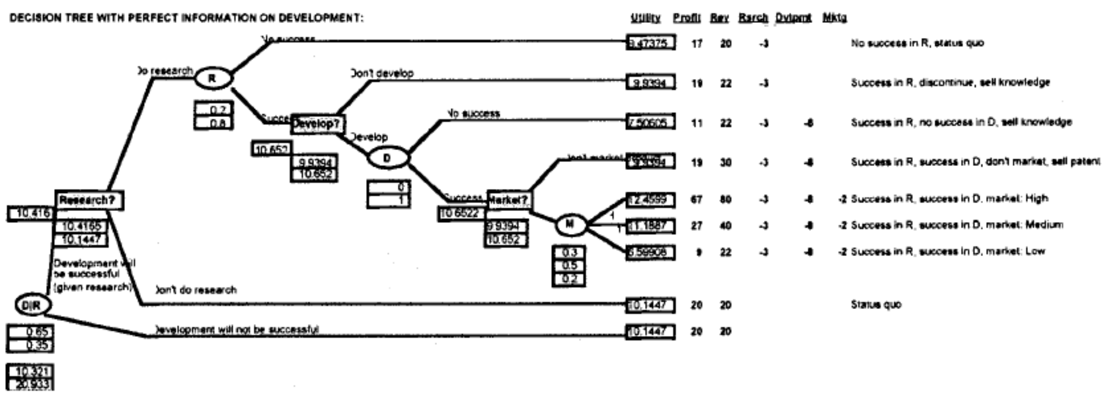
(g) - (h) - (i) The decision tree with expected utilities is displayed. The expected utilities are calculated in the following way: for each of the outcome branches of the decision tree (e.g., profit of $\$6,700,000$), the corresponding utility is computed (e.g., 12.45992). Once this is done, the expected utilities are calculated. The best course of action is to not do research (expected utility of 10.14469 vs. 9.846267 in the case of doing research).



(j) The expected utility for perfect information on research equals 9.939397, which is still less than the expected utility of not doing research (10.14469). Therefore, the best course of action is to not do research, implying a value of zero for perfect information on the outcome of the research effort.

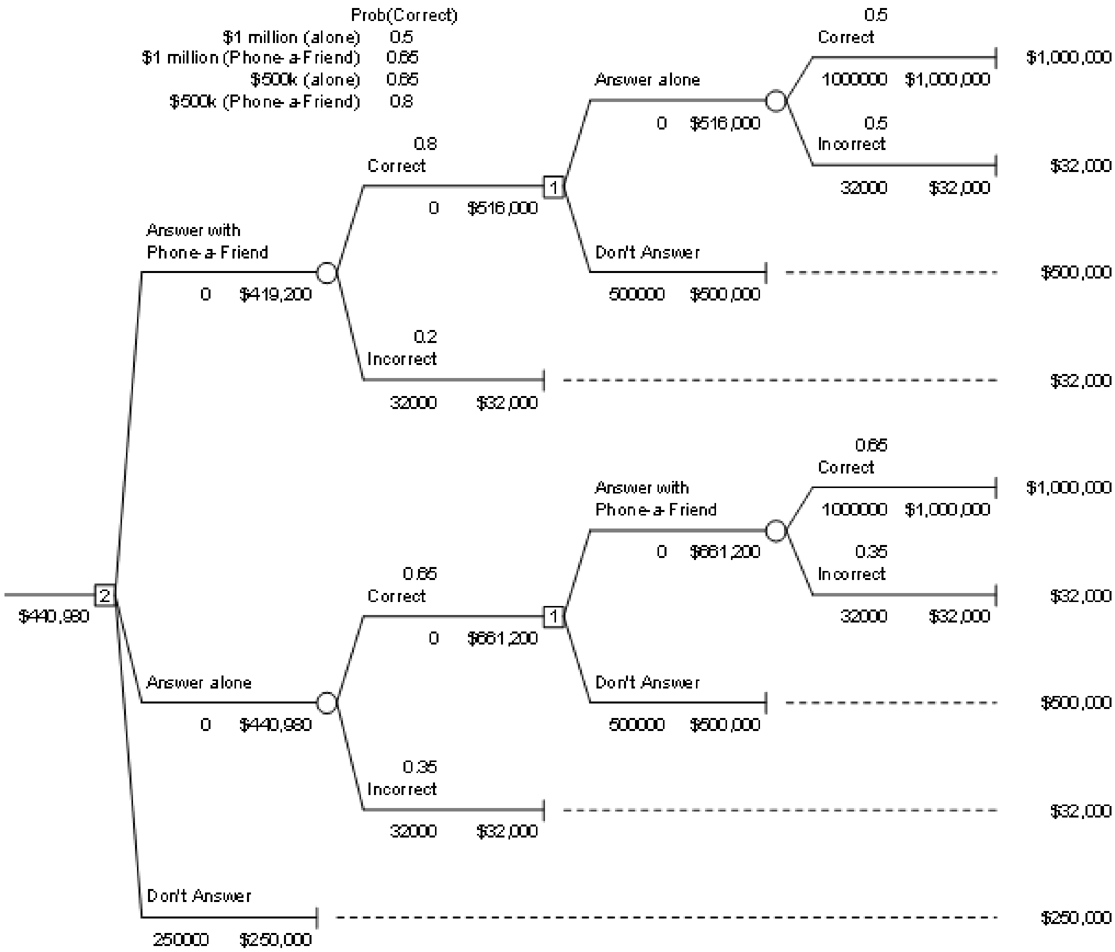


(k) The expected utility for perfect information on development equals 10.321347, which is more than the expected utility without information (10.14469). The value of perfect information on development is the difference between the inverses of these two utility values, $\mathrm{U}^{-1}(10.321347) - \mathrm{U}^{-1}(10.14469) = 20.93274 - 20 = 0.93274$. The value of perfect information on the outcome of the development effort is \$$93.274$.



CASE 15.3 Who Wants to be a Millionaire

(a) The course of action that maximizes the expected payoff is to answer \$500,000 question alone. If you get the question correct, then use the phone-a-friend lifeline to help answer \$1 million question. The expected payoff is \$440,980.



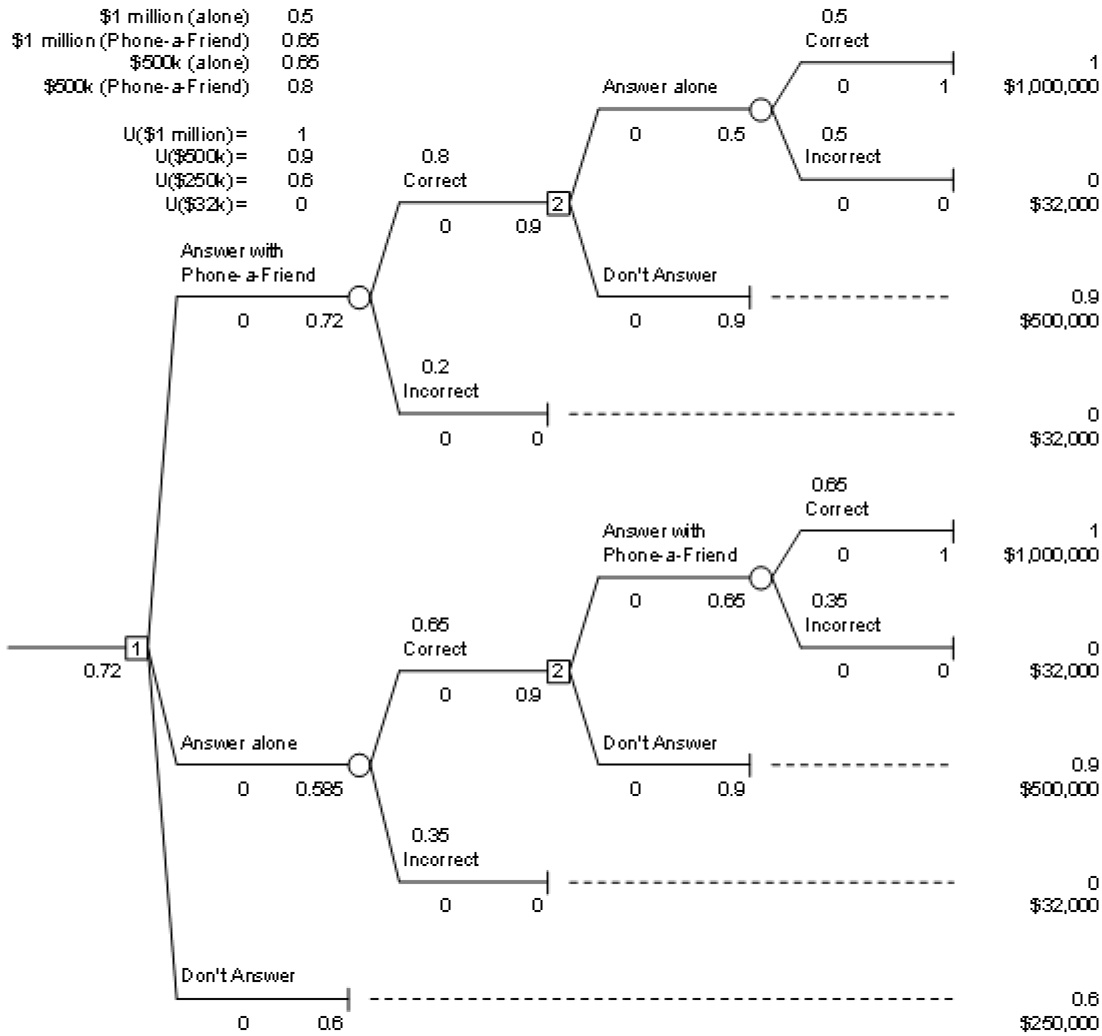
(b) Answers will vary depending on your level of risk aversion. One possible solution is obtained by setting

$$U(\textit{Maximum}) = U(\$1 \text{ million}) = 1 \text{ and } U(\textit{Minimum}) = U(\$32{,}000) = 0.$$

If getting \$250,000 for sure is equivalent to a 60% chance of getting \$1 million vs. a 40% chance of getting \$32,000, then $U(\$250{,}000) = p = 0.6$.

If getting \$500,000 for sure is equivalent to a 90% chance of getting \$1 million vs. a 10% chance of getting \$32,000, then $U(\$500{,}000) = p = 0.9$.

(c) With the utilities derived in part (b), the decision changes to using the phone-a-friend lifeline to help answer the \$500,000 question, and then walk away.


$1 million (alone) 0.5
$1 million (Phone-a-Friend) 0.65
$500k (alone) 0.65
$500k (Phone-a-Friend) 0.8
U($1 million) = 1
U($500k) = 0.9
U($250k) = 0.6
U($32k) = 0
Answer with Phone-a-Friend
0 0.72
0.8 Correct
0 0.9
Answer alone
0 0.5
0.5 Correct
0 1
1 $1,000,000
0.5 Incorrect
0 0
0 $32,000
Don't Answer
0 0.9
0.9 $500,000
0.2 Incorrect
0 0
0 $32,000
0.72
Answer alone
0 0.585
0.65 Correct
0 0.9
Answer with Phone-a-Friend
0 0.65
0.65 Correct
0 1
1 $1,000,000
0.35 Incorrect
0 0
0 $32,000
Don't Answer
0 0.9
0.9 $500,000
0.35 Incorrect
0 0
0 $32,000
Don't Answer
0 0.6
0.6 $250,000

CASE 15.4 University Toys and the Business Professor Action Figures

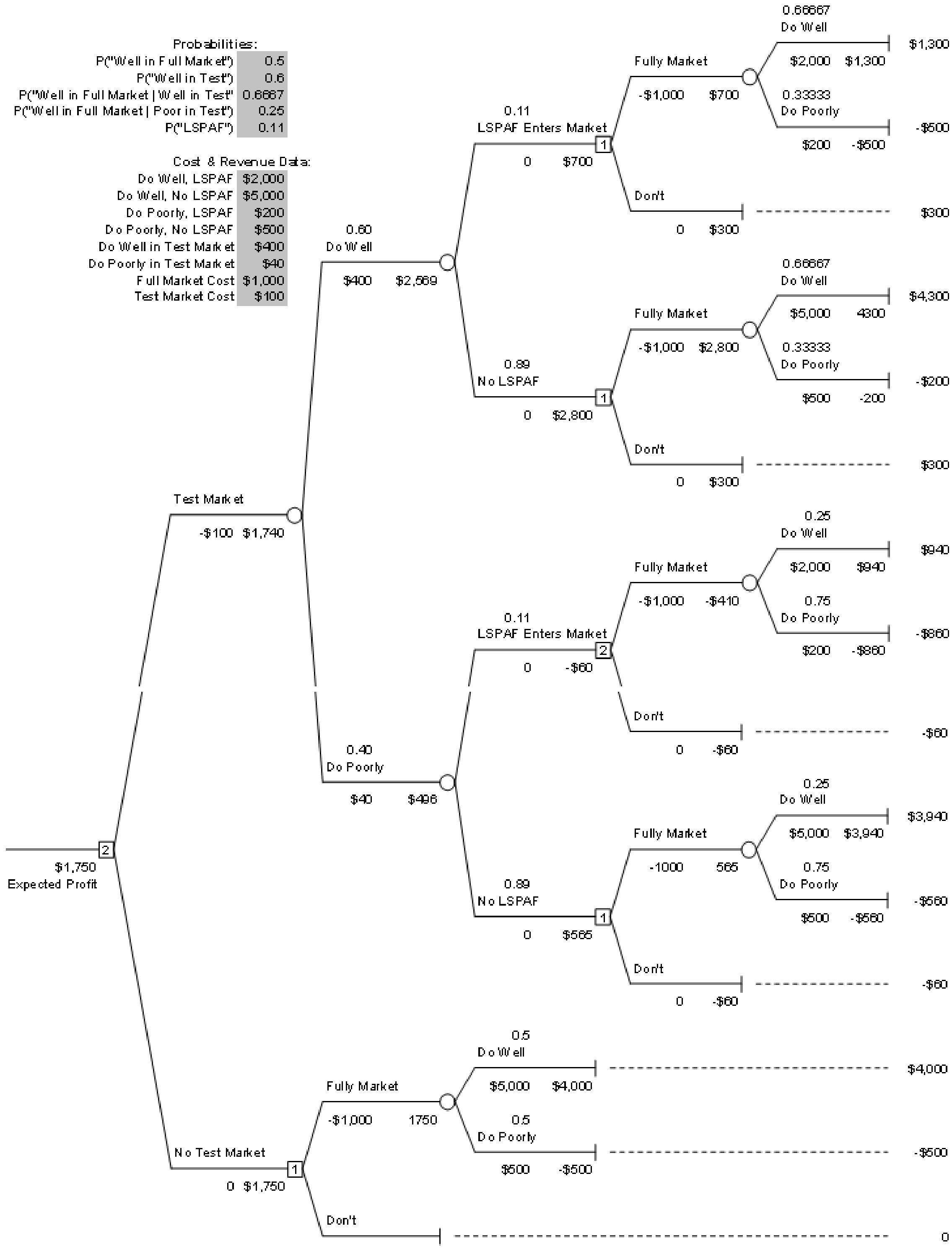
(a)

Template for Posterior Probabilities

Data:		P(Finding | State)	
State of	Prior	Finding	
Nature	Probability	Well in Test	Poor in Test
Well in Full Market	0.5	0.8	0.2
Poor in Full Market	0.5	0.4	0.6

Posterior		P(State | Finding)	
Probabilities:		State of Nature	
Finding	P(Finding)	Well in Full Market	Poor in Full Market
Well in Test	0.6	0.666666667	0.333333333
Poor in Test	0.4	0.25	0.75

(b) The best course of action is to skip the test market, and immediately market the product fully. The expected payoff is $1750.



(c) If the probability that the LSPAFs enter the market before the test marketing would be completed increases this would make the test market even less desirable, so it would still not be worthwhile to do. However, if the probability decreases, this would make the test market more desirable. It might reach the point where the test market is worthwhile.

(d) Let p denote the probability that the LSPAFs will enter and EP the expected payoff.

p	EP	Test Market?
	\$1,750	No
0.0	\$1,906	Yes
0.1	\$1,755	Yes
0.2	\$1,750	No
0.3	\$1,750	No
0.4	\$1,750	No
0.5	\$1,750	No
0.6	\$1,750	No
0.7	\$1,750	No
0.8	\$1,750	No
0.9	\$1,750	No
1.0	\$1,750	No

(e) It is better to perform the test market if the probability that the LSPAFs will enter the market is 10% or less. It is better to skip the test market if this probability is greater than 10%.

CHAPTER 16: MARKOV CHAINS

16.2-1.

In this study, Markov chains are used to model the changes in credit ratings of corporations that work with Merrill Lynch Bank USA. The bank manages a portfolio of revolving credit-line commitments worth billions of dollars. A corporation that has a credit line can withdraw a significant amount of money from the bank on short notice. The risk associated with the bank's ability to meet these cash requests is referred to as the liquidity risk. Merrill Lynch developed a model to assess this risk and to evaluate various scenarios like financial stress. The model consists of a mix of multiple OR techniques. The core of the model is a Monte Carlo simulation of revolving credit lines. In doing this, the monthly changes in credit ratings for each company are modeled as a discrete-time Markov chain. A company's rating in a month is assumed to depend only on its rating in the previous month and the transition probabilities used in forming the credit-migration matrix are assumed to be stationary.

The model provided Merrill Lynch Bank a systematic way to measure and to manage the liquidity risk. After the implementation of the model, required liquidity reserves have been decreased by 30% and $4 billion that is freed up consequently can now be used in more profitable investments. During the first 21 months, the bank's portfolio has increased from $8 billion to $13 billion and from 80 companies to 100. The evaluation of different scenarios enabled the bank to ensure liquidity even during financial crises. The basic model is now run once every month and is also used in long-term planning.

16.2-2.

(a) Since the probability of rain tomorrow is only dependent on the weather today, Markovian property holds for the evolution of the weather.

(b) Let the two states be $0 =$ Rain and $1 =$ No Rain. Then the transition matrix is

$$\mathrm{P} = \mathrm{P}^{(1)} = \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{pmatrix}.$$

16.2-3.

(a) Let $1 =$ increased today and yesterday,
$2 =$ increased today and decreased yesterday,
$3 =$ decreased today and increased yesterday,
$4 =$ decreased today and yesterday.

$$\mathrm{P} = \mathrm{P}^{(1)} = \begin{pmatrix} \alpha_1 & 0 & 1-\alpha_1 & 0 \\ \alpha_2 & 0 & 1-\alpha_2 & 0 \\ 0 & \alpha_3 & 0 & 1-\alpha_3 \\ 0 & \alpha_4 & 0 & 1-\alpha_4 \end{pmatrix}$$

(b) The state space is properly defined to include information about changes yesterday and today. This is the only information needed to determine the next state, namely changes today and tomorrow.

16.2-4.

Yes, it can be formulated as a Markov chain with the following $8\ (=2^3)$ states.

State	Today	1 Day Ago	2 Days Ago
1	inc	inc	inc
2	inc	inc	dec
3	inc	dec	inc
4	inc	dec	dec
5	dec	inc	inc
6	dec	inc	dec
7	dec	dec	inc
8	dec	dec	dec

These states include all the information needed to predict the change in the stock tomorrow whereas the states in Prob. 16.2-2 do not consider the day before yesterday, so they do not contain all necessary information to predict the change tomorrow.

16.3-1.

(a)

$$\text{P}^{(2)} = \begin{pmatrix} 0.3 & 0.7 \\ 0.14 & 0.86 \end{pmatrix} \qquad \text{P}^{(5)} = \begin{pmatrix} 0.175 & 0.825 \\ 0.165 & 0.835 \end{pmatrix}$$

$$\text{P}^{(10)} = \begin{pmatrix} 0.167 & 0.833 \\ 0.167 & 0.833 \end{pmatrix} \qquad \text{P}^{(20)} = \begin{pmatrix} 0.167 & 0.833 \\ 0.167 & 0.833 \end{pmatrix}$$

(b)

$$\text{P}(\text{Rain } n \text{ days from now} \mid \text{Rain today}) = \text{P}_{11}^{(n)}$$

$$\text{P}(\text{Rain } n \text{ days from now} \mid \text{No rain today}) = \text{P}_{21}^{(n)}.$$

If the probability it will rain today is 0.5,

$$\text{P}(\text{Rain } n \text{ days from now}) = p_n = 0.5\text{P}_{11}^{(n)} + 0.5\text{P}_{21}^{(n)}.$$

Hence, $p_2 = 0.22$, $p_5 = 0.17$, $p_{10} = 0.167$, $p_{20} = 0.167$.

(c) We find $\pi_1 = 0.167$ and $\pi_2 = 0.833$. As n grows large, $\text{P}^{(n)}$ approaches

$$\begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix},$$

the stationary probabilities. Indeed,

$$\text{P}^{(10)} = \text{P}^{(20)} = \begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix}.$$

16.3-2.

(a) Let states 0 and 1 denote that a 0 and a 1 have been recorded respectively. Then the transition matrix is

$$\mathrm{P} = \begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix},$$

where $q = 0.005$.

(b)

$$\mathrm{P}^{(10)} = \begin{pmatrix} 0.952 & 0.048 \\ 0.048 & 0.952 \end{pmatrix}$$

The probability that a digit will be recorded accurately after the last transmission is 0.952.

(c)

$$\mathrm{P}^{(10)} = \begin{pmatrix} 0.98 & 0.02 \\ 0.02 & 0.98 \end{pmatrix}$$

The probability that a digit will be recorded accurately after the last transmission is 0.98.

16.3-3.

(a)

$$\mathrm{P} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{pmatrix}.$$

(b)

$$P^5 = \begin{bmatrix} 0.062 & 0.312 & 0.156 & 0.156 & 0.312 \\ 0.312 & 0.062 & 0.312 & 0.156 & 0.156 \\ 0.156 & 0.312 & 0.062 & 0.312 & 0.156 \\ 0.156 & 0.156 & 0.312 & 0.062 & 0.312 \\ 0.312 & 0.156 & 0.156 & 0.312 & 0.062 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.248 & 0.161 & 0.215 & 0.215 & 0.161 \\ 0.161 & 0.248 & 0.161 & 0.215 & 0.215 \\ 0.215 & 0.161 & 0.248 & 0.161 & 0.215 \\ 0.215 & 0.215 & 0.161 & 0.248 & 0.161 \\ 0.161 & 0.215 & 0.215 & 0.161 & 0.248 \end{bmatrix}$$

$$P^{20} = \begin{bmatrix} 0.206 & 0.195 & 0.202 & 0.202 & 0.195 \\ 0.195 & 0.206 & 0.195 & 0.202 & 0.202 \\ 0.202 & 0.195 & 0.206 & 0.195 & 0.202 \\ 0.202 & 0.202 & 0.195 & 0.206 & 0.195 \\ 0.195 & 0.202 & 0.202 & 0.195 & 0.206 \end{bmatrix}$$

$$P^{40} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

$$P^{80} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

(c) $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 0.2$.

16.4-1.

(a) P has one recurrent communicating class: $\{0, 1, 2, 3\}$.

(b) P has 3 communicating classes: $\{0\}$ absorbing, so recurrent; $\{1, 2\}$ recurrent and $\{3\}$ transient.

16.4-2.

(a) P has one recurrent communicating class: $\{0, 1, 2, 3\}$.

(b) P has one recurrent communicating class: $\{0, 1, 2\}$.

16.4-3.

P has 3 communicating classes: $\{0, 1\}$ recurrent, $\{2\}$ transient and $\{3, 4\}$ recurrent.

16.4-4.

P has one communicating class, so each state has the same period 4.

16.4-5.

(a) P has two classes: $\{0, 1, 2, 4\}$ transient and $\{3\}$ recurrent.

(b) The period of $\{0, 1, 2, 4\}$ is 2 and the period of $\{3\}$ is 1.

16.5-1.

$$\mathrm{P} = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}$$

$\pi \mathrm{P} = \pi \Rightarrow \alpha\pi_1 + (1-\beta)\pi_2 = \pi_1$ and $\pi_1 + \pi_2 = 1$

$\Rightarrow \pi = \left(\frac{1-\beta}{2-\alpha-\beta}, \frac{1-\alpha}{2-\alpha-\beta}\right)$.

16.5-2.

We need to show that $\pi_j = \frac{1}{M+1}$ for $j = 0, 1, \ldots, M$ satisfies the steady-state equations: $\pi_j = \sum_{i=0}^{M} \pi_i P_{ij}$ and $\sum_{i=0}^{M} \pi_i = 1$. These are easily verified, using $\sum_{i=0}^{M} P_{ij} = 1$ for every j. The chain is irreducible, aperiodic and positive recurrent , so this is the unique solution.

16.5-3.

$M = 5 \Rightarrow \pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 1/5 = 0.2$
The steady-state probabilities do not change if the probabilities for moving steps change.

16.5-4.

$\pi = (0.511, 0.289, 0.2)$

The steady-state market share for A and B are 0.511 and 0.289 respectively.

16.5-5.

(a) Assuming demand occurs after delivery of orders:

$$\mathrm{P} = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 & 0 & 0 \\ 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0 & 0.1 & 0.2 & 0.7 \end{pmatrix}$$

(b) $\pi \mathrm{P} = \pi$ and $\sum_j \pi_j = 1 \;\Rightarrow\; \pi = (0.139\;\; 0.139\;\; 0.139\;\; 0.138\;\; 0.141\;\; 0.130\;\; 0.174)$.

(c) The steady-state probability that a pint of blood is to be discarded is

$$\mathrm{P}(D = 0) \cdot \mathrm{P}(\text{state} = 7) = 0.4 \text{ x } 0.174 = 0.0696.$$

(d)
$$\begin{aligned} \mathrm{P}(\text{need for emergency delivery}) &= \textstyle\sum_{i=1}^{2} \mathrm{P}(\text{state} = i) \cdot \mathrm{P}(D > i) \\ &= 0.139 \text{ x } (0.2 + 0.1) + 0.139 \text{ x } 0.1 \\ &= 0.0556 \end{aligned}$$

16.5-6.

For an (s, S) policy with $s = 2$ and $S = 3$:

$$c(x_{t-1}, D_t) = \begin{cases} 10 + 25(3 - x_{t-1}) + 50\max(D_t - 3, 0) & \text{for } x_{t-1} < 2 \\ 50\max(D_t - x_{t-1}, 0) & \text{for } x_{t-1} \geq 2. \end{cases}$$

$K(0) = E[c(0, D_t)] = 85 + 50[\sum_{j=4}^{\infty} (j - 3) \cdot P(D_t = j)] \simeq 86.2,$

$K(1) = E[c(1, D_t)] = 60 + 50[\sum_{j=4}^{\infty} (j - 3) \cdot P(D_t = j)] \simeq 61.2,$

$K(2) = E[c(2, D_t)] = 0 + 50[\sum_{j=4}^{\infty}(j-2) \cdot P(D_t = j)] \simeq 5.2,$

$K(3) = E[c(3, D_t)] = 0 + 50[\sum_{j=4}^{\infty}(j-2) \cdot P(D_t = j)] \simeq 1.2.$

$$x_{t+1} = \begin{cases} \max(3 - D_{t+1}, 0) & \text{for } x_t < 2 \\ \max(x_t - D_{t+1}, 0) & \text{for } x_t \geq 2 \end{cases}$$

$$\mathrm{P} = \begin{pmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.080 & 0.184 & 0.368 & 0.368 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{pmatrix}$$

Solving the steady-state equations gives $(\pi_0, \pi_1, \pi_2, \pi_3) = (0.148, 0.252, 0.368, 0.232)$. Then the long-run average cost per week is $\sum_{j=0}^{3} K(j) \cdot \pi_j = 30.37$.

16.5-7.

(a)

$$x_{t+1} = \begin{cases} \max(x_t + 2 - D_{t+1}, 0) & \text{for } x_t \leq 1 \\ \max(x_t - D_{t+1}, 0) & \text{for } x_t \geq 2 \end{cases}$$

$$\mathrm{P} = \begin{pmatrix} 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{pmatrix}$$

Solving the steady-state equations gives $(\pi_0, \pi_1, \pi_2, \pi_3) = (0.182, 0.285, 0.368, 0.165)$.

(b) $\lim_{n\to\infty} E\left(\frac{1}{n}\sum_{t=1}^{n} c(x_t)\right) = 0 \cdot \pi_0 + 2 \cdot \pi_1 + 8 \cdot \pi_2 + 18 \cdot \pi_3 = 6.48.$

16.5-8.

(a) $\mathrm{P}_{11} = \mathrm{P}(D_{n+1} = 0) + \mathrm{P}(D_{n+1} = 2) + \mathrm{P}(D_{n+1} = 4) = 3/5$

$\mathrm{P}_{12} = \mathrm{P}(D_{n+1} = 1) + \mathrm{P}(D_{n+1} = 3) = 2/5$

$\mathrm{P}_{21} = \mathrm{P}(D_{n+1} = 1) + \mathrm{P}(D_{n+1} = 3) = 2/5$

$\mathrm{P}_{22} = \mathrm{P}(D_{n+1} = 0) + \mathrm{P}(D_{n+1} = 2) + \mathrm{P}(D_{n+1} = 4) = 3/5$

$$\mathrm{P} = \begin{pmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{pmatrix}$$

(b) $\pi = \pi\mathrm{P}$ and $\pi_1 + \pi_2 = 1 \;\Rightarrow\; \pi_1 = \pi_2 = 1/2$.

(c) P is doubly stochastic and there are two states, so $\pi_1 = \pi_2 = 1/2$.

(d)
$$\begin{aligned} K(1) &= E[c(1, D_n)] \\ &= (2/5)[3 + 2(1)] + (2/5)[3 + 2(2)] + (1/5)(1) + (4/5)[1 + 2 + 3] \\ &= 9.8, \end{aligned}$$

$$\begin{aligned} K(2) &= E[c(2, D_n)] \\ &= (2/5)[3+2(1)] + (1/5)[3+2(2)] + (1/5)(2+1) + (4/5)[1+2] \\ &= 6.4. \end{aligned}$$

So the long-run average cost per unit time is $9.8(1/2) + 6.4(1/2) = 8.1$.

16.5-9.

(a) P(the unit will be inoperable after n periods) $= \mathrm{P}_{02}^{(n)}$

$$\mathrm{P}^2 = \begin{bmatrix} 0.64 & 0.16 & 0.04 & 0.16 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0.64 & 0.16 & 0.04 & 0.16 \end{bmatrix} \qquad \mathrm{P}^5 = \begin{bmatrix} 0.62 & 0.195 & 0.037 & 0.148 \\ 0.594 & 0.174 & 0.046 & 0.186 \\ 0.64 & 0.232 & 0.026 & 0.102 \\ 0.62 & 0.195 & 0.037 & 0.148 \end{bmatrix}$$

$$\mathrm{P}^{10} = \begin{bmatrix} 0.615 & 0.192 & 0.039 & 0.154 \\ 0.616 & 0.193 & 0.038 & 0.153 \\ 0.614 & 0.191 & 0.039 & 0.156 \\ 0.615 & 0.192 & 0.039 & 0.154 \end{bmatrix} \qquad \mathrm{P}^{20} = \begin{bmatrix} 0.615 & 0.192 & 0.038 & 0.154 \\ 0.615 & 0.192 & 0.038 & 0.154 \\ 0.615 & 0.192 & 0.038 & 0.154 \\ 0.615 & 0.192 & 0.038 & 0.154 \end{bmatrix}$$

$n = 2$: $\mathrm{P}_{02}^{(n)} = 0.04$; $n = 5$: $\mathrm{P}_{02}^{(n)} = 0.037$;

$n = 10$: $\mathrm{P}_{02}^{(n)} = 0.039$; $n = 20$: $\mathrm{P}_{02}^{(n)} = 0.038$.

(b) $\pi_0 = 0.615$, $\pi_1 = 0.192$, $\pi_2 = 0.038$, and $\pi_3 = 0.154$.

(c) Long-run average cost per period is $30,000\pi_3 = 4,620$.

16.6-1.

(a)

$$\mathrm{P} = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}$$

(b)
$$\begin{aligned} \mu_{00} &= 1 + 0.05\mu_{10} \\ \mu_{01} &= 1 + 0.95\mu_{01} \\ \mu_{10} &= 1 + 0.50\mu_{10} \\ \mu_{11} &= 1 + 0.50\mu_{01} \end{aligned}$$

$\Rightarrow\ \mu_{00} = 1.1, \mu_{01} = 20, \mu_{10} = 2, \mu_{11} = 11$

16.6-2.

(a) States: $0 =$ Operational, $1 =$ Down, $2 =$ Repaired.

$$\mathrm{P} = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0 & 0 & 1 \\ 0.9 & 0.1 & 0 \end{pmatrix}$$

(b) We need to solve $\mu_{ij} = 1 + \sum_{k \neq j} \text{P}_{ik} \mu_{kj}$ for every i and j.

$$\mu_{00} = 1 + 0.1\mu_{10}$$
$$\mu_{10} = 1 + \mu_{20}$$
$$\mu_{20} = 1 + 0.1\mu_{10}$$

$\Rightarrow \mu_{00} = 11/9, \mu_{10} = 20/9, \mu_{20} = 11/9$

$$\mu_{01} = 1 + 0.9\mu_{01}$$
$$\mu_{11} = 1 + \mu_{21}$$
$$\mu_{21} = 1 + 0.9\mu_{01}$$

$\Rightarrow \mu_{01} = 10, \mu_{11} = 11, \mu_{21} = 10$

$$\mu_{02} = 1 + 0.9\mu_{02} + 0.1\mu_{12}$$
$$\mu_{12} = 1 + 0$$
$$\mu_{22} = 1 + 0.9\mu_{02} + 0.1\mu_{12}$$

$\Rightarrow \mu_{02} = 11, \mu_{12} = 1, \mu_{22} = 11$

The expected number of full days that the machine will remain operational before the next breakdown after a repair is completed is $\mu_{01} = 10$.

(c) It remains the same because of the Markovian property. The expected number of days the machine will remain operational starting operational does not depend on how long the machine remained operational in the past.

16.6-3.

(a) We order the states as $(1,1)$, $(0,1)$ and $(1,0)$ and write the transition matrix:

$$\text{P} = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.9 & 0 & 0.1 \\ 0.9 & 0.1 & 0 \end{pmatrix}.$$

(b) $\mu_{33} = 1/\pi_3$. From $\pi = \pi\text{P}$ and $\pi \cdot 1 = 1$, we get $\pi_3 = 1/110$, so the expected recurrence time for the state $(1,0)$ is $\mu_{33} = 110$.

16.6-4.

(a)

$$\text{P} = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \end{pmatrix}$$

(b)

$$\text{P}^{(2)} = \begin{pmatrix} 0.5 & 0.375 & 0.125 \\ 0.375 & 0.438 & 0.188 \\ 0.5 & 0.375 & 0.125 \end{pmatrix}$$

$$P^{(5)} = \begin{pmatrix} 0.449 & 0.4 & 0.15 \\ 0.451 & 0.399 & 0.149 \\ 0.449 & 0.4 & 0.15 \end{pmatrix}$$

$$P^{(10)} = \begin{pmatrix} 0.45 & 0.4 & 0.15 \\ 0.45 & 0.4 & 0.15 \\ 0.45 & 0.4 & 0.15 \end{pmatrix}$$

(c) $\mu_{00} = 1 + 0.5\mu_{10} + 0.25\mu_{20}$
$\mu_{10} = 1 + 0.25\mu_{10}$
$\mu_{20} = 1 + 0.5\mu_{10} + 0.25\mu_{20}$

$\Rightarrow \mu_{00} = 20/9, \mu_{10} = 4/3, \mu_{20} = 20/9$

$\mu_{01} = 1 + 0.25\mu_{01} + 0.25\mu_{21}$
$\mu_{11} = 1 + 0.75\mu_{01}$
$\mu_{21} = 1 + 0.25\mu_{01} + 0.25\mu_{21}$

$\Rightarrow \mu_{01} = 2, \mu_{11} = 2\frac{1}{2}, \mu_{21} = 2$

$\mu_{02} = 1 + 0.25\mu_{02} + 0.5\mu_{12}$
$\mu_{12} = 1 + 0.75\mu_{02} + 0.25\mu_{12}$
$\mu_{22} = 1 + 0.25\mu_{02} + 0.5\mu_{12}$

$\Rightarrow \mu_{02} = 20/3, \mu_{12} = 8, \mu_{22} = 20/3$

(d) The steady-state probability vector is $(0.45 \;\; 0.4 \;\; 0.15)$.

(e) $\pi \cdot C = 0(0.45) + 2(0.4) + 8(0.15) =$ \$ 2 / week

16.6-5.

(a)

$$P = \begin{pmatrix} 0 & 0.875 & 0.062 & 0.062 \\ 0 & 0.75 & 0.125 & 0.125 \\ 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\pi_0 = 0.154$, $\pi_1 = 0.538$, $\pi_2 = 0.154$, and $\pi_3 = 0.154$

(b) $\pi \cdot C = 1(0.538) + 3(0.154) + 6(0.154) =$ \$ 1923.08

(c) $\mu_{00} = 1 + 0.875\mu_{10} + 0.0625\mu_{20} + 0.0625\mu_{30}$
$\mu_{10} = 1 + 0.75\mu_{10} + 0.125\mu_{20} + 0.125\mu_{30}$
$\mu_{20} = 1 + 0.5\mu_{20} + 0.5\mu_{30}$
$\mu_{30} = 1 + 0$

So the expected recurrence time for state 0 is $\mu_{00} = 6.5$.

16.7-1.

(a) $\text{P}_{00} = \text{P}_{TT} = 1$; $\text{P}_{i,i-1} = q$; $\text{P}_{i,i+1} = p$; $\text{P}_{i,k} = 0$ else.

$$\text{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & & \\ q & 0 & p & 0 & \cdots & & & \\ & \vdots & & \ddots & & & & \\ & & & & q & 0 & p & 0 \\ & & & & 0 & q & 0 & p \\ & & & & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Class 1: $\{0\}$ absorbing
Class 2: $\{T\}$ absorbing
Class 3: $\{1, 2, \ldots, T-1\}$ transient

(c) Let $f_{iK} = \text{P}(\text{absorption at } K \text{ starting at } i)$. Then $f_{00} = f_{33} = 1$, $f_{30} = f_{03} = 0$. Since $\text{P}_{ij} = 0$ for $|i-j| \neq 1$ and $\text{P}_{i,i+1} = p$, $\text{P}_{i,i-1} = q$, we get:

$$f_{10} = q + pf_{20}$$
$$f_{13} = 1 - f_{10}$$
$$f_{20} = qf_{10}$$
$$f_{23} = 1 - f_{20}$$

Solving this system gives

$$f_{10} = \frac{q}{1-pq} = 0.886,\ f_{13} = 0.114,\ f_{20} = 0.62,\ f_{23} = 0.38.$$

(d) Plugging in $p = 0.7$ in the formulas in part (c), we obtain

$$f_{10} = 0.38,\ f_{13} = 0.62,\ f_{20} = 0.114,\ f_{23} = 0.886.$$

Observe that when $p > 1/2$, the drift is towards T and when $p < 1/2$, it is towards 0.

16.7-2.

(a) $0 =$ Have to honor warranty
$1 =$ Reorder in 1st year
$2 =$ Reorder in 2nd year
$3 =$ Reorder in 3rd year

$$\text{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.01 & 0 & 0.99 & 0 \\ 0.05 & 0 & 0 & 0.95 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) The probability that the manufacturer has to honor the warranty is $f_{10.}$

$$f_{10} = 0.01f_{00} + 0f_{10} + 0.99f_{20} + 0f_{30}$$
$$f_{20} = 0.05f_{00} + 0f_{10} + 0f_{20} + 0.95f_{30}$$
$f_{00} = 1$ and $f_{30} = 0$
$\Rightarrow f_{10} = 0.01 + 0.99f_{20}$ and $f_{20} = 0.05$
$\Rightarrow f_{10} = 0.0595 = 5.95\%.$

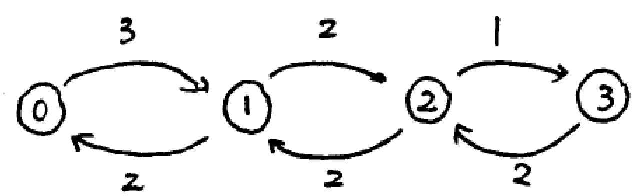
16.8-1.

In 1998, the new management of PSA Peugeot Citroën set new goals regarding the amount of production, the introduction of new models and the profit in the following years. To achieve these goals, the car-body shops, which were the bottlenecks of production, needed to be redesigned. A new architecture was needed to process diverse models on the same platform and to introduce new models quickly. To evaluate the performance of various designs, PSA adopted a combination of simulation and analytic models. The states of a machine or of a worker in a production line are modeled as a continuous time Markov chain. A machine can be either up or down. When it is up, it can go down with the average failure rate λ. When it is down, it can be repaired with the average repair rate μ. If the machine can fail only when it is processing a car part, an additional state is included to represent an idle machine. The operators are also treated as machines. An operator's state is "up" when he is working regular time and "down" when he is working overtime.

As a result of this study, a software called DispO is developed. A conservative estimate of the additional profit generated using this software is $130 million, which is around 6.5% of the total profit. PSA acquired a 4% increase in productivity in 2001. Additionally, the new model enabled PSA to understand its assembly lines better and to correct "some incorrect but deeply ingrained beliefs and practices" [p. 46]. The ability to compare the estimates obtained from analysis with the actual values convinced the personnel about the reliability of the methods. As a consequence, OR gained importance throughout the company. The efficiency in improving production-line designs enhanced throughput without overloading workers. The social climate and the quality of production are both improved. The software developed is used also by the suppliers of PSA. This, in turn, reduces the time to negotiate schedules.

16.8-2.

(a)



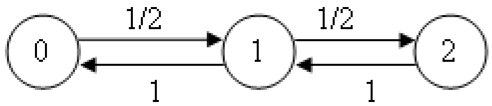
(b) Steady-state equations:

$$
\begin{aligned}
&3\pi_0 = 2\pi_1 \\
&4\pi_1 = 3\pi_0 + 2\pi_2 \\
&3\pi_2 = 2\pi_1 + 2\pi_3 \\
&2\pi_3 = \pi_2 \\
&\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1
\end{aligned}
$$

(c) Solving the steady-state equations gives $\pi = \left(\frac{4}{19}, \frac{6}{19}, \frac{6}{19}, \frac{3}{19}\right)$.

16.8-3.

(a) Let the state be the number of jobs at the work center.



(b) Steady-state equations:

$$\frac{1}{2}\pi_0 = \pi_1$$

$$\frac{3}{2}\pi_1 = \frac{1}{2}\pi_0 + \pi_2$$

$$\pi_2 = \frac{1}{2}\pi_1$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

(c) Solving the steady-state equations gives $\pi = \left(\frac{4}{7}, \frac{2}{7}, \frac{1}{7}\right)$.

CHAPTER 17: QUEUEING THEORY

17.2-1.

A typical barber shop is a queueing system with input source being the population having hair, customers being the people who want haircut and servers being the barbers. The queue forms as customers wait for a barber to serve them. The customers are served usually with the first-come-first-served discipline. The service mechanism involves the barbers and equipment.

17.2-2.

(a) Average number of customers in the shop, including those getting their haircut:

$$L = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 2\left(\frac{6}{16}\right) + 3\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right) = 2$$

(b)

n	# in queue	probability	product
0	0		
1	0		
2	0		
3	1	0.25	0.25
4	2	0.0625	0.125

Average number of customers waiting in the shop: $L_q = 0.375$

(c) Expected number of customers being served: $\frac{4}{16} + 2\left(\frac{6}{16} + \frac{4}{16} + \frac{1}{16}\right) = \frac{13}{8}$

(d) $W = \frac{L}{\lambda} = \frac{2}{4} = 0.5$ hours $= 30$ minutes

$W_q = \frac{L_q}{\lambda} = \frac{0.375}{4} = 0.094$ hours $= 5.625$ minutes

Hence, each customer will be in the shop for half an hour on the average. This includes the time to get a haircut. The average waiting time for a customer before getting a haircut is 5.625 minutes.

(e) $W - W_q = 0.406$ hours $= 24.36$ minutes

17.2-3.

(a) A parking lot is a queueing system for providing parking. The customers are the cars and the servers are the parking spaces. The service time is the amount of time a car stays parked in a space and the queue capacity is zero.

(b) $L = 0(0.1) + 1(0.2) + 2(0.4) + 3(0.3) = 1.9$ cars

$L_q = 0$ cars

$W = \frac{L}{\lambda} = \frac{1.9}{2} = 0.95$ hours

$W_q = \frac{L_q}{\lambda} = \frac{0}{2} = 0$ hours

(c) A car spends an average of 57 minutes in a parking space.

17.2-4.

(a) FALSE. The queue is where customers wait before being served.

(b) FALSE. Queueing models conventionally assume infinite capacity.

(c) TRUE. The most common is first-come-first-served.

17.2-5.

(a) A bank is a queueing system with people as the customers and tellers as the servers.

(b) $W_q = 1$ minute

$W = W_q + \frac{1}{\mu} = 1 + 2 = 3$ minutes

$L_q = \lambda W_q = \frac{40}{60}(1) = 0.667$ customers

$L = \lambda W = \frac{40}{60}(3) = 2$ customers

17.2-6.

The utilization factor ρ represents the fraction of time that the server is busy. The server is busy except when there is nobody in the system. P_0 is the probability of having zero customers in the system, so $\rho = 1 - P_0$.

17.2-7.

$\lambda_2 = 2\lambda_1, \mu_2 = 2\mu_1, L_2 = 2L_1 \Rightarrow \frac{W_1}{W_2} = \frac{L_1/\lambda_1}{L_2/\lambda_2} = 1$

17.2-8.

(a)

$$L = \begin{cases} L_q & \text{when nobody is in the system} \\ L_q + 1 & \text{otherwise} \end{cases}$$

$\Rightarrow L = P_0 L_q + (1 - P_0)(L_q + 1) = L_q + (1 - P_0)$

(b) $L = \lambda W = \lambda(W_q + 1/\mu) = \lambda W_q + \lambda/\mu = L_q + \rho$

(c) $L = L_q + \rho = L_q + (1 - P_0) \Rightarrow \rho = (1 - P_0)$

17.2-9.

$$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{s-1} nP_n + \sum_{n=s}^{\infty} nP_n = \sum_{n=0}^{s-1} nP_n + \sum_{n=s}^{\infty} (n-s)P_n + \sum_{n=s}^{\infty} sP_n$$

$$= \sum_{n=0}^{s-1} nP_n + L_q + s\sum_{n=s}^{\infty} P_n = \sum_{n=0}^{s-1} nP_n + L_q + s\left(1 - \sum_{n=0}^{s-1} P_n\right)$$

17.3-1.

Part	Customers	Servers
(a)	Customers waiting for checkout	Checkers
(b)	Fires	Firefighting units
(c)	Cars	Toll collectors
(d)	Broken bicycles	Bicycle repairpersons
(e)	Ships to be loaded or unloaded	Longshoremen & equipment
(f)	Machines needing operator	Operator
(g)	Materials to be handled	Handling equipment
(h)	Calls for plumbers	Plumbers
(i)	Custom orders	Customized process
(j)	Typing requests	Typists

17.4-1.

$\lambda_n = 1/2$ for $n \geq 0$ and $\mu_n = \begin{cases} 1/2 & \text{for } n = 1 \\ 1 & \text{for } n \geq 2 \end{cases}$

(a) $P\{\text{next arrival before 1:00}\} = 1 - e^{-1/2} = 0.393$

$P\{\text{next arrival between 1:00 and 2:00}\} = (1 - e^{-(1/2)\cdot 2})(1 - e^{-1/2}) = 0.239$

$P\{\text{next arrival after 2:00}\} = e^{-(1/2)\cdot 2} = 0.368$

(b) Probability that the next arrival will occur between 1:00 and 2:00 given no arrivals between 12:00 and 1:00 is $(1 - e^{-1/2}) = 0.393$.

(c) $P\{\text{no arrivals between 1:00 and 2:00}\} = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-1/2} = 0.607$

$P\{\text{one arrival between 1:00 and 2:00}\} = \frac{(\lambda t)^1 e^{-\lambda t}}{1!} = \frac{1}{2} e^{-1/2} = 0.303$

$P\{\text{two or more arrivals between 1:00 and 2:00}\} = 1 - e^{-1/2} - \frac{1}{2} e^{-1/2} = 0.09$

(d) $P\{\text{none served by 2:00}\} = e^{-1} = 0.368$

$P\{\text{none served by 1:10}\} = e^{-1(1/10)} = 0.846$

$P\{\text{none served by 1:01}\} = e^{-1(1/60)} = 0.983$

17.4-2.

$\lambda_n = 2$ for $n \geq 0 \Rightarrow P\{n \text{ arrivals in an hour}\} = \frac{2^n e^{-2}}{n!}$

(a) $P\{0 \text{ arrivals in an hour}\} = \frac{2^0 e^{-2}}{0!} = 0.135$

(b) $P\{2 \text{ arrivals in an hour}\} = \frac{2^2 e^{-2}}{2!} = 0.270$

(c) $P\{5 \text{ or more arrivals in an hour}\} = 1 - \sum_{n=0}^{4} P\{n \text{ arrivals in an hour}\}$

$= 1 - e^{-2} - 2e^{-2} - (4/3)e^{-2} - (2/3)e^{-2} = 0.527$

17.4-3.

Expected pay: $100 \cdot P\{T < 2\} + 80 \cdot P\{T > 2\} = 100 - 20 \cdot P\{T > 2\}$

$$P\{T_{\text{old}} > 2\} = e^{-\frac{1}{4}\cdot 2} = 0.607$$

$$P\{T_{\text{special}} > 2\} = e^{-\frac{1}{2}\cdot 2} = 0.368$$

Expected increase in pay: $20[P\{T_{\text{old}} > 2\} - P\{T_{\text{special}} > 2\}] = 4.78$

17.4-4.

Given the memoryless property, the system becomes a two-server after the first completion occurs. Let T be the amount of time after $t = 1$ until the next service completion occurs.

$$P\{T < t\} = P\{\min(T_2, T_3) < t\}$$

By Property 3, T has an exponential distribution with mean $0.5/2 = 0.25$.

17.4-5.

By memoryless property, $U = \min(T_1, T_2, T_3)$, where $T_1 \sim \text{Exp}(1/30)$, $T_2 \sim \text{Exp}(1/20)$, and $T_3 \sim \text{Exp}(1/15)$. By Property 3

$$U \sim \text{Exp}\left(\frac{1}{30} + \frac{1}{20} + \frac{1}{15}\right) = \text{Exp}(0.15).$$

Then, the expected waiting time is $1/0.15 \approx 6.67$ minutes.

17.4-6.

(a) From aggregation property of Poisson process, the arrival process does still have a Poisson distribution with mean rate 10 per hour, so the distribution of the time between consecutive arrivals is exponential with a mean of 0.1 hours $= 6$ minutes.

(b) The waiting time of this type 2 customer is the minimum of two exponential random variables, so by Property 3, it is exponentially distributed with a mean of 5 minutes.

17.4-7.

(a) This customer's waiting time is exponentially distributed with a mean of 5 minutes.

(b) The total waiting time of the customer in the system is $\mathcal{W} = \mathcal{W}_q + T_s$, where $\mathcal{W}_q$ and T_s are independent from each other.

$$E(\mathcal{W}) = E(\mathcal{W}_q) + E(T_s) = 5 + 10 = 15 \text{ minutes} = 1/4 \text{ hour}$$

$$\text{var}(\mathcal{W}) = \text{var}(\mathcal{W}_q) + \text{var}(T_s) = \left(\frac{1}{12}\right)^2 + \left(\frac{1}{6}\right)^2 = 0.0347$$

(c) $\overline{\mathcal{W}} = 5 + \mathcal{W} \Rightarrow E(\overline{\mathcal{W}}) = 20$ minutes, $\text{var}(\overline{\mathcal{W}}) = 0.0347$

17.4-8.

(a) FALSE. $E(T) = 1/\alpha$ and $\text{var}(T) = 1/\alpha^2$, p.775.

(b) FALSE. "The exponential distribution clearly does not provide a close approximation to the service-time distribution for this type of situation," second paragraph, p.776.

(c) FALSE. A new arrival would have an expected waiting time, before entering service of $1/n\mu$, second last paragraph, p.777.

17.4-9.

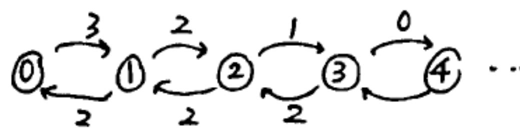
Let $U = \min\{T_1, \ldots, T_n\}$.

$$P\{U = T_j\} = \int_0^\infty P\{T_j < T_i \text{ for all } i \neq j | T_j = t\} \alpha_j e^{-\alpha_j t} dt$$

$$= \int_0^\infty e^{-t\sum_{i=1}^{n}\alpha_i} e^{\alpha_j t} \alpha_j e^{-\alpha_j t} dt = \alpha_j \int_0^\infty e^{-t\sum_{i=1}^{n}\alpha_i} dt = \frac{\alpha_j}{\sum_{i=1}^{n}\alpha_i}$$

17.5-1.

(a)



(b)

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{3}{2} P_0$$

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{3}{2} P_0$$

$$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0 = \frac{3}{4} P_0$$

$$P_4 = P_5 = \cdots = 0$$

$$P_0 + P_1 + P_2 + P_3 = \left(1 + \frac{3}{2} + \frac{3}{2} + \frac{3}{4}\right) P_0 = 1$$

$$\Rightarrow P_0 = \frac{4}{19}, P_1 = P_2 = \frac{12}{38}, P_3 = \frac{6}{38}$$

(c)

$$L = \sum_{n=0}^{\infty} n P_n = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 = \frac{27}{19} = 1.421$$

$$L_q = \sum_{n=1}^{\infty} (n-1) P_n = 0 \cdot P_1 + 1 \cdot P_2 + 2 \cdot P_3 = \frac{12}{19} = 0.632$$

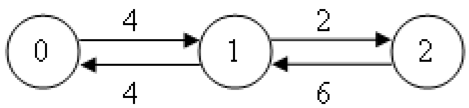
$$\overline{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n = 3 \cdot P_0 + 2 \cdot P_1 + 1 \cdot P_2 + 0 \cdot P_3 = \frac{30}{19} = 1.579$$

$$W = \frac{L}{\overline{\lambda}} = \frac{27/19}{30/19} = 0.9$$

$$W_q = \frac{L_q}{\overline{\lambda}} = \frac{12/19}{30/19} = 0.4$$

17.5-2.

(a)



(b) $4P_0 = 4P_1, 6P_1 = 4P_0 + 6P_2, 6P_2 = 2P_1, P_0 + P_1 + P_2 = 1$

(c) $P_0 = P_1 = \frac{3}{7}, P_2 = \frac{1}{7}$

(d)

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = P_0, P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{1}{3} P_0$$

$$P_0 + P_1 + P_2 = \left(1 + 1 + \frac{1}{3}\right) P_0 = 1 \Rightarrow P_0 = P_1 = \frac{3}{7}, P_2 = \frac{1}{7}$$

$$L = \sum_{n=0}^{\infty} n P_n = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 = \frac{5}{7}$$

$$L_q = \sum_{n=1}^{\infty} (n-1) P_n = 0 \cdot P_1 + 1 \cdot P_2 = \frac{1}{7}$$

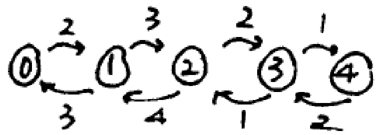
$$\overline{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n = 4 \cdot P_0 + 2 \cdot P_1 = \frac{18}{7}$$

$$W = \frac{L}{\overline{\lambda}} = \frac{5}{18}$$

$$W_q = \frac{L_q}{\overline{\lambda}} = \frac{1}{18}$$

17.5-3.

(a)



(b) (1) $2P_0 = 3P_1$
(2) $2P_0 + 4P_2 = 6P_1$
(3) $3P_1 + P_3 = 6P_2$
(4) $2P_2 + 2P_4 = 2P_3$
(5) $P_3 = 2P_4$
(6) $P_0 + P_1 + P_2 + P_3 + P_4 = 1$

(c)

(1) $\Rightarrow P_1 = \frac{2}{3}P_0$

(2) $\Rightarrow P_2 = \left(6 \cdot \frac{2}{3}P_0 - 2P_0\right)/4 = \frac{1}{2}P_0$

(3) $\Rightarrow P_3 = \left(6 \cdot \frac{1}{2}P_0 - 3 \cdot \frac{2}{3}P_0\right) = P_0$

(4) $\Rightarrow P_4 = \left(2P_0 - 2 \cdot \frac{1}{2}P_0\right)/2 = \frac{1}{2}P_0$

(5) $\Rightarrow P_0 + \frac{2}{3}P_0 + \frac{1}{2}P_0 + P_0 + \frac{1}{2}P_0 = 1$

$\Rightarrow P_0 = P_3 = \frac{3}{11}, P_1 = \frac{2}{11}, P_2 = P_4 = \frac{3}{22}$

(d)

$P_1 = \frac{\lambda_0}{\mu_1}P_0 = \frac{2}{3}P_0$

$P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0 = \frac{1}{2}P_0$

$P_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}P_0 = P_0$

$P_4 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4}P_0 = \frac{1}{2}P_0$

$P_0 + P_1 + P_2 + P_3 = 1 \Rightarrow P_0 = P_3 = \frac{3}{11}, P_1 = \frac{2}{11}, P_2 = P_4 = \frac{3}{22}$

$L = \sum_{n=0}^{\infty} nP_n = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + 4 \cdot P_4 = \frac{20}{11}$

$L_q = \sum_{n=1}^{\infty} (n-1)P_n = 0 \cdot P_1 + 1 \cdot P_2 + 2 \cdot P_3 + 3 \cdot P_4 = \frac{12}{11}$

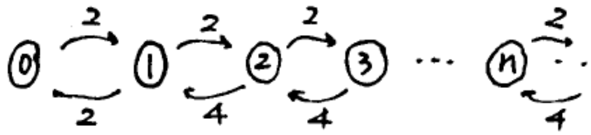
$\overline{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n = 2 \cdot P_0 + 3 \cdot P_1 + 2 \cdot P_2 + 1 \cdot P_3 = \frac{18}{11}$

$W = \frac{L}{\overline{\lambda}} = \frac{10}{9}$

$W_q = \frac{L_q}{\overline{\lambda}} = \frac{2}{3}$

17.5-4.

(a)



(b)

$P_1 = \frac{\lambda_0}{\mu_1} P_0 = P_0$

$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{1}{2} P_0$

$\vdots$

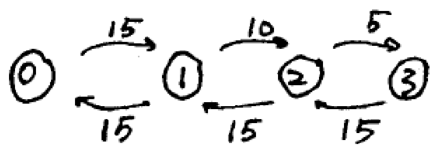
$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0 = \left(\frac{1}{2}\right)^{n-1} P_0$

$\sum_{n=0}^{\infty} P_n = P_0 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} P_0 = 3P_0 = 1 \Rightarrow P_0 = \frac{1}{3}, P_n = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{n-1}$

(c) The mean arrival rate to the system and the mean service rate for each server when it is busy serving customers are both 2.

17.5-5.

(a)



(b) (1) $15P_0 = 15P_1$
(2) $15P_0 + 15P_2 = 25P_1$
(3) $10P_1 + 15P_3 = 20P_2$
(4) $5P_2 = 15P_3$
(5) $P_0 + P_1 + P_2 + P_3 = 1$

(c) (1) $\Rightarrow P_1 = P_0$

(2) $\Rightarrow P_2 = (2/3)P_0$

(3) $\Rightarrow P_3 = (2/9)P_0$

(5) $\Rightarrow P_0 = P_1 = \frac{9}{26}, P_2 = \frac{3}{13}, P_3 = \frac{1}{13}$

The same equations can be obtained as follows:

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = P_0, \;\; P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{2}{3} P_0, \;\; P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0 = \frac{2}{9} P_0.$$

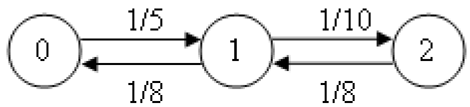
(d) $L = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 = \frac{27}{26} = 1.04$

$\overline{\lambda} = 15 \cdot P_0 + 10 \cdot P_1 + 5 \cdot P_2 = \frac{255}{26} = 9.81$

$W = \frac{L}{\overline{\lambda}} = \frac{9}{85} = 0.106$ hours

17.5-6.

(a) Let the state represent the number of machines that are broken down.



(b)

$$P_1 = \frac{8}{5}P_0, P_2 = \frac{32}{25}P_0, P_0 + P_1 + P_2 = 1$$

$$\Rightarrow P_0 = \frac{25}{97}, P_1 = \frac{40}{97}, P_2 = \frac{32}{97}$$

(c)

$$\overline{\lambda} = \frac{1}{5} \cdot P_0 + \frac{1}{10} \cdot P_1 = \frac{9}{97} = 0.093$$

$$L = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 = \frac{104}{97} = 1.072$$

$$L_q = 0 \cdot P_1 + 1 \cdot P_2 = \frac{32}{97} = 0.330$$

$$W = \frac{L}{\overline{\lambda}} = \frac{104}{9} \approx 11.556 \text{ hours}$$

$$W_q = \frac{L_q}{\overline{\lambda}} = \frac{32}{9} \approx 3.556 \text{ hours}$$

(d)

$$P_1 + P_2 = \frac{72}{97} = 0.742$$

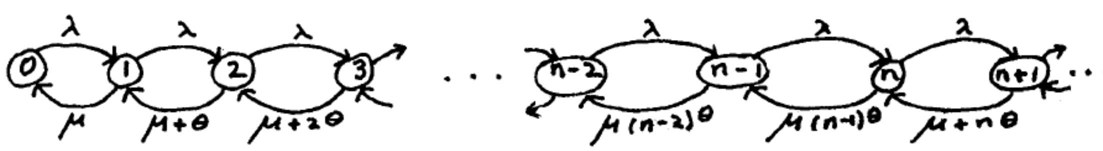
(e)

$$P_0 + \frac{1}{2}P_1 = \frac{45}{97} = 0.464$$

(f) The birth-and-death process is a special case of continuous time Markov chains.

17.5-7.

(a)



(b) $\mu P_1 = \lambda P_0$

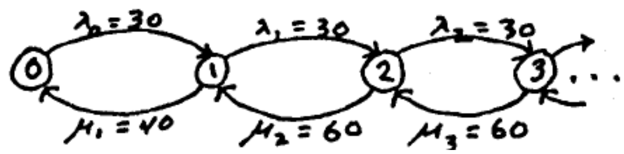
$$\lambda P_0 + (\mu + \theta)P_2 = (\mu + \lambda)P_1$$

$$\vdots$$

$$\lambda P_{n-1} + (\mu + n\theta)P_{n+1} = (\mu + \lambda + (n-1)\theta)P_n$$

17.5-8.

(a)



(b)

$$P_0 = \left[1+\sum_{n=1}^{\infty}\frac{\lambda^n}{\mu_1\mu_2^{n-1}}\right]^{-1} = \left[1+\frac{\lambda}{\mu_1}\sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu_2}\right)^{n-1}\right]^{-1}$$

$$= \left[1+\frac{\lambda}{\mu_1}\left(\frac{1}{1-\frac{\lambda}{\mu_2}}\right)\right]^{-1} = \left[1+\frac{3}{4}\left(\frac{1}{1-\frac{1}{2}}\right)\right]^{-1} = 0.4$$

$$P_n = \frac{\lambda^n}{\mu_1\mu_2^{n-1}}P_0 = \frac{3}{5}\left(\frac{1}{2}\right)^n \text{ for } n \geq 1$$

(c)

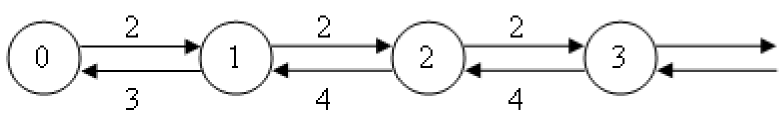
$$L = \sum_{n=0}^{\infty} nP_n = \frac{3}{5}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{3}{5}\cdot\frac{1}{2}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{6}{5}$$

$$L_q = L-(1-P_0) = \frac{3}{5}$$

$$W = \frac{L}{\lambda} = \frac{1}{25},\; W_q = \frac{L_q}{\lambda} = \frac{1}{50}$$

17.5-9.

(a) Let the state represent the number of documents received, but not completed.



(b) P_n below corresponds to the steady-state probability that n documents are received but not completed.

$$P_1 = \frac{2}{3}P_0, P_2 = \frac{2}{3}\left(\frac{1}{2}\right)P_0, \cdots, P_n = \frac{2}{3}\left(\frac{1}{2}\right)^{n-1}P_0$$

$$\sum_{n=0}^{\infty} P_n = \left(1+\sum_{n=1}^{\infty}\frac{2}{3}\left(\frac{1}{2}\right)^{n-1}\right)P_0 = \frac{7}{3}P_0 = 1 \Rightarrow P_0 = \frac{3}{7}, P_n = \frac{4}{7}\left(\frac{1}{2}\right)^n$$

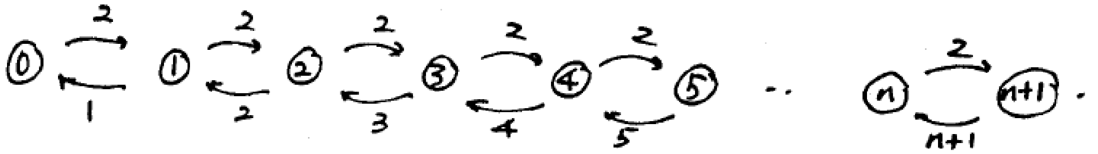
(c)

$$L = \sum_{n=0}^{\infty} nP_n = \frac{4}{7}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{4}{7}\cdot\frac{1}{2}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{8}{7}$$

$$L_q = \sum_{n=1}^{\infty}(n-1)P_n = L-(1-P_0) = \frac{4}{7}$$

$$W = \frac{L}{\lambda} = \frac{4}{7},\;\; W_q = \frac{L_q}{\lambda} = \frac{2}{7}$$

17.5-10.



$P_1 = \frac{\lambda_0}{\mu_1} P_0 = 2P_0$

$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = 2P_0$

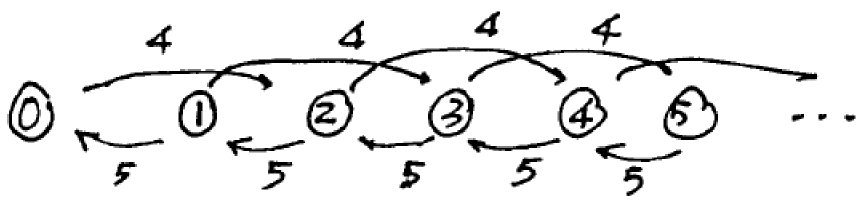
$\vdots$

$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0 = \frac{2^n}{n!} P_0$

$\sum_{n=0}^{\infty} P_n = e^2 \cdot P_0 = 1 \Rightarrow P_0 = e^{-2}, P_n = 2e^{-2}$ for $n \geq 1$

17.5-11.

(a)



(b) $5P_1 = 4P_0$

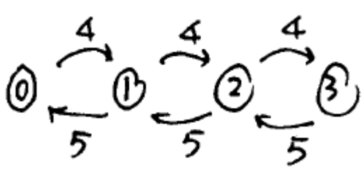
$5P_2 = 9P_1$

$5P_3 + 4P_0 = 9P_2$

$\vdots$

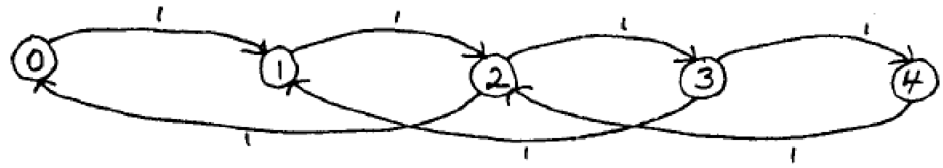
$5P_{n+1} + 4P_{n-2} = 9P_n$

(c)



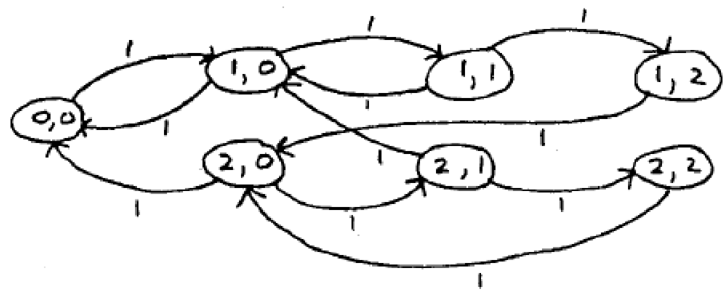
17.5-12.

(a) Let n be the number of customers in the system.



Balance equations: $P_0 = P_2, P_1 = P_0 + P_3, 2P_2 = P_1 + P_4, 2P_3 = P_2, P_4 = P_3$

(b) Let the state (s, q) be the number of customers in service and in queue respectively.

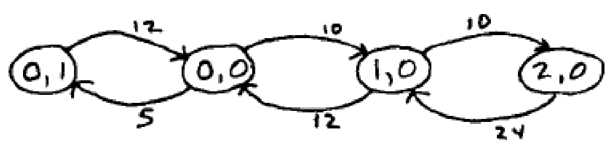


Balance equations:

$$P_{00} = P_{10} + P_{20}$$
$$2P_{10} = P_{00} + P_{11} + P_{21}$$
$$2P_{11} = P_{10}$$
$$2P_{20} = P_{12} + P_{22}$$
$$2P_{21} = P_{20}, P_{21} = P_{22}$$

17.5-13.

(a) Let the state (n_1, n_2) be the number of type 1 and type 2 customers in the systems.



(b) Balance equations:

$$12P_{01} = 5P_{00}$$
$$15P_{00} = 12(P_{01} + P_{10})$$
$$22P_{10} = 10P_{00} + 24P_{20}$$
$$24P_{20} = 10P_{10}$$
$$P_{00} + P_{10} + P_{01} + P_{20} = 1$$

(c) $P_{00} = \frac{72}{187}, P_{10} = \frac{60}{187}, P_{01} = \frac{30}{187}, P_{20} = \frac{25}{187}$

(d) Type 1 customers are blocked when the system is in state $(2, 0)$ or $(0, 1)$, so the fraction of type 1 customers who cannot enter the system is $P_{20} + P_{01} = 55/187$. Type 2 customers are blocked when the system is in state $(2, 0)$, $(0, 1)$ or $(1, 0)$, so the fraction of type 2 arrivals that are blocked is $P_{20} + P_{10} + P_{01} = 115/187$.

17.6-1.

KeyCorp deploys queueing theory as part of its Service Excellence Management System (SEMS) to improve productivity and service in its branches. The main objective of this study is to enhance customer satisfaction by reducing wait times without increasing the staffing costs. To do this, first a system that collects data about various phases of customer transactions is developed. Then, a preliminary analysis is conducted to determine the number of tellers required for at most 10% of customers to wait more than five minutes. The underlying model is an M/M/k queue with an average service time of 246 seconds. The arrival and service rates, λ and μ are estimated from the data. By using steady state equations, measures such as average queue length, average waiting time, and probability of having zero customers waiting are computed. The analysis revealed that with the current service time, the bank needed over 500 new employees. Hiring so many new tellers was too costly and physically impossible. Alternatively, the bank could achieve its goal by reducing the average service time. The investigation of the collected data helped to identify potential improvements in service. Accordingly, customer processing is reengineered, proficiency of tellers is improved and efficient schedules are obtained. Heuristic algorithms are incorporated in the model to make it more realistic.

The model allowed KeyCorp to reduce the processing time by 53%. As a result of this, the customer wait time has decreased and the percentage of customers who wait more than five minutes is reduced to 4%. In addition to increased customer satisfaction, the new system resulted in the reduction of operating costs. Savings from personnel expenses is estimated to be $98 million over five years whereas the cost of the new system was only half a million dollars. The reports generated from the data are used in obtaining better schedules and identifying service components that are open to improvement. Efficient scheduling and reduced personnel released 15% of the capacity, which can now be used for more profitable investments. KeyCorp also gained more credibility by using a systematic approach in making decisions. KeyCorp management, customers, employees and shareholders all benefit from this study.

17.6-2.

(a) M/M/1 queue with $\lambda = 2, \mu = 4 \Rightarrow \rho = 1/2$

$\Rightarrow P_0 = 1 - \rho = 1/2$ and $P_n = (1-\rho)\rho^n = (1/2)^{n+1}$

Proportion of the time the storage space will be adequate: $\sum_{n=0}^{4} P_n = 31/32 = 0.97$

(b)

$P_0 =$	0.5
$P_1 =$	0.25
$P_2 =$	0.125
$P_3 =$	0.0625
$P_4 =$	0.03125

Total $= 0.97$

17.6-3.

$\lambda = 30, \mu = 50 \Rightarrow \rho = 0.6, P_0 = 1 - \rho = 0.4$ (proportion of time no one is waiting)

17.6-4.

(a) $\mathcal{W} \sim \text{Exp}(\mu - \lambda), W = \frac{1}{\mu-\lambda}, P\{\mathcal{W} > W\} = (\mu - \lambda)e^{-\frac{\mu-\lambda}{\mu-\lambda}} = (\mu - \lambda)/e$

(b) $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

$$\mathcal{W}_q(t) = \begin{cases} 1 - \rho & \text{if } t \leq 0 \\ 1 - \rho e^{-\mu(1-\rho)t} & \text{if } t > 0 \end{cases}$$

$$P\{\mathcal{W}_q > W_q\} = 1 - \mathcal{W}_q(W_q) = \rho e^{-\frac{\mu(1-\rho)\lambda}{\mu(\mu-\lambda)}} = \frac{\lambda}{\mu}e^{-\lambda/\mu}$$

17.6-5.

Use the equalities $P_0 = 1 - \frac{\lambda}{\mu}$ and $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$.

$$\frac{(1-P_0)^2}{W_q P_0} = \frac{\left(\frac{\lambda}{\mu}\right)^2}{\frac{\lambda}{\mu(\mu-\lambda)}(1-\frac{\lambda}{\mu})} = \frac{\left(\frac{\lambda}{\mu}\right)^2}{\frac{\lambda}{\mu^2}} = \lambda$$

$$\frac{1-P_0}{W_q P_0} = \frac{\frac{\lambda}{\mu}}{\frac{\lambda}{\mu(\mu-\lambda)}(1-\frac{\lambda}{\mu})} = \frac{\frac{\lambda}{\mu}}{\frac{\lambda}{\mu^2}} = \mu$$

17.6-6.

The system without the storage restriction is an M/M/1 queue with $\lambda = 4$ and $\mu = 5$. The proportion of the time that n square feet floor space is adequate for waiting jobs is $\sum_{i=0}^{n+1} P_i$. Hence, the goal is to find n_j such that $\sum_{i=0}^{n_j+1} P_i \geq q_j$ for $j = 1, 2, 3$ and $q_1 = 0.5$, $q_2 = 0.9$, $q_3 = 0.99$.

$$\sum_{i=0}^{n_j+1} P_i \geq q_j \Leftrightarrow \sum_{i=0}^{n_j+1} (1-\rho)\rho^i \geq q_j \Leftrightarrow (1-\rho)\left(\frac{1-\rho^{n_j+2}}{1-\rho}\right) \geq q_j \Leftrightarrow \rho^{n_j+2} \leq 1 - q_j$$

$$\Leftrightarrow (n_j + 2)\ln\rho \leq \ln(1 - q_j) \Leftrightarrow n_j \geq \frac{\ln(1-q_j)}{\ln\rho} - 2,\ \rho = 0.8$$

Part	q_j	$\frac{\ln(1-q_j)}{\ln\rho} - 2$	Floor space required
(a)	0.5	1.106	2
(b)	0.9	8.319	9
(c)	0.99	18.638	19

17.6-7.

(a) TRUE. A customer does not wait before the service begins if and only if there is no one in the system, so the long-run probability that the customer does not wait is $1 - P_0 = \rho$.

(b) FALSE. The expected number of customers in the system is $L = \rho/(1-\rho)$, so it is not proportional to ρ.

(c) FALSE. When ρ is increased from 0.9 to 0.99, L increases from 9 to 99. When it is increased from 0.99 to 0.999, L increases from 99 to 999.

17.6-8.

(a) FALSE. A temporary return to the state where no customers are present is possible.

(b) TRUE. Since $\lambda > \mu$, the queue grows without bound.

(c) TRUE. Since $\lambda < 2\mu$, the system can reach steady-state conditions.

17.6-9.

(a) TRUE. "$\mathcal{W}$ has an exponential distribution with parameter $\mu(1-\rho)$," p.787.

(b) FALSE. "$\mathcal{W}_q$ does not quite have an exponential distribution, because $P\{\mathcal{W}_q = 0\} > 0$," p.787.

(c) TRUE. "S_{n+1} represents the conditional waiting time given n customers already in the system. As discussed in Sec. 17.7, S_{n+1} is known to have an Erlang distribution," p.787.

17.6-10.

(a) $L = \frac{\lambda}{\mu-\lambda} = \frac{20}{30-20} = 2$ customers, $W = \frac{1}{\mu-\lambda} = \frac{1}{30-20} = 0.1$ hours

$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{20}{30(30-20)} = \frac{1}{15}$ hours, $L_q = \lambda W_q = 20 \cdot \frac{1}{15} = \frac{4}{3}$ customers

$P_0 = 1-\rho = 1-\frac{2}{3} = \frac{1}{3}$, $P_1 = (1-\rho)\rho = \frac{2}{9}$, $P_2 = (1-\rho)\rho^2 = \frac{4}{27}$

There is a 29.6% chance of having more than 2 customers at the checkout stand.

(b) Time in minutes:

	Data	
λ =	0.333333	(mean arrival rate)
μ=	0.5	(mean service rate)
s =	1	(# servers)
Pr($\mathcal{W}$ > t) =	0.311403	
when t =	7	
Prob($\mathcal{W}_q$ > t) =	0.289732	
when t =	5	

	Results
L =	2
L_q =	1.333333333
W =	6
W_q =	4
ρ =	0.666666667

n	P_n
0	0.333333333
1	0.222222222
2	0.148148148
3	0.098765432
4	0.065843621
5	0.043895748

Time in hours:

	Data	
λ =	20	(mean arrival rate)
μ=	30	(mean service rate)
s =	1	(# servers)
Pr($\mathcal{W}$ > t) =	0.311403	
when t =	0.116667	
Prob($\mathcal{W}_q$ > t) =	0.289732	
when t =	0.083333	

	Results
L =	2
L_q =	1.333333333
W =	0.1
W_q =	0.066666667
ρ =	0.666666667

n	P_n
0	0.333333333
1	0.222222222
2	0.148148148
3	0.098765432
4	0.065843621
5	0.043895748

(c) $L = \frac{\lambda}{\mu-\lambda} = \frac{20}{40-20} = 1$ customer, $W = \frac{1}{\mu-\lambda} = \frac{1}{40-20} = 0.05$ hrs

$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{20}{40(40-20)} = 0.025$ hrs, $L_q = \lambda W_q = 20 \cdot 0.025 = 0.5$ customers

$P_0 = 1 - \rho = 1 - 0.5 = 0.5$, $P_1 = (1-\rho)\rho = 0.25$, $P_2 = (1-\rho)\rho^2 = 0.125$

There is a 12.5% chance of having more than 2 customers at the checkout stand.

(d) Time in hours:

	Data	
λ =	20	(mean arrival rate)
μ =	40	(mean service rate)
s =	1	(# servers)
Pr(W > t) =	0.096972	
when t =	0.116667	
Prob(W_q > t) =	0.094438	
when t =	0.083333	

	Results
L =	1
L_q =	0.5
W =	0.05
W_q =	0.025
ρ =	0.5

n	P_n
0	0.5
1	0.25
2	0.125
3	0.0625
4	0.03125
5	0.015625

(e) The manager should hire another person to help the cashier by bagging the groceries.

17.6-11.

(a) All the criteria are currently satisfied.

	Data	
λ =	10	(mean arrival rate)
μ =	20	(mean service rate)
s =	1	(# servers)
Pr(W > t) =	0.006738	
when t =	0.5	
Prob(W_q > t) =	0.003369	
when t =	0.5	

	Results
L =	1
L_q =	0.5
W =	0.1
W_q =	0.05
ρ =	0.5

n	P_n
0	0.5
1	0.25
2	0.125
3	0.0625
4	0.03125
5	0.015625

$P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 0.984$

(b) None of the criteria are satisfied.

	Data	
λ =	15	(mean arrival rate)
μ =	20	(mean service rate)
s =	1	(# servers)
Pr(W > t) =	0.082085	
when t =	0.5	
Prob(W_q > t) =	0.061564	
when t =	0.5	

	Results
L =	3
L_q =	2.25
W =	0.2
W_q =	0.15
ρ =	0.75

n	P_n
0	0.25
1	0.1875
2	0.140625
3	0.10546875
4	0.079101563
5	0.059326172

$P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 0.822$

(c) The first and third criteria are satisfied, but the second is not.

	Data	
λ =	25	(mean arrival rate)
μ =	20	(mean service rate)
s =	2	(# servers)
Pr(W > t) =	0.001022	
when t =	0.5	
Prob(W_q > t) =	0.000266	
when t =	0.5	

	Results
L =	2.051282051
L_q =	0.801282051
W =	0.082051282
W_q =	0.032051282
ρ =	0.625

n	P_n
0	0.230769231
1	0.288461538
2	0.180288462
3	0.112680288
4	0.07042518
5	0.044015738

$P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 0.927$

17.6-12.

(a) All the guidelines are currently met.

	Data	
λ =	2	(mean arrival rate)
μ =	1	(mean service rate)
s =	4	(# servers)
Pr(W > t) =	0.007902	
when t =	5	
Prob(W_q > t) =	7.9E-06	
when t =	5	

	Results
L =	2.173913043
L_q =	0.173913043
W =	1.086956522
W_q =	0.086956522
ρ =	0.5

n	P_n
0	0.130434783
1	0.260869565
2	0.260869565
3	0.173913043
4	0.086956522
5	0.043478261
6	0.02173913
7	0.010869565
8	0.005434783
9	0.002717391

$\sum_{i=0}^{9} P_i = 0.997$

(b) The first two guidelines will not be satisfied in a year, but the third will be.

	Data	
λ =	3	(mean arrival rate)
μ =	1	(mean service rate)
s =	4	(# servers)
Pr(W > t) =	0.023901	
when t =	5	
Prob(W_q > t) =	0.003433	
when t =	5	

	Results
L =	4.528301887
L_q =	1.528301887
W =	1.509433962
W_q =	0.509433962
ρ =	0.75

n	P_n
0	0.037735849
1	0.113207547
2	0.169811321
3	0.169811321
4	0.127358491
5	0.095518868
6	0.071639151
7	0.053729363
8	0.040297022
9	0.030222767

$\sum_{i=0}^{9} P_i = 0.909$

(c) Five tellers are needed in a year.

17.6-13.

(a)

λ	L	L_q	W	W_q	$P\{\mathcal{W} > 5\}$
0.5	1	0.50	2	1	0.082
0.9	9	8.10	10	9	0.607
0.99	99	98.01	100	99	0.951

(b)

λ	λ/μ	ρ	P_0	L	L_q	W	W_q	$P\{\mathcal{W} > 5\}$
0.5	1	0.5	0.3333	1.333	0.333	2.667	0.667	0.150
0.9	1.8	0.9	0.0526	9.474	7.674	10.526	8.526	0.641
0.99	1.98	0.99	0.0050	99.497	97.517	100.509	98.503	0.956

17.6-14.

	Data	
λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	1	(# servers)
Pr(W > t) =	2.06E-09	
when t =	10	
Prob(W_q > t) =	0.833333	
when t =	0	

	Results
L =	5
L_q =	4.166666667
W =	0.5
W_q =	0.416666667
ρ =	0.833333333

n	P_n
0	0.166666667
1	0.138888889
2	0.115740741
3	0.096450617

$P_0 + P_1 = 0.306$

	Data	
λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	2	(# servers)
Pr(W > t) =	1.89E-52	
when t =	10	
Prob(W_q > t) =	0.245098	
when t =	0	

	Results
L =	1.008403361
L_q =	0.175070028
W =	0.100840336
W_q =	0.017507003
ρ =	0.416666667

n	P_n
0	0.411764706
1	0.343137255
2	0.142973856
3	0.05957244

$P_0 + P_1 + P_2 = 0.898$

	Data	
λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	3	(# servers)
Pr(W > t) =	8.05E-53	
when t =	10	
Prob(W_q > t) =	0.05771	
when t =	0	

	Results
L =	0.855529512
L_q =	0.022196179
W =	0.085552951
W_q =	0.002219618
ρ =	0.277777778

n	P_n
0	0.432132964
1	0.360110803
2	0.150046168
3	0.041679491

$P_0 + P_1 + P_2 + P_3 = 0.984$

	Data	
λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	4	(# servers)
Pr(W > t) =	7.71E-53	
when t =	10	
Prob(W_q > t) =	0.011024	
when t =	0	

	Results
L =	0.836234411
L_q =	0.002901077
W =	0.083623441
W_q =	0.000290108
ρ =	0.208333333

n	P_n
0	0.434331675
1	0.361943063
2	0.150809609
3	0.041891558
4	0.008727408

$P_0 + P_1 + P_2 + P_3 + P_4 = 0.998$

	Data	
λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	5	(# servers)
Pr(W > t) =	7.67E-53	
when t =	10	
Prob(W_q > t) =	0.001746	
when t =	0	

	Results
L =	0.833682622
L_q =	0.000349289
W =	0.083368262
W_q =	3.49289E-05
ρ =	0.166666667

n	P_n
0	0.434571213
1	0.362142678
2	0.150892782
3	0.041914662
4	0.008732221
5	0.00145537

$P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 0.9997$

Part	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Number of servers	2	3	2	1	5	1	3

17.6-15.

M/M/1 queue with $\lambda = 20, \mu = 30$

$P\{$An arriving customer does not have to wait before service$\} = P_0 = 1 - \frac{\lambda}{\mu} = \frac{1}{3}$

Expected price of gasoline per gallon: $4 \times \frac{1}{3} + 3.5 \times \frac{2}{3} = \$3.667$

17.6-16.

Expected cost per customer: $\sum_{n=0}^{\infty} n \cdot P_n = \sum_{n=1}^{\infty} n \cdot (1-\rho)\rho^n = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

17.6-17.

Let $G(t) = P\{\mathcal{W} \leq t\}$ and $g(t) = \frac{dG(t)}{dt}$.

$$\begin{aligned} 1 - G(t) &= P\{\mathcal{W} > t\} = \sum_{n=0}^{\infty} P_n \cdot P\{S_{n+1} > t\} \\ &= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[\int_t^{\infty} \frac{\mu^{n+1} x^n e^{-\mu x}}{n!} dx\right] \\ &= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[1 - \int_0^{t} \frac{\mu^{n+1} x^n e^{-\mu x}}{n!} dx\right] \end{aligned}$$

Differentiate both sides of the equation.

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty}(1-\rho)\rho^n \tfrac{\mu^{n+1}t^n e^{-\mu t}}{n!} = (1-\rho)\mu e^{-\mu t}\sum_{n=0}^{\infty}\tfrac{(\lambda t)^n}{n!} \\ &= (1-\rho)\mu e^{-\mu t}e^{\lambda t} = (1-\rho)\mu e^{-\mu(1-\rho)t} \end{aligned}$$

Integrate to get $P\{\mathcal{W} > t\}$.

$P\{\mathcal{W} > t\} = 1 - \int_0^t g(x)dx = e^{-\mu(1-\rho)t}$

17.6-18.

(a) Let $G(t) = P\{\mathcal{W} \leq t\}$ and $g(t) = dG(t)/dt$.

$1 - G(t) = P\{\mathcal{W} > t\} = \sum_{n=1}^{\infty} P_n \cdot P\{S_n > t\} = \sum_{n=1}^{\infty}(1-\rho)\rho^n\left[1 - \int_0^t \frac{\mu^n x^{n-1}e^{-\mu x}}{(n-1)!}dx\right]$

Differentiate both sides of the equation.

$$\begin{aligned} g(t) &= \sum_{n=1}^{\infty}(1-\rho)\rho^n \tfrac{\mu^n t^{n-1}e^{-\mu t}}{(n-1)!} = (1-\rho)\lambda e^{-\mu t}\sum_{n=1}^{\infty}\tfrac{(\lambda t)^{n-1}}{(n-1)!} \\ &= (1-\rho)\lambda e^{-\mu t}e^{\lambda t} = \left(\tfrac{\lambda}{\mu}\right)(\mu-\lambda)e^{-(\mu-\lambda)t} \\ W_q &= \left(\tfrac{\lambda}{\mu}\right)\int_0^{\infty} t(\mu-\lambda)e^{-(\mu-\lambda)t}dt = \tfrac{\lambda}{\mu(\mu-\lambda)} \end{aligned}$$

(b) Let $G(t) = P\{\mathcal{W} \leq t\}$ and $g(t) = dG(t)/dt$.

$1 - G(t) = P\{\mathcal{W} > t\} = \sum_{n=s}^{\infty} P_n \cdot P\{S_{n-s+1} > t\} = \sum_{n=s}^{\infty} P_n\left[1 - \int_0^t \frac{(s\mu)^{n-s+1}x^{n-s}e^{-(s\mu)x}}{(n-s)!}dx\right]$

$P_n = \frac{(\lambda/\mu)^n}{s!s^{n-s}}P_0$ for $n \geq s$

Differentiate both sides of the equation.

$$\begin{aligned} g(t) &= \sum_{n=s}^{\infty}\left[\tfrac{(\lambda/\mu)^n P_0}{s!s^{n-s}}\right]\left[\tfrac{(s\mu)^{n-s+1}t^{n-s}e^{-(s\mu)t}}{(n-s)!}\right] \\ &= \tfrac{P_0(s\mu)(\lambda/\mu)^s}{s!}e^{-s\mu t}\sum_{n=s}^{\infty}\tfrac{(\lambda t)^{n-s}}{(n-s)!} = \tfrac{P_0(s\mu)(\lambda/\mu)^s}{s!}e^{-s\mu t}e^{\lambda t} \\ &= \tfrac{P_0(s\mu)(\lambda/\mu)^s}{s!}e^{-(s\mu)(1-\rho)t} \\ W_q &= \tfrac{P_0(s\mu)(\lambda/\mu)^s}{s!}\int_0^{\infty} t(s\mu)e^{-(s\mu)(1-\rho)t}dt \\ &= \tfrac{P_0(s\mu)(\lambda/\mu)^s}{s!(1-\rho)}\int_0^{\infty} t(s\mu)(1-\rho)e^{-(s\mu)(1-\rho)t}dt = \tfrac{P_0(\lambda/\mu)^s}{s!(1-\rho)^2(s\mu)} = \tfrac{P_0(\lambda/\mu)^s\rho}{s!(1-\rho)^2\lambda} = \tfrac{L_q}{\lambda} \end{aligned}$$

17.6-19.

$\lambda = 3, \mu = 2, s = 2 \Rightarrow P_0 = \frac{1}{7}, P_1 = \frac{3}{14}, P_2 = \frac{9}{56}$

Mean rate at which service completion occurs during the periods when no customers are waiting in the queue:

$$\frac{\mu_0 P_0 + \mu_1 P_1 + \mu_2 P_2}{P_0 + P_1 + P_2} = \frac{0P_0 + 2P_1 + 4P_2}{P_0 + P_1 + P_2} = \frac{60}{29} = 2.07$$

17.6-20.

	Data	
λ =	4	(mean arrival rate)
μ =	6	(mean service rate)
s =	2	(# servers)
Pr(W > t) =	1	
when t =	0	
Prob(W_q > t) =	0.003053	
when t =	0.5	

	Results
L =	0.75
L_q =	0.083333333
W =	0.1875
W_q =	0.020833333
ρ =	0.333333333

n	P_n
0	0.5
1	0.333333333
2	0.111111111
3	0.037037037
4	0.012345679
5	0.004115226
6	0.001371742
7	0.000457247
8	0.000152416
9	5.08053E-05
10	1.69351E-05

$$\begin{aligned} P\{\mathcal{W}_q > 0.5 \mid \text{number of customers} \geq 2\} &= \frac{P\{\mathcal{W}_q > 0.5, \text{number of customers} \geq 2\}}{P\{\text{number of customers} \geq 2\}} \\ &= \frac{P\{\mathcal{W}_q > 0.5\}}{1 - P_0 - P_1} = \frac{0.003}{1 - 0.5 - 0.3333} = 0.018 \end{aligned}$$

17.6-21.

(a) $W = (\mu - \lambda)^{-1}$

$W_{\text{Clara}} = \frac{1}{20-16} = 1/4$ hours $= 15$ minutes

$W_{\text{Clarence}} = \frac{1}{20-14} = 1/6$ hours $= 10$ minutes

$$\begin{aligned} W_{\text{total}} &= P\{\text{Clara}\}W_{\text{Clara}} + P\{\text{Clarence}\}W_{\text{Clarence}} = \frac{16}{30} \cdot 15 + \frac{14}{30} \cdot 10 \\ &= 12.67 \text{ minutes} = 0.211 \text{ hours} \end{aligned}$$

(b) It is an M/M/2 queue, $\lambda = 16 + 14 = 30$, $\mu = 20$, and $s = 2$. OR Courseware gives $W = 0.114$ hours.

(c)

μ	W
$60/3.5$	0.249
$60/3.4$	0.204
$60/3.45$	0.225
$60/3.425$	0.214
$60/3.419$	0.212
$60/3.4185$	0.211

An expected processing time of 3.485 minutes results in the same expected waiting time.

17.6-22.

(a) Current system: $\lambda = 10, \mu = 7.5, s = 2$
$\Rightarrow L = 2.4, L_q = 1.067, W = 0.24, W_q = 0.107$
Next year's system: $\lambda = 5, \mu = 7.5, s = 1$
$\Rightarrow L = 2, L_q = 1.333, W = 0.4, W_q = 0.267$

The next year's system yields smaller L, but larger L_q, W and W_q.

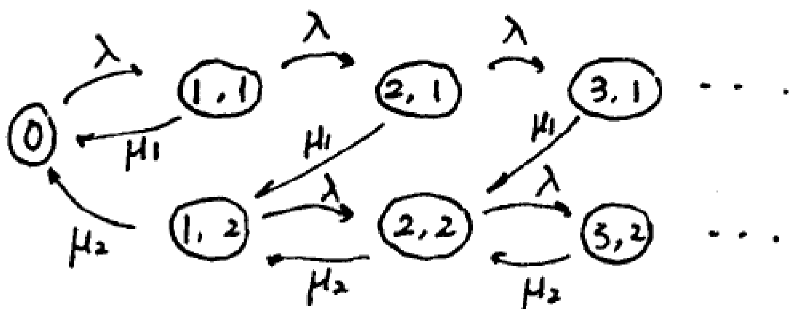
(b) $W = (\mu - \lambda)^{-1} \Rightarrow \mu = W^{-1} + \lambda = 0.24^{-1} + 5 = 9.17$

(c) $W_q = \frac{\lambda}{\mu(\mu - \lambda)} \Rightarrow \mu = \frac{\lambda W_q \pm \sqrt{(\lambda W_q)^2 + 4\lambda W_q}}{2W_q} \Rightarrow \mu = 9.78$

17.6-23.

(a) The future evolution of the queueing system is affected by whether the parameter of the service time distribution for the customer currently in service is μ_1 or μ_2. Therefore, the current state of the system needs to include this information from the history of the process. Let the state (n, s) be the number of customers in the system and the index of the current service rate. Note that the state $n = 0$ does not need an index of service rate.

$$s = \begin{cases} 1 & \text{if the current parameter is } \mu_1, \\ 2 & \text{if the current parameter is } \mu_2. \end{cases}$$



(b) $\lambda P_0 = \mu_1 P_{1,1} + \mu_2 P_{1,2}$

$(\lambda + \mu_1) P_{1,1} = \lambda P_0$

$(\lambda + \mu_1) P_{n,1} = \lambda P_{n-1,1}$ for $n \geq 2$

$(\lambda + \mu_2) P_{1,2} = \mu_1 P_{2,1} + \mu_2 P_{2,2}$

$(\lambda + \mu_2) P_{n,2} = \lambda P_{n-1,2} + \mu_1 P_{n+1,2} + \mu_2 P_{n+1,2}$ for $n \geq 2$

(c) Truncate the balance equations at a very large n and then solve the resulting finite system of equations numerically. The resulting approximation of the stationary distribution should be good if the steady-state probability that the number of customers in the original system exceeds n is negligible.

(d) $L = \sum_{n=1}^{\infty} n(P_{n,1} + P_{n,2}), W = \frac{L}{\lambda}, L_q = \sum_{n=1}^{\infty} (n-1)(P_{n,1} + P_{n,2}), W_q = \frac{L_q}{\lambda}$

(e) Because the input is Poisson, the distribution of the state of the system is the same just before an arrival and at an arbitrary point in time.

$$\begin{aligned} P\{\mathcal{W} \leq t\} &= P\{\mathcal{W} \leq t | \text{A new arrival finds the system in state } 0\} P_0 \\ &+ \sum_{n=1}^{\infty} P\{\mathcal{W} \leq t | \text{A new arrival finds the system in state } (n,1)\} P_{n,1} \\ &+ \sum_{n=1}^{\infty} P\{\mathcal{W} \leq t | \text{A new arrival finds the system in state } (n,2)\} P_{n,2} \end{aligned}$$

The three conditional distributions of $\mathcal{W}$ are (1) Exp(μ_1), (2) a convolution of Exp(μ_1) and Erlang$(n/\mu_2, n)$, (3) Erlang$((n+1)\mu_2, n+1)$ respectively.

$$P\{\mathcal{W} \leq t\} = (1 - e^{-\mu_1 t}) P_0 + \sum_{n=1}^{\infty} \left[\int_0^t \left(1 - e^{-\mu_1 (t - t_1)}\right) \frac{\mu_2^n t_1^{n-1} e^{-\mu_2 t}}{(n-1)!} dt_1 \right] P_{n,1} + \sum_{n=1}^{\infty} \left[\int_0^t \frac{\mu_2^{n+1} x^n e^{-\mu_2 x}}{n!} dx \right] P_{n,2}$$

17.6-24.

(a) (0) $\lambda P_0 = \mu P_1$

(1) $\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1$

$\vdots$

(n) $\lambda P_{n-1} + \mu P_{n+1} = (\lambda + \mu) P_n$

The solution given in Sec. 17.6 is: $P_n = (1-\rho)\rho^n$ for $n = 0, 1, 2, \ldots$. Substitute this in the balance equations.

(0) $\lambda(1-\rho) = \mu(1-\rho)\rho \Leftrightarrow \lambda = \mu \cdot \rho = \mu \cdot \frac{\lambda}{\mu} = \lambda$

(n) $\lambda(1-\rho)\rho^{n-1} + \mu(1-\rho)\rho^{n+1} = (\lambda+\mu)(1-\rho)\rho^n \Leftrightarrow \lambda + \mu\rho^2 = (\lambda+\mu)\rho$

$\Leftrightarrow \lambda + \mu\left(\frac{\lambda}{\mu}\right)^2 = (\lambda+\mu)\frac{\lambda}{\mu}$

Hence, the solution satisfies the balance equations.

(b) $\lambda P_0 = \mu P_1$

$\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1$

$\lambda P_1 = \mu P_2$

The solution given in Sec. 17.6 is: $P_n = \left(\frac{1-\rho}{1-\rho^3}\right)\rho^n$ for $n = 0, 1, 2$. Substitute this in the balance equations.

$\lambda\left(\frac{1-\rho}{1-\rho^3}\right) = \mu\left(\frac{1-\rho}{1-\rho^3}\right)\rho \Leftrightarrow \lambda = \mu \cdot \rho = \mu \cdot \frac{\lambda}{\mu} = \lambda$

$\lambda\left(\frac{1-\rho}{1-\rho^3}\right) + \mu\left(\frac{1-\rho}{1-\rho^3}\right)\rho^2 = (\lambda+\mu)\left(\frac{1-\rho}{1-\rho^3}\right)\rho \Leftrightarrow \lambda + \mu\rho^2 = (\lambda+\mu)\rho$

$\Leftrightarrow \lambda + \mu\left(\frac{\lambda}{\mu}\right)^2 = (\lambda+\mu)\frac{\lambda}{\mu}$

$\lambda\left(\frac{1-\rho}{1-\rho^3}\right)\rho = \mu\left(\frac{1-\rho}{1-\rho^3}\right)\rho^2 \Leftrightarrow \lambda\rho = \mu\rho^2 \Leftrightarrow \lambda \cdot \frac{\lambda}{\mu} = \mu\left(\frac{\lambda}{\mu}\right)^2$

Hence, the solution satisfies the balance equations.

(c) $2\lambda P_0 = \mu P_1$

$2\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1$

$\lambda P_1 = \mu P_2$

The solution given in Sec. 17.6 is:

$$P_0 = \left[\sum_{n=0}^{2} \frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} = \left[1 + 2\left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2\right]^{-1}$$

$$P_n = \frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0 \text{ for } n = 1, 2.$$

Substitute this in the balance equations.

$$\frac{2\lambda}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} = \frac{\mu\cdot 2\left(\frac{\lambda}{\mu}\right)}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} \Leftrightarrow 2\lambda = \mu \cdot 2\left(\frac{\lambda}{\mu}\right)$$

$$\frac{2\lambda}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} + \frac{\mu\cdot 2\left(\frac{\lambda}{\mu}\right)^2}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} = \frac{(\lambda+\mu)2\left(\frac{\lambda}{\mu}\right)}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} \Leftrightarrow 2\lambda + 2\mu\left(\frac{\lambda}{\mu}\right)^2 = 2(\lambda+\mu)\left(\frac{\lambda}{\mu}\right)$$

$$\frac{\lambda\cdot 2\left(\frac{\lambda}{\mu}\right)}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} = \frac{\mu\cdot 2\left(\frac{\lambda}{\mu}\right)^2}{1+2\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2} \Leftrightarrow 2\frac{\lambda^2}{\mu} = 2\frac{\lambda^2}{\mu}$$

Hence, the solution satisfies the balance equations.

17.6-25.

(a)

	Data	
λ =	6	(mean arrival rate)
μ =	4	(mean service rate)
s =	3	(# servers)
Pr(W > t) =	0.025817	
when t =	1	
Prob(W_q > t) =	0.236842	
when t =	0	

	Results
L =	1.736842105
L_q =	0.236842105
W =	0.289473684
W_q =	0.039473684
ρ =	0.5

n	P_n
0	0.210526316
1	0.315789474
2	0.236842105
3	0.118421053
4	0.059210526
5	0.029605263
6	0.014802632
7	0.007401316
8	0.003700658
9	0.001850329
10	0.000925164
11	0.000462582

(b) $P\{\text{A phone is answered immediately}\} = 1 - P\{\mathcal{W}_q > 0\} = 0.763$

Or $P\{\text{A phone is answered immediately}\} = P\{\text{At least one server is free}\}$

$= P_0 + P_1 + P_2 = 0.21053 + 0.31579 + 0.23684 = 0.763$

(c)

$$P\{n \text{ calls on hold}\} = \begin{cases} P_{n+3} & \text{if } n \geq 1 \\ P_0 + P_1 + P_2 + P_3 & \text{if } n = 0 \end{cases}$$

(d) Finite Queue Variation

	Data	
λ =	6	(mean arrival rate)
μ =	4	(mean service rate)
s =	3	(# servers)
K =	3	(max customers)

	Results
L =	1.29851
L_q =	0
W =	0.2500
W_q =	0
ρ =	0.5

n	P_n
0	0.23881
1	0.35821
2	0.26866
3	0.13433
4	0
5	0

$P\{\text{An arriving call is lost}\} = P\{\text{All three servers are busy}\} = P_3 = 0.13433$

17.6-26.

These form M/M/1/K queues with $K = 1$, 3 and 5 respectively, $\lambda = 1/4$ and $\mu = 1/3$, so $\rho = 3/4$ and the fraction of customers lost is

$$P_K = \frac{(1-\rho)}{(1-\rho^{K+1})} \cdot \rho^K.$$

(a) Zero spaces:

$$P_1 = \frac{(1-3/4)}{(1-(3/4)^2)} \cdot (3/4) = 3/7 = 0.429$$

(b) Two spaces:

$$P_3 = \frac{(1-3/4)}{(1-(3/4)^4)} \cdot (3/4)^3 = 27/175 = 0.154$$

(c) Four spaces:

$$P_5 = \frac{(1-3/4)}{(1-(3/4)^6)} \cdot (3/4)^5 = 243/3367 = 0.072$$

17.6-27.

M/M/s/K model

$$\begin{aligned} L_q \quad &= \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{K}(n-s)\frac{(\lambda/\mu)^n}{s!s^{n-s}}P_0 = \frac{P_0(\lambda/\mu)^{s+1}}{s!s}\sum_{n=s}^{K}(n-s)\left(\frac{\lambda}{s\mu}\right)^{n-s-1} \\ &= \frac{P_0(\lambda/\mu)^s\rho}{s!}\sum_{j=0}^{K-s} j\rho^{j-1} = \frac{P_0(\lambda/\mu)^s\rho}{s!}\sum_{j=0}^{K-s}\frac{d(\rho^j)}{d\rho} = \frac{P_0(\lambda/\mu)^s\rho}{s!}\frac{d}{d\rho}\left(\sum_{j=0}^{K-s}\rho^j\right) \\ &= \frac{P_0(\lambda/\mu)^s\rho}{s!}\frac{d}{d\rho}\left(\frac{1-\rho^{K-s+1}}{1-\rho}\right) = \frac{P_0(\lambda/\mu)^s\rho}{s!}\left[\frac{1-\rho^{K-s}-(K-s)\rho^{K-s}(1-\rho)}{(1-\rho)^2}\right] \end{aligned}$$

17.6-28.

$\mathcal{W}$ and $\mathcal{W}_q$ represent the waiting times of arriving customers who enter the system. The probability that such a customer finds n customers in the system already is:

$$P\{n \text{ customers in system}|\text{system not full}\} = \begin{cases} \frac{P_n}{1-P_K} & \text{for } 0 \le n \le K-1 \\ 0 & \text{for } n = K. \end{cases}$$

(a)

$$P\{\mathcal{W} > t\} = \frac{1}{1-P_K}\sum_{n=0}^{K-1} P_n P\{S_{n+1} > t\}$$

(b)

$$P\{\mathcal{W}_q > t\} = \frac{1}{1-P_K}\sum_{n=0}^{K-1} P_n P\{S_n > t\}$$

17.6-29.

(a) - (b)

	Data	
λ =	20	(mean arrival rate)
μ =	30	(mean service rate)
s =	1	(# servers)
K =	2	(max customers)

Results	
L =	0.73684
L_q =	0.21053
W =	0.0467
W_q =	0.01333
ρ =	0.66667

	Data	
λ =	20	(mean arrival rate)
μ =	30	(mean service rate)
s =	1	(# servers)
K =	3	(max customers)

Results	
L =	1.01538
L_q =	0.43077
W =	0.0579
W_q =	0.02456
ρ =	0.66667

	Data	
λ =	20	(mean arrival rate)
μ =	30	(mean service rate)
s =	1	(# servers)
K =	4	(max customers)

Results	
L =	1.24171
L_q =	0.62559
W =	0.0672
W_q =	0.03385
ρ =	0.66667

	Data	
λ =	20	(mean arrival rate)
μ =	30	(mean service rate)
s =	1	(# servers)
K =	5	(max customers)

Results	
L =	1.42256
L_q =	0.78797
W =	0.0747
W_q =	0.04139
ρ =	0.66667

(c)

Spaces	Rate P_K at which customers are lost	Change in P_K	Profit / hour $\$4\lambda(1-P_K)$	Change in Profit / hour
2	0.21		\$63.20	
3	0.12	0.09	\$70.40	\$7.20
4	0.08	0.04	\$73.60	\$3.20
5	0.05	0.03	\$76.00	\$2.40

(d) Since it costs \$200 per month per car length rented, each additional space must bring at least \$200 per month (or \$1 per hour) in additional profit. Five spaces still bring more than that, so five should be provided.

17.6-30.

(a) The M/M/s model with finite calling population fits this queueing system.

(b) The probabilities that there are 0, 1, 2, or 3 machines not running are P_0, P_1, P_2, and P_3 respectively. The mean of this distribution is $L = 0.718$.

	Data	
λ =	0.111111	(exponential parameter)
μ=	0.5	(mean service rate)
s =	1	(# servers)
N =	3	(size of population)

	Results
L =	0.7180527
L_q =	0.2109533
W =	2.832
W_q =	0.832
ρ=	0.6666667
λ-bar =	0.2535497

n	P_n
0	0.4929006
1	0.3286004
2	0.1460446
3	0.0324544
4	0

(c) $W = L/\lambda = 0.718/0.253 = 2.832$ hours

(d) The expected fraction of time that the repair technician will be busy is the system utilization, which is $\rho = 0.667$.

(e) M/M/s model

	Data	
λ =	0.333333	(mean arrival rate)
μ=	0.5	(mean service rate)
s =	1	(# servers)

	Results
L =	2
L_q =	1.333333333
W =	6
W_q =	4
ρ =	0.666666667

M/M/s/K model

	Data	
λ =	0.333333	(mean arrival rate)
μ=	0.5	(mean service rate)
s =	1	(# servers)
K =	3	(max customers)

	Results
L =	1.01538
L_q =	0.43077
W =	3.4737
W_q =	1.47368
ρ =	0.66667

(f)

	Data	
λ =	0.111111	(exponential parameter)
μ=	0.5	(mean service rate)
s =	2	(# servers)
N =	3	(size of population)

	Results
L =	0.552809
L_q =	0.0089888
W =	2.0330579
W_q =	0.0330579
ρ=	0.3333333
λ-bar =	0.2719101

n	P_n
0	0.5460674
1	0.3640449
2	0.0808989
3	0.0089888
4	0

The probabilities that there are 0, 1, 2, or 3 machines not running are P_0, P_1, P_2, and P_3 respectively. The mean of this distribution is $L = 0.553$. The expected fraction of time that the repair technician will be busy is the system utilization, $\rho = 0.333$.

17.6-31.

(a) This is an M/M/s model with a finite calling population, with $\lambda = 1$, $\mu = 2$, $s = 1$, and $N = 3$.

(b)

	Data	
λ =	1	(exponential parameter)
μ=	2	(mean service rate)
s =	1	(# servers)
N =	3	(size of population)

	Results
L =	1.4210526
L_q =	0.6315789
W =	0.9
W_q =	0.4
ρ=	1.5
λ-bar =	1.5789474

n	P_n
0	0.2105263
1	0.3157895
2	0.3157895
3	0.1578947
4	0
5	0

17.6-32.

(a) Alternative 1:

	Data	
λ =	0.4	(exponential parameter)
μ=	4	(mean service rate)
s =	1	(# servers)
N =	3	(size of population)

	Results
L =	0.3206442
L_q =	0.0527086
W =	0.2991803
W_q =	0.0491803

Three machines are the maximum that can be assigned to an operator while still achieving the required production rate. The average number of machines not running is $L = 0.32$, so $1 - (0.32/3) = 89.7\%$ of the machines are running on the average. The utilization of servers is $(\overline{\lambda}/s\mu) = 1.072/(1 \cdot 4) = 0.268$.

(b) Alternative 2:

	Data	
λ =	0.4	(exponential parameter)
μ=	4	(mean service rate)
s =	3	(# servers)
N =	12	(size of population)

	Results
L =	1.1246521
L_q =	0.0371173
W =	0.2585324
W_q =	0.0085324

Three operators are needed to achieve the required production rate. The average number of machines not running is $L = 1.125$, so $1 - (1.125/12) = 90.6\%$ of the machines are running on the average. The utilization of servers is $(\overline{\lambda}/s\mu) = 4.350/(3 \cdot 4) = 0.363$.

(c) Alternative 3:

	Data	
λ =	0.4	(exponential parameter)
μ=	8	(mean service rate)
s =	1	(# servers)
N =	12	(size of population)

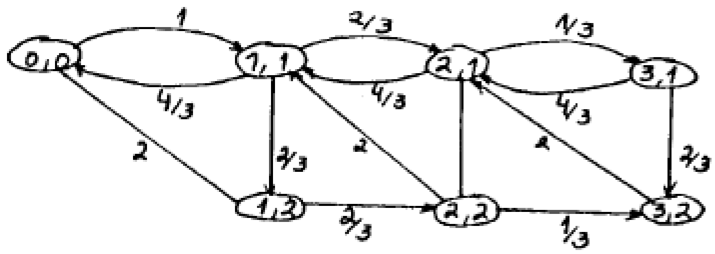
	Results
L =	1.0357708
L_q =	0.4875593
W =	0.2361705
W_q =	0.1111705

Two operators are needed to achieve the required production rate. The average number of machines not running is $L = 1.035$, so $1 - (1.035/12) = 91.4\%$ of the machines are running on the average. The utilization of servers is $(\overline{\lambda}/s\mu) = 4.386/(1 \cdot 8) = 0.548$.

17.6-33.

(a) Let the state (n, i) be the number of failed machines $(n = 0, 1, 2, 3)$ and the stage of service for the machine under repair ($i = 0$ if all machines are running properly, 1 or 2 otherwise).

(b)



(c)

State	Balance Equation
$(0,0)$	$\frac{4}{3}P_{1,1} + 2P_{1,2} = P_{0,0}$
$(1,1)$	$P_{0,0} + \frac{4}{3}P_{2,1} + 2P_{2,2} = \left(\frac{4}{3} + \frac{2}{3} + \frac{1}{3}\right)P_{1,1}$
$(2,1)$	$\frac{2}{3}P_{1,1} + \frac{4}{3}P_{3,1} + 2P_{3,2} = \left(\frac{4}{3} + \frac{2}{3} + \frac{1}{3}\right)P_{2,1}$
$(3,1)$	$\frac{1}{3}P_{2,1} = \left(\frac{4}{3} + \frac{2}{3}\right)P_{3,1}$
$(1,2)$	$\frac{2}{3}P_{1,1} = \left(2 + \frac{2}{3}\right)P_{1,2}$
$(2,2)$	$\frac{2}{3}(P_{1,2} + P_{2,1}) = \left(2 + \frac{1}{3}\right)P_{2,2}$
$(3,2)$	$\frac{1}{3}P_{2,2} + \frac{2}{3}P_{3,1} = 2P_{3,2}$

17.7-1.

(a)

(i) Exponential: $W_q^{\text{Exp}} = \frac{\lambda}{\mu(\mu-\lambda)}$

(ii) Constant: $W_q^{\text{C}} = \frac{1}{2} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$

(iii) Erlang: $\sigma = \frac{1}{2}\left(0 + \frac{1}{\mu}\right) = \frac{1}{2\mu} \Rightarrow \sigma^2 = \frac{1}{4\mu^2} \Rightarrow K = 4$

$W_q^{\text{Erlang}} = \frac{1+4}{8} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \frac{5}{8} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$

$\Rightarrow W_q^{\text{Exp}} = 2W_q^{\text{C}} = (8/5)W_q^{\text{Erlang}}$

(b) Let $B = 1, (1/2), (5/8)$ when the distribution is exponential, constant and Erlang respectively. Now, $\lambda^{(2)} = 2\lambda^{(1)}$ and $\mu^{(2)} = 2\mu^{(1)}$.

$$W_q^{(2)} = B\left[\frac{2\lambda^{(1)}}{2\mu^{(1)}(2\mu^{(1)} - 2\lambda^{(1)})}\right] = \frac{W_q^{(1)}}{2}$$

$$L_q^{(2)} = \lambda^{(2)}W_q^{(2)} = 2\lambda^{(1)}W_q^{(1)}/2 = \lambda^{(1)}W_q^{(1)} = L_q^{(1)}$$

Hence, the expected waiting time is reduced by 50% and the expected queue length remained the same.

17.7-2.

(a)

	Data			Results
λ =	0.2	(mean arrival rate)	L =	4.000
$1/\mu$ =	4	(expected service time)	L_q =	3.200
σ=	4	(standard deviation)		
s =	1	(# servers)	W =	20.000
			W_q =	16.000

	Data			Results
λ =	0.2	(mean arrival rate)	L =	3.300
$1/\mu$ =	4	(expected service time)	L_q =	2.500
σ=	3	(standard deviation)		
s =	1	(# servers)	W =	16.500
			W_q =	12.500

	Data			Results
λ =	0.2	(mean arrival rate)	L =	2.800
$1/\mu$ =	4	(expected service time)	L_q =	2.000
σ=	2	(standard deviation)		
s =	1	(# servers)	W =	14.000
			W_q =	10.000

	Data			Results
λ =	0.2	(mean arrival rate)	L =	2.500
$1/\mu$ =	4	(expected service time)	L_q =	1.700
σ=	1	(standard deviation)		
s =	1	(# servers)	W =	12.500
			W_q =	8.500

	Data			Results
λ =	0.2	(mean arrival rate)	L =	2.400
$1/\mu$ =	4	(expected service time)	L_q =	1.600
σ=	0	(standard deviation)		
s =	1	(# servers)	W =	12.000
			W_q =	8.000

(b) If $\sigma = 0$, L_q is half of the value with $\sigma = 4$, so it is quite important to reduce the variability of the service times.

(c)

σ	L_q	Change
4	3.2	
3	2.5	0.7 largest reduction
2	2	0.5
1	1.7	0.3
0	1.6	0.1 smallest reduction

(d) μ needs to be increased by 0.05 to achieve the same L_q.

17.7-3.

M/G/1 with $\rho < 1$: $L = \rho + \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}, L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}, W = \frac{L}{\lambda}, W_q = \frac{L_q}{\lambda}$

(a) FALSE. When L and L_q increase, both W and W_q increase too provided that λ is fixed.

(b) FALSE. Smaller μ and σ^2 do not necessarily imply a smaller L_q. For example, let $\lambda = 1, \mu_1 = 2, \sigma_1^2 = 1, \mu_2 = 5, \sigma_2^2 = 1.6$. Even though $\mu_1 < \mu_2$ and $\sigma_1^2 < \sigma_2^2$, $L_{q,1} = 1.25 > 1.025 = L_{q,2}$.

(c) TRUE. If the service time is exponential, $\sigma^2 = 1/\mu^2$ so that $L_q = \frac{2\rho^2}{2(1-\rho)}$. If it is constant, $\sigma^2 = 0$ and $L_q = \frac{\rho^2}{2(1-\rho)}$.

(d) FALSE. It is possible to find a distribution with $\sigma^2 > 1/\mu^2$.

17.7-4.

(a) $\lambda = 25, \mu = 40 \Rightarrow \rho = 0.625 < 1$

$\sigma = \frac{1}{\mu} = 0.025$

$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = 1.042$

$L = \rho + L_q = 1.667$

$W_q = \frac{L_q}{\lambda} = 0.042$ hours

$W = W_q + \frac{1}{\mu} = 0.067$ hours

(b) $\lambda = 25, \mu = 40 \Rightarrow \rho = 0.625 < 1$

$\sigma = 0$

$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = 0.521$

$L = \rho + L_q = 1.146$

$W_q = \frac{L_q}{\lambda} = 0.021$ hours

$W = W_q + \frac{1}{\mu} = 0.046$ hours

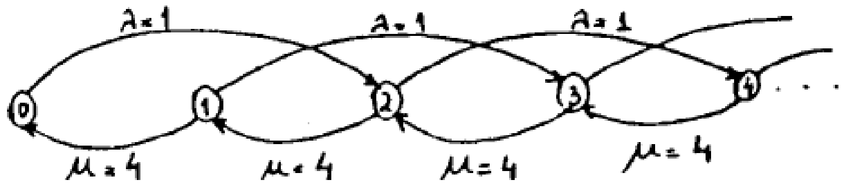
(c) L_q in (b) is half of L_q in (a).

(d) $\mu = 40 \Rightarrow L_q = 1.042$
$\mu = 80 \Rightarrow L_q = 0.142$
$\mu = 60 \Rightarrow L_q = 0.298$
$\mu = 50 \Rightarrow L_q = 0.5$
$\mu = 49 \Rightarrow L_q = 0.531$

Marsha needs to serve about 50 customers per hour, assuming the distribution of the service time will still be exponential. This means that she should reduce her expected service time to 72 seconds.

17.7-5.

(a)



$\mu P_1 = \lambda P_0$

$\mu P_2 = (\lambda + \mu) P_1$

$\lambda P_0 + \mu P_3 = (\lambda + \mu) P_2$

$\vdots$

$\lambda P_{n-2} + \mu P_{n+1} = (\lambda + \mu) P_n$

(b) Poisson input with $\lambda = 1$ and Erlang service times with $\mu = 4/2 = 2$, $k = 2$.

(c) $L = \rho + L_q = \rho + \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = 0.5 + \frac{1^2 0.354^2+0.5^2}{2(1-0.5)} = 0.875$

(d) $W = \frac{1}{\mu} + W_q = \frac{1}{\mu} + \frac{L_q}{\lambda} = \frac{1}{2} + \frac{0.875-0.5}{1} = 0.875$

(e)

	Data				Results
λ =	1	(mean arrival rate)		L =	0.875
μ =	2	(mean service rate)		L_q =	0.375
k =	2	(shape parameter)			
s =	1	(# servers)		W =	0.875
				W_q =	0.375

17.7-6.

(a) Current Policy:

	Data	
λ =	1	(mean arrival rate)
μ =	2	(mean service rate)
s =	1	(# servers)

	Results
L =	1
L_q =	0.5
W =	1
W_q =	0.5

Proposal:

	Data				Results
λ =	0.25	(mean arrival rate)		L =	0.8125
μ =	0.5	(mean service rate)		L_q =	0.3125
k =	4	(shape parameter)			
s =	1	(# servers)		W =	3.25
				W_q =	1.25

Under the current policy, an airplane loses one day of flying time as opposed to 3.25 days under the proposed policy.

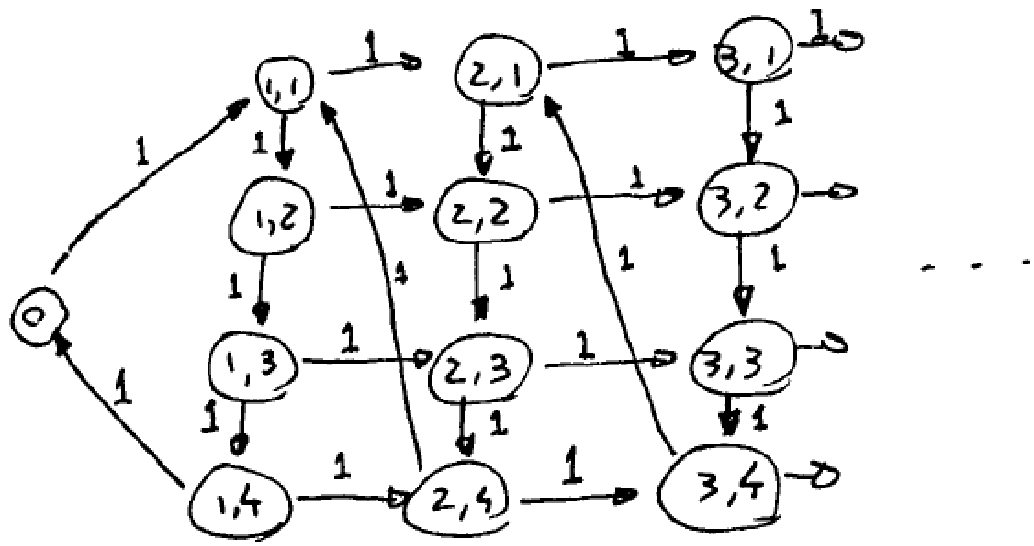
(b) Under the current policy, one airplane is losing flying time each day as opposed to 0.8125 airplanes under the proposed policy.

(c) The comparison in (b) is the appropriate one for making the decision, since it takes into account that airplanes will not have to come in for service as often.

17.7-7.

(a) Let the state (n, s) be the number of airplanes at the base and the stage of service of the airplane being overhauled.

(b)



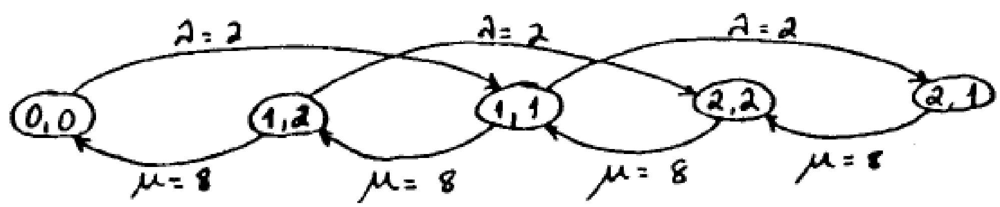
17.7-8.

For the current arrangement, $\lambda = 18$ and $\mu = 20$, so $\rho = 0.9$. For the proposal, $\lambda = 36$, $\mu = 20$ and $s = 2$, so $\rho = 0.9$.

	Current			Proposal	
Model	L at each crib	Total L	$W = L/\lambda$	L	$W = L/\lambda$
Fig. 17.6	9.0	18.0	0.500	9.47	0.263
Fig. 17.8	4.95	9.9	0.275	8	0.222
Fig. 17.10	6.975	13.95	0.388	7	0.194
Fig. 17.11	5.5	11	0.061	6	0.167

17.7-9.

(a) Let the state (i, j) denote i calling units in the system with the calling unit being served at the jth stage of its service. Then, the state space is $\{(0,0), (1,2), (1,1), (2,2), (2,1)\}$.



Note that this analysis is possible because an Erlang distribution with parameters $1/\mu = 1/4$ and $k = 2$ is equivalent to the distribution of the sum of two independent exponential random variables each with parameter $1/\mu = 1/8$. The steady-state equations are:

$$8P_{1,2} = 2P_{0,0} \qquad 8P_{1,1} = 10P_{1,2}$$

$$2P_{0,0} + 8P_{2,2} = 10P_{1,1} \qquad 2P_{1,2} + 8P_{2,1} = 8P_{2,2}\, 2P_{1,1} = 8P_{2,1}.$$

(b) The solution of the steady-state equations:

$$(P_{0,0}, P_{1,2}, P_{1,1}, P_{2,2}, P_{2,1}) = \left(\frac{64}{114}, \frac{16}{114}, \frac{20}{114}, \frac{9}{114}, \frac{5}{114}\right)$$

$$\Rightarrow P_0 = \frac{64}{114} = 0.561, P_1 = \frac{16+20}{114} = 0.316, P_2 = \frac{9+5}{114} = 0.123$$

$$\Rightarrow L = \frac{18+14}{52} = 0.561$$

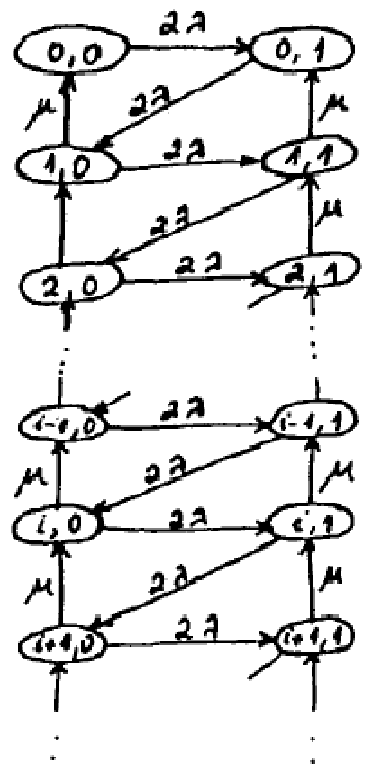
(c) If the service time is exponential, then the system is an M/M/1 queue with capacity $K = 2$, $\lambda = 2$ and $\mu = 4$.

$$P_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{1/2}{1-1/8} = 0.571, P_1 = \frac{1}{2}P_0 = 0.286, P_2 = \left(\frac{1}{2}\right)^2 P_0 = 0.143$$

$$L = \frac{2+2}{7} = 0.571$$

17.7-10.

Let the state (n, i) represent the number of customers in the system $(n \geq 0)$ and the number of completed arrival stages for currently arriving customer $(i = 0, 1)$.



17.7-11.

(a) Let T be the repair time.

$$\begin{aligned} E(T) &= E(T|\text{minor repair needed}) \cdot 0.9 + E(T|\text{major repair needed}) \cdot 0.1 \\ &= \tfrac{1}{2} \cdot 0.9 + 5 \cdot 0.1 = 0.95 \text{ hours} \end{aligned}$$

Now let X be a Bernoulli random variable with

$$P\{X = 1\} = p = 0.9 \text{ and } P\{X = 0\} = q = 0.1,$$

Y_i be an exponential random variable with mean $1/\lambda_i$ for $i = 1, 2$, where $\lambda_1 = 2$ and $\lambda_2 = 1/5$.

$$T = Y_1 \cdot X + Y_2 \cdot (1 - X),$$

where X, Y_1, Y_2 are independent.

$$\text{var}(T|X) = \text{var}(Y_1) \cdot X + \text{var}(Y_2) \cdot (1 - X) = \tfrac{1}{\lambda_1^2} \cdot X + \tfrac{1}{\lambda_2^2} \cdot (1 - X)$$

$$E(\text{var}(T|X)) = \tfrac{p}{\lambda_1^2} + \tfrac{q}{\lambda_2^2}$$

$$\begin{aligned} E(T|X) &= E(Y_1) \cdot X + E(Y_2) \cdot (1 - X) = \tfrac{1}{\lambda_1} \cdot X + \tfrac{1}{\lambda_2} \cdot (1 - X) \\ &= \tfrac{1}{\lambda_2} + \left(\tfrac{1}{\lambda_1} - \tfrac{1}{\lambda_2}\right) X \end{aligned}$$

$$\text{var}(E(T|X)) = \left(\tfrac{1}{\lambda_1} - \tfrac{1}{\lambda_2}\right)^2 \text{var}(X) = \left(\tfrac{1}{\lambda_1} - \tfrac{1}{\lambda_2}\right)^2 pq$$

$$\text{var}(T) = \tfrac{p}{\lambda_1^2} + \tfrac{q}{\lambda_2^2} + \left(\tfrac{1}{\lambda_1} - \tfrac{1}{\lambda_2}\right)^2 pq = 4.5475$$

Observe that T has a much larger variance than $(0.95)^2 = 0.9025$, the variance of an exponential random variable with the same mean.

(b) M/G/1 queue with $\mu = 1/0.95, \lambda = 1 \Rightarrow \rho = 0.95$

$$P_0 = 1 - \rho = 0.05$$

$$L_q = \tfrac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} = \tfrac{1^2 4.5475^2 + 0.95^2}{2(1-0.95)} = 215.82$$

$$L = \rho + L_q = 216.77$$

$$W_q = \tfrac{L_q}{\lambda} = 215.82$$

$$W = \tfrac{1}{\mu} + W_q = 216.77$$

(c) $W|\text{major repair needed} = W_q + 5 = 220.82$

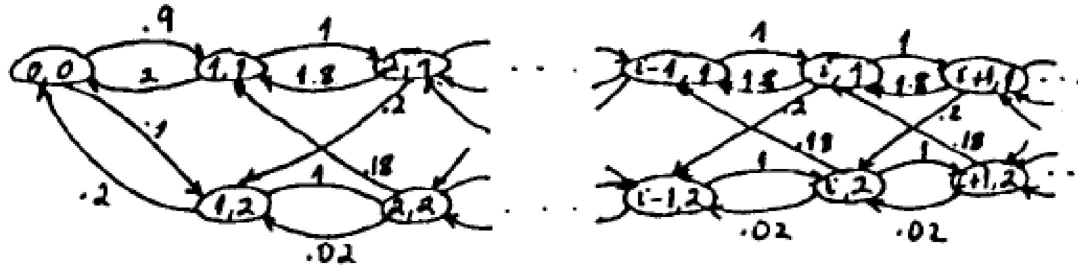
$$W|\text{minor repair needed} = W_q + 0.5 = 216.32$$

$$L_{\text{major repair machines}} = (\lambda)(0.1)(220.82) = 22.082$$

$$L_{\text{minor repair machines}} = (\lambda)(0.9)(216.32) = 194.688$$

(d) Let the state (n, i) denote the number of failed machines and the type of repair being done on the machine under repair ($i = 1$ represents minor repair and $i = 2$ represents major repair).

(e)



17.7-12.

(a)

$$X_{n+1} = \begin{cases} X_n - 1 + A_{n+1} & \text{if } X_n \geq 1 \\ A_{n+1} & \text{if } X_n = 0 \end{cases} \text{ and } X_{n+1} \leq 3,$$

where A_{n+1} is the number of arrivals in 10 minutes.

$$P\{A = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} = a_n \text{ and } \lambda t = \frac{60}{50} \cdot \frac{10}{60} = 0.2$$

$$\Rightarrow P = \begin{pmatrix} a_0 & a_1 & a_2 & 1 - a_0 - a_1 - a_2 \\ a_0 & a_1 & a_2 & 1 - a_0 - a_1 - a_2 \\ 0 & a_0 & a_1 & 1 - a_0 - a_1 \\ 0 & 0 & a_0 & 1 - a_0 \end{pmatrix} = \begin{pmatrix} 0.819 & 0.164 & 0.016 & 0.001 \\ 0.819 & 0.164 & 0.016 & 0.001 \\ 0 & 0.819 & 0.164 & 0.017 \\ 0 & 0 & 0.819 & 0.181 \end{pmatrix}$$

(b) Using the OR Courseware: $P_0 = 0.801, P_1 = 0.177, P_2 = 0.02, P_3 = 0.002$

(c) $L = P_1 + 2 \cdot P_2 + 3 \cdot P_3 = 0.223$

M/D/1 model: $L^\infty = \rho + \frac{\rho^2}{2(1-\rho)} = 0.2 + \frac{0.2^2}{2(1-0.2)} = 0.225 > 0.223 = L$

17.8-1.

(a) This system is an example of a nonpreemptive priority queueing system.

(b) $n = 2, \mu = 20, s = 1$

			Results		
	λ_i	L	Lq	W	Wq
Priority Class 1	2	0.1666667	0.0666667	0.0833333	0.0333333
Priority Class 2	10	1.3333333	0.8333333	0.1333333	0.0833333

λ =	12
ρ =	0.6

(c) $\frac{W_{q,1}}{W_{q,2}} = \frac{0.033}{0.083} = 0.4$

(d) $\rho = 0.6$ (12 hours) $= 7.2$ hours

17.8-2.

s	μ	$W_{q,1}$	$L_{q,1}$	W_1	L_1	$W_{q,2}$	$L_{q,2}$	W_2	L_2
1	6	0.208	0.417	0.375	0.75	1.25	3.75	1.417	4.25
2	3	0.189	0.379	0.523	1.045	1.136	3.409	1.47	4.409

If W_1 is the primary concern, one should choose the alternative with one fast server. If $W_{q,1}$ is the primary concern, one should choose the alternative with two slow servers.

17.8-3.

(a)

	u	a	b	W
0			1	
1	2.5	0.16	0.6	0.67
2	3.33	0.25	0.3	1.69
3	5	0.29	0.1	9.87

(b)

u	3.33
r	0.30
A	4.44

	B	W
0	1	
1	0.7	0.62
2	0.4	1.10
3	0.1	5.93

The approximation is not good for W_2 and W_3.

17.8-4.

$\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 2, \lambda = \sum_{i=1}^{3} \lambda_i = 8, \mu = 10$

(a) First-come-first-served: $W = (\mu - \lambda)^{-1} = 0.5$ days

(b) Nonpreemptive priority:

$A = \frac{\mu^2}{\lambda} = \frac{25}{2}$

$B_1 = 1 - \frac{\lambda_1}{\mu} = \frac{4}{5}, B_2 = 1 - \frac{\lambda_1+\lambda_2}{\mu} = \frac{2}{5}, B_3 = 1 - \frac{\lambda}{\mu} = \frac{1}{5}$

$W_1 = \frac{1}{AB_1} + \frac{1}{\mu} = \frac{1}{5} = 0.2$ days

$W_2 = \frac{1}{AB_1B_2} + \frac{1}{\mu} = \frac{7}{20} = 0.35$ days

$W_3 = \frac{1}{AB_2B_3} + \frac{1}{\mu} = \frac{11}{10} = 1.1$ days

(c) Preemptive priority: $W_1 = \frac{1/\mu}{B_1} = \frac{1}{8} = 0.125$ days

$W_2 = \frac{1/\mu}{B_1B_2} = \frac{5}{16} = 0.3125$ days

$W_3 = \frac{1/\mu}{B_2B_3} = \frac{5}{4} = 1.25$ days

17.8-5.

$$\lambda_1 = 0.1, \lambda_2 = 0.4, \lambda_3 = 1.5, \lambda = \sum_{i=1}^{3} \lambda_i = 2, \mu = 3$$

	Preemptive		Nonpreemptive	
	$s=1$	$s=2$	$s=1$	$s=2$
A	$\cdots$	$\cdots$	4.5	36
B_1	0.967	$\cdots$	0.967	0.983
B_2	0.833	$\cdots$	0.833	0.917
B_3	0.333	$\cdots$	0.333	0.667
$W_1 - \frac{1}{\mu}$	0.011	0.00009	0.230	0.028
$W_2 - \frac{1}{\mu}$	0.080	0.00289	0.276	0.031
$W_3 - \frac{1}{\mu}$	0.867	0.05493	0.800	0.045

17.8-6.

(a) The expected number of customers would not change since customers of both types have exactly the same arrival pattern and service times. The change of the priority would not affect the total service rate from the server's view and thus, the total queue size stays the same.

(b) Using the template for M/M/s nonpreemptive priorities queueing model:

	Data	
n =	2	(# of priority classes)
μ =	6	(mean service rate)
s =	2	(# servers)

λ =	10
ρ =	0.8333333

	λ_i	L	Lq	W	Wq
		Results			
Priority Class 1	5	1.3744589	0.5411255	0.2748918	0.1082251
Priority Class 2	5	4.0800866	3.2467532	0.8160173	0.6493506

$L_p = L_1 + L_2 = 5.45455$

Using the template for M/M/s queueing model:

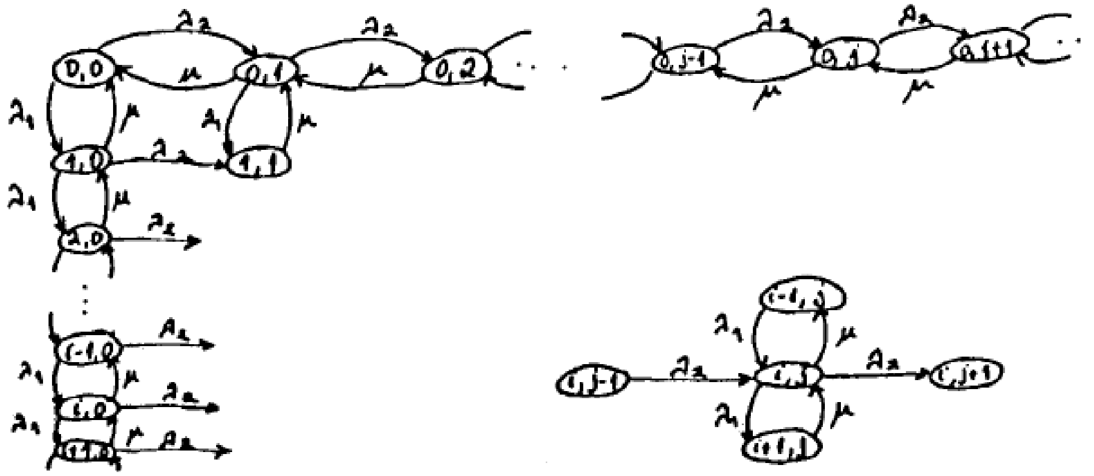
	Data	
λ =	10	(mean arrival rate)
μ =	6	(mean service rate)
s =	2	(# servers)

	Results
L =	5.454545455
L_q =	3.787878788
W =	0.545454545
W_q =	0.378787879
ρ =	0.833333333

Hence, $L_p = L$.

17.8-7.

Let the state (i, j) denote i jobs of high priority and j jobs of low priority.



State	Balance Equation
$(0,0)$	$\mu(P_{0,1}+P_{1,0})=(\lambda_1+\lambda_2)P_{0,0}$
$(i,0)$ for $i \geq 1$	$\mu P_{i+1,0}+\lambda_1 P_{i-1,0}=(\mu+\lambda_1+\lambda_2)P_{i,0}$
$(0,j)$ for $j \geq 1$	$\mu(P_{i,j}+P_{0,j+1})+\lambda_2 P_{0,j-1}=(\mu+\lambda_1+\lambda_2)P_{0,j}$
(i,j) for $i,j \geq 1$	$\mu P_{i+1,j}+\lambda_1 P_{i-1,j}+\lambda_2 P_{i,j-1}=(\mu+\lambda_1+\lambda_2)P_{i,j}$

17.9-1.

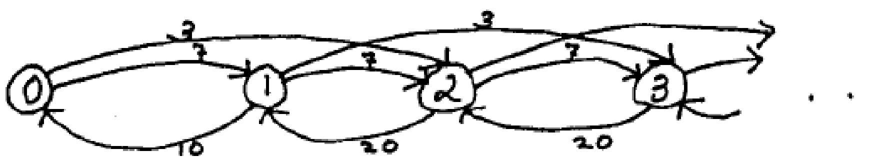
GM launched this project to improve the throughput of its production lines. A sequence of stations through which parts move sequentially until completion is called a production line. These stations are separated by finite-capacity buffers. Since machines may have unequal speeds and fail randomly, analyzing even simple production lines is not easy. To overcome the difficulties in measuring throughput and identifying bottlenecks, GM developed a throughput-analysis tool named C-MORE. The analysis assumes unreliable stations with deterministic speeds, exponential failure and repair times. Analytic decomposition and simulation methods are deployed. Analytic decomposition is based on first solving the two-station problem and then extending the results to multiple stations. Each station is modeled as a single-server queueing system with constant interarrival and service times. The server at each station can fail randomly. The first station is blocked and shuts down if its buffer is full and the second station is starved and shuts down if there are no jobs completed by the first station. The state of the system includes information about blocked and starved stations, downtimes, and buffer contents. Closed-form expressions for the steady-state distribution of buffer contents when both stations are up are obtained. The output includes throughput, system-time and work-in-process averages, average state of the system, bottleneck and sensitivity analysis.

The results of this study include enhanced throughput, lowered overtime and increased sales of high-demand products. These improvements translated into savings of more than \$2.1 billion. The use of a systematic approach enabled GM to make reliable decisions about equipment purchases, product launch times and maintenance schedules while meeting its production targets. Consequently, unprofitable investments and unfruitful improvement efforts are avoided. Alternatives are evaluated efficiently and questions are answered accurately. Continuous improvement of productivity is made possible. Overall, this study provided GM a competitive advantage in the industry. Following this study, OR has been widely adopted throughout the organization.

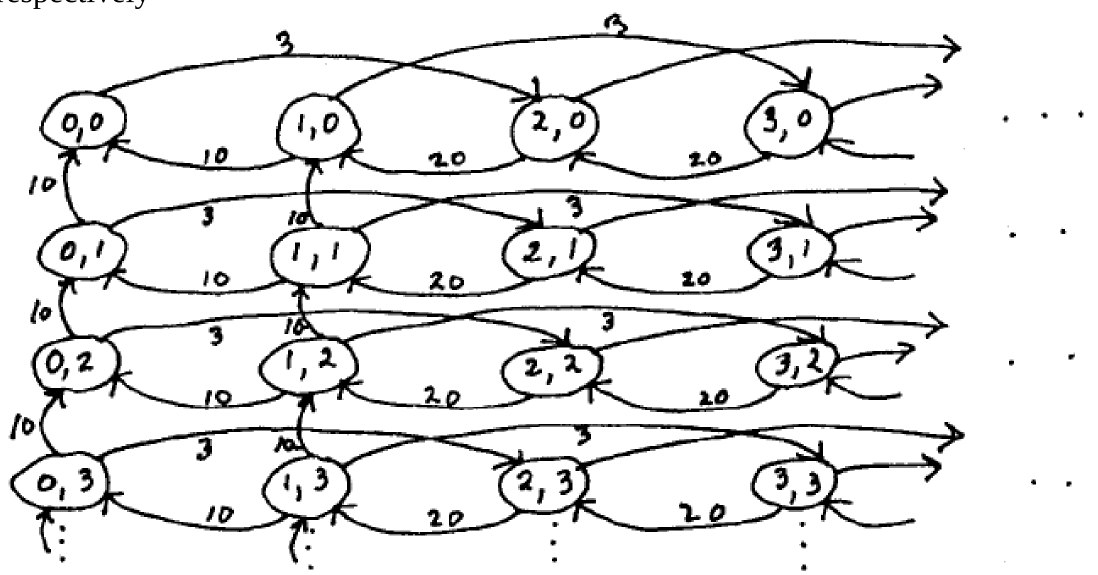
17.9-2.

(a) Let the state n_1 be the number of type 1 customers in the system.

(b) Let the state n be the number of customers in the system.



(c) Let the state (n_1, n_2) be the number of type 1 and type 2 customers in the system respectively



17.9-3.

(a) $P_{n_1} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n_1}, P_{n_2} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n_2}$

$P\{(N_1, N_2) = (n_1, n_2)\} = P_{n_1} \cdot P_{n_2} = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)^{n_1}\left(\frac{2}{3}\right)^{n_2}$

(b) $P\{(N_1, N_2) = (0, 0)\} = \frac{1}{6}$

(c) $L = L_1 + L_2 = 1 + 2 = 3$
$W = W_1 + W_2 = \frac{1}{10} + \frac{2}{10} = 0.3$ hour $= 18$ minutes

17.9-4.

In a system of infinite queues in series, customers are served at m service facilities in a fixed order. Each facility has an infinite queue capacity. The arrivals from outside the system to the first facility form a Poisson process with rate $a_1 = \lambda$. There are no arrivals from outside the system to other facilities, so $a_i = 0$ for $i > 1$, this is a Poisson process with parameter 0. From the equivalence property, under steady-state conditions, the arrivals to each facility i have a Poisson distribution with rate λ. Facility i has s_i servers whose service time is exponentially distributed with rate μ_i. A customer leaving facility i is routed to facility $i+1$ with probability 1 if $i < m$ and leaves the system if $i = m$, so for $i < m$,

$$p_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{else,} \end{cases}$$

and $q_m = 1$. It is assumed that $s_i \mu_i > \lambda$ so that the queue does not grow without bound.

17.9-5.

(a) $\lambda_1 = a_1 + 0 \cdot a_1 + 0.5 \cdot a_2 + 0.4 \cdot a_3 = 11.6$

$\lambda_2 = a_2 + 0.2 \cdot a_1 + 0 \cdot a_2 + 0.3 \cdot a_3 = 10.4$

$\lambda_3 = a_3 + 0.4 \cdot a_1 + 0.3 \cdot a_2 + 0 \cdot a_3 = 8.8$

(b)

$$\rho_i = \frac{\lambda_i}{s_i \mu_i} = \begin{cases} 0.464 & \text{for } i = 1 \\ 0.347 & \text{for } i = 2 \\ 0.440 & \text{for } i = 3 \end{cases}$$

$P_{n_1} = (0.536)(0.464)^{n_1}$ for facility 1

$P_{n_2} = (0.653)(0.347)^{n_2}$ for facility 2

$P_{n_3} = (0.560)(0.440)^{n_3}$ for facility 3

$P\{(N_1, N_2, N_3) = (n_1, n_2, n_3)\} = P_{n_1} P_{n_2} P_{n_3} = 0.196(0.464)^{n_1}(0.347)^{n_2}(0.440)^{n_3}$

(c) $P\{(N_1, N_2, N_3) = (0, 0, 0)\} = 0.196$

(d) $L_1 = 0.866, L_2 = 0.531, L_3 = 0.786 \Rightarrow L = L_1 + L_2 + L_3 = 2.182$

(e)

$$W = \frac{L}{a_1 + a_2 + a_3} = \frac{2.182}{6+8+4} = 0.121$$

17.10-1.

(a) The optimal number of servers is one.

	Data	
λ =	8	(mean arrival rate)
μ =	10	(mean service rate)
s =	1	(# servers)
Pr(W > t) =	0.904837	
when t =	0.05	
Prob(W_q > t) =	0.72387	
when t =	0.05	
Economic Analysis:		
Cs =	$100.00	(cost / server / unit time)
Cw =	$10.00	(waiting cost / unit time)
Cost of Service	$100.00	
Cost of Waiting	$40.00	
Total Cost	$140.00	

(b) The optimal number of servers is two.

	Data	
λ =	8	(mean arrival rate)
μ =	10	(mean service rate)
s =	2	(# servers)
Pr(W > t) =	0.672495	
when t =	0.05	
Prob(W_q > t) =	0.125443	
when t =	0.05	
Economic Analysis:		
Cs =	$100.00	(cost / server / unit time)
Cw =	$100.00	(waiting cost / unit time)
Cost of Service	$200.00	
Cost of Waiting	$95.24	
Total Cost	$295.24	

(c) The optimal number of servers is three.

	Data	
λ =	8	(mean arrival rate)
μ =	10	(mean service rate)
s =	3	(# servers)
Pr(W > t) =	0.618397	
when t =	0.05	
Prob(W_q > t) =	0.01732	
when t =	0.05	
Economic Analysis:		
Cs =	$10.00	(cost / server / unit time)
Cw =	$100.00	(waiting cost / unit time)
Cost of Service	$30.00	
Cost of Waiting	$81.89	
Total Cost	$111.89	

17.10-2.

Jim should operate four cash registers during the lunch hour.

	Data	
λ =	66	(mean arrival rate)
μ =	30	(mean service rate)
s =	4	(# servers)
Pr(W > t) =	0.267335	
when t =	0.05	
Prob(W_q > t) =	0.015242	
when t =	0.05	
Economic Analysis:		
Cs =	$9.00	(cost / server / unit time)
Cw =	$18.00	(waiting cost / unit time)
Cost of Service	$36.00	
Cost of Waiting	$44.59	
Total Cost	$80.59	

17.10-3.

Note that if there are less than three copiers, $\rho > 1$, so the queue for the copier explodes.

Number of Copiers	3	4	5	6	7	8
Total Cost	\$367.87	\$144.94	\$124.06	\$120.65	\$121.19	\$122.80

The company needs a total of six machines to minimize its expected total cost per hour.

	Data	
λ =	40	(mean arrival rate)
μ =	15	(mean service rate)
s =	6	(# servers)
Pr(W > t) =	0.482728	
when t =	0.05	
Prob(W_q > t) =	0.005085	
when t =	0.05	
Economic Analysis:		
Cs =	\$2.00	(cost / server / unit time)
Cw =	\$40.00	(waiting cost / unit time)
Cost of Service	\$12.00	
Cost of Waiting	\$108.65	
Total Cost	\$120.65	

17.11-1.

Answers will vary.

17.11-2.

Answers will vary.

CASES

17.1 a) Status quo at the presses – 7.52 sheets of in-process inventory.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	7	(mean arrival rate)		L =	7.517372837
5		μ =	1	(mean service rate)		L_q =	0.517372837
6		s =	10	(# servers)			

Status quo at the inspection station – 3.94 wing sections of in-process inventory.

	A	B	C	D	E	F	G
1	**Template for M/D/1 Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	7	(mean arrival rate)		L =	3.9375
5		μ =	8	(mean service rate)		L_q =	3.0625
6		s =	1	(# servers)			

Inventory cost = (7.52 + 3.94)($8/hour) = $91.68 / hour
Machine cost = (10)($7/hour) = $70 / hour
Inspector cost = $17 / hour

Total cost = $178.68 / hour

b) Proposal 1 will increase the in-process inventory at the presses to 11.05 sheets since the mean service rate has decreased.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	7	(mean arrival rate)		L =	11.04740664
5		μ =	0.83333333	(mean service rate)		L_q =	2.647406638
6		s =	10	(# servers)			

The in-process inventory at the inspection station will not change.

Inventory cost = (11.05 + 3.94)($8/hour) = $119.92 / hour
Machine cost = (10)($6.50) = $65 / hour
Inspector cost = $17 / hour

Total cost = $201.92 / hour

This total cost is higher than for the status quo so should not be adopted. The main reason for the higher cost is that slowing down the machines won't change in-process inventory for the inspection station.

c) Proposal 2 will increase the in-process inventory at the inspection station to 4.15 wing sections since the variability of the service rate has increased.

	B	C	D	E	F	G
3		**Data**				**Results**
4	λ =	7	(mean arrival rate)		L =	4.1475
5	μ =	8.33333333	(mean service rate)		L_q =	3.3075
6	k =	2	(shape parameter)			
7	s =	1	(# servers)		W =	0.5925
8					W_q =	0.4725

The in-process inventory at the presses will not change.

Inventory cost = (7.52 + 4.15)($8/hour) = $93.36 / hour
Machine cost = (10)($7/hour) = $70 / hour
Inspector cost = $17 / hour

Total cost = $180.36 / hour

This total cost is higher than for the status quo so should not be adopted. The main reason for the higher cost is the increase in the service rate variability (Erlang rather than constant) and the resulting increase in the in-process inventory.

d) They should consider *increasing* power to the presses (increasing there cost to $7.50 per hour but reducing their average time to form a wing section to 0.8 hours). This would decrease the in-process inventory at the presses to 5.69.

	A	B	C	D	E	G	H
1	**Template for the M/M/s Queueing Model**						
2							
3			**Data**				**Results**
4		λ =	7	(mean arrival rate)		L =	5.688419945
5		μ =	1.25	(mean service rate)		L_q =	0.088419945
6		s =	10	(# servers)			

Inventory cost = (5.69 + 3.94)($8/hour) = $77.04 / hour
Machine cost = (10)($7.50/hour) = $75 / hour
Inspector cost = $17 / hour

Total cost = $169.04 / hour

This total cost is lower than the status quo and both proposals.

Case

17.2 The operations of the records and benefits call center can be modeled as an M/M/s queueing system. We, therefore, use the template for the M/M/s queueing model throughout this case. The mean arrival rate equals 70 per hour, and the mean service rate of every representative equals 6 per hour. Mark needs at least s = 12 representatives answering phone calls to ensure that the queue does not grow indefinitely.

a) In order to solve this problem we have to determine the number of servers by "trial and error" until we find a number s such that the probability of waiting more than 4 minutes in the queue is above 35%.

For 13 servers we obtain the following results:

Template for M/M/s Queueing Model

Data

l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	13	(# servers)

Pr(w>t) =	0.825608
when t =	0.066667

Prob(w_q>t) =	0.362914
when t =	0.066667

Results

L =	17.07963527
L_q =	5.4129686
W =	0.24399479
W_q =	0.077328123
r =	0.897435897
P_0 =	5.32592E-06
P_1 =	6.21358E-05
P_2 =	0.000362459
P_3 =	0.001409561
P_4 =	0.004111221
P_5 =	0.009592849
P_6 =	0.018652761
P_7 =	0.031087935
P_8 =	0.045336573
P_9 =	0.058769631
P_{10} =	0.06856457
P_{11} =	0.072719998
P_{12} =	0.070699998
P_{13} =	0.063448716
P_{14} =	0.056941166
P_{15} =	0.051101037
P_{16} =	0.045859905
P_{17} =	0.041156325
P_{18} =	0.036935163
P_{19} =	0.033146942
P_{20} =	0.029747255
P_{21} =	0.026696255
P_{22} =	0.023958177
P_{23} =	0.021500928
P_{24} =	0.019295705
P_{25} =	0.017316658

For 13 servers, the probability that a customer has to wait more than 4 minutes equals 36.3%.

If there are 12 servers, this probability would be 78%:

Template for M/M/s Queueing Model

Data

l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	12	(# servers)

Pr(w>t) =	0.944173
when t =	0.066667

Prob(w_q>t) =	0.779968
when t =	0.066667

If there are 14 servers, this probability would be less than 16.4%:

Template for M/M/s Queueing Model

Data

l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	14	(# servers)

Pr(w>t) =	0.75683
when t =	0.066667

Prob(w_q>t) =	0.163704
when t =	0.066667

It appears that Mark currently employs 13 servers.

b) Using the same procedure as in part (a) we find that for s = 18 servers the probability of waiting more than 1 minute drops below 5%:

Template for M/M/s Queueing Model

Data

l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	18	(# servers)

Pr(w>t) =	0.909075
when t =	0.016667

Prob(w_q>t) =	0.032078
when t =	0.016667

Results

L =	11.77798802
L_q =	0.111321353
W =	0.168256972
W_q =	0.001590305
r =	0.648148148
P_0 =	8.49029E-06
P_1 =	9.90534E-05
P_2 =	0.000577812
P_3 =	0.002247045
P_4 =	0.006553882
P_5 =	0.015292391
P_6 =	0.029735204
P_7 =	0.049558673
P_8 =	0.072273065
P_9 =	0.093687307
P_{10} =	0.109301858
P_{11} =	0.115926213
P_{12} =	0.11270604
P_{13} =	0.101146446
P_{14} =	0.084288705
P_{15} =	0.065557882
P_{16} =	0.047802622
P_{17} =	0.032805721
P_{18} =	0.021262967
P_{19} =	0.013781553
P_{20} =	0.008932488
P_{21} =	0.005789576
P_{22} =	0.003752503
P_{23} =	0.002432178
P_{24} =	0.001576411
P_{25} =	0.001021748

c) Using the same "trial and error" method as before, we find the minimal number of servers necessary to ensure that 80% of customers wait one minute or less to be s = 15

Template for M/M/s Queueing Model

	Data	
l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	15	(# servers)

Pr(w>t) =	0.926712
when t =	0.016667

Prob(w_q>t) =	0.194213
when t =	0.016667

The minimal number of servers to ensure that 95% of customers wait 90 seconds or less is s = 17.

Template for M/M/s Queueing Model

	Data	
l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	17	(# servers)

Pr(w>t) =	0.870524
when t =	0.025

Prob(w_q>t) =	0.046459
when t =	0.025

When an employee of Cutting Edge calls the benefits center from work and has to wait on the phone, the company loses valuable work time for this customer. Mark should try to estimate the amount of work time employees lose when they have to wait on the phone. Then he could determine the cost of this waiting time and try to choose the number of representatives in such a fashion that he reaches a reasonable trade-off between the cost of employees waiting on the phone and the cost of adding new representatives.

Clearly, Mark's criteria would be different if he were dealing with external customers. While the internal customers might become disgruntled when they have to wait on the phone, they cannot call somewhere else. Effectively, the benefits center holds monopolistic power. On the contrary, if Mark were running a call center dealing with external customers, these customers could decide to do business with a competitor if they become angry from waiting on the phone.

d) If the representatives can only handle 6 calls per hour, then Mark needs to employ 18 representatives (see part b). If a representative can handle 8 calls per hour, then the minimal number of representatives equals 14:

Template for M/M/s Queueing Model

Data

l =	70	(mean arrival rate)
m =	8	(mean service rate)
s =	14	(# servers)

Pr(w>t) =	0.881748
when t =	0.016667

Prob(w_q>t) =	0.036649
when t =	0.016667

The cost of training 14 employees equals 14*$2500 = $35000 and saves Mark 4*$30000 = $120000 in annual salary. In the first year alone Mark would save $85000 if he chose to train all his employees so that they can handle 8 instead of 6 phone calls per hour.

e) Mark needs to carefully check the number of calls arriving at the call center per hour. In this case we have made the simplifying assumption that the arrival rate is constant. That assumption is unrealistic; clearly we would expect more calls during certain times of the day, during certain days of the week, and during certain weeks of the year. We might want to collect data on the number of calls received depending on the time. This data could then be used to forecast the number of calls the center will receive in the near future, which in turn would help to forecast the number of representatives needed.

Also, Mark should carefully check the number of phone calls a representative can answer per hour. Clearly, the length of a call will depend on the issue the caller wants to discuss. We might want to consider training representatives for special issues. These representatives could then always answer those particular calls. Using specialized representatives might increase the number of phone calls the entire center can handle.

Finally, using an M/M/s model is clearly a great simplification. We need to evaluate whether the assumptions for an M/M/s model are at least approximately satisfied. If this is not the case, we should consider more general models such as M/G/s or G/G/s.

CHAPTER 18: INVENTORY THEORY

18.3-1.

(a)

$$K = 15,\, h = 0.30,\, d = 30 \Rightarrow Q^* = \sqrt{\tfrac{(2)(30)(15)}{0.30}} = 54.77$$

$$t^* = Q^*/d = 1.83 \text{ months}$$

(b)

$$p = 3 \Rightarrow Q^* = \sqrt{\tfrac{2(30)(15)}{0.30}}\sqrt{\tfrac{3+0.30}{3}} = 57.45$$

$$S^* = \sqrt{\tfrac{2(30)(15)}{0.30}}\sqrt{\tfrac{3}{3+0.30}} = 52.22$$

$$t^* = Q^*/d = 1.91 \text{ months}$$

18.3-2.

(a)

$$K = 40,\, h = 0.10,\, d = 1,000 \Rightarrow Q^* = \sqrt{\tfrac{2(1,000)(40)}{0.10}} = 894.43$$

$$t^* = Q^*/d = 0.89443 \text{ weeks}$$

(b)

$$p = 3 \Rightarrow Q^* = \sqrt{\tfrac{2(1,000)(40)}{0.10}}\sqrt{\tfrac{3+0.10}{3}} = 909.21$$

$$S^* = \sqrt{\tfrac{2(1,000)(40)}{0.10}}\sqrt{\tfrac{3}{3+0.10}} = 879.88$$

$$t^* = Q^*/d = 0.90921 \text{ weeks}$$

18.3-3.

(a)

	Data				Results
d =	676	(demand/year)		Reorder Point	6.48219178
K =	$75	(setup cost)			
h =	$600.00	(unit holding cost)		Annual Setup Cost	$10,140.00
L =	3.5	(lead time in days)		Annual Holding Cost	$1,500.00
WD =	365	(working days/year)		Total Variable Cost	$11,640.00
	Decision				
Q =	5				

(b)

Q	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
5	$10,140	$1,500	$11,640
7	$7,243	$2,100	$9,343
9	$5,633	$2,700	$8,333
11	$4,609	$3,300	$7,909
13	$3,900	$3,900	$7,800
15	$3,380	$4,500	$7,880
17	$2,982	$5,100	$8,082
19	$2,668	$5,700	$8,368
21	$2,414	$6,300	$8,714
23	$2,204	$6,900	$9,104
25	$2,028	$7,500	$9,528

(c)

	Data				Results
d =	676	(demand/year)		Reorder Point	6.48219178
K =	$75	(setup cost)			
h =	$600.00	(unit holding cost)		Annual Setup Cost	$3,900.00
L =	3.5	(lead time in days)		Annual Holding Cost	$3,900.00
WD =	365	(working days/year)		Total Variable Cost	$7,800.00
	Decision				
Q =	13				

(d)

	Data				Results
d =	676	(demand/year)		Reorder Point	6.48219178
K =	$75	(setup cost)			
h =	$600.00	(unit holding cost)		Annual Setup Cost	$3,900.00
L =	3.5	(lead time in days)		Annual Holding Cost	$3,900.00
WD =	365	(working days/year)		Total Variable Cost	$7,800.00
	Decision				
Q =	13	(optimal order quantity)			

The results are the same as those obtained in (c).

(e)

$$Q^* = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2(75)(676)}{0.2(3000)}} = 13 \text{ computers purchased with each order}$$

(f)

Number of order per year: $\frac{D}{Q} = \frac{676}{13} = 52$

$$ROP = D(LT) = (13)\left(\frac{1}{2}\right) = 6.5 \text{ inventory level when each order is placed}$$

(g) The optimal policy reduces the total variable inventory cost by $\$3,840$ per year, which is a 33% reduction.

18.3-4.

(a)

	Data				Results
d =	120000	(demand/year)		Reorder Point	0
K =	$2,000	(setup cost)			
h =	$0.48	(unit holding cost)		Annual Setup Cost	$24,000.00
L =	0	(lead time in days)		Annual Holding Cost	$2,400.00
WD =	365	(working days/year)		Total Variable Cost	$26,400.00
	Decision				
Q =	10000				

(b)

Month	Q	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
1	10,000	$24,000	$2,400	$26,400
2	20,000	$12,000	$4,800	$16,800
3	30,000	$8,000	$7,200	$15,200
4	40,000	$6,000	$9,600	$15,600
5	50,000	$4,800	$12,000	$16,800
6	60,000	$4,000	$14,400	$18,400
7	70,000	$3,429	$16,800	$20,229
8	80,000	$3,000	$19,200	$22,200
9	90,000	$2,667	$21,600	$24,267
10	100,000	$2,400	$26,400	$28,800

(c)

	Data				Results
d =	120000	(demand/year)		Reorder Point	0
K =	$2,000	(setup cost)			
h =	$0.48	(unit holding cost)		Annual Setup Cost	$7,589.47
L =	0	(lead time in days)		Annual Holding Cost	$7,589.47
WD =	365	(working days/year)		Total Variable Cost	$15,178.93
	Decision				
Q =	31622.78				

If Q is required to be integer:

	Data				Results
d =	120000	(demand/year)		Reorder Point	0
K =	$2,000	(setup cost)			
h =	$0.48	(unit holding cost)		Annual Setup Cost	$7,589.41
L =	0	(lead time in days)		Annual Holding Cost	$7,589.52
WD =	365	(working days/year)		Total Variable Cost	$15,178.93
	Decision				
Q =	31623				

(d)

	Data				Results
d =	120000	(demand/year)		Reorder Point	0
K =	$2,000	(setup cost)			
h =	$0.48	(unit holding cost)		Annual Setup Cost	$7,589.47
L =	0	(lead time in days)		Annual Holding Cost	$7,589.47
WD =	365	(working days/year)		Total Variable Cost	$15,178.93
	Decision				
Q =	31622.8	(optimal order quantity)			

The results are the same as those in (c).

(e)

$Q^* = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2(2{,}000)(10{,}000)}{0.04}} = 31,622.78$ gallons purchased with each order

18.3-5.

(a) Q^* will decrease by half.

(b) Q^* will double.

(c) Q^* remains the same.

(d) Q^* will double.

(e) Q^* remains the same.

18.3-6.

(a)

$$Q^* = \sqrt{\frac{2KD}{h}} \Rightarrow 50 = \sqrt{\frac{2(75)(50)}{h}} \Rightarrow h = \$3 \text{ per month,}$$

which is 15% of the acquisition cost.

Basic EOQ Model (Solver Version)					
	Data				Results
d =	600	(demand/year)		Reorder Point	0
K =	$75	(setup cost)			
h =	$36.00	(unit holding cost)		Annual Setup Cost	$900.00
L =	0	(lead time in days)		Annual Holding Cost	$900.00
WD =	365	(working days/year)		Total Variable Cost	$1,800.00
	Decision				
Q =	50				

(b) Optimal Order Quantity

Basic EOQ Model (Solver Version)					
	Data				Results
d =	600	(demand/year)		Reorder Point	0
K =	$75	(setup cost)			
h =	$48.00	(unit holding cost)		Annual Setup Cost	$1,039.23
L =	0	(lead time in days)		Annual Holding Cost	$1,039.23
WD =	365	(working days/year)		Total Variable Cost	$2,078.46
	Decision				
Q =	43.30127				

Current Order Quantity

Basic EOQ Model (Solver Version)					
	Data				Results
d =	600	(demand/year)		Reorder Point	0
K =	$75	(setup cost)			
h =	$48.00	(unit holding cost)		Annual Setup Cost	$900.00
L =	0	(lead time in days)		Annual Holding Cost	$1,200.00
WD =	365	(working days/year)		Total Variable Cost	$2,100.00
	Decision				
Q =	50				

(c)

Basic EOQ Model (Solver Version)					
	Data				Results
d =	600	(demand/year)		Reorder Point	10
K =	$75	(setup cost)			
h =	$48.00	(unit holding cost)		Annual Setup Cost	$1,039.23
L =	5	(lead time in days)		Annual Holding Cost	$1,039.23
WD =	300	(working days/year)		Total Variable Cost	$2,078.46
	Decision				
Q =	43.30127				

(d) $ROP = 5 + (50)(5/25) = 15$ hammers, which adds $5 \times \$4 = \$20$ to TVC every month, $240 per year.

18.3-7.

$K = 12,000$, $h = 0.30$, $d = 8,000$, $p = 5$

$$Q^* = \sqrt{\frac{2(8000)(12000)}{0.30}}\sqrt{\frac{5+0.30}{5}} = 26,046$$

$$S^* = \sqrt{\frac{2(8000)(12000)}{0.30}}\sqrt{\frac{5}{5+0.30}} = 24,572$$

$t^* = Q^*/d = 3.26$ months

18.3-8.

(a)

	Data				Results
d =	6000	(demand/year)		Reorder Point	0
K =	$1,000	(setup cost)			
h =	$100.00	(unit holding cost)		Annual Setup Cost	$17,320.51
L =	0	(lead time in days)		Annual Holding Cost	$17,320.51
WD =	365	(working days/year)		Total Variable Cost	$34,641.02
	Decision				
Q =	346.41	(optimal order quantity)			

(b)

	Data				Results
d =	6000	(demand/year)		Max Inventory Level	268.33
K =	$1,000	(setup cost)			
h =	$100.00	(unit holding cost)		Annual Setup Cost	$13,416.41
p =	$150.00	(unit shortage cost)		Annual Holding Cost	$8,049.84
				Annual Shortage Cost	$5,366.56
	Decision			Total Variable Cost	$26,832.82
Q =	447.2136	(optimal order quantity)			
S =	178.8854	(optimal maximum shortage)			

18.3-9.

(a)

	Data				Results
d =	676	(demand/year)		Max Inventory Level	6.00
K =	$75	(setup cost)			
h =	$600.00	(unit holding cost)		Annual Setup Cost	$1,950.00
p =	$200.00	(unit shortage cost)		Annual Holding Cost	$415.38
				Annual Shortage Cost	$1,538.46
	Decision			Total Variable Cost	$3,903.85
Q =	26	(order quantity)			
S =	20	(maximum shortage)			

This TVC is almost half of the optimal value found for Problem 18.3-3.

(b)

Q	Annual Setup Cost	Annual Holding Cost	Annual Shortage Cost	Total Variable Cost
	$1,950	$415	$1,538	$3,904
15	$3,380	$500	$2,667	$6,547
17	$2,982	$159	$2,353	$5,494
19	$2,668	$16	$2,105	$4,789
21	$2,414	$14	$1,905	$4,333
23	$2,204	$117	$1,739	$4,061
25	$2,028	$300	$1,600	$3,928
27	$1,878	$544	$1,481	$3,904
29	$1,748	$838	$1,379	$3,966
31	$1,635	$1,171	$1,290	$4,097
33	$1,536	$1,536	$1,212	$4,285
35	$1,449	$1,929	$1,143	$4,520

(c)

S	Annual Setup Cost	Annual Holding Cost	Annual Shortage Cost	Total Variable Cost
	$1,950	$415	$1,538	$3,904
10	$1,950	$2,954	$385	$5,288
12	$1,950	$2,262	$554	$4,765
14	$1,950	$1,662	$754	$4,365
16	$1,950	$1,154	$985	$4,088
18	$1,950	$738	$1,246	$3,935
20	$1,950	$415	$1,538	$3,904
22	$1,950	$185	$1,862	$3,996
24	$1,950	$46	$2,215	$4,212
26	$1,950	$0	$2,600	$4,550
28	$1,950	$46	$3,015	$5,012
30	$1,950	$185	$3,462	$5,596

18.3-10.

$\frac{p}{h}$	$Q^* = \sqrt{\frac{h+p}{p}}\sqrt{\frac{2KD}{h}}$	Maximum Inventory Level	Maximum Shortage
1/3	2,000	500	1,500
1	1,414	707	707
2	1,225	816	408
3	1,155	866	289
5	1,095	913	183
10	1,049	953	95

18.3-11.

(a)

Maximum inventory: $\frac{(b-a)Q}{b}$
Length of interval I: $\frac{Q}{b}$
Average inventory in interval I: $\frac{(b-a)Q}{2b}$
Length of interval II: $\frac{Q}{a} - \frac{Q}{b}$
Average inventory in interval II: $\frac{(b-a)Q}{2b}$

Average inventory per cycle: $\frac{(b-a)Q}{2b}$

Holding cost per cycle: $\frac{(b-a)Q}{2ah}$

$$\Rightarrow T = -\frac{aK}{Q} + \frac{(b-a)hQ}{2b} + ac$$

(b)

$$\frac{dT}{dQ} = -\frac{aK}{Q^2} + \frac{(b-a)h}{2b} = 0 \Rightarrow Q^* = \sqrt{\frac{2abk}{(b-a)h}}$$

18.3-12.

(a) $D = 5200, K = \$50, I = 0.2, N = 3$

		Range of order quantities				Annual Purchase Cost	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
Category	Price	Lower Limit	Upper Limit	EOQ	Q*				
1	$100.00	0	99	161	99	$520,000	$2,626	$990	$523,616
2	$95.00	100	499	165	165	$494,000	$1,572	$1,572	$497,143
3	$90.00	500	10000000	170	500	$468,000	$520	$4,500	$473,020
			Results						
		Optimal Q	500						
		Total Variable Cost	$473,020						

(b)

Orders placed per year: $\frac{D}{Q} = \frac{5200}{500} = 10.4$
Time interval between orders: $\frac{Q}{D} = \frac{500}{5200} = 0.096$ years ≈ 5 weeks

18.3-13.

(a) $D = 365, K = \$10, I = 0.1, N = 3$

						Annual	Annual	Annual	Total
		Range of order quantities				Purchase	Setup	Holding	Variable
Category	Price	Lower Limit	Upper Limit	EOQ	Q*	Cost	Cost	Cost	Cost
1	$5.00	1	49	121	49	$1,825	$74	$12	$1,912
2	$4.85	50	99	123	99	$1,770	$37	$24	$1,831
3	$4.70	100	10000000	125	125	$1,716	$29	$29	$1,774

	Results
Optimal Q	124.6271
Total Variable Cost	$1,774

(b)

Orders placed per year: $\frac{D}{Q} = \frac{365}{124.63} = 2.93$

Time interval between orders: $\frac{Q}{D} = \frac{124.63}{365} = 0.341$ years ≈ 17.76 weeks

18.3-14.

(a)

Discount Category	$TVC = cD + K\frac{D}{Q} + h\frac{Q}{2}$
1	$TVC = (8.50)(400) + (80)\left(\frac{400}{Q}\right) + (0.2)(8.50)\left(\frac{Q}{2}\right)$
2	$TVC = (8.00)(400) + (80)\left(\frac{400}{Q}\right) + (0.2)(8.00)\left(\frac{Q}{2}\right)$
3	$TVC = (7.50)(400) + (80)\left(\frac{400}{Q}\right) + (0.2)(7.50)\left(\frac{Q}{2}\right)$

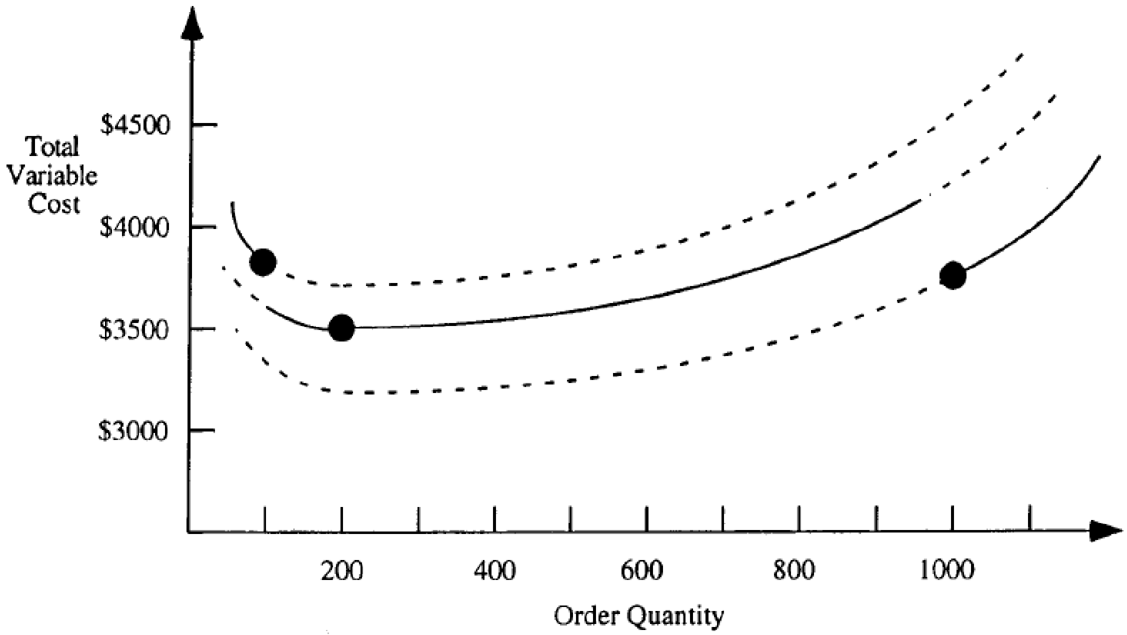
(b)

Discount Category	$Q^* = \sqrt{\frac{2KD}{h}}$
1	$Q^* = \sqrt{\frac{2(80)(400)}{0.2(8.50)}} = 194$
2	$Q^* = \sqrt{\frac{2(80)(400)}{0.2(8.00)}} = 200$
3	$Q^* = \sqrt{\frac{2(80)(400)}{0.2(7.50)}} = 207$

(c)

Discount Category	Feasible Q	$TVC = cD + K\frac{D}{Q} + h\frac{Q}{2}$
1	99	$\$3,807.38$
2	200	$\$3,520.00$
3	1000	$\$3,782.00$

(d)



(e) $Q^* = 200$ with a TVC of $\$3,520$

(f) $D = 400, K = \$80, I = 0.2, N = 3$

						Annual	Annual	Annual	Total
		Range of order quantities				Purchase	Setup	Holding	Variable
Category	Price	Lower Limit	Upper Limit	EOQ	Q*	Cost	Cost	Cost	Cost
1	$8.50	0	99	194	99	$3,400	$323	$84	$3,807
2	$8.00	100	999	200	200	$3,200	$160	$160	$3,520
3	$7.50	1000	10000000	207	1000	$3,000	$32	$750	$3,782
			Results						
		Optimal Q	200						
		Total Variable Cost	$3,520						

(g) Since the value of Q that minimizes TVC for discount category 2 is feasible, this order quantity minimizes the annual setup and holding costs. Then, category 1 cannot have lower annual setup and holding costs. Furthermore, since the purchase price per case is higher for category 1, it cannot have lower purchasing costs. Hence, category 1 can be eliminated as a candidate for providing the optimal order quantity.

(h)

Orders placed per year: $\frac{D}{Q} = \frac{400}{200} = 2$

Time interval between orders: $\frac{Q}{D} = \frac{200}{400} = 0.5 \text{ years} = 6 \text{ months}$

18.3-15.

(a)

Discount Category	$TVC = cD + K\frac{D}{Q} + h\frac{Q}{2}$
1	$TVC = (1.00)(2400) + (4)\left(\frac{2400}{Q}\right) + (0.17)(1.00)\left(\frac{Q}{2}\right)$
2	$TVC = (0.95)(2400) + (4)\left(\frac{2400}{Q}\right) + (0.17)(0.95)\left(\frac{Q}{2}\right)$
3	$TVC = (0.90)(2400) + (4)\left(\frac{2400}{Q}\right) + (0.17)(0.90)\left(\frac{Q}{2}\right)$

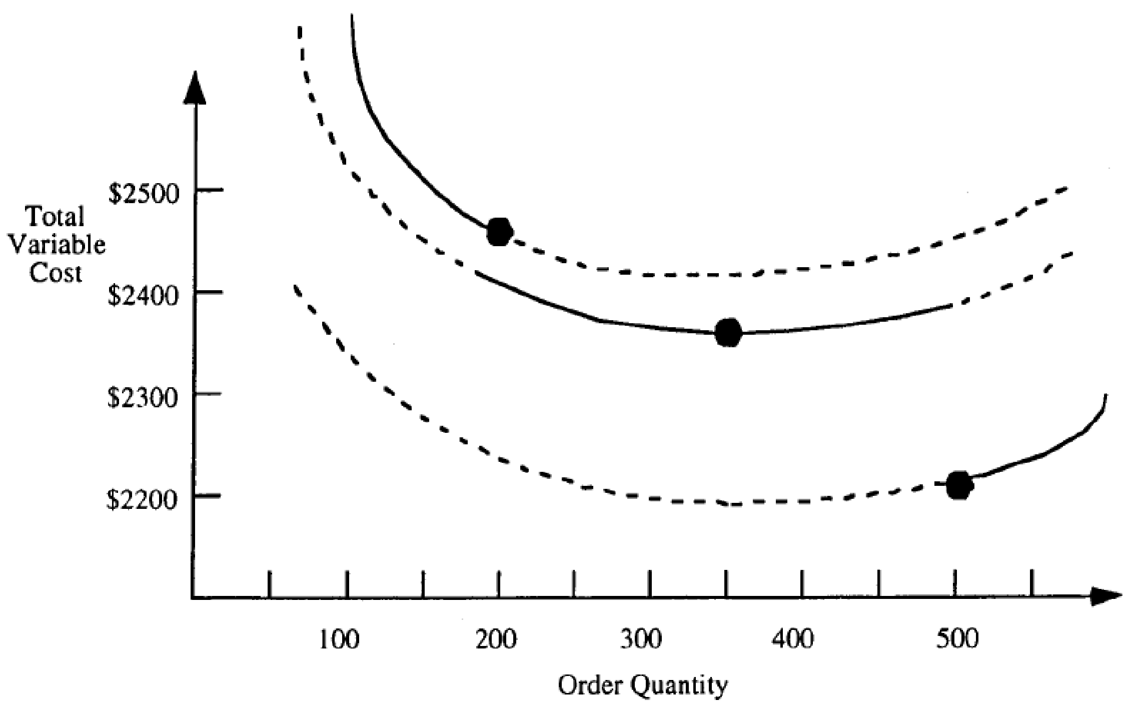
(b)

Discount Category	$Q^* = \sqrt{\frac{2KD}{h}}$
1	$Q^* = \sqrt{\frac{2(4)(2400)}{0.17(1.00)}} = 336$
2	$Q^* = \sqrt{\frac{2(4)(2400)}{0.17(0.95)}} = 345$
3	$Q^* = \sqrt{\frac{2(4)(2400)}{0.17(0.90)}} = 354$

(c)

Discount Category	Feasible Q	$TVC = cD + K\frac{D}{Q} + h\frac{Q}{2}$
1	199	$\$2,465.16$
2	345	$\$2,335.68$
3	500	$\$2,217.45$

(d)



(e) $Q^* = 500$ with a TVC of $\$2,217.45$

(f) $D = 2400, K = \$4, I = 0.17, N = 3$

						Annual	Annual	Annual	Total
		Range of order quantities				Purchase	Setup	Holding	Variable
Category	Price	Lower Limit	Upper Limit	EOQ	Q*	Cost	Cost	Cost	Cost
1	$1.00	0	199	336	199	$2,400	$48	$17	$2,465
2	$0.95	200	499	345	345	$2,280	$28	$28	$2,336
3	$0.90	500	10000000	354	500	$2,160	$19	$38	$2,217
			Results						
		Optimal Q	500						
	Total Variable Cost		$2,217						

(g) Since the value of Q that minimizes TVC for discount category 2 is feasible, this order quantity minimizes the annual setup and holding costs. Then, category 1 cannot have lower annual setup and holding costs. Furthermore, since the purchase price per bag is higher for category 1, it cannot have lower purchasing costs. Hence, category 1 can be eliminated as a candidate for providing the optimal order quantity.

(h)

Orders placed per year: $\frac{D}{Q} = \frac{2400}{500} = 4.8$

Time interval between orders: $\frac{Q}{D} = \frac{500}{2400} = 0.21$ years ≈ 2.5 months

18.4-1.

$C_5 = C_6 + 9 = 0 + 9 = 9$

$C_4^{(4)} = C_5 + 9 = 9 + 9 = 18$

$C_4^{(5)} = C_6 + 9 + 0.8(r_5) = 0 + 9 + 0.8(20) = 25$

$C_4 = \min\{18, 25\} = 18$

$C_3^{(3)} = C_4 + 9 = 18 + 9 = 27$

$C_3^{(4)} = C_5 + 9 + 0.8(r_4) = 9 + 9 + 0.8(10) = 26$

$C_3^{(5)} = C_6 + 9 + 0.8(r_4 + 2r_5) = 0 + 9 + 0.8(10 + 40) = 49$

$C_3 = \min\{27, 26, 49\} = 26$

$C_2^{(2)} = C_3 + 9 = 26 + 9 = 35$

$C_2^{(3)} = C_4 + 9 + 0.8(r_3) = 18 + 9 + 0.8(15) = 39$

$C_2^{(4)} = C_5 + 9 + 0.8(r_3 + 2r_4) = 9 + 9 + 0.8(15 + 20) = 46$

$C_2^{(5)} = C_6 + 9 + 0.8(r_3 + 2r_4 + 3r_5) = 0 + 9 + 0.8(15 + 20 + 60) = 85$

$C_2 = \min\{35, 39, 46, 85\} = 35$

$C_1^{(1)} = C_2 + 9 = 35 + 9 = 44$

$C_1^{(2)} = C_3 + 9 + 0.8(r_2) = 26 + 9 + 0.8(25) = 55$

$C_1^{(3)} = C_4 + 9 + 0.8(r_2 + 2r_3) = 18 + 9 + 0.8(25 + 30) = 71$

$C_1^{(4)} = C_5 + 9 + 0.8(r_2 + 2r_3 + 3r_4) = 9 + 9 + 0.8(25 + 30 + 30) = 86$

$C_1^{(5)} = C_6 + 9 + 0.8(r_2 + 2r_3 + 3r_4 + 4r_5) = 0 + 9 + 0.8(25 + 30 + 30 + 80) = 141$

$C_1 = \min\{44, 55, 71, 86, 141\} = 44$

The optimal production schedule is to produce 10 in the first month, 25 in the second, 25 in the third and 20 in the last month. The total variable cost associated with this schedule is \$$44,000$. The total cost including the production cost is \$$284,000$.

18.4-2.

$C_4 = C_5 + 2 = 2$

$C_3^{(3)} = C_4 + 2 = 2 + 2 = 4$

$C_3^{(4)} = C_5 + 2 + 0.2(r_4) = 0 + 2 + 0.2(3) = 2.6$

$C_3 = \min\{4, 2.6\} = 2.6$

$C_2^{(2)} = C_3 + 2 = 2.6 + 2 = 4.6$

$C_2^{(3)} = C_4 + 2 + 0.2(r_3) = 2 + 2 + 0.2(4) = 4.8$

$C_2^{(4)} = C_5 + 2 + 0.2(r_3 + 2r_4) = 0 + 2 + 0.2(4 + 6) = 4$

$C_2 = \min\{4.6, 4.8, 4\} = 4$

$C_1^{(1)} = C_2 + 2 = 4 + 2 = 6$

$C_1^{(2)} = C_3 + 2 + 0.2(r_2) = 2.6 + 2 + 0.2(3) = 5.2$

$C_1^{(3)} = C_4 + 2 + 0.2(r_2 + 2r_3) = 2 + 2 + 0.2(3 + 8) = 6.2$

$C_1^{(4)} = C_5 + 2 + 0.2(r_2 + 2r_3 + 3r_4) = 0 + 2 + 0.2(3 + 8 + 9) = 6$

$C_1 = \min\{6, 5.2, 6.2, 6\} = 5.2$

The optimal production schedule is to produce 7 units in the first and third periods at a total variable cost of \$$5.2$ million.

18.4-3.

x_4	z_4	$C_4^*(x_4)$	z_4^*
0	2	4	2
1	1	3	1
2	0	0	0

	$C_3(x_3, z_3)$							
x_3	0	1	2	3	4	5	$C_3^*(x_3)$	z_3^*
0	–	–	–	10.2	10.8	9.4	9.4	5
1	–	–	8.8	9.4	8.0	–	8.0	4
2	–	7.4	8.0	6.6	–	–	6.6	3
3	4.0	6.6	5.2	–	–	–	4.0	0
4	3.2	3.8	–	–	–	–	3.2	0
5	0.4	–	–	–	–	–	0.4	0

	$C_2(x_2, z_2)$									
x_2	0	1	2	3	4	5	6	7	$C_2^*(x_2)$	z_2^*
0	–	–	13.4	13.2	14.0	11.6	13.0	10.4	10.4	7
1	–	12.4	12.2	13.0	10.6	12.0	9.4	–	9.4	6
2	9.4	11.2	12.0	9.6	11.0	8.4	–	–	8.4	5
3	8.2	11.0	8.6	10.0	7.4	–	–	–	7.4	4
4	7.0	7.6	9.0	6.4	–	–	–	–	6.4	3
5	4.6	8.0	5.4	–	–	–	–	–	4.6	0
6	4.0	4.4	–	–	–	–	–	–	4.0	0
7	1.4	–	–	–	–	–	–	–	1.4	0

	$C_1(x_1, z_1)$									
x_1	3	4	5	6	7	8	9	10	$C_1^*(x_1)$	z_1^*
0	16.8	17.2	17.8	18.4	19.0	18.8	19.8	18.8	16.8	3

The optimal production schedule is to produce 3 units in period 1 and 7 units in period 2, with a cost of \$16.8 million.

18.4-4.

$h = 2$

$$B(x_n, z_n) = \begin{cases} k_n + c_n z_n + 2\max\{0, z_n - 3\} + h(x_n + z_n - r_n) & \text{for } 0 < z_n \leq 4 \\ h(x_n - z_n) & \text{for } z_n = 0 \end{cases}$$

x_3	z_3	$C_3^*(x_3)$	z_3^*
0	4	47	4
1	3	36	3
2	2	27	2
3	1	18	1
4	0	4	0

	$C_2(x_2, z_2)$						
x_2	0	1	2	3	4	$C_2^*(x_2)$	z_2^*
0	–	–	–	87	90	87	3
1	–	–	77	78	83	77	2
2	–	67	68	71	76	67	1
3	47	58	61	64	64	47	0
4	38	51	54	52	–	38	0

	$C_1(x_1, z_1)$						
x_1	0	1	2	3	4	$C_1^*(x_1)$	z_1^*
0	87	92	92	82	85	82	3

The optimal production schedule is to produce 3 units in period 1 and 4 units in period 3, with a cost of \$82 thousand.

18.5-1.

Deere & Company uses inventory theory to determine optimal inventory levels ensuring product availability, on-time delivery, and customer satisfaction. In doing this, the multistage inventory planning and optimization (MIPO) tool developed by SmartOps is deployed. The underlying model is a stochastic, capacitated, multiechelon, multiproduct production and inventory model. In MIPO, the material flow in the supply chain is represented as an acyclic-directed graph. The recommended stock levels are found by minimizing the inventory costs among periodic-review replenishment policies with a certain service level. The demand is stochastic and its probability distribution is nonstationary over time. The latter allows to model seasonality of demand. The capacities and supply paths can be nonstationary. Lower bounds on service levels and other constraints can be encapsulated in the model. The main decision variables are safety stocks. Once the optimal stock levels are found, what-if analyses are performed to evaluate the impact of changes.

After the implementation of the results, on-time deliveries have increased from 63% to 92% with a 90% customer service level. The reduction in inventory provided a savings of $890 million between 2001 and 2003 and a $107 million increase in annual shareholder value added. Estimated savings by the end of 2004 exceed $1 billion. The new system also allows Deere to reduce the amount of aged inventory and to offer customers newer models. This, in turn, avoids discounts and saves Deere over $10 million per year. Other benefits from this study include enhanced manufacturing flexibility, improved service levels, accurate predictions, ability to respond to changes quickly and trust in the supply chain.

18.5-2.

$K_1 = \$25,000, K_2 = \$1,500, h_1 = \$30, h_2 = \$35, d = 4,000$

Optimizing separately:

$$Q_2^* = \sqrt{\frac{2dK_2}{h_2}} = 586$$

$$C_2^* = \sqrt{2dK_2h_2} = \$20,493.9$$

$$n^* = \sqrt{\frac{K_1h_2}{K_2h_1}} = 4.41, \frac{n^*}{[n^*]} \leq \frac{[n^*]+1}{n^*} \Rightarrow n = 4$$

$$Q_1^* = nQ_2^* = 2344$$

$$C_1^* = \frac{dK_1}{nQ_2} + \frac{h_1(n-1)Q_2}{2} = \$69,032.12$$

$$C^* = C_1^* + C_2^* = \$89,526.02$$

Optimizing simultaneously:

$$e_1 = h_1 = 30, e_2 = h_2 - h_1 = 5$$

$$n^* = \sqrt{\frac{K_1e_2}{K_2e_1}} = 1.67, \frac{n^*}{[n^*]} > \frac{[n^*]+1}{n^*} \Rightarrow n = 2$$

$$Q_2^* = \sqrt{\frac{2d\left(\frac{K_1}{n}+K_2\right)}{ne_1+e_2}} = 1313$$

$$Q_1^* = nQ_2^* = 2626$$

$$C^* = \sqrt{2d\left(\frac{K_1}{n} + K_2\right)(ne_1 + e_2)} = \$85,322.92$$

Quantity	Separate Optimization	Simultaneous Optimization
Q_2^*	586	1313
n^*	4.41	1.67
n	4	2
Q_1^*	2344	2626
C^*	$\$89,526$	$\$85,323$

The increase in the total variable cost per unit time if the results from separate optimization were to be used instead of the ones from simultaneous optimization is almost 5%.

18.5-3.

(a) $h_1 = \$25, h_2 = \$250, d = 2,500$

Quantity	$(\$25000, \$1000)$	$(\$10000, \$2500)$	$(\$5000, \$5000)$
Q_2^*	149	236	333
n^*	15	6	3
n	15	6	3
Q_1^*	2236	1414	1000

(b) $K_1 = \$10,000, K_2 = \$2500, d = 2,500$

Quantity	$(\$10, \$500)$	$(\$25, \$250)$	$(\$50, \$100)$
Q_2^*	160	236	500
n^*	14	6	2
n	14	6	2
Q_1^*	2236	1414	1000

(c) $K_1 = \$10,000, K_2 = \$2500, h_1 = \$25, h_2 = \$250$

Quantity	1000	2500	5000
Q_2^*	149	236	333
n^*	6	6	6
n	6	6	6
Q_1^*	894	1414	2000

18.5-4.

$K_1 = \$5,000, K_2 = \$200, h_1 = \$10, h_2 = \$11, d = 100$

Quantity	Separate Optimization (a)	Simultaneous Optimization (b)
Q_2^*	60	160
n^*	5.24	1.58
n	5	2
Q_1^*	302	321
C^*	3528	3367

(c) The decrease in the total variable cost per unit time C^* by using the approach in (b) rather than the one in (a) is 5%.

18.5-5.

$K_1 = \$50,000, K_2 = \$500, h_1 = \$50, h_2 = \$60, d = 500$

Quantity	Separate Optimization (a)	Simultaneous Optimization (b)
Q_2^*	91	249
C_2^*	5477	8469
n^*	10.95	4.47
n	11	4
Q_1^*	1004	995
C_1^*	47718	43780
C^*	53195	52249

(c) The assembly plant will lose money ($-\$2,992$) by using the joint inventory policy obtained in (b) whereas the supplier will make money ($\$3,938$) by doing so. One possible financial agreement between the supplier and the assembly plant is that the supplier will compensate for the loss of the plant so that the plant agrees to a supply contract inducing the inventory policy in (b). By using this policy instead of separately optimal ones, the total saving is $-\$2,992 + \$3,938 = \$946$.

18.5-6.

$K_1 = \$50,000, K_2 = \$2,000, K_3 = \$360, h_1 = \$1, h_2 = \$2, h_3 = \$10, d = 5,000$

Installation i	Solution of Relaxed Problem		Initial Solution of Revised Problem		Final Solution of Revised Problem	
	Q_i	C_i	Q_i^*	C_i	Q_i^*	C_i
1	10000	10000	9600	10008	9628	10007
2	2000	2000	2400	2033	2407	2034
3	300	2400	300	2400	301	2400
	$\underline{C}$ = 14400		C = 14442		$\overline{C}$ = 14442	

The cost $\overline{C}$ is about 0.29% above the optimal cost $\underline{C}$ of the relaxed problem. Since the latter is a lower bound on the optimal cost C^* of the original problem, the optimal cost $\overline{C}$ of the revised problem can exceed C^* at most by 0.29%.

18.5-7.

$K_1 = \$125,000, K_2 = \$20,000, K_3 = \$6,000, K_4 = \$10,000, K_5 = \$250$
$h_1 = \$2, h_2 = \$10, h_3 = \$15, h_4 = \$20, h_5 = \$30, d = 1,000$
$e_1 = \$2, e_2 = \$8, e_3 = \$5, e_4 = \$5, e_5 = \$10$

Since $(K_3/e_3) = 1200 < 2000 = (K_4/e_4)$, we need to merge the installation 3 and 4 as a new installation 3' with $K_{3'} = \$16,000$ and $e_{3'} = \$10$.

Installation i	Solution of Relaxed Problem		Initial Solution of Revised Problem		Final Solution of Revised Problem	
	Q_i	C_i	Q_i^*	C_i	Q_i^*	C_i
1	11180	22361	14311	23045	13954	22912
2	2236	17889	1789	18336	1744	18443
3' (3+4)	1789	17889	1789	17889	1744	17894
5	224	2236	224	2236	218	2237
	$\underline{C} = 60374$		$C = 61506$		$\overline{C} = 61486$	

The cost $\overline{C}$ is about 1.84% above the optimal cost $\underline{C}$ of the relaxed problem. Since the latter is a lower bound on the optimal cost C^* of the original problem, the optimal cost $\overline{C}$ of the revised problem can exceed C^* at most by 1.84%.

18.5-8.

$K_1 = \$1,000, K_2 = \$5, K_3 = \$75, K_4 = \$80$
$h_1 = \$0.5, h_2 = \$0.55, h_3 = \$3.55, h_4 = \$7.55, d = 4,000$

Installation i	Solution of Relaxed Problem		Initial Solution of Revised Problem		Final Solution of Revised Problem	
	Q_i	C_i	Q_i^*	C_i	Q_i^*	C_i
1	4000	2000	3200	2050	3200	2050
2	894	45	800	45	800	45
3	447	1342	400	1350	400	1350
4	400	1600	400	1600	400	1600
	$\underline{C} = 4986$		$C = 5045$		$\overline{C} = 5045$	

The cost $\overline{C}$ is about 1.18% above the optimal cost $\underline{C}$ of the relaxed problem. Since the latter is a lower bound on the optimal cost C^* of the original problem, the optimal cost $\overline{C}$ of the revised problem can exceed C^* at most by 1.18%.

18.5-9.

$K_1 = \$60,000, K_2 = \$6,000, K_3 = \$400, h_1 = \$3, h_2 = \$7, h_3 = \$9, d = 10,000$

Installation i	Solution of Relaxed Problem		Initial Solution of Revised Problem		Final Solution of Revised Problem	
	Q_i	C_i	Q_i^*	C_i	Q_i^*	C_i
1	20000	60000	16000	61500	20257	60005
2	5477	21909	4000	23000	5064	21976
3	2000	4000	2000	4000	2532	4112
	$\underline{C} = 85909$		$C = 88500$		$\overline{C} = 86093$	

The cost $\overline{C}$ is about 0.21% above the optimal cost $\underline{C}$ of the relaxed problem. Since the latter is a lower bound on the optimal cost C^* of the original problem, the optimal cost $\overline{C}$ of the revised problem can exceed C^* at most by 0.21%.

18.6-1.

(a)

$$Q=\sqrt{\frac{h+p}{p}}\sqrt{\frac{2KD}{h}}=\sqrt{\frac{3000+1000}{1000}}\sqrt{\frac{2(1500)(900)}{3000}}=60$$

(b) $R=\mu+K_L\sigma=50+0.675(15)=60$

(c)

	Data				Results
d =	900	(average demand/unit time)		Q =	60
K =	$1,500	(setup cost)		R =	60
h =	$3,000.00	(unit holding cost)			
p =	$1,000	(unit shortage cost)			
L =	0.75	(service level)			
Demand During Lead Time					
Distribution	Normal				
mean =	50				
stand. dev. =	15				

(d) Safety Stock: $R-\text{mean}=60-50=10$

(e) If demand during the delivery time exceeds the order quantity 60, then the reorder point will be hit again before the order arrives, triggering another order.

18.6-2.

(a)

$$Q=\sqrt{\frac{2DK}{h}}\sqrt{\frac{h+p}{p}}=\sqrt{\frac{2(80)(100)}{15}}\sqrt{\frac{15+3}{3}}=80$$

$$R=a+L(b-a)=10+0.8(30-10)=26$$

(b)

	Data				Results
d =	80	(average demand/unit time)		Q =	80
K =	$100	(setup cost)		R =	26
h =	$15.00	(unit holding cost)			
p =	$3	(unit shortage cost)			
L =	0.8	(service level)			
Demand During Lead Time					
Distribution	Uniform				
a =	10	(lower endpoint)			
b =	30	(upper endpoint)			

(c) Average number of orders per year: $(80)(12)/80=12$
Probability of a stock-out before the order is received: $1-0.8=0.2$
Average number of stock-outs per year: $12(0.2)=2.4$

18.6-3.

(a)

	Case 1	Case 2	Case 3	Case 4
L	$h = \$1, \sigma = 1$	$h = \$100, \sigma = 1$	$h = \$1, \sigma = 100$	$h = \$100, \sigma = 100$
0.5	0	0	0	0
0.75	0.675	67.5	67.5	6750
0.9	1.282	128.2	128.2	$12,820$
0.95	1.645	164.5	164.5	$16,450$
0.99	2.327	232.7	232.7	$23,270$
0.999	3.098	309.8	309.8	$30,980$

(b)

	Case 1	Case 2	Case 3	Case 4
ΔL	$h = \$1, \sigma = 1$	$h = \$100, \sigma = 1$	$h = \$1, \sigma = 100$	$h = \$100, \sigma = 100$
0.5	0.675	67.5	67.5	6750
0.15	0.607	60.7	60.7	6070
0.05	0.363	36.3	36.3	3630
0.04	0.682	68.2	68.2	6820
0.009	0.771	77.1	77.1	7710

(c) As the service level gets higher, increasing the service level further costs more for smaller increases. Thus, there will be diminishing returns when raising the service level further and further. A manager should balance the cost of the safety stock with the cost of stock-outs to determine the best service level.

18.6-4.

(a) $C = hK_L\sigma = (100)(1.282)(100) = \$12,820$

(b) $\sigma = \sqrt{d}\sigma_1 \Rightarrow 100 = \sqrt{4}\sigma_1 \Rightarrow \sigma_1 = 50$

If the lead time were one day: $C = hK_L\sigma_1 = (100)(1.282)(50) = \$6,410$. This is a 50% reduction in the cost of the safety stock.

(c) $\sigma = \sqrt{d}\sigma_1 = \sqrt{8}(50) = 141.4$, $C = hK_L\sigma_1 = (100)(1.282)(141.4) = \$18,127$

This is a 41% increase in the cost of the safety stock.

(d) The lead time would need to quadruple to 16 days.

18.6-5.

(a) The safety stock drops to zero.

(b) The safety stock decreases.

(c) The safety stock remains the same for a given service level. However, with higher shortage costs, there will be an incentive to increase the service level, which induces a higher level of safety stock.

(d) The safety stock increases.

(e) The safety stock doubles.

(f) The safety stock doubles.

18.6-6.

(a) Ground Chuck

	Data				Results
d =	26000	(average demand/unit time)		Q =	2,183
K =	$25	(setup cost)		R =	145
h =	$0.30	(unit holding cost)			
p =	$3	(unit shortage cost)			
L =	0.95	(service level)			
Demand During Lead Time					
Distribution	Uniform				
a =	50	(lower endpoint)			
b =	150	(upper endpoint)			

Chuck Wagon

	Data				Results
d =	26000	(average demand/unit time)		Q =	6,175
K =	$200	(setup cost)		R =	829
h =	$0.30	(unit holding cost)			
p =	$3	(unit shortage cost)			
L =	0.95	(service level)			
Demand During Lead Time					
Distribution	Normal				
mean =	500				
stand. dev. =	200				

(b) Ground Chuck: $R = a + L(b - a) = 50 + 0.95(150 - 50) = 145$

Chuck Wagon: $R = \mu + K_L\sigma = 500 + 1.645(200) = 829$

(c) Ground Chuck: safety stock $R -$ mean $= 145 - 100 = 45$

Chuck Wagon: safety stock $R -$ mean $= 829 - 500 = 329$

(d) Ground Chuck:

Annual average holding cost: $(0.30)\left(\frac{45+(2183+45)}{2}\right) = \$340.95$

Chuck Wagon:

Annual average holding cost: $(0.30)\left(\frac{329+(6175+329)}{2}\right) = \$3,416.50$

(e) Ground Chuck:

Annual shipping cost: $K\left(\frac{D}{Q}\right) = 25\left(\frac{26,000}{2183}\right) = \$297.76$

Annual purchasing cost: $(26,000)(1.49) = \$38,740$

Average annual acquisition cost: $\$297.76 + \$38,740 = \$39,037.76$

Chuck Wagon:

Annual shipping cost: $K\left(\frac{D}{Q}\right)+0.10D=200\left(\frac{26{,}000}{6175}\right)+0.10(26,000)$

$= \$3442.11$

Annual purchasing cost: $(26,000)(1.35)=\$35,100$

Average annual acquisition cost: $\$3442.11+\$35,100=\$38,542.11$

(f) Ground Chuck: $\$340.95+\$39,037.76=\$39,378.71$

Chuck Wagon: $\$3,416.50+\$38,542.11=\$41,958.61$

Jed should choose Ground Chuck as their supplier.

(g) If Jed would like to use the beef within a month of receiving it, then Ground Chuck is the best choice. The order quantity with Ground Chuck is roughly one month's supply whereas with Chuck Wagon, it is roughly three months' supply.

18.7-1.

In this study, inventory theory is applied to the three-echelon distribution problem faced by Time Inc., the largest magazine publisher in the US. For each issue of each magazine, Time Inc. needs to solve three subproblems. The first is to determine the total number D of copies to be printed and shipped. The second is to find an allocation $D_1,\ldots,D_N$ of these D copies among N wholesalers. The third subproblem is to decide on the distribution d_{ij} of D_j copies among n_j retailers of wholesaler j for every j. Complicated cost and revenue structures, timing and constraints on available information complicate these problems. The overall objective is to maximize the expected total profit. The problem is solved backwards by using readily available results from the literature of newsvendor problem under ideal conditions. The solution found is then adjusted to incorporate deviations from the ideal.

To solve the store-level allocation problem, first the distribution of demand is estimated using statistical analysis. If $F(k|\mu_{ij})$ is the probability that the demand in store i of wholesaler j is at least k, then the optimal allocation to this retailer is determined as $d_{ij}=F^{-1}(\lambda|\mu_{ij})$ or the best approximation to this. With this allocation, the probability of selling out is $1-\lambda$ for each store and the solution satisfies $\sum_i d_{ij}=D_i$. Similarly, the wholesaler-level allocation is found from the equation $m_j(D_j)=m$, where $m_j(\,\cdot\,)$ is the probability that wholesaler j will sell the last copy shipped and m is chosen such that $\sum_j D_j=D$. Finally, a lower bound on the national print order is determined from $M(D_0)=c/r$, where $M(\,\cdot\,)$ is the probability of selling the last copy printed and shipped, c and r are the marginal cost and revenue respectively. Because of the complications in identifying c and r, Time Inc. aims at producing more than D_0.

The new system increased Time Inc.'s annual profits by over $3.5 million. The benefits include improvement of wholesaler and retailer allocations, and increase of sales stimulation effect by over 1%.

18.7-2.

F$(S^*)=\frac{S^*-50}{25}=\frac{p-c}{p+h}=\frac{0.75-0.55}{0.75+0.01}\Rightarrow S^*=50+\frac{5}{0.76}\approx 57$

18.7-3.

(a) Freddie's most profitable alternative is to order 16 copies.

	State of Nature				Expected
Alternative	15	16	17	18	Payoff
Order 15 copies	15	15	15	15	\$15.00
Order 16 copies	14	16	16	16	\$15.20 Maximum
Order 17 copies	13	15	17	17	\$15.00
Order 18 copies	12	14	16	18	\$14.20
Prior Probability	0.4	0.2	0.3	0.1	

(b) Freddie's most profitable alternative is to order 16 copies.

	State of Nature				Expected
Alternative	15	16	17	18	Cost
Order 15 copies	0	1	2	3	\$1.10
Order 16 copies	1	0	1	2	\$0.90 Minimum
Order 17 copies	2	1	0	1	\$1.10
Order 18 copies	3	2	1	0	\$1.90
Prior Probability	0.4	0.2	0.3	0.1	

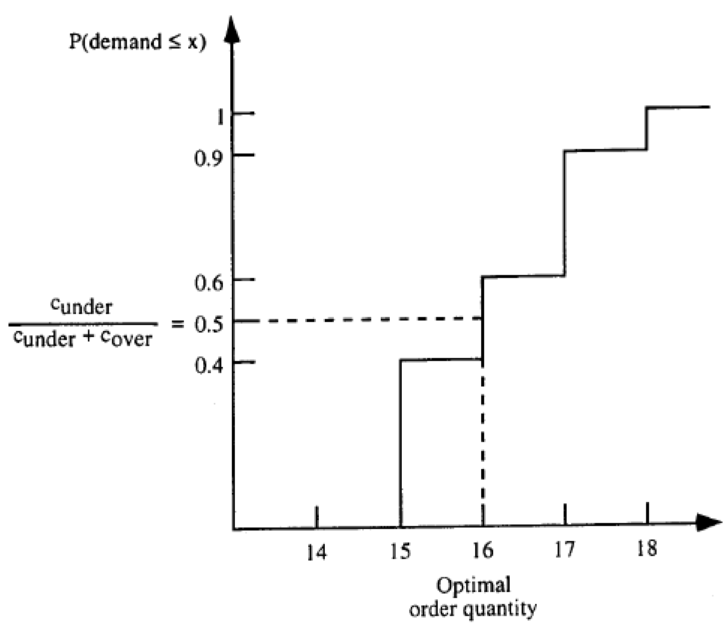
(c)

Alternative	Service Level
Order 15 copies	0.4
Order 16 copies	0.6
Order 17 copies	0.9
Order 18 copies	1

Optimal service level: $\frac{C_{\text{under}}}{C_{\text{under}}+C_{\text{over}}} = \frac{1}{1+1} = 0.5$

Freddie should order 16 copies.

(d)



18.7-4.

(a) $C_{\text{under}} = \$3 - \$1 = \$2$, $C_{\text{over}} = \$1 - \$0.50 = \$0.50$

(b) Prepare 4 doughnuts everyday to minimize the costs.

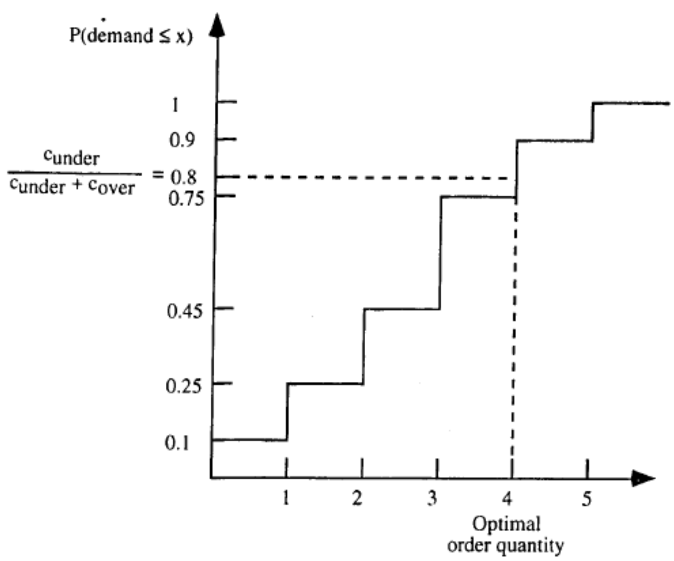
Alternative	State of Nature (Purchase Requests) 0	1	2	3	4	5	Expected Cost	
Make 0	0.0	2.0	4.0	6.0	8.0	10.0	5.10	
Make 1	0.5	0.0	2.0	4.0	6.0	8.0	3.35	
Make 2	1.0	0.5	0.0	2.0	4.0	6.0	1.98	
Make 3	1.5	1.0	0.5	0.0	2.0	4.0	1.10	
Make 4	2.0	1.5	1.0	0.5	0.0	2.0	0.98	Minimum
Make 5	2.5	2.0	1.5	1.0	0.5	0.0	1.23	
Prior Probability	0.1	0.15	0.2	0.3	0.15	0.1		

(c)

Alternative	Service Level
Make 0	0.1
Make 1	0.25
Make 2	0.45
Make 3	0.75
Make 4	0.9
Make 5	1

Optimal service level: $\frac{C_{\text{under}}}{C_{\text{under}}+C_{\text{over}}} = \frac{2}{2+0.5} = 0.8$

Prepare 4 doughnuts everyday.



(d) The probability of running short is $1 - 0.9 = 10\%$.

(e) Before 5 doughnuts are prepared, the optimal service level needs to exceed 0.9. Let g be the cost of lost customer goodwill. Then $C_{\text{under}} = 2 + g$.

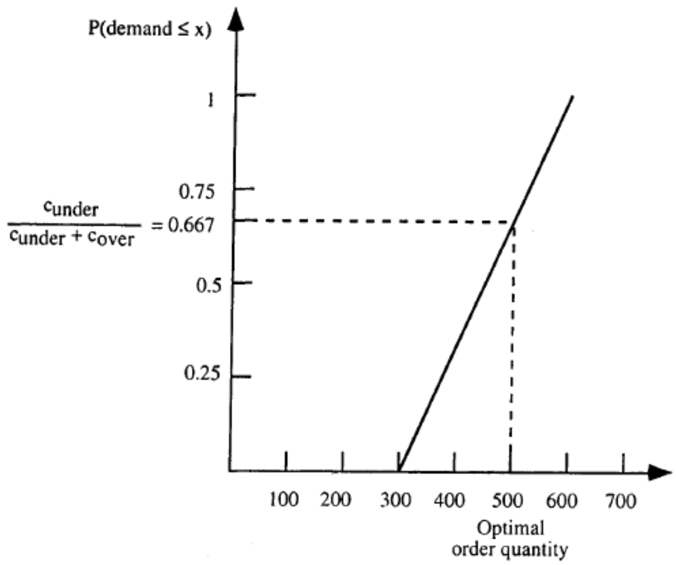
$$\frac{C_{\text{under}}}{C_{\text{under}}+C_{\text{over}}} > 0.9 \Leftrightarrow \frac{2+g}{2+g+0.5} > 0.9 \Leftrightarrow g > 2.50$$

The goodwill cost should be at least $\$2.50$ before 5 doughnuts are prepared.

18.7-5.

(a) Optimal service level: $\frac{C_{\text{under}}}{C_{\text{under}}+C_{\text{over}}} = \frac{1}{1+0.5} = 0.667$

(b)



(c) $Q^* = 300 + 0.667(600 - 300) = 500$

(d) The probability of running short is $1 - 0.667 = 33.3\%$.

(e) Optimal service level:

$$\frac{C_{\text{under}}}{C_{\text{under}}+C_{\text{over}}} = \frac{1+1.5}{1+1.5+0.5} = 0.833$$

$$Q^* = 300 + 0.833(600 - 300) = 550$$

The probability of running short is $1 - 0.833 = 16.7\%$.

18.7-6.

(a) Revenue (with shortages): $500(3) = \$1,500$

(b) Average number of loaves sold (without shortages): $300 + \frac{500-300}{2} = 400$

Average daily revenue (without shortages): $400(3.00) = \$1,200$

(c) With shortages: $1,500 \times 0.333 = \$500$

Without shortages: $1,200 \times 0.667 = \$800$

Average daily revenue over all days: $\$500 + \$800 = \$1,300$

(d) Average number of loves not sold: $\frac{200-0}{2} = 100$

Average number of day-old loaves obtained over all days: $100 \times 0.667 = 66.7$

Average daily revenue from day-old loaves: $66.7(1.50) = \$100$

(e) Average total daily revenue: $\$1,300 + \$100 = \$1,400$

Average daily profit: $\$1,400 - \$2(500) = \$400$

(f) Average daily profit with 600 loaves: $3(450) - 2(600) + 1.50(150) = \$375$

(g) Average daily profit with 550 loaves: $375 + \frac{400-375}{2} = \$387.50$

(h) Average size of shortage with 550 loaves: $\frac{600-550}{2} = 25$ loaves

Average daily shortage over all days: $25 \times 0.167 = 4.167$

Average daily cost of lost goodwill: $4.167 \times 1.50 = \$6.25$

Average daily profit with 550 loaves and lost goodwill: $\$387.50 - \$6.25 = \$381.25$

(i) Average size of shortage with 500 loaves: $\frac{100-0}{2} = 50$ loaves

Average daily shortage over all days: $50 \times 0.333 = 16.67$

Average daily cost of lost goodwill: $16.67 \times 1.50 = \$25$

Average daily profit with 500 loaves and lost goodwill: $\$400 - \$25 = \$375$

18.7-7.

(a) $Q^* = a + (\text{service level})(b-a) = a + (0.667)(75) = a + 50$

(b) Probability of incurring shortage: $1 - 0.667 = 33.3\%$ (same as in 18.7-4)

(c) Maximum shortage: $b - (a+50) = 25$

Maximum number of loaves that will not be sold: 50

The corresponding numbers for 18.7-5 are 100 and 200 respectively, which are four times the amounts in this problem.

(d) The average daily costs of underordering and overordering for the new plan are 25% of the original costs, so it is quite valuable to obtain as much information as possible about the demand before placing the final order for a perishable product.

(e) $Q^* = a + (\text{service level})(b-a) = a + (0.833)(75) = a + 62.5$

Probability of incurring shortage: $1 - 0.833 = 16.67\%$

Maximum shortage: $b - (a+62.5) = 12.5$

Maximum number of loaves that will not be sold: 62.5

18.7-8.

(a)

$$S^* = -\lambda \ln\left(\frac{c+h}{p+h}\right) = -50\ln\left(\frac{1000+300}{10000+300}\right) \approx 103$$

(b) $C(y) = c(y-I) + L(y) = cy - cI + L(y)$

Taking the derivative with respect to y, the term involving the initial inventory I vanishes, so the optimal policy is the same as in (a), i.e., to order up to 103 or equivalently to order $103 - 23 = 80$ parts.

(c) $\mathrm{P}\{D \leq S\} = \mathrm{F}(S) = 1 - e^{-\frac{S}{50}} = 0.9 \Rightarrow S = -50\ln(0.1) \approx 115$

(d)

$$\frac{p-c}{p+h} = 0.9 \Rightarrow \frac{p-1000}{p+300} = 0.9 \Rightarrow p = \$12,700$$

18.7-9.

(a) Optimal service level:

$$\frac{C_{\text{under}}}{C_{\text{under}}+C_{\text{over}}} = \frac{3000}{3000+1000} = 0.75$$

(b) $Q = \mu + K_L\sigma = 50 + (0.675)(15) = 60$

18.7-10.

$L(y) = \frac{1}{20}\left[\int_0^y (y-x)dx + 3\int_y^{20}(x-y)dx\right] = \frac{y^2}{10} - 3y + 30$

$cy + L(y) = 2y + \frac{y^2}{10} - 3y + 30 = \frac{y^2}{10} - y + 30$

Taking the derivative with respect to y: $\frac{y}{5} - 1 = 0 \Rightarrow S = 5$. We could have used the result $\text{P}\{D \leq S\} = (p-c)/(p+h)$ directly:

$$\text{P}\{D \leq S\} = S/20 = (p-c)/(p+h) = (3-2)/(3+1) = 0.25 \Rightarrow S = 5.$$

$$C(s) = K + cS + L(S) \Rightarrow cs + L(s) = K + cS + L(S)$$

$$\Rightarrow \frac{s^2}{10} - s + 30 = 1.50 + \frac{5^2}{10} - 5 + 30 \Rightarrow \frac{s^2}{10} - s + 1 = 0$$

$$\Rightarrow s = 5 - \sqrt{15} \approx 1.13$$

The $(s, S) = (1.13, 5)$ policy is optimal.

18.7-11.

Single-period model with a setup cost:

Demand density is exponential with $\lambda = 25$. Per unit production/purchasing cost is $c = 1$. Per unit inventory holding cost is $h = 0.4$ and per unit shortage cost is $p = 1.5$. The setup cost is $K = 10$. The optimal policy is an (s, S) policy with $s = -11.e3$ and $S = 7.63454$.

18.8-1.

In each case, $L = 200, p_1 = \$1000$ and D has a normal distribution with mean 60 and standard deviation 20.

$p_2 = \$300 \Rightarrow F(x^*) = 1 - \frac{p_2}{p_1} = 0.7 \Rightarrow\ x^* = 60 + K_{0.3}(20) = 60 + 0.52(20) = 70.4$

When the discount fare is \$300, 70 seats should be reserved for class 1 customers and the request to make a sale to the class 2 customer should be accepted if there are 71 or more seats remaining.

$p_2 = \$400 \Rightarrow F(x^*) = 1 - \frac{p_2}{p_1} = 0.6 \Rightarrow\ x^* = 60 + K_{0.4}(20) = 60 + 0.25(20) = 65$

$p_2 = \$500 \Rightarrow F(x^*) = 1 - \frac{p_2}{p_1} = 0.5 \Rightarrow\ x^* = 60 + K_{0.5}(20) = 60 + 0(20) = 60$

$p_2 = \$600 \Rightarrow F(x^*) = 1 - \frac{p_2}{p_1} = 0.4 \Rightarrow\ x^* = 60 + K_{0.6}(20)$

$= 60 - K_{0.4}(20) = 60 - 0.25(20) = 55$

As the discount fare increases, the optimal number x^* of reservation slots for class 1 customers decreases.

18.8-2.

The capacity L is $1000,$ the price p_1 paid by luxury-seeking customers is $\$20,000$ and the discount fare is $p_2 = \$12,000$. The demand D by luxury-seeking customers has a normal distribution with mean 400 and standard deviation 100.

$F(x^*) = 1 - \frac{p_2}{p_1} = 0.4$

$\Rightarrow \; x^* \; = 400 + K_{0.6}(100) = 400 - K_{0.4}(100) = 400 - 0.25(100) = 150$

$\Rightarrow \; L - x^* = 1000 - 150 = 850$

Hence, the maximum number of cabins that should be sold at the discount fare is 850.

18.8-3.

$L = 100, p_1 = 300, p_2 = 100$

The demand D for full-fare tickets has a uniform distribution on integers between 31 and 50.

$$p_2 \leq p_1 P(D \geq x^*) \Leftrightarrow \frac{p_2}{p_1} = \frac{1}{3} \leq \frac{50-x^*+1}{20} \Leftrightarrow x^* \leq 51 - \frac{20}{3} = 44.33$$

$$p_2 > p_1 P(D \geq x^* + 1) \Leftrightarrow \frac{p_2}{p_1} = \frac{1}{3} > \frac{50-x^*}{20} \Leftrightarrow x^* > 50 - \frac{20}{3} = 43.33$$

Thus $x^* = 44$ slots should be reserved to full-fare customers.

18.8-4.

$L = 150, p = 0.8, r = \$300, s = \$1500$

$P\{D(n^*) \geq 150\} = \frac{r}{sp} = 0.25$

$D(n)$ is normally distributed with mean $0.8n$ and standard deviation $0.4\sqrt{n}$.

$K_{0.25} = \frac{150-0.8n}{0.4\sqrt{n}} \Rightarrow 0.67 = \frac{150-0.8n}{0.4\sqrt{n}} \Rightarrow 0.8n + 0.268\sqrt{n} - 150 = 0$

$\Rightarrow \sqrt{n} = \frac{-0.268+\sqrt{(0.268)^2-4(0.8)(-150)}}{1.6} = 13.527 \Rightarrow n^* = (13.527)^2 = 183$

We chose the smallest integer that is greater than $(13.527)^2$ to determine n^*. Hence, the number of reservations to accept for this flight is 183.

18.8-5.

$L = 125, r = \$250, s = 300 + 300 = \$600$

Finding the optimal overbooking requires finding the smallest integer n with $\Delta E(P(n))$ nonpositive.

$$\Delta E(P(n)) \qquad = 250 - 600\left[\sum_{d=126}^{n}(d-125)[P\{D(n+1)=d\} - P\{D(n)=d\}]\right]$$

Let X denote the random variable associated with no-shows.

$$\begin{aligned}\Delta E(P(n)) \qquad &= 250 - 600\left[\sum_{k=0}^{n-126}(n-k-125)[P\{X=k+1\} - P\{X=k\}]\right] \\ &= 250 - 600\left[\sum_{k=0}^{n-126} P\{X=k+1\}\right] = 250 - 600P\{X \leq n-125\}\end{aligned}$$

Then the problem is to find the smallest n such that

$$P\{X \leq n - 125\} \geq \frac{250}{600} = 0.417.$$

x	0	1	2	3	4	5	6	7	8	9
$P\{X \leq x\}$	0	0.05	0.15	0.25	0.4	0.6	0.75	0.85	0.95	1

From the cumulative distribution of X, n^* is found to be $125 + 5 = 130$, so 5 reservations can be accepted in addition to the capacity.

18.8-6.

$L = 3, p = 0.5, r = \$1000, s = \$5000$

To determine the optimal number of reservations to accept, we need to find the smallest integer n such that

$$spP\{D(n) \geq 3\} \geq r \Leftrightarrow P\{D(n) \geq 3\} \geq 0.4 \Leftrightarrow P\{D(n) \leq 2\} \leq 0.6$$

$$\Leftrightarrow \left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2}\right]0.5^n \leq 0.6$$

$$\Leftrightarrow (n^2 + n + 2)0.5^{n+1} \leq 0.6$$

A first guess can be $n = 6$, since then the average number of customers with reservation and who actually show up is $L = 3$. It satisfies

$$(6^2 + 6 + 2)0.5^7 < 0.6,$$

so $spP\{D(6) \geq 3\} \geq r$. This suggests $n^* \leq 6$. Now consider $n = 5$.

$$(5^2 + 5 + 2)0.5^6 < 0.6,$$

so $spP\{D(5) \geq 3\} \geq r$. Then $n^* \leq 5$. For $n = 4$,

$$(4^2 + 4 + 2)0.5^5 > 0.6,$$

so $spP\{D(4) \geq 3\} < r$. Hence the optimal number of reservations to accept is 5.

18.8-7.

$L = 100, p = 0.9, r = \$3000, s = \$20000$

$P\{D(n^*) \geq 100\} = \frac{r}{sp} = \frac{1}{6} \approx 0.167$

$D(n^*)$ is normally distributed with mean $0.9n$ and standard deviation $0.3\sqrt{n}$.

$K_{0.167} = \frac{100 - 0.9n}{0.3\sqrt{n}} \Rightarrow 0.97 = \frac{100 - 0.9n}{0.3\sqrt{n}}$

$\Rightarrow \sqrt{n} = \frac{-0.291 + \sqrt{(0.291)^2 - 4(0.9)(-100)}}{1.8} \approx 10.38 \Rightarrow n^* = (10.38)^2 = 108$

The number of reservations to accept is 108, so 8 reservations should be overbooked.

18.9-1.

Answers will vary.

18.9-2.

Answers will vary.

CASES

CASE 18.1 Brushing Up on Inventory Control

(a) Robert's problem can be solved using the basic EOQ model, with the data:

$$D = 12(250) = 3,000,\ K = 18.75/3 = 6.25,$$

$$h = 0.12(1.25) = 0.15,\ L = 0,\ WD = 12(30) = 360$$

Data

D =	3000	(demand/year)
K =	\$6.25	(setup cost)
h =	\$0.15	(unit holding cost)
L =	0	(lead time in days)
WD =	360	(working days/year)

Results

Reorder Point =	0
Annual Setup Cost =	\$37.50
Annual Holding Cost =	\$37.50
Total Variable Cost =	\$75.00

Decision

Q =	500.00	(order quantity)

Robert should order 500 toothbrushes 6 times per year.

(b) EOQ model with $L = 6$ days

Data

D =	3000	(demand/year)
K =	\$6.25	(setup cost)
h =	\$0.15	(unit holding cost)
L =	6	(lead time in days)
WD =	360	(working days/year)

Results

Reorder Point =	50
Annual Setup Cost =	\$37.50
Annual Holding Cost =	\$37.50
Total Variable Cost =	\$75.00

Decision

Q =	500.00	(order quantity)

Whenever the inventory drops down to 50, Robert should place an order for 500 toothbrushes. He needs to place 6 orders per year.

(c) Planned shortages with $p = \$1.50$/unit

Data

D =	3000	(demand/year)
K =	\$6.25	(setup cost)
h =	\$0.15	(unit holding cost)
p =	\$1.50	(unit shortage cost)

Results

Max Inventory Level =	476.73
Annual Setup Cost =	\$35.75
Annual Holding Cost =	\$32.50
Annual Shortage Cost =	\$3.25
Total Variable Cost =	\$71.51

Decision

Q =	524.40	(order quantity)
S =	47.67	(maximum shortage)

Robert should order about 524 toothbrushes. Since the lead time is 6 days, the reorder point is $-47.67 + 6(3000/360) = 2.33$. The maximum shortage size is approximately 48.

(d) Two extreme cases: $p = \$0.85$/unit and $p = \$25$/unit

Data

D =	3000	(demand/year)
K =	$6.25	(setup cost)
h =	$0.15	(unit holding cost)
p =	$0.85	(unit shortage cost)

Results

Max Inventory Level =	460.98
Annual Setup Cost =	$34.57
Annual Holding Cost =	$29.39
Annual Shortage Cost =	$5.19
Total Variable Cost =	$69.15

Decision

Q =	542.33	(order quantity)
S =	81.35	(maximum shortage)

The reorder point when $p = \$0.85$/unit is $-81.35 + 6(3000/360) = -31.35$.

Data

D =	3000	(demand/year)
K =	$6.25	(setup cost)
h =	$0.15	(unit holding cost)
p =	$25.00	(unit shortage cost)

Results

Max Inventory Level =	498.51
Annual Setup Cost =	$37.39
Annual Holding Cost =	$37.17
Annual Shortage Cost =	$0.22
Total Variable Cost =	$74.78

Decision

Q =	501.50	(order quantity)
S =	2.99	(maximum shortage)

The reorder point when $p = \$25$/unit is $-2.99 + 6(3000/360) = 47.01$. This suggests that as the shortage cost increases, the reorder point increases.

(e) EOQ model with quantity discounts, with three prices $\$1.25$, $\$1.15$ and $\$1.00$ and holding cost rate $I = 0.12$.

Data

D =	3000	(demand/year)
K =	$6.25	(setup cost)
I =	0.12	(inventory holding cost rate)
N =	3	(number of discount categories)

Ctgry	Price	Range of order quantities Lower Limit	Upper Limit	EOQ	Q*	Annual Purchase Cost	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
1	$1.25	0	500	500.00	500.00	$3,750.00	$37.50	$37.50	$3,825.00
2	$1.15	501	999	521.29	521.29	$3,450.00	$35.97	$35.97	$3,521.94
3	$1.00	1000	10000000	559.02	1000.00	$3,000.00	$18.75	$60.00	$3,078.75

Results

Optimal Q =	1000
TVC=	$3,078.75

The optimal order quantity is $Q = 1,000$ and Robert should order 3 times a year.

CASE 18.2 TNT: Tackling Newsboy's Teaching

For the analysis of this case, we use the template for perishable products.

(a) First we need to determine the optimal service level for Howie. The unit sale price is $5, the unit purchase cost is $3, and the unit salvage value is $0.5 \times \$3 = \$1$.

Data	
Unit sale price =	5
Unit purchase cost =	3
Unit salvage value =	1

Results	
Cost of overordering =	2
Cost of underordering =	2
Optimal Service Level =	0.5

Since Talia assumes that the demand is uniformly distributed between 120 and 420 sets, Howie should order $120 + 0.5 \times 300 = 270$ sets.

(b) If Leisure Limited refunds 75% of the purchase cost, then the unit salvage value for a returned set becomes $0.75 \times \$3 - \$0.5 = \$1.75$. We determine the new optimal service level.

Data	
Unit sale price =	5
Unit purchase cost =	3
Unit salvage value =	1.75

Results	
Cost of overordering =	1.25
Cost of underordering =	2
Optimal Service Level =	0.615385

The order quantity is now $120 + 0.615385 \times 300 = 304.62$. Note that Howie can now order more sets at one time than he could under the scenario of part (a) because he is not punished as severely as before when he fails to sell all sets.

When the refund is 25%, the unit salvage cost is $0.25.

Data	
Unit sale price =	5
Unit purchase cost =	3
Unit salvage value =	0.25

Results	
Cost of overordering =	2.75
Cost of underordering =	2
Optimal Service Level =	0.421053

Consequently, the order quantity is reduced to $120 + 0.421053 \times 300 = 246.32$. In this case, Howie should purchase fewer sets at one time (compared to previous scenarios), since he is punished more severely for failing to sell all the sets.

(c) The unit sale price is now $6 and there is a 50% refund on returned firecracker sets.

Data	
Unit sale price =	6
Unit purchase cost =	3
Unit salvage value =	1

Results	
Cost of overordering =	2
Cost of underordering =	3
Optimal Service Level =	0.6

However, if Howie raises the price of a firecracker set, one would expect a decrease in the demand for his sets, so Talia should not use the same uniform demand distribution that she used for her previous calculations of the optimal order quantity.

(d) Talia's strategy for estimating the demand is overly simplistic. She makes the very simplifying assumption that the demand is uniformly distributed between 120 and 420 sets. However, she does not take into account that the demand depends on the price of a firecracker set. She should expect that stands charging less than the average price of $5 per set typically sell more sets than stands charging more. Talia should call Buddy again to try to obtain more detailed information such as the range of sales and the average sale of stands charging $5 or $6 per set.

Talia should also reevaluate her assumption that the demand is uniformly distributed. She should check how her forecasts change if she uses other demand distribution like normal distribution.

CASE 18.3 Jettisoning Surplus Stock

(a) We can use Excel to compute the sample mean and variance.

Observations													Mean	Std. Dev.
25	31	18	22	40	19	38	21	25	36	34	28	27	28	7.29154

Hence, the sample mean is 28 and the sample variance is $7.29154^2 \approx 53.1667$.

(b) Based on the findings of Scarlett Windermere, American Aerospace can use an (R, Q) policy for the inventory of part 10003487. The assumptions of the model are satisfied.

1- The part is a stable product.

2- Its inventory level is under continuous review.

3- While the production of the part itself has no lead time, it is typically delayed by the lead time of 1.5 months of the little steel part. Assume the lead time is 1.5 months.

4- The demand for the part is the same as for the jet engine MX332, since it is used only for this particular engine. Hence, assume that the demand is approximately normally distributed with mean 28 and variance 53.1667.

5- Excess demand is backlogged.

6- There is a fixed setup cost $K = \$5,800$, a holding cost $h = \$750$ and a shortage cost $p = \$3,250$.

Note that the average demand per year is $12 \times 28 = 336$, the average demand during the lead time is $1.5 \times 28 = 42$ and it has a standard deviation of $1.5 \times 7.29154 = 10.93732$.

Data

D=	336	(average demand/unit time)
K=	5800	(setup cost)
h=	750	(unit holding cost)
p=	3250	(unit shortage cost)
L=	0.85	(service level)

Results

Q=	79.9753808
R=	53.3357995

Demand During Lead Time

Distribution=	N	(U=uniform; N=Normal)
mean=	42	
stand. dev. =	10.9373214	

American Aerospace should implement the (R, Q) policy with $R = 53.34$ and $Q = 79.98$. These can be rounded to $R = 53$ and $Q = 80$, since the order for the part should be integer-valued.

(c) The average inventory just before an order arrives is $53 - 42 = 11$ and the one just after an order has arrived is $11 + 80 = 91$. Then, the average inventory is $(11 + 91)/2 = 51$, with an average holding cost of $51(750) = \$38,250$ per year. The average number of setups in a year is $336/80 = 4.2$, with a resulting average setup cost of $4.2(5,800) = \$24,360$ per year.

(d) The new service level is $L = 0.95$.

Data

D =	336	(average demand/unit time)
K =	5800	(setup cost)
h =	750	(unit holding cost)
p =	3250	(unit shortage cost)
L =	0.95	(service level)

Results

Q =	79.9753808
R =	59.990286

Demand During Lead Time

Distribution =	N	(U=uniform; N=Normal)
mean =	42	
stand. dev. =	10.9373214	

Round these values up to get $R = 60$ and $Q = 80$. The average inventory just before an order arrives is $80 - 60 = 20$ and just after an order has arrived is $20 + 80 = 100$, so the average inventory is 60 and the resulting average inventory holding cost is $60(750) = \$45,000$ per year. Note that the average holding cost has increased substantially. This is a consequence of increasing the safety stock to 20 from 11. The average number of setups per year is still 4.2 and the average setup cost is $\$24,360$ per year.

(e) Scarlett's independent analysis of the stationary part 10003487 can be justified since there is only one jet engine that needs this part and this part appears to be the bottleneck in the production process. However, in general, a stationary part is used for several jet engines, so the demand for stationary parts depends on the demand for several jet engines and a stock-out in one stationary part affects the demand for other parts. These interdependencies cannot be captured by an independent analysis of each part; therefore, Scarlett's approach is most likely to result in rather inaccurate inventory policies for many other stationary parts.

(f) Scarlett could try to forecast the demand for jet engines based on sales data from previous years.

SUPPLEMENT 1 TO CHAPTER 18
DERIVATION OF THE OPTIMAL POLICY FOR THE STOCHASTIC SINGLE-PERIOD MODEL FOR PERISHABLE PRODUCTS

18S1-1.

$C(\underline{S}) = c\underline{S} + h\int_0^{\underline{S}}(\underline{S} - x)\underline{f}(x)dx + p\int_{\underline{S}}^{\infty}(x - \underline{S})\underline{f}(x)dx + kP\{D \geq \underline{S}\}$

D is uniformly distributed on $[a, b]$, so $P\{D \geq \underline{S}\} = \frac{b-\underline{S}}{b-a}$.

$C(\underline{S}) = c\underline{S} + k\frac{b-\underline{S}}{b-a} + L(\underline{S}) \Rightarrow \frac{dC(\underline{S})}{d\underline{S}} = c - \frac{k}{b-a} + h\underline{F}(\underline{S}) - p\{1 - \underline{F}(\underline{S})] = 0$

$\Rightarrow \underline{F}(\underline{S}) = \frac{p+\frac{k}{b-a}-c}{p+h}$

Let $p = c + 2, k = 14, h = -(c - 1), a = 40, b = 60$.

$\Rightarrow \underline{F}(\underline{S}) = \frac{\underline{S}-40}{20} = \frac{2.7}{3} = 0.9 \Rightarrow \underline{S} = 58$

18S1-2.

(a)

$$C(\underline{I}, \underline{S}) = c(\underline{S} - \underline{I}) + pP\{D > \underline{S}\} = c(\underline{S} - \underline{I}) + pe^{-\underline{S}}$$
$$\Rightarrow \frac{\partial C(\underline{I},\underline{S})}{\partial \underline{S}} = c - pe^{-\underline{S}} = 0 \Rightarrow \underline{S} = -\text{ln}\,(c/p)$$

Order up to $\underline{S}$ if $\underline{I} < \underline{S}$, do not order otherwise.

(b)

$$C(\underline{I}, \underline{S}) = \begin{cases} K + c(\underline{S} - \underline{I}) + pe^{-\underline{S}} & \text{if } \underline{I} < \underline{S} \\ pe^{-\underline{I}} & \text{if } \underline{I} = \underline{S} \end{cases}$$

An (s, S) policy is optimal with $S = -\text{ln}\,(c/p)$ and s being the smallest value such that

$$cs + pe^{-s} = K - c\text{ln}\,(c/p) + c.$$

SUPPLEMENT 2 TO CHAPTER 18
STOCHASTIC PERIODIC-REVIEW MODELS

18S2-1.

(a) Single-period model with no setup cost:

Demand density is exponential with $\lambda = 25$. Per unit production/purchasing cost is $c = 10$. Per unit inventory holding cost is $h = 6$ and per unit shortage cost is $p = 15$. The optimal one-period inventory level is $S(0) = 6.79834$.

(b) Two-period model with no setup cost:

Demand density is exponential with $\lambda = 25$. Per unit production/purchasing cost is $c = 10$. Per unit inventory holding cost is $h = 6$ and per unit shortage cost is $p = 15$. The optimal two-period policy consists of the inventory levels $S_1(0) = 23.2932$ and $S_2(0) = 6.79834$.

18S2-2.

(a) Single-period model with no setup cost:

Demand density is uniform on $[0, 50]$. Per unit production/purchasing cost is $c = 10$. Per unit inventory holding cost is $h = 8$ and per unit shortage cost is $p = 15$. The optimal one-period inventory level is $S^* = 10.8696$. It is optimal to order up to S^* if the initial inventory is below S^* and not to order otherwise.

(b) Two-period model with no setup cost:

Demand density is uniform on $[0, 50]$. Per unit production/purchasing cost is $c = 10$. Per unit inventory holding cost is $h = 8$ and per unit shortage cost is $p = 15$. The optimal two-period policy consists of the inventory levels $S_1^* = 9.26156$ and $S_2^* = 10.8696$. It is optimal to order up to S_i^* if the initial inventory is below S_i^* in period i and not to order otherwise.

18S2-3.

Two-period model with no setup cost:

Demand density is exponential with $\lambda = 25$. Per unit production/purchasing cost is $c = 1$. Per unit inventory holding cost is $h = 0.25$ and per unit shortage cost is $p = 2$. The discount factor is 0.9. The optimal two-period policy is the same as the one for the infinite-period model, so consists of the inventory level $S(0) = 46.5188$.

18S2-4.

Two-period model with no setup cost:

Demand density is exponential with $\lambda = 25$. Per unit production/purchasing cost is $c = 1$. Per unit inventory holding cost is $h = 0.25$ and per unit shortage cost is $p = 2$. The optimal two-period policy consists of the inventory levels $S_1(0) = 36.521$ and $S_2(0) = 14.6947$.

18S2-5.

Infinite-period model with no setup cost:

Demand density is exponential with $\lambda = 25$. Per unit production/purchasing cost is $c = 1$. Per unit inventory holding cost is $h = 0.25$ and per unit shortage cost is $p = 2$. The discount factor is 0.9. The optimal policy consists of the inventory level $S(0) = 46.5188$.

18S2-6.

Infinite-period model with no setup cost:

Demand density is exponential with $\lambda = 1$. Per unit production/purchasing cost is $c = 2$. Per unit inventory holding cost is $h = 1$ and per unit shortage cost is $p = 5$. The discount factor is 0.95. The optimal policy consists of the inventory level $S(0) = 1.69645$.

18S2-7.

12-period model with no setup cost:

The answer is the same as in 18S2-6, so the optimal policy consists of the inventory level $S(0) = 1.69645$.

18S2-8.

Infinite-period model with no setup cost:

Demand density is uniform on $[2000, 3000]$. Per unit production/purchasing cost is $c = 150$. Per unit inventory holding cost is $h = 2$ and per unit shortage cost is $p = 30$. The discount factor is 0.9. The optimal policy consists of the inventory level $S(0) = 2,468.75$.

18S2-9.

Infinite-period model with no setup cost:

Demand density is exponential with $\lambda = 1000$. Per unit production/purchasing cost is $c = 80$. Per unit inventory holding cost is $h = 0.70$ and per unit shortage cost is $p = 2$. The discount factor is 0.998. The optimal policy consists of the inventory level $S(0) = 497$.

18S2-10.

$h = 0.3, p = 2.5$

$G(\underline{S}) = 0.3\int_0^{\underline{S}} \frac{(\underline{S}-x)}{25} e^{-\underline{x}/25} d\underline{x} + 2.5\int_{\underline{S}}^{\infty} \frac{(x-\underline{S})}{25} e^{-\underline{x}/25} d\underline{x} = 0.3\underline{S} + 70e^{-\underline{S}/25} - 7.5$
$G'(\underline{S}) = 0.3 - 2.8e^{-\underline{S}/25} = 0 \Rightarrow \underline{S} = 55.84$
$G''(\underline{S}) = \frac{2.8}{25} e^{-\underline{S}/25} > 0 \Rightarrow \underline{S} = 55.84$ minimizes $G(\underline{S})$.

$G(k) = G(k + 100) \Leftrightarrow 0.3k + 70e^{-k/25} = 0.3(k + 100) + 70e^{-(k+100)/25}$

$\Leftrightarrow 70e^{-k/25}(1 - e^{-4}) = 30 \Leftrightarrow k = 20.72 \approx 21$

$k = 21 < \underline{S} = 55.84 < 121 = k + 100$ and $G(21) \approx G(121)$

Hence, the optimal policy is a $(k, Q) = (21, 100)$ policy.

18S2-11.

Since $c = 0$, the answer is identical to that for 18.S2-10, viz., $(k, Q) = (21, 100)$ is optimal.

18S2-12.

$L(\underline{S}) = \int_0^{\underline{S}} h(\underline{S} - \underline{x})\underline{f}(\underline{x})d\underline{x} + \int_{\underline{S}}^{\infty} p(\underline{x} - \underline{S})\underline{f}(\underline{x})d\underline{x}$

$\frac{dL(\underline{S})}{d\underline{S}} = \int_0^{\underline{S}} h\underline{f}(\underline{x})d\underline{x} + \int_{\underline{S}}^{\infty} -p\underline{f}(\underline{x})d\underline{x} = h\underline{F}(\underline{S}) - p[1 - \underline{F}(\underline{S})]$

$\frac{dL(\underline{S})}{d\underline{S}} + c(1-\alpha) = 0 \Rightarrow -p + p\underline{F}(\underline{S}) + h\underline{F}(\underline{S}) + c(1-\alpha) = 0$

$\Rightarrow \underline{F}(\underline{S}) = \frac{p - c(1-\alpha)}{p+h}$

CHAPTER 19: MARKOV DECISION PROCESSES

19.2-1.

Bank One, one of the major credit card issuers in the United States has developed the portfolio control and optimization (PORTICO) system to manage APR and credit-line changes of its card holders. Customers prefer low APR and high credit lines, which can reduce the bank's profitability and increase the risk. Consequently, the bank faces the need to find a balance between revenue growth and risk. PORTICO formulates the problem as a Markov decision process. The state variables are chosen in a way to satisfy Markovian assumption as closely as possible while keeping the dimension of the state space at a tractable level. The resulting variables are (x, y), where x corresponds to the credit line and APR level and y represents the behavior variables. The transition probabilities are estimated from the available data. The objective is to maximize the expected net present value of the cash flows over a 36-month horizon. The dynamic programming equation for the decision periods of the problem is

$$V_t(x,y) = \max_{a \in A(x,y)} \left\{ r(x \pm a, y) + \beta \sum_{j \in \mathcal{S}} p(x \pm a, y; j) V_{t+1}(x \pm a, j) \right\},$$

where $r(\,\cdot\,)$ denotes the immediate net cash flow and β is the discount factor. The solution obtained is then adjusted to conform to business rules.

Benchmark tests are performed to evaluate the output policy. These tests suggest that the new policy improves profitability. By adopting this policy, Bank One is expected to increase its annual profit by more than $75 million.

19.2-2.

(a) Let the states $i = 0, 1, 2$ be the number of customers at the facility. There are two possible actions when the facility has one or two customers. Let decision 1 be to use the slow configuration and decision 2 be to use the fast configuration. Also let C_{ij} denote the expected net immediate cost of using decision j in state i. Then,

$$C_{11} = C_{21} = 3 - \tfrac{3}{5} \times 50 = -27$$
$$C_{12} = C_{22} = 9 - \tfrac{4}{5} \times 50 = -31$$
$$C_{01} = 3$$
$$C_{02} = 9$$

(b) In state 0, the configuration chosen does not affect the transition probabilities, so it is best to choose the slow configuration when there are no customers in line. Consequently, the number of stationary policies is four.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
1	1	1	2	2
2	1	2	1	2

Policy	Transition Matrix	Expected Average Cost
R_1	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \\ 0 & \frac{3}{5} & \frac{2}{5} \end{pmatrix}$	$C_1 = 3\pi_0 - 27\pi_1 - 27\pi_2$
R_2	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \\ 0 & \frac{4}{5} & \frac{1}{5} \end{pmatrix}$	$C_2 = 3\pi_0 - 27\pi_1 - 31\pi_2$
R_3	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{10} \\ 0 & \frac{3}{5} & \frac{2}{5} \end{pmatrix}$	$C_3 = 3\pi_0 - 31\pi_1 - 27\pi_2$
R_4	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{10} \\ 0 & \frac{4}{5} & \frac{1}{5} \end{pmatrix}$	$C_4 = 3\pi_0 - 31\pi_1 - 31\pi_2$

(c)

Policy	π_0	π_1	π_2	Average Cost
R_1	0.3103	0.5172	0.1724	$C_1 = -17.69$
R_2	0.3243	0.5405	0.1351	$C_2 = -17.81$
R_3	0.4068	0.5085	0.0847	$C_3 = -16.83$
R_4	0.416	0.519	0.065	$C_4 = -16.87$

C_2 is the minimum, so the optimal policy is R_2, i.e., to use slow configuration when no customer or only one customer is present and fast configuration when there are two customers.

19.2-3.

(a) Let the states represent whether the student's car is dented, $i = 1$, or not, $i = 0$.

Decision	Action	State	Immediate Cost
1	Park on street in one space	0	$C_{01} = 0$
2	Park on street in two spaces	0	$C_{02} = 4.5$
3	Park in lot	0	$C_{03} = 5$
4	Have it repaired	1	$C_{14} = 50$
5	Drive dented	1	$C_{15} = 9$

(b) Assuming the student's car has no dent initially, once she decides to park in lot, state 1 will never be entered. In that case, the decision chosen in state 1 does not affect the expected average cost. Hence, it is enough to consider five stationary deterministic policies.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$	$d_i(R_5)$
0	1	1	2	2	3
1	4	5	4	5	$-$

Policy	Transition Matrix	Expected Average Cost
R_1	$\begin{pmatrix} 0.9 & 0.1 \\ 1 & 0 \end{pmatrix}$	$C_1 = 0\pi_0 + 50\pi_1$
R_2	$\begin{pmatrix} 0.9 & 0.1 \\ 0 & 1 \end{pmatrix}$	$C_2 = 0\pi_0 + 9\pi_1$
R_3	$\begin{pmatrix} 0.98 & 0.02 \\ 1 & 0 \end{pmatrix}$	$C_3 = 4.5\pi_0 + 50\pi_1$
R_4	$\begin{pmatrix} 0.98 & 0.02 \\ 0 & 1 \end{pmatrix}$	$C_4 = 4.5\pi_0 + 9\pi_1$
R_5	$\begin{pmatrix} 1 & 0 \\ - & - \end{pmatrix}$	$C_5 = 5\pi_0$

(c)

Policy	π_0	π_1	Average Cost
R_1	0.909	0.091	4.55
R_2	0	1	9
R_3	0.98	0.02	5.41
R_4	0	1	9
R_5	1	0	5 (if initially not dented)

The policy R_1 has the minimum cost, so it is optimal to park on the street in one space if not dented and to have it repaired if dented.

19.2-4.

(a) Let states 0 and 1 denote the good and the bad mood respectively. The decision in each state is between providing refreshments or not.

Decision	Action	State	Immediate Cost
1	Provide refreshments	0	$C_{01} = 14$
2	Not provide refreshments	0	$C_{02} = 0$
1	Provide refreshments	1	$C_{11} = 14$
2	Not provide refreshments	1	$C_{12} = 75$

(b) There are four possible stationary policies.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

Policy	Transition Matrix	Expected Average Cost
R_1	$\begin{pmatrix} 0.875 & 0.125 \\ 0.875 & 0.125 \end{pmatrix}$	$C_1 = 14\pi_0 + 14\pi_1$
R_2	$\begin{pmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{pmatrix}$	$C_2 = 14\pi_0 + 75\pi_1$
R_3	$\begin{pmatrix} 0.125 & 0.875 \\ 0.875 & 0.125 \end{pmatrix}$	$C_3 = 14\pi_1$
R_4	$\begin{pmatrix} 0.125 & 0.875 \\ 0.125 & 0.875 \end{pmatrix}$	$C_4 = 75\pi_1$

(c)

Policy	π_0	π_1	Average Cost
R_1	0.875	0.125	$C_1 = 14$
R_2	0.5	0.5	$C_2 = 44.5$
R_3	0.5	0.5	$C_3 = 7$
$R4$	0.125	0.875	$C_4 = 65.625$

The optimal policy is R_3, i.e., to provide refreshments only if the group begins the night in a bad mood.

19.2-5.

(a) Let state 0 denote point over, two serves to go on next point and state 1 denote one serve left. The decision in each state is to attempt an ace or a lob.

Decision	Action	State	Immediate Cost
1	Attempt ace	0	$C_{01} = \frac{3}{8}\left(\frac{2}{3}(-1) + \frac{1}{3}(1)\right) = -\frac{1}{8}$
2	Attempt lob	0	$C_{02} = \frac{7}{8}\left(\frac{1}{3}(-1) + \frac{2}{3}(1)\right) = \frac{7}{24}$
1	Attempt ace	1	$C_{11} = \frac{3}{8}\left(\frac{2}{3}(-1) + \frac{1}{3}(1)\right) + \frac{5}{8}(1) = \frac{1}{2}$
2	Attempt lob	1	$C_{12} = \frac{7}{8}\left(\frac{1}{3}(-1) + \frac{2}{3}(1)\right) + \frac{1}{8}(1) = \frac{5}{12}$

(b) There are four possible stationary deterministic policies.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

Policy	Transition Matrix	Expected Average Cost
R_1	$\begin{pmatrix} 3/8 & 5/8 \\ 1 & 0 \end{pmatrix}$	$C_1 = (-1/8)\pi_0 + (1/2)\pi_1$
R_2	$\begin{pmatrix} 3/8 & 5/8 \\ 1 & 0 \end{pmatrix}$	$C_2 = (-1/8)\pi_0 + (5/12)\pi_1$
R_3	$\begin{pmatrix} 7/8 & 1/8 \\ 1 & 0 \end{pmatrix}$	$C_3 = (7/24)\pi_0 + (1/2)\pi_1$
R_4	$\begin{pmatrix} 7/8 & 1/8 \\ 1 & 0 \end{pmatrix}$	$C_4 = (7/24)\pi_0 + (5/12)\pi_1$

(c)

Policy	π_0	π_1	Average Cost
R_1	0.615	0.385	$C_1 = 0.270$
R_2	0.615	0.385	$C_2 = 0.237$
R_3	0.889	0.111	$C_3 = 0.315$
$R4$	0.889	0.111	$C_4 = 0.306$

The optimal policy is R_3, i.e., to attempt lob in state 0 and ace in state 1.

19.2-6.

(a) Let states $i = 0, 1, 2$ represent the state of the market, $13,000$, $14,000$ and $15,000$ respectively. The decision is between two funds, namely the Go-Go Fund and the Go-Slow Mutual Fund. All the costs are expressed in thousand dollars.

Decision	Action	State	Immediate Cost
1	Invest in the Go-Go	0	$C_{01} = 0.4(-25) + 0.2(-60) = -22$
2	Invest in the Go-Slow	0	$C_{02} = 0.4(-10) + 0.2(-25) = -9$
1	Invest in the Go-Go	1	$C_{11} = 0.3(25) + 0.3(-60) = -10.5$
2	Invest in the Go-Slow	1	$C_{12} = 0.3(10) + 0.3(-25) = -4.5$
1	Invest in the Go-Go	2	$C_{21} = 0.1(60) + 0.4(25) = 16$
2	Invest in the Go-Slow	2	$C_{22} = 0.1(25) + 0.4(10) = 6.5$

(b) There are eight possible stationary policies.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$	$d_i(R_5)$	$d_i(R_6)$	$d_i(R_7)$	$d_i(R_8)$
0	1	1	1	1	2	2	2	2
1	1	1	2	2	2	1	1	2
2	1	2	2	1	1	2	1	2

All R_i's have the same transition matrix: $\begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}$.

Policy	Expected Average Cost
R_1	$C_1 = -22\pi_0 - 10.5\pi_1 + 16\pi_2$
R_2	$C_2 = -22\pi_0 - 10.5\pi_1 + 6.5\pi_2$
R_3	$C_3 = -22\pi_0 - 4.5\pi_1 + 6.5\pi_2$
R_4	$C_4 = -22\pi_0 - 4.5\pi_1 + 16\pi_2$
R_5	$C_5 = -9\pi_0 - 4.5\pi_1 + 16\pi_2$
R_6	$C_6 = -9\pi_0 - 10.5\pi_1 + 6.5\pi_2$
R_7	$C_7 = -9\pi_0 - 10.5\pi_1 + 16\pi_2$
R_8	$C_8 = -9\pi_0 - 4.5\pi_1 + 6.5\pi_2$

(c) $\pi = (0.257, 0.4, 0.343)$

Policy	Average Cost
R_1	-4.371
R_2	-7.629
R_3	-5.229
R_4	-1.971
R_5	1.371
R_6	-4.286
R_7	-1.029
R_8	-1.886

The optimal policy is R_5, i.e. to invest in the Go-Go Fund in states 0 and 1, in the Go-Slow Fund in state 2.

19.2-7.

(a) Let states 0 and 1 represent whether the machine is broken down or is running respectively. The decision is between Buck and Bill.

Decision	Action	State	Immediate Cost
1	Buck	0	$C_{01} = 0$
2	Bill	0	$C_{02} = 0$
1	Buck	1	$C_{11} = -1200$
2	Bill	1	$C_{12} = -1200$

(b) There are four possible stationary deterministic policies.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

Policy	Transition Matrix	Expected Average Cost
R_1	$\begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}$	$C_1 = -1200\pi_1$
R_2	$\begin{pmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{pmatrix}$	$C_2 = -1200\pi_1$
R_3	$\begin{pmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{pmatrix}$	$C_3 = -1200\pi_1$
R_4	$\begin{pmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{pmatrix}$	$C_4 = -1200\pi_1$

(c)

Policy	π_0	π_1	Average Cost
R_1	0.5	0.5	$C_1 = -600$
R_2	0.4	0.6	$C_2 = -720$
R_3	0.545	0.455	$C_3 = -546$
$R4$	0.444	0.556	$C_4 = -667.2$

The largest expected average profit is given by R_2.

19.2-8.

(a) Let the states be the number of items in inventory at the beginning of the period and the decision be the number of items ordered. To conform to the software package, one needs to relabel the decisions as $1, 2, 3$ respectively. The cost matrix is:

c_{ik}	1	2	3
0	$40/3$	$56/3$	24
1	4	19	$-$
2	4	$-$	$-$

Let R_3 denote the policy to order 2 items when the inventory level is initially 0 and not to order when the inventory level is initially either 0 or 1. In other words, $d_0(R_3) = 3$ and $d_1(R_3) = d_2(R_3) = 1$.

$$P(R_3) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \Rightarrow \pi = (4/9, 3/9, 2/9)$$

Expected average cost: $(4/9)C_{03} + (3/9)C_{11} + (2/9)C_{21} = 116/9 \approx$ \$12.89/period

(b) There are $3^3 = 27$ stationary policies, since one can order $0, 1$ or 2 items in each state. However, only six of these are feasible. The remaining 21 policies are infeasible and the decision at least in one of the states leads to over capacity.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$	$d_i(R_5)$	$d_i(R_6)$
0	1	2	3	1	2	3
1	1	1	1	2	2	2
2	1	1	1	1	1	1

19.3-1.

(a) minimize $3y_{01} + 9y_{02} + 3y_{11} + 9y_{12} + 28y_{21} + 34y_{22}$

subject to $y_{01} + y_{02} + y_{11} + y_{12} + y_{21} + y_{22} = 1$

$$y_{01} + y_{02} - \left(\tfrac{1}{2}y_{01} + \tfrac{1}{2}y_{02} + \tfrac{3}{10}y_{11} + \tfrac{2}{5}y_{12}\right) = 0$$

$$y_{11} + y_{12} - \left(\tfrac{1}{2}y_{01} + \tfrac{1}{2}y_{02} + \tfrac{1}{2}y_{11} + \tfrac{1}{2}y_{12} + \tfrac{3}{5}y_{21} + \tfrac{4}{5}y_{22}\right) = 0$$

$$y_{21} + y_{22} - \left(\tfrac{2}{10}y_{11} + \tfrac{1}{10}y_{12} + \tfrac{2}{5}y_{21} + \tfrac{1}{5}y_{22}\right) = 0$$

$y_{ik} \geq 0$ for $i = 0, 1, 2$ and $k = 1, 2$

(b) Using the simplex method, we find $y_{01} = 0.32432, y_{11} = 0.54054, y_{22} = 0.13514$ and the remaining y_{ik}'s are zero. Hence, the optimal policy uses decision 1 in states 0 and 1, decision 2 in state 2.

19.3-2.

(a) minimize $4.5y_{02} + 5y_{03} + 50y_{14} + 9y_{15}$

subject to $y_{01} + y_{02} + y_{03} + y_{14} + y_{15} = 1$

$$y_{01} + y_{02} + y_{03} - \left(\tfrac{9}{10}y_{01} + \tfrac{49}{50}y_{02} + y_{03} + y_{14}\right) = 0$$

$$y_{14} + y_{15} - \left(\tfrac{1}{10}y_{01} + \tfrac{1}{50}y_{02} + y_{15}\right) = 0$$

$y_{01}, y_{02}, y_{03}, y_{14}, y_{15} \geq 0$

(b) Using the simplex method, all y_{ik}'s turn out to be zero except that $y_{01} = 0.90909$ and $y_{14} = 0.09091$, so the policy that uses decision 1 in state 0 and decision 4 in state 1 is optimal.

19.3-3.

(a) minimize $14y_{01} + 14y_{11} + 75y_{12}$

subject to

$$y_{01} + y_{02} + y_{11} + y_{12} = 1$$
$$y_{01} + y_{02} - \left(\tfrac{7}{8}y_{01} + \tfrac{1}{8}y_{02} + \tfrac{7}{8}y_{11} + \tfrac{1}{8}y_{12}\right) = 0$$
$$y_{11} + y_{12} - \left(\tfrac{1}{8}y_{01} + \tfrac{7}{8}y_{02} + \tfrac{1}{8}y_{11} + \tfrac{7}{8}y_{12}\right) = 0$$
$$y_{ik} \geq 0 \text{ for } i = 0, 1 \text{ and } k = 1, 2$$

(b) Using the simplex method, we find $y_{02} = y_{11} = 0.5, y_{01} = y_{12} = 0$, so the optimal policy is to use decision 2 in state 0 and decision 1 in state 1.

19.3-4.

(a) minimize $-\frac{1}{8}y_{01} + \frac{7}{24}y_{02} + \frac{1}{2}y_{11} + \frac{5}{12}y_{12}$

subject to

$$y_{01} + y_{02} + y_{11} + y_{12} = 1$$
$$y_{01} + y_{02} - \left(\tfrac{3}{8}y_{01} + \tfrac{7}{8}y_{02} + y_{11} + y_{12}\right) = 0$$
$$y_{11} + y_{12} - \left(\tfrac{5}{8}y_{01} + \tfrac{1}{8}y_{02}\right) = 0$$
$$y_{ik} \geq 0 \text{ for } i = 0, 1 \text{ and } k = 1, 2$$

(b) Using the simplex method, we find $y_{02} = 0.8889, y_{11} = 0.1111, y_{01} = y_{12} = 0$, so the optimal policy is to use decision 2 (lob) in state 0 and decision 1 (ace) in state 1.

19.3-5.

(a) minimize $-22y_{01} - 9y_{02} - 10.5y_{11} - 4.5y_{12} + 16y_{21} + 6.5y_{22}$

subject to

$$y_{01} + y_{02} + y_{11} + y_{12} + y_{21} + y_{22} = 1$$
$$y_{01} + y_{02} - \left(\tfrac{4}{10}y_{01} + \tfrac{4}{10}y_{02} + \tfrac{3}{10}y_{11} + \tfrac{3}{10}y_{12} + \tfrac{1}{10}y_{21} + \tfrac{1}{10}y_{22}\right) = 0$$
$$y_{11} + y_{12} - \left(\tfrac{4}{10}y_{01} + \tfrac{4}{10}y_{02} + \tfrac{4}{10}y_{11} + \tfrac{4}{10}y_{12} + \tfrac{4}{10}y_{21} + \tfrac{4}{10}y_{22}\right) = 0$$
$$y_{21} + y_{22} - \left(\tfrac{2}{10}y_{01} + \tfrac{2}{10}y_{02} + \tfrac{3}{10}y_{11} + \tfrac{3}{10}y_{12} + \tfrac{5}{10}y_{21} + \tfrac{5}{10}y_{22}\right) = 0$$
$$y_{ik} \geq 0 \text{ for } i = 0, 1, 2 \text{ and } k = 1, 2$$

(b) Using the simplex method, we find $y_{01} = 0.257, y_{11} = 0.4, y_{22} = 0.343$ and the remaining y_{ik}'s are zero. Hence, the optimal policy uses decision 1 (the Go-Go Fund) in states 0 and 1, decision 2 in state 2 (the Go-Slow Fund).

19.3-6.

(a) minimize $-1200y_{11} - 1200y_{12}$

subject to

$$y_{01} + y_{02} + y_{11} + y_{12} = 1$$

$$y_{01} + y_{02} - \left(0.4y_{01} + 0.5y_{02} + 0.6y_{11} + 0.4y_{12}\right) = 0$$

$$y_{11} + y_{12} - \left(0.6y_{01} + 0.5y_{02} + 0.4y_{11} + 0.6y_{12}\right) = 0$$

$y_{ik} \geq 0$ for $i = 0, 1$ and $k = 1, 2$

(b) Using the simplex method, we find $y_{01} = 0.4, y_{12} = 0.6, y_{02} = y_{11} = 0$, so the optimal policy is to use decision 1 (Buck) in state 0 and decision 2 (Bill) in state 1.

19.3-7.

(a) minimize $\frac{40}{3}y_{01} + \frac{56}{3}y_{02} + 24y_{03} + 4y_{11} + 19y_{12} + 4y_{21}$

subject to

$$y_{01} + y_{02} + y_{03} + y_{11} + y_{12} + y_{21} = 1$$

$$y_{01} + y_{02} - \left(y_{01} + \frac{2}{3}y_{02} + \frac{1}{3}y_{03} + \frac{2}{3}y_{11} + \frac{1}{3}y_{12} + \frac{1}{3}y_{21}\right) = 0$$

$$y_{11} + y_{12} - \left(\frac{1}{3}y_{02} + \frac{1}{3}y_{03} + \frac{1}{3}y_{11} + \frac{1}{3}y_{12} + \frac{1}{3}y_{21}\right) = 0$$

$$y_{21} - \left(\frac{1}{3}y_{03} + \frac{1}{3}y_{11} + \frac{1}{3}y_{12} + \frac{1}{3}y_{21}\right) = 0$$

$y_{ik} \geq 0$ for $i = 0, 1, 2$ and $k = 1, 2, 3$

(b) Using the simplex method, we find $y_{03} = 0.4444, y_{11} = 0.3333, y_{21} = 0.2222$ and the remaining y_{ik}'s are zero. Hence, the optimal policy is to order 2 items in state 0 and not to order in states 1 and 2.

19.4-1.

Number of states: 3

Number of decisions: 2

Cost Matrix, C_{ik}: $\begin{bmatrix} 0 & 0 \\ -27 & -31 \\ -27 & -31 \end{bmatrix}$

$p_{ij}(1) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.6 & 0.4 \end{bmatrix}$ $p_{ij}(2) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.8 & 0.2 \end{bmatrix}$

Initial Policy:

$d_0(R_1) = 1$
$d_1(R_1) = 1$
$d_2(R_1) = 2$ Discount Factor = 1

Iteration # 1

Value Determination:

$$g(R_1) = 0 + 0.5v_0(R_1) + 0.5v_1(R_1) + 0v_2(R_1) - v_0(R_1)$$
$$g(R_1) = -27 + 0.3v_0(R_1) + 0.5v_1(R_1) + 0.2v_2(R_1) - v_1(R_1)$$
$$g(R_1) = -31 + 0v_0(R_1) + 0.8v_1(R_1) + 0.2v_2(R_1) - v_2(R_1)$$

Solution of Value Determination Equations:

$g(R_1) = -18.8$
$v_0(R_1) = 52.84$
$v_1(R_1) = 15.27$
$v_2(R_1) = 0$

Policy Improvement:

State 0:
$0 + 0.5(52.84) + 0.5(15.27) + (0) - (52.84) = -18.8$
$0 + 0.5(52.84) + 0.5(15.27) + (0) - (52.84) = -18.8$

State 1:
$-27 + 0.3(52.84) + 0.5(15.27) + (0) - (15.27) = -18.8$
$-31 + 0.4(52.84) + 0.5(15.27) + (0) - (15.27) = -17.5$

State 2:
$-27 + 0(52.84) + 0.6(15.27) + (0) - (0) = -17.8$
$-31 + 0(52.84) + 0.8(15.27) + (0) - (0) = -18.8$

Optimal Policy:

$d_0(R_2) = 1$ $g(R_2) = -18.8$
$d_1(R_2) = 1$ $v_0(R_2) = 52.84$
$d_2(R_2) = 2$ $v_1(R_2) = 15.27$
$v_2(R_2) = 0$

19.4-2.

Number of states = 2

Number of decisions = 5

Cost Matrix, C(ik):

$$\begin{bmatrix} 0 & 4.5 & 5 & ---- & ---- \\ ---- & ---- & ---- & 50 & 9 \end{bmatrix}$$

Transition Matrix, p(ij)[1]:

$$\begin{bmatrix} 0.9 & 0.1 \\ 0 & 0 \end{bmatrix}$$

Transition Matrix, p(ij)[2]:

$$\begin{bmatrix} 0.98 & 0.02 \\ 0 & 0 \end{bmatrix}$$

Transition Matrix, p(ij)[3]:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Transition Matrix, p(ij)[4]:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Transition Matrix, p(ij)[5]:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Initial Policy:

d0(R1) = 1
d1(R1) = 4

Discount Factor = 1

ITERATION # 1

Value Determination:

```
g(R1) = 0     +  0.9v0(R1) +   0.1v1(R1) - v0(R1)
g(R1) = 50    +    1v0(R1) +     0v1(R1) - v1(R1)
```

Solution of Value Determination Equations:

```
g(R1)  = 4.545
v0(R1) = -45.5
v1(R1) = 0
```

Policy Improvement:

State 0:

```
0    + 0.9  (-45.5) + 0.1  (0) - (-45.5) = 4.545
4.5  + 0.98 (-45.5) + 0.02 (0) - (-45.5) = 5.409
5    + 1    (-45.5) + 0    (0) - (-45.5) = 5
---  + 0    (-45.5) + 0    (0) - (-45.5) = ---
---  + 0    (-45.5) + 0    (0) - (-45.5) = ---
```

State 1:

```
---  + 0    (-45.5) + 0    (0) - (0) = ---
---  + 0    (-45.5) + 0    (0) - (0) = ---
---  + 0    (-45.5) + 0    (0) - (0) = ---
50   + 1    (-45.5) + 0    (0) - (0) = 4.545
9    + 0    (-45.5) + 1    (0) - (0) = 9
```

```
Optimal Policy: g(R1)  = 4.545
  d0(R2) = 1    v0(R1) = -45.5
  d1(R2) = 4    v1(R1) = 0
```

19.4-3.

```
Number of states = 2                 Cost Matrix, C(ik):
                                     |‾ 0        14     ‾|
Number of decisions = 2              |_ 75       14     _|

Transition Matrix, p(ij)[1]:         Transition Matrix, p(ij)[2]:
|‾ 0.125  0.875 ‾|                   |‾ 0.875  0.125 ‾|
|_ 0.125  0.875 _|                   |_ 0.875  0.125 _|


Initial Policy:

d0(R1) = 1
d1(R1) = 2

Discount Factor = 1

ITERATION # 1

Value Determination:

g(R1) = 0     +0.125v0(R1) + 0.875v1(R1) - v0(R1)
g(R1) = 14    +0.875v0(R1) + 0.125v1(R1) - v1(R1)

Solution of Value Determination Equations:
g(R1)  = 7
v0(R1) = -8
v1(R1) = 0

Policy Improvement:

State 0:
0     + 0.125(   -8) + 0.875(0) - (-8    ) = 7
14    + 0.875(   -8) + 0.125(0) - (-8    ) = 15

State 1:
75    + 0.125(   -8) + 0.875(0) - (0) = 74
14    + 0.875(   -8) + 0.125(0) - (0) = 7

Optimal Policy: g(R1)  = 7
  d0(R2) = 1    v0(R1) = -8
  d1(R2) = 2    v1(R1) = 0
```

19.4-4.

Number of states: 2

Number of decisions: 2

Cost Matrix, C_{ik}: $\begin{bmatrix} -0.12 & 0.292 \\ 0.5 & 0.417 \end{bmatrix}$

$p_{ij}(1) = \begin{bmatrix} 0.375 & 0.625 \\ 1 & 0 \end{bmatrix}$ $p_{ij}(2) = \begin{bmatrix} 0.875 & 0.125 \\ 1 & 0 \end{bmatrix}$

Initial Policy:

$d_0(R_1) = 1$
$d_1(R_1) = 1$

Iteration # 1

Value Determination:

$$g(R_1) = -0.12+0.375v_0(R_1) + 0.625v_1(R_1) - v_0(R_1)$$
$$g(R_1) = 0.5 + 1v_0(R_1) + 0v_1(R_1) - v_1(R_1)$$

Solution of Value Determination Equations:

$g(R_1) = 0.115$
$v_0(R_1) = -0.38$
$v_1(R_1) = 0$

Policy Improvement:

State 0:
$-0.12+ 0.375(-0.38) +(0) - (-0.38) = 0.115$
$0.292+ 0.875(-0.38) +(0) - (-0.38) = 0.34$

State 1:
$0.5 + 1(-0.38) +(0) - (0) = 0.115$
$0.417+ 1(-0.38) +(0) - (0) = 0.032$

New Policy:
$d_0(R_2) = 1$
$d_1(R_2) = 2$

Iteration # 2

Value Determination:

$$g(R_2) = -0.12+0.375v_0(R_2) + 0.625v_1(R_2) - v_0(R_2)$$
$$g(R_2) = 0.417+ 1v_0(R_2) + 0v_1(R_2) - v_1(R_2)$$

Solution of Value Determination Equations:

$g(R_2) = 0.083$
$v_0(R_2) = -0.33$
$v_1(R_2) = 0$

Policy Improvement:

State 0:
$-0.12+ 0.375(-0.33) +(0) - (-0.33) = 0.083$
$0.292+ 0.875(-0.33) +(0) - (-0.33) = 0.333$

State 1:
$0.5 + 1(-0.33) +(0) - (0) = 0.167$
$0.417+ 1(-0.33) +(0) - (0) = 0.083$

New Policy:

$d_0(R_3) = 2$
$d_1(R_3) = 1$

Iteration # 3

Value Determination:

$$g(R_3) = -0.12+0.375v_0(R_3) + 0.625v_1(R_3) - v_0(R_3)$$
$$g(R_3) = 0.5 + 1v_0(R_3) + 1v_1(R_3) - v_1(R_3)$$

Solution of Value Determination Equations:

$g(R_3) = 0.115$
$v_0(R_3) = -0.38$
$v_1(R_3) = 0$

Policy Improvement:

State 0:
$-0.12+ 0.375(-0.38) +(0) - (-0.38) = 0.115$
$0.292+ 0.875(-0.38) +(0) - (-0.38) = 0.34$

State 1:
$0.5 + 1(-0.38) +(0) - (0) = 0.115$
$0.417+ 1(-0.38) +(0) - (0) = 0.032$

Optimal Policy: $g(R_4) = 0.115$
$d_0(R_4) = 2$ $v_0(R_4) = -0.38$
$d_1(R_4) = 1$ $v_1(R_4) = 0$

19.4-5.

Initial Policy:

d0(R1) = 1
d1(R1) = 1
d2(R1) = 1

ITERATION # 1

Value Determination:

$$g(R1) = -22 + 0.4v0(R1) + 0.4v1(R1) + 0.2v2(R1) - v0(R1)$$
$$g(R1) = -10.5+ 0.3v0(R1) + 0.4v1(R1) + 0.3v2(R1) - v1(R1)$$
$$g(R1) = 16 + 0.1v0(R1) + 0.4v1(R1) + 0.5v2(R1) - v2(R1)$$

Solution of Value Determination Equations:
g(R1) = -4.37
v0(R1) = -54.3
v1(R1) = -37.4
v2(R1) = 0

```
Policy Improvement:

State 0:
-22   + 0.4  (-54.3) + 0.4  (-37.4) + 0.2  (0) - (-54.3) = -4.37
-9    + 0.4  (-54.3) + 0.4  (-37.4) + 0.2  (0) - (-54.3) = 8.629

State 1:
-10.5+ 0.3  (-54.3) + 0.4  (-37.4) + 0.3  (0) - (-37.4) = -4.37
-4.5 + 0.3  (-54.3) + 0.4  (-37.4) + 0.3  (0) - (-37.4) = 1.629

State 2:
16   + 0.1  (-54.3) + 0.4  (-37.4) + 0.5  (0) - (0) = -4.37
6.5  + 0.1  (-54.3) + 0.4  (-37.4) + 0.5  (0) - (0) = -13.9

New Policy:
  d0(R2) = 1
  d1(R2) = 1
  d2(R2) = 2

ITERATION # 2

Value Determination:

g(R2) = -22   +  0.4v0(R2) +   0.4v1(R2) +   0.2v2(R2) - v0(R2)
g(R2) = -10.5+  0.3v0(R2) +   0.4v1(R2) +   0.3v2(R2) - v1(R2)
g(R2) = 6.5   +  0.1v0(R2) +   0.4v1(R2) +   0.5v2(R2) - v2(R2)

Solution of Value Determination Equations:
g(R2)  = -7.63
v0(R2) = -40.7
v1(R2) = -25.1
v2(R2) = 0
Policy Improvement:

State 0:
-22   + 0.4  (-40.7) + 0.4  (-25.1) + 0.2  (0) - (-40.7) = -7.63
-9    + 0.4  (-40.7) + 0.4  (-25.1) + 0.2  (0) - (-40.7) = 5.371

State 1:
-10.5+ 0.3  (-40.7) + 0.4  (-25.1) + 0.3  (0) - (-25.1) = -7.63
-4.5 + 0.3  (-40.7) + 0.4  (-25.1) + 0.3  (0) - (-25.1) = -1.63

State 2:
16   + 0.1  (-40.7) + 0.4  (-25.1) + 0.5  (0) - (0) = 1.871
6.5  + 0.1  (-40.7) + 0.4  (-25.1) + 0.5  (0) - (0) = -7.63

Optimal Policy:
  d0(R3) = 1
  d1(R3) = 1
  d2(R3) = 2
```

19.4-6.

```
Number of states = 2                       Cost Matrix, C(ik):

Number of decisions = 2                    |‾ 0       0      ‾|
                                           |_ -1200   -1200  _|

Transition Matrix, p(ij)[1]:               Transition Matrix, p(ij)[2]:

|‾ 0.4     0.6    ‾|                       |‾ 0.5     0.5    ‾|
|_ 0.6     0.4    _|                       |_ 0.4     0.6    _|

 Initial Policy:

 d0(R1) = 1                 Discount Factor = 1
 d1(R1) = 1
ITERATION # 1

Value Determination:

g(R1) = 0     +  0.4v0(R1) +   0.6v1(R1) - v0(R1)
g(R1) = -1200+  0.6v0(R1) +   0.4v1(R1) - v1(R1)

Solution of Value Determination Equations:
g(R1)  = -600
v0(R1) = 1000
v1(R1) = 0

Policy Improvement:

State 0:
0     + 0.4  ( 1000) + 0.6  (0) - (1000 ) = -600
0     + 0.5  ( 1000) + 0.5  (0) - (1000 ) = -500

State 1:
-1200+ 0.6  ( 1000) + 0.4  (0) - (0) = -600
-1200+ 0.4  ( 1000) + 0.6  (0) - (0) = -800

New Policy:
  d0(R2) = 1
  d1(R2) = 2

ITERATION # 2

Value Determination:

g(R2) = 0     +  0.4v0(R2) +   0.6v1(R2) - v0(R2)
g(R2) = -1200+  0.4v0(R2) +   0.6v1(R2) - v1(R2)

Solution of Value Determination Equations:
g(R2)  = -720
v0(R2) = 1200
v1(R2) = 0

Policy Improvement:

State 0:
0     + 0.4  ( 1200) + 0.6  (0) - (1200 ) = -720
0     + 0.5  ( 1200) + 0.5  (0) - (1200 ) = -600

-1200+ 0.6  ( 1200) + 0.4  (0) - (0) = -480
-1200+ 0.4  ( 1200) + 0.6  (0) - (0) = -720

Optimal Policy: g(R2)  = -720
  d0(R3) = 1    v0(R2) = 1200
  d1(R3) = 2    v1(R2) = 0
```

19.4-7.

```
Markovian Decision Processes Model:

Number of states = 3            Cost Matrix, C(ik):

Number of decisions = 3         |‾ 13.33   18.67   24    ‾|
                                |  4       19      ----   |
                                |_ 4       ----    ----  _|

Transition Matrix, p(ij)[1]:    Transition Matrix, p(ij)[2]:    Transition Matrix, p(ij)[3]:

|‾ 1      0      0     ‾|       |‾ 0.667  0.333  0     ‾|       |‾ 0.333  0.333  0.333 ‾|
|  0.667  0.333  0      |       |  0.333  0.333  0.333  |       |  0      0      0      |
|_ 0.333  0.333  0.333 _|       |_ 0      0      0     _|       |_ 0      0      0     _|

 Initial Policy:

 d0(R1) = 3
 d1(R1) = 1                     Discount Factor = 1
 d2(R1) = 1

Average Cost Policy Improvement Algorithm:

ITERATION # 1

Value Determination:

g(R1) = 24    +0.333v0(R1) + 0.333v1(R1) + 0.333v2(R1) - v0(R1)
g(R1) = 4     +0.667v0(R1) + 0.333v1(R1) +     0v2(R1) - v1(R1)
g(R1) = 4     +0.333v0(R1) + 0.333v1(R1) + 0.333v2(R1) - v2(R1)

Solution of Value Determination Equations:
g(R1)  = 12.89
v0(R1) = 20
v1(R1) = 6.667
v2(R1) = 0

Policy Improvement:

State 0:
13.33+ 1     (   20) + 0     (6.667) + 0     (0) - (20    ) = 13.33
18.67+ 0.667(    20) + 0.333(6.667) + 0     (0) - (20    ) = 14.22
24   + 0.333(    20) + 0.333(6.667) + 0.333(0) - (20    ) = 12.89

State 1:
4    + 0.667(    20) + 0.333(6.667) + 0     (0) - (6.667) = 12.89
19   + 0.333(    20) + 0.333(6.667) + 0.333(0) - (6.667) = 21.22
---  + 0     (   20) + 0     (6.667) + 0     (0) - (6.667) = ---

State 2:
4    + 0.333(    20) + 0.333(6.667) + 0.333(0) - (0) = 12.89
---  + 0     (   20) + 0     (6.667) + 0     (0) - (0) = ---
---  + 0     (   20) + 0     (6.667) + 0     (0) - (0) = ---

New Policy:
  d0(R2) = 3
  d1(R2) = 1
  d2(R2) = 1
```

19.4-8.

When the number of pints of blood delivered can be specified at the time of delivery, the starting number of pints including the delivery will never exceed the largest possible demand in a period, so we can restrict our attention to states $i = 0, 1, 2, 3$. The admissible actions in state i are to order $0 \leq k \leq 3 - i$. Given a decision k, the transition probabilities and the immediate cost are computed as follows:

$$p_{ij}(k) = P\{D = i + k - j\} \text{ if } j \geq 1$$

$$p_{i0}(k) = P\{D \geq i + k\}$$

$$C_{ik} = 50k + E[100(i + k - D)^+].$$

Initialization: $d_i(R_1) = 1$ for $i = 0, 1, 2$ and $d_3(R_1) = 0$

$$\mathrm{P}(R_1) = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{pmatrix} \quad C(R_1) = \begin{pmatrix} 90 \\ 60 \\ 50 \\ 0 \end{pmatrix}$$

Iteration 1:

Step 1: Value determination:

$$g(R_1) = 90 + 0.6v_0(R_1) + 0.4v_1(R_1) - v_0(R_1)$$

$$g(R_1) = 60 + 0.3v_0(R_1) + 0.3v_1(R_1) + 0.4v_2(R_1) - v_1(R_1)$$

$$g(R_1) = 50 + 0.1v_0(R_1) + 0.2v_1(R_1) + 0.3v_2(R_1) + 0.4v_3(R_1) - v_2(R_1)$$

$$g(R_1) = \ \ 0 + 0.1v_0(R_1) + 0.2v_1(R_1) + 0.3v_2(R_1) + 0.4v_3(R_1) - v_3(R_1)$$

$$v_3(R_1) = 0$$

$$\Rightarrow g(R_1) = 57.8, v_0(R_1) = 196.3, v_1(R_1) = 115.9, v_2(R_1) = 50, v_3(R_1) = 0$$

Step 2: Policy improvement:

$$\text{minimize} \begin{pmatrix} 100 + v_0(R_1) - v_0(R_1) = 100 \\ 90 + 0.6v_0(R_1) + 0.4v_1(R_1) - v_0(R_1) = 57.8 \\ 110 + 0.3v_0(R_1) + 0.3v_1(R_1) + 0.4v_2(R_1) - v_0(R_1) = 27.36 \\ 150 + 0.1v_0(R_1) + 0.2v_1(R_1) + 0.3v_2(R_1) + 0.4v_3(R_1) - v_0(R_1) = \mathbf{11.51} \end{pmatrix}$$

$$\Rightarrow d_0(R_2) = 3$$

$$\text{minimize} \begin{pmatrix} 40 + 0.6v_0(R_1) + 0.4v_1(R_1) - v_1(R_1) = 88.24 \\ 60 + 0.3v_0(R_1) + 0.3v_1(R_1) + 0.4v_2(R_1) - v_1(R_1) = 57.8 \\ 100 + 0.1v_0(R_1) + 0.2v_1(R_1) + 0.3v_2(R_1) + 0.4v_3(R_1) - v_1(R_1) = \mathbf{41.91} \end{pmatrix}$$

$$\Rightarrow d_1(R_2) = 2$$

$$\text{minimize} \begin{pmatrix} 10 + 0.3v_0(R_1) + 0.3v_1(R_1) + 0.4v_2(R_1) - v_2(R_1) = 73.66 \\ 50 + 0.1v_0(R_1) + 0.2v_1(R_1) + 0.3v_2(R_1) + 0.4v_3(R_1) - v_2(R_1) = \mathbf{57.8} \end{pmatrix}$$

$$\Rightarrow d_2(R_2) = 1$$

R_2 is not identical to R_1, so optimality test fails.

Iteration 2:

Step 1: Value determination:

$$g(R_2) = 150 + 0.1v_0(R_2) + 0.2v_1(R_2) + 0.3v_2(R_2) + 0.4v_3(R_2) - v_0(R_2)$$
$$g(R_2) = 100 + 0.1v_0(R_2) + 0.2v_1(R_2) + 0.3v_2(R_2) + 0.4v_3(R_2) - v_1(R_2)$$
$$g(R_2) = 50 + 0.1v_0(R_2) + 0.2v_1(R_2) + 0.3v_2(R_2) + 0.4v_3(R_2) - v_2(R_2)$$
$$g(R_2) = \ \ 0 + 0.1v_0(R_2) + 0.2v_1(R_2) + 0.3v_2(R_2) + 0.4v_3(R_2) - v_3(R_2)$$
$$v_3(R_2) = 0$$

$\Rightarrow g(R_2) = 50, v_0(R_2) = 150, v_1(R_2) = 100, v_2(R_2) = 50, v_3(R_2) = 0$

Step 2: Policy improvement:

$$\text{minimize} \begin{pmatrix} 100 + v_0(R_2) - v_0(R_2) = 100 \\ 90 + 0.6v_0(R_2) + 0.4v_1(R_2) - v_0(R_2) = 70 \\ 110 + 0.3v_0(R_2) + 0.3v_1(R_2) + 0.4v_2(R_2) - v_0(R_2) = 55 \\ 150 + 0.1v_0(R_2) + 0.2v_1(R_2) + 0.3v_2(R_2) + 0.4v_3(R_2) - v_0(R_2) = \mathbf{50} \end{pmatrix}$$

$\Rightarrow d_0(R_3) = 3$

$$\text{minimize} \begin{pmatrix} 40 + 0.6v_0(R_2) + 0.4v_1(R_2) - v_1(R_2) = 70 \\ 60 + 0.3v_0(R_2) + 0.3v_1(R_2) + 0.4v_2(R_2) - v_1(R_2) = 55 \\ 100 + 0.1v_0(R_2) + 0.2v_1(R_2) + 0.3v_2(R_2) + 0.4v_3(R_2) - v_1(R_2) = \mathbf{50} \end{pmatrix}$$

$\Rightarrow d_1(R_3) = 2$

$$\text{minimize} \begin{pmatrix} 10 + 0.3v_0(R_1) + 0.3v_1(R_1) + 0.4v_2(R_1) - v_2(R_1) = 55 \\ 50 + 0.1v_0(R_1) + 0.2v_1(R_1) + 0.3v_2(R_1) + 0.4v_3(R_1) - v_2(R_1) = \mathbf{50} \end{pmatrix}$$

$\Rightarrow d_2(R_3) = 1$

R_3 is identical to R_2, so it is optimal to start every period with 3 pints of blood after delivery of the order.

19.5-1.

Let states 0, 1 and 2 denote \$600, \$800 and \$1000 offers respectively and let state 3 designate the case that the car has already been sold (state ∞ of the hint). Let decisions 1 and 2 be to reject and to accept the offer respectively.

$C_{01} = C_{11} = C_{21} = 60$, $C_{02} = 600$, $C_{12} = -800$ and $C_{22} = -1000$

$$P(1) = \begin{pmatrix} 5/8 & 1/4 & 1/8 & 0 \\ 5/8 & 1/4 & 1/8 & 0 \\ 5/8 & 1/4 & 1/8 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P(2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Start with the policy to reject only the \$600 offer. The relevant equations are:

$$V_0 = 60 + 0.95\left(\tfrac{5}{8}V_0 + \tfrac{1}{4}V_1 + \tfrac{1}{8}V_2\right)$$
$$V_1 = -800 + 0.95V_3$$
$$V_2 = -1000 + 0.95V_3$$
$$V_3 = 0.95V_3,$$

which admit the unique solution $(V_0, V_1, V_2, V_3) = (-7960/13, -800, -1000, 0)$.

Policy improvement:

State 0 with decision 2: $-600 + 0.95V_3 = -600 > V_0$

State 1 with decision 1: $60 + 0.95[(5/8)V_0 + (1/4)V_1 + (1/8)V_2] = -7960/13 > V_1$

State 2 with decision 1: $60 + 0.95[(5/8)V_0 + (1/4)V_1 + (1/8)V_2] = -7960/13 > V_2$

Hence, the policy to reject the \$600 offer and to accept \$800 and \$1000 offers is optimal.

19.5-2.

(a) minimize $\quad 60y_{01} - 600y_{02} + 60y_{11} - 800y_{12} + 60y_{21} - 1000y_{22}$

subject to

$$y_{01} + \quad y_{02} - 0.95\left(\tfrac{5}{8}\right)\left(y_{01} + y_{11} + y_{21}\right) = \tfrac{1}{3}$$

$$y_{11} + \quad y_{12} - 0.95\left(\tfrac{1}{4}\right)\left(y_{01} + y_{11} + y_{21}\right) = \tfrac{1}{3}$$

$$y_{21} + \quad y_{22} - 0.95\left(\tfrac{1}{8}\right)\left(y_{01} + y_{11} + y_{21}\right) = \tfrac{1}{3}$$

$y_{ik} \geq 0$ for $i = 0, 1, 2$ and $k = 1, 2$

(b) Using the simplex method, we find $y_{01} = 0.81979, y_{12} = 0.5277, y_{22} = 0.43056$ and the remaining y_{ik}'s are zero. Hence, the optimal policy is to reject the \$600 offer and to accept the \$800 and \$1000 offers.

19.5-3.

$V_i^n = \min\{60 + 0.95((5/8)V_0^{n-1} + (1/4)V_1^{n-1} + (1/8)V_2^{n-1}), -(\text{offer})\}$ for $i = 0, 1, 2$

$V_i^0 = 0$ for $i = 0, 1, 2$

Iteration 1: $\quad V_i^1 = \min\{60, -(\text{offer})\} = -(\text{offer})$ for $i = 0, 1, 2 \Rightarrow$ Accept

Iteration 2: $\quad V_0^2 = \min\{-605, -600\} = -605 \Rightarrow$ Reject

$V_1^2 = \min\{-605, -800\} = -800 \Rightarrow$ Accept

$V_2^2 = \min\{-605, -1000\} = -1000 \Rightarrow$ Accept

Iteration 3: $\quad V_0^3 = \min\{-607.97, -600\} = -607.97 \Rightarrow$ Reject

$V_1^3 = \min\{-607.97, -800\} = -800 \Rightarrow$ Accept

$V_2^3 = \min\{-607.97, -1000\} = -1000 \Rightarrow$ Accept

The approximate optimal solution is to reject the \$600 offer and to accept the \$800 and \$1000 offers. This policy is indeed optimal, as found in Problem 19.5-1 and 19.5-2.

19.5-4.

Let states 0, 1 and 2 denote the selling price of \$10, \$20 and \$30 respectively and let state 3 designate the case that the stock has already been sold. Let decisions 1 and 2 be to hold and to sell the stock respectively.

$C_{01} = C_{11} = C_{21} = 0$, $C_{02} = -10$, $C_{12} = -20$ and $C_{22} = -30$

$$P(1) = \begin{pmatrix} 4/5 & 1/5 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P(2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Start with the policy to sell only when the price is \$30. The relevant equations are:

$$V_0 = 0 + 0.9\left(\tfrac{4}{5}V_0 + \tfrac{1}{5}V_1\right)$$

$$V_1 = 0 + 0.9\left(\tfrac{1}{4}V_0 + \tfrac{1}{4}V_1 + \tfrac{1}{2}V_2\right)$$

$$V_2 = -30 + 0.9V_3$$

$$V_3 = 0 + 0.9V_3,$$

which admit the unique solution $(V_0, V_1, V_2, V_3) = (-4860/353, -7560/353, -30, 0)$.

Policy improvement:

State 0 with decision 2: $-10 + 0.9V_3 = -10 > V_0$

State 1 with decision 2: $-20 + 0.9V_3 = -20 > V_1$

State 2 with decision 1: $0 + 0.9[(3/4)V_1 + (1/4)V_2] = -21.21 > V_2$

Hence, the policy to hold the stock when the price is \$10 and \$20, and to sell it when the price is \$30.

19.5-5.

(a) minimize $\quad -10y_{02} - 20y_{12} - 30y_{22}$

subject to

$$y_{01} + y_{02} - 0.9\left(\tfrac{4}{5}y_{01} + \tfrac{1}{4}y_{11}\right) = \tfrac{1}{3}$$

$$y_{11} + y_{12} - 0.9\left(\tfrac{1}{5}y_{01} + \tfrac{1}{4}y_{11} + \tfrac{3}{4}y_{21}\right) = \tfrac{1}{3}$$

$$y_{21} + y_{22} - 0.9\left(\tfrac{1}{2}y_{11} + \tfrac{1}{4}y_{21}\right) = \tfrac{1}{3}$$

$y_{ik} \geq 0$ for $i = 0, 1, 2$ and $k = 1, 2$

(b) Using the simplex method, we find $y_{01} = 1.96059, y_{11} = 0.95851, y_{22} = 0.76463$ and the remaining y_{ik}'s are zero. Hence, the optimal policy is to hold the stock at the prices \$10 and \$20 and to sell it at the price \$30.

19.5-6.

$V_0^n = \min\{0.9((4/5)V_0^{n-1} + (1/5)V_1^{n-1}), -10\}$

$V_1^n = \min\{0.9((1/4)V_0^{n-1} + (1/4)V_1^{n-1} + (1/2)V_2^{n-1}), -20\}$

$V_2^n = \min\{0.9((3/4)V_1^{n-1} + (1/4)V_2^{n-1}), -30\}$

$V_i^0 = 0$ for $i = 0, 1, 2$

<u>Iteration 1:</u> $\quad V_0^1 = \min\{0, -10\} = -10 \Rightarrow$ Sell

$V_1^1 = \min\{0, -20\} = -20 \Rightarrow$ Sell

$V_2^1 = \min\{0, -30\} = -30 \Rightarrow$ Sell

Iteration 2: $V_0^2 = \min\{-10.8, -10\} = -10.8 \Rightarrow$ Hold

$V_1^2 = \min\{-20.25, -20\} = -20.25 \Rightarrow$ Hold

$V_2^2 = \min\{-20.25, -30\} = -30 \Rightarrow$ Sell

Iteration 3: $V_0^3 = \min\{-11.42, -10\} = -11.42 \Rightarrow$ Hold

$V_1^3 = \min\{-20.49, -20\} = -20.49 \Rightarrow$ Hold

$V_2^3 = \min\{-20.42, -30\} = -30 \Rightarrow$ Sell

The approximate optimal solution is to sell if the price is $30 and to hold otherwise. This policy is indeed optimal, as found in Problem 19.5-3 and 19.5-4.

19.5-7.

(a) Let states 0 and 1 be the chemical produced this month, $C1$ and $C2$ respectively, and decisions 1 and 2 refer to the process to be used next month, A and B respectively. There are four stationary deterministic policies.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

The transition matrix is the same for every decision, viz.

$$P = \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix}.$$

The costs C_{ik} correspond to the expected amount of pollution using the process k in the next period.

$$C_{01} = 0.3(15) + 0.7(2) = 5.9,$$

$$C_{02} = 0.3(3) + 0.7(8) = 6.5,$$

$$C_{11} = 0.4(15) + 0.6(2) = 7.2,$$

$$C_{12} = 0.4(3) + 0.6(8) = 6.$$

(b)

```
Initial Policy:

d0(R1) = 1
d1(R1) = 1

Discount Factor = 0.5
Discounted Cost Policy Improvement Algorithm:

ITERATION # 1

Value Determination:

g(R1) = 5.9  + (0.5) [  0.3V0(R1) +   0.7V1(R1) ]
g(R1) = 7.2  + (0.5) [  0.4V0(R1) +   0.6V1(R1) ]

Solution of Value Determination Equations:
V1(R1) = 12.67
V2(R1) = 13.9
```

```
Policy Improvement:

State 0:
5.9  + (0.5) [0.3  (12.67) + 0.7  ( 13.9) ] = 12.67
6.5  + (0.5) [0.3  (12.67) + 0.7  ( 13.9) ] = 13.27

State 1:
7.2  + (0.5) [0.4  (12.67) + 0.6  ( 13.9) ] = 13.9
6    + (0.5) [0.4  (12.67) + 0.6  ( 13.9) ] = 12.7

New Policy:
  d0(R2) = 1
  d1(R2) = 2

ITERATION # 2

Value Determination:

g(R2) = 5.9  + (0.5) [  0.3V0(R2) +   0.7V1(R2) ]
g(R2) = 6    + (0.5) [  0.4V0(R2) +   0.6V1(R2) ]

Solution of Value Determination Equations:
V1(R2) = 11.87
V2(R2) = 11.96

Policy Improvement:

State 0:
5.9  + (0.5) [0.3  (11.87) + 0.7  (11.96) ] = 11.87
6.5  + (0.5) [0.3  (11.87) + 0.7  (11.96) ] = 12.47

State 1:
7.2  + (0.5) [0.4  (11.87) + 0.6  (11.96) ] = 13.16
6    + (0.5) [0.4  (11.87) + 0.6  (11.96) ] = 11.96

Optimal Policy:
  d0(R3) = 1
  d1(R3) = 2
```

19.5-8.

(a) minimize $5.9y_{01} + 6.5y_{02} + 7.2y_{11} + 6y_{12}$

subject to

$$y_{01} + \quad y_{02} - \tfrac{1}{2}\left(\tfrac{3}{10}y_{01} + \tfrac{4}{10}y_{11} + \tfrac{3}{10}y_{02} + \tfrac{4}{10}y_{12}\right) = \tfrac{1}{2}$$

$$y_{11} + \quad y_{12} - \tfrac{1}{2}\left(\tfrac{7}{10}y_{01} + \tfrac{6}{10}y_{11} + \tfrac{7}{10}y_{02} + \tfrac{6}{10}y_{12}\right) = \tfrac{1}{2}$$

$y_{ik} \geq 0$ for $i = 0, 1$ and $k = 1, 2$

(b) Using the simplex method, we find $y_{01} = 0.857, y_{12} = 1.143$ and $y_{02} = y_{11} = 0$. Hence, the optimal policy is to use process A if $C1$ is produced and B if $C2$ is produced this month.

19.5-9.

```
Discount Factor = 0.5
Method of Successive Approximations:

Initial V(i):

  v(1) = 0
  v(2) = 0

ITERATION #1

New Policy and New V(i):
  d0(R1) = 1,          V(0) = 5.9
  d1(R1) = 2,          V(1) = 6
 ITERATION # 2

 State 0:
 5.9  + (0.5) [0.3  (  5.9) + 0.7  (    6) ] = 8.885
 6.5  + (0.5) [0.3  (  5.9) + 0.7  (    6) ] = 9.485

 State 1:
 7.2  + (0.5) [0.4  (  5.9) + 0.6  (    6) ] = 10.18
 6    + (0.5) [0.4  (  5.9) + 0.6  (    6) ] = 8.98

New Policy and New V(i):
  d0(R2) = 1,          V(0) = 8.885
  d1(R2) = 2,          V(1) = 8.98

ITERATION # 3

State 0:
5.9  + (0.5) [0.3  (8.885) + 0.7  ( 8.98) ] = 10.38
6.5  + (0.5) [0.3  (8.885) + 0.7  ( 8.98) ] = 10.98

State 1:
7.2  + (0.5) [0.4  (8.885) + 0.6  ( 8.98) ] = 11.67
6    + (0.5) [0.4  (8.885) + 0.6  ( 8.98) ] = 10.47

 New Policy and New V(i):
   d0(R3) = 1,          V(0) = 10.38
   d1(R3) = 2,          V(1) = 10.47
```

19.5-10.

The three iterations of successive approximations in Problem 19.5-9 gives the optimal policy for the three-period problem. The optimal policy is, therefore, to use the process A if $C1$ is produced and B if $C2$ is produced in all periods.

19.5-11.

$V_0^n = \min\{0 + 0.90((7/8)V_1^{n-1} + (1/16)V_2^{n-1} + (1/16)V_3^{n-1}), 4000 + 0.90V_1^{n-1}, 6000 + 0.90V_0^{n-1}\}$

$V_1^n = \min\{1000 + 0.90((3/4)V_1^{n-1} + (1/8)V_2^{n-1} + (1/8)V_3^{n-1}), 4000 + 0.90V_1^{n-1}, 6000 + 0.90V_0^{n-1}\}$

$V_2^n = \min\{3000 + 0.90((1/2)V_2^{n-1} + (1/2)V_3^{n-1}), 4000 + 0.90V_1^{n-1}, 6000 + 0.90V_0^{n-1}\}$

$V_3^n = 6000 + 0.90V_0^{n-1}$

$V_i^0 = 0$ for $i = 0, 1, 2, 3$

Iteration 1: $V_0^1 = \min\{0, 4000, 6000\} = 0 \Rightarrow$ Do nothing

$V_1^1 = \min\{1000, 4000, 6000\} = 1000 \Rightarrow$ Do nothing

$V_2^1 = \min\{3000, 4000, 6000\} = 3000 \Rightarrow$ Do nothing

$V_3^1 = 6000 \Rightarrow$ Replace

Iteration 2: $V_0^2 = \min\{1293.75, 4900, 6000\} = 1293.75 \Rightarrow$ Do nothing

$V_1^2 = \min\{2687.5, 4900, 6000\} = 2687.5 \Rightarrow$ Do nothing

$V_2^2 = \min\{7050, 4900, 6000\} = 4900 \Rightarrow$ Overhaul

$V_3^2 = 6000 \Rightarrow$ Replace

Iteration 3: $V_0^3 = \min\{2729.53, 6418.75, 7164.38\} = 2729.53 \Rightarrow$ Do nothing

$V_1^3 = \min\{4040.31, 6418.75, 7164.38\} = 4040.31 \Rightarrow$ Do nothing

$V_2^3 = \min\{7905, 6418.75, 7164.38\} = 6418.75 \Rightarrow$ Overhaul

$V_3^3 = 7164.38 \Rightarrow$ Replace

Iteration 4: $V_0^4 = \min\{3945.80, 7636.28, 8456.58\} = 3945.80 \Rightarrow$ Do nothing

$V_1^4 = \min\{5255.31, 7636.28, 8456.58\} = 5255.31 \Rightarrow$ Do nothing

$V_2^4 = \min\{9112.41, 7636.28, 8456.58\} = 7636.28 \Rightarrow$ Overhaul

$V_3^4 = 8456.58 \Rightarrow$ Replace

The optimal policy is to do nothing in states $0, 1$ and to replace in state 3 in all periods. When in state 2, it is best to overhaul in periods $1, 2, 3$ and to do nothing in period 4.

CHAPTER 20: SIMULATION

20.1-1.

(a) 0.0000 to 0.4999 correspond to tails.
0.5000 to 0.9999 correspond to heads.

Random observations: 0.6961 = heads, 0.2086 = tails, 0.1457 = tails, 0.3098 = tails, 0.6996 = heads, 0.9617 = heads

(b) 0.0000 to 0.5999 correspond to strikes.
0.6000 to 0.9999 correspond to balls.

Random observations: 0.6961 = ball, 0.2086 = strike, 0.1457 = strike, 0.3098 = strike, 0.6996 = ball, 0.9617 = ball

(c) 0.0000 to 0.3999 correspond to green lights.
0.4000 to 0.4999 correspond to yellow lights.
0.5000 to 0.9999 correspond to red lights.

Random observations: 0.6961 = red, 0.2086 = green, 0.1457 = green, 0.3098 = green, 0.6996 = red, 0.9617 = red

20.1-2.

(a) If it is raining: 0.0000 to 0.5999 correspond to rain next day,
0.6000 to 0.9999 correspond to clear next day.

If it is clear: 0.0000 to 0.7999 correspond to clear next day,
0.8000 to 0.9999 correspond to rain next day.

Day	Random Number	Weather
1	0.6996	Clear
2	0.9617	Rain
3	0.6117	Clear
4	0.3948	Clear
5	0.7769	Clear
6	0.5750	Clear
7	0.6271	Clear
8	0.2017	Clear
9	0.7760	Clear
10	0.9918	Rain

(b)

Day	Random Number	Weather
1	0.8212	Rain
2	0.1449	Rain
3	0.1762	Rain
4	0.7318	Clear
5	0.9218	Rain
6	0.1237	Rain
7	0.2881	Rain
8	0.8235	Clear
9	0.5954	Clear
10	0.8405	Rain

20.1-3.

(a)

$$P(2) = \frac{4}{25}, P(3) = \frac{7}{25}, P(4) = \frac{8}{25}, P(5) = \frac{5}{25}, P(6) = \frac{1}{25}$$

(b)

Mean: $(2)\frac{4}{25} + (3)\frac{7}{25} + (4)\frac{8}{25} + (5)\frac{5}{25} + (6)\frac{1}{25} = 3.68$ stoves

(c) 0.0000 to 0.1599 correspond to 2 stoves being sold.
0.1600 to 0.4399 correspond to 3 stoves being sold.
0.4400 to 0.7599 correspond to 4 stoves being sold.
0.7600 to 0.9599 correspond to 5 stoves being sold.
0.9600 to 0.9999 correspond to 6 stoves being sold.

(d) $0.4476 \Rightarrow 4$ stoves, $0.9713 \Rightarrow 6$ stoves, $0.0629 \Rightarrow 2$ stoves

The average of these is $(4 + 6 + 2)/3 = 4$, which exceeds the mean in (b) by 0.32.

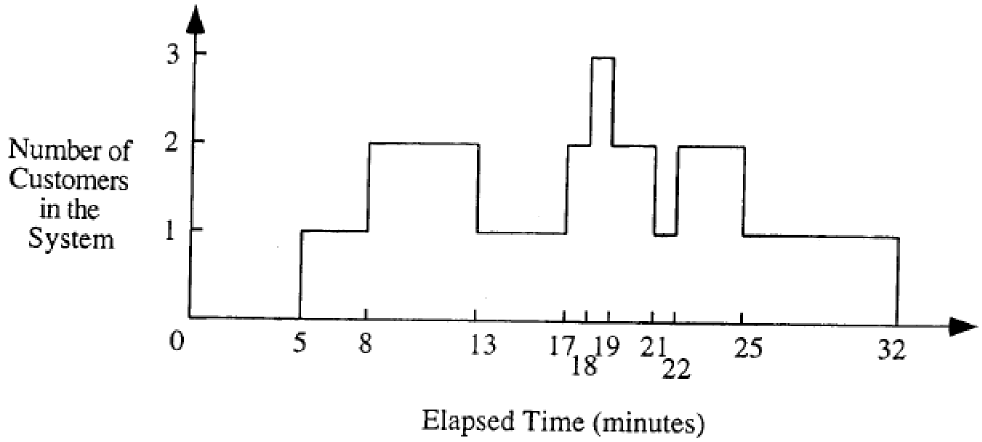
(e) Answers will vary. The following 300-day simulation yielded an average demand of 3.72.

Day	Random Number	Demand
1	0.5475	4
2	0.3597	3
3	0.6539	4
4	0.6263	4
5	0.9576	5
6	0.8396	5
7	0.1005	2
297	0.5809	4
298	0.3673	3
299	0.4453	4
300	0.1361	2

Distribution of Demand		
Probability	Cumulative	Demand
0.16	0	2
0.28	0.16	3
0.32	0.44	4
0.20	0.76	5
0.04	0.96	6

20.1-4.

(a)



(b)

$\text{Est}\{P_0\} = \frac{5}{32} = 0.156$ $\text{Est}\{P_1\} = \frac{3+4+1+7}{32} = 0.469$

$\text{Est}\{P_2\} = \frac{5+1+2+3}{32} = 0.344$ $\text{Est}\{P_3\} = \frac{1}{32} = 0.031$

$$\text{Est}\{L\} = \sum_{n=0}^{3} nP_n = 0 \cdot 0.156 + 1 \cdot 0.469 + 2 \cdot 0.344 + 3 \cdot 0.031 = 1.25 \text{ customers}$$

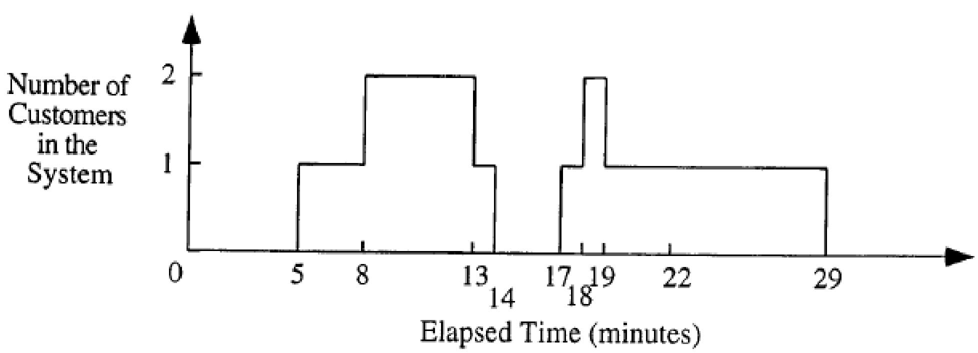
$$\text{Est}\{L_q\} = \sum_{n=1}^{3} (n-1)P_n = 0 \cdot 0.469 + 1 \cdot 0.344 + 2 \cdot 0.031 = 0.406 \text{ customers}$$

Customers	Arrival Time	Service Time	Departure Time	System Time	Wait Time
1	5	8	13	8	0
2	8	6	19	11	5
3	17	2	21	4	2
4	18	4	25	7	3
5	22	7	32	10	3

$$\text{Est}\{W\} = \frac{\text{sum of observed system times}}{\text{number of observed system times}} = \frac{40}{5} = 8 \text{ minutes}$$

$$\text{Est}\{W_q\} = \frac{\text{sum of observed waiting times}}{\text{number of observed waiting times}} = \frac{32}{5} = 2.6 \text{ minutes}$$

(c)



(d)

$$\text{Est}\{P_0\} = \frac{5+3}{29} = 0.276 \qquad \text{Est}\{P_1\} = \frac{3+1+1+3+7}{29} = 0.517$$

$$\text{Est}\{P_2\} = \frac{5+1}{29} = 0.207$$

$$\text{Est}\{L\} = \sum_{n=0}^{2} nP_n = 0 \cdot 0.276 + 1 \cdot 0.517 + 2 \cdot 0.207 = 0.931 \text{ customers}$$

$$\text{Est}\{L_q\} = \sum_{n=1}^{2} (n-1)P_n = 0 \cdot 0.517 + 1 \cdot 0.207 = 0.207 \text{ customers}$$

Customers	Arrival Time	Service Time	Departure Time	System Time	Wait Time
1	5	8	13	8	0
2	8	6	14	6	0
3	17	2	19	2	0
4	18	4	22	4	0
5	22	7	29	7	0

$$\text{Est}\{W\} = \frac{\text{sum of observed system times}}{\text{number of observed system times}} = \frac{27}{5} = 5.4 \text{ minutes}$$

$$\text{Est}\{W_q\} = \frac{\text{sum of observed waiting times}}{\text{number of observed waiting times}} = \frac{0}{5} = 0 \text{ minutes}$$

20.1-5.

(a) Interarrival Time ~Exp$(1/6$ per minute$)$, Service Time ~Exp$(1/5$ per minute$)$

Next interarrival time: $-6\ln r_A$

Next service time: $-5\ln r_D$

Let t and $N(t)$ denote the time in minutes and the number of customers in the system at time t respectively. In the table below, N.I.T. stands for Next Interarrival Time and N.S.T. for Next Service Time.

t	$N(t)$	r_A	N.I.T.	r_D	N.S.T.	Next Arriv.	Next Dep.	Next Event
0	0	0.096	14.060	–	–	14.060	–	Arrival
14.060	1	0.569	3.383	0.665	2.040	17.443	16.100	Departure
16.100	0	–	–	–	–	17.443	–	Arrival
17.443	1	0.764	1.615	0.842	0.860	19.058	18.303	Departure
18.303	0	–	–	–	–	19.058	–	Arrival
19.058	1							

(b)

$$P\{\text{arrival in two-minute period}\} = 1 - e^{-\frac{1}{3}} = 0.283$$

$$P\{\text{departure in two-minute period}\} = 1 - e^{-\frac{2}{5}} = 0.330$$

$r_A < 0.283 \Rightarrow$ arrival occurred, $r_A \geq 0.283 \Rightarrow$ arrival did not occur.

$r_D < 0.330 \Rightarrow$ departure occurred, $r_D \geq 0.330 \Rightarrow$ departure did not occur.

Let t and $N(t)$ denote the time in minutes and the number of customers in the system at time t respectively.

t	$N(t)$	r_A	Arrival?	r_D	Departure?
0	0				
2	1	0.096	Yes	–	–
4	1	0.569	No	0.665	No
6	1	0.764	No	0.842	No
8	0	0.492	No	0.224	Yes
10	0	0.950	No	–	–
12	0	0.610	No	–	–
14	1	0.145	Yes	–	–
16	1	0.484	No	0.552	No
18	1	0.350	No	0.590	No
20	0	0.430	No	0.041	Yes

(c) Interarrival Time ~Exp$(1/10)$, Service Time ~Exp$(1/12)$

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	1.59516	0.05791
0.05791	0	No	1.59516	---
1.59516	0	Yes	2.01515	2.32053
2.01515	1	Yes	2.18621	2.32053
2.18621	2	Yes	2.52934	2.32053
2.32053	1	Yes	2.52934	2.57411
2.52934	2	Yes	4.67015	2.57411
2.57411	1	Yes	4.67015	3.44101
3.44101	0	Yes	4.67015	3.5073
3.5073	0	No	4.67015	---
4.67015	0	Yes	5.45971	4.84881
4.84881	0	No	5.45971	---
5.45971	0	Yes	8.54887	5.55357
5.55357	0	No	8.54887	---
8.54887	0	Yes	8.59868	10.9813
8.59868	1	Yes	9.33448	10.9813
9.33448	2	Yes	11.3485	10.9813
10.9813	1	Yes	11.3485	11.7831
11.3485	2	Yes	11.5231	11.7831
11.5231	3	Yes	11.8163	11.7831
11.7831	2	Yes	11.8163	12.4319
11.8163	3	Yes	12.7794	12.4319
12.4319	2	Yes	12.7794	16.7805
12.7794	3	Yes	12.8719	16.7805
12.8719	4	Yes	16.5715	16.7805
16.5715	5	Yes	17.4063	16.7805
16.7805	4	Yes	17.4063	17.1525
17.1525	3	Yes	17.4063	18.7851
17.4063	4	Yes	17.7437	18.7851
17.7437	5	Yes	18.1435	18.7851

```
 18.1435|          6          |          Yes          |        18.4939        |          18.7851
 18.4939|          7          |          Yes          |        20.2767        |          18.7851
 18.7851|          6          |          Yes          |        20.2767        |          19.0576
 19.0576|          5          |          Yes          |        20.2767        |            19.16
   19.16|          4          |          Yes          |        20.2767        |          20.9374
 20.2767|          5          |          Yes          |         20.706        |          20.9374
  20.706|          6          |          Yes          |        21.3904        |          20.9374
 20.9374|          5          |          Yes          |        21.3904        |          21.2357
 21.2357|          4          |          Yes          |        21.3904        |          22.0278
 21.3904|          5          |          Yes          |        24.1124        |          22.0278
 22.0278|          4          |          Yes          |        24.1124        |          23.6643
 23.6643|          3          |          Yes          |        24.1124        |          23.7054
```

Average number waiting to begin service: 2.33652
Average number waiting for or in service: 3.167162
Average waiting time excluding service: 2.30529
Average waiting time including service: 3.21887

(d)

		Results	
	Point	95% Confidence Interval	
	Estimate	Low	High
L =	5.7749101	4.34834384	7.201476355
L_q =	4.942339	3.537019642	6.347658336
W =	0.580131	0.444220564	0.71604147
W_q =	0.4964933	0.361637568	0.631349023
P_0 =	0.1674289	0.14167657	0.193181214
P_1 =	0.1414463	0.12109788	0.161794649
P_2 =	0.1147575	0.098821547	0.130693466
P_3 =	0.0883648	0.076687015	0.100042668
P_4 =	0.0746175	0.064766646	0.084468439
P_5 =	0.0623432	0.052996622	0.071689766
P_6 =	0.0492943	0.041116902	0.05747167
P_7 =	0.0387165	0.031563519	0.045869425
P_8 =	0.0334397	0.026950751	0.039928583
P_9 =	0.0280708	0.022245604	0.03389596
P_{10} =	0.0224129	0.017018014	0.027807755

(e)

	Data	
λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	1	(# servers)
Pr(W > t) =	0.135335	
when t =	1	
Prob(W_q > t) =	0.112779	
when t =	1	

	Results
L =	5
L_q =	4.166666667
W =	0.5
W_q =	0.416666667
ρ =	0.833333333

Every measure is inside the 95% confidence level.

20.1-6.

(a) The system is a single-server queueing system with the crew being servers and the machines being customers. The service time has a uniform distribution between 0 and twice the mean. The interarrival time is exponentially distributed with mean being 5 hours. A simulation clock records the amount of simulated time that elapses. The state $N(t)$ of the system at time t is the number of machines that need repair at time t. The breakdowns and repairs that occur over time are randomly generated by generating random observations from the distributions of interarrival and service times. The state of the system needs to be adjusted when a breakdown or repair occurs:

$$\text{Reset } N(t) = \begin{cases} N(t)+1 & \text{if a breakdown occurs at time } t, \\ N(t)-1 & \text{if a repair occurs at time } t. \end{cases}$$

The time on the simulation clock is adjusted by using the next-event time advance procedure. The time t is in hours.

(b) The random numbers r_A and r_D are obtained from Table 20.3 starting from the front of the first row. N.I.T. stands for Next Interarrival Time and N.S.T. for Next Service Time. Interarrival times are computed as $-5\ln r_A$ and service times correspond to $8\,r_D$. Initially there is one broken machine in the system.

t	$N(t)$	r_A	N.I.T.	r_D	N.S.T.	Next Arriv.	Next Dep.	Next Event
0	1	0.096	11.717	0.569	4.552	11.717	4.552	Departure
4.552	0	–	–	–	–	11.717	–	Arrival
11.717	1	0.665	2.040	0.764	6.112	13.757	17.829	Arrival
13.757	2	0.842	0.860	–	–	14.617	17.829	Arrival
14.617	3	0.492	3.546	–	–	18.163	17.829	Departure
17.829	2	–	–	0.224	1.792	18.163	19.621	Arrival
18.163	3	0.950	0.256	–	–	18.420	19.621	Arrival
18.420	4	0.610	2.471	–	–	20.891	19.621	Departure
19.621	3	–	–	0.145	1.160	20.891	20.781	Departure

(c)

$$P\{\text{arrival in one-hour period}\} = 1 - e^{-1/5} = 0.181$$

$$P\{\text{departure in one-hour period}\} = 1/8 = 0.125$$

$r_A < 0.181 \Rightarrow$ arrival occurred, $r_A \geq 0.181 \Rightarrow$ arrival did not occur.

$r_D < 0.125 \Rightarrow$ departure occurred, $r_D \geq 0.125 \Rightarrow$ departure did not occur.

Let t and $N(t)$ denote the time in hours and the number of broken machines in the system at time t respectively. r_A and r_D are obtained from Table 20.3 starting from the front of the first row.

t	$N(t)$	r_A	Arrival?	r_D	Departure?
0	1				
0	2	0.096	Yes	0.569	No
1	2	0.665	No	0.764	No
2	2	0.842	No	0.492	No
3	2	0.224	No	0.950	No
4	2	0.610	No	0.145	No
5	2	0.484	No	0.552	No
6	2	0.350	No	0.590	No
7	1	0.430	No	0.041	Yes
8	1	0.802	No	0.471	No
9	1	0.255	No	0.799	No
10	1	0.608	No	0.577	No
11	1	0.347	No	0.933	No
12	1	0.581	No	0.173	No
13	0	0.603	No	0.040	Yes
14	0	0.605	No	–	–
15	0	0.842	No	–	–
16	0	0.720	No	–	–
17	0	0.449	No	–	–
18	1	0.076	Yes	–	–
19	1	0.407	No	0.202	No
20	1	0.963	No	0.412	No

(d) Crew size = 2

Current Time	Number of Customers in Queue	Customer Being Served?	Next Arrival	Next Service Completion
0.00000	0	Yes	0.45442	5.06774
0.45442	1	Yes	23.52844	5.06774
5.06774	0	Yes	23.52844	12.56525
12.56525	0	No	23.52844	-
23.52844	0	Yes	24.13347	29.98968
24.13347	1	Yes	35.10738	29.98968
29.98968	0	Yes	35.10738	32.23639
32.23639	0	No	35.10738	-
35.10738	0	Yes	41.89761	39.87832
39.87832	0	No	41.89761	-
41.89761	0	Yes	45.97317	44.93853
44.93853	0	No	45.97317	-
45.97317	0	Yes	48.46326	50.40101
48.46326	1	Yes	51.81284	50.40101
50.40101	0	Yes	51.81284	55.84630
51.81284	1	Yes	52.94219	55.84630
52.94219	2	Yes	89.09479	55.84630
55.84630	1	Yes	89.09479	61.63057
61.63057	0	Yes	89.09479	63.08379
63.08379	0	No	89.09479	-
89.09479	0	Yes	99.09964	94.10255

Average waiting time excluding service: 3.141 hours
Average waiting time including service: 7.982 hours
Average number waiting to begin service: 0.282
Average number waiting or in service: 0.717

Crew size $= 3$

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0.00000	0	Yes	3.23986	5.22623
3.23986	1	Yes	7.47514	5.22623
5.22623	0	Yes	7.47514	10.29107
7.47514	1	Yes	15.15030	10.29107
10.29107	0	Yes	15.15030	15.57362
15.15030	1	Yes	27.53296	15.57362
15.57362	0	Yes	27.53296	16.06349
16.06349	0	No	27.53296	-
27.53296	0	Yes	42.72952	29.37910
29.37910	0	No	42.72952	-
42.72952	0	Yes	46.23502	46.66759
46.23502	1	Yes	48.75186	46.66759
46.66759	0	Yes	48.75186	48.69142
48.69142	0	No	48.75186	-
48.75186	0	Yes	50.75080	54.60197
50.75080	1	Yes	50.88372	54.60197
50.88372	2	Yes	55.78357	54.60197
54.60197	1	Yes	55.78357	59.86150
55.78357	2	Yes	56.25391	59.86150

Average waiting time excluding service: 1.057 hours
Average waiting time including service: 4.943 hours
Average number waiting to begin service: 0.258
Average number waiting or in service: 0.812

Crew size $= 4$

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0.00000	0	Yes	28.44578	3.45477
3.45477	0	No	28.44578	-
28.44578	0	Yes	29.78728	29.41541
29.41541	0	No	29.78728	-
29.78728	0	Yes	32.76097	32.96767
32.76097	1	Yes	38.62356	32.96767
32.96767	0	Yes	38.62356	36.41909
36.41909	0	No	38.62356	-
38.62356	0	Yes	48.10272	40.47197
40.47197	0	No	48.10272	-
48.10272	0	Yes	54.56103	51.69710
51.69710	0	No	54.56103	-
54.56103	0	Yes	55.07481	57.13491
55.07481	1	Yes	57.38123	57.13491
57.13491	0	Yes	57.38123	57.30586
57.30586	0	No	57.38123	-
57.38123	0	Yes	58.73878	58.40348
58.40348	0	No	58.73878	-
58.73878	0	Yes	62.16265	59.34633
59.34633	0	No	62.16265	-
62.16265	0	Yes	65.06976	64.07583

Average waiting time excluding service: 0.227 hours
Average waiting time including service: 2.314 hours
Average number waiting to begin service: 0.036
Average number waiting or in service: 0.372

(e) Crew size = 2

Queueing Simulator						
	Data				Results	
Number of Servers =	1			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			L =	2.640024	2.391023	2.889024
Distribution =	Exponential		L_q =	1.846050	1.608886	2.083214
Mean =	5		W =	13.172814	12.069683	14.275946
			W_q =	9.211156	8.123435	10.298876
Service Times			P_0 =	0.206026	0.189914	0.222138
Distribution =	Uniform		P_1 =	0.210335	0.196666	0.224004
Minimum Value =	0		P_2 =	0.176818	0.166556	0.187079
Maximum Value =	8		P_3 =	0.129046	0.120984	0.137108
			P_4 =	0.092375	0.084072	0.100678
Length of Simulation Run			P_5 =	0.059729	0.052306	0.067152
Number of Arrivals =	10,000		P_6 =	0.040612	0.033736	0.047488
			P_7 =	0.025701	0.019924	0.031478
			P_8 =	0.019530	0.014201	0.024858
			P_9 =	0.013836	0.008579	0.019092
Run Simulation			P_{10} =	0.008875	0.004711	0.013039

Crew size = 3

Queueing Simulator						
	Data				Results	
Number of Servers =	1			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			L =	1.178656	1.111742	1.245570
Distribution =	Exponential		L_q =	0.581427	0.524765	0.638089
Mean =	5		W =	5.919185	5.650535	6.187836
			W_q =	2.919913	2.666936	3.172891
Service Times			P_0 =	0.402771	0.389561	0.415981
Distribution =	Uniform		P_1 =	0.287935	0.280079	0.295791
Minimum Value =	0		P_2 =	0.161810	0.155143	0.168477
Maximum Value =	6		P_3 =	0.079673	0.073661	0.085684
			P_4 =	0.036517	0.031748	0.041286
Length of Simulation Run			P_5 =	0.017571	0.013323	0.021820
Number of Arrivals =	10,000		P_6 =	0.007611	0.005156	0.010066
			P_7 =	0.002871	0.001559	0.004182
			P_8 =	0.001559	0.000479	0.002639
			P_9 =	0.001021	0.000143	0.001898
Run Simulation			P_{10} =	0.000540	-0.000050	0.001129

Crew size $= 4$

Queueing Simulator					
	Data		**Results**		
Number of Servers =	1		Point Estimate	95% Confidence Interval Low	High
Interarrival Times		$L =$	0.581799	0.557904	0.605694
Distribution =	Exponential	$L_q =$	0.184646	0.168481	0.200810
Mean =	5	$W =$	2.917323	2.835377	2.999270
		$W_q =$	0.925872	0.855488	0.996256
Service Times		$P_0 =$	0.602847	0.593328	0.612366
Distribution =	Uniform	$P_1 =$	0.265406	0.259908	0.270904
Minimum Value =	0	$P_2 =$	0.093625	0.089000	0.098250
Maximum Value =	4	$P_3 =$	0.027373	0.024388	0.030358
		$P_4 =$	0.007793	0.006052	0.009534
Length of Simulation Run		$P_5 =$	0.002086	0.001273	0.002900
Number of Arrivals =	10,000	$P_6 =$	0.000670	0.000185	0.001156
		$P_7 =$	0.000198	-0.000025	0.000421
		$P_8 =$	0.000001	-0.000001	0.000004
		$P_9 =$	0.000000	0.000000	0.000000
Run Simulation		$P_{10} =$	0.000000	0.000000	0.000000

According to these simulation runs, a crew size of 3 is enough to get the average waiting time before repair below 3 hours. If the high end of the 95% confidence interval is required to be less than 3 hours, then a crew size of 4 should be chosen.

(f) λ, $1/\mu$, σ^2, and s denote the mean breakdown rate, the expected repair time, the variance of the repair time, and the number of servers respectively. The variance of a random variable uniformly distributed between a and b is $(b-a)^2/12$.

Crew size $= 2$: $\lambda = 0.2, \frac{1}{\mu} = 4, a = 0, b = 8, \sigma^2 = 5.333$

$\rho = \frac{\lambda}{\mu} = 0.8$

$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = 2.133$, $L = \rho + L_q = 2.933$

$W_q = \frac{L_q}{\lambda} = 10.667$, $W = W_q + \frac{1}{\mu} = 14.667$

Crew size $= 3$: $\lambda = 0.2, \frac{1}{\mu} = 3, a = 0, b = 6, \sigma^2 = 3$

$\rho = \frac{\lambda}{\mu} = 0.6$

$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = 0.6, \quad L = \rho + L_q = 1.2$

$W_q = \frac{L_q}{\lambda} = 3$, $W = W_q + \frac{1}{\mu} = 6$

Crew size = 4: $\lambda = 0.2, \frac{1}{\mu} = 2, a = 0, b = 4, \sigma^2 = 1.333$

$$\rho = \frac{\lambda}{\mu} = 0.4$$

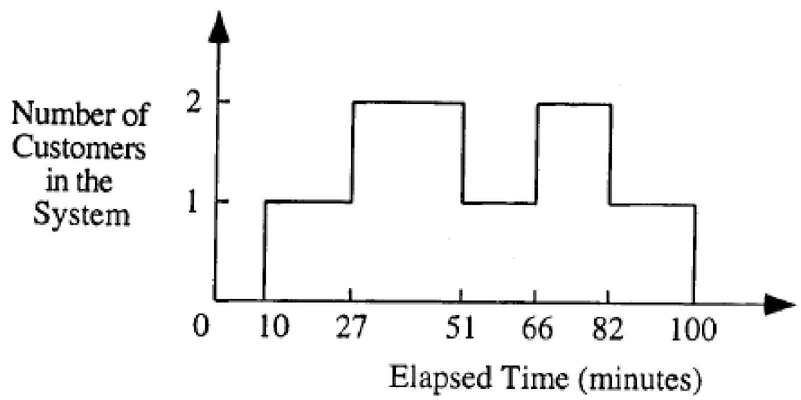
$$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} = 0.178,\ L = \rho + L_q = 0.578$$

$$W_q = \frac{L_q}{\lambda} = 0.889,\ W = W_q + \frac{1}{\mu} = 2.889$$

A crew size of 3 is enough to have the average waiting time before repair begins no more than 3 hours.

20.1-7.

(a)



(b)

$$\text{Est}\{P_0\} = \frac{10}{100} = 0.1 \qquad \text{Est}\{P_1\} = \frac{17+15+18}{100} = 0.4$$

$$\text{Est}\{P_2\} = \frac{24+16}{100} = 0.4 \qquad \text{Est}\{P_3\} = \frac{0}{100} = 0$$

(c)

$$\text{Est}\{L\} = \sum_{n=0}^{3} nP_n = 0\cdot 0.1 + 1\cdot 0.4 + 2\cdot 0.4 + 3\cdot 0 = 1.2 \text{ customers}$$

$$\text{Est}\{L_q\} = \sum_{n=1}^{3} (n-1)P_n = 0\cdot 0.4 + 1\cdot 0.4 + 2\cdot 0 = 0.4 \text{ customers}$$

(d)

$$\text{Est}\{W\} = \frac{\text{sum of observed system times}}{\text{number of observed system times}} = \frac{41+55+34}{3} = 43.33 \text{ minutes}$$

$$\text{Est}\{W_q\} = \frac{\text{sum of observed waiting times}}{\text{number of observed waiting times}} = \frac{0+24+16}{3} = 13.33 \text{ minutes}$$

20.1-8.

(a)

Distr. of interarrival times: Translated Exp. Min = 0.5 Mean = 1
Distr. of service times: Erlang Mean = 1.5 k = 4

Current Time	Number of Customers in Queue	Customer Being Served Server 1	Customer Being Served Server 2	Next Arrival	Next Service Completion Server 1	Next Service Completion Server 2
0	0	Yes	No	1.6685	3.00911	---
1.6685	0	Yes	Yes	2.2903	3.00911	4.06113
2.2903	1	Yes	Yes	3.45204	3.00911	4.06113
3.00911	0	Yes	Yes	3.45204	4.31305	4.06113
3.45204	1	Yes	Yes	3.99204	4.31305	4.06113
3.99204	2	Yes	Yes	5.08213	4.31305	4.06113
4.06113	1	Yes	Yes	5.08213	4.31305	4.82208
4.31305	0	Yes	Yes	5.08213	7.01408	4.82208
4.82208	0	Yes	No	5.08213	7.01408	---
5.08213	0	Yes	Yes	5.80875	7.01408	6.56763
5.80875	1	Yes	Yes	6.42612	7.01408	6.56763
6.42612	2	Yes	Yes	8.45996	7.01408	6.56763
6.56763	1	Yes	Yes	8.45996	7.01408	7.793
7.01408	0	Yes	Yes	8.45996	9.25094	7.793
7.793	0	Yes	No	8.45996	9.25094	---
8.45996	0	Yes	Yes	8.45996	9.25094	8.95073
8.45996	1	Yes	Yes	9.01185	9.25094	8.95073
8.95073	0	Yes	Yes	9.01185	9.25094	10.3732
9.01185	1	Yes	Yes	10.7538	9.25094	10.3732
9.25094	0	Yes	Yes	10.7538	11.0051	10.3732
10.3732	0	Yes	No	10.7538	11.0051	---
10.7538	0	Yes	Yes	11.8319	11.0051	11.9901
11.0051	0	No	Yes	11.8319	---	11.9901
11.8319	0	Yes	Yes	12.5131	13.3238	11.9901
11.9901	0	Yes	No	12.5131	13.3238	---
12.5131	0	Yes	Yes	15.8697	13.3238	14.986
13.3238	0	No	Yes	15.8697	---	14.986
14.986	0	No	No	15.8697	---	---
15.8697	0	Yes	No	18.1124	16.8485	---
16.8485	0	No	No	18.1124	---	---
18.1124	0	Yes	No	19.0569	18.8949	---
18.1124	0	Yes	No	19.0569	18.8949	---
18.8949	0	No	No	19.0569	---	---
19.0569	0	Yes	No	19.8234	21.8863	---
19.8234	0	Yes	Yes	20.7688	21.8863	21.3164
20.7688	0	Yes	Yes	---	21.8863	21.3164

Average number waiting to begin service: 0.186891
Average number waiting for or in service: 1.522408
Average waiting time excluding service: 0.18669
Average waiting time including service: 1.87597

(b) Two Tellers

Queueing Simulator						
	Data				Results	
Number of Servers =	2			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			L =	1.8916169	1.797799179	1.98543456
Distribution =	Translated Exponential		L_q =	0.3821604	0.314682897	0.449637826
Minimum Value =	0.5		W =	1.8883612	1.806278584	1.970443834
Mean =	1		W_q =	0.3815026	0.316252552	0.446752699
Service Times			P_0 =	0.0831748	0.074555785	0.091793811
Distribution =	Erlang		P_1 =	0.3241939	0.30548793	0.342899863
Mean =	1.5		P_2 =	0.3399268	0.325527051	0.354326542
k =	4		P_3 =	0.1618758	0.14916797	0.174583587
			P_4 =	0.0619756	0.049305086	0.07464612
Length of Simulation Run			P_5 =	0.0208701	0.012224227	0.029515958
Number of Arrivals =	5,000		P_6 =	0.006311	0.002235605	0.010386453
			P_7 =	0.0015531	-0.00011068	0.003216788
			P_8 =	0.000119	-0.00011327	0.000351178
			P_9 =	0	0	0
Run Simulation			P_{10} =	0	0	0

(c) Three Tellers

Queueing Simulator						
	Data				Results	
Number of Servers =	3			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			L =	1.5159924	1.484880165	1.547104562
Distribution =	Translated Exponential		L_q =	0.0118407	0.009081176	0.014600248
Minimum Value =	0.5		W =	1.5116824	1.489782703	1.533582175
Mean =	1		W_q =	0.011807	0.009089235	0.014524863
Service Times			P_0 =	0.1018135	0.093997406	0.109629589
Distribution =	Erlang		P_1 =	0.4119282	0.400147287	0.423709211
Mean =	1.5		P_2 =	0.3665514	0.35560916	0.377493556
k =	4		P_3 =	0.1082202	0.099939007	0.116501306
			P_4 =	0.0111328	0.008716351	0.013549183
Length of Simulation Run			P_5 =	0.000354	4.87263E-05	0.000659219
Number of Arrivals =	5,000		P_6 =	0	0	0
			P_7 =	0	0	0
			P_8 =	0	0	0
			P_9 =	0	0	0
Run Simulation			P_{10} =	0	0	0

(d) Two Tellers

Queueing Simulator

	Data			**Results**	
Number of Servers =	2		Point	95% Confidence Interval	
			Estimate	Low	High
Interarrival Times		$L =$	2.3908912	2.241805543	2.539976948
Distribution =	Translated Exponential	$L_q =$	0.7236811	0.597902598	0.849459547
Minimum Value =	0.5	$W =$	2.1487678	2.026740049	2.270795538
Mean =	0.9	$W_q =$	0.6503945	0.540866912	0.759922142
Service Times		$P_0 =$	0.0448497	0.038617135	0.051082232
Distribution =	Erlang	$P_1 =$	0.2430905	0.223049896	0.263131024
Mean =	1.5	$P_2 =$	0.3214538	0.301005391	0.34190214
k =	4	$P_3 =$	0.2008161	0.187347516	0.214284666
		$P_4 =$	0.1044243	0.087200271	0.121648324
Length of Simulation Run		$P_5 =$	0.0500277	0.036075878	0.063979621
Number of Arrivals =	5,000	$P_6 =$	0.02027	0.010643612	0.029896401
		$P_7 =$	0.0089975	0.003725713	0.014269206
		$P_8 =$	0.0046276	-0.00016476	0.009419952
		$P_9 =$	0.0014429	-0.00067057	0.00355635
Run Simulation		$P_{10} =$	0	0	0

Three Tellers

Queueing Simulator

	Data			**Results**	
Number of Servers =	3		Point	95% Confidence Interval	
			Estimate	Low	High
Interarrival Times		$L =$	1.7050114	1.670548122	1.739474655
Distribution =	Translated Exponential	$L_q =$	0.0222237	0.017359202	0.027088241
Minimum Value =	0.5	$W =$	1.5209078	1.496862881	1.544952626
Mean =	0.9	$W_q =$	0.019824	0.015539702	0.024108397
Service Times		$P_0 =$	0.0688074	0.062324314	0.075290568
Distribution =	Erlang	$P_1 =$	0.35062	0.337180867	0.364059144
Mean =	1.5	$P_2 =$	0.40955	0.398544607	0.420555391
k =	4	$P_3 =$	0.1499384	0.139825152	0.160051587
		$P_4 =$	0.0200063	0.016017091	0.023995492
Length of Simulation Run		$P_5 =$	0.0010163	0.000432052	0.001600449
Number of Arrivals =	5,000	$P_6 =$	6.164E-05	-5.9202E-05	0.000182488
		$P_7 =$	0	0	0
		$P_8 =$	0	0	0
		$P_9 =$	0	0	0
Run Simulation		$P_{10} =$	0	0	0

(e) Let λ denote the average time between customer arrivals. Some performance measures are given for two-teller and three-teller systems in the following tables.

	Two Tellers	Three Tellers
L	1.892	1.516
L_q	0.382	0.012
W	1.888	1.512
W_q	0.382	0.012
Idle	0.407	0.881

$\lambda = 1$

	Two Tellers	Three Tellers
L	2.391	1.705
L_q	0.724	0.022
W	2.149	1.521
W_q	0.650	0.020
Idle	0.288	0.830

$\lambda = 0.9$

The last row corresponds to the probability that at least one of the tellers is idle. For the two-teller system it is $P_0 + P_1$ and for the three-teller system it is $P_0 + P_1 + P_2$. There is a big difference between the idle-time ratios of the two-teller and three-teller systems for both λ values. For this reason, it may be better to hire two tellers. Two tellers also provide reasonable wait times, $W_q = 0.382$ minutes for $\lambda = 1$ and $W_q = 0.650$ minutes for $\lambda = 0.9$. A thorough analysis would also incorporate the cost of hiring and the profit from the completion of each job. Another consideration can be the robustness of the system and its sensitivity to the uncertainty in λ. The following table gives the percent changes in the performance measures when λ decreases from 1 to 0.9 minute.

% Change	Two Tellers	Three Tellers
L	26.4	12.5
L_q	89.5	83.3
W	13.8	0.5
W_q	70.2	66.7
Idle	29.2	5.8

20.1-9.

Priority Class 1 (higher priority) customers
Distr. of interarrival times: Uniform Min = 1 Max = 3
Distr. of service times: Erlang Mean = 1.5 k = 4

Priority Class 2 (lower priority) customers
Distr. of interarrival times: Translated Exp. Min = 0.5 Mean = 1
Distr. of service times: Erlang Mean = 1.5 k = 4

Current Time	# of Customers in Line		Class of Customer Being Served		Next Arrival		Next Service Completion	
	Class 1	Class 2	Server 1	Server 2	Class 1	Class 2	Server 1	Server 2
0	0	0	1	idle	1.19323	0.59076	4.05587	---
0.59076	0	0	1	2	1.19323	1.98438	4.05587	1.75598
1.19323	1	0	1	2	4.03947	1.98438	4.05587	1.75598
1.75598	0	0	1	1	4.03947	2.57211	4.05587	2.6547
2.57211	0	1	1	1	4.03947	4.35852	4.05587	2.6547
2.6547	0	0	1	2	4.03947	4.35852	4.05587	5.02524
4.03947	1	0	1	2	6.60605	4.35852	4.05587	5.02524
4.05587	0	0	1	2	6.60605	4.35852	6.31076	5.02524
4.35852	0	1	1	2	6.60605	5.60471	6.31076	5.02524
5.02524	0	0	1	2	6.60605	5.60471	6.31076	6.5263
5.60471	0	1	1	2	6.60605	6.32351	6.31076	6.5263
6.31076	0	0	2	2	6.60605	6.32351	7.80267	6.5263
6.32351	0	1	2	2	6.60605	7.67972	7.80267	6.5263
6.5263	0	0	2	2	6.60605	7.67972	7.80267	7.41307
6.60605	1	0	2	2	7.66733	7.67972	7.80267	7.41307
7.66733	2	0	2	2	9.48954	7.67972	7.80267	7.41307
7.41307	1	0	2	1	9.48954	7.67972	7.80267	8.0084
7.67972	1	1	2	1	9.48954	8.7606	7.80267	8.0084
7.80267	0	1	1	1	9.48954	8.7606	9.39632	8.0084
8.0084	0	0	1	2	9.48954	8.7606	9.39632	10.085
8.7606	0	1	1	2	9.48954	9.54627	9.39632	10.085
9.39632	0	0	2	2	9.48954	9.54627	11.8025	10.085
9.48954	1	0	2	2	11.0103	9.54627	11.8025	10.085
9.54627	1	1	2	2	11.0103	10.484	11.8025	10.085
10.085	0	1	2	1	11.0103	10.484	11.8025	10.6113
10.484	0	2	2	1	11.0103	11.2066	11.8025	10.6113
10.6113	0	1	2	2	11.0103	11.2066	11.8025	11.1199
11.0103	1	1	2	2	13.2226	11.2066	11.8025	11.1199
11.1199	0	1	2	1	13.2226	11.2066	11.8025	12.4916
11.2066	0	2	2	1	13.2226	11.804	11.8025	12.4916
11.804	0	3	2	1	13.2226	12.6438	11.8025	12.4916
11.8025	0	2	2	1	13.2226	12.6438	13.5255	12.4916
12.4916	0	1	2	2	13.2226	12.6438	13.5255	13.1055
12.6438	0	2	2	2	13.2226	13.1919	13.5255	13.1055
13.1055	0	1	2	2	13.2226	13.1919	13.5255	14.1874
13.1919	0	2	2	2	13.2226	14.1007	13.5255	14.1874
13.2226	1	2	2	2	15.2685	14.1007	13.5255	14.1874
13.5255	0	2	1	2	15.2685	14.1007	15.9147	14.1874

Class 1 Customers:
Average number waiting to begin service: 0.209215
Average number waiting for or in service: 1.007613
Average waiting time excluding service: 1.07381
Average waiting time including service: 2.59826

Class 2 Customers:
Average number waiting to begin service: 1.406468
Average number waiting for or in service: 2.575706
Average waiting time excluding service: 0.38188
Average waiting time including service: 1.91684

20.1-10.

(a) For parts (a) through (f), each type of car corresponds to an M/M/1 system and they are independent of each other. For parts (g) through (i), the system is an M/M/2 system. Both interarrival and service times are exponentially distributed. A simulation clock records the amount of simulated time that elapses. The state of the system at time t consists of the number $N_J(t)$ of Japanese cars that need to be repaired at time t and the number $N_G(t)$ of German cars that need to be repaired at time t. The breakdowns and repairs that occur over time are generated by random observations with exponential distributions. The state of the system follows the dynamics:

$$N_J(t) = \begin{cases} N_J(t)+1 & \text{if a Japanese car arrives to the shop,} \\ N_J(t)-1 & \text{if a Japanese car is repaired,} \end{cases}$$

$$N_G(t) = \begin{cases} N_G(t)+1 & \text{if a German car arrives to the shop,} \\ N_G(t)-1 & \text{if a German car is repaired.} \end{cases}$$

The time is advanced using the next-event time advance procedure.

(b)

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	0.03044	0.04731
0.03044	1	Yes	0.16674	0.04731
0.04731	0	Yes	0.16674	0.0818
0.0818	0	No	0.16674	—
0.16674	0	Yes	0.32435	0.33876
0.32435	1	Yes	—	0.33876
0.32435	2	Yes	—	0.33876
0.32435	2	Yes	1.47007	0.33954
1.47007	2	Yes	1.73755	1.71047
1.73755	2	Yes	2.05826	1.92858
2.05826	2	Yes	2.15076	2.17713
2.15076	2	Yes	2.5451	2.16401
2.5451	2	Yes	2.64143	2.84944
2.64143	2	Yes	2.67262	2.69043

(c) German Cars

Queueing Simulator					
	Data			Results	
Number of Servers =	1			Point	95% Confidence Interval
				Estimate	Low	High
Interarrival Times			L =	4.4538246	3.761705456	5.145943841
Distribution =	Exponential		L_q =	3.6316608	2.954275722	4.309045972
Mean =	0.25		W =	1.1282513	0.964537526	1.291965065
			W_q =	0.9199792	0.757831377	1.082127031
Service Times			P_0 =	0.1778362	0.157439907	0.198232489
Distribution =	Exponential		P_1 =	0.1474538	0.132348578	0.162559057
Mean =	0.2		P_2 =	0.1176296	0.106278491	0.128980671
			P_3 =	0.0995436	0.090603325	0.10848388
			P_4 =	0.0790584	0.07099058	0.087126172
Length of Simulation Run			P_5 =	0.0654922	0.057986283	0.072998158
Number of Arrivals =	10,000		P_6 =	0.0578711	0.051080135	0.064662027
			P_7 =	0.048298	0.041318086	0.055277946
			P_8 =	0.0417734	0.034304357	0.049242459
			P_9 =	0.0335036	0.027074919	0.039932367
Run Simulation			P_{10} =	0.0270032	0.020815767	0.033190534

(d) Japanese Cars

Queueing Simulator						
	Data			Results		
Number of Servers =	1			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			L =	0.6351417	0.606304224	0.663979234
Distribution =	Exponential		L_q =	0.239814	0.219494995	0.260132923
Mean =	0.5		W =	0.3205858	0.308944485	0.332227141
			W_q =	0.1210454	0.11183276	0.13025794
Service Times			P_0 =	0.6046722	0.594213424	0.615131035
Distribution =	Exponential		P_1 =	0.2406602	0.234997801	0.246322589
Mean =	0.2		P_2 =	0.0987168	0.093620357	0.103813177
			P_3 =	0.0365163	0.032896287	0.040136323
			P_4 =	0.0128289	0.010707379	0.014950515
Length of Simulation Run			P_5 =	0.0043899	0.00309776	0.005682118
Number of Arrivals =	10,000		P_6 =	0.0014922	0.000769328	0.002215104
			P_7 =	0.0005442	9.5186E-05	0.000993248
			P_8 =	0.0001419	5.59536E-06	0.000278139
			P_9 =	3.732E-05	-1.5451E-05	9.00845E-05
Run Simulation			P_{10} =	0	0	0

(e)

Current Time	Number of Customers in Queue	Customer Being Served Server 1	Server 2	Next Arrival	Next Service Completion Server 1	Server
0	0	Yes	No	0.03044	0.04731	—
0.03044	0	Yes	Yes	0.76195	0.04731	0.06493
0.04731	0	No	Yes	0.76195	—	0.06493
0.06493	0	No	No	0.76195	—	—
0.76195	0	Yes	No	0.83716	1.01054	—
0.83716	1	Yes	No	0.85615	1.05115	1.07757
0.85615	1	Yes	No	1.17686	1.04719	0.93015
1.17686	1	Yes	No	1.32545	1.49234	1.19012
1.32545	1	Yes	No	1.42178	1.62979	1.3504
1.42178	1	Yes	No	1.48302	1.42802	1.53588
1.48302	1	Yes	No	1.81541	1.57642	1.68985
1.81541	1	Yes	No	1.8777	2.5102	2.03288

(f) German Cars

Queueing Simulator

	Data				**Results**	
Number of Servers =	2			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			L =	0.9567591	0.915925579	0.997592612
Distribution =	Exponential		L_q =	0.1634491	0.140317796	0.186580344
Mean =	0.25		W =	0.2388033	0.230574713	0.247031974
			W_q =	0.0407963	0.03529884	0.046293663
Service Times			P_0 =	0.432362	0.420452565	0.444271339
Distribution =	Exponential		P_1 =	0.3419661	0.334264013	0.349668127
Mean =	0.2		P_2 =	0.1312902	0.125703432	0.136876905
			P_3 =	0.0557089	0.051360155	0.060057572
			P_4 =	0.0215388	0.01873507	0.024342577
Length of Simulation Run			P_5 =	0.0093106	0.007264255	0.011356854
Number of Arrivals =	10,000		P_6 =	0.004544	0.003111739	0.005976258
			P_7 =	0.0021068	0.001110676	0.003102828
			P_8 =	0.0007196	0.000243238	0.00119591
			P_9 =	0.000107	1.94028E-06	0.00021203
Run Simulation			P_{10} =	0.0001615	-0.00015081	0.000473864

(g) This option significantly decreases the waiting time for German cars without the added cost of an additional mechanic.

Queueing Simulator					
	Data		**Results**		
Number of Servers =	2		Point	95% Confidence Interval	
			Estimate	Low	High
Interarrival Times		L =	2.3907479	2.245678563	2.535817333
Distribution =	Exponential	L_q =	1.065088	0.940183481	1.189992432
Mean =	0.1667	W =	0.3926979	0.371240571	0.414155241
		W_q =	0.1749485	0.155447354	0.194449682
Service Times		P_0 =	0.2023543	0.192371626	0.212337051
Distribution =	Exponential	P_1 =	0.2696313	0.26015233	0.279110333
Mean =	0.22	P_2 =	0.1731333	0.167267547	0.178999127
		P_3 =	0.1146076	0.109533387	0.119681718
		P_4 =	0.0807285	0.076124011	0.085333063
Length of Simulation Run		P_5 =	0.0536806	0.049303681	0.058057479
Number of Arrivals =	20,000	P_6 =	0.0345925	0.030785253	0.038399827
		P_7 =	0.0241606	0.020850571	0.027470673
		P_8 =	0.0160514	0.012976417	0.019126365
		P_9 =	0.011016	0.008413719	0.013618288
Run Simulation		P_{10} =	0.0069186	0.004757367	0.009079852

(h)

Part	Est$\{W\}$	W
(c)	1.128	1.000
(d)	0.321	0.333
(f)	0.238	0.238
(g)	0.393	0.390

The results of the simulation were quite accurate.

(i) Answers will vary. The option of training the two current mechanics significantly decreases the waiting time for German cars, without a significant impact on the wait for German cars, and does so without the added cost of a third mechanic. Adding a third mechanic reduces the average wait for German cars even more, but comes with the added cost of a third mechanic.

20.1-11.

(a) There are two independent G/M/1 systems: printers and monitors. For printers, the arrival stream is deterministic; for monitors, the arrival process is uniformly distributed between 10 and 20. The inspection time is exponentially distributed with a mean of 10 minutes. A simulation clock records the amount of simulated time that elapses. The state of the system at time t consists of the number $N_M(t)$ of monitors in the inspection station at time t and the number $N_P(t)$ of printers in the inspection station at time t. The arrivals to the stations and the inspection times are generated by sampling distributions according to interarrival and service time distributions. The system evolves according to the law:

$$N_M(t) = \begin{cases} N_M(t)+1 & \text{if a monitor arrives to the inspection station,} \\ N_M(t)-1 & \text{if a monitor is repaired,} \end{cases}$$

$$N_P(t) = \begin{cases} N_P(t)+1 & \text{if a printer arrives to the inspection station,} \\ N_P(t)-1 & \text{if a printer is repaired.} \end{cases}$$

The time is advanced using the next-event time advance procedure.

(b)

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	18.8535	2.3654
2.3654	0	No	18.8535	—
18.8535	0	Yes	30.1903	20.578
20.578	0	No	30.1903	—
30.1903	0	Yes	45.5138	38.7912
38.7912	0	No	45.5138	40.7702
45.5138	0	Yes	62.9157	57.9432
62.9157	0	Yes	73.018	63.6754
73.018	0	Yes	86.4483	85.0383
86.4483	0	Yes	99.2208	96.0002
99.2208	0	Yes	116.128	105.164
116.128	0	Yes	128.193	116.791
128.193	0	Yes	144.996	143.41
144.996	0	Yes	163.823	147.445

(c)

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	15	1.21772
1.21772	0	No	15	—
15	0	Yes	30	20.4518
20.4518	0	No	30	—
30	0	Yes	45	50.1234
45	0	Yes	60	46.7245
60	0	Yes	75	60.6191
75	0	Yes	90	80.4591
90	0	Yes	105	96.3044
105	0	Yes	120	113.601
120	0	Yes	135	127.452
135	0	Yes	150	136.979

(d) Monitors

Queueing Simulator						
	Data				Results	
Number of Servers =	1			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			L =	1.1451012	1.066454729	1.223747677
Distribution =	Uniform		L_q =	0.4846536	0.416642611	0.552664538
Minimum Value =	10		W =	17.151366	15.98246531	18.3202666
Maximum Value =	20		W_q =	7.2591582	6.244300569	8.274015886
Service Times			P_0 =	0.3395524	0.326131941	0.352972802
Distribution =	Exponential		P_1 =	0.3820527	0.371544309	0.392561116
Mean =	10		P_2 =	0.1623795	0.154529275	0.170229821
			P_3 =	0.064742	0.057617043	0.071866998
			P_4 =	0.0292033	0.023308637	0.035097966
Length of Simulation Run			P_5 =	0.0121333	0.008145886	0.016120745
Number of Arrivals =	10,000		P_6 =	0.0054888	0.002494257	0.00848335
			P_7 =	0.0028845	0.000585366	0.005183675
			P_8 =	0.0009369	1.27914E-05	0.001860937
			P_9 =	0.0003059	-0.0002381	0.00084993
Run Simulation			P_{10} =	0.000316	-0.00030101	0.000932933

Printers

Queueing Simulator						
	Data				Results	
Number of Servers =	1			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			L =	1.1396349	1.070247268	1.20902263
Distribution =	Constant		L_q =	0.4690206	0.409350919	0.528690326
Value =	15		W =	17.094524	16.05370902	18.13533945
			W_q =	7.0353093	6.140263782	7.930354892
Service Times			P_0 =	0.3293857	0.316571485	0.342199862
Distribution =	Exponential		P_1 =	0.3896143	0.379366603	0.399861927
Mean =	10		P_2 =	0.1686762	0.160479858	0.176872479
			P_3 =	0.0682924	0.060680329	0.075904496
			P_4 =	0.0264922	0.021086762	0.031897547
Length of Simulation Run			P_5 =	0.0103902	0.006984888	0.013795445
Number of Arrivals =	10,000		P_6 =	0.0035434	0.001709995	0.005376718
			P_7 =	0.0015595	0.00023512	0.002883797
			P_8 =	0.0009594	-0.00045006	0.002368941
			P_9 =	0.0008664	-0.00077164	0.002504371
Run Simulation			P_{10} =	0.0002034	-0.00019399	0.000600852

(e) Monitors

Queueing Simulator					
	Data			Results	
Number of Servers =	1		Point	95% Confidence Interval	
			Estimate	Low	High
Interarrival Times		L =	0.7509682	0.736907128	0.76502925
Distribution =	Uniform	L_q =	0.0850358	0.076444155	0.093627449
Minimum Value =	10	W =	11.255076	11.05317253	11.45697848
Maximum Value =	20	W_q =	1.2744673	1.146677882	1.402256631
Service Times		P_0 =	0.3340676	0.327214214	0.340921011
Distribution =	Erlang	P_1 =	0.5855458	0.580168141	0.590923496
Mean =	10	P_2 =	0.075767	0.069970369	0.081563664
k =	4	P_3 =	0.0045899	0.002570327	0.006609419
		P_4 =	2.968E-05	-5.9577E-06	6.53172E-05
Length of Simulation Run		P_5 =	0	0	0
Number of Arrivals =	10,000	P_6 =	0	0	0
		P_7 =	0	0	0
		P_8 =	0	0	0
		P_9 =	0	0	0
Run Simulation		P_{10} =	0	0	0

Printers

Queueing Simulator					
	Data			Results	
Number of Servers =	1		Point	95% Confidence Interval	
			Estimate	Low	High
Interarrival Times		L =	0.733283	0.720311183	0.746254753
Distribution =	Constant	L_q =	0.0677194	0.060150124	0.075288764
Value =	15	W =	10.999245	10.80466775	11.1938213
		W_q =	1.0157917	0.902251861	1.129331454
Service Times		P_0 =	0.3344365	0.327790024	0.341082927
Distribution =	Erlang	P_1 =	0.6008208	0.595646493	0.605995151
Mean =	10	P_2 =	0.0617782	0.056011597	0.067544899
k =	4	P_3 =	0.0029522	0.001566686	0.004337647
		P_4 =	1.229E-05	-6.0691E-06	3.06443E-05
Length of Simulation Run		P_5 =	0	0	0
Number of Arrivals =	10,000	P_6 =	0	0	0
		P_7 =	0	0	0
		P_8 =	0	0	0
		P_9 =	0	0	0
Run Simulation		P_{10} =	0	0	0

The new inspection equipment would drastically reduce the average waiting time for both monitors (from 7.3 minutes to 1.3 minutes) and printers (from 7 minutes to 1 minute).

20.2-1.

Merrill Lynch launched the Management Science Group to deal with the issues raised by the rise of electronic trading in the late 1990s. The group studied various product structure and pricing alternatives. They focused on two main pricing options, viz., an asset-based pricing option and a direct online pricing option. Monte Carlo simulation is applied to simulate the behavior of the clients who choose between the two product and pricing options in the light of economic and qualitative factors. In the simulation model, "the observed system data consist of every revenue-generating component of every account of every client at Merrill Lynch. The output measures are the resulting revenue at the firm level, the compensation impact on each FA, and the percentage of clients considered adverse selectors" [p. 13]. Sensitivity analysis is performed to evaluate various scenarios.

"The benefits were significant and fell into four areas: seizing the marketplace initiative, finding the pricing sweet spot, improving financial performance, and adopting the approach in other strategic initiatives in other strategic initiatives" [p. 15]. As a result of this study, Merrill Lynch also acquired new clients.

20.2-2.

Answers will vary.

20.3-1.

(a)

n	x_n	x_n+3	$\frac{x_n+3}{10}$	x_{n+1}
0	2	5	$\frac{5}{10}$	5
1	5	8	$\frac{8}{10}$	8
2	8	11	$1\frac{1}{10}$	1
3	1	4	$\frac{4}{10}$	4
4	4	7	$\frac{7}{10}$	7
5	7	10	$1\frac{0}{10}$	0
6	0	3	$\frac{3}{10}$	3
7	3	6	$\frac{6}{10}$	6
8	6	9	$\frac{9}{10}$	9
9	9	12	$1\frac{2}{10}$	2

(b)

n	x_n	$5x_n+1$	$\frac{5x_n+1}{8}$	x_{n+1}
0	1	6	$\frac{6}{8}$	6
1	6	31	$3\frac{7}{8}$	7
2	7	36	$4\frac{4}{8}$	4
3	4	21	$2\frac{5}{8}$	5
4	5	26	$3\frac{2}{8}$	2
5	2	11	$1\frac{3}{8}$	3
6	3	16	$1\frac{0}{8}$	0
7	0	1	$\frac{1}{8}$	1

(c)

n	x_n	$61x_n+27$	$\frac{61x_n+27}{100}$	x_{n+1}
0	10	637	$6\frac{37}{100}$	37
1	37	2284	$22\frac{84}{100}$	84
2	84	5151	$51\frac{51}{100}$	51
3	51	3138	$31\frac{38}{100}$	38
4	38	2345	$23\frac{45}{100}$	45

20.3-2.

(a)

$$U_{n+1} = \frac{x_{n+1}+\frac{1}{2}}{10}, n = 0, 1, \ldots, 9$$

(b)

$$U_{n+1} = \frac{x_{n+1}+\frac{1}{2}}{8}, n = 0, 1, \ldots, 7$$

(c)

$$U_{n+1} = \frac{x_{n+1}+\frac{1}{2}}{100}, n = 0, 1, \ldots, 99$$

20.3-3.

n	x_n	$11x_n + 23$	$\frac{11x_n+23}{100}$	x_{n+1}
0	52	595	$5\frac{95}{100}$	95
1	95	1068	$10\frac{68}{100}$	68
2	68	771	$7\frac{71}{100}$	71
3	71	804	$8\frac{4}{100}$	4
4	4	67	$\frac{67}{100}$	67

20.3-4.

n	x_n	$201x_n + 503$	$\frac{201x_n+503}{1000}$	x_{n+1}
0	485	97988	$97\frac{988}{1000}$	988
1	988	199091	$199\frac{91}{1000}$	91
2	91	18794	$18\frac{794}{1000}$	794

20.3-5.

(a)

n	x_n	$13x_n + 15$	$\frac{13x_n+15}{32}$	x_{n+1}
0	14	197	$6\frac{5}{32}$	5
1	5	80	$2\frac{16}{32}$	16
2	16	223	$6\frac{31}{32}$	31
3	31	418	$13\frac{2}{32}$	2
4	2	41	$1\frac{9}{32}$	9

(b)

$$U_{n+1} = \frac{x_{n+1}+\frac{1}{2}}{32}, n = 0, 1, \ldots, 4 \Rightarrow (0.1719, 0.5156, 0.9844, 0.0781, 0.2696)$$

20.3-6.

(a) $x_1 = 7, x_2 = 10, x_3 = 5, x_4 = 9, x_5 = 11, x_6 = 12,$

$x_7 = 6, x_8 = 3, x_9 = 8, x_{10} = 4, x_{11} = 2, x_{12} = 1$

(b) Each integer appears only once in part (a).

(c) $x_{13}, x_{14}, \ldots$ will repeat the cycle $x_1, \ldots, x_{12}$ with length 12.

20.4-1.

(a) Answers will vary.

(b) The formula in cell D10 is $=$ VLOOKUP(C10, \$J\$8:\$K\$9, 2).

	B	C	D	E	F	G	H	I	J	K
3	Summary of Game							Distribution of		
4	Number of Flips =			2				Coin Flips		
5	Winnings =			$6						Result
6										(0=Tails,
7			Result					Probability	Cumulative	1=Heads)
8		Random	(0=Tails,	Total	Total			0.5	0	0
9	Flip	Number	1=Heads)	Heads	Tails	Stop?		0.5	0.5	1
10	1	0.1921	0	0	1					
11	2	0.4894	0	0	2					
12	3	0.4593	0	0	3	Stop				
13	4	0.1960	0	0	4	NA				
14	5	0.4498	0	0	5	NA				
58	49	0.4746	0	21	28	NA				
59	50	0.7349	1	22	28	NA				

(c) A simulation with 14 replications:

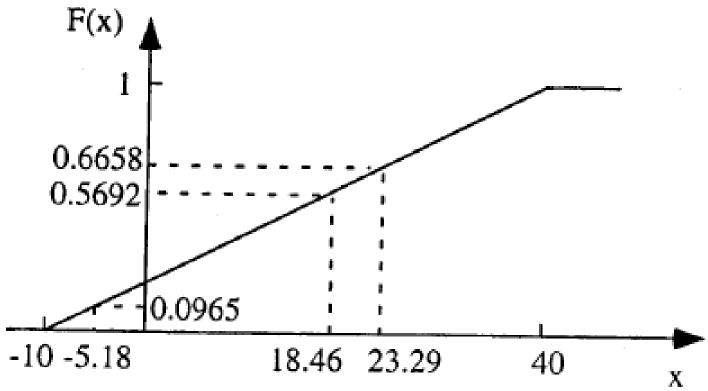
	Number	
Play	of Flips	Winnings
	7	$1
1	11	-$3
2	5	$3
3	5	$3
4	9	-$1
5	7	$1
6	7	$1
7	5	$3
8	3	$5
9	17	-$9
10	5	$3
11	5	$3
12	3	$5
13	9	-$1
14	7	$1
Avg.	7	$1

(d) A simulation with 1000 replications:

	Number	
Play	of Flips	Winnings
	11	-$3
1	5	$3
2	13	-$5
3	15	-$7
4	5	$3
5	19	-$11
6	11	-$3
7	3	$5
8	3	$5
9	3	$5
10	7	$1
995	5	$3
996	3	$5
997	3	$5
998	3	$5
999	21	-$13
1000	5	$3
Avg.	9.15	$1.15

20.4-2.

(a)



(b) $F(x) = \frac{x+10}{50} \Rightarrow F(-5.18) = 0.0965, F(18.46) = 0.5692, F(23.29) = 0.6658$

(c) If cell A1 contains the uniform random number, then the Excel function is " $= 50*\text{A1} - 10$."

20.4-3.

(a) $r = P\{X \leq x\} = \int_{25}^{x} \frac{dt}{50} = \frac{x-25}{50} \Rightarrow x = 50r + 25$

r	X
0.096	29.80
0.569	53.45
0.665	58.25

(b) $r = P\{X \leq x\} = \int_{-1}^{x} \frac{(t+1)^3}{4} dt = \frac{(x+1)^4}{16} \Rightarrow x = 2r^{1/4} - 1$

r	X
0.096	0.113
0.569	0.737
0.665	0.806

(c) $r = P\{X \leq x\} = \int_{40}^{x} \frac{(t-40)}{200} dt = \frac{(x-40)^2}{400} \Rightarrow x = 20(2 + \sqrt{r})$

r	X
0.096	46.197
0.569	55.086
0.665	56.310

20.4-4.

(a) To determine whether $X = 0$ or X is distributed uniformly between -5 and 15, look at a three-digit random number from Table 20.3.

$000 \leq r \leq 499 \Rightarrow X = 0.$
$500 \leq r \leq 999 \Rightarrow X$ is uniformly distributed.

If $X = 0$, nothing else need to be done. Otherwise, use the next three-digit random number as a decimal to generate X.

$$r = P\{X \leq x\} = \int_{-5}^{x} \frac{dt}{20} = \frac{x+5}{20} \Rightarrow x = 20r - 5$$

r	
0.096	$X_1 = 0$
0.569	$X_2 \sim U(-5, 15)$
0.665	$X_2 = 20(0.665) - 5 = 8.3$
0.764	$X_3 \sim U(-5, 15)$
0.842	$X_3 = 20(0.842) - 5 = 11.84$

Hence, the sequence is $(0, 8.3, 11.84)$.

(b)

$$P\{1 \leq X \leq 2\} = \int_1^2 (t-1)dt = \frac{1}{2},\ P\{2 \leq X \leq 3\} = \int_2^3 (3-t)dt = \frac{1}{2}$$

For $0 \leq r \leq ½$, $r = \int_1^x (t-1)dt = \frac{(x-1)^2}{2} \Rightarrow x = \sqrt{2r} + 1$.

For $½ \leq r \leq 1$, $r = \frac{1}{2} + \int_2^x (3-t)dt = \frac{1}{2} - \frac{(3-x)^2}{2} \Rightarrow x = 3 - \sqrt{2-2r}$.

r	X
0.096	1.438
0.569	2.072
0.665	2.181

(c) Let Z be a Bernoulli random variable with $p = 1/3$, i.e., $P\{Z = 1\} = 1/3$ and $P\{Z = 0\} = 2/3$. Then, X is a random variable denoting the number of trials until the Bernoulli random variable takes the value 1.

$000 \leq r \leq 332 \Rightarrow Z = 1.$
$333 \leq r \leq 999 \Rightarrow Z = 0.$

r	Z	X
096	1	1
569	0	
665	0	
764	0	
842	0	
492	0	
224	1	6
950	0	
610	0	
145	1	3

Hence, the sequence is $(1, 6, 3)$.

20.4-5.

(a) Answers will vary.

(b) 0.0000 to 0.4999 correspond to heads.
0.5000 to 0.9999 correspond to tails.

Group 1: HHH, Group 2: THH, Group 3: HTT, Group 4: THT,
Group 5: THH, Group 6: HHT, Group 7: THT, Group 8: TTH
Number of groups with 0 heads: 0
Number of groups with 1 heads: 4
Number of groups with 2 heads: 3
Number of groups with 3 heads: 1

(c)

Flip	Random Number	Result
1	0.6447	Heads
2	0.6897	Heads
3	0.1961	Tails

Total number of heads: 2

(d) Answers will vary. The following eight replications have two replications with no heads $(1/4)$, four replications with one heads $(1/2)$, one replication with two heads $(1/8)$, and one replication with three heads $(1/8)$. This is not very close to the expected probability distribution.

Replication	1	2	3	4	5	6	7	8
Number of Heads	1	3	2	0	1	1	1	0

(e) Answers will vary. Among the following 800 replications, 93 have no heads $(93/800)$, 303 have one heads $(303/800)$, 309 have two heads $(309/800)$, and 95 have three heads $(95/800)$. This is quite close to the expected probability distribution.

Replication	1	2	3	4	5	798	799	800
Number of Heads	1	2	1	1	1	0	2	1

20.4-6.

(a)

Summary of Results:

Win? (1=Yes, 0=No)	0
Number of Tosses =	3

Simulated Tosses

Toss	Die 1	Die 2	Sum
1	4	2	6
2	3	2	5
3	6	1	7
4	5	2	7
5	4	4	8
6	1	4	5
7	2	6	8

Results

Win?	Lose?	Continue?
0	0	Yes
0	0	Yes
0	1	No
NA	NA	No
NA	NA	No
NA	NA	No
NA	NA	No

(b) Answers will vary. Below is the results from a 25-replication simulation.

Game	1	2	3	4	5	6	7	8	$9,\ldots,15$	16	17	18	19	$20,\ldots,24$	25
Win?	0	0	1	0	0	0	0	1	0	1	0	1	0	1	0

(c) 9 wins and 16 loses $\Rightarrow P\{\text{win}\} = 9/25$ and $P\{\text{lose}\} = 16/25$

(d)

$$\frac{\overline{X}-0.493}{0.5/\sqrt{n}} \sim N(0,1) \Rightarrow P\left\{\frac{\overline{X}-0.493}{0.5/\sqrt{n}} \leq 1.64\right\} = 0.95$$

$$\Rightarrow P\left\{\overline{X} \leq \frac{0.82}{\sqrt{n}} + 0.493\right\} = 0.95$$

$$\frac{0.82}{\sqrt{n}} + 0.493 = 0.5 \Rightarrow n = 13.689$$

20.4-7.

$$r = P\{X \leq x\} = P\left\{\frac{X-1}{2} \leq \frac{x-1}{2}\right\} = 1 - \Phi\left(\frac{x-1}{2}\right) \Rightarrow x = 2\Phi^{-1}(1-r) + 1$$

We can use r directly instead of $1-r$, since both have uniform distribution. The following values $\Phi^{-1}(r)$ are obtained in Excel using the function NORMINV$(r, 0, 1)$.

r	$\Phi^{-1}(r)$	x
0.096	−1.305	−1.609
0.569	0.174	1.348
0.665	0.426	1.852
0.764	0.719	2.438
0.842	1.003	3.005
0.492	−0.020	0.960
0.224	−0.759	−0.518
0.950	1.645	4.290
0.610	0.279	1.559
0.145	−1.058	−1.116

Average: 1.221

20.4-8.

(a)

r_i^1	r_i^2	r_i^3
0.096	0.764	0.224
0.569	0.842	0.950
0.665	0.492	0.610

$\sum_{i=1}^{3} r_i^k$	$x_k = 20\left(\sum_{i=1}^{3} r_i^k\right) - 25$
1.330	1.6
2.098	17.0
1.784	10.7

(b) $x = 5\Phi^{-1}(r) + 10$

r	$\Phi^{-1}(r)$	x
0.096	−1.305	3.475
0.569	0.174	10.870
0.665	0.426	12.130

20.4-9.

(a)

r_i^1	r_i^2	r_i^3	r_i^4
0.096	0.764	0.224	0.145
0.569	0.842	0.950	0.484
0.665	0.492	0.610	0.552

$\sum_{i=1}^{3} r_i^k$	$x_k = 2\left(\sum_{i=1}^{3} r_i^k\right) - 3$
1.330	-0.340
2.098	1.196
1.784	0.568
1.181	-0.638

Let z_i denote the chi-square observations, for $i = 1, 2$. Then

$$z_1 = x_1^2 + x_2^2 = 1.546 \text{ and } z_2 = x_3^2 + x_4^2 = 0.730.$$

(b)

r	$\Phi^{-1}(r)$
0.096	-1.305
0.569	0.174
0.665	0.426
0.764	0.719

(c) $Y = X_1^2 + X_2^2$

From (a), $Y_1 = 1.546$ and $Y_2 = 0.730$.
From (b), $Y_1 = 1.733$ and $Y_2 = 0.698$.

20.4-10.

(a)

r	$x = -10\ln(r)$
0.096	23.434
0.569	5.639

(b)

r_1	r_2	$x = -5\ln(r_1 r_2)]$
0.096	0.569	14.536
0.665	0.764	3.386

(c)

r_i^1	r_i^2
0.096	0.224
0.569	0.950
0.665	0.610
0.764	0.145
0.842	0.484
0.492	0.552

$\sum_{i=1}^{6} r_i^k$	$x_k = 4\left(\sum_{i=1}^{6} r_i^k\right) - 2$
3.428	11.71
2.965	9.86

20.4-11.

(a)

Uniform Random Number	Random Observation
0.2655	9.22
0.3472	9.49
0.0248	7.25
0.9205	12.21
0.6130	10.38

(b) If cell C4 contains the uniform random number, then the Excel function would be:
$=$ IF(C4<0.2, 7+(2/0.2)*C4, IF(C4<0.8, 9+(2/0.6)*(C4-0.2), 11+(2/0.2)*(C4-0.8))).

20.4-12.

r	$x = -20\ln(r)$
0.096	46.868
0.569	11.278
0.665	8.159
0.764	5.384

Hence, the Erlang observation is $\sum_{i=1}^{4} x_k = 71.689$.

20.4-13.

(a) TRUE. Both r_i and $1 - r_i$ are uniformly distributed.

(b) FALSE. Numerically, $\prod r_i \neq \prod (1 - r_i) \Rightarrow \sum x_i \neq \sum y_i$.

(c) TRUE. The sum of independent exponential random variables each with the same mean has Erlang distribution.

20.4-14.

(a) It is not valid, since $P\{x_i = 8\} = P\{8/8 \leq r_i < 9/8\} = 0$. Replace n by $n - 1$ to make it a valid method. Generate uniform random numbers r_i and set $x_i = n$ where n satisfies $(n-1)/8 \leq r_i < n/8$.

(b) It is valid. When $(n-1)/8 \leq r_i < n/8$, $n \leq 1 + 8r_i < n + 1$.

(c) It is not valid, since $x_0' = 4$, $x_1' = 3$, $x_2' = 6$, $x_3' = 5$, $x_4' = 0$, $x_5' = 7$, $x_6' = 2$, $x_7' = 1$, and $x_8' = 4$, so this method does not cover the number 8. Instead, let $x_i = x_i' + 1$, then it is a valid method.

20.4-15.

r_1	x	r_2	$f(x)$	Accept?
0.096	0.192	0.569	0.192	No
0.665	1.330	0.764	0.670	No
0.842	1.684	0.492	0.316	No
0.224	0.448	0.950	0.448	No
0.610	1.220	0.145	0.780	Yes
0.484	0.968	0.552	0.968	Yes
0.350	0.700	0.590	0.700	Yes

The three samples from the triangular distribution are $1.220, 0.968$, and 0.700.

20.4-16.

Let $x = 10r_1 + 10$.

r_1	x	r_2	$f(x)$	Accept?
0.096	10.96	0.569	0.0192	No
0.665	16.65	0.764	0.1350	No
0.842	18.42	0.492	0.1684	No
0.224	12.24	0.950	0.0448	No
0.610	16.10	0.145	0.1220	No
0.484	14.84	0.552	0.0968	No
0.350	13.50	0.590	0.0700	No
0.430	14.30	0.041	0.0860	Yes
0.802	18.02	0.471	0.1604	No
0.255	12.55	0.799	0.0510	No
0.608	16.08	0.577	0.1216	No
0.347	13.47	0.933	0.0694	No
0.581	15.81	0.173	0.1162	No
0.603	16.03	0.040	0.1206	Yes
0.605	16.05	0.842	0.1210	No
0.720	17.20	0.449	0.1440	No
0.076	10.76	0.407	0.0152	No
0.202	12.02	0.963	0.0404	No
0.412	14.12	0.369	0.0824	No
0.976	19.76	0.171	0.1952	Yes

The three samples from the given distribution are $14.30, 16.03$, and 19.76.

20.4-17.

$$\text{size of risk} = \begin{cases} 0 & \text{if } 0 \leq U < 0.7 \\ 1 & \text{if } 0.7 \leq U < 0.9 \\ 2 & \text{if } 0.9 \leq U < 1 \end{cases}$$

$$\text{size of loss } x = \begin{cases} (20U)^2 & \text{if } 0 \leq U < ½ \\ 200U & \text{if } U \geq ½ \end{cases}$$

Run 1		Run 2	
U	size	U	size
0.096	0	0.492	0
0.569	0	0.224	0
0.665	0	0.950	2
0.764	1	0.610	0

U	x
0.842	164.4

U	x
0.145	8.41
0.484	91.09

Total loss: $\sum_{i=1}^{4} I_{(\text{size}>0)} \sum_{j=1}^{\text{size}} x_{ij}$

Two simulation runs give 164.4 and 99.5. Actually, 100 runs give 145.

20.4-18.

Since the number N of employees incurring medical expenses has a binomial distribution with $p = 0.9$ and $n = 3$:

$$P\{N = 0\} = C_3^0 \cdot 0.9^0 \cdot 0.1^3 = 0.001,$$

$$P\{N = 1\} = C_3^1 \cdot 0.9^1 \cdot 0.1^2 = 0.027,$$

$$P\{N = 2\} = C_3^2 \cdot 0.9^2 \cdot 0.1^1 = 0.243,$$

$$P\{N = 3\} = C_3^3 \cdot 0.9^3 \cdot 0.1^0 = 0.729.$$

Let $p_0 = 0, p_1 = 0.001, p_2 = 0.028, p_3 = 0.271, p_4 = 1.$

$$N = i \text{ if } p_i \leq U < p_{i+1}$$

$$0.01 \Rightarrow N = 1, 0.20 \Rightarrow N = 2$$

$$\text{Total amount} = \begin{cases} 100 & \text{if } 0 \leq U < 0.9 \\ 10,000 & \text{if } 0.9 \leq U < 1 \end{cases}$$

Only 0.95 causes an actual payment from the insurance company and the total payment is $\$5,000$.

20.5-1.

AT&T uses a discrete event simulation model to simulate inbound call centers. "The call processing simulator (CAPS) simulates the interactive behavior of the operational variables in inbound call centers. AT&T uses CAPS to propose optimal staffing, trunking (number of phone lines), network routing, and premises routing. CAPS can demonstrate cost/benefit trade-offs and can show the implications of good versus bad service levels. It can also show the effects of proposed operational changes in an inbound call center, using what-if scenarios" [p. 9]. Simulation allows modeling complex systems and evaluating different modes of operation without changing the actual operation of the call center. Three steps of the CAPS process are data collection, data case simulation and alternative scenario simulation.

As a result of this study, "AT&T has increased, protected, and regained more than $1 billion from a business customer base of about 2,000 accounts per year. Much of its effective market and revenue-share management results from using CAPS to demonstrate advanced 800 network features. CAPS is vital to the marketability of such new and exclusive AT&T 800 network offerings" [p. 20]. The revenues are increased also from equipment sales to business customers whose need for new equipment is shown by CAPS studies. This study improved AT&T's consultative profile, its credibility and the profitability of 800 services. It also reduced the access charges and the overhead. The benefits of the study for business customers include reduced overhead costs due to the reduction in labor costs, increased call completion, reduced queue time, optimal utilization of operators, increased revenues, higher customer satisfaction and repeated sales.

20.6-1.

(a) Answers will vary. A typical set of 5 runs: $(45.72, 44.24, 46.68, 46.24, 47.90)$

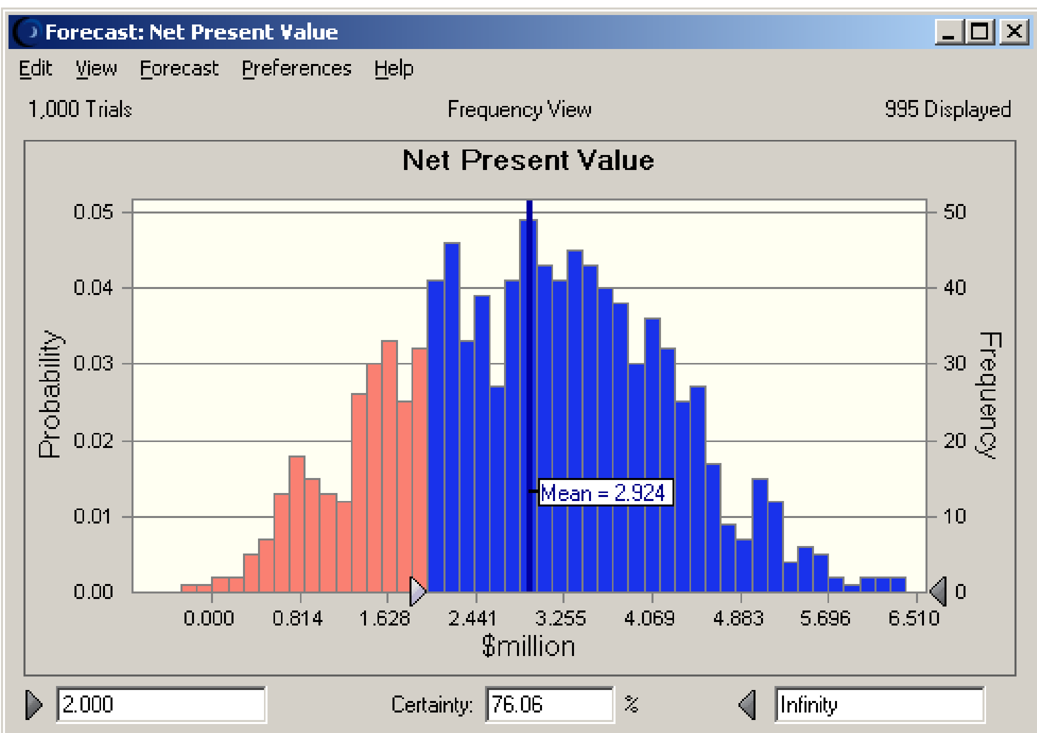
(b) Answers will vary. A typical set of 5 runs: $(46.60, 47.06, 46.67, 46.76, 46.84)$

(c) The mean profits in part (b) seem to be more consistent.

20.6-2.

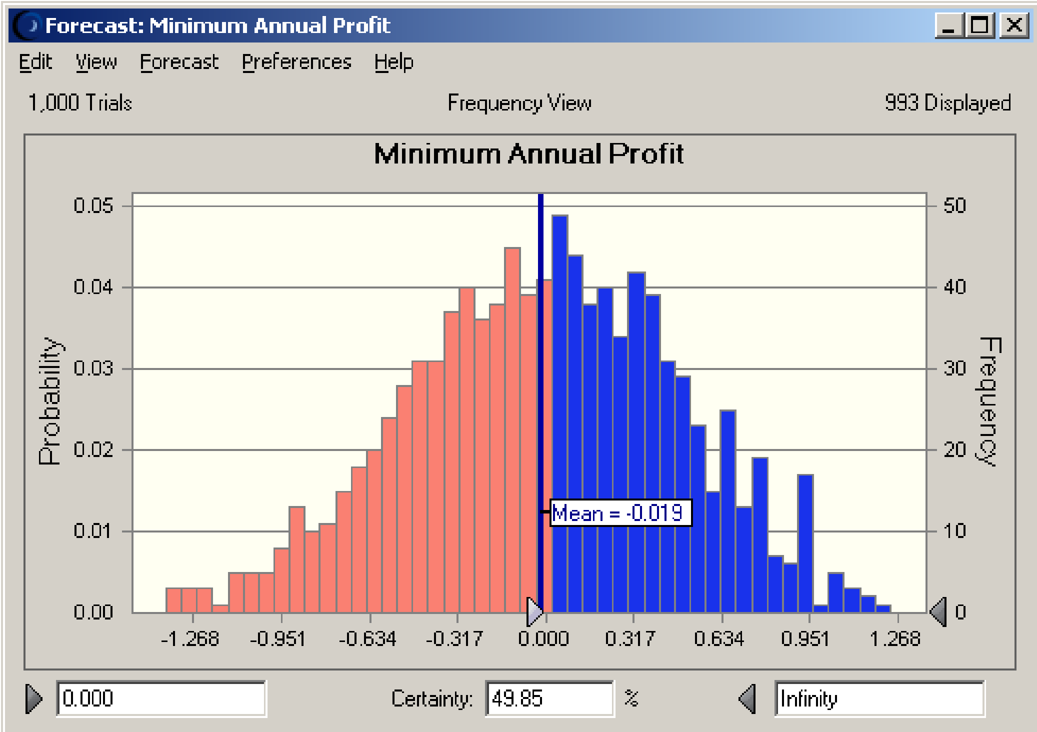
	A	B	C	D	E	F	G	H	I	J	K
1						Now	Year 1	Year 2	Year 3	Year 4	Year 5
2	Land Purchase	Fixed				-1					
3	Construction Cost	Triangular(min,likely,max)	-2.4	-2	-1.6		-2				
4	Operating Profit	Normal(mean,s.dev.)	0.7	0.7				0.7	0.7	0.7	0.7
5	Selling Price	Uniform(min,max)	4	8							6
6											
7					Total Cash Flow	-1	-2	0.7	0.7	0.7	6.7
8											
9					Discount Factor	10%					
10											
11					Net Present Value ($million)	2.925					
12											
13					Minimum Annual Operating Profit ($million in y2-y5)	0.700					

(a) The mean NPV is approximately $2.9 million.



(b) The probability that the NPV will be at least $2 million is approximately 77%.

(c) The mean value of the minimum annual operating profit is approximately zero.

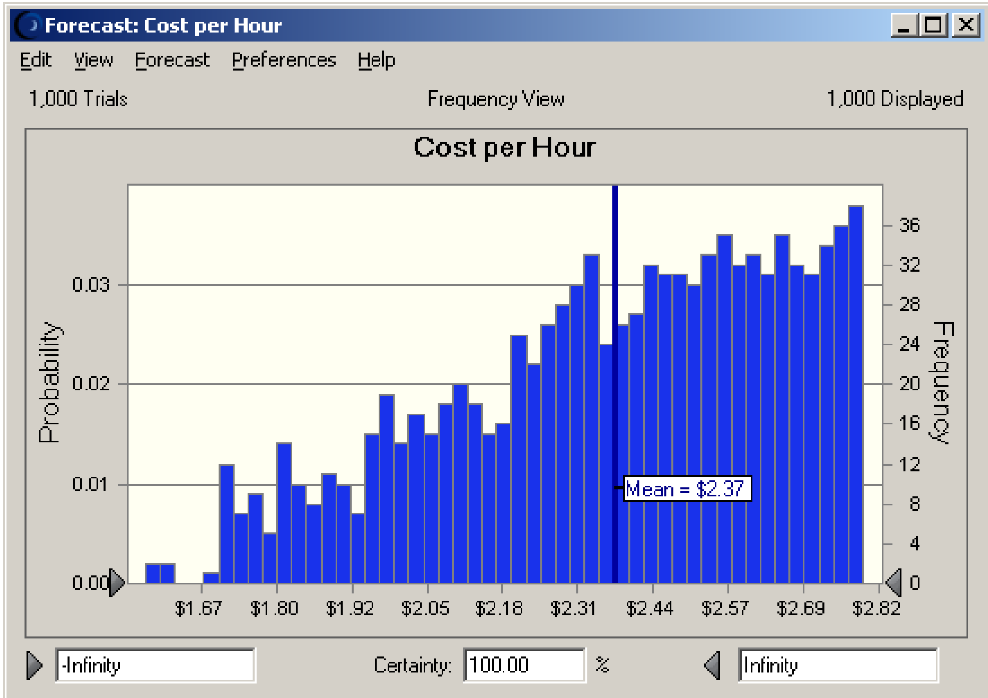


(d) The probability that the minimum annual operating profit will be at least zero in all four years of operation is approximately 49.9%.

20.6-3.

The expected cost with the proposed system of replacing all relays whenever any one of them fails is approximately \$2.37 per hour. This is cheaper than the current system of replacing each relay as it fails. Therefore, they should replace all four relays with the first failure.

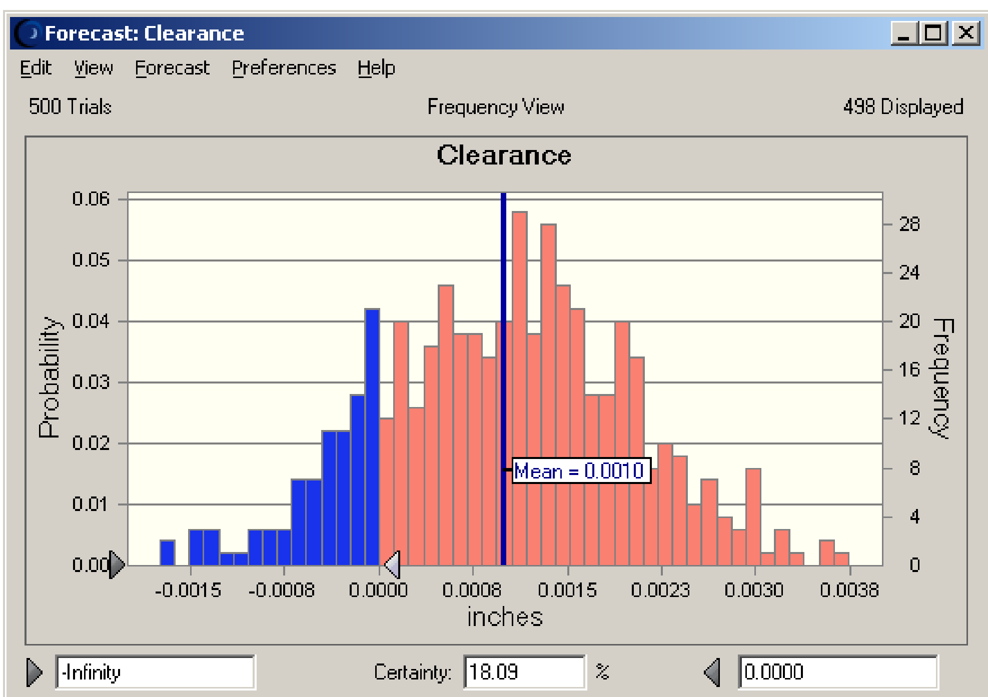
	A	B	C	D	E
1		Time to			
2		Failure			
3		(hours)		Min	Max
4	Relay 1	1,500	Uniform	1,000	2,000
5	Relay 2	1,500	Uniform	1,000	2,000
6	Relay 3	1,500	Uniform	1,000	2,000
7	Relay 4	1,500	Uniform	1,000	2,000
8					
9	Time to First Failure	1,500			
10	Time to End of Shutdown	1,502			
11	Total Cost	\$2,800			
12	Cost per Hour	\$1.86			



20.6-4.

The chance of negative clearance is approximately 18.4%.

	A	B	C	D	E	F
1	Shaft Radius	1.001	Triangular(min,likely,max)	1.000	1.001	1.002
2	Bushing Radius	1.002	Normal(mean,st.dev.)	1.002	0.001	
3						
4	Clearance	0.0010				



20.6-5.

	A	B	C	D	E	F	G	H	I
1	Toss	Die 1	Die 2	Sum	Win?	Lose?	Continue?		Win
2	1	4	4	7	Yes	No	No		Game?
3	2	4	4	7	#N/A	#N/A	#N/A		(1=yes,0=no)
4	3	4	4	7	#N/A	#N/A	#N/A		1
5	4	4	4	7	#N/A	#N/A	#N/A		
6	5	4	4	7	#N/A	#N/A	#N/A		
7	6	4	4	7	#N/A	#N/A	#N/A		
8	7	4	4	7	#N/A	#N/A	#N/A		
9	8	4	4	7	#N/A	#N/A	#N/A		
10	9	4	4	7	#N/A	#N/A	#N/A		
11	10	4	4	7	#N/A	#N/A	#N/A		
12	11	4	4	7	#N/A	#N/A	#N/A		
13	12	4	4	7	#N/A	#N/A	#N/A		
14	13	4	4	7	#N/A	#N/A	#N/A		
15	14	4	4	7	#N/A	#N/A	#N/A		
16	15	4	4	7	#N/A	#N/A	#N/A		
17	16	4	4	7	#N/A	#N/A	#N/A		
18	17	4	4	7	#N/A	#N/A	#N/A		
19	18	4	4	7	#N/A	#N/A	#N/A		
20	19	4	4	7	#N/A	#N/A	#N/A		
21	20	4	4	7	#N/A	#N/A	#N/A		
22	21	4	4	7	#N/A	#N/A	#N/A		
23	22	4	4	7	#N/A	#N/A	#N/A		
24	23	4	4	7	#N/A	#N/A	#N/A		
25	24	4	4	7	#N/A	#N/A	#N/A		
26	25	4	4	7	#N/A	#N/A	#N/A		
27	26	4	4	7	#N/A	#N/A	#N/A		
28	27	4	4	7	#N/A	#N/A	#N/A		
29	28	4	4	7	#N/A	#N/A	#N/A		
30	29	4	4	7	#N/A	#N/A	#N/A		
31	30	4	4	7	#N/A	#N/A	#N/A		

(a) Answers will vary. The mean standard error is approximately 0.05, so the typical values should be between 0.44 and 0.54.

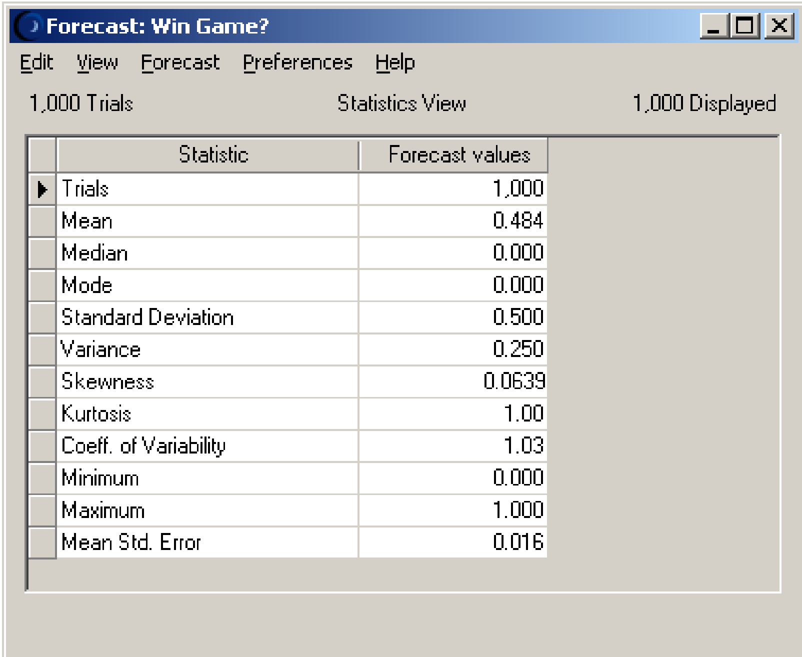
Forecast: Win Game?

Edit View Forecast Preferences Help

100 Trials Statistics View 100 Displayed

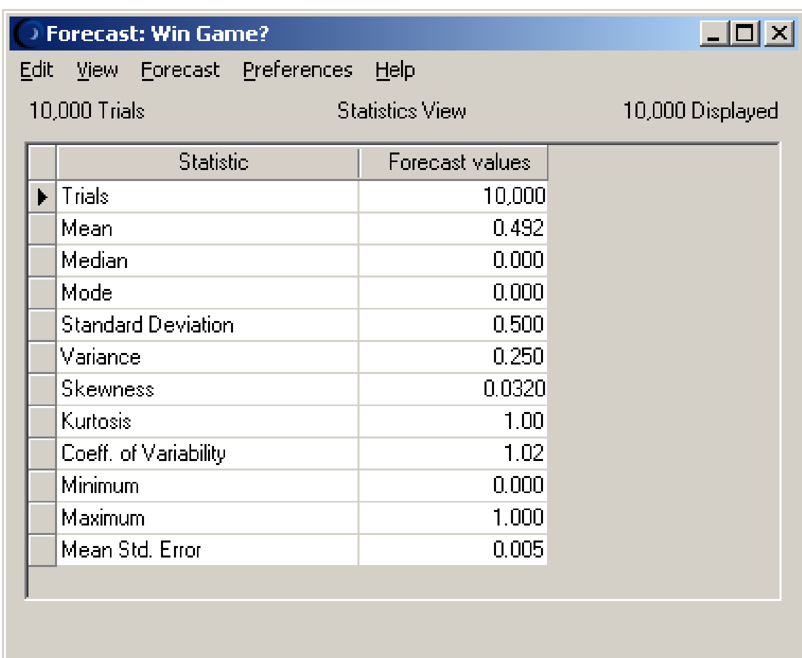
Statistic	Forecast values
Trials	100
Mean	0.450
Median	0.000
Mode	0.000
Standard Deviation	0.500
Variance	0.250
Skewness	0.1980
Kurtosis	1.02
Coeff. of Variability	1.11
Minimum	0.000
Maximum	1.000
Mean Std. Error	0.050

(b) Answers will vary. The mean standard error is approximately 0.016, so the typical values should be between 0.476 and 0.509.



Statistic	Forecast values
Trials	1,000
Mean	0.484
Median	0.000
Mode	0.000
Standard Deviation	0.500
Variance	0.250
Skewness	0.0639
Kurtosis	1.00
Coeff. of Variability	1.03
Minimum	0.000
Maximum	1.000
Mean Std. Error	0.016

(c) Answers will vary. The mean standard error is approximately 0.005, so the typical values should be between 0.487 and 0.497.



Statistic	Forecast values
Trials	10,000
Mean	0.492
Median	0.000
Mode	0.000
Standard Deviation	0.500
Variance	0.250
Skewness	0.0320
Kurtosis	1.00
Coeff. of Variability	1.02
Minimum	0.000
Maximum	1.000
Mean Std. Error	0.005

(d) Answers will vary. There is a fair amount of variability in the number of wins, so a large number of iterations, say $10,000$, is necessary to predict the true probability. With $10,000$ iterations, the mean standard error is less than 0.007.

20.6-6.

The order quantity that maximizes the mean profit is approximately 55.

Order Quantity (50)	Order Quantity (51)	Order Quantity (52)	Order Quantity (53)	Order Quantity (54)	Order Quantity (55)	Order Quantity (56)	Order Quantity (57)	Order Quantity (58)	Order Quantity (59)	Order Quantity (60)
$46.67	$46.97	$47.20	$47.36	$47.46	$47.50	$47.46	$47.36	$47.20	$46.96	$46.66

20.7-1.

Answers will vary.

20.7-2.

Answers will vary.

Cases

20-1 a) Status quo at the presses – 7.5 sheets of in-process inventory.

	A	B	C	D	E	F	G	H
1	**Template for Queueing Simulation**							
2								
3			**Data**				**Results**	
4		Number of Servers =	10			Point	95% Confidence Interval	
5						Estimate	Low	High
6		**Interarrival Times**			$L =$	7.48596004	7.122474949	7.849445126
7		Distribution =	Exponential		$L_q =$	0.55020043	0.347368991	0.753031867
8		Mean =	0.142857143		$W =$	1.0770836	1.036422591	1.117744603
9					$W_q =$	0.07916311	0.050901621	0.107424593
10								
11		**Service Times**			$P_0 =$	0.00110924	0.000312762	0.001905718
12		Distribution =	Exponential		$P_1 =$	0.00582387	0.003292739	0.008355008
13		Mean =	1		$P_2 =$	0.02306409	0.018701971	0.027426208
14					$P_3 =$	0.05166684	0.043052172	0.060281501
15					$P_4 =$	0.0866959	0.077527167	0.09586463
16		**Length of Simulation Run**			$P_5 =$	0.12118604	0.112124348	0.130247735
17		Number of Arrivals =	10,000		$P_6 =$	0.14062225	0.13442836	0.14681614
18					$P_7 =$	0.14294653	0.134902634	0.150990419
19					$P_8 =$	0.12452751	0.11900339	0.130051626
20					$P_9 =$	0.08806336	0.084082813	0.092043901
21		Run Simulation			$P_{10} =$	0.06192446	0.055935883	0.067913035

Status quo at the inspection station – 3.6 wing sections of in-process inventory.

	A	B	C	D	E	F	G	H
1	**Template for Queueing Simulation**							
2								
3			**Data**				**Results**	
4		Number of Servers =	1			Point	95% Confidence Interval	
5						Estimate	Low	High
6		**Interarrival Times**			$L =$	3.57765981	3.096884037	4.058435589
7		Distribution =	Exponential		$L_q =$	2.71234549	2.244158962	3.180532014
8		Mean =	0.142857143		$W =$	0.51681506	0.454627294	0.57900283
9					$W_q =$	0.39181506	0.329627294	0.45400283
10								
11		**Service Times**			$P_0 =$	0.13468567	0.118076335	0.151295015
12		Distribution =	Constant		$P_1 =$	0.18444199	0.164766618	0.204117359
13		Value =	0.125		$P_2 =$	0.16054199	0.145653686	0.175430299
14					$P_3 =$	0.12577666	0.114607169	0.136946159
15					$P_4 =$	0.09279878	0.083029162	0.102568391
16		**Length of Simulation Run**			$P_5 =$	0.07546784	0.065828648	0.085107034
17		Number of Arrivals =	10,000		$P_6 =$	0.0548405	0.045754492	0.063926513
18					$P_7 =$	0.04326737	0.033313657	0.053221074
19					$P_8 =$	0.03643173	0.026094365	0.046769093
20					$P_9 =$	0.02983638	0.020206033	0.039466733
21		Run Simulation			$P_{10} =$	0.02245891	0.014710033	0.030207788

Inventory cost = (7.5 + 3.6)($8/hour) = $88.80 / hour
Machine cost = (10)($7/hour) = $70 / hour
Inspector cost = $17 / hour

Total cost = $175.80 / hour

b) Proposal 1 will increase the in-process inventory at the presses to 10.6 sheets since the mean service rate has decreased.

	A	B	C	D	E	F	G	H
1	**Template for Queueing Simulation**							
2								
3			**Data**				**Results**	
4		Number of Servers =	10			Point	95% Confidence Interval	
5						Estimate	Low	High
6		**Interarrival Times**			$L =$	10.6124208	10.07045277	11.15438883
7		Distribution =	Exponential		$L_q =$	2.34034351	1.812410733	2.868276277
8		Mean =	0.142857143		$W =$	1.5192496	1.422904809	1.615594383
9					$W_q =$	0.33503816	0.255248897	0.41482742
10								
11		**Service Times**			$P_0 =$	0.00034416	-0.000135983	0.000824295
12		Distribution =	Exponential		$P_1 =$	0.00330079	0.002146705	0.004454878
13		Mean =	1.2		$P_2 =$	0.00683624	0.005191338	0.008481139
14					$P_3 =$	0.0225304	0.017788623	0.027272181
15					$P_4 =$	0.0437143	0.041059108	0.0463695
16		**Length of Simulation Run**			$P_5 =$	0.06530488	0.058747044	0.071862716
17		Number of Arrivals =	10,000		$P_6 =$	0.08305601	0.074729232	0.091382794
18					$P_7 =$	0.09066307	0.081970997	0.099355138
19					$P_8 =$	0.09495054	0.09393376	0.095967318
20					$P_9 =$	0.09944674	0.090813615	0.108079863
21		Run Simulation			$P_{10} =$	0.08672109	0.077951648	0.095490525

The in-process inventory at the inspection station will not change.

Inventory cost = (10.6 + 3.6)($8/hour) = $113.60 / hour
Machine cost = (10)($6.50) = $65 / hour
Inspector cost = $17 / hour

Total cost = $195.60 / hour

This total cost is higher than for the status quo so should not be adopted. The main reason for the higher cost is that slowing down the machines won't change in-process inventory for the inspection station.

c) Proposal 2 will increase the in-process inventory at the inspection station to 4.2 wing sections since the variability of the service rate has increased.

	A	B	C	D	E	F	G	H
1	**Template for Queueing Simulation**							
2								
3			**Data**				**Results**	
4		Number of Servers =	1			Point	95% Confidence Interval	
5						Estimate	Low	High
6		**Interarrival Times**			L =	4.15349196	3.51945922	4.787524705
7		Distribution =	Exponential		L_q =	3.31066782	2.691612222	3.929723426
8		Mean =	0.142857143		W =	0.58953022	0.506614288	0.672446148
9					W_q =	0.46990309	0.387653637	0.552152552
10								
11		**Service Times**			P_0 =	0.15717586	0.13797724	0.176374483
12		Distribution =	Erlang		P_1 =	0.16164362	0.143938659	0.179348578
13		Mean =	0.12		P_2 =	0.1417251	0.127306603	0.156143599
14		k =	2		P_3 =	0.11157869	0.100074725	0.123082653
15					P_4 =	0.08340382	0.074497166	0.092310469
16		**Length of Simulation Run**			P_5 =	0.0729656	0.064546969	0.081384232
17		Number of Arrivals =	10,000		P_6 =	0.05422094	0.04655526	0.061886616
18					P_7 =	0.04033746	0.033104015	0.047570898
19					P_8 =	0.03068653	0.023437928	0.037935133
20					P_9 =	0.02468793	0.018553583	0.030822285
21		Run Simulation			P_{10} =	0.02288346	0.016278465	0.029488445

The in-process inventory at the presses will not change.

Inventory cost = (7.5 + 4.2)($8/hour) = $93.60 / hour
Machine cost = (10)($7/hour) = $70 / hour
Inspector cost = $17 / hour

Total cost = $180.60 / hour

This total cost is higher than for the status quo so should not be adopted. The main reason for the higher cost is the increase in the service rate variability (Erlang rather than constant) and the resulting increase in the in-process inventory.

d) They should consider *increasing* power to the presses (increasing there cost to $7.50 per hour but reducing their average time to form a wing section to 0.8 hours). This would decrease the in-process inventory at the presses to 5.7.

	A	B	C	D	E	F	G	H
1	**Template for Queueing Simulation**							
2								
3			**Data**				**Results**	
4		Number of Servers =	10			Point	95% Confidence Interval	
5						Estimate	Low	High
6		**Interarrival Times**			$L =$	5.74237458	5.581608211	5.903140941
7		Distribution =	Exponential		$L_q =$	0.11624317	0.076181593	0.156304743
8		Mean =	0.142857143		$W =$	0.81487258	0.801697429	0.828047729
9					$W_q =$	0.01649551	0.011001805	0.021989206
10								
11		**Service Times**			$P_0 =$	0.00445475	0.002433487	0.00647602
12		Distribution =	Exponential		$P_1 =$	0.0241519	0.019051394	0.0292524
13		Mean =	0.8		$P_2 =$	0.06075455	0.0522877	0.069221409
14					$P_3 =$	0.10828334	0.096234	0.120332681
15					$P_4 =$	0.14577459	0.138731319	0.152817867
16		**Length of Simulation Run**			$P_5 =$	0.1580859	0.148929657	0.167242144
17		Number of Arrivals =	10,000		$P_6 =$	0.14882682	0.137378613	0.160275035
18					$P_7 =$	0.12347465	0.116102784	0.13084652
19					$P_8 =$	0.0909915	0.084900257	0.097082738
20					$P_9 =$	0.05514285	0.050413495	0.059872212
21		Run Simulation			$P_{10} =$	0.03360049	0.029185971	0.038015016

Inventory cost = (5.7 + 3.6)($8/hour) = $74.40 / hour
Machine cost = (10)($7.50/hour) = $75 / hour
Inspector cost = $17 / hour

Total cost = $166.40 / hour

This total cost is lower than the status quo and both proposals.

CASE 20.2 Action Adventures

(a) The spreadsheet model is spread over the next several pages:

	A	B	C	D	E	F	G	H	I
1	Cost & Revenue Data					Interest Rate Data			
2	Selling Price	$10			Initial Prime Rate	5%			
3	Replacement Part Cost	$5,000			Loan Rate Prime Gap	2%			
4	Monthly Fixed Cost	$15,000			Loan Rate Maximum	9%			
5	Minimum Balance	$20,000			Savings Rate Prime Gap	-2%			
6	Starting Balance	$25,000			Savings Rate Minimum	2%			
7									
8	**Sales**	Dec	Jan	Feb	Mar	Apr	May	June	July
9	Seasonality Index	1.18	0.79	0.88	0.95	1.05	1.09	0.84	0.74
10	Base Sales	6,000	6,000	6,000	6,000	6,000	6,000	6,000	6,000
11	Actual Sales	7,080	4,740	5,280	5,700	6,300	6,540	5,040	4,440
12	Fraction Cash Customers	42%	39%	39%	39%	39%	39%	39%	39%
13									
14	**Interest Rates**								
15	Prime Rate Change		0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
16	Prime Rate	5.00%	5.00%	5.00%	5.00%	5.00%	5.00%	5.00%	5.00%
17	Loan Interest Rate	7.00%	7.00%	7.00%	7.00%	7.00%	7.00%	7.00%	7.00%
18	Savings Interest Rate	3.00%	3.00%	3.00%	3.00%	3.00%	3.00%	3.00%	3.00%
19									
20	**Manufacturing Costs**								
21	Replacement Parts Needed		0.8	0.8	0.8	0.8	0.8	0.8	0.8
22									
23	Variable Cost		$7	$7	$7	$7	$7	$7	$7
24									
25	**Cash Flows**								
26	Beginning Balance		$25,000	$32,962	$27,479	$23,827	$20,762	$20,533	$26,469
27	Cash Receipts		$18,328	$20,416	$22,040	$24,360	$25,288	$19,488	$17,168
28	30-Day Credit Receipts		$41,064	$29,072	$32,384	$34,960	$38,640	$40,112	$30,912
29	Fixed Cost		-$15,000	-$15,000	-$15,000	-$15,000	-$15,000	-$15,000	-$15,000
30	Total Variable Cost		-$33,180	-$36,960	-$39,900	-$44,100	-$45,780	-$35,280	-$31,080
31	Repair Cost		-$4,000	-$4,000	-$4,000	-$4,000	-$4,000	-$4,000	-$4,000
32	Loan Payoff		$0	$0	$0	$0	$0	$0	$0
33	Loan Interest		$0	$0	$0	$0	$0	$0	$0
34	Savings Interest		$750	$989	$824	$715	$623	$616	$794
35	Balance Before Loan		$32,962	$27,479	$23,827	$20,762	$20,533	$26,469	$25,263
36	New Loan		$0	$0	$0	$0	$0	$0	$0
37	Ending Balance	$25,000	$32,962	$27,479	$23,827	$20,762	$20,533	$26,469	$25,263
38			>=	>=	>=	>=	>=	>=	>=
39	Minimum Balance		$20,000	$20,000	$20,000	$20,000	$20,000	$20,000	$20,000
40									
41	Ending Net Worth	$57,681							
42									
43	Maximum Loan	$7,507							

	A	B	C
1		**Cost & Revenue Data**	
2	Selling Price	10	
3	Replacement Part Cost	5000	
4	Monthly Fixed Cost	15000	
5	Minimum Balance	20000	
6	Starting Balance	25000	
7			
8	**Sales**	Dec	Jan
9	Seasonality Index	1.18	0.79
10	Base Sales	6000	6000
11	Actual Sales	=SeasonalityIndex*BaseSales	=SeasonalityIndex*BaseSales
12	Fraction Cash Customers	0.42	0.386666666666667
13			
14	**Interest Rates**		
15	Prime Rate Change		1.6784487245436E-19
16	Prime Rate	=InitialPrimeRate	=B16+PrimeRateChange
17	Loan Interest Rate	=MIN(PrimeRate+LoanRateGap,LoanRateMax)	=MIN(PrimeRate+LoanRateGap,LoanRateMax)
18	Savings Interest Rate	=MAX(PrimeRate+SavingsRateGap,SavingsRateMin)	=MAX(PrimeRate+SavingsRateGap,SavingsRateMin)
19			
20	**Manufacturing Costs**		
21	Replacement Parts Needed		0.8
22			
23	Variable Cost		7
24			
25	**Cash Flows**		
26	Beginning Balance		=B37
27	Cash Receipts		=ActualSales*FractionCashCustomers*SellingPrice
28	30-Day Credit Receipts		=B11*(1-B12)*SellingPrice
29	Fixed Cost		=-MonthlyFixedCost
30	Total Variable Cost		=-VariableCost*ActualSales
31	Repair Cost		=-ReplacementPartsNeeded*ReplacementPartCost
32	Loan Payoff		=-B36
33	Loan Interest		=-B36*B17
34	Savings Interest		=B37*B18
35	Balance Before Loan		=SUM(C26:C34)
36	New Loan		=IF(BalanceBeforeLoan<=MinimumBalance,MinimumBalance-BalanceBeforeLoan,0)
37	Ending Balance	=StartingBalance	=BalanceBeforeLoan+NewLoan
38			>=
39	Minimum Balance		=MinimumBalance
40			
41	Ending Net Worth	=O35	
42			
43	Maximum Loan	=MAX(NewLoan)	

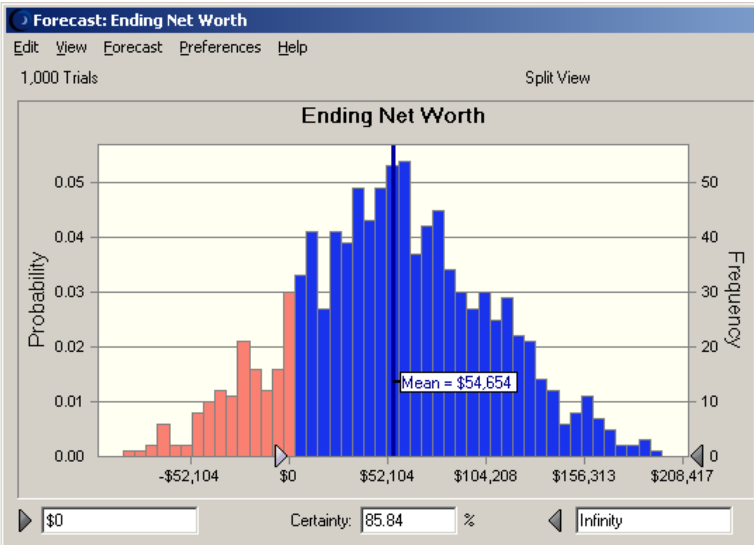
	A	J	K	L	M	N	O	P	Q	R
1										
2	Selling Price									
3	Replacement Part Cost									
4	Monthly Fixed Cost									
5	Minimum Balance									
6	Starting Balance									
7										
8	**Sales**	August	Sept	October	November	December	January			
9	Seasonality Index	0.98	1.06	1.1	1.16	1.18				
10	Base Sales	6,000	6,000	6,000	6,000	6,000	*Normal*	prev mo.	500	
11	Actual Sales	5,880	6,360	6,600	6,960	7,080				
12	Fraction Cash Customers	39%	39%	39%	39%	39%	*Triangular*	28%	40%	48%
13										
14	**Interest Rates**									
15	Prime Rate Change	0.00%	0.00%	0.00%	0.00%	0.00%	*Custom*	-0.50%	0.05	
16	Prime Rate	5.00%	5.00%	5.00%	5.00%	5.00%		-0.25%	0.1	
17	Loan Interest Rate	7.00%	7.00%	7.00%	7.00%	7.00%		0%	0.7	
18	Savings Interest Rate	3.00%	3.00%	3.00%	3.00%	3.00%		0.25%	0.1	
19								0.50%	0.05	
20	**Manufacturing Costs**									
21	Replacement Parts Needed	0.8	0.8	0.8	0.8	0.8	*Binomial*	10%	8	
22										
23	Variable Cost	$7	$7	$7	$7	$7	*Uniform*	$6	$8	
24										
25	**Cash Flows**									
26	Beginning Balance	$25,263	$20,000	$20,000	$20,000	$20,000	$20,000			
27	Cash Receipts	$22,736	$24,592	$25,520	$26,912	$27,376				
28	30-Day Credit Receipts	$27,232	$36,064	$39,008	$40,480	$42,688	$43,424			
29	Fixed Cost	-$15,000	-$15,000	-$15,000	-$15,000	-$15,000				
30	Total Variable Cost	-$41,160	-$44,520	-$46,200	-$48,720	-$49,560				
31	Repair Cost	-$4,000	-$4,000	-$4,000	-$4,000	-$4,000				
32	Loan Payoff	$0	-$4,171	-$6,727	-$7,270	-$7,507	-$5,928			
33	Loan Interest	$0	-$292	-$471	-$509	-$525	-$415			
34	Savings Interest	$758	$600	$600	$600	$600	$600			
35	Balance Before Loan	$15,829	$13,273	$12,730	$12,493	$14,072	$57,681			
36	New Loan	$4,171	$6,727	$7,270	$7,507	$5,928				
37	Ending Balance	$20,000	$20,000	$20,000	$20,000	$20,000				
38		>=	>=	>=	>=	>=				
39	Minimum Balance	$20,000	$20,000	$20,000	$20,000	$20,000				
40										
41	Ending Net Worth									
42										
43	Maximum Loan									

	A	N	O
1			
2	Selling Price		
3	Replacement Part Cost		
4	Monthly Fixed Cost		
5	Minimum Balance		
6	Starting Balance		
7			
8	**Sales**	December	January
9	Seasonality Index	1.18	
10	Base Sales	6000	*Normal*
11	Actual Sales	=SeasonalityIndex*BaseSales	
12	Fraction Cash Customers	0.386666666666667	*Triangular*
13			
14	**Interest Rates**		
15	Prime Rate Change	1.6784487245436E-19	*Custom*
16	Prime Rate	=M16+PrimeRateChange	
17	Loan Interest Rate	=MIN(PrimeRate+LoanRateGap,LoanRateMax)	
18	Savings Interest Rate	=MAX(PrimeRate+SavingsRateGap,SavingsRateMin)	
19			
20	**Manufacturing Costs**		
21	Replacement Parts Needed	0.8	*Binomial*
22			
23	Variable Cost	7	*Uniform*
24			
25	**Cash Flows**		
26	Beginning Balance	=M37	=N37
27	Cash Receipts	=ActualSales*FractionCashCustomers*SellingPrice	
28	30-Day Credit Receipts	=M11*(1-M12)*SellingPrice	=N11*(1-N12)*SellingPrice
29	Fixed Cost	=-MonthlyFixedCost	
30	Total Variable Cost	=-VariableCost*ActualSales	
31	Repair Cost	=-ReplacementPartsNeeded*ReplacementPartCost	
32	Loan Payoff	=-M36	=-N36
33	Loan Interest	=-M36*M17	=-N36*N17
34	Savings Interest	=M37*M18	=N37*N18
35	Balance Before Loan	=SUM(N26:N34)	=SUM(O26:O34)
36	New Loan	=IF(BalanceBeforeLoan<=MinimumBalance,MinimumBalance-BalanceBeforeLoan,0)	
37	Ending Balance	=BalanceBeforeLoan+NewLoan	
38		>=	
39	Minimum Balance	=MinimumBalance	
40			
41	Ending Net Worth		
42			
43	Maximum Loan		

The range names are as follows:

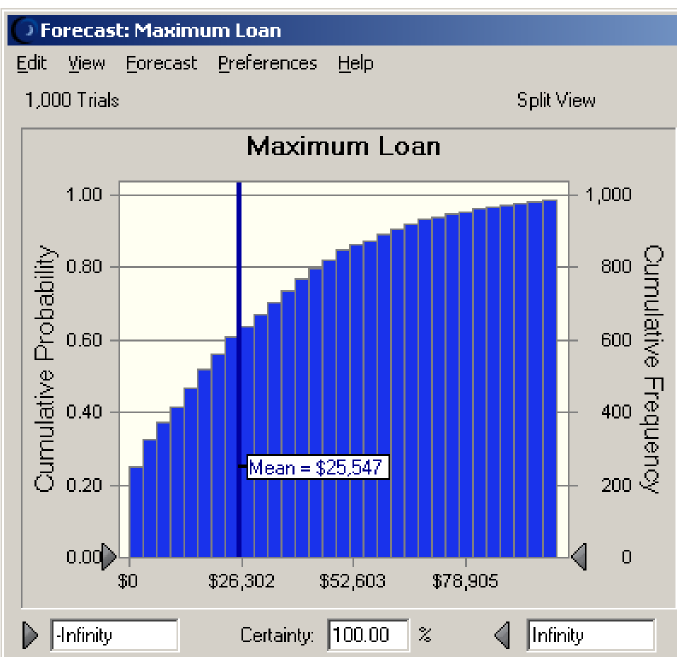
Range Name	Cells
ActualSales	B11:N11
BalanceBeforeLoan	C35:N35
BaseSales	B10:N10
BeginningBalance	C26:N26
CashReceipts	C27:N27
CreditReceipts	C28:N28
EndingBalance	C37:N37
EndingNetWorth	B41
FixedCost	C29:N29
FractionCashCustomers	B12:N12
InitialPrimeRate	G2
LoanInterest	C33:N33
LoanPayoff	C32:N32
LoanRate	B17:N17
LoanRateGap	G3
LoanRateMax	G4
MaximumLoan	B43
MinimumBalance	B5
MonthlyFixedCost	B4
NewLoan	C36:N36
PrimeRate	B16:N16
PrimeRateChange	C15:N15
RepairCost	C31:N31
ReplacementPartCost	B3
ReplacementPartsNeeded	C21:N21
SavingsInterest	C34:N34
SavingsRate	B18:N18
SavingsRateGap	G5
SavingsRateMin	G6
SeasonalityIndex	B9:N9
SellingPrice	B2
StartingBalance	B6
TotalVariableCost	C30:N30
VariableCost	C23:N23

(b) The mean ending net worth is approximately $54.7 thousand. The probability that it will be greater than $0 is approximately 85.8%.



Statistic	Forecast values
Trials	1,000
Mean	$54,654
Median	$53,606
Mode	---
Standard Deviation	$53,168
Variance	$2,826,791,758
Skewness	-0.0360
Kurtosis	3.22
Coeff. of Variability	0.9728
Minimum	-$141,172
Maximum	$228,421
Mean Std. Error	$1,681

(c) The maximum short-term loan is forecasted in cell B43. The cumulative chart and percentile chart follow. These charts indicate that the maximum short-term loan averages just over $25 thousand. However, to be fairly sure that the credit limit is high enough, it should probably be set quite a bit higher. The cumulative chart shows the probability that any given credit limit will be large enough. For example, a $75 thousand credit limit has about a 95% chance of being sufficient.



Percentile	Forecast values
0%	$0
10%	$0
20%	$187
30%	$5,480
40%	$11,381
50%	$18,293
60%	$25,249
70%	$34,896
80%	$45,442
90%	$63,223
100%	$165,801

20.3 Current Situation: A simulation run (shown below) indicates that the average number of jobs in the system is 2.0. Of these, half will be platen castings (1) and half will be housing castings (1). The waiting cost is therefore (\$200)(1) + (\$100)(1) = \$300 / hour.

	A	B	C	D	E	F	G	H
1	**Template for Queueing Simulation**							
2								
3			**Data**				**Results**	
4		Number of Servers =	2			Point	95% Confidence Interval	
5						Estimate	Low	High
6		**Interarrival Times**			L =	1.98365641	1.870700578	2.096612244
7		Distribution =	Exponential		L_q =	0.66628639	0.575783306	0.756789465
8		Mean =	15		W =	30.0811805	28.73655618	31.4258049
9					W_q =	10.1039076	8.845870432	11.36194473
10								
11		**Service Times**			P_0 =	0.19988054	0.188799674	0.210961405
12		Distribution =	Translated Exponential		P_1 =	0.2828689	0.271597628	0.294140162
13		Minimum Value =	10		P_2 =	0.21948306	0.211376682	0.227589435
14		Mean =	20		P_3 =	0.13257277	0.125756108	0.13938943
15					P_4 =	0.0722497	0.0660523	0.078447105
16		**Length of Simulation Run**			P_5 =	0.04178641	0.036150923	0.047421901
17		Number of Arrivals =	10,000		P_6 =	0.02261418	0.018147395	0.027080961
18					P_7 =	0.0129863	0.009197547	0.016775062
19					P_8 =	0.00771744	0.004659116	0.010775773
20					P_9 =	0.003861	0.001884639	0.005837354
21		Run Simulation			P_{10} =	0.00185903	0.000591296	0.003126755

Proposal 1: A simulation run (shown below) indicates that the average number of jobs in the system with three planers is approximately 1.4. Of these, half will be platen castings (0.7) and half will be housing castings (0.7). The waiting cost is therefore (\$200)(0.7) + (\$100)(0.7) = \$210 / hour. The savings (\$90 / hour) is substantially more than the added cost of the third planer (\$30 / hour), so this looks to be worthwhile. The net savings would be \$60 / hour.

	A	B	C	D	E	F	G	H
1	**Template for Queueing Simulation**							
2								
3			**Data**				**Results**	
4		Number of Servers =	3			Point	95% Confidence Interval	
5						Estimate	Low	High
6		**Interarrival Times**			L =	1.42409865	1.380256124	1.467941167
7		Distribution =	Exponential		L_q =	0.09771456	0.076609893	0.11881923
8		Mean =	15		W =	21.4712624	21.07385924	21.86866564
9					W_q =	1.47325117	1.168148546	1.778353785
10								
11		**Service Times**			P_0 =	0.25534157	0.245577077	0.265106061
12		Distribution =	Translated Exponential		P_1 =	0.33796761	0.329893149	0.346042064
13		Minimum Value =	10		P_2 =	0.231656	0.224819899	0.238492092
14		Mean =	20		P_3 =	0.11158027	0.106409547	0.116750986
15					P_4 =	0.04233244	0.038546771	0.046118111
16		**Length of Simulation Run**			P_5 =	0.01406273	0.011836939	0.016288531
17		Number of Arrivals =	10,000		P_6 =	0.00409905	0.002819395	0.005378708
18					P_7 =	0.00133435	0.000425396	0.002243309
19					P_8 =	0.00080672	-0.000131529	0.001744969
20					P_9 =	0.00036429	-0.000147459	0.000876045
21		Run Simulation			P_{10} =	0.00025729	-0.000188128	0.000702701

Proposal 2: A simulation run (shown below) indicates that the average number of jobs in the system with constant interarrival times is approximately 1.4. Of these, half will be platen castings (0.7) and half will be housing castings (0.7). The waiting cost is therefore ($200)(0.7) + ($100)(0.7) = $210 / hour. The savings ($90 / hour) is somewhat more than the added cost of changing the preceding production cost ($60 / hour). The net savings ($30) is less than for proposal 1, so this option is less worthwhile.

	A	B	C	D	E	F	G	H
1	**Template for Queueing Simulation**							
2								
3			**Data**				**Results**	
4		Number of Servers =	2			Point	95% Confidence Interval	
5						Estimate	Low	High
6		**Interarrival Times**			L =	1.40396168	1.383412164	1.424511205
7		Distribution =	Constant		L_q =	0.06455913	0.055139154	0.073979097
8		Value =	15		W =	21.0594253	20.75118246	21.36766807
9					W_q =	0.96838689	0.82708731	1.109686461
10								
11		**Service Times**			P_0 =	0.05142644	0.049243293	0.053609594
12		Distribution =	Translated Exponential		P_1 =	0.55774455	0.547877528	0.56761158
13		Minimum Value =	10		P_2 =	0.33345643	0.325678768	0.341234095
14		Mean =	20		P_3 =	0.05060063	0.045262513	0.055938746
15					P_4 =	0.00635733	0.004037203	0.008677449
16		**Length of Simulation Run**			P_5 =	0.00041461	2.30036E-06	0.000826928
17		Number of Arrivals =	10,000		P_6 =	0	0	0
18					P_7 =	0	0	0
19					P_8 =	0	0	0
20					P_9 =	0	0	0
21		Run Simulation			P_{10} =	0	0	0

Proposal 1 and 2: A simulation run (shown below) indicates that the average number of jobs in the system with both three planers and constant interarrival times is approximately 1.33. Of these, half will be platen castings (0.665) and half will be housing castings (0.665). The waiting cost is therefore ($200)(0.665) + ($100)(0.665) = $200 / hour. The savings ($85 / hour) is less than the combined cost of adding a third planer and changing the preceding production cost ($90 / hour), so this combined option does not appear to be worthwhile.

	A	B	C	D	E	F	G	H
1	**Template for Queueing Simulation**							
2								
3			**Data**				**Results**	
4		Number of Servers =	3			Point	95% Confidence Interval	
5						Estimate	Low	High
6		**Interarrival Times**			L =	1.32985569	1.316690172	1.343021211
7		Distribution =	Constant		L_q =	0.00052554	0.000184596	0.00086648
8		Value =	15		W =	19.9478354	19.75035259	20.14531816
9					W_q =	0.00788307	0.002768944	0.012997201
10								
11		**Service Times**			P_0 =	0.05754771	0.055474824	0.059620587
12		Distribution =	Translated Exponential		P_1 =	0.58946474	0.581401676	0.5975278
13		Minimum Value =	10		P_2 =	0.31909725	0.311531281	0.326663225
14		Mean =	20		P_3 =	0.03336476	0.03022801	0.036501519
15					P_4 =	0.00052554	0.000184596	0.00086648
16		**Length of Simulation Run**			P_5 =	0	0	0
17		Number of Arrivals =	10,000		P_6 =	0	0	0
18					P_7 =	0	0	0
19					P_8 =	0	0	0
20					P_9 =	0	0	0
21		Run Simulation			P_{10} =	0	0	0

Overall recommendation: Proposal 1 appears to be the most worthwhile with a net savings of about $36/hour over the current situation. Other proposals that may be worth looking into should include giving priority to platen castings, because of the higher waiting cost for that type of job.

CASE 20.4 Pricing under Pressure

(a) Before we begin the formal problem, we must first calculate the mean μ and standard deviation σ of the normally distributed random variable N. We are told that the annual interest rate will be used to estimate μ and the historical annual volatility will be used to estimate σ. Because the case is simulating weekly – not yearly – change, we must convert these yearly values to weekly values.

We first convert the annual interest rate $r = 8\%$ to a weekly interest rate w with the following formula:

$$\begin{aligned} w &= (1 + r)^{(1/52)} - 1 \\ &= (1 + 0.08)^{(1/52)} - 1 \\ &= (1.08)^{(1/52)} - 1 \\ &= 0.00148 \end{aligned}$$

We next convert the annual volatility $V_a = 0.30$ to a weekly volatility V_w with the following formula:

$$\begin{aligned} V_w &= V_a / \sqrt{52} \\ &= 0.30 / \sqrt{52} \\ &= 0.0416 \end{aligned}$$

Once we have the weekly interest rate and volatility, we can calculate μ and σ.

$$\begin{aligned} \mu &= w - 0.5(V_w)^2 \\ &= 0.00148 - 0.5(0.0416)^2 \\ &= 0.0006 \end{aligned}$$

$$\begin{aligned} \sigma &= V_w \\ &= 0.0416 \end{aligned}$$

1. One component appears in this system: the stock price. The stock price in the previous week is used to calculate the stock price in the next week. The relationship between the stock price in the previous week and the stock price in the next week is given by $s_n = e^N s_c$.

2. State of the system: $P(t)$ = price of the stock at time t.

3. This simulation requires generating a series of random observations from the normal distribution. Each random observation is a normally distributed random variable that determines the increase or decrease of the stock price at the end of next week. The random variable is substituted for N in the following equation:

$$s_n = e^N s_c$$

To generate a series of random variables, we define an assumption cell with normal distribution, where $\mu = 0.0006$ and $\sigma = 0.0416$.

4. The formula $s_n = e^N s_c$ gives us a procedure for changing the price (the state of the system) when an event occurs.

5. In this simulation, the time periods are fixed. We have a twelve-week period, and we need to calculate the change in the stock price each week. We have a formula $s_n = e^N s_c$ that relates the stock price at the end of the next week to the stock price at the end of the previous week. Thus, we do not have to worry about advancing the clock. We simply have to generate N for each of the twelve weeks.

6. We need to build a spreadsheet using the Crystal Ball. We start with the current stock price of \$42.00. We then use the formula $s_n = e^N s_c$ to calculate the stock price at the end of each of the twelve weeks. We substitute a Crystal Ball assumption cell with normal distribution (with mean $\mu = 0.0006$, and standard deviation $\sigma = 0.0416$) for N.

We then use the stock price at the end of the twelfth week to calculate the value of the option at the end of the twelfth week. If the stock price at the end of the twelfth week is greater than the exercise price of \$44.00, the value of the option is the difference between the value of the stock at the end of the twelfth week and the exercise price. If the stock price at the end of the twelfth week is less than or equal to the exercise price of \$44.00, the value of the option is \$0.

Finally, we need to discount the value of the option at the end of the twelfth week to the value of the option in today's dollars using the following formula:

$$(\text{Value of the option at the end of the twelfth week}) / (1.00148)^{12}$$

The spreadsheet model is shown below. The assumption cells are the N values (B8:B19), and the forecast cell is the price of the option today (C22).

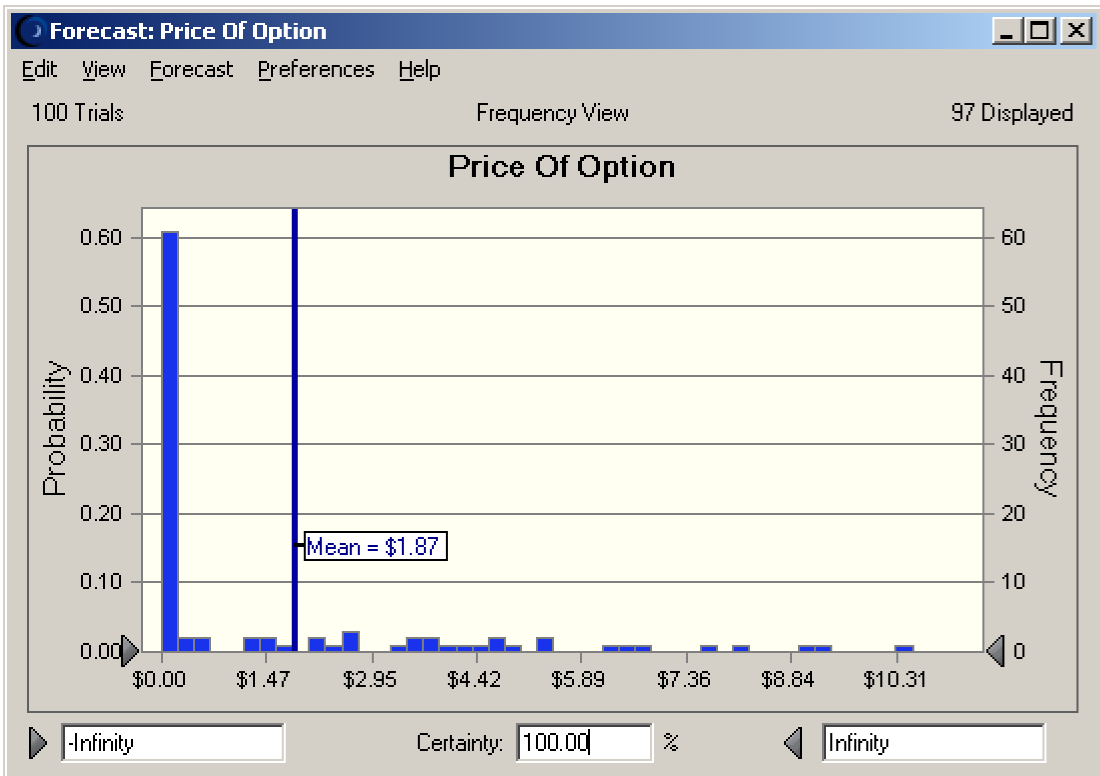
	A	B	C	D	E	F
1	**Simulation Model to Estimate Option Value**					
2						
3		Current Stock Price	$42.00		Annual Interest Rate	8%
4		Exercise Price	$44.00		Weekly Interest Rate	0.148%
5						
6			Stock Price at		Annual Volatility	30%
7	Week	N	End of Week		Weekly Volatility	4.160%
8	1	0.000615731	$42.03			
9	2	0.000615731	$42.05		μ =	0.0006
10	3	0.000615731	$42.08		σ =	0.0416
11	4	0.000615731	$42.10			
12	5	0.000615731	$42.13			
13	6	0.000615731	$42.16			
14	7	0.000615731	$42.18			
15	8	0.000615731	$42.21			
16	9	0.000615731	$42.23			
17	10	0.000615731	$42.26			
18	11	0.000615731	$42.29			
19	12	0.000615731	$42.31			
20						
21	Price of Option at end of Week 12		$0.00			
22	Price of Option Today		$0.00			

	A	B	C
3		Current Stock Price	42
4		Exercise Price	44
5			
6			Stock Price at
7	eek	N	End of Week
8	1	0.000615731176617215	=CurrentStockPrice*EXP(B8)
9	2	0.000615731176617215	=EXP(B9)*C8
10	3	0.000615731176617215	=EXP(B10)*C9
11	4	0.000615731176617215	=EXP(B11)*C10
12	5	0.000615731176617215	=EXP(B12)*C11
13	6	0.000615731176617215	=EXP(B13)*C12
14	7	0.000615731176617215	=EXP(B14)*C13
15	8	0.000615731176617215	=EXP(B15)*C14
16	9	0.000615731176617215	=EXP(B16)*C15
17	10	0.000615731176617215	=EXP(B17)*C16
18	11	0.000615731176617215	=EXP(B18)*C17
19	12	0.000615731176617215	=EXP(B19)*C18
20			
21		Price of Option at end of Week 12	=IF(C19>ExercisePrice,C19-ExercisePrice,0)
22		Price of Option Today	=C21/(1+WeeklyInterestRate)^12

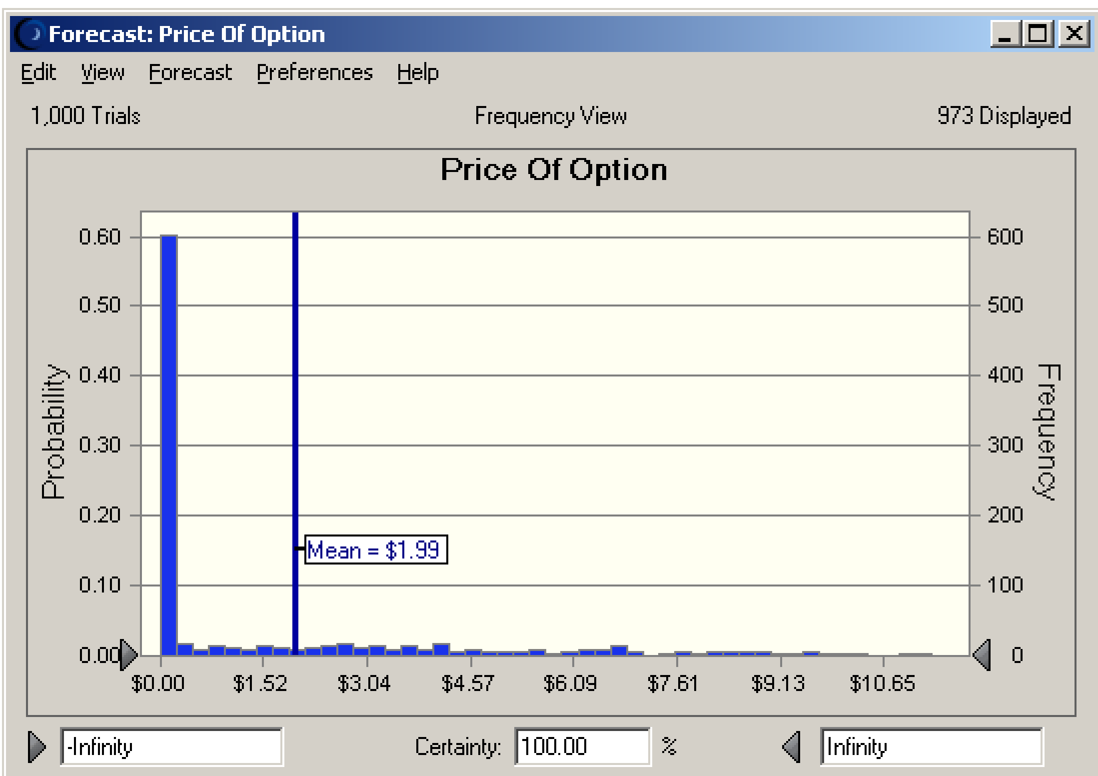
Range Name	Cells
AnnualInterestRate	F3
AnnualVolatility	F6
CurrentStockPrice	C3
ExercisePrice	C4
Mean	F9
PriceOfOption	C22
StandardDeviation	F10
WeeklyInterestRate	F4
WeeklyVolatility	F7

	E	F
3	Annual Interest Rate	0.08
4	Weekly Interest Rate	=((1+AnnualInterestRate)^(1/52))-1
5		
6	Annual Volatility	0.3
7	Weekly Volatility	=AnnualVolatility/SQRT(52)
8		
9	μ =	=WeeklyInterestRate-0.5*(WeeklyVolatility^2)
10	σ =	=WeeklyVolatility

The mean of the "Price of Option Today" is the price of the option in today's dollars. The simulation results after 100, 500, and 1,000 trials are shown below.




Forecast: Price Of Option
Edit View Forecast Preferences Help
500 Trials
Frequency View
487 Displayed
Price Of Option
Probability
Frequency
0.60
0.50
0.40
0.30
0.20
0.10
0.00
280
240
200
160
120
80
40
0
Mean = $1.99
$0.00 $1.54 $3.08 $4.62 $6.16 $7.70 $9.24 $10.78
-Infinity
Certainty: 100.00 %
Infinity


Forecast: Price Of Option
Edit View Forecast Preferences Help
1,000 Trials
Frequency View
973 Displayed
Price Of Option
Probability
Frequency
0.60
0.50
0.40
0.30
0.20
0.10
0.00
600
500
400
300
200
100
0
Mean = $1.99
$0.00 $1.52 $3.04 $4.57 $6.09 $7.61 $9.13 $10.65
-Infinity
Certainty: 100.00 %
Infinity

(b) Using the Black-Scholes Formula, the price of the option is $1.88. The spreadsheet used to calculate the Black-Scholes Formula in Excel follows:

	A	B	C	D	E	F
1	**Black-Scholes Calculation of Option Value**					
2						
3		Current Stock Price	$42.00		**Black-Scholes**	
4					d1 =	-0.127503153
5		Weeks to exercise date	12		d2 =	-0.271618491
6		Exercise Price	$44.00			
7		Exercise Price Present Value	$43.23		N[d1] =	0.449271051
8					N[d2] =	0.39295775
9		Annual Interest Rate	8%			
10		Weekly Interest Rate	0.148%		Value =	$1.88
11						
12		Annual Volatility	30%			
13		Weekly Volatility	4.160%			
14						
15		μ =	0.0006			
16		σ =	0.0416			

	E	F
3	**Black-Scholes**	
4	d1 =	=LN(CurrentStockPrice/ExercisePricePV)/(StandardDeviation*SQRT(WeeksToExerciseDate))+StandardDe
5	d2 =	=d_1-StandardDeviation*SQRT(WeeksToExerciseDate)
6		
7	N[d1] =	=NORMSDIST(d_1)
8	N[d2] =	=NORMSDIST(d_2)
9		
10	Value =	=Nd1*CurrentStockPrice-Nd2*ExercisePricePV

Range Name	Cells
AnnualInterestRate	C9
AnnualVolatility	C12
CurrentStockPrice	C3
d_1	F4
d_2	F5
ExercisePrice	C6
ExercisePricePV	C7
Mean	C15
Nd1	F7
Nd2	F8
StandardDeviation	C16
Value	F10
WeeklyInterestRate	C10
WeeklyVolatility	C13
WeeksToExerciseDate	C5

The price of the option obtained by simulation and the price of the option obtained by the Black-Scholes formula are fairly close. The 1,000-iteration simulation price is off by just thirteen cents.

(c) No, a random walk does not completely describe the price movement of the stock because the random walk assumes a consistent lognormal increase or decrease in the price of the stock. The price of the stock could change according to a different distribution, however, especially if an event occurs to trigger a dramatic increase or decrease in the stock. In this case, the European Space Agency may award Fellare the International Space Station contract. The award notice would most likely trigger a dramatic movement in the stock. The random walk does not take into account this dramatic event.

SUPPLEMENT 1 TO CHAPTER 20
VARIANCE-REDUCING TECHNIQUES

20S1-1.

(a)

$$r = P\{X \leq x\} = \int_1^x \frac{dt}{t^2} = 1 - \frac{1}{x} \Rightarrow x = \frac{1}{1-r}$$

r	$x = 1/(1-r)$
0.096	1.106
0.569	2.320
0.665	2.985
0.764	4.237
0.842	6.329
0.492	1.969
0.224	1.289
0.950	20.000
0.610	2.564
0.145	1.170

$$\widehat{\mu} = \frac{43.969}{10} = 4.3969$$

(b) Stratum 1: $r' = 0.0 + 0.6r$
Stratum 2: $r' = 0.6 + 0.3r$
Stratum 3: $r' = 0.9 + 0.1r$

Let w denote the sampling weight.

Stratum	r	r'	$x = 1/(1-r')$	w	x/w
1	0.096	0.058	1.062	½	2.124
1	0.569	0.341	1.517	½	3.034
1	0.665	0.399	1.664	½	3.328
2	0.764	0.829	5.848	1	5.848
2	0.842	0.853	6.803	1	6.803
2	0.492	0.748	3.968	1	3.968
3	0.224	0.922	12.821	4	3.205
3	0.950	0.995	200.000	4	50.000
3	0.610	0.961	25.641	4	6.410
3	0.145	0.915	11.765	4	2.941

$$\widehat{\mu} = \frac{87.661}{10} = 8.7661$$

(c)

$r' = 1 - r \Rightarrow x' = \frac{1}{1-r'} = \frac{1}{r}$

r	$x = 1/(1-r')$	$x' = 1/r$
0.096	1.106	10.417
0.569	2.320	1.757
0.665	2.985	1.504
0.764	4.237	1.309
0.842	6.329	1.188
0.492	1.969	2.033
0.224	1.289	4.464
0.950	20.000	1.053
0.610	2.564	1.639
0.145	1.170	6.897
Sum	43.969	32.261
$\widehat{\mu}$	4.3969	3.2261

$\widehat{\mu} = \frac{4.3969+3.2261}{2} = 3.8115$

20S1-2.

Stratum	x	x^2	w	x/w	x^2/w
1	8	64	18/10	80/18	640/18
1	5	25	18/10	50/18	250/18
1	1	1	18/10	10/18	10/18
1	6	36	18/10	60/18	360/18
1	3	9	18/10	30/18	90/18
1	7	49	18/10	70/18	490/18
2	3	9	9/10	60/18	180/18
2	5	25	9/10	100/18	500/18
2	2	4	9/10	40/18	80/18
3	2	4	3/10	120/18	240/18

$\widehat{\mu} = \frac{620/18}{10} = 3\frac{4}{9}, E[X^2] = \frac{2840/18}{10} = 15\frac{7}{9}$

20S1-3.

(a)

$$X = \begin{cases} 0 & \text{if } 0.1 \le r_i < 1 \\ 100r_i + 5 & \text{if } 0 \le r_i < 0.1 \end{cases}$$

r_i	0.096	0.665	0.842	0.224	0.610
X	14.5	0	0	0	0

$\widehat{\mu} = 2.9$

(b) Stratum 1: $r^* = 0.0 + 0.9r$
Stratum 2: $r^* = 0.9 + 0.1r,\ x = 100(r^* - 0.9) + 5$

Stratum	r	r^*	x	w	x/w
1	0.096	0.0864	0.00	$2/9$	0.00
2	0.665	0.9665	11.65	8	1.46
2	0.842	0.9842	13.42	8	1.68
2	0.224	0.9224	7.24	8	0.905
2	0.610	0.9610	11.10	8	1.39

$\widehat{\mu} = \frac{5.435}{5} = 1.087$

20S1-4.

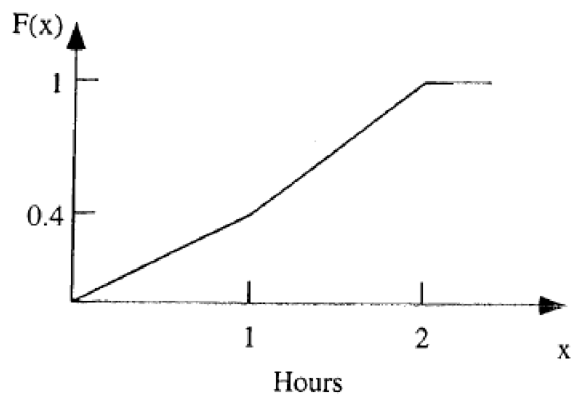
(a) 0.0000 to 0.3999 correspond to a minor repair.
0.4000 to 0.9999 correspond to a major repair.

Random observations: $0.7256 =$ major, $0.0817 =$ minor, $0.4392 =$ major

Using random numbers, generate length of each repair: 1.2243 hours, 0.9503 hours, 1.6104 hours. Then the average repair time is

$$(1.2243 + 0.9503 + 1.6104)/3 = 1.26 \text{ hours.}$$

(b)



(c)

$$F(x) = \begin{cases} 0.4x & \text{if } 0 \le x \le 1 \\ 0.4 + 0.6(x-1) & \text{if } x \ge 1 \end{cases}$$

$F(x) = 0.2243 \Rightarrow x = 0.561$ hours
$F(x) = 0.9503 \Rightarrow x = 1.917$ hours
$F(x) = 0.6104 \Rightarrow x = 1.351$ hours

Average repair time: $(0.561 + 1.917 + 1.351)/3 = 1.28$ hours

(d) $F(x) = 0.7757 \Rightarrow x = 1.626$ hours
$F(x) = 0.0497 \Rightarrow x = 0.124$ hours
$F(x) = 0.3896 \Rightarrow x = 0.974$ hours

Average repair time: $(1.626 + 0.124 + 0.974)/3 = 0.91$ hours

(e) Average repair time:

$$(0.561 + 1.917 + 1.351 + 1.626 + 0.124 + 0.974)/6 = 1.09 \text{ hours}$$

(f) The method of complementary random numbers in (e) gave the closest estimate. It performs well because using complements helps counteract rather extreme random numbers such as 0.9503.

(g) Results will vary. The following 300-day simulation using the method of complementary random numbers yielded an overall average service time of 1.095 minutes. This is very close to the true mean, which is 1.1 minutes.

Day	Random Number	Service Time	Complimentary Random Number	Complimentary Service Time
1	0.1348	0.337	0.8652	1.775
2	0.6798	1.466	0.3202	0.800
3	0.7941	1.657	0.2059	0.515
4	0.1825	0.456	0.8175	1.696
5	0.6502	1.417	0.3498	0.874
6	0.1088	0.272	0.8912	1.819
7	0.1153	0.288	0.8847	1.808
297	0.5456	1.243	0.4544	1.091
298	0.3514	0.878	0.6486	1.414
299	0.8990	1.832	0.1010	0.253
300	0.1544	0.386	0.8456	1.743
	Average	1.102		1.088
	Overall Average		1.095	

(h) We get $0.7256, 0.2744, 0.0817, 0.9183, 0.4382, 0.5608$ for minor repair times and $1.2243, 1.7757, 1.9503, 1.0497, 1.6104, 1.3896$ for major repair times. The weight for minor repair times is $(6/12)/0.4 = 1.25$ and the weight for major repair times is $(6/12)/0.6 = 1.089$. By dividing each sample by its corresponding weight, we obtain 1.1 minutes as the estimate of the mean of the overall distribution of repair times.

20S1-5.

(a) 0.0000 to 0.3999 correspond to no claims filled.
0.4000 to 0.7999 correspond to small claims filled.
0.8000 to 0.9999 correspond to large claims filled.

Random observations: $0.7256 =$ small, $0.0817 =$ no, $0.4392 =$ small

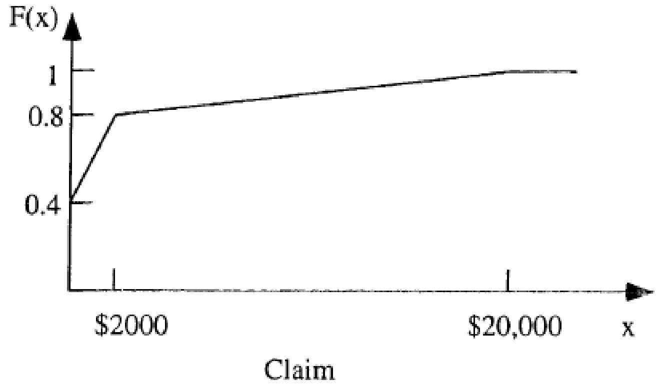
Using random numbers, generate size of each claim:

$$0.2243 \cdot 2,000 = \$448.60, \$0, 0.6104 \cdot 2,000 = \$1,220.80.$$

Then the average claim size is

$$(448.60 + 0 + 1,220.80)/3 = \$556.47.$$

(b)



(c)

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.4 + 0.4\frac{x}{2{,}000} & \text{if } 0 \leq x \leq 2,000 \\ 0.8 + 0.2\frac{(x-2{,}000)}{18{,}000} & \text{if } 2,000 \leq x \leq 20,000 \end{cases}$$

$F(x) = 0.2243 \Rightarrow x = \$0$
$F(x) = 0.9503 \Rightarrow x = \$15,527$
$F(x) = 0.6104 \Rightarrow x = \$1,052$

Average claim size: $(\$0 + \$15,527 + \$1,052)/3 = \$5,526.33$

(d) $F(x) = 0.7757 \Rightarrow x = \$1,880$
$F(x) = 0.0497 \Rightarrow x = \$0$
$F(x) = 0.3896 \Rightarrow x = \$0$

Average claim size: $(\$1,880 + \$0 + \$0)/3 = \$626.67$

(e) Average claim size: $(\$0 + \$15,527 + \$1,052 + \$1,880 + \$0 + \$0)/6 = \$3,076.50$

(f) The method of complementary random numbers in (e) gave the closest estimate. It performs well because using complements helps counteract rather extreme random numbers such as 0.9503.

(g) Results will vary. The following 300-day simulation using the method of complementary random numbers yielded an overall average claim size of $\$2,547.15$. This is very close to the true mean, which is $\$2,600$.

Day	Random Number	Size of Claim	Complimentary Random Number	Complimentary Size of Claim
1	0.2837	$0.00	0.7163	$1,581.27
2	0.4067	$33.50	0.5933	$966.50
3	0.4202	$101.18	0.5798	$898.82
4	0.3473	$0.00	0.6527	$1,263.54
5	0.9728	$17,550.20	0.0272	$0.00
6	0.8839	$9,547.73	0.1161	$0.00
7	0.6365	$1,182.61	0.3635	$0.00
297	0.1141	$0.00	0.8859	$9,734.86
298	0.3657	$0.00	0.6343	$1,171.49
299	0.7641	$1,820.30	0.2359	$0.00
300	0.0532	$0.00	0.9468	$15,215.71
	Average =	$2,648.11		$2,446.19
		Overall Average =	$2,547.15	

(h) We get 1451.2, 163.4, 878.4, 548.8, 1836.6, 1121.6 for small claims and 6037.4, 19105.4, 12987.2, 15962.6, 2894.6, 9012.8 for large claims. The weight for small claims is $(6/12)/0.4 = 1.25$ and the weight for large claims is $(6/12)/0.2 = 2.5$. By dividing each sample by its corresponding weight, we obtain $\$2,600$ as the estimate of the mean of the overall distribution of claim sizes.

20S1-6.

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{2}(x+1)^2 & \text{if } -1 \le x < 0 \\ 1 - \frac{1}{2}(1-x)^2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

$$\Rightarrow x = \begin{cases} \sqrt{2r} - 1 & \text{if } 0 \le r < \frac{1}{2} \\ 1 - \sqrt{2(1-r)} & \text{if } \frac{1}{2} \le r < 1 \end{cases}$$

r	x	$1-r$	x
0.096	-0.5618	0.904	0.5618
0.569	0.0716	0.431	-0.0716

$\Rightarrow$ sample mean: 0

20S1-7.

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0 & \text{if } x < -1 \\ \frac{x^3+1}{2} & \text{if } -1 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

$$\Rightarrow x = \sqrt[3]{2r-1}$$

r	x	$1-r$	x
0.096	-0.9314	0.904	0.9314
0.569	0.5168	0.431	-0.5168

$\Rightarrow$ sample mean: 0

20S1-8.

(a)

$$P\{X = k\} = \begin{cases} 0.125 & \text{if } k = 0 \\ 0.375 & \text{if } k = 1 \\ 0.375 & \text{if } k = 2 \\ 0.125 & \text{if } k = 3 \end{cases} \qquad X = \begin{cases} 0 & \text{if } 0 \le r < 0.125 \\ 1 & \text{if } 0.125 \le r < 0.5 \\ 2 & \text{if } 0.5 \le r < 0.875 \\ 3 & \text{if } 0.875 \le r < 1 \end{cases}$$

r	x	$1-r$	x
0.096	0	0.904	3
0.569	2	0.431	1
0.665	2	0.335	1

$\Rightarrow$ sample mean: $(0+2+2)/3 = 1.33$

(b) Sample mean: $(4+5)/6 = 1.5$

(c)

$$X = \begin{cases} \text{head} & \text{if } 0 \le r < \frac{1}{2} \\ \text{tail} & \text{if } \frac{1}{2} \le r < 1 \end{cases}$$

$$r_1 = \{0.096, 0.569, 0.665\} \Rightarrow X_1 = 1$$
$$r_2 = \{0.764, 0.842, 0.492\} \Rightarrow X_2 = 1$$
$$r_3 = \{0.224, 0.950, 0.610\} \Rightarrow X_3 = 1$$

sample mean: $3/3 = 1$

(d)

$$r_1^* = \{0.904, 0.431, 0.335\} \Rightarrow X_1^* = 2$$
$$r_2^* = \{0.236, 0.158, 0.508\} \Rightarrow X_2^* = 2$$
$$r_3^* = \{0.776, 0.050, 0.390\} \Rightarrow X_3^* = 2$$

sample mean: $(3+6)/3 = 3$

20S1-9.

(a)

Shaft radius: $r_s = \int_1^s 400e^{-400(t-1)}dt = 1 - e^{-400(s-1)} \Rightarrow s = 1 + \frac{\ln(1-r_s)}{-400}$

Bushing radius: $r_b = \int_1^b 100dt = 100(b-1) \Rightarrow b = 1 + \frac{r_b}{100}$

r_s	s	r_b	b	$s > b$?
0.096	1.000252	0.569	1.00569	No
0.665	1.002734	0.764	1.00764	No
0.842	1.004613	0.492	1.00492	No
0.224	1.000634	0.950	1.00950	No
0.610	1.002354	0.145	1.00145	Yes
0.484	1.001654	0.552	1.00552	No
0.350	1.001077	0.590	1.00590	No
0.430	1.001405	0.041	1.00041	Yes
0.802	1.001405	0.471	1.00471	No
0.255	1.000736	0.799	1.00799	No

When $s > b$, interference occurs, so the probability of interference is estimated as $2/10 = 20\%$.

(b)

Stratum	Portion of Distribution	Stratum Random Number	Size	Weight
1	$0.0 \leq F(b) \leq 0.2$	$r'_b = 0.2r_b$	6	$1/3$
2	$0.2 \leq F(b) \leq 0.6$	$r'_b = 0.2 + 0.4r_b$	2	$1/2$
3	$0.6 \leq F(b) \leq 1.0$	$r'_b = 0.6 + 0.4r_b$	2	$1/2$

Stratum	r_s	s	r_b	r'_b	b	Interference Weight
1	0.096	1.000252	0.569	0.114	1.00114	0
1	0.665	1.002734	0.764	0.153	1.00153	$1/3$
1	0.842	1.004613	0.492	0.098	1.00098	$1/3$
1	0.224	1.000634	0.950	0.190	1.00190	0
1	0.610	1.002354	0.145	0.029	1.00029	$1/3$
1	0.484	1.001654	0.552	0.110	1.00110	$1/3$
2	0.350	1.001077	0.590	0.436	1.00436	0
2	0.430	1.001405	0.041	0.216	1.00216	0
3	0.802	1.001405	0.471	0.788	1.00788	0
3	0.255	1.000736	0.799	0.920	1.00920	0

Estimated probability of interference: $4/30 = 2/15$

(c)

r_s	s	r_b	b	$s > b$?	s'	b'	$s' > b'$
0.096	1.000252	0.569	1.00569	No	1.005859	1.00431	Yes
0.665	1.002734	0.764	1.00764	No	1.001020	1.00236	No
0.842	1.004613	0.492	1.00492	No	1.000430	1.00508	No
0.224	1.000634	0.950	1.00950	No	1.003740	1.00050	Yes
0.610	1.002354	0.145	1.00145	Yes	1.001236	1.00855	No
0.484	1.001654	0.552	1.00552	No	1.001814	1.00448	No
0.350	1.001077	0.590	1.00590	No	1.002625	1.00410	No
0.430	1.001405	0.041	1.00041	Yes	1.002110	1.00959	No
0.802	1.004048	0.471	1.00471	No	1.000552	1.00529	Yes
0.255	1.000736	0.799	1.00799	No	1.003416	1.00201	Yes

Estimated probability of interference: $\frac{1}{2}\left(\frac{1}{5}+\frac{2}{5}\right) = 30\%$

Summary:

Method:	Monte Carlo	Stratified Sampling	Complementary RNs
Interference Probability:	$1/5$	$2/15$	$3/10$

SUPPLEMENT 2 TO CHAPTER 20

REGENERATIVE METHOD OF STATISTICAL ANALYSIS

20S2-1.

(a)

$$\begin{aligned} y_1 &= 0+5+4 &&= 9; \quad z_1 = 3 \\ y_2 &= 0+2 &&= 2; \quad z_2 = 2 \\ y_3 &= 0+3+1+6 &&= 10; \; z_3 = 4 \end{aligned}$$

$\overline{y} = 21/3 = 7;$ $\qquad\qquad \overline{z} = 9/3 = 3$

$\text{Est}\{W_q\} = \frac{7}{3} = 2\frac{1}{3}$

$s_{11}^2 = (81+4+100)/2 - (9+2+10)^2/6 = 19$

$s_{22}^2 = (9+4+16)/2 - (3+2+4)^2/6 = 1$

$s_{12}^2 = (27+4+40)/2 - (21)(9)/6 = 4$

$s^2 = 19 - (2)(7/3)(4) + (7/3)^2 = 5.778 \Rightarrow s = 2.404$

$1 - 2\alpha = 0.90 \Rightarrow \alpha = 0.05 \Rightarrow K_\alpha = 1.645$

$P\{1.572 \leq W_q \leq 3.094\} = 0.90$

(b)

$$\begin{aligned} y_1 &= 0+3+2 &&= 5; \quad z_1 = 3 \\ y_2 &= 0+3+1+5 &&= 9; \quad z_2 = 4 \\ y_3 &= 0 &&= 0; \quad z_3 = 1 \\ y_4 &= 0+2+4 &&= 6; \quad z_4 = 3 \\ y_5 &= 0+3+5+2 &&= 10; \; z_5 = 4 \end{aligned}$$

$\overline{y} = 30/5 = 6;$ $\qquad\qquad \overline{z} = 15/5 = 3$

$\text{Est}\{W_q\} = \frac{6}{3} = 2$

$s_{11}^2 = (25+81+36+100)/4 - (10+6+0+9+5)^2/20 = 15\frac{1}{2}$

$s_{22}^2 = (9+16+1+9+16)/4 - (3+4+1+3+4)^2/20 = 1\frac{1}{2}$

$s_{12}^2 = (15+36+0+18+40)/4 - (30)(15)/20 = 4\frac{3}{4}$

$s^2 = 15\frac{1}{2} - (2)(2)\left(4\frac{3}{4}\right) + (2)^2\left(1\frac{1}{2}\right) = 2\frac{1}{2} \Rightarrow s = 1.581$

$1 - 2\alpha = 0.90 \Rightarrow \alpha = 0.05 \Rightarrow K_\alpha = 1.645$

$P\{1.612 \leq W_q \leq 2.388\} = 0.90$

20S2-2.

When a service completion occurs, t minutes have passed since the last arrival, where $0 \leq t \leq 25$. The time until the next arrival is uniformly distributed between $\bar{t}$ and $25 - t$, where $\bar{t} = \max(0, 5 - t)$. Thus, the probabilistic structure of when future arrivals will occur depends on the history, so this cannot be a regeneration point.

20S2-3.

(a) For any new tube, the time of the next failure is given by "current time $+ 1000 + 1000r$," where r is a random number from Table 20.3. At each shutdown, one hour is added to the time of the next failure for all tubes when simulating the status quo and two hours are added when simulating the proposal.

Simulation of the status quo:

					Time of Failure of			
Time	r_1	r_2	r_3	r_4	Tube 1	Tube 2	Tube 3	Tube 4
0	0.096	0.569	0.665	0.764	1096	1569	1665	1764
1096	0.842	–	–	–	2939	1570	1666	1765
1570	–	0.492	–	–	2940	3063	1667	1766
1667	–	–	0.224	–	2941	3064	2892	1767
1767	–	–	–	0.950	2942	3065	2893	3718
2893	–	–	0.610	–	2943	3066	4504	3719
2943	0.145	–	–	–	4089	3067	4505	3720
3067	–	0.484	–	–	4090	4552	4506	3721
3721	–	–	–	0.552	4091	4553	4507	5274
4091	0.350	–	–	–	5442	4554	4508	5275
4508	–	–	0.590	–	5443	4555	6099	5276
4555	–	0.430	–	–	5444	5986	6100	5277
5000	–	–	–	–	5444	5986	6100	5277

Estimated cost of the status quo: $11 \times \$1,200 = \$13,200$

Simulation of the proposal:

Time	r_1	r_2	r_3	r_4	First Tube to Fail	Time of Failure
0	0.096	0.569	0.665	0.764	Tube 1	1096
1096	0.842	0.492	0.224	0.950	Tube 3	2322
2322	0.610	0.145	0.484	0.552	Tube 2	3469
3469	0.350	0.590	0.430	0.041	Tube 4	4512
4512	0.802	0.471	0.255	0.799	Tube 3	5769

Estimated cost of the proposal: $4 \times \$2,800 = \$11,200$

(b) Based on the simulation results in part (a), the proposal should be accepted.

(c) For the proposed policy, each shutdown is a regeneration point because all tubes are replaced and the process begins a new. For the status quo, the process never repeats itself because each tube is replaced when it fails.

(d)

Cycle	Cycle Cost	Cycle Length
1	$\$2,800$	1096
2	$\$2,800$	1226
3	$\$2,800$	1147
4	$\$2,800$	1043

$\overline{y} = \$2,800, \overline{z} = 1128$

$\text{Est}\{\text{cost/hour}\} = 2800/1128 = \$2.482$

$s_{11}^2 = \frac{(4 \times 2800^2)}{3} - \frac{(4 \times 2800)^2}{12} = 0$

$s_{22}^2 = \frac{(1086^2+1226^2+1147^2+1043^2)}{3} - \frac{(1086+1226+1147+1043)^2}{12} = 6071\frac{1}{3}$

$s_{12}^2 = \frac{(2800)(1086+1226+1147+1043)}{3} - \frac{(4)(2800)(1086+1226+1147+1043)}{12} = 0$

$s^2 = 0 - (2.482)(0)(2) + (2.482)^2\left(6071\frac{1}{3}\right) = 37410 \Rightarrow s = 193.4$

$1 - 2\alpha = 0.95 \Rightarrow \alpha = 0.025 \Rightarrow K_\alpha = 1.96$

$P\{2.314 \leq \text{cost/hour} \leq 2.650\} = 0.95$

20S2-4.

(a)

(i)

	Data
Number of Servers =	1
Interarrival Times	
Distribution =	Exponential
Mean =	1.25
Service Times	
Distribution =	Exponential
Mean =	1
Length of Simulation Run	
Number of Arrivals =	10,000

Run Simulation

	Results		
	Point	95% Confidence Interval	
	Estimate	Low	High
L =	4.8790378	3.344529192	6.413546353
L_q =	4.0690374	2.552881224	5.585193514
W =	6.0870551	4.231161152	7.942948994
W_q =	5.076504	3.233440493	6.91956742
P_0 =	0.1899996	0.16425264	0.215746552
P_1 =	0.151018	0.132489844	0.169546091
P_2 =	0.1253097	0.111337121	0.139282371
P_3 =	0.0954104	0.084720833	0.106099913
P_4 =	0.076206	0.066901918	0.085509995
P_5 =	0.0622451	0.054038618	0.070451571
P_6 =	0.0562059	0.047527927	0.064883901
P_7 =	0.0420322	0.035157883	0.048906526
P_8 =	0.0311873	0.025046424	0.037328273
P_9 =	0.0271509	0.020825618	0.033476206
P_{10} =	0.0229525	0.016668188	0.029236719

(ii)

	Data				Results	
Number of Servers =	1			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			$L =$	2.7233857	2.426711315	3.020060108
Distribution =	Exponential		$L_q =$	1.929419	1.646668271	2.212169662
Mean =	1.25		$W =$	3.4255718	3.099322189	3.751821491
			$W_q =$	2.4268921	2.104137696	2.749646512
Service Times			$P_0 =$	0.2060333	0.188347397	0.223719112
Distribution =	Erlang		$P_1 =$	0.2123885	0.196737339	0.228039706
Mean =	1		$P_2 =$	0.1691555	0.157922441	0.180388584
k =	4		$P_3 =$	0.1203942	0.111814695	0.128973778
			$P_4 =$	0.0820109	0.074401677	0.089620118
Length of Simulation Run			$P_5 =$	0.0604059	0.053059888	0.067751862
Number of Arrivals =	10,000		$P_6 =$	0.0464817	0.038778844	0.054184579
			$P_7 =$	0.0345098	0.02699144	0.042028079
			$P_8 =$	0.0237711	0.017106209	0.030436065
			$P_9 =$	0.0150995	0.009781903	0.020417036
Run Simulation			$P_{10} =$	0.0107493	0.005500231	0.015998293

(iii)

	Data				Results	
Number of Servers =	1			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			$L =$	2.3610223	2.133069489	2.588975073
Distribution =	Exponential		$L_q =$	1.5623811	1.346964384	1.777797733
Mean =	1.25		$W =$	2.956299	2.715448043	3.197150042
			$W_q =$	1.956299	1.715448043	2.197150042
Service Times			$P_0 =$	0.2013588	0.18542182	0.217295736
Distribution =	Constant		$P_1 =$	0.2465909	0.230719649	0.262462131
Value =	1		$P_2 =$	0.1855193	0.175252934	0.195785632
			$P_3 =$	0.1236632	0.115231596	0.1320948
			$P_4 =$	0.0861806	0.078092827	0.094268471
Length of Simulation Run			$P_5 =$	0.0560059	0.048413787	0.063597968
Number of Arrivals =	10,000		$P_6 =$	0.038352	0.030657389	0.04604656
			$P_7 =$	0.0256758	0.018972941	0.032378704
			$P_8 =$	0.0158616	0.010224825	0.021498383
			$P_9 =$	0.0096232	0.005462589	0.013783841
Run Simulation			$P_{10} =$	0.0054207	0.002088955	0.00875247

$L_{q2}/L_{q1} = 1.93/4.07 = 0.47, L_{q3}/L_{q1} = 1.56/4.07 = 0.38$

(b)

$L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}, L = \rho + L_q, W_q = \frac{L_q}{\lambda}, W = W_q + \frac{1}{\mu}$

(i) $L_{q1} = \frac{0.64+0.64}{2\times0.2} = 3.2, L_1 = 4, W_{q1} = 4, W_1 = 5$

(ii) $L_{q2} = \frac{0.64\times0.25+0.64}{2\times0.2} = 2, L_2 = 2.8, W_{q2} = 2.5, W_2 = 3.5$

(iii) $L_{q3} = \frac{0.64}{2\times 0.2} = 1.6, L_3 = 2.4, W_{q3} = 2, W_3 = 3$

$L_{q2}/L_{q1} = 0.675, L_{q3}/L_{q1} = 1.6/3.2 = 0.5$

They all fall into 95% confidence intervals in (a).

20S2-5.

(i)

	Data
Number of Servers =	2
Interarrival Times	
Distribution =	Exponential
Mean =	0.625
Service Times	
Distribution =	Exponential
Mean =	1
Length of Simulation Run	
Number of Arrivals =	10,000

Run Simulation

	Results		
	Point	95% Confidence Interval	
	Estimate	Low	High
L =	4.4169819	3.762741932	5.071221805
L_q =	2.8189572	2.196408584	3.441505892
W =	2.7745954	2.374659281	3.174531514
W_q =	1.7707715	1.386379475	2.155163601
P_0 =	0.1130209	0.098946991	0.127094867
P_1 =	0.1759335	0.157735081	0.194131942
P_2 =	0.1490888	0.135041965	0.163135721
P_3 =	0.1132411	0.103224222	0.123258055
P_4 =	0.0904506	0.081666149	0.099235032
P_5 =	0.0700615	0.062387405	0.077735635
P_6 =	0.054781	0.047780672	0.061781305
P_7 =	0.0470371	0.040150589	0.053923562
P_8 =	0.0388488	0.032048832	0.045648736
P_9 =	0.0316042	0.024904844	0.038303496
P_{10} =	0.0238166	0.01786118	0.029771984

(ii)

	Data
Number of Servers =	2
Interarrival Times	
Distribution =	Erlang
Mean =	0.625
k =	4
Service Times	
Distribution =	Exponential
Mean =	1
Length of Simulation Run	
Number of Arrivals =	10,000

Run Simulation

	Results		
	Point	95% Confidence Interval	
	Estimate	Low	High
L =	3.0227816	2.758114081	3.287449176
L_q =	1.4278496	1.189613949	1.666085338
W =	1.8907137	1.7298684	2.051558948
W_q =	0.8931028	0.746329377	1.039876295
P_0 =	0.0856493	0.076363741	0.094934929
P_1 =	0.2337693	0.215976216	0.251562476
P_2 =	0.2148076	0.202324676	0.227290474
P_3 =	0.1528126	0.143082694	0.162542553
P_4 =	0.1019164	0.093254948	0.110577874
P_5 =	0.0700971	0.061808978	0.078385177
P_6 =	0.0446838	0.036937765	0.052429909
P_7 =	0.0292769	0.022588535	0.035965318
P_8 =	0.0229797	0.016760364	0.029198939
P_9 =	0.0139757	0.009611176	0.018340127
P_{10} =	0.008787	0.005231761	0.012342204

(iii)

	Data				Results	
Number of Servers =	2			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			$L =$	2.7171114	2.486527419	2.94769536
Distribution =	Constant		$L_q =$	1.1162964	0.911632383	1.320960417
Value =	0.625		$W =$	1.6981946	1.554079637	1.8423096
			$W_q =$	0.6976853	0.56977024	0.825600261
Service Times			$P_0 =$	0.070462	0.063259787	0.077664229
Distribution =	Exponential		$P_1 =$	0.258261	0.23901565	0.277506339
Mean =	1		$P_2 =$	0.2538447	0.238401452	0.269287903
			$P_3 =$	0.1564562	0.146459176	0.166453132
			$P_4 =$	0.0955554	0.085682	0.10542875
Length of Simulation Run			$P_5 =$	0.0565784	0.047834224	0.065322534
Number of Arrivals =	10,000		$P_6 =$	0.0382855	0.029977035	0.04659395
			$P_7 =$	0.0272313	0.019025996	0.035436574
			$P_8 =$	0.0183559	0.011310208	0.025401589
			$P_9 =$	0.0118822	0.006451832	0.017312476
Run Simulation			$P_{10} =$	0.0060134	0.002734597	0.009292269

$$L_{q2}/L_{q1} = 1.43/2.82 = 0.51, L_{q3}/L_{q1} = 1.12/2.82 = 0.4$$

20S2-6.

	Data				Results	
Number of Servers =	1			Point	95% Confidence Interval	
				Estimate	Low	High
Interarrival Times			$L =$	4.8790378	3.344529192	6.413546353
Distribution =	Exponential		$L_q =$	4.0690374	2.552881224	5.585193514
Mean =	1		$W =$	4.8696441	3.384928921	6.354359196
			$W_q =$	4.0612032	2.586752395	5.535653936
Service Times			$P_0 =$	0.1899996	0.16425264	0.215746552
Distribution =	Exponential		$P_1 =$	0.151018	0.132489844	0.169546091
Mean =	0.8		$P_2 =$	0.1253097	0.111337121	0.139282371
			$P_3 =$	0.0954104	0.084720833	0.106099913
			$P_4 =$	0.076206	0.066901918	0.085509995
Length of Simulation Run			$P_5 =$	0.0622451	0.054038618	0.070451571
Number of Arrivals =	10,000		$P_6 =$	0.0562059	0.047527927	0.064883901
			$P_7 =$	0.0420322	0.035157883	0.048906526
			$P_8 =$	0.0311873	0.025046424	0.037328273
			$P_9 =$	0.0271509	0.020825618	0.033476206
Run Simulation			$P_{10} =$	0.0229525	0.016668188	0.029236719

Data			Results		
Number of Servers =	1		Point	95% Confidence Interval	
			Estimate	Low	High
Interarrival Times		L =	2.723309	2.194934622	3.251683349
Distribution =	Erlang	L_q =	1.9243222	1.410008106	2.438636196
Mean =	1	W =	2.7189869	2.198334959	3.239638917
k =	4	W_q =	1.9212681	1.412587093	2.429949189
Service Times		P_0 =	0.2010132	0.181930867	0.220095464
Distribution =	Exponential	P_1 =	0.242906	0.222885871	0.262926189
Mean =	0.8	P_2 =	0.1663002	0.153868897	0.178731601
		P_3 =	0.1134182	0.103373945	0.123462432
		P_4 =	0.0799261	0.071322345	0.088529927
Length of Simulation Run		P_5 =	0.0564405	0.048182688	0.064698214
Number of Arrivals =	10,000	P_6 =	0.0391303	0.031599012	0.046661643
		P_7 =	0.0296395	0.022208383	0.037070549
		P_8 =	0.0217429	0.014858914	0.02862694
		P_9 =	0.0143098	0.008863696	0.019755947
Run Simulation		P_{10} =	0.0095029	0.004996216	0.014009554

Data			Results		
Number of Servers =	1		Point	95% Confidence Interval	
			Estimate	Low	High
Interarrival Times		L =	2.1663884	1.965340222	2.367436515
Distribution =	Constant	L_q =	1.3576707	1.168672404	1.546669
Value =	1	W =	2.1663884	1.965340222	2.367436515
		W_q =	1.3576707	1.168672404	1.546669
Service Times		P_0 =	0.1912823	0.175681523	0.206883144
Distribution =	Exponential	P_1 =	0.2902171	0.27140572	0.30902839
Mean =	0.8	P_2 =	0.1844177	0.174531753	0.194303653
		P_3 =	0.1274616	0.117413441	0.137509763
		P_4 =	0.0809442	0.070654919	0.091233476
Length of Simulation Run		P_5 =	0.0511365	0.042240076	0.060032961
Number of Arrivals =	10,000	P_6 =	0.0318558	0.023999424	0.039712241
		P_7 =	0.0199732	0.013343687	0.026602654
		P_8 =	0.0105021	0.00624001	0.014764245
		P_9 =	0.0050794	0.002389521	0.007769261
Run Simulation		P_{10} =	0.0021602	0.000370531	0.003949808

$$L_{q2}/L_{q1} = 1.92/4.07 = 0.47, L_{q3}/L_{q1} = 1.36/4.07 = 0.33$$

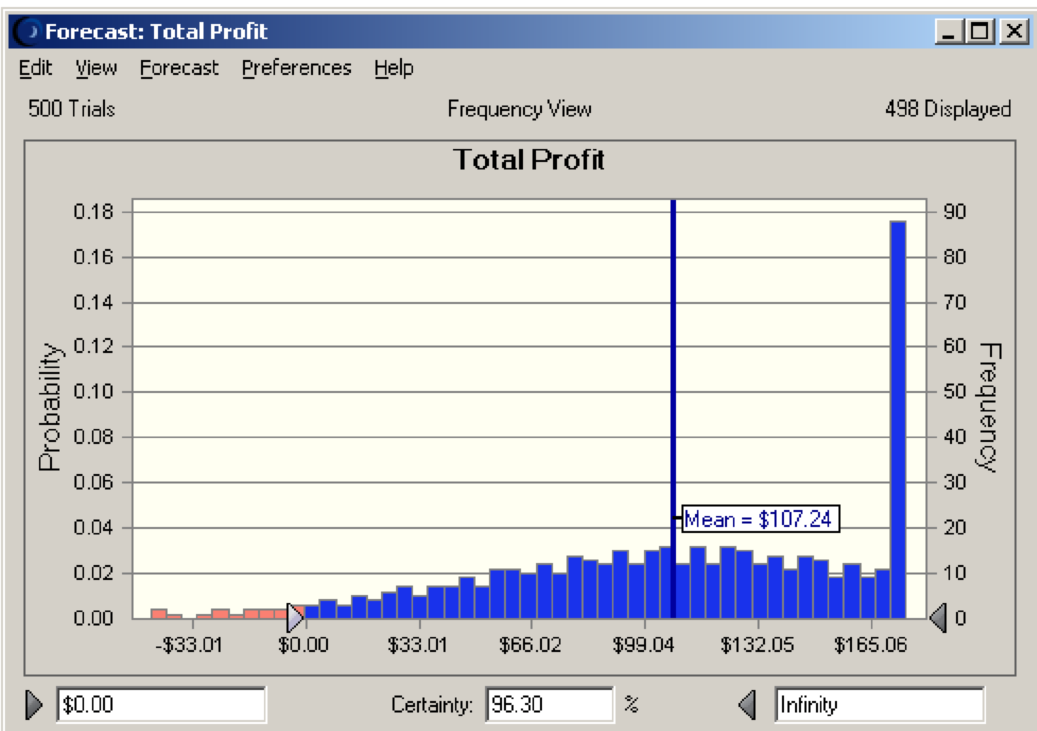
SUPPLEMENT 3 TO CHAPTER 20

OPTIMIZING WITH OPTQUEST

20S3-1.

	A	B	C	D	E
1	Purchase Price	$0.75			
2	Selling Price	$1.25			
3					
4	Order Quantity	350			
5				Mean	St. Dev.
6	Demand	300	*Normal*	300	50
7	Rounded Demand	300			
8					
9	Revenue	$375.00			
10	Purchase Cost	$262.50			
11	Total Profit	$112.50			

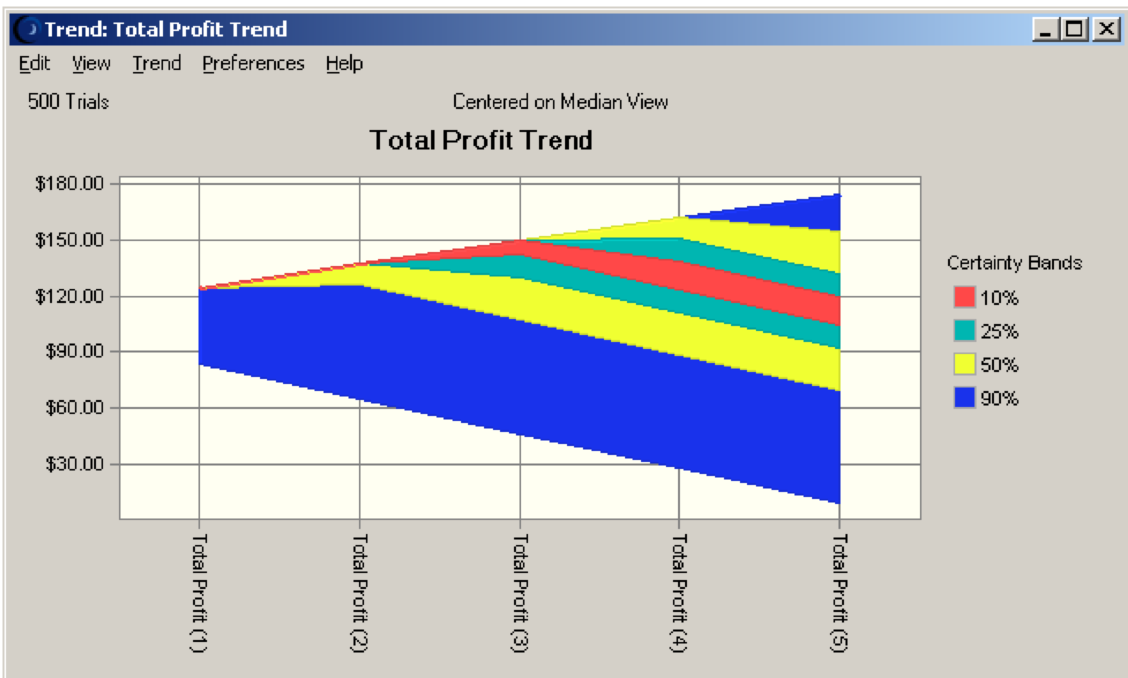
(a) The mean profit is approximately $107. The chance of making a nonnegative profit is approximately 96.3%.



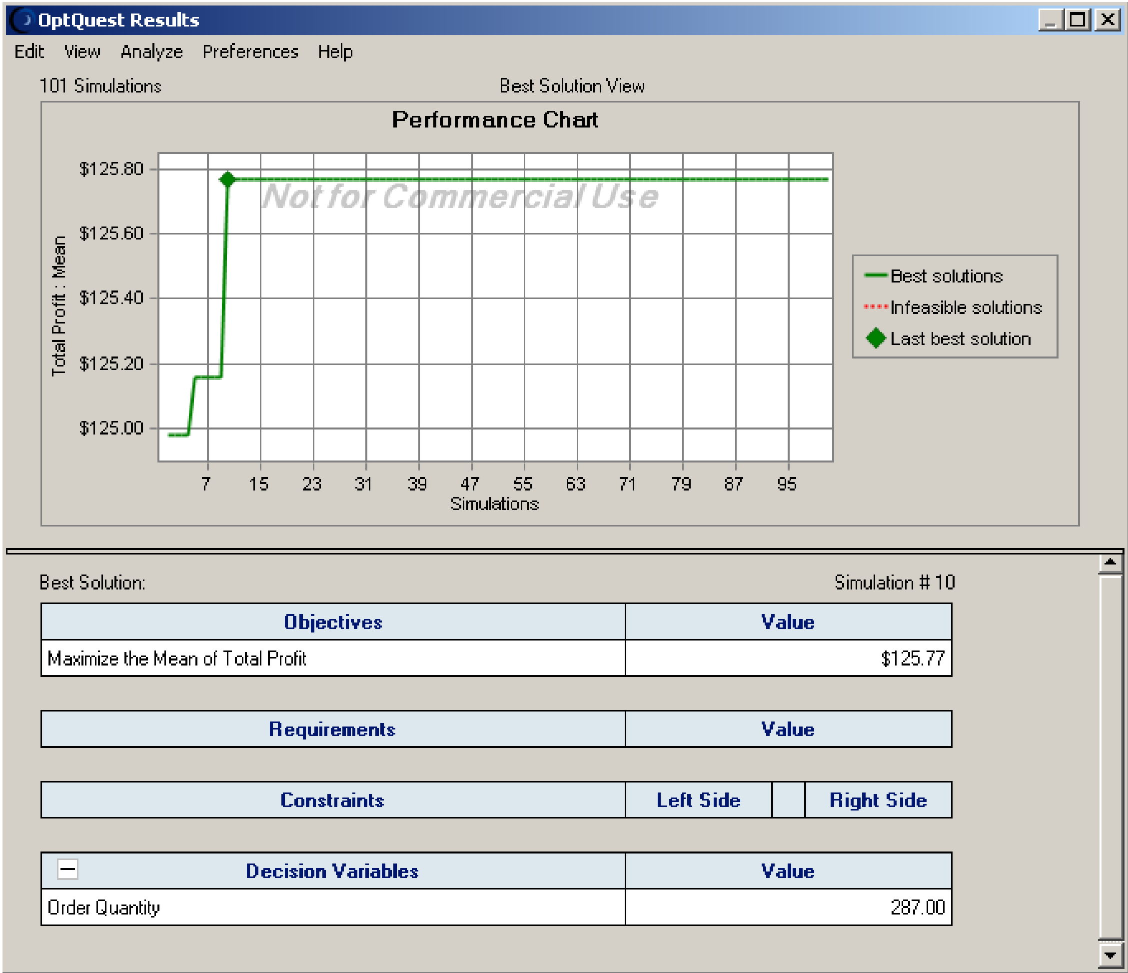
(b) The order quantities 275 and 300 are very close to maximizing the mean profit, so the order quantity that actually maximizes the mean profit is probably somewhere between these two quantities.

Order Quantity (250)	Order Quantity (275)	Order Quantity (300)	Order Quantity (325)	Order Quantity (350)
$119.74	$125.08	$125.02	$118.83	$107.24

(c)



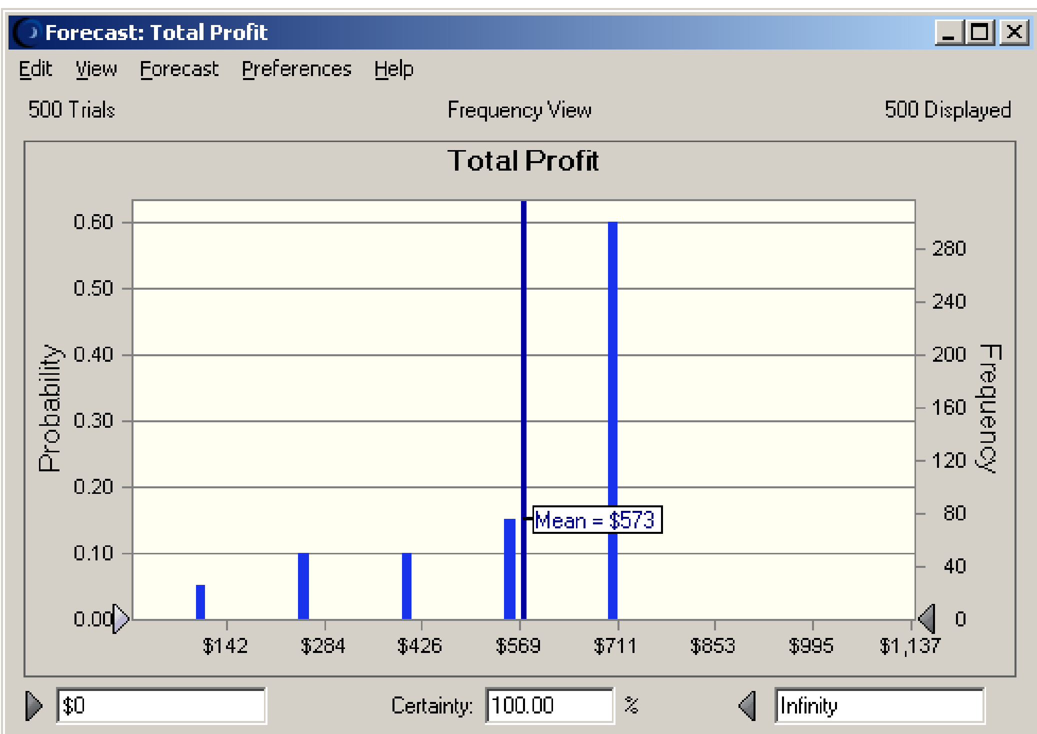
(d) An order quantity of approximately 287 maximizes Michael's mean profit.



20S3-2.

	A	B	C	D	E
1	Purchase Price	$100			
2	Selling Price	$150			
3					
4	Order Quantity	14			
5				Value	Probability
6	Demand	14	*Discrete (Custom)*	10	0.05
7				11	0.1
8	Revenue	$2,100		12	0.1
9	Purchase Cost	$1,400		13	0.15
10	Total Profit	$700		14	0.2
11				15	0.15
12				16	0.1
13				17	0.1
14				18	0.05

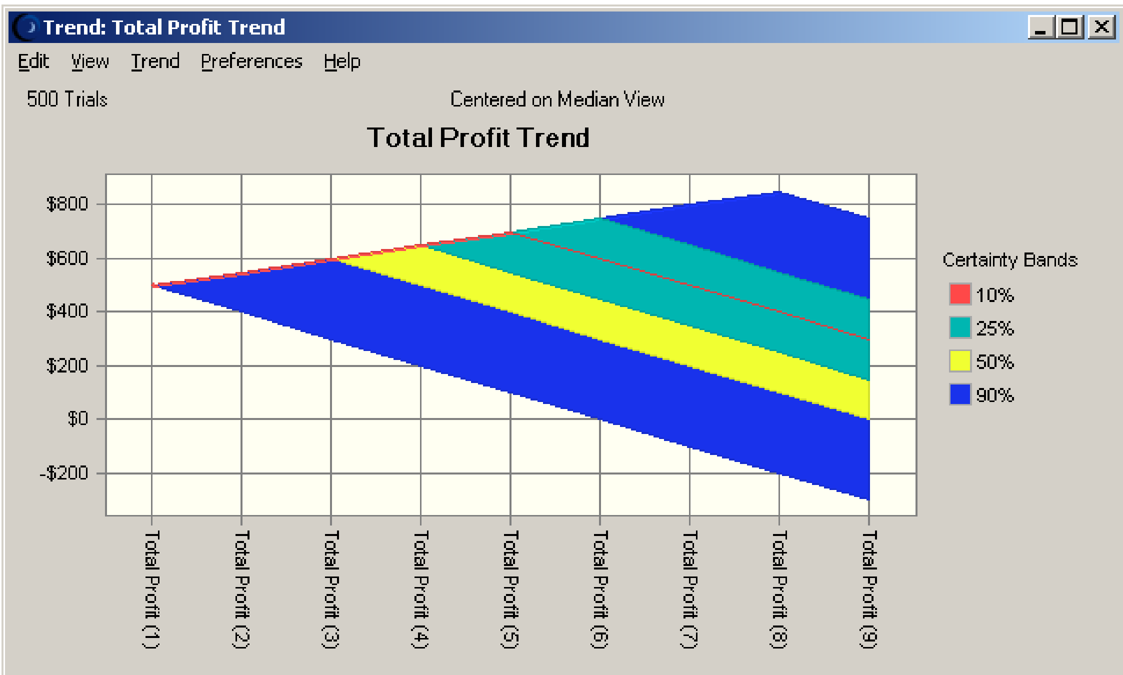
(a) The mean profit is approximately $573. There is a 100% chance of making a nonnegative profit.



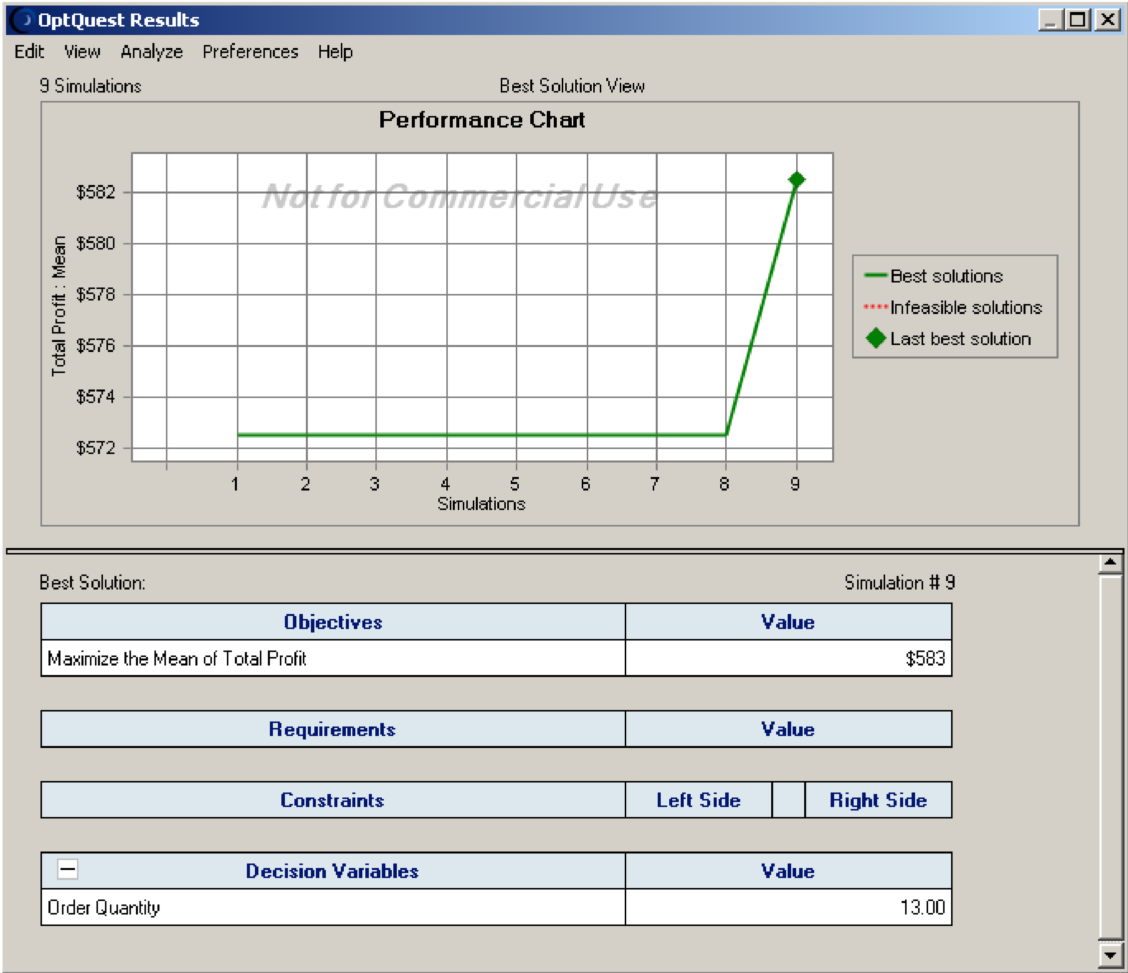
(b) Susan's mean profit is maximized with 13 tickets.

Order Quantity (10)	Order Quantity (11)	Order Quantity (12)	Order Quantity (13)	Order Quantity (14)	Order Quantity (15)	Order Quantity (16)	Order Quantity (17)	Order Quantity (18)
$500	$543	$570	$583	$573	$533	$470	$393	$300

(c)



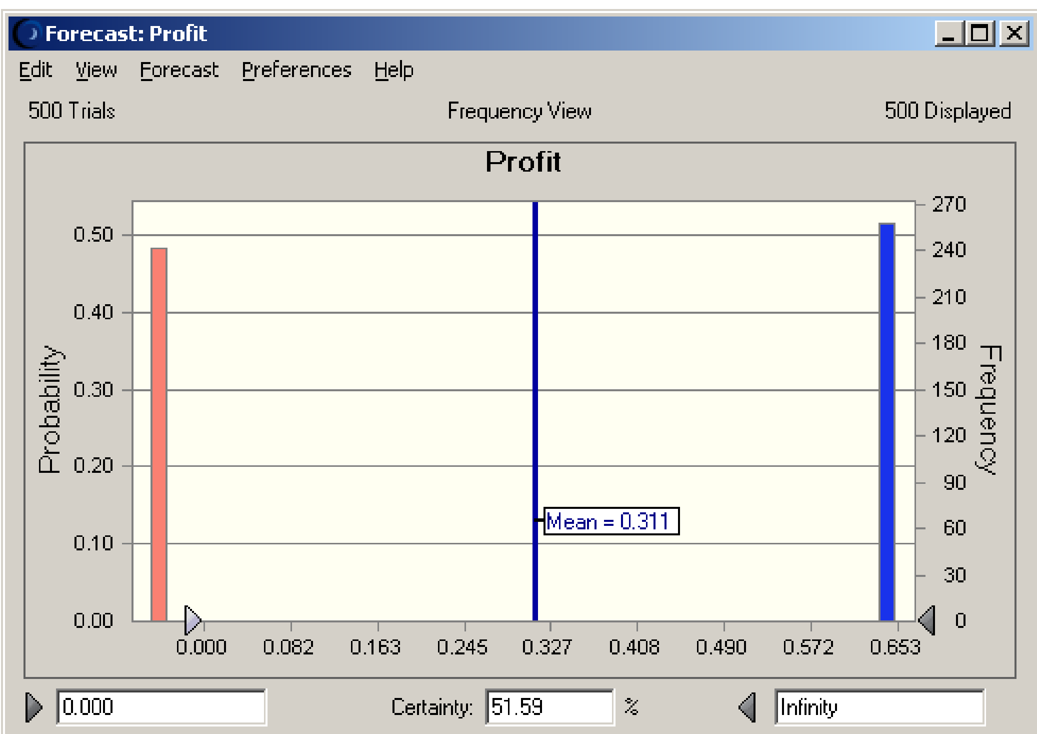
(d) The optimal order quantity found by OptQuest is 13.



20S3-3.

	A	B	C	D	E
1	**Data**				
2	Our Project Cost ($million)	5.000			
3	Our Bid Cost ($million)	0.050			
4					
5	**Competitor Bids**	Competitor 1	Competitor 2	Competitor 3	Competitor 4
6	Bid ($million)	6.083	6.083	6.083	6.083
7					
8	Distribution	*Triangular*	*Triangular*	*Triangular*	*Triangular*
9					
10	**Competitor Distribution Parameters (Proportion of Our Project Cost)**				
11	Minimum	105%	105%	105%	105%
12	Most Likely	120%	120%	120%	120%
13	Maximum	140%	140%	140%	140%
14					
15	**Competitor Distribution Parameters ($millions)**				
16	Minimum	5.250	5.250	5.250	5.250
17	Most Likely	6.000	6.000	6.000	6.000
18	Maximum	7.000	7.000	7.000	7.000
19					
20	**Minimum Competitor**				
21	**Bid ($million)**	6.083			
22					
23	**Our Bid ($million)**	5.700			
24					
25	**Win Bid?**	1	(1=yes, 0=no)		
26					
27	**Profit ($million)**	0.650			

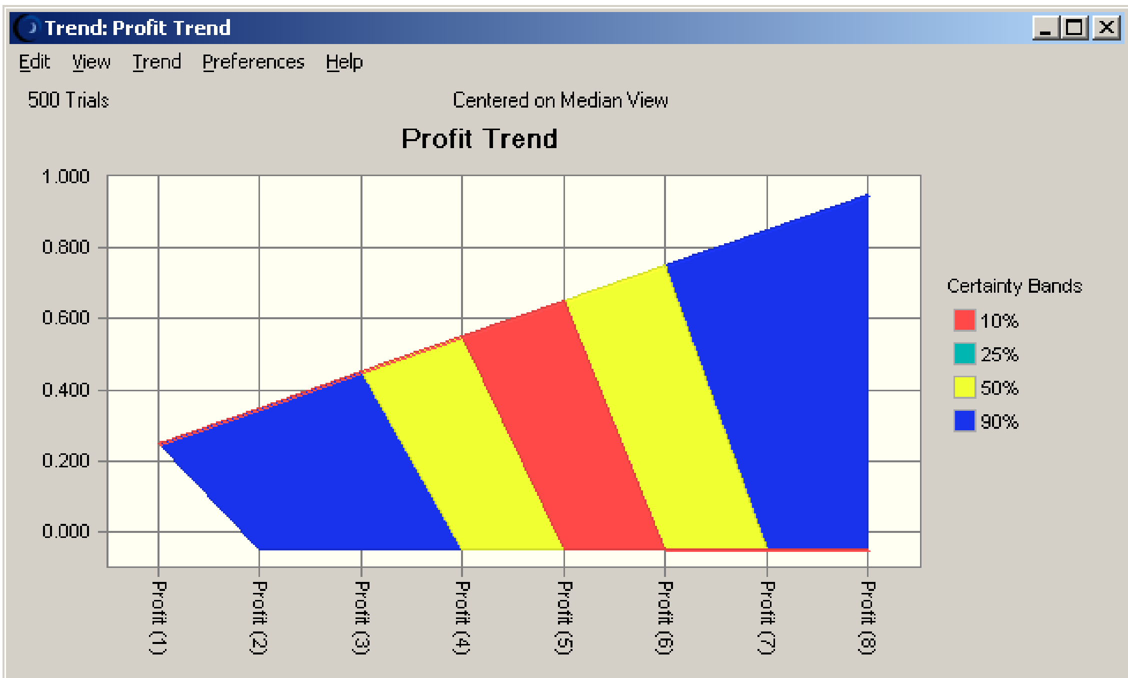
(a) The mean profit is approximately $0.31 million. The probability of winning the bid is approximately 51.6%.



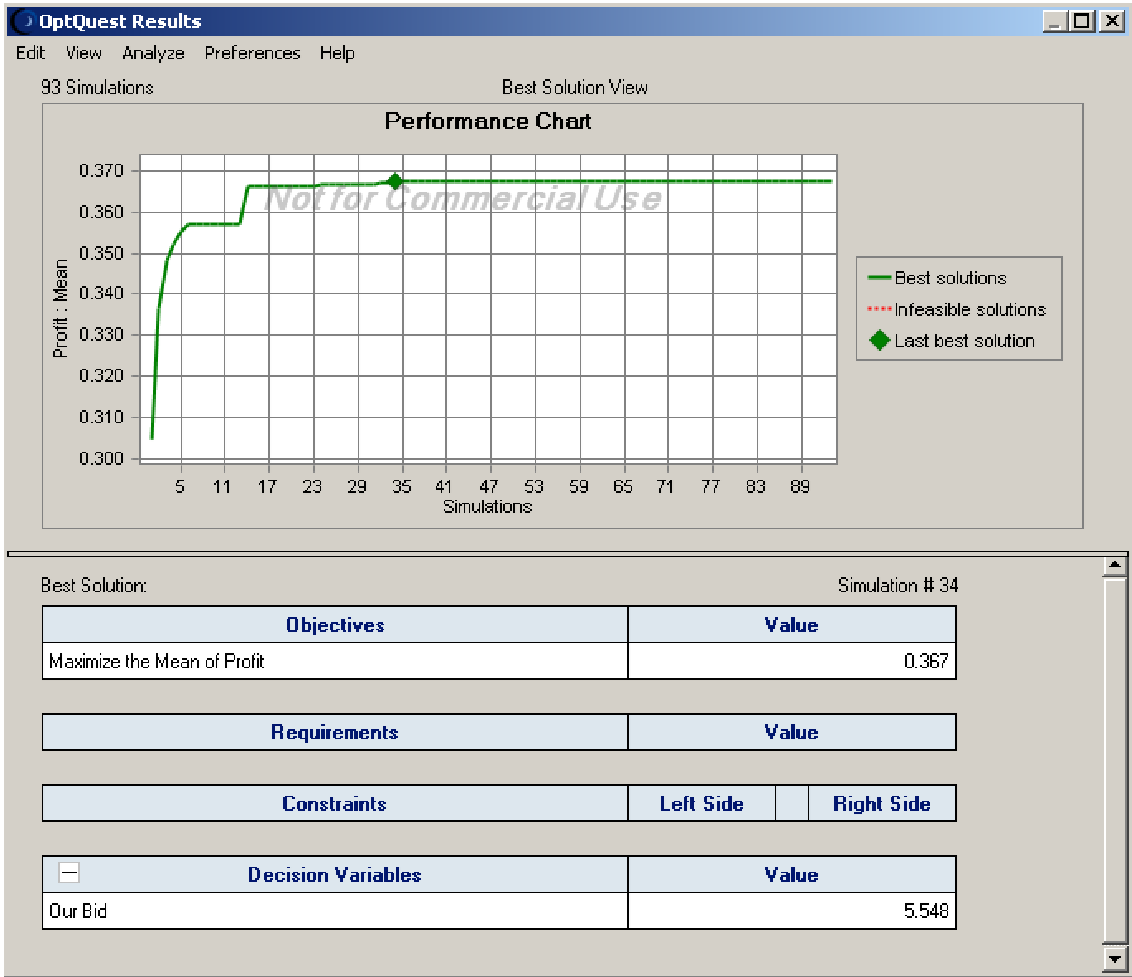
(b) RPI's mean profit is maximized with a bid of approximately $5.5 million.

Our Bid (5.300)	Our Bid (5.400)	Our Bid (5.500)	Our Bid (5.600)	Our Bid (5.700)	Our Bid (5.800)	Our Bid (5.900)	Our Bid (6.000)
0.248	0.324	0.361	0.354	0.311	0.222	0.141	0.060

(c)



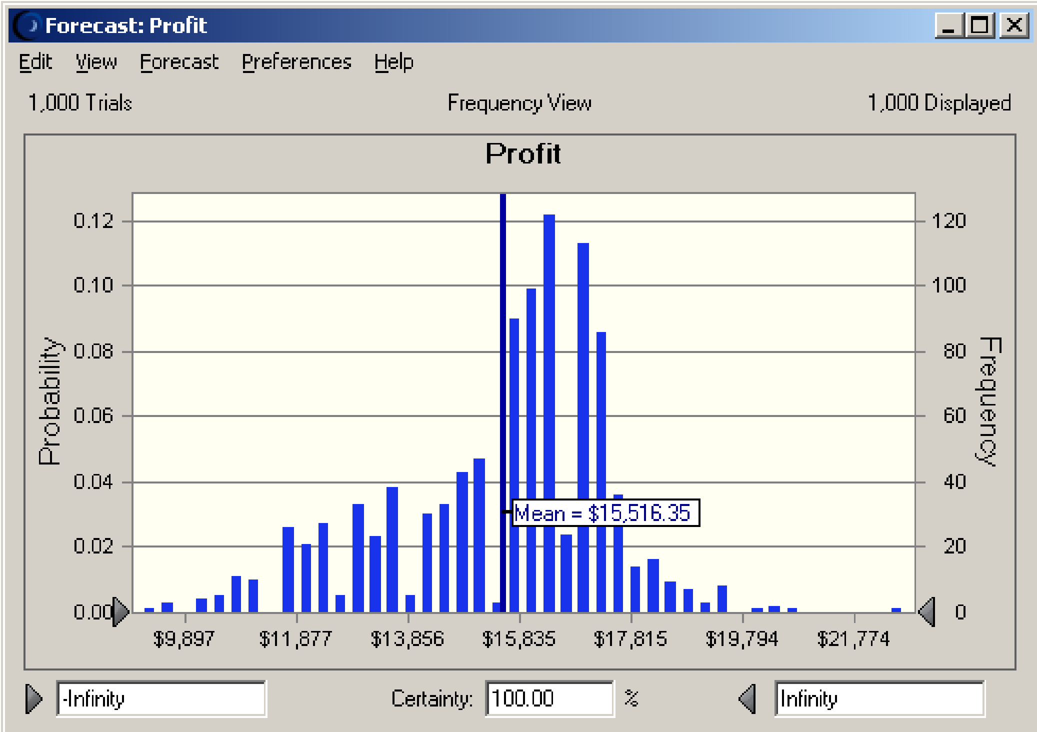
(d) The optimal bid found by OptQuest is approximately \$5.55 million.

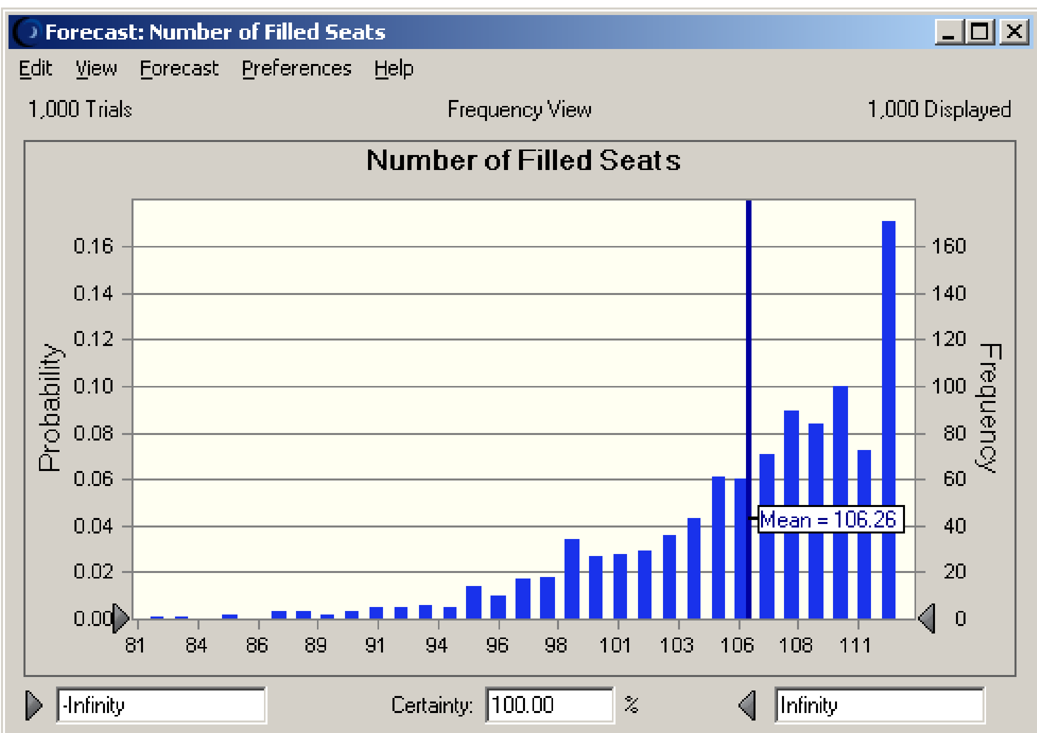


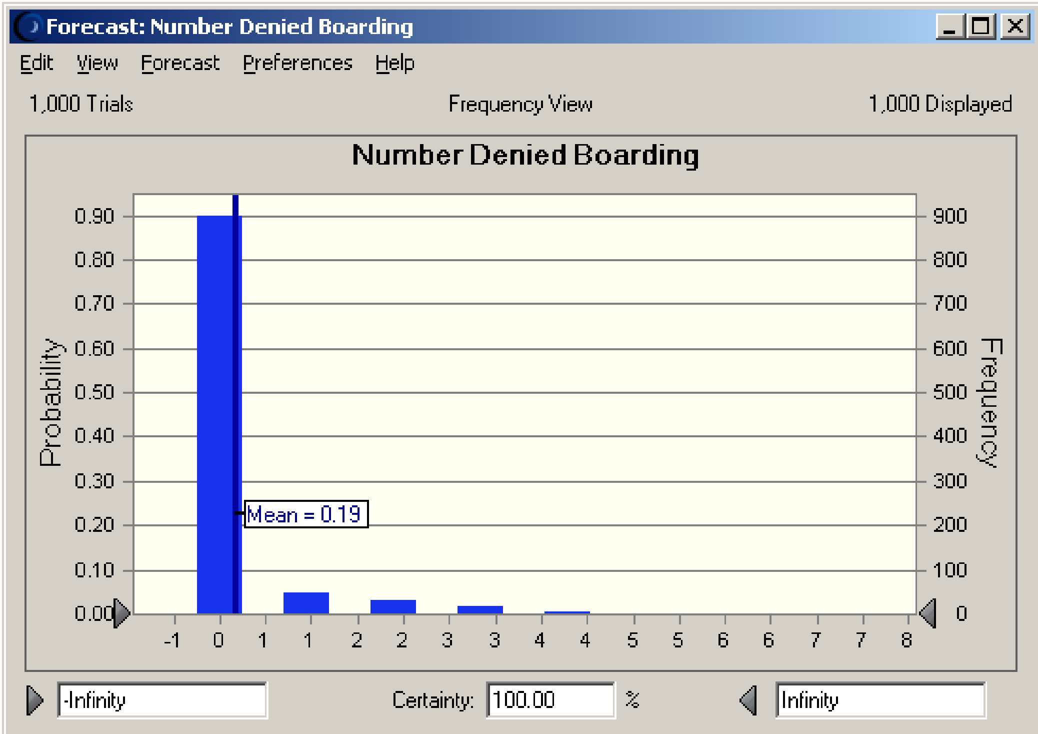
20S3-4.

	A	B	C	D	E	F
1	**Airline Overbooking**					
2					**Discount**	
3		**Data**			**Reservations**	
4		Seats Available	112		**to Accept**	75
5		Fixed Cost	$10,000			
6		Discount Fare	$150		**Total**	
7		Full Coach Fare	$400		**Reservations**	
8		Cost of Bumping	$600		**to Accept**	120
9						
10		**Discount Ticket Demand (Triangular)**				
11		Minimum	50		Discount-Fare Demand	96.66666667
12		Most Likely	90		Rounded	97
13		Maximum	150		Tickets Purchased	75
14		Probability to Show Up	95%		Number that Show	71.25
15						
16		**Full-Coach Ticket Demand (Uniform)**			Full Coach Demand	50
17		Minimum	30		Rounded	50
18		Maximum	70		Tickets Purchased	45
19		Probability to Show Up	85%		Number that Show	38.25
20						
21					Number Denied Boarding	0
22					Number of Filled Seats	109.5
23						
24					Revenue (Discount Fare)	$11,250
25					Revenue (Full Coach)	$15,300
26					Bumping Cost	$0
27					Fixed Cost	$10,000
28					**Profit**	$16,550.00

(a)


Forecast: Profit
Edit View Forecast Preferences Help
1,000 Trials
Frequency View
1,000 Displayed
Profit
Probability
0.12
0.10
0.08
0.06
0.04
0.02
0.00
Frequency
120
100
80
60
40
20
0
Mean = $15,516.35
$9,897
$11,877
$13,856
$15,835
$17,815
$19,794
$21,774
-Infinity
Certainty: 100.00 %
Infinity

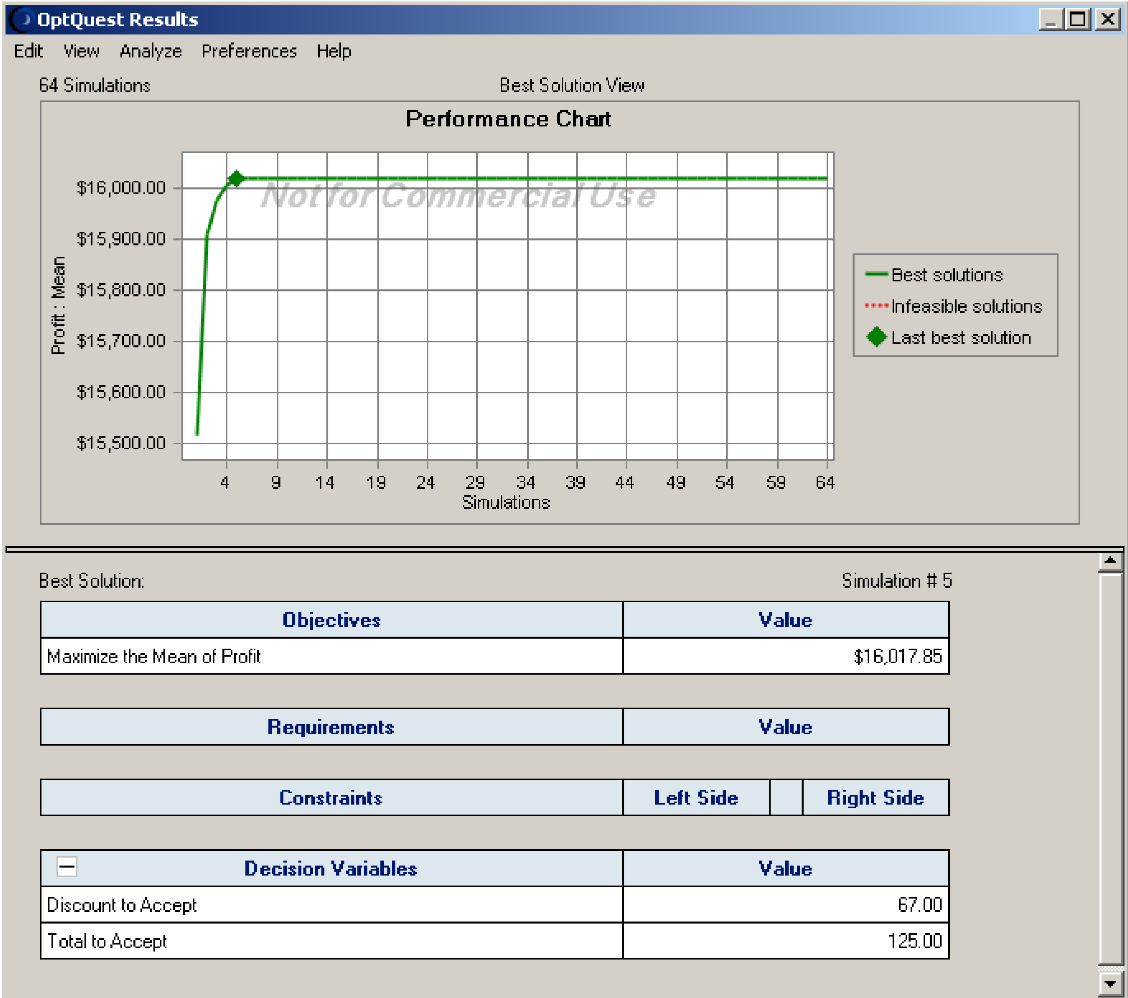

Forecast: Number of Filled Seats
Edit View Forecast Preferences Help
1,000 Trials
Frequency View
1,000 Displayed
Number of Filled Seats
Probability
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
Frequency
160
140
120
100
80
60
40
20
0
Mean = 106.26
81
84
86
89
91
94
96
98
101
103
106
108
111
-Infinity
Certainty: 100.00 %
Infinity



(b)

Trend Chart Overlay Chart Forecast Charts	Discount to Accept (50)	Discount to Accept (60)	Discount to Accept (70)	Discount to Accept (80)	Discount to Accept (90)
Total to Accept (112)	$14,234.40	$14,627.05	$14,202.90	$13,117.10	$11,818.25
Total to Accept (117)	$14,467.20	$15,284.45	$15,263.50	$14,518.70	$13,348.45
Total to Accept (122)	$14,503.80	$15,668.65	$15,909.10	$15,334.70	$14,161.65
Total to Accept (127)	$14,503.80	$15,725.25	$15,948.30	$15,263.50	$13,920.25
Total to Accept (132)	$14,503.80	$15,715.05	$15,789.50	$14,923.30	$13,423.25

(c) They should accept approximately 67 discount reservations and up to approximately 125 total in order to maximize the mean profit, as found by OptQuest.



21.1.

	A	B	C	D	E	F	G	H	I	J	K	L
1	LT Rate	7%										
2	ST Rate	10%										
3	Savings Interest	3%										
4												
5	Start Balance	1		(all cash figures in millions of dollars)								
6	Minimum Cash	0.5										
7												
8		Cash	LT	ST	LT	ST	LT	ST	Savings			Minimum
9	Year	Flow	Loan	Loan	Interest	Interest	Payback	Payback	Interest	Balance		Balance
10	2010	-8	7.50	0.00						0.50	>=	0.50
11	2011	-2		2.51	-0.53	0.00		0.00	0.015	0.50	>=	0.50
12	2012	-4		7.27	-0.53	-0.25		-2.51	0.015	0.50	>=	0.50
13	2013	3		5.51	-0.53	-0.73		-7.27	0.015	0.50	>=	0.50
14	2014	6		0.57	-0.53	-0.55		-5.51	0.015	0.50	>=	0.50
15	2015	3		0	-0.53	-0.06		-0.57	0.015	2.36	>=	0.50
16	2016	-4		2.59	-0.53	0		0	0.07093	0.50	>=	0.50
17	2017	7		0	-0.53	-0.26		-2.59	0.015	4.14	>=	0.50
18	2018	-2		0	-0.53	0		0	0.12423	1.74	>=	0.50
19	2019	10		0	-0.53	0		0	0.05221	11.27	>=	0.50
20	2020				-0.53	0	-7.50	0	0.33803	3.58	>=	0.50

21.2.

(a) The COO will need to know how many of each product to produce. Thus, the decisions are how many end tables, how many coffee tables, and how many dining room tables to produce. The objective is to maximize total profit.

(b) Pine wood used = (3 end tables)(8 pounds/end table)
+ (3 dining room tables)(80 pounds/dining room table)
= 264 pounds

Labor used = (3 end tables)(1 hour/end table)
+ (3 dining room tables)(4 hours/dining room table)
= 15 hours

(c)

	A	B	C	D	E	F	G
1		End Tables	Coffee Tables	Dining Room Tables			
2	Unit Profit						
3							
4		Resource Used per unit Produced			Total Used		Available
5	Pine Wood					<=	
6	Labor					<=	
7							
8		End Tables	Coffee Tables	Dining Room Tables			Total Profit
9	Units Produced						

(d)

	A	B	C	D	E	F	G
1		End Tables	Coffee Tables	Dining Room Tables			
2	Unit Profit	$50	$100	$220			
3							
4		Resource Used per unit Produced			Total Used		Available
5	Pine Wood	8	15	80	3000	<=	3000
6	Labor	1	2	4	200	<=	200
7							
8		End Tables	Coffee Tables	Dining Room Tables			Total Profit
9	Units Produced	0	40	30			$10,600

21.3.

(a) Top management will need to know how much to produce in each quarter. Thus, the decisions are the production levels in quarters 1, 2, 3, and 4. The objective is to maximize the net profit.

(b)

$$\begin{aligned}
\text{Ending Inventory(Q1)} &= \text{Starting Inventory(Q1)} + \text{Production(Q1)} - \text{Sales(Q1)} \\
&= 1,000 + 5,000 - 3,000 = 3,000 \\
\text{Ending Inventory(Q2)} &= \text{Starting Inventory(Q2)} + \text{Production(Q2)} - \text{Sales(Q2)} \\
&= 3,000 + 5,000 - 4,000 = 4,000 \\
\text{Profit from Sales(Q1)} &= \text{Sales(Q1)} \times (\$20) = 3,000 \times (\$20) = \$60,000 \\
\text{Profit from Sales(Q2)} &= \text{Sales(Q2)} \times (\$20) = 4,000 \times (\$20) = \$80,000
\end{aligned}$$

$$\begin{aligned}
\text{Inventory Cost(Q1)} &= \text{Ending Inventory(Q1)} \times (\$8) = 3,000 \times (\$8) = \$24,000 \\
\text{Inventory Cost(Q2)} &= \text{Ending Inventory(Q2)} \times (\$8) = 4,000 \times (\$8) = \$32,000
\end{aligned}$$

(c)

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Inventory Holding Cost												
2	Gross Profit from Sales												
3													
4		Starting			Maximum		Demand/	Ending				Inventory	Gross Profit
5		Inventory	Production		Production		Sales	Inventory				Cost	from Sales
6	Quarter 1			<=					>=				
7	Quarter 2			<=					>=				
8	Quarter 3			<=					>=				
9	Quarter 4			<=					>=				
10													
11													
12												Net Profit	

(d)

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Inventory Holding Cost		$8										
2	Gross Profit from Sales		$20										
3													
4		Starting			Maximum		Demand/	Ending				Inventory	Gross Profit
5		Inventory	Production		Production		Sales	Inventory				Cost	from Sales
6	Quarter 1	1,000	2,000	<=	6,000		3,000	0	>=	0		$0	$60,000
7	Quarter 2	0	4,000	<=	6,000		4,000	0	>=	0		$0	$80,000
8											Totals	$0	$140,000
9													
10												Net Profit	$140,000

(e)

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Inventory Holding Cost		$8										
2	Gross Profit from Sales		$20										
3													
4		Starting			Maximum		Demand/	Ending				Inventory	Gross Profit
5		Inventory	Production		Production		Sales	Inventory				Cost	from Sales
6	Quarter 1	1,000	3,000	<=	6,000		3,000	1,000	>=	0		$8,000	$60,000
7	Quarter 2	1,000	6,000	<=	6,000		4,000	3,000	>=	0		$24,000	$80,000
8	Quarter 3	3,000	6,000	<=	6,000		8,000	1,000	>=	0		$8,000	$160,000
9	Quarter 4	1,000	6,000	<=	6,000		7,000	0	>=	0		$0	$140,000
10											Totals	$40,000	$440,000
11													
12												Net Profit	$400,000

21.4.

(a) Fairwinds needs to know how much to participate in each of the three projects and what their ending balances will be. The decisions to be made are how much to participate in each of the three projects. The objective is to maximize the ending balance at the end of six years.

(b)

$$\begin{aligned}\text{Ending Balance(Y1)} &= \text{Starting Balance} + \text{Project A} + \text{Project C} + \text{Other Projects}\\ &= 10 + (100\%)(-4) + (50\%)(-10) + 6 = \$7 \text{ million}\end{aligned}$$

$$\begin{aligned}\text{Ending Balance(Y2)} &= \text{Starting Balance} + \text{Project A} + \text{Project C} + \text{Other Projects}\\ &= 7 + (100\%)(-6) + (50\%)(-7) + 6 = \$3.5 \text{ million}\end{aligned}$$

(c)

	A	B	C	D	E	F	G	H	I
1	Starting Cash								
2									
3					Total				
4		Cash Flow (at full participation, $million)			Cash Flow	Other	Ending		Minimum
5	Year	Project A	Project B	Project C	From ABC	Projects	Balance		Balance
6	1							>=	
7	2							>=	
8	3							>=	
9	4							>=	
10	5							>=	
11	6							>=	
12									
13	Participation								
14		<=	<=	<=					
15		100%	100%	100%					

(d)

	A	B	C	D	E	F	G	H	I
1	Starting Cash	10		all cash numbers are in $millions					
2									
3					Total				
4		Cash Flow (at full participation, $million)			Cash Flow	Other	Ending		Minimum
5	Year	Project A	Project B	Project C	From ABC	Projects	Balance		Balance
6	1	-4	-8	-10	0	6	16	>=	1
7	2	-6	-8	-7	0	6	22	>=	1
8									
9									
10									
11									
12									
13	Participation	0%	0%	0%					
14		<=	<=	<=					
15		100%	100%	100%					

(e)

	A	B	C	D	E	F	G	H	I
1	Starting Cash	10		all cash numbers are in $millions					
2									
3					Total				
4		Cash Flow (at full participation, $million)			Cash Flow	Other	Ending		Minimum
5	Year	Project A	Project B	Project C	From ABC	Projects	Balance		Balance
6	1	-4	-8	-10	-10.75	6	5.25	>=	1
7	2	-6	-8	-7	-8.125	6	3.125	>=	1
8	3	-6	-4	-7	-8.125	6	1	>=	1
9	4	24	-4	-5	-0.5	6	6.5	>=	1
10	5	0	30	-3	-3	6	9.5	>=	1
11	6	0	0	44	44	6	59.5	>=	1
12									
13	Participation	18.75%	0%	100%					
14		<=	<=	<=					
15		100%	100%	100%					

21.5.

(a) Web Mercantile needs to know each month how many square feet to lease and for how long. The decisions therefore are for each month how many square feet to lease for one month, two months, three months, etc. The objective is to minimize the overall leasing cost.

(b) Total Cost = (30,000 sq feet)($190/sq foot) + (20,000 sq feet)($100/sq foot) = $7.7 million

(c)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1									Month Covered by Lease?								Total		Space
2	Month of Lease:	1	1	1	1	1	2	2	2	2	3	3	3	4	4	5	Leased		Required
3	Length of Lease:	1	2	3	4	5	1	2	3	4	1	2	3	1	2	1	(sq. ft.)		(sq. ft.)
4	Month 1																	>=	
5	Month 2																	>=	
6	Month 3																	>=	
7	Month 4																	>=	
8	Month 5																	>=	
9																			
10	Cost of Lease																		
11	(per sq. ft.)																		
12																			Total Cost
13	Lease (sq. ft.)																		

(d)

	A	B	C	D	E	F	G
1		Month Covered by Lease?			Total		Space
2	Month of Lease:	1	1	2	Leased		Required
3	Length of Lease:	1	2	1	(sq. ft.)		(sq. ft.)
4	Month 1	1	1		30,000	>=	30,000
5	Month 2		1	1	20,000	>=	20,000
6							
7	Cost of Lease	$65	$100	$65			
8	(per sq. ft.)						
9							Total Cost
10	Lease (sq. ft.)	10,000	20,000	0			$2,650,000

(e)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1							Month Covered by Lease?										Total		Space
2	Month of Lease:	1	1	1	1	1	2	2	2	2	3	3	3	4	4	5	Leased		Required
3	Length of Lease:	1	2	3	4	5	1	2	3	4	1	2	3	1	2	1	(sq. ft.)		(sq. ft.)
4	Month 1	1	1	1	1	1											30,000	>=	30,000
5	Month 2		1	1	1	1	1	1	1	1							30,000	>=	20,000
6	Month 3			1	1	1		1	1	1	1	1	1				40,000	>=	40,000
7	Month 4				1	1			1	1		1	1	1	1		30,000	>=	10,000
8	Month 5					1				1			1		1	1	50,000	>=	50,000
9																			
10	Cost of Lease	$65	$100	$135	$160	$190	$65	$100	$135	$160	$65	$100	$135	$65	$100	$65			
11	(per sq. ft.)																		
12																			Total Cost
13	Lease (sq. ft.)	0	0	0	0	30,000	0	0	0	0	10,000	0	0	0	0	20,000			$7,650,000

21.6.

(a) Larry needs to know how many employees should work each possible shift. Therefore, the decision variables are the number of employees that work each shift. The objective is to minimize the total cost of the employees.

(b) Working 8 A.M.-noon: 3 FT morning $+ 3$ PT $= 6$
Working Noon-4 P.M.: 3 FT morning $+ 2$ FT afternoon $+ 3$ PT $= 8$
Working 4 P.M-8 P.M: 2 FT afternoon $+ 4$ FT evening $+ 3$ PT $= 9$
Working 8 P.M-midnight: 4 FT evening $+ 3$ PT $= 7$

Total cost per day $= (9\text{ FT})(8\text{ hrs})(\$40/\text{hr}) + (12\text{ PT})(4\text{ hrs})(\$30/\text{hr}) = \$4,320$

(c)

	A	B	C	D	E	F	G	H	I	J	K
1		Full Time	Full Time	Full Time	Part Time	Part Time	Part Time	Part Time			
2		8am-4pm	noon-8pm	4pm-midnight	8am-noon	noon-4pm	4pm-8pm	8pm-midnight			
3	Cost per Shift										
4									Total		Total
5				Shift Covers Time of Day? (1=yes, 0=no)					Working		Needed
6	8am-noon									>=	
7	noon-4pm									>=	
8	4pm-8pm									>=	
9	8pm-midnight									>=	
10											
11	Workers per Shift										
12											
13											
14		Total		Times Total				Total			
15	Time of Day	Full Time		Part Time				Cost			
16	8am-noon		>=								
17	noon-4pm		>=								
18	4pm-8pm		>=								
19	8pm-midnight		>=								

(d)

	Full Time	Full Time	Full Time	Part Time	Part Time	Part Time	Part Time			
	8am-4pm	noon-8pm	4pm-midnight	8am-noon	noon-4pm	4pm-8pm	8pm-midnight			
Cost per Shift	$320	$320	$320	$120	$120	$120	$120			
								Total		Total
			Shift Covers Time of Day? (1=yes, 0=no)					Working		Needed
8am-noon	1			1				4	>=	4
noon-4pm	1	1			1			8	>=	8
4pm-8pm		1	1			1		10	>=	10
8pm-midnight			1				1	6	>=	6
Workers per Shift	2.6666667	2.6666667	4	1.3333333	2.6666667	3.3333333	2			
			2							
	Total		Times Total				Total			
Time of Day	Full Time		Part Time				Cost			
8am-noon	2.6666667	>=	2.666666667				$4,107			
noon-4pm	5.3333333	>=	5.333333333							
4pm-8pm	6.6666667	>=	6.666666667							
8pm-midnight	4	>=	4							

21.7.

(a) Al will need to know how much to invest in each possible investment each year. Thus, the decisions are how much to invest in investment A in year 1, 2, 3, and 4; how much to invest in B in year 1, 2, and 3; how much to invest in C in year 2; and how much to invest in D in year 5. The objective is to accumulate the maximum amount of money by the beginning of year 6.

(b)

Ending Cash(Y1) = ($60,000)(Starting Balance)-($20,000)(A in Y1) = $40,000
Ending Cash(Y2) = ($40,000)(Starting Balance)-($20,000)(B in Y2)-($20,000)(C in Y2) = $0
Ending Cash(Y3) = ($0)(Starting Balance)+($20,000)(1.4)(investment A) = $28,000
Ending Cash(Y4) = ($28,000)(Starting Balance)
Ending Cash(Y5) = ($28,000)(Starting Balance)+($20,000)(1.7)(investment B) = $62,000
Ending Cash(Y6) = ($62,000)(Starting Balance)+($20,000)(1.9)(investment C) = $100,000

(c)

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Beginning Balance												
2	Minimum Balance												
3													
4	Investment	A	A	A	A	B	B	B	C	D	Ending		Minimum
5	Year	1	2	3	4	1	2	3	2	5	Balance		Balance
6	Year 1											>=	
7	Year 2											>=	
8	Year 3											>=	
9	Year 4											>=	
10	Year 5											>=	
11	Year 6											>=	
12													
13	Dollars Invested												

(d)

	A	B	C	D	E	F	G	H	I	J	K
1	Beginning Balance	$60,000									
2	Minimum Balance	$0									
3											
4	Investment	A	A	A	B	B	B	C	Ending		Minimum
5	Year	1	2	3	1	2	3	2	Balance		Balance
6	Year 1	-1			-1				$0	>=	$0
7	Year 2		-1			-1		-1	$0	>=	$0
8	Year 3	1.4		-1			-1		$84,000	>=	$0
9											
10	Dollars Invested	$60,000	$0	$0	$0	$0	$0	$0			

(e)

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Beginning Balance	$60,000											
2	Minimum Balance	$0											
3													
4	Investment	A	A	A	A	B	B	B	C	D	Ending		Minimum
5	Year	1	2	3	4	1	2	3	2	5	Balance		Balance
6	Year 1	-1				-1					$0	>=	$0
7	Year 2		-1				-1		-1		$0	>=	$0
8	Year 3	1.4		-1				-1			$0	>=	$0
9	Year 4		1.4		-1	1.7					$0	>=	$0
10	Year 5			1.4			1.7			-1	$0	>=	$0
11	Year 6				1.4			1.7	1.9	1.3	$152,880	>=	$0
12													
13	Dollars Invested	$60,000	$0	$84,000	$0	$0	$0	$0	$0	$117,600			

21.8.

In the poor formulation, the data are not separated from the formula - they are buried inside the equations in column C. In contrast, the spreadsheet in Figure 21.6 separates all of the data in their own cells, and then the formulas for hours used and total profit refer to these data cells.

In the poor formulation, no range names are used. The spreadsheet in Figure 21.6 uses range names for UnitProfit, HoursUsed, TotalProfit, etc.

The poor formulation uses no borders, shading, or colors to distinguish between cell types. The spreadsheet in Figure 21.6 uses borders and shading to distinguish the data cells, changing cells, and target cell.

The poor formulation does not show the entire model on the spreadsheet. There is no indication of the constraints on the spreadsheet (they are only displayed in the Solver dialogue box). Furthermore, the right-hand-sides of the constraints are not on the spreadsheet, but buried in the Solver dialogue box. The spreadsheet in Figure 21.6 shows all of the constraints of the model in three adjacent cells on the spreadsheet.

21.9.

Cell F16 has -0.47 for LT Interest, rather than -LTRate*LTLoan.

Cell G14 for the 2013 ST Interest uses the LT Loan amount rather than the ST Loan amount.

Cell H21 for the LT Payback refers to the 2010 ST Loan rather than the LT Loan to determine the payback amount.

21.10.

Cell G21 for the 2020 ST Interest uses LTRate instead of STRate.

Cell H21 for the LT Payback in 2020 has -6.649 instead of -LTLoan.

Cell I15 for ST Payback in 2014 has -LTLoan instead of -E14 (STLoan for 2013).

CASES

CASE 21.1 Prudent Provisions for Pensions

(a) PFS needs to know how many units of each of the four bonds to purchase, how much to invest in the money market, and their ending balance in the money market fund each year after paying the pensions. The decisions are how many units of each bond to purchase, as well as the initial investment in 2007 in the money market. The objective is to minimize the overall initial investment necessary in 2007 in order to meet the pension payments through 2016.

(b)

Payment received from Bond 1 (2008) = (10,000 units)($1,000 face value)
+ (10,000 units)($1,000 face value)(0.04)
= $10.4 million

Payment received from Bond 1 (2009) = $0

Payment received from Bond 2 (2008) = (10,000 units)($1,000 face value)(0.02)
= $0.2 million

Payment received from Bond 2 (2009) = (10,000 units)($1,000 face value)(0.02)
= $0.2 million

Balance in money market fund (2007) = $28 million (initial investment)
− $8 million (pension payment)
= $20 million

Balance in money market fund (2008) = $20 million (starting balance)
+ $10.4 million (payment from Bond 1)
+ $0.2 million (payment from Bond 2)
− $12 million (pension payment)
+ $1 million (money market interest)
= $19.6 million

Balance in money market fund (2009) = $19.6 million (starting balance)
+ $0.2 million (payment from Bond 2)
− $13 million (pension payment)
+ $0.98 million (money market interest)
= $7.78 million

(c) PFS will need to track the flow of cash from bond investments, the initial investment, the required pension payments, interest from the money market, and the money market balance. The decisions are the number of units to purchase of each bond. Data for the problem include the yearly cash flows from the bonds (per unit purchased), the money market rate, and the minimum required balance in the money market fund at the end of each year. A sketch of a spreadsheet model might appear as follows.

	A	B	C	D	E	F	G	H	I	J	K	L
1							Money Market Rate					
2							Minimum Required Balance					
3												
4								Required	Money	Money		
5		Bond Cash Flows (per unit)				Bond	Initial	Pension	Market	Market		
6		Bond 1	Bond 2	Bond 3	Bond 4	Flow	Investment	Flow	Interest	Balance		
7	2010										>=	0
8	2011										>=	0
9	2012										>=	0
10	2013										>=	0
11	2014										>=	0
12	2015										>=	0
13	2016										>=	0
14	2017										>=	0
15	2018										>=	0
16	2019										>=	0
17												
18	Units Purchased											

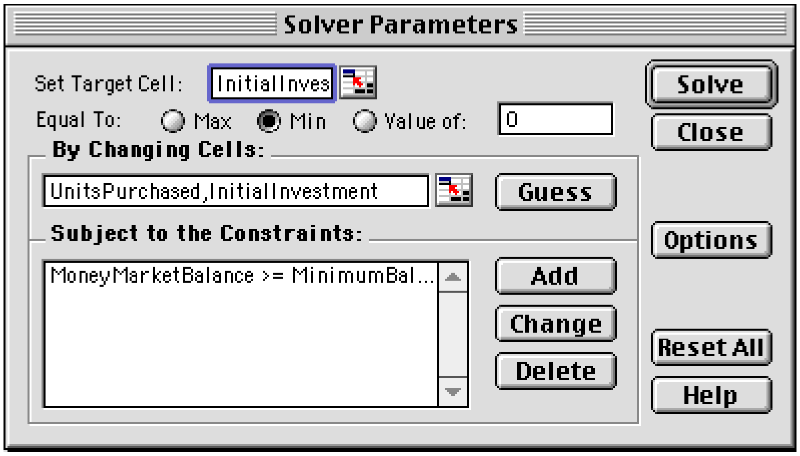
(d) The bond cash flows (per unit) are calculated in B7:E9. For example, one unit of Bond 1 costs $0.98 in 2007, and returns the face value ($1) plus the coupon rate ($0.04) in 2008. The total cash flow from bonds is then calculated in column F. The Initial Investment (G7) is both a decision variable and the target cell. It includes all money invested on January 1, 2007 (including enough to pay for the bonds and pension payment in 2007, as well as any initial investment in the money market).

If just years 2007 through 2009 are considered, then 23.44 thousand units of Bond 1 should be purchased at a cost of $22.97 million, along with an initial $8 million investment in the money market fund on January 1, 2007.

	A	B	C	D	E	F	G	H	I	J	K	L
1							Money Market Rate		5%			
2							Minimum Required Balance		0			
3												
4								Required	Money	Money		
5		Bond Cash Flows (per unit)				Bond	Initial	Pension	Market	Market		
6		Bond 1	Bond 2	Bond 3	Bond 4	Flow	Investment	Flow	Interest	Balance		
7	2010	-0.98	-0.92	-0.75	-0.80	-22.97	30.97	-8		0.00	>=	0
8	2011	1.04	0.02		0.03	24.38		-12	0.00	12.38	>=	0
9	2012		0.02		0.03	0.00		-13	0.62	0.00	>=	0
10												
11	Units Purchased	23.44	0	0	0		all cash figures in $millions					
12	(thousands)											
13												
14	Cost of Bonds	0.98	0.92	0.75	0.8							

	F
5	Bond
6	Flow
7	=SUMPRODUCT(B7:E7,UnitsPurchased)
8	=SUMPRODUCT(B8:E8,UnitsPurchased)
9	=SUMPRODUCT(B9:E9,UnitsPurchased)

	I	J
4	Money	Money
5	Market	Market
6	Interest	Balance
7		=SUM(F7:I7)
8	=MoneyMarketRate*J7	=J7+SUM(F8:I8)
9	=MoneyMarketRate*J8	=J8+SUM(F9:I9)



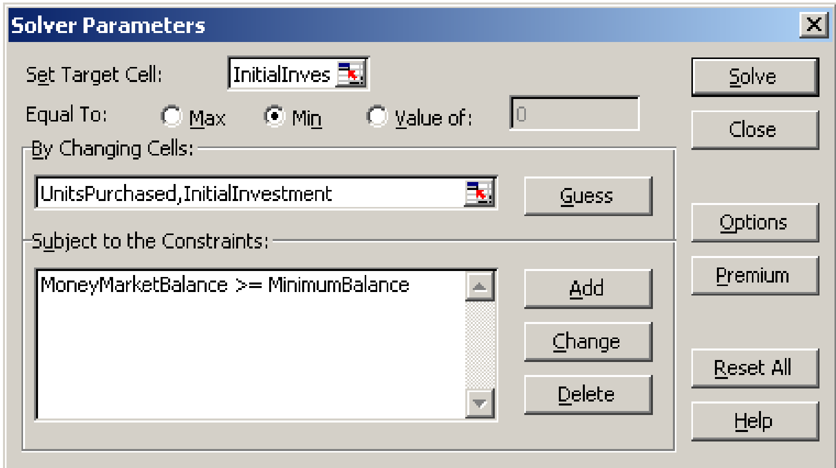
Range Name	Cells
BondFlow	F7:F9
InitialInvestment	G7
MinimumBalance	L7:L9
MinimumRequiredBalance	I2
MoneyMarketBalance	J7:J9
MoneyMarketInterest	I7:I9
MoneyMarketRate	I1
PensionFlow	H7:H9
UnitsPurchased	B11:E11

(e) Expanded to consider all years through 2016, the spreadsheet is as shown below. PFS should purchase 44.27 thousand units of Bond 1, 51.36 thousand units of Bond 3, and 43.55 thousand units of Bond 4 (at a cost of $116.74 million), and invest an additional $8 million in the money market on January 1, 2007.

	A	B	C	D	E	F	G	H	I	J	K	L
1								Money Market Rate	5%			
2								Minimum Required Balance	0			
3												
4								Required	Money	Money		
5		Bond Cash Flows (per unit)				Bond	Initial	Pension	Market	Market		
6		Bond 1	Bond 2	Bond 3	Bond 4	Flow	Investment	Flow	Interest	Balance		
7	2010	-0.98	-0.92	-0.75	-0.80	-116.74	124.74	-8		0.00	>=	0
8	2011	1.04	0.02		0.03	47.34		-12	0.00	35.34	>=	0
9	2012		0.02		0.03	1.31		-13	1.77	25.42	>=	0
10	2013		1.02		0.03	1.31		-14	1.27	13.99	>=	0
11	2014				0.03	1.31		-16	0.70	0.00	>=	0
12	2015			1.00	0.03	52.67		-17	0.00	35.67	>=	0
13	2016				0.03	1.31		-20	1.78	18.76	>=	0
14	2017				0.03	1.31		-21	0.94	0.00	>=	0
15	2018				1.03	44.86		-22	0.00	22.86	>=	0
16	2019					0.00		-24	1.14	0.00	>=	0
17												
18	Units Purchased	44.27	0	51.36	43.55		all cash figures in $millions					
19	(thousands)											

	F
5	Bond
6	Flow
7	=SUMPRODUCT(B7:E7,UnitsPurchased)
8	=SUMPRODUCT(B8:E8,UnitsPurchased)
9	=SUMPRODUCT(B9:E9,UnitsPurchased)
10	=SUMPRODUCT(B10:E10,UnitsPurchased)
11	=SUMPRODUCT(B11:E11,UnitsPurchased)
12	=SUMPRODUCT(B12:E12,UnitsPurchased)
13	=SUMPRODUCT(B13:E13,UnitsPurchased)
14	=SUMPRODUCT(B14:E14,UnitsPurchased)
15	=SUMPRODUCT(B15:E15,UnitsPurchased)
16	=SUMPRODUCT(B16:E16,UnitsPurchased)

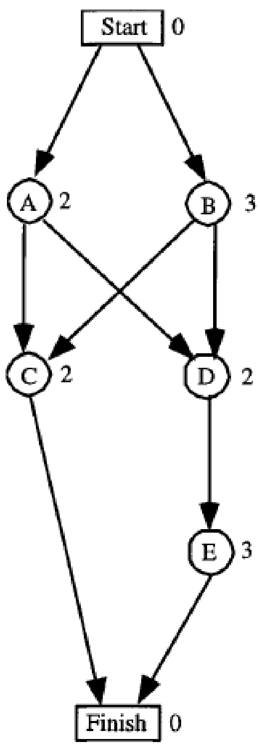
	I	J
4	Money	Money
5	Market	Market
6	Interest	Balance
7		=SUM(F7:I7)
8	=MoneyMarketRate*J7	=J7+SUM(F8:I8)
9	=MoneyMarketRate*J8	=J8+SUM(F9:I9)
10	=MoneyMarketRate*J9	=J9+SUM(F10:I10)
11	=MoneyMarketRate*J10	=J10+SUM(F11:I11)
12	=MoneyMarketRate*J11	=J11+SUM(F12:I12)
13	=MoneyMarketRate*J12	=J12+SUM(F13:I13)
14	=MoneyMarketRate*J13	=J13+SUM(F14:I14)
15	=MoneyMarketRate*J14	=J14+SUM(F15:I15)
16	=MoneyMarketRate*J15	=J15+SUM(F16:I16)



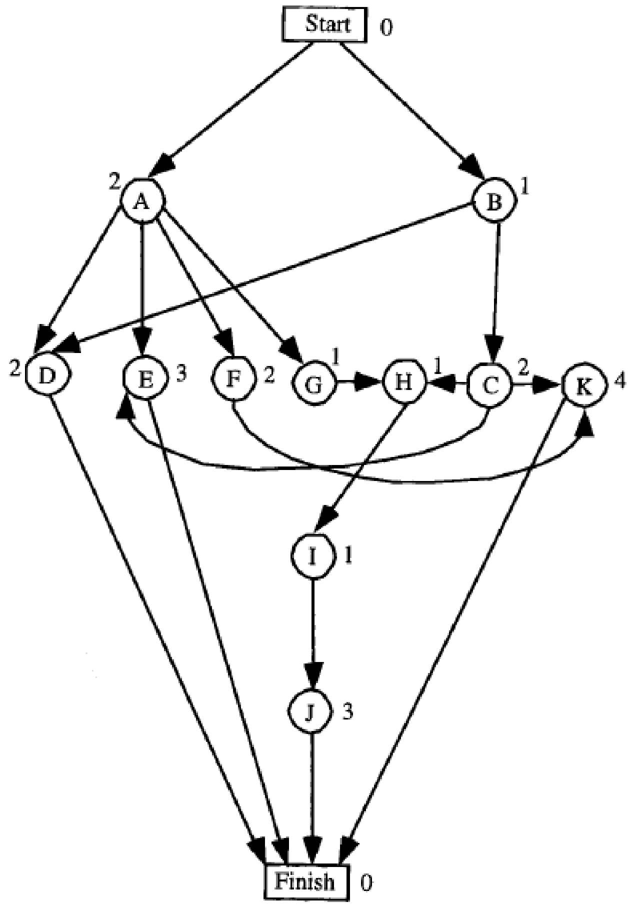
Range Name	Cells
BondFlow	F7:F16
InitialInvestment	G7
MinimumBalance	L7:L16
MinimumRequiredBalance	I2
MoneyMarketBalance	J7:J16
MoneyMarketInterest	I7:I16
MoneyMarketRate	I1
PensionFlow	H7:H16
UnitsPurchased	B18:E18

CHAPTER 22: PROJECT MANAGEMENT WITH PERT/CPM

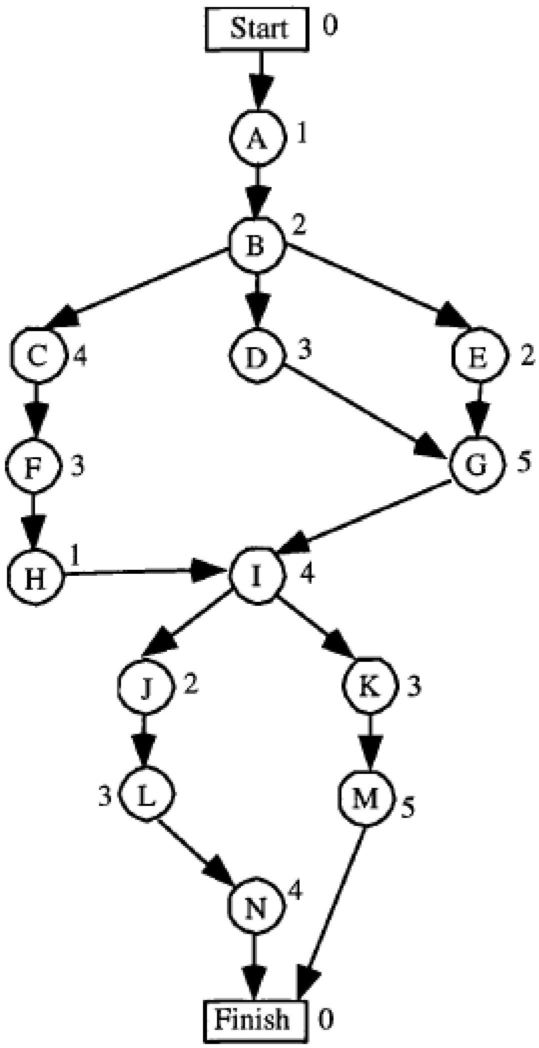
22.2-1.



22.2-2.

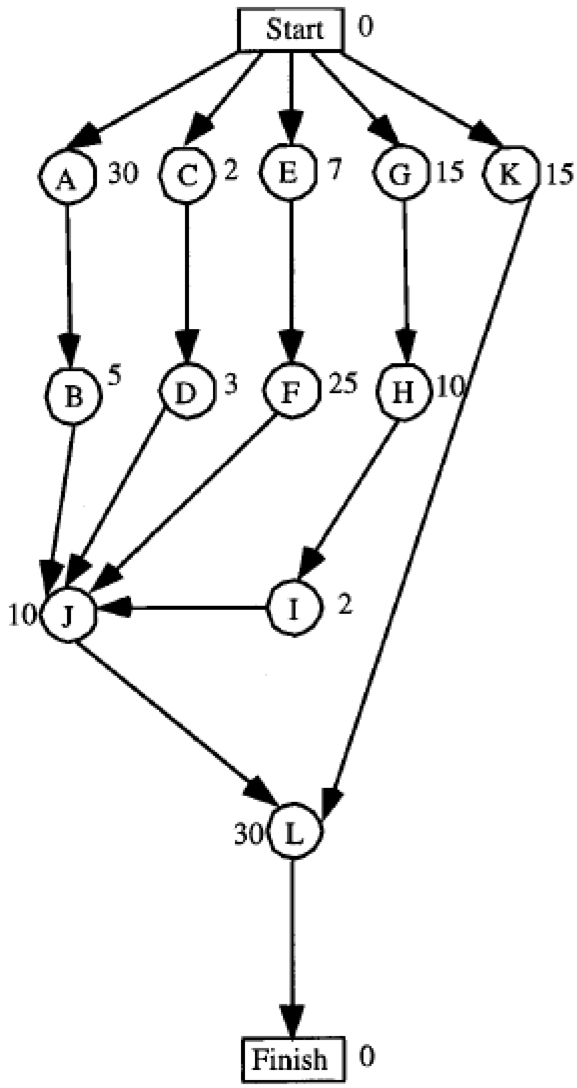


22.2-3.



22.3-1.

(a)



(b) Start $\to A \to B \to J \to L \to$ Finish   Length $= 75$ minutes
Start $\to C \to D \to J \to L \to$ Finish   Length $= 45$ minutes
Start $\to E \to F \to J \to L \to$ Finish   Length $= 72$ minutes
Start $\to G \to H \to I \to J \to L \to$ Finish   Length $= 67$ minutes
Start $\to K \to L \to$ Finish   Length $= 45$ minutes

Hence, Start $\to A \to B \to J \to L \to$ Finish is the critical path.

(c) - (d) - (e)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	30	0	30	0	Yes
B	30	35	30	35	0	Yes
C	0	2	30	32	30	No
D	2	5	32	35	30	No
E	0	7	3	10	3	No
F	7	32	10	35	3	No
G	0	15	8	23	8	No
H	15	25	23	33	8	No
I	25	27	33	35	8	No
J	35	45	35	45	0	Yes
K	0	15	30	45	30	No
L	45	75	45	75	0	Yes
Finish	75	75	75	75	0	Yes

Critical Path: Start $\to A \to B \to J \to L \to$ Finish

(f) Dinner will be delayed three minutes because of the phone call. If the food processor is used, dinner will not be delayed, since there was a slack of three minutes, five minutes of cutting time is saved and the call used only six minutes of these eight minutes.

22.3-2.

(a)	Start $\to A \to C \to$ Finish	Length $= 4$ weeks
	Start $\to A \to D \to E \to$ Finish	Length $= 7$ weeks
	Start $\to B \to C \to$ Finish	Length $= 5$ weeks
	Start $\to B \to D \to E \to$ Finish	Length $= 8$ weeks

Hence, Start $\to B \to D \to E \to$ Finish is the critical path.

(b)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	2	1	3	1	No
B	0	3	0	3	0	Yes
C	3	5	6	8	3	No
D	3	5	3	5	0	Yes
E	5	8	5	8	0	Yes
Finish	8	8	8	8	0	Yes

Critical Path: Start $\to B \to D \to E \to$ Finish

(c) No, this will not shorten the length of the project because the activity is not on the critical path.

22.3-3.

(a) Start $\rightarrow A \rightarrow D \rightarrow$ Finish Length $= 4$ weeks
Start $\rightarrow A \rightarrow E \rightarrow$ Finish Length $= 5$ weeks
Start $\rightarrow A \rightarrow F \rightarrow K \rightarrow$ Finish Length $= 8$ weeks
Start $\rightarrow A \rightarrow G \rightarrow H \rightarrow I \rightarrow J \rightarrow$ Finish Length $= 8$ weeks
Start $\rightarrow B \rightarrow D \rightarrow$ Finish Length $= 3$ weeks
Start $\rightarrow B \rightarrow C \rightarrow E \rightarrow$ Finish Length $= 6$ weeks
Start $\rightarrow B \rightarrow C \rightarrow H \rightarrow I \rightarrow J \rightarrow$ Finish Length $= 8$ weeks
Start $\rightarrow B \rightarrow C \rightarrow K \rightarrow$ Finish Length $= 7$ weeks

Critical Paths: Start $\rightarrow A \rightarrow F \rightarrow K \rightarrow$ Finish
Start $\rightarrow A \rightarrow G \rightarrow H \rightarrow I \rightarrow J \rightarrow$ Finish
Start $\rightarrow B \rightarrow C \rightarrow H \rightarrow I \rightarrow J \rightarrow$ Finish

(b)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	2	0	2	0	Yes
B	0	1	0	1	0	Yes
C	1	3	1	3	0	Yes
D	2	4	6	8	4	No
E	3	6	5	8	2	No
F	2	4	2	4	0	Yes
G	2	3	2	3	0	Yes
H	3	4	3	4	0	Yes
I	4	5	4	5	0	Yes
J	5	8	5	8	0	Yes
K	4	8	4	8	0	Yes
Finish	8	8	8	8	0	Yes

Critical Paths: Start $\rightarrow A \rightarrow F \rightarrow K \rightarrow$ Finish
Start $\rightarrow A \rightarrow G \rightarrow H \rightarrow I \rightarrow J \rightarrow$ Finish
Start $\rightarrow B \rightarrow C \rightarrow H \rightarrow I \rightarrow J \rightarrow$ Finish

(c) No, this will not shorten the length of the project because A is not on all of the critical paths.

22.3-4.

(a) Start $\rightarrow A \rightarrow D \rightarrow H \rightarrow M \rightarrow$ Finish Length $= 19$ weeks
Start $\rightarrow B \rightarrow E \rightarrow J \rightarrow M \rightarrow$ Finish Length $= 20$ weeks
Start $\rightarrow C \rightarrow F \rightarrow K \rightarrow N \rightarrow$ Finish Length $= 16$ weeks
Start $\rightarrow A \rightarrow I \rightarrow M \rightarrow$ Finish Length $= 17$ weeks
Start $\rightarrow C \rightarrow G \rightarrow L \rightarrow N \rightarrow$ Finish Length $= 20$ weeks

Critical Paths: Start $\rightarrow B \rightarrow E \rightarrow J \rightarrow M \rightarrow$ Finish
Start $\rightarrow C \rightarrow G \rightarrow L \rightarrow N \rightarrow$ Finish

(b)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	6	1	7	1	No
B	0	3	0	3	0	Yes
C	0	4	0	4	0	Yes
D	6	10	7	11	1	No
E	3	10	3	10	0	Yes
F	4	8	8	12	4	No
G	4	10	4	10	0	Yes
H	10	13	11	14	1	No
I	6	11	9	14	3	No
J	10	14	10	14	0	Yes
K	8	11	12	15	4	No
L	10	15	10	15	0	Yes
M	14	20	14	20	0	Yes
N	15	20	15	20	0	Yes
Finish	20	20	20	20	0	Yes

Ken will be able to meet his deadline.

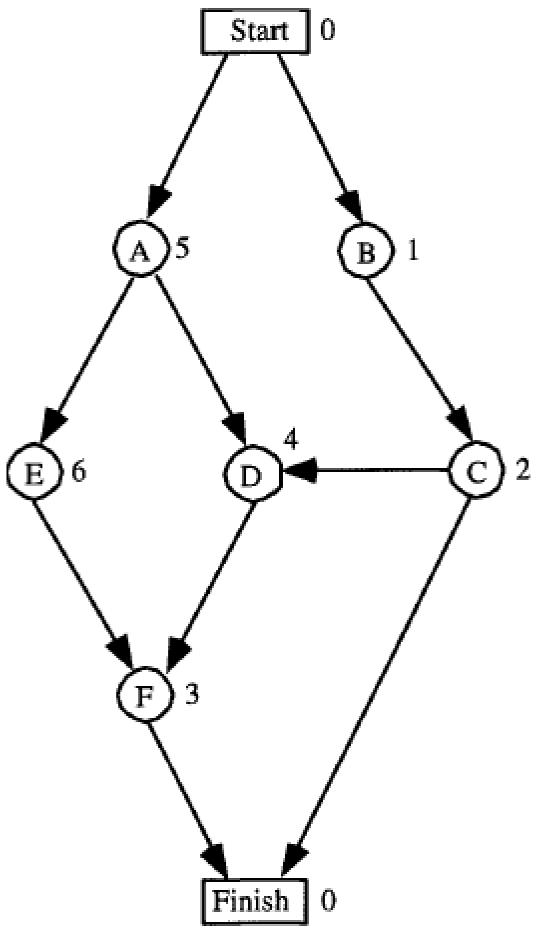
(c) Critical Paths: Start $\rightarrow B \rightarrow E \rightarrow J \rightarrow M \rightarrow$ Finish
Start $\rightarrow C \rightarrow G \rightarrow L \rightarrow N \rightarrow$ Finish

Focus attention on activities with no slack.

(d) If activity I takes two more weeks, there will be no delay because its slack is three. If activity H takes two extra weeks, then there will be a delay of one week because its slack is only one week. If activity J takes two more weeks, there will be a delay of two weeks, since it has no slack.

22.3-5.

(a)



(b)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	5	0	5	0	Yes
B	0	1	11	12	11	No
C	1	3	12	14	11	No
D	5	9	7	11	2	No
E	5	11	5	11	0	Yes
F	11	14	11	14	0	Yes
Finish	14	14	14	14	0	Yes

Critical Path: Start $\rightarrow A \rightarrow E \rightarrow F \rightarrow$ Finish

(c) 6 months

22.3-6.

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	3	0	3	0	Yes
B	3	11	3	11	0	Yes
C	11	29	11	29	0	Yes
D	29	39	29	39	0	Yes
E	29	34	30	35	1	No
F	34	44	35	45	1	No
G	39	50	39	50	0	Yes
H	50	67	50	67	0	Yes
I	29	38	36	45	7	No
J	44	53	45	54	1	No
K	53	57	57	61	4	No
L	53	60	54	61	1	No
M	67	70	67	70	0	Yes
N	60	69	61	70	1	No
Finish	70	70	70	70	0	Yes

Critical Path: Start $\rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow G \rightarrow H \rightarrow M \rightarrow$ Finish
Total duration: 70 weeks

22.3-7.

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	1	0	1	0	Yes
B	1	3	1	3	0	Yes
C	3	9	3	9	0	Yes
D	9	13	11	15	2	No
E	9	10	9	10	0	Yes
F	10	14	10	14	0	Yes
G	13	18	15	20	2	No
H	18	23	20	25	2	No
I	9	12	11	14	2	No
J	14	17	14	17	0	Yes
K	17	21	17	21	0	Yes
L	17	18	20	21	3	No
M	23	24	25	26	2	No
N	21	26	21	26	0	Yes
Finish	26	26	26	26	0	Yes

Critical Path: Start $\rightarrow A \rightarrow B \rightarrow C \rightarrow E \rightarrow F \rightarrow J \rightarrow K \rightarrow N \rightarrow$ Finish
Total duration: 26 weeks

22.3-8.

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	1	0	1	0	Yes
B	1	3	1	3	0	Yes
C	3	10	3	10	0	Yes
D	10	14	13	17	3	No
E	10	13	10	13	0	Yes
F	13	16	13	16	0	Yes
G	14	18	17	21	3	No
H	18	24	21	27	3	No
I	10	15	11	16	1	No
J	16	22	16	22	0	Yes
K	22	25	22	25	0	Yes
L	22	25	22	25	0	Yes
M	24	25	27	28	3	No
N	25	28	25	28	0	Yes
Finish	28	28	28	28	0	Yes

Critical Path: Start $\to A \to B \to C \to E \to F \to J \to K \to N \to$ Finish
Start $\to A \to B \to C \to E \to F \to J \to L \to N \to$ Finish

Total duration: 28 weeks

22.4-1.

$$\mu = \frac{o+4m+p}{6} = \frac{30+4(36)+48}{6} = 37$$

$$\sigma^2 = \left(\frac{p-o}{6}\right)^2 = \left(\frac{48-30}{6}\right)^2 = 9$$

22.4-2.

(a) Start $\to A \to E \to I \to$ Finish — Length $=17$ months
Start $\to A \to C \to F \to I \to$ Finish — Length $=17$ months
Start $\to B \to D \to G \to J \to$ Finish — Length $=17$ months
Start $\to B \to H \to J \to$ Finish — Length $=18$ months

Critical Path: Start $\to B \to H \to J \to$ Finish

(b) $\frac{d-\mu_p}{\sqrt{\sigma_p}} = \frac{22-18}{\sqrt{31}} = 0.718 \Rightarrow \mathrm{P}\{T \leq 22\} \approx 0.77$

(c) Start $\to A \to E \to I \to$ Finish: $\frac{d-\mu_p}{\sqrt{\sigma_p}} = \frac{22-17}{\sqrt{25}} = 1 \Rightarrow \mathrm{P}\{T \leq 22\} \approx 0.84$

Start $\to A \to C \to F \to I \to$ Finish: $\frac{d-\mu_p}{\sqrt{\sigma_p}} = \frac{22-17}{\sqrt{27}} = 0.962 \Rightarrow \mathrm{P}\{T \leq 22\} \approx 0.84$

Start $\to B \to D \to G \to J \to$ Finish: $\frac{d-\mu_p}{\sqrt{\sigma_p}} = \frac{22-17}{\sqrt{28}} = 0.945 \Rightarrow \mathrm{P}\{T \leq 22\} \approx 0.84$

(d) There is approximately a 77% chance that the drug will be ready in 22 weeks.

22.4-3.

Start → B → H → J → Finish

Mean Critical Path

μ =	**18.4166667**
σ^2 =	**31.2013889**
P(TŠd)=	**0.73940284**
where d =	**22**

Start → A → E → I → Finish

Mean Critical Path

μ =	**17.0833333**
σ^2 =	**25.3402778**
P(TŠd)=	**0.83564332**
where d =	**22**

Start → A → C → F → I → Finish

Mean Critical Path

μ =	**17.5833333**
σ^2 =	**27.3680556**
P(TŠd)=	**0.80073605**
where d =	**22**

Start → B → D → G → J → Finish

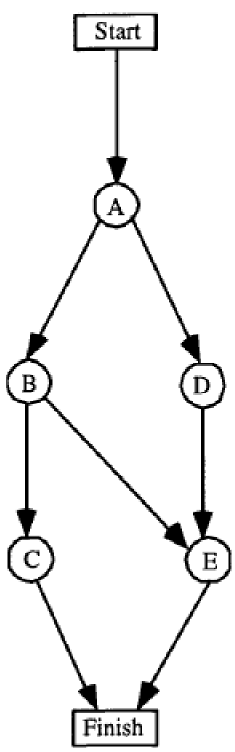
Mean Critical Path

μ =	**17.8333333**
σ^2 =	**28.0416667**
P(TŠd)=	**0.78431252**
where d =	**22**

There is approximately a 73% chance that the drug will be ready in 22 weeks.

22.4-4.

(a)



(b)

Activity	μ	σ^2
A	4	0.111
B	2	0
C	4.83	0.25
D	3	0.444
E	3.17	0.25

(c) Start $\rightarrow A \rightarrow B \rightarrow C \rightarrow$ Finish Length $= 10.83$ weeks
Start $\rightarrow A \rightarrow B \rightarrow E \rightarrow$ Finish Length $= 9.17$ weeks
Start $\rightarrow A \rightarrow D \rightarrow E \rightarrow$ Finish Length $= 10.17$ weeks

Critical Path: Start $\rightarrow A \rightarrow B \rightarrow C \rightarrow$ Finish

(d) $\frac{d-\mu_p}{\sqrt{\sigma_p}} = \frac{11-10.83}{\sqrt{0.361}} = 0.028 \Rightarrow \mathrm{P}\{T \leq 11\} = 0.6$

(e) Make the bid, since there is approximately a 60% chance that the project will be completed in 11 weeks or less.

22.4-5.

(a)

Activity	μ	σ^2
A	12	0
B	23	16
C	15	1
D	27	9
E	18	4
F	6	4

(b) Start $\rightarrow A \rightarrow C \rightarrow E \rightarrow F \rightarrow$ Finish Length $= 51$ days
Start $\rightarrow B \rightarrow D \rightarrow$ Finish Length $= 50$ days

Critical Path: Start $\rightarrow A \rightarrow C \rightarrow E \rightarrow F \rightarrow$ Finish

(c) $\frac{d-\mu_p}{\sqrt{\sigma_p}} = \frac{57-51}{\sqrt{9}} = 2 \Rightarrow \mathrm{P}\{T \leq 57\} = 0.9772$ (Normal Distribution table)

(d) $\frac{d-\mu_p}{\sqrt{\sigma_p}} = \frac{57-50}{\sqrt{25}} = 1.4 \Rightarrow \mathrm{P}\{T \leq 57\} = 0.9192$ (Normal Distribution table)

(e) $(0.9772)(0.9192) = 0.8982$, so the procedure used in (c) overestimates the probability of completing the project within 57 days.

22.4-6.

(a)

Activity	μ	σ^2
A	32	1.78
B	27.7	2.78
C	36	11.1
D	16	0.444
E	32	0
F	53.7	32.1
G	16.7	4
H	20.3	2.78
I	34	7.11
J	17.7	9

(b)

Start $\rightarrow A \rightarrow C \rightarrow J \rightarrow$ Finish	Length $= 85.7$ weeks
Start $\rightarrow B \rightarrow F \rightarrow J \rightarrow$ Finish	Length $= 99.1$ weeks
Start $\rightarrow B \rightarrow E \rightarrow H \rightarrow$ Finish	Length $= 80$ weeks
Start $\rightarrow B \rightarrow E \rightarrow I \rightarrow$ Finish	Length $= 93.7$ weeks
Start $\rightarrow B \rightarrow D \rightarrow G \rightarrow H \rightarrow$ Finish	Length $= 80.7$ weeks
Start $\rightarrow B \rightarrow D \rightarrow G \rightarrow I \rightarrow$ Finish	Length $= 94.4$ weeks

Critical Path: Start $\rightarrow B \rightarrow F \rightarrow J \rightarrow$ Finish

(c) $\frac{d-\mu_p}{\sqrt{\sigma_p}} = \frac{100-99.1}{\sqrt{43.89}} = 0.136 \Rightarrow \mathrm{P}\{T \leq 100\} = 0.4443$ (Normal Distribution table)

(d) Higher

22.4-7.

(a) TRUE. The optimistic and pessimistic estimates lie at the extremes of what is possible, p.33.

(b) FALSE. The probability distribution is a Beta distribution, p.33.

(c) FALSE. The mean critical path will turn out to be the longest path in the project network.

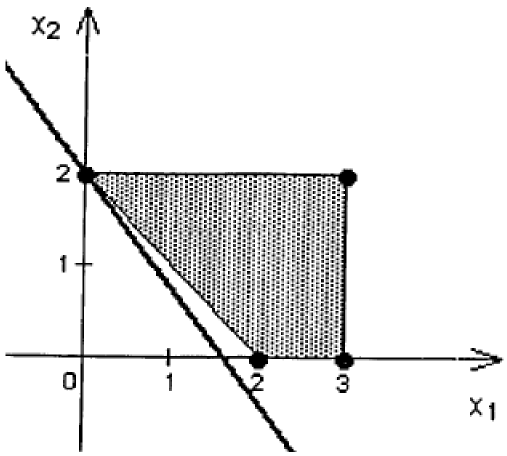
22.5-1.

		Length of Path	
Activity to Crash	Crash Cost	$A-C$	$B-D$
		14	16
B	$\$5,000$	14	15
B	$\$5,000$	14	15
D	$\$6,000$	14	14
C	$\$4,000$	13	14
D	$\$6,000$	13	13
C	$\$4,000$	12	13
D	$\$6,000$	12	12

22.5-2.

(a) Let x_A and x_C be the reduction in A and C respectively, due to crashing.

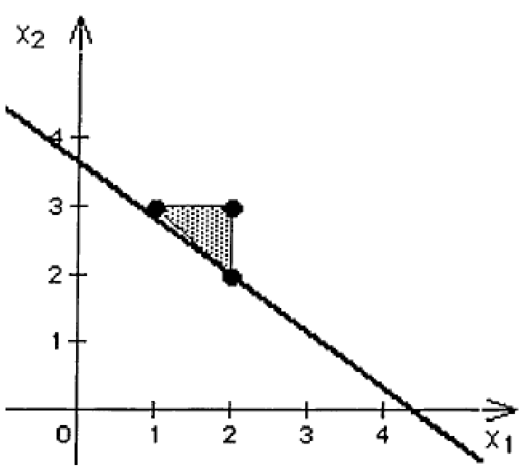
minimize	$C = 5000x_A + 4000x_C$
subject to	$x_A \leq 3$
	$x_C \leq 2$
	$x_A + x_C \geq 2$
and	$x_A, x_C \geq 0$



Optimal Solution: $(x_A, x_C) = (0, 2)$ and $C^* = 8,000$.

(b) Let x_B and x_D be the reduction in B and D respectively, due to crashing.

minimize	$C = 5000x_B + 6000x_D$
subject to	$x_B \leq 2$
	$x_D \leq 3$
	$x_B + x_D \geq 4$
and	$x_B, x_D \geq 0$



Optimal Solution: $(x_B, x_D) = (2, 2)$ and $C^* = 22,000$.

(c) Let x_A, x_B, x_C, and x_D be the reduction in the duration of A, B, C, and D respectively, due to crashing.

$$\begin{array}{ll} \text{minimize} & C = 5000x_A + 5000x_B + 4000x_C + 6000x_D \\ \text{subject to} & x_A \leq 3 \\ & x_B \leq 2 \\ & x_C \leq 2 \\ & x_D \leq 3 \\ & x_A + x_C \geq 2 \\ & x_B + x_D \geq 4 \\ \text{and} & x_A, x_B, x_C, x_D \geq 0 \end{array}$$

Optimal Solution: $(x_A, x_B, x_C, x_D) = (0, 2, 2, 2)$ and $C^* = 30,000$.

(d) Let x_j be the reduction in the duration of activity j due to crashing for $j = A, B, C, D$. Also let y_j denote the start time of activity j for $j = C, D$ and y_{FINISH} the project duration.

$$\begin{array}{ll} \text{minimize} & C = 5000x_A + 5000x_B + 4000x_C + 6000x_D \\ \text{subject to} & x_A \leq 3, x_B \leq 2, x_C \leq 2, x_D \leq 3 \\ & y_C \geq 0 + 8 - x_A \\ & y_D \geq 0 + 9 - x_B \\ & y_{\text{FINISH}} \geq y_C + 6 - x_C \\ & y_{\text{FINISH}} \geq y_D + 7 - x_D \\ & y_{\text{FINISH}} \leq 12 \\ \text{and} & x_A, x_B, x_C, x_D, y_C, y_D, y_{\text{FINISH}} \geq 0 \end{array}$$

(e)

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	8	5	\$25000	\$40000	3	\$5000	0	0	8
B	9	7	\$20000	\$30000	2	\$5000	0	2	7
C	6	4	\$16000	\$24000	2	\$4000	8	2	12
D	7	4	\$27000	\$45000	3	\$6000	7	2	12

Finish Time = 12
Total Cost = \$118000

(f) The solution found using LINGO agrees with the solution in (e), i.e., it is optimal to reduce the duration of activities B, C, and D by two months. Then the entire project takes 12 months and costs $25 + 30 + 24 + (27 + 12) = 118$ thousand dollars.

```
Variable           Value        Reduced Cost
      XA        0.000000            0.000000
      XB        2.000000            0.000000
      XC        2.000000            0.000000
      XD        2.000000            0.000000
```

```
Row     Slack or Surplus       Dual Price
  1         30000.00           -1.000000
  2         3.000000            0.000000
  3         0.000000            1000.000
  4         0.000000            1000.000
  5         1.000000            0.000000
  6         0.000000           -5000.000
  7         0.000000           -6000.000
```

(g) Deadline of 11 months

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	8	5	$25000	$40000	3	$5000	0	1	7
B	9	7	$20000	$30000	2	$5000	0	2	7
C	6	4	$16000	$24000	2	$4000	7	2	11
D	7	4	$27000	$45000	3	$6000	7	3	11

Finish Time = 11
Total Cost = $129000

Deadline of 13 months

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	8	5	$25000	$40000	3	$5000	0	0	8
B	9	7	$20000	$30000	2	$5000	0	2	7
C	6	4	$16000	$24000	2	$4000	8	1	13
D	7	4	$27000	$45000	3	$6000	7	1	13

Finish Time = 13
Total Cost = $108000

22.5-3.

(a)

Activity to Crash	Crash Cost	Length of Path B-D
		50
B	$10,000	49
B	$10,000	48
B	$10,000	47

(b)

Activity to Crash	Crash Cost	Length of Path A-C-E-F
		51
C	$10,000	50
C	$10,000	49
C	$10,000	48
E	$15,000	47

(c)

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	12	9	$210000	$270000	3	$20000	**0**	**0**	12
B	23	18	$410000	$460000	5	$10000	**0**	**3**	20
C	15	12	$290000	$320000	3	$10000	**12**	**3**	24
D	27	21	$440000	$500000	6	$10000	**20**	**0**	47
E	18	14	$350000	$410000	4	$15000	**24**	**1**	41
F	6	4	$160000	$210000	2	$25000	**41**	**0**	47

Finish Time = 47
Total Cost = $ 193,500

22.5-4.

(a)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	3	0	3	0	Yes
B	3	7	4	8	1	No
C	3	8	3	8	0	Yes
D	7	10	9	12	2	No
E	8	12	8	12	0	Yes
Finish	12	12	12	12	0	Yes

Critical Path: Start $\rightarrow A \rightarrow C \rightarrow E \rightarrow$ Finish
Total Duration: 12 weeks

(b) $7,834 is saved by the new plan given below.

		Length of Path		
Activity to Crash	Crash Cost	$A-B-D$	$A-B-E$	$A-C-E$
		10	11	12
C	$\$1,333$	10	11	11
E	$\$2,500$	10	10	10
D & E	$\$4,000$	9	9	9
B & C	$\$4,333$	8	8	8

Activity	Duration	Cost
A	3 weeks	$\$54,000$
B	3 weeks	$\$65,000$
C	3 weeks	$\$58,666$
D	2 weeks	$\$41,500$
E	2 weeks	$\$80,000$

(c)

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**0**	3
B	4	3	$62000	$65000	1	$3000	**4**	**0**	8
C	5	2	$66000	$70000	3	$1333	**3**	**0**	8
D	3	1	$40000	$43000	2	$1500	**9**	**0**	12
E	4	2	$75000	$80000	2	$2500	**8**	**0**	12

Finish Time = 12
Total Cost = $ 297,000

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**0**	3
B	4	3	$62000	$65000	1	$3000	**3**	**0**	7
C	5	2	$66000	$70000	3	$1333	**3**	**1**	7
D	3	1	$40000	$43000	2	$1500	**8**	**0**	11
E	4	2	$75000	$80000	2	$2500	**7**	**0**	11

Finish Time = 11
Total Cost = $ 298,333

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**0**	3
B	4	3	$62000	$65000	1	$3000	**3**	**0**	7
C	5	2	$66000	$70000	3	$1333	**3**	**1**	7
D	3	1	$40000	$43000	2	$1500	**7**	**1.22E-15**	10
E	4	2	$75000	$80000	2	$2500	**7**	**1**	10

Finish Time = 10
Total Cost = $ 300,833

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**0**	3
B	4	3	$62000	$65000	1	$3000	**3**	**4.66E-12**	7
C	5	2	$66000	$70000	3	$1333	**3**	**1**	7
D	3	1	$40000	$43000	2	$1500	**7**	**1**	9
E	4	2	$75000	$80000	2	$2500	**7**	**2**	9

Finish Time = 9
Total Cost = $ 304,833

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**3.66E-11**	3
B	4	3	$62000	$65000	1	$3000	**3**	**1**	6
C	5	2	$66000	$70000	3	$1333	**3**	**2**	6
D	3	1	$40000	$43000	2	$1500	**6**	**1**	8
E	4	2	$75000	$80000	2	$2500	**6**	**2**	8

Finish Time = 8
Total Cost = $ 309,167

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	\$54000	\$60000	1	\$6000	**0**	**1**	2
B	4	3	\$62000	\$65000	1	\$3000	**2**	**1**	5
C	5	2	\$66000	\$70000	3	\$1333	**2**	**2**	5
D	3	1	\$40000	\$43000	2	\$1500	**5**	**1**	7
E	4	2	\$75000	\$80000	2	\$2500	**5**	**2**	7

Finish Time = 7
Total Cost = \$ 315,167

Crash to 8 weeks.

22.5-5.

(a) Let x_j be the reduction in the duration of activity j and y_j be the start time of activity j.

minimize $\quad C = 6x_A + 12x_B + 4x_C + 6.67x_D + 10x_E + 7.33x_F + 5.75x_G + 8x_H$

subject to

$$\begin{array}{llll}
0 \le x_A \le 2 & 0 \le x_B \le 1 & 0 \le x_C \le 2 & 0 \le x_D \le 3 \\
0 \le x_E \le 1 & 0 \le x_F \le 3 & 0 \le x_G \le 4 & 0 \le x_H \le 2 \\
y_A + 5 - x_A \le y_C & & y_A + 5 - x_A \le y_D & \\
y_B + 3 - x_B \le y_E & & y_B + 3 - x_B \le y_F & \\
y_C + 4 - x_C \le y_G & & y_D + 6 - x_D \le y_H & \\
y_E + 5 - x_E \le y_G & & y_F + 7 - x_F \le y_H & \\
y_G + 9 - x_G \le y_{\text{FINISH}} & & y_H + 8 - x_H \le y_{\text{FINISH}} & \\
0 \le y_{\text{FINISH}} \le 15 & & & \\
y_j \ge 0 & & &
\end{array}$$

(b) Finish Time: 15 weeks, total crashing cost: \$$45.75$ million, total cost: \$$259.75$ million.

Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Maximum Time Reduction	Crash Cost per Week Saved	Start Time	Time Reduction	Finish Time
A	5	3	24	36	2	6.00	0	2	3
B	3	2	13	25	1	12.00	0	1	2
C	4	2	21	29	2	4.00	3	0	7
D	6	3	30	50	3	6.67	3	0	9
E	5	4	26	36	1	10.00	2	0	7
F	7	4	35	57	3	7.33	2	0	9
G	9	5	30	53	4	5.75	7	1	15
H	8	6	35	51	2	8.00	9	2	15

22.5-6.

(a) Let x_j be the reduction in the duration of activity j and y_j be the start time of activity j.

minimize $\quad C = 5x_A + 7x_B + 8x_C + 4x_D + 5x_E + 6x_F + 3x_G + 4x_H + 9x_I + 2x_J$

subject to

$0 \le x_A \le 4 \quad 0 \le x_B \le 3 \quad 0 \le x_C \le 5 \quad 0 \le x_D \le 3 \quad 0 \le x_E \le 5$
$0 \le x_F \le 7 \quad 0 \le x_G \le 2 \quad 0 \le x_H \le 3 \quad 0 \le x_I \le 4 \quad 0 \le x_J \le 2$
$y_A + 32 - x_A \le y_C \qquad y_B + 28 - x_B \le y_D$
$y_B + 28 - x_B \le y_E \qquad y_B + 28 - x_B \le y_F$
$y_C + 36 - x_C \le y_J \qquad y_D + 16 - x_D \le y_G$
$y_E + 32 - x_E \le y_H \qquad y_E + 32 - x_E \le y_I$
$y_F + 54 - x_F \le y_J \qquad y_G + 17 - x_G \le y_H$
$y_G + 17 - x_G \le y_I \qquad y_H + 20 - x_H \le y_{\text{FINISH}}$
$y_I + 34 - x_I \le y_{\text{FINISH}} \qquad y_J + 18 - x_J \le y_{\text{FINISH}}$
$0 \le y_{\text{FINISH}} \le 92$
$y_j \ge 0$

(b) Finish Time: 92 weeks, total crashing cost: \$43 million, total cost: \$1.388 billion.

Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Maximum Time Reduction	Crash Cost per Week Saved	Start Time	Time Reduction	Finish Time
A	32	28	160	180	4	5	8	0	40
B	28	25	125	146	3	7	0	3	25
C	36	31	170	210	5	8	40	0	76
D	16	13	60	72	3	4	25	0	41
E	32	27	135	160	5	5	26	0	58
F	54	47	215	257	7	6	25	3	76
G	17	15	90	96	2	3	41	0	58
H	20	17	120	132	3	4	58	0	78
I	34	30	190	226	4	9	58	0	92
J	18	16	80	84	2	2	76	2	92

22.6-1.

(a)

Activity	ES	EF
Start	0	0
A	0	3
B	3	6
C	3	6
D	6	8
E	6	8
Finish	8	8

Total Duration: 8 weeks

(b) - (c) - (d)

Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost per week of it's duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	3	$54,000	0	$ 18,000	18000	18000	18000	0	0	0	0	0
B	3	$65,000	3	$ 21,667	0	0	0	21667	21667	21667	0	0
C	3	$68,666	3	$ 22,889	0	0	0	22889	22889	22889	0	0
D	2	$41,500	6	$ 20,750	0	0	0	0	0	0	20750	20750
E	2	$80,000	6	$ 40,000	0	0	0	0	0	0	40000	40000
					0	0	0	0	0	0	0	0
				Weekly Project Cost	1800	1800	1800	44555	44555	44555	60750	60750
				Cumulative Project Cost	1800	3600	1800	5400	98555	14311	167666	309166

(e)

Activity	Budgeted Cost	Percent Completed	Value Completed	Actual Cost To Date	Cost Overrun To Date
A	$54,000	100%	$54,000	$65,000	$11,000
B	$65,000	100%	$65,000	$55,000	-$10,000
C	$68,666	33%	$22,660	$44,000	$21,340
Total	$187,666		$141,660	$164,000	$22,340

Michael should concentrate his efforts on activity C, since it is not yet completed.

22.6-2.

(a)

Activity	ES	EF	LS	LF	Slack
Start	0	0	0	0	0
A	0	6	0	6	0
B	0	2	4	6	4
C	6	10	9	13	3
D	6	11	6	11	0
E	10	17	13	20	3
F	11	20	11	20	0
Finish	20	20	20	20	0

The earliest finish time for this project is 20 weeks.

(b)

Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost per week of it's duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	6	$420,000	0	$ 70,000	7000	7000	7000	7000	7000	7000	0	0
B	2	$180,000	0	$ 90,000	9000	9000	0	0	0	0	0	0
C	4	$540,000	6	$ 135,000	0	0	0	0	0	0	135000	135000
D	5	$360,000	6	$ 72,000	0	0	0	0	0	0	72000	72000
E	7	$590,000	10	$ 84,286	0	0	0	0	0	0	0	0
F	9	$630,000	11	$ 70,000	0	0	0	0	0	0	0	0
					0	0	0	0	0	0	0	0
				Weekly Project Cost	160000	160000	70000	70000	70000	70000	207000	207000
				Cumulative Project Cost	160000	320000	390000	460000	530000	600000	807000	1014000

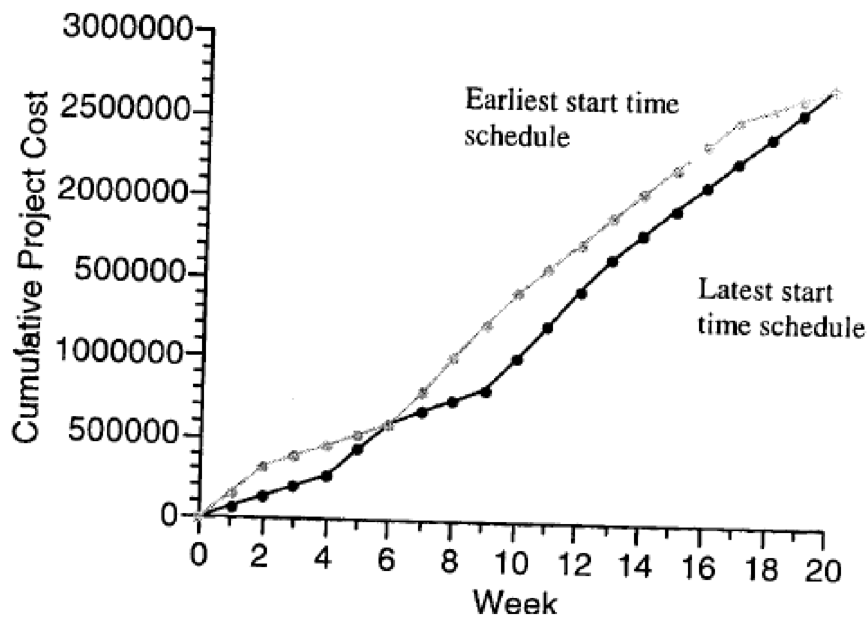
Week 9	Week 10	Week 11	Week 12	Week 13	Week 14	Week 15	Week 16	Week 17	Week 18	Week 19	Week 20
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
135000	135000	0	0	0	0	0	0	0	0	0	0
72000	72000	72000	0	0	0	0	0	0	0	0	0
0	0	84286	84285.7	84285.7	84285.7	84285.7	84285.7	84285.7	0	0	0
0	0	0	7000	7000	7000	7000	7000	7000	7000	7000	7000
0	0	0	0	0	0	0	0	0	0	0	0
207000	207000	156285.7	154286	154286	154286	154286	154286	154286	70000	70000	70000
1221000	1428000	1584286	1738571	1892857	2047143	2201429	2355714	2510000	2580000	2650000	2720000

(c)

Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost Per Week of Its Duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	6	$420,000	0	$70,000	70000	70000	70000	70000	70000	70000	0	0
B	2	$180,000	4	$90,000	0	0	0	0	90000	90000	0	0
C	4	$540,000	9	$135,000	0	0	0	0	0	0	0	0
D	5	$360,000	6	$72,000	0	0	0	0	0	0	72000	72000
E	7	$590,000	13	$84,286	0	0	0	0	0	0	0	0
F	9	$630,000	11	$70,000	0	0	0	0	0	0	0	0
					0	0	0	0	0	0	0	0

Week 9	Week 10	Week 11	Week 12	Week 13	Week 14	Week 15	Week 16	Week 17	Week 18	Week 19	Week 20
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	135000	135000	135000	135000	0	0	0	0	0	0	0
72000	72000	72000	0	0	0	0	0	0	0	0	0
0	0	0	0	0	84285.7	84285.7	84285.7	84285.7	84285.7	84285.7	84285.7
0	0	0	70000	70000	70000	70000	70000	70000	70000	70000	70000
0	0	0	0	0	0	0	0	0	0	0	0
72000	207000	207000	205000	205000	154286	154286	154286	154286	154286	154286	154286
816000	1023000	1230000	1435000	1640000	1794286	1948571	2102857	2257143	2411429	2565714	2720000

(d)



(e)

Activit y	Budgeted Cost	Percent Complete d	Value Completed	Actual Cost To Date	Cost Overrun To Date
A	$420,000	50%	$210,000	$200,000	-$10,000
B	$180,000	100%	$180,000	$200,000	$20,000
D	$360,000	50%	$180,000	$210,000	$30,000
Total	$960,000		$570,000	$610,000	$40,000

The project manager should focus attention on activity *D*, since it is not yet finished and they are running over budget.

22.6-3.

(a)

Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost Per Week of Its Duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	6	$180,000	0	$30,000	30000	30000	30000	30000	30000	30000	0	0
B	3	$75,000	0	$25,000	25000	25000	25000	0	0	0	0	0
C	4	$120,000	0	$30,000	30000	30000	30000	30000	0	0	0	0
D	4	$140,000	6	$35,000	0	0	0	0	0	0	35000	35000
E	7	$175,000	3	$25,000	0	0	0	25000	25000	25000	25000	25000
F	4	$80,000	4	$20,000	0	0	0	0	20000	20000	20000	20000
G	6	$210,000	4	$35,000	0	0	0	0	35000	35000	35000	35000
H	3	$45,000	10	$15,000	0	0	0	0	0	0	0	0
I	5	$125,000	6	$25,000	0	0	0	0	0	0	25000	25000
J	4	$100,000	10	$25,000	0	0	0	0	0	0	0	0
K	3	$60,000	8	$20,000	0	0	0	0	0	0	0	0
L	5	$50,000	10	$10,000	0	0	0	0	0	0	0	0
M	6	$90,000	14	$15,000	0	0	0	0	0	0	0	0
N	5	$150,000	15	$30,000	0	0	0	0	0	0	0	0
					0	0	0	0	0	0	0	0
				Weekly Project Cost	85000	85000	85000	85000	110000	110000	140000	140000
				Cumulative Project Cost	85000	170000	255000	340000	450000	560000	700000	840000

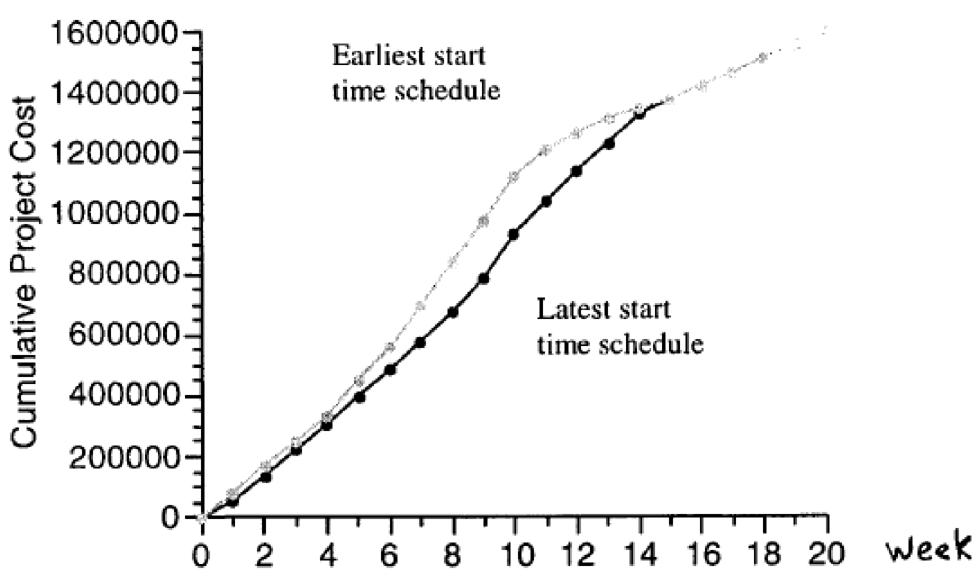
Week 9	Week 10	Week 11	Week 12	Week 13	Week 14	Week 15	Week 16	Week 17	Week 18	Week 19	Week 20
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
35000	35000	0	0	0	0	0	0	0	0	0	0
25000	25000	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
35000	35000	0	0	0	0	0	0	0	0	0	0
0	0	15000	15000	15000	0	0	0	0	0	0	0
25000	25000	25000	0	0	0	0	0	0	0	0	0
0	0	25000	25000	25000	25000	0	0	0	0	0	0
20000	20000	20000	0	0	0	0	0	0	0	0	0
0	0	10000	10000	10000	10000	10000	0	0	0	0	0
0	0	0	0	0	0	15000	15000	15000	15000	15000	15000
0	0	0	0	0	0	0	30000	30000	30000	30000	30000
0	0	0	0	0	0	0	0	0	0	0	0
140000	140000	95000	50000	50000	35000	25000	45000	45000	45000	45000	45000
980000	1120000	1215000	1265000	1315000	1350000	1375000	1420000	1465000	1510000	1555000	1600000

(b)

Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost Per Week of Its Duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	6	$180,000	1	$30,000	0	30000	30000	30000	30000	30000	30000	0
B	3	$75,000	0	$25,000	25000	25000	25000	0	0	0	0	0
C	4	$120,000	0	$30,000	30000	30000	30000	30000	0	0	0	0
D	4	$140,000	7	$35,000	0	0	0	0	0	0	0	35000
E	7	$175,000	3	$25,000	0	0	0	25000	25000	25000	25000	25000
F	4	$80,000	8	$20,000	0	0	0	0	0	0	0	0
G	6	$210,000	4	$35,000	0	0	0	0	35000	35000	35000	35000
H	3	$45,000	11	$15,000	0	0	0	0	0	0	0	0
I	5	$125,000	9	$25,000	0	0	0	0	0	0	0	0
J	4	$100,000	10	$25,000	0	0	0	0	0	0	0	0
K	3	$60,000	12	$20,000	0	0	0	0	0	0	0	0
L	5	$50,000	10	$10,000	0	0	0	0	0	0	0	0
M	6	$90,000	14	$15,000	0	0	0	0	0	0	0	0
N	5	$150,000	15	$30,000	0	0	0	0	0	0	0	0
					0	0	0	0	0	0	0	0
				Weekly Project Cost	55000	85000	85000	85000	90000	90000	90000	95000
				Cumulative Project Cost	55000	140000	225000	310000	400000	490000	580000	675000

Week 9	Week 10	Week 11	Week 12	Week 13	Week 14	Week 15	Week 16	Week 17	Week 18	Week 19	Week 20
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
35000	35000	35000	0	0	0	0	0	0	0	0	0
25000	25000	0	0	0	0	0	0	0	0	0	0
20000	20000	20000	20000	0	0	0	0	0	0	0	0
35000	35000	0	0	0	0	0	0	0	0	0	0
0	0	0	15000	15000	15000	0	0	0	0	0	0
0	25000	25000	25000	25000	25000	0	0	0	0	0	0
0	0	25000	25000	25000	25000	0	0	0	0	0	0
0	0	0	0	20000	20000	20000	0	0	0	0	0
0	0	10000	10000	10000	10000	10000	0	0	0	0	0
0	0	0	0	0	0	15000	15000	15000	15000	15000	15000
0	0	0	0	0	0	0	30000	30000	30000	30000	30000
0	0	0	0	0	0	0	0	0	0	0	0
115000	140000	115000	95000	95000	95000	45000	45000	45000	45000	45000	45000
790000	930000	1045000	1140000	1235000	1330000	1375000	1420000	1465000	1510000	1555000	1600000

(c)



(d)

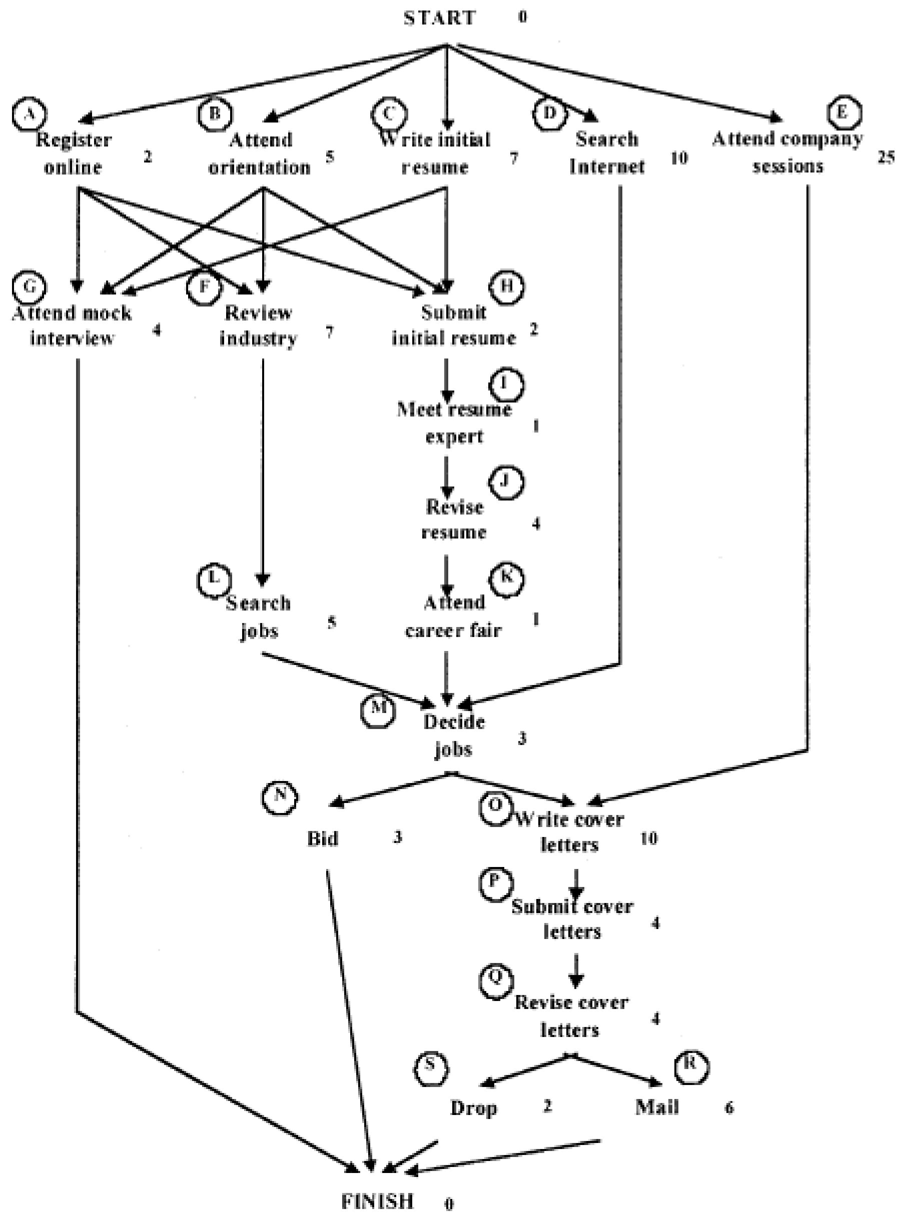
Activity	Budgeted Cost	Percent Completed	Value Completed	Actual Cost To Date	Cost Overrun To Date
A	\$180,000	100%	\$180,000	\$190,000	\$10,000
B	\$75,000	100%	\$75,000	\$70,000	-\$5,000
C	\$120,000	100%	\$120,000	\$150,000	\$30,000
D	\$140,000	40%	\$56,000	\$70,000	\$14,000
E	\$175,000	50%	\$87,500	\$100,000	\$12,500
F	\$80,000	60%	\$48,000	\$45,000	-\$3,000
G	\$210,000	25%	\$52,500	\$50,000	-\$2,500
I	\$125,000	20%	\$25,000	\$35,000	\$10,000
Total	\$1,105,000		\$644,000	\$710,000	\$66,000

The project manager should investigate activities D, E and I, since they are not yet finished and they are running over budget.

CASE

CASE 22.1 "School's Out Forever ..." Alice Cooper

(a)



The estimated project duration equals the length of the longest path in the project network. To calculate this length, we use the layout of the Excel spreadsheets for Reliable's project in this chapter. We need to modify the spreadsheet to reflect the network unique to this case.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2	0	=E4+D4	=H4-D4	=MIN(G9,G10,G1	=H4-F4	=IF(I4=0,"Yes","No")
5		B	Attend orientation	5	0	=E5+D5	=H5-D5	=MIN(G9,G10,G1	=H5-F5	=IF(I5=0,"Yes","No")
6		C	Write initial resume	7	0	=E6+D6	=H6-D6	=MIN(G11,G10)	=H6-F6	=IF(I6=0,"Yes","No")
7		D	Search Internet	10	0	=E7+D7	=H7-D7	=G16	=H7-F7	=IF(I7=0,"Yes","No")
8		E	Attend company sessions	25	0	=E8+D8	=H8-D8	=G18	=H8-F8	=IF(I8=0,"Yes","No")
9		F	Review industry, etc.	7	=MAX(F4,F5)	=E9+D9	=H9-D9	=G15	=H9-F9	=IF(I9=0,"Yes","No")
10		G	Attend mock interview	4	=MAX(F4,F5,F6)	=E10+D10	=H10-D10	=F25	=H10-F10	=IF(I10=0,"Yes","No")
11		H	Submit initial resume	2	=MAX(F4,F5,F6)	=E11+D11	=H11-D11	=G12	=H11-F11	=IF(I11=0,"Yes","No")
12		I	Meet resume expert	1	=F11	=E12+D12	=H12-D12	=G13	=H12-F12	=IF(I12=0,"Yes","No")
13		J	Revise resume	4	=F12	=E13+D13	=H13-D13	=G14	=H13-F13	=IF(I13=0,"Yes","No")
14		K	Attend career fair	1	=F13	=E14+D14	=H14-D14	=G16	=H14-F14	=IF(I14=0,"Yes","No")
15		L	Search jobs	5	=F9	=E15+D15	=H15-D15	=G16	=H15-F15	=IF(I15=0,"Yes","No")
16		M	Decide jobs	3	=MAX(F15,F14,F7)	=E16+D16	=H16-D16	=MIN(G17,G18)	=H16-F16	=IF(I16=0,"Yes","No")
17		N	Bid	3	=F16	=E17+D17	=H17-D17	=F25	=H17-F17	=IF(I17=0,"Yes","No")
18		O	Write cover letters	10	=MAX(F16,F8)	=E18+D18	=H18-D18	=G19	=H18-F18	=IF(I18=0,"Yes","No")
19		P	Submit cover letters	4	=F18	=E19+D19	=H19-D19	=G20	=H19-F19	=IF(I19=0,"Yes","No")
20		Q	Revise cover letters	4	=F19	=E20+D20	=H20-D20	=MIN(G21,G22)	=H20-F20	=IF(I20=0,"Yes","No")
21		R	Mail	6	=F20	=E21+D21	=H21-D21	=F25	=H21-F21	=IF(I21=0,"Yes","No")
22		S	Drop	2	=F20	=E22+D22	=H22-D22	=F25	=H22-F22	=IF(I22=0,"Yes","No")
23										
24										
25					Project Duration =	=MAX(F22,F21,F17,F10)				

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2	0	2	8	10	8	No
5		B	Attend orientation	5	0	5	5	10	5	No
6		C	Write initial resume	7	0	7	7	14	7	No
7		D	Search Internet	10	0	10	12	22	12	No
8		E	Attend company sessions	25	0	25	0	25	0	Yes
9		F	Review industry, etc.	7	5	12	10	17	5	No
10		G	Attend mock interview	4	7	11	45	49	38	No
11		H	Submit initial resume	2	7	9	14	16	7	No
12		I	Meet resume expert	1	9	10	16	17	7	No
13		J	Revise resume	4	10	14	17	21	7	No
14		K	Attend career fair	1	14	15	21	22	7	No
15		L	Search jobs	5	12	17	17	22	5	No
16		M	Decide jobs	3	17	20	22	25	5	No
17		N	Bid	3	20	23	46	49	26	No
18		O	Write cover letters	10	25	35	25	35	0	Yes
19		P	Submit cover letters	4	35	39	35	39	0	Yes
20		Q	Revise cover letters	4	39	43	39	43	0	Yes
21		R	Mail	6	43	49	43	49	0	Yes
22		S	Drop	2	43	45	47	49	4	No
23										
24										
25					Project Duration =	49				

Brent can start the interviews in 49 days. The critical steps in the process are:
Start $\rightarrow E \rightarrow O \rightarrow P \rightarrow Q \rightarrow R \rightarrow$ Finish.

(b) We substitute first the pessimistic, then the optimistic estimates for the time values used in part (a).

Pessimistic Estimates:

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	4	0	4	6	10	6	No
5		B	Attend orientation	10	0	10	0	10	0	Yes
6		C	Write initial resume	14	0	14	4	18	4	No
7		D	Search Internet	12	0	12	20	32	20	No
8		E	Attend company sessions	30	0	30	6	36	6	No
9		F	Review industry, etc.	12	10	22	10	22	0	Yes
10		G	Attend mock interview	8	14	22	66	74	52	No
11		H	Submit initial resume	6	14	20	18	24	4	No
12		I	Meet resume expert	1	20	21	24	25	4	No
13		J	Revise resume	6	21	27	25	31	4	No
14		K	Attend career fair	1	27	28	31	32	4	No
15		L	Search jobs	10	22	32	22	32	0	Yes
16		M	Decide jobs	4	32	36	32	36	0	Yes
17		N	Bid	8	36	44	66	74	30	No
18		O	Write cover letters	12	36	48	36	48	0	Yes
19		P	Submit cover letters	7	48	55	48	55	0	Yes
20		Q	Revise cover letters	9	55	64	55	64	0	Yes
21		R	Mail	10	64	74	64	74	0	Yes
22		S	Drop	3	64	67	71	74	7	No
23										
24										
25					Project Duration =	74				

Under the worst-case scenario, Brent will require 74 days before he is ready to start interviewing. The critical path is:

Start $\rightarrow B \rightarrow F \rightarrow L \rightarrow M \rightarrow O \rightarrow P \rightarrow Q \rightarrow R \rightarrow$ Finish.

Optimistic Estimates:

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	1	0	1	9	10	9	No
5		B	Attend orientation	3	0	3	7	10	7	No
6		C	Write initial resume	5	0	5	7	12	7	No
7		D	Search Internet	7	0	7	11	18	11	No
8		E	Attend company sessions	20	0	20	0	20	0	Yes
9		F	Review industry, etc.	5	3	8	10	15	7	No
10		G	Attend mock interview	3	5	8	29	32	24	No
11		H	Submit initial resume	1	5	6	12	13	7	No
12		I	Meet resume expert	1	6	7	13	14	7	No
13		J	Revise resume	3	7	10	14	17	7	No
14		K	Attend career fair	1	10	11	17	18	7	No
15		L	Search jobs	3	8	11	15	18	7	No
16		M	Decide jobs	2	11	13	18	20	7	No
17		N	Bid	2	13	15	30	32	17	No
18		O	Write cover letters	3	20	23	20	23	0	Yes
19		P	Submit cover letters	2	23	25	23	25	0	Yes
20		Q	Revise cover letters	3	25	28	25	28	0	Yes
21		R	Mail	4	28	32	28	32	0	Yes
22		S	Drop	1	28	29	31	32	3	No
23										
24										
25					Project Duration =	32				

Under the best-case scenario, Brent will require 32 days before he is ready to begin interviewing. The critical path remains the same as in (a).

(c) The mean critical path is the path in the project network that would be critical path if the duration of each activity equals its mean. To compute the mean duration of each activity, we use the Excel spreadsheet named PERT.

	A	B	C	D	E	F	G
3			Time Estimates				
4		Activity	o	m	p	μ	σ^2
5		A	1	2	4	=IF(C5="","",(C5+4*D5+E5)/6)	=IF(C5="","",((E5-C5)/6)^2)
6		B	3	5	10	=IF(C6="","",(C6+4*D6+E6)/6)	=IF(C6="","",((E6-C6)/6)^2)
7		C	5	7	14	=IF(C7="","",(C7+4*D7+E7)/6)	=IF(C7="","",((E7-C7)/6)^2)
8		D	7	10	12	=IF(C8="","",(C8+4*D8+E8)/6)	=IF(C8="","",((E8-C8)/6)^2)
9		E	20	25	30	=IF(C9="","",(C9+4*D9+E9)/6)	=IF(C9="","",((E9-C9)/6)^2)
10		F	5	7	12	=IF(C10="","",(C10+4*D10+E10)/6)	=IF(C10="","",((E10-C10)/6)^2)
11		G	3	4	8	=IF(C11="","",(C11+4*D11+E11)/6)	=IF(C11="","",((E11-C11)/6)^2)
12		H	1	2	6	=IF(C12="","",(C12+4*D12+E12)/6)	=IF(C12="","",((E12-C12)/6)^2)
13		I	1	1	1	=IF(C13="","",(C13+4*D13+E13)/6)	=IF(C13="","",((E13-C13)/6)^2)
14		J	3	4	6	=IF(C14="","",(C14+4*D14+E14)/6)	=IF(C14="","",((E14-C14)/6)^2)
15		K	1	1	1	=IF(C15="","",(C15+4*D15+E15)/6)	=IF(C15="","",((E15-C15)/6)^2)
16		L	3	5	10	=IF(C16="","",(C16+4*D16+E16)/6)	=IF(C16="","",((E16-C16)/6)^2)
17		M	2	3	4	=IF(C17="","",(C17+4*D17+E17)/6)	=IF(C17="","",((E17-C17)/6)^2)
18		N	2	3	8	=IF(C18="","",(C18+4*D18+E18)/6)	=IF(C18="","",((E18-C18)/6)^2)
19		O	3	10	12	=IF(C19="","",(C19+4*D19+E19)/6)	=IF(C19="","",((E19-C19)/6)^2)
20		P	2	4	7	=IF(C20="","",(C20+4*D20+E20)/6)	=IF(C20="","",((E20-C20)/6)^2)
21		Q	3	4	9	=IF(C21="","",(C21+4*D21+E21)/6)	=IF(C21="","",((E21-C21)/6)^2)
22		R	4	6	10	=IF(C22="","",(C22+4*D22+E22)/6)	=IF(C22="","",((E22-C22)/6)^2)
23		S	1	2	3	=IF(C23="","",(C23+4*D23+E23)/6)	=IF(C23="","",((E23-C23)/6)^2)

	A	B	C	D	E	F	G
3			Time Estimates				
4		Activity	o	m	p	μ	σ^2
5		A	1	2	4	2.167	0.25
6		B	3	5	10	5.5	1.36
7		C	5	7	14	7.833	2.25
8		D	7	10	12	9.833	0.69
9		E	20	25	30	25	2.77
10		F	5	7	12	7.5	1.36
11		G	3	4	8	4.5	0.69
12		H	1	2	6	2.5	0.69
13		I	1	1	1	1	0
14		J	3	4	6	4.167	0.25
15		K	1	1	1	1	0
16		L	3	5	10	5.5	1.36
17		M	2	3	4	3	0.11
18		N	2	3	8	3.667	1
19		O	3	10	12	9.167	2.25
20		P	2	4	7	4.167	0.69
21		Q	3	4	9	4.667	1
22		R	4	6	10	6.333	1
23		S	1	2	3	2	0.11

Now, substitute the mean duration of each activity for the time values.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2.17	0.00	2.17	6.83	9.00	6.833	No
5		B	Attend orientation	5.5	0.00	5.50	3.50	9.00	3.5	No
6		C	Write initial resume	7.83	0.00	7.83	5.50	13.33	5.5	No
7		D	Search Internet	9.83	0.00	9.83	12.17	22.00	12.17	No
8		E	Attend company sessions	25	0.00	25.00	0.00	25.00	0	Yes
9		F	Review industry, etc.	7.5	5.50	13.00	9.00	16.50	3.5	No
10		G	Attend mock interview	4.5	7.83	12.33	44.83	49.33	37	No
11		H	Submit initial resume	2.5	7.83	10.33	13.33	15.83	5.5	No
12		I	Meet resume expert	1	10.33	11.33	15.83	16.83	5.5	No
13		J	Revise resume	4.17	11.33	15.50	16.83	21.00	5.5	No
14		K	Attend career fair	1	15.50	16.50	21.00	22.00	5.5	No
15		L	Search jobs	5.5	13.00	18.50	16.50	22.00	3.5	No
16		M	Decide jobs	3	18.50	21.50	22.00	25.00	3.5	No
17		N	Bid	3.67	21.50	25.17	45.67	49.33	24.17	No
18		O	Write cover letters	9.17	25.00	34.17	25.00	34.17	0	Yes
19		P	Submit cover letters	4.17	34.17	38.33	34.17	38.33	0	Yes
20		Q	Revise cover letters	4.67	38.33	43.00	38.33	43.00	0	Yes
21		R	Mail	6.33	43.00	49.33	43.00	49.33	0	Yes
22		S	Drop	2	43.00	45.00	47.33	49.33	4.333	No
23										
24										
25					Project Duration =	49.33				

The mean critical path is the same as in (a). To compute the variance of the project duration, we use the PERT template again.

	J	K
5		Mean Critical
6		Path
7	μ =	=SUMIF(H5:H23,"*",F5:F23)
8	σ² =	=SUMIF(H5:H23,"*",G5:G23)

The mean and the variance of the mean critical path are $\mu = 49.333$ and $\sigma_2 = 7.722$.

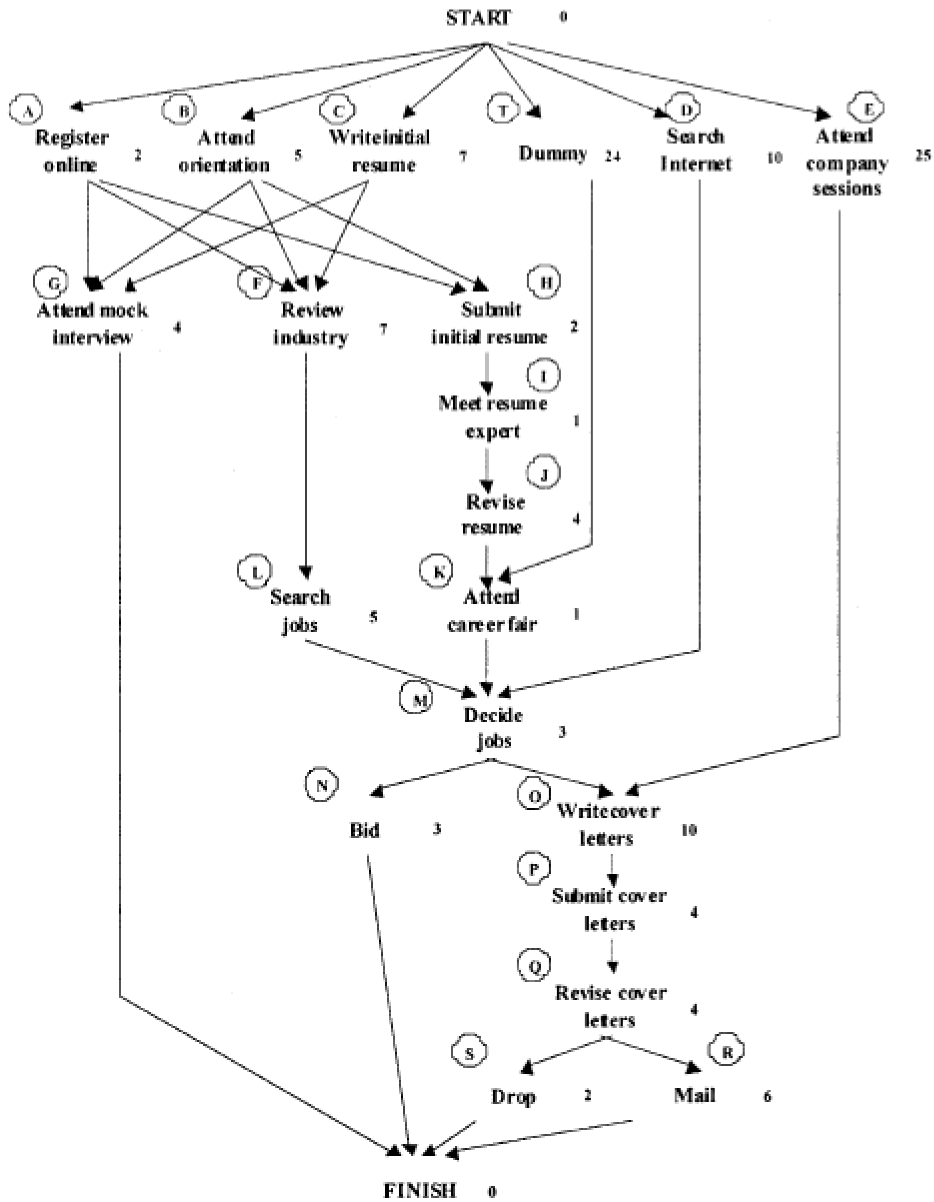
(d) We use the PERT template as in part (c). We substitute the value 60 for d in cell K12.

	J	K
5		Mean Critical
6		Path
7	μ =	=SUMIF(H5:H23,"*",F5:F23)
8	σ² =	=SUMIF(H5:H23,"*",G5:G23)
9		
10	P(T=d) =	=NORMDIST(K12,K7,SQRT(K8),1)
11	where	
12	d =	60

	J	K
5	Mean Critical	
6	Path	
7	μ =	49.3333
8	σ² =	7.72222
9		
10	P(T=d) =	0.99994
11	where	
12	d =	60

Brent will be ready for his interviews within 60 days with probability 99.994%.

(e) The earliest start time for the career fair is day 24 and the career fair itself still lasts one day. To ensure that the earliest start time for the career fair is day 24, we add a dummy node T with duration 24 days to the project network, directly following the START node and preceding the career fair node K.



(f) To obtain the mean critical path for the new network and the probability that Brent will complete the project within 60 days, we first use the PERT template to compute the mean duration for each activity. We add the new node T to the list of activities.

	A	B	C	D	E	F	G
3		Time Estimates					
4		Activity	o	m	p	μ	σ^2
5		A	1	2	4	2.17	0.25
6		B	3	5	10	5.50	1.36111111
7		C	5	7	14	7.83	2.25
8		D	7	10	12	9.83	0.69444444
9		E	20	25	30	25.00	2.77777777
10		T	24	24	24	24.00	0
11		F	5	7	12	7.50	1.36111111
12		G	3	4	8	4.50	0.69444444
13		H	1	2	6	2.50	0.69444444
14		I	1	1	1	1.00	0
15		J	3	4	6	4.17	0.25
16		K	1	1	1	1.00	0
17		L	3	5	10	5.50	1.36111111
18		M	2	3	4	3.00	0.11111111
19		N	2	3	8	3.67	1
20		O	3	10	12	9.17	2.25
21		P	2	4	7	4.17	0.69444444
22		Q	3	4	9	4.67	1
23		R	4	6	10	6.33	1
24		S	1	2	3	2.00	0.11111111

We next substitute these mean duration values for the time values to find the critical path. We need to add node T to the spreadsheet used in (a).

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2.166	0	=E4+D4	=H4-D4	=MIN(G10,G11,G12)	=H4-F4	=IF(I4=0,"Yes","No")
5		B	Attend orientation	5.5	0	=E5+D5	=H5-D5	=MIN(G10,G11,G12)	=H5-F5	=IF(I5=0,"Yes","No")
6		C	Write initial resume	7.833	0	=E6+D6	=H6-D6	=MIN(G12,G11)	=H6-F6	=IF(I6=0,"Yes","No")
7		D	Search internet	9.833	0	=E7+D7	=H7-D7	=G17	=H7-F7	=IF(I7=0,"Yes","No")
8		E	Attend company sessions	25	0	=E8+D8	=H8-D8	=G19	=H8-F8	=IF(I8=0,"Yes","No")
9		T	Dummy for career fair	24	0	=E9+D9	=H9-D9	=G15	=H9-F9	=IF(I9=0,"Yes","No")
10		F	Review industry, etc	7.5	=MAX(F4,F5)	=E10+D10	=H10-D10	=G16	=H10-F10	=IF(I10=0,"Yes","No")
11		G	Attend mock interview	4.5	=MAX(F4,F5,F6)	=E11+D11	=H11-D11	=F26	=H11-F11	=IF(I11=0,"Yes","No")
12		H	Submit initial resume	2.5	=MAX(F4,F5,F6)	=E12+D12	=H12-D12	=G13	=H12-F12	=IF(I12=0,"Yes","No")
13		I	Meet resume expert	1	=F12	=E13+D13	=H13-D13	=G14	=H13-F13	=IF(I13=0,"Yes","No")
14		J	Revise resume	4.166	=F13	=E14+D14	=H14-D14	=G15	=H14-F14	=IF(I14=0,"Yes","No")
15		K	Attend career fair	1	=MAX(F14,F9)	=E15+D15	=H15-D15	=G17	=H15-F15	=IF(I15=0,"Yes","No")
16		L	Search jobs	5.5	=F10	=E16+D16	=H16-D16	=G17	=H16-F16	=IF(I16=0,"Yes","No")
17		M	Dedde jobs	3	=MAX(F16,F15,F7)	=E17+D17	=H17-D17	=MIN(G18,G19)	=H17-F17	=IF(I17=0,"Yes","No")
18		N	Bid	3.666	=F17	=E18+D18	=H18-D18	=F26	=H18-F18	=IF(I18=0,"Yes","No")
19		O	Write cover letters	9.166	=MAX(F17,F8)	=E19+D19	=H19-D19	=G20	=H19-F19	=IF(I19=0,"Yes","No")
20		P	Submit cover letters	4.166	=F19	=E20+D20	=H20-D20	=G21	=H20-F20	=IF(I20=0,"Yes","No")
21		Q	Revise cover letters	4.666	=F20	=E21+D21	=H21-D21	=MIN(G22,G23)	=H21-F21	=IF(I21=0,"Yes","No")
22		R	Mail	6.333	=F21	=E22+D22	=H22-D22	=F26	=H22-F22	=IF(I22=0,"Yes","No")
23		S	Drop	2	=F21	=E23+D23	=H23-D23	=F26	=H23-F23	=IF(I23=0,"Yes","No")
24										
25										
26					Project Duration =	=MAX(F23,F22,F18,F11)				

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2.17	0.00	2.17	9.83	12.00	9.833	No
5		B	Attend orientation	5.50	0.00	5.50	6.50	12.00	6.5	No
6		C	Write initial resume	7.83	0.00	7.83	8.50	16.33	8.5	No
7		D	Search Internet	9.83	0.00	9.83	15.17	25.00	15.17	No
8		E	Attend company sessions	25.00	0.00	25.00	3.00	28.00	3	No
9		T	Dummy for career fair	24.00	0.00	24.00	0.00	24.00	0	Yes
10		F	Review industry, etc.	7.50	5.50	13.00	12.00	19.50	6.5	No
11		G	Attend mock interview	4.50	7.83	12.33	47.83	52.33	40	No
12		H	Submit initial resume	2.50	7.83	10.33	16.33	18.83	8.5	No
13		I	Meet resume expert	1.00	10.33	11.33	18.83	19.83	8.5	No
14		J	Revise resume	4.17	11.33	15.50	19.83	24.00	8.5	No
15		K	Attend career fair	1.00	24.00	25.00	24.00	25.00	0	Yes
16		L	Search jobs	5.50	13.00	18.50	19.50	25.00	6.5	No
17		M	Decide jobs	3.00	25.00	28.00	25.00	28.00	0	Yes
18		N	Bid	3.67	28.00	31.67	48.67	52.33	20.67	No
19		O	Write cover letters	9.17	28.00	37.17	28.00	37.17	0	Yes
20		P	Submit cover letters	4.17	37.17	41.33	37.17	41.33	0	Yes
21		Q	Revise cover letters	4.67	41.33	46.00	41.33	46.00	0	Yes
22		R	Mail	6.33	46.00	52.33	46.00	52.33	0	Yes
23		S	Drop	2.00	46.00	48.00	50.33	52.33	4.333	No
24										
25										
26					Project Duration	=52.33				

The mean project duration is now 52.33 days and the new mean critical path is:

Start $\rightarrow T \rightarrow K \rightarrow M \rightarrow O \rightarrow P \rightarrow Q \rightarrow R \rightarrow$ Finish.

We specify this new critical path in the PERT spreadsheet to obtain the probability that Brent will complete the project within 60 days.

	J	K
5	Mean Critical	
6	Path	
7	$\mu=$	52.3333
8	$\sigma^2=$	5.0555
9		
10	P(T=d) =	0.9996
11	where	
12	d =	60

Brent will be ready for his interviews within 60 days with probability 99.967%, which is slightly less than the probability computed in part (d). This decrease is a result of the increase in the mean project duration. However, since the variance of the project duration is smaller than the one found in (d), the probability decreases only slightly.

CHAPTER 23: ADDITIONAL SPECIAL TYPES OF LINEAR PROGRAMMING PROBLEMS

23.1-1.

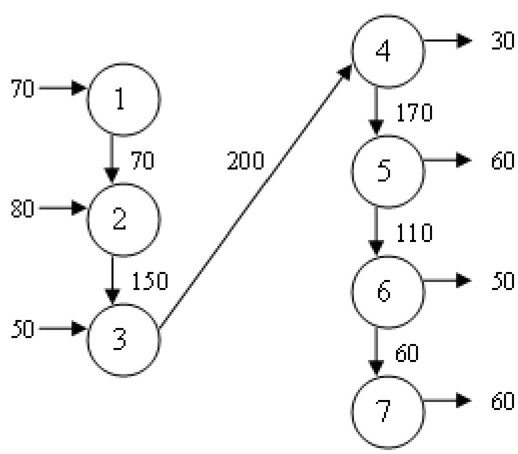
(a) Locations 1, 2, 3 are supply centers and locations 4, 5, 6, 7 are receiving centers. Shipments can be sent via intermediate points.

(b)

	1	2	3	4	5	6	7	s_i
1	0	21	50	62	93	77	M	270
2	29	0	17	54	67	M	48	280
3	50	17	0	60	98	67	25	250
4	62	54	60	0	27	M	38	200
5	93	67	98	27	0	47	42	200
6	77	M	67	M	47	0	35	200
7	M	48	25	38	42	35	0	200
d_j	200	200	200	230	260	250	260	

(c)

	1	2	3	4	5	6	7	s_i
1	200	70						270
2		130	150					280
3			50	200				250
4				30	170			200
5					90	110		200
6						140	60	200
7							200	200
d_j	200	200	200	230	260	250	260	



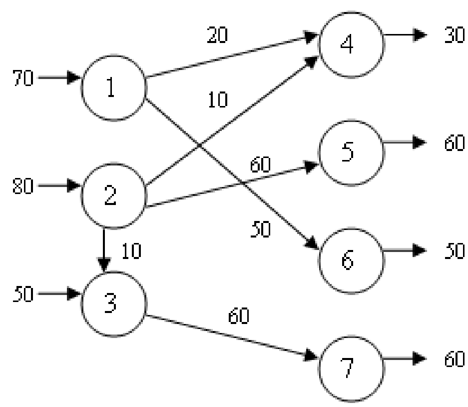
The shipping pattern obtained with the northwest corner rule forms a chain where location i ships only to location $i+1$.

(d)

Optimal Solution: The main body of the table shows the optimal number of units (if not zero) to be sent from each source to each destination.

	Destination 1	2	3	4	5	6	7	Supply
Source 1	200			20		50		270
2		200	10	10	60			280
3			190				60	250
4				200				200
5					200			200
6						200		200
7							200	200
Demand	200	200	200	230	260	250	260	Cost is 11320

Shipping pattern:



23.1-2.

(a) Let the supply center be year 0 with a supply of 1 and the receiving center be year 3 with a demand of 1. Years 1 and 2 are transshipment points. The parameter table is as follows:

Years	0	1	2	3	Supply
0	0	13	28	48	1
1	M	0	17	33	0
2	M	M	0	20	0
3	M	M	M	0	0
Demand	0	0	0	1	

(b) The transportation problem is the same as above except that all supplies and demands are increased by one.

Cost Per Unit Distributed

	Destination 1	2	3	4	Supply
Source 1	0	13	28	48	2
2	1M	0	17	33	1
3	1M	1M	0	20	1
4	1M	1M	1M	0	1
Demand	1	1	1	2	

(c) Vogel's approximation

```
          |           Destination           |
          |   1    |   2    |   3    |   4   |Supply  u[i]
__________|________|________|________|_______|______________
          |     0| |    13| |    28| |   48| |        |
        1 |---- B|---- B|----   |----      |        |
          |      1|      1|      2|      2|      2|    13
          |________|________|________|_______|        |
          |     M| |     0| |    17| |   33| |        |
        2 |----   |---- B|----   |---- B|        |
          | 1M+13|      0|      4|      1|      1|     0
          |________|________|________|_______|        |
          |     M| |     M| |     0| |   20| |        |
        3 |----   |----   |---- B|---- B|        |
          | 1M+26| 1M+13|      1|      0|      1|   -13
          |________|________|________|_______|        |
          |     M| |     M| |     M| |    0| |        |
        4 |----   |----   |----   |---- B|        |
          | 1M+46| 1M+33| 1M+20|      1|      1|   -33
          |________|________|________|_______|________|______
Demand    |      1        1        1       2|
__________|_________________________________|
   v[j]   |    -13        0       13      33|
          |                                  | Z = 46
```

(d) Vogel's approximation prices out optimal.

23.1-3.

(a) Let c_{ij}^k be the cost of buying a very old car $(k=1)$ or a moderately old car $(k=2)$ at the beginning of year i and trading it in at the end of year j. This cost is the difference of the purchase price, operating and maintenance costs for years $1, 2, \ldots, j{-}i{+}1$ from the trade in value after $j{-}i{+}1$ years.

c_{ij}^1

	1	2	3	4
1	2400	4800	7400	10300
2	M	2400	4800	7400
3	M	M	2400	4800
4	M	M	M	2400

c_{ij}^2

	1	2	3	4
1	3000	5000	7200	10700
2	M	3000	5000	7200
3	M	M	3000	5000
4	M	M	M	3000

Let $c_{i,j+1} = \min\ \{c_{ij}^1, c_{ij}^2\}$. Let the supply center be year 1 with unit supply and the demand center be year 5 with unit demand. Years 2, 3, 4 are transshipment points. $c_{ii} = 0$, $c_{i1} =$ M for $i > 1$ and $c_{5j} =$ M for $j < 5$. The following is the parameter table of this transshipment problem:

	Year j					
Year i	1	2	3	4	5	Supply
1	0	2400	4800	7200	10300	1
2	M	0	2400	4800	7200	0
3	M	M	0	2400	4800	0
4	M	M	M	0	2400	0
5	M	M	M	M	0	0
Demand	0	0	0	0	1	

(b) The cost and requirements table of the equivalent transportation problem is identical to the one in (a) except that all supplies and demands need to be increased by one.

(c)

	1	2	3	4	5	Supply
1	1	1				2
2					1	1
3			1			1
4				1		1
5					1	1
Demand	1	1	1	1	2	Cost: $9,600$

The optimal solution is to purchase a very old car for year 1 and a moderately old one for years 2, 3, and 4. The cost of this is $\$9,600$.

23.1-4.

Suppose there are m supply centers, n receiving centers and p transshipment points.

$$\text{minimize} \quad \sum_{i=1}^{m+n+p} \sum_{j=1}^{m+n+p} c_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{m+n+p} (x_{ij} - x_{ji}) = \begin{cases} s_i & \text{for } i = 1, 2, \ldots, m \\ -d_i & \text{for } i = m{+}1, \ldots, m{+}n \\ 0 & \text{for } i = m{+}n{+}1, \ldots, m{+}n{+}p \end{cases}$$

$$x_{ij} \geq 0, \text{ for all } i \neq j$$

This model has the special structure that each decision variable appears in exactly two constraints, once with a coefficient of $+1$ and once with a coefficient of -1. The table of constraint coefficients is:

x_{12}	x_{13}	$\cdots$	$x_{1,m+n+p}$	x_{21}	x_{23}	$\cdots$	$x_{2,m+n+p}$	$\cdots$	$x_{m+n+p,1}$	$x_{m+n+p,2}$	$\cdots$	$x_{m+n+p,m+n+p-1}$
1	1	$\cdots$	1	-1	0	$\cdots$	0	$\cdots$	-1	0	$\cdots$	0
-1	0	$\cdots$	0	1	1	$\cdots$	1	$\cdots$	0	-1	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
0	0	$\cdots$	-1	0	0	$\cdots$	-1	$\cdots$	1	1	$\cdots$	1

23.2-1.

(a)

$$\begin{array}{lrl}
\text{Maximize} & 2x_1 + 4x_2 + 3x_3 + 2x_4 - 5x_5 + 3x_6 & \\
\text{Master Problem} & 5x_1 - 2x_2 + 3x_3 + 4x_4 + 2x_5 + x_6 & \leq 20 \\
 & 2x_1 + 4x_2 \qquad + 2x_4 \qquad + 3x_6 & \leq 60 \\
\text{Subproblem 1} & 3x_1 + 2x_2 + 3x_3 & \leq 30 \\
 & 5x_1 \qquad - x_3 & \leq 30 \\
 & -x_1 + 2x_2 + x_3 & \geq 20 \\
\text{Subproblem 2} & x_4 & \leq 15 \\
 & x_4 & \geq 3 \\
\text{Subproblem 3} & 2x_5 - x_6 & \leq 20 \\
 & 2x_5 + 3x_6 & \leq 40
\end{array}$$

$x_j \geq 0$, for $j = 1, 2, \ldots, 6$

(b) After converting $\geq$ inequalities to $\leq$ inequalities, the coefficient table becomes:

x_1	x_2	x_3	x_4	x_5	x_6
5	-2	3	4	2	1
2	4		2		3
3	2	3			
5		-1			
1	-2	-1			
			1		
			-1		
				2	-1
				2	3

23.2-2.

(a)

	Constraint	x_1	x_4	x_2	x_5	x_7	x_3	x_6
Master Problem	3	4	2	3	4	1	-2	0
	6	0	0	5	1	4	3	-2
Subproblem 1	2	0	1					
	5	1	1					
	9	2	4					
Subproblem 2	1			1	1	1		
	8			2	1	3		
Subproblem 3	4						2	4
	7						0	1

(b) The first constraint of Subproblem 1 and the second constraint of Subproblem 3 are the upper-bound constraints. The second constraint of Subproblem 1 and the first constraint of Subproblem 2 are the GUB constraints.

23.2-3.

(a) maximize $7x_1 + 3x_2 + 5x_3 + 4x_4 + 7x_5 + 5x_6$

subject to
$$16x_1 + 7x_2 + 13x_3 + 8x_4 + 20x_5 + 10x_6 \leq 150$$
$$10x_1 + 3x_2 + 7x_3 \leq 50$$
$$4x_1 + 2x_2 + 5x_3 \leq 30$$
$$6x_4 + 13x_5 + \ \ 9x_6 \leq 45$$
$$3x_4 + \ \ 8x_5 + \ \ 2x_6 \leq 25$$
$xj \geq 0$, for $j = 1, 2, \dots, 6$

(b)

x_1	x_2	x_3	x_4	x_5	x_6
16	7	13	8	20	10
10	3	7			
4	2	5			
			6	13	9
			3	8	2

23.3-1.

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 0 & 2 \end{pmatrix},\ A_1 = (3),\ A_2 = (2),\ A_3 = (1),\ A_4 = (2)$$

$c_1 = (3)$, $c_2 = (5)$, $\vec{x_1} = (x_1)$, $\vec{x_2} = (x_2)$, $b = 18$, $b_1 = 4$, $b_2 = 12$

Subproblem 1: maximize $z_1 = 3x_1$
subject to $x_1 \le 4$, $x_1 \ge 0$

$x_{11}^* = 0 \ \to \rho_{11}$, $x_{12}^* = 4 \ \to \rho_{12}$

Subproblem 2: maximize $z_2 = 5x_2$
subject to $2x_2 \le 12$, $x_2 \ge 0$

$x_{21}^* = 0 \ \to \rho_{21}$, $x_{22}^* = 6 \ \to \rho_{22}$

Reformulate: maximize $12\rho_{12} + 30\rho_{22}$
subject to

$$\begin{aligned} &12\rho_{12} + 12\rho_{22} + x_5 = 18 \\ &\rho_{11} + \rho_{12} = 1 \\ &\rho_{21} + \rho_{22} = 1 \\ &\rho \ge 0, x_5 \ge 0 \end{aligned}$$

(1) Start with $x_B = \begin{pmatrix} x_5 \\ \rho_{11} \\ \rho_{21} \end{pmatrix}$, $B = I = B^{-1}$, $B^{-1}b = \begin{pmatrix} 18 \\ 1 \\ 1 \end{pmatrix}$

<u>$j = 1$:</u> minimize $w_1 = -3x_1$
subject to $x_1 \le 4$, $x_1 \ge 0 \to x_1^* = 4 = x_{12}^*$, $w_1^* = -12$

<u>$j = 2$:</u> minimize $w_2 = -5x_2$
subject to $2x_2 \le 12$, $x_2 \ge 0 \to x_2^* = 6 = x_{22}^*$, $w_2^* = -30$

Not optimal, $w_2^* < w_1^*$, so ρ_{22} enters the basis.

$A_k' = \begin{pmatrix} 12 \\ 0 \\ 1 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 18 \\ 1 \\ 1 \end{pmatrix}$, minimum ratio: $1/1$, so ρ_{21} leaves the basis.

(2) $x_B = \begin{pmatrix} x_5 \\ \rho_{11} \\ \rho_{22} \end{pmatrix}$, $c_B = \begin{pmatrix} 0 & 0 & 30 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & 0 & -12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$w_1 = -3x_1$, $x_1^* = 4 = x_{12}^*$, $w_1^* = -12$

$w_2 = -5x_2 + 30$, $x_2^* = 6 = x_{22}^*$, $w_2^* = 0$

Not optimal, $w_1^* < w_2^*$, so ρ_{12} enters the basis.

$A_k' = \begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$, minimum ratio: $6/12$, so x_5 leaves the basis.

(3) $x_B = \begin{pmatrix} \rho_{12} \\ \rho_{11} \\ \rho_{22} \end{pmatrix}$, $c_B = (\,12 \quad 0 \quad 30\,)$, $B = \begin{pmatrix} 12 & 0 & 12 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$B^{-1} = \begin{pmatrix} 1/12 & 0 & -1 \\ -1/12 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$w_1 = 0x_1 + 0$$

$$w_2 = -3x_2 + 18,\ x_2^* = 6 = x_{22}^*,\ w_2^* = 0$$

$c_B B^{-1} = 1 > 0$, so the solution is optimal, stop.

$$x_B = \begin{pmatrix} \rho_{12} \\ \rho_{11} \\ \rho_{22} \end{pmatrix} = B^{-1}b = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

$\Rightarrow x_1 = 0(1/2) + 4(1/2) = 2$, $x_2 = 0(0) + 6(1) = 6$, $z = 36$

23.3-2.

(a) Reformulate:

Subproblem 1: $x_{11}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x_{12}^* = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $x_{13}^* = \begin{pmatrix} 5/2 \\ 15/2 \end{pmatrix}$, $x_{14}^* = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

Subproblem 2: $x_{21}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x_{22}^* = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $x_{23}^* = \begin{pmatrix} 10/3 \\ 10/3 \end{pmatrix}$, $x_{14}^* = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

maximize $\quad 50\rho_{12} + \frac{125}{2}\rho_{13} + 50\rho_{14} + 40\rho_{22} + 50\rho_{23} + 35\rho_{24}$

subject to $\quad 30\rho_{12} + \frac{105}{2}\rho_{13} + 50\rho_{14} + 20\rho_{22} + \frac{100}{3}\rho_{23} + 30\rho_{24} + x_5 = 40$

$\qquad \rho_{11} + \rho_{12} + \rho_{13} + \rho_{14} = 1$

$\qquad \rho_{21} + \rho_{22} + \rho_{23} + \rho_{24} = 1$

$\qquad \rho \geq 0, x_5 \geq 0$

(b) Start with $x_B = \begin{pmatrix} x_5 \\ \rho_{11} \\ \rho_{21} \end{pmatrix}$, $B = I = B^{-1}$, $B^{-1}b = \begin{pmatrix} 40 \\ 1 \\ 1 \end{pmatrix}$, $c_B = 0$

<u>$j = 1$:</u> minimize $\quad w_1 = 10x_1 - 5x_2$
subject to $\quad 3x_1 + x_2 \leq 15$, $x_1 + x_2 \leq 10$, $x_1, x_2 \geq 0$

$x_{13}^* = \begin{pmatrix} 5/2 \\ 15/2 \end{pmatrix}$ is optimal, $w_1^* = -125/2$.

<u>$j = 2$:</u> minimize $\quad w_2 = -8x_3 - 7x_4$
subject to $\quad x_3 + 2x_4 \leq 10$, $2x_3 + x_4 \leq 10$, $x_3, x_4 \geq 0$

$x_{23}^* = \begin{pmatrix} 10/3 \\ 10/3 \end{pmatrix}$ is optimal, $w_2^* = -50$.

Not optimal, $w_1^* < w_2^*$, so ρ_{13} enters the basis.

$A'_k = \begin{pmatrix} 105/2 \\ 1 \\ 0 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 40 \\ 1 \\ 1 \end{pmatrix}$, minimum ratio: $80/105$, so x_5 leaves the basis.

(2) $x_B = \begin{pmatrix} \rho_{13} \\ \rho_{11} \\ \rho_{21} \end{pmatrix}$, $c_B = \begin{pmatrix} 125/2 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 105/2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$B^{-1} = \begin{pmatrix} 2/105 & 0 & 0 \\ -2/105 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$w_1 = -\frac{20}{7}x_1 + \frac{20}{21}x_2$, x_{12}^* is optimal, $w_1^* = -14.28$.

$w_2 = -\frac{68}{21}x_3 + \frac{1}{7}x_4$, x_{22}^* is optimal, $w_2^* = -16.19$.

Not optimal, $w_2^* < w_1^*$, so ρ_{22} enters the basis.

$A'_k = \begin{pmatrix} 40/105 \\ -40/105 \\ 1 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 80/105 \\ 25/105 \\ 1 \end{pmatrix}$, minimum ratio: $1/1$, so ρ_{21} leaves the basis.

(3) $x_B = \begin{pmatrix} \rho_{13} \\ \rho_{11} \\ \rho_{22} \end{pmatrix}$, $c_B = \begin{pmatrix} 125/2 & 0 & 40 \end{pmatrix}$, $B = \begin{pmatrix} 105/2 & 0 & 20 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$B^{-1} = \begin{pmatrix} 2/105 & 0 & -40/105 \\ -2/105 & 1 & 40/105 \\ 0 & 0 & 1 \end{pmatrix}$$

$w_1 = -\frac{20}{7}x_1 + \frac{20}{21}x_2$, x_{12}^* is optimal, $w_1^* = -14.28$.

$w_2 = -\frac{68}{21}x_3 + \frac{1}{7}x_4 - \frac{500}{21} + 40$, x_{22}^* is optimal, $w_2^* = 0$.

Not optimal, $w_1^* < w_2^*$, so ρ_{12} enters the basis.

$A'_k = \begin{pmatrix} 60/105 \\ 55/105 \\ 0 \end{pmatrix}$, $B^{-1}b = \begin{pmatrix} 40/105 \\ 65/105 \\ 1 \end{pmatrix}$, minimum ratio: $40/60$, so ρ_{13} leaves the basis.

(4) $x_B = \begin{pmatrix} \rho_{12} \\ \rho_{11} \\ \rho_{22} \end{pmatrix}$, $c_B = \begin{pmatrix} 50 & 0 & 40 \end{pmatrix}$, $B = \begin{pmatrix} 30 & 0 & 20 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$B^{-1} = \begin{pmatrix} 1/30 & 0 & -2/3 \\ -1/30 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}$$

$w_1 = \frac{10}{3}x_2$, x_{11}^* and x_{12}^* are both optimal, $w_1^* = 0$.

$w_2 = -\frac{4}{3}x_3 + 3x_4 - \frac{100}{3} + 40$, x_{22}^* is optimal, $w_2^* = 0$.

$c_B B^{-1} = 5/3 > 0$, so optimality test holds, stop.

$$x_B = \begin{pmatrix} \rho_{12} \\ \rho_{11} \\ \rho_{22} \end{pmatrix} = B^{-1}b = \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{x}_1 = \tfrac{1}{3}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 0 \end{pmatrix}, \vec{x}_2 = 1\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 = 10/3, x_2 = 0, x_3 = 5, x_4 = 0, z = 220/3$$

23.3-3.

The problem has three subproblems and two linking constraints.

(1) Initial basis: $x_B = \begin{pmatrix} x_{51} \\ x_{52} \\ \rho_{11} \\ \rho_{21} \\ \rho_{31} \end{pmatrix}$, $B = B^{-1} = I$, $c_B = 0$

$\underline{j = 1:}$ minimize $\quad -8x_1 - 5x_2 - 6x_3$

subject to
$$\begin{aligned} &2x_1 + 4x_2 + 3x_3 \le 10 \\ &7x_1 + 3x_2 + 6x_3 \le 15 \\ &5x_1 \qquad\;\; + 3x_3 \le 12 \\ &x_1, x_2, x_3 \ge 0 \end{aligned}$$

$x^*_{1k} = \begin{pmatrix} 15/11 \\ 20/11 \\ 0 \end{pmatrix}$ is optimal, $w^*_1 = -20$.

$\underline{j = 2:}$ minimize $\quad -9x_4 - 7x_5 - 9x_6$

subject to
$$\begin{aligned} &3x_4 + \;\; x_5 + 2x_6 \le 7 \\ &2x_4 + 4x_5 + 3x_6 \le 9 \\ &x_4, x_5, x_6 \ge 0 \end{aligned}$$

$x^*_{2k} = \begin{pmatrix} 3/5 \\ 0 \\ 13/5 \end{pmatrix}$ is optimal, $w^*_2 = -28.8$.

$\underline{j = 3:}$ minimize $\quad -6x_7 - 5x_8$

subject to
$$\begin{aligned} &8x_7 + 5x_8 \le 25 \\ &7x_7 + 9x_8 \le 30 \\ &6x_7 + 4x_8 \le 20 \\ &x_7, x_8 \ge 0 \end{aligned}$$

$x^*_{3k} = \begin{pmatrix} 75/37 \\ 65/37 \end{pmatrix}$ is optimal, $w^*_2 = -20.95$.

w^*_2 is smallest, so ρ_{2k} enters the basis.

$$A'_k = \begin{pmatrix} A_2 x^*_{2k} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 2 & 0 & 3 \\ 3 & 7 & 0 \end{pmatrix} \begin{pmatrix} 3/5 \\ 0 \\ 13/5 \end{pmatrix} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 9/5 \\ 0 \\ 1 \\ 0 \end{pmatrix}, B^{-1}b = \begin{pmatrix} 30 \\ 20 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

minimum ratio: $1/1$, so ρ_{21} leaves.

(2) $x_B = \begin{pmatrix} x_{51} \\ x_{52} \\ \rho_{11} \\ \rho_{2k} \\ \rho_{31} \end{pmatrix}$, $c_B = \begin{pmatrix} 0 & 0 & 0 & 144/5 & 0 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & 0 & 0 & -9 & 0 \\ 0 & 1 & 0 & -9/5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

w_1 same, $w^*_1 = -20$
$w_2 = \begin{pmatrix} -9 & -7 & -9 \end{pmatrix} \vec{x_2} + 144/5$, $w^*_2 = 0$
w_3 same, $w^*_3 = -20.95$

w^*_3 is smallest, so ρ_{3k} enters the basis.

$$A'_k = \begin{pmatrix} A_3 x^*_{3k} \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 4 & 6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 75/37 \\ 65/37 \end{pmatrix} \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 18.65 \\ 2.03 \\ 0 \\ 0 \\ 1 \end{pmatrix}, B^{-1}b = \begin{pmatrix} 21 \\ 91/5 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

minimum ratio: $1/1$, so ρ_{31} leaves.

Let $x_B = \begin{pmatrix} x_{51} \\ x_{52} \\ \rho_{11} \\ \rho_{2k} \\ \rho_{3k} \end{pmatrix}$ and continue. This suggests that in the next iteration, ρ_{11} will be replaced by ρ_{1k}.

23.4-1.

Constraint	x_3	x_6	x_7	x_1	x_2	x_4	x_5	x_8	x_9	x_{10}
1	0	0	0	3	1					
2	−1	0	0	1	2					
3	0	0	0			1	5			
4	1	−1	−1			2	−1			
5	0	0	0			0	1			
6	1	1	1					1	3	2
7	0	0	0					2	−1	1

23.4-2.

(a) Let x_{ij} denote the number of units of product i to be produced in year j for $i = 1, 2$ and $j = 1, 2, 3$. Let y_{ij} denote the number of units of product i to be sold in year j for $i = 1, 2$ and $j = 1, 2, 3$. Let z_{ijk} denote the number of units of product i to be produced and stored in year j and sold in year k, for $i = 1, 2$, $j = 1, 2, 3$, and $k = j{+}1, j{+}2, \ldots, 3$.

$$
\begin{array}{ll}
\text{maximize} & 3y_{11} + 5y_{21} + 4y_{12} + 4y_{22} + 5y_{13} + 8y_{23} \\
& -\, 2z_{112} - 2z_{212} - 4z_{113} - 4z_{213} - 2z_{123} - 2z_{223} \\
\text{subject to} & x_{11} \leq 4 \\
& 2x_{21} \leq 12 \\
& 3x_{11} + 2x_{21} \leq 18 \\
& x_{11} - y_{11} - z_{112} - z_{113} = 0 \\
& x_{21} - y_{21} - z_{212} - z_{213} = 0 \\
& x_{12} \leq 6 \\
& 2x_{22} \leq 12 \\
& 3x_{12} + 2x_{22} \leq 24 \\
& z_{112} + x_{12} - y_{12} - z_{123} = 0 \\
& z_{112} - y_{12} \leq 0 \\
& z_{212} + x_{22} - y_{22} - z_{223} = 0 \\
& z_{212} - y_{22} \leq 0 \\
& x_{13} \leq 3 \\
& 2x_{23} \leq 10 \\
& 3x_{13} + 2x_{23} \leq 15 \\
& z_{113} + z_{123} + x_{13} - y_{13} = 0 \\
& z_{213} + z_{223} + x_{23} - y_{23} = 0 \\
& x_{ij} \geq 0, y_{ij} \geq 0, z_{ijk} \geq 0, \text{ for all } i, j, k.
\end{array}
$$

(b) Table of constraint coefficients:

z_{112}	z_{113}	z_{212}	z_{213}	z_{123}	z_{223}	x_{11}	y_{11}	x_{21}	y_{21}	x_{12}	y_{12}	x_{22}	y_{22}	x_{13}	y_{13}	x_{23}	y_{23}
0	0	0	0	0	0	1	0	0	0								
0	0	0	0	0	0	0	0	2	0								
0	0	0	0	0	0	3	0	2	0								
-1	-1	0	0	0	0	1	-1	0	0								
0	0	-1	-1	0	0	0	0	1	-1								
0	0	0	0	0	0					1	0	0	0				
0	0	0	0	0	0					0	0	2	0				
0	0	0	0	0	0					3	0	2	0				
1	0	0	0	-1	0					1	-1	0	0				
1	0	0	0	0	0					0	-1	0	0				
0	0	1	0	0	-1					0	0	1	-1				
0	0	1	0	0	0					0	0	0	-1				
0	0	0	0	0	0									1	0	0	0
0	0	0	0	0	0									0	0	2	0
0	0	0	0	0	0									3	0	2	0
0	1	0	0	1	0									1	-1	0	0
0	0	0	1	0	1									0	0	1	-1

23.5-1.

Constraint	x_2	x_8	x_1	x_4	x_3	x_7	x_5	x_9	x_{10}	x_6
3	-1	0	5	-1	2	-3	-1	0	4	0
7	1	1	2	3	0	0	0	-1	0	2
1	0	1	2	3						
6	0	0	1	1						
2	1	2			1	2				
8	-1	-1			2	1				
5	-1	-2					2	5	3	
9	0	0					1	2	1	
10	-1	0					4	1	5	
4	0	-1								1

23.5-2.

(a) Let types 1 and 2 denote raw lumber and plywood respectively. Let x_{ij} be the thousand board feet of type i to be purchased in season j, for $i = 1, 2$ and $j = 1, 2, 3, 4$. Let y_{ij} be the thousand board feet of type i to be sold in season j, for $i = 1, 2$ and $j = 1, 2, 3, 4$. Let z_{ijk} be the thousand board feet of type i to be purchased and stored in season j and sold in season k, for $i = 1, 2$, $j = 1, 2, 3, 4$, and $k = j+1, j+2, \ldots, 4$.

maximize
$$
\begin{aligned}
&-410x_{11} + 425y_{11} - 17z_{112} - 27z_{113} - 37z_{114} \\
&-680x_{21} + 705y_{21} - 24z_{212} - 42z_{213} - 60z_{214} \\
&-430x_{12} + 440y_{12} - 17z_{123} - 27z_{124} \\
&-715x_{22} + 730y_{22} - 24z_{223} - 42z_{224} \\
&-460x_{13} + 465y_{13} - 17z_{134} - 760x_{23} + 770y_{23} - 24z_{234} \\
&-450x_{14} + 455y_{14} - 740x_{24} + 750y_{24}
\end{aligned}
$$

subject to
$$
\begin{aligned}
&x_{11} - y_{11} - z_{112} - z_{113} - z_{114} = 0 \\
&x_{21} - y_{21} - z_{212} - z_{213} - z_{214} = 0 \\
&x_{11} + x_{21} \le 2000 \\
&y_{11} \le 1000 \\
&y_{21} \le 800 \\
&z_{112} + x_{12} - y_{12} - z_{123} - z_{124} = 0 \\
&z_{112} - y_{12} \le 0 \\
&z_{212} + x_{22} - y_{22} - z_{223} - z_{224} = 0 \\
&z_{212} - y_{22} \le 0 \\
&z_{112} + z_{113} + z_{114} + z_{212} + z_{213} + z_{214} + x_{12} + x_{22} \le 2000 \\
&y_{12} \le 1400 \\
&y_{22} \le 1200 \\
&z_{113} + z_{123} + x_{13} - y_{13} - z_{134} = 0 \\
&z_{113} + z_{123} - y_{13} \le 0 \\
&z_{213} + z_{223} + x_{23} - y_{23} - z_{234} = 0 \\
&z_{213} + z_{223} - y_{23} \le 0 \\
&z_{113} + z_{114} + z_{123} + z_{124} + z_{213} + z_{214} + z_{223} + z_{224} + x_{13} + x_{23} \le 2000 \\
&y_{13} \le 2000 \\
&y_{23} \le 1500 \\
&z_{114} + z_{124} + z_{134} + x_{14} - y_{14} = 0 \\
&z_{214} + z_{224} + z_{234} + x_{24} - y_{24} = 0 \\
&z_{114} + z_{124} + z_{134} + z_{214} + z_{224} + z_{234} + x_{14} + x_{24} \le 2000 \\
&y_{14} \le 1600 \\
&y_{24} \le 100 \\
&x_{ij} \ge 0, y_{ij} \ge 0, z_{ijk} \ge 0, \text{ for all } i, j, k.
\end{aligned}
$$

(b)

z_{112}	z_{113}	z_{114}	z_{212}	z_{213}	z_{214}	z_{123}	z_{124}	z_{223}	z_{224}	z_{134}	z_{234}	x_{11}	y_{11}	x_{21}	y_{21}	x_{12}	y_{12}	x_{22}	y_{22}	x_{13}	y_{13}	x_{23}	y_{23}	x_{14}	y_{14}	x_{24}	y_{24}	RHS
0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	≤ 2000
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	≤ 2000
0	1	1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	≤ 2000
0	0	1	0	0	1	0	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	≤ 2000
-1	-1	-1	0	0	0	0	0	0	0	0	0	1	-1															$= 0$
0	0	0	0	0	0	0	0	0	0	0	0	0	1															≤ 1000
0	0	0	-1	-1	-1	0	0	0	0	0	0			1	-1													$= 0$
0	0	0	0	0	0	0	0	0	0	0	0			0	1													≤ 800
1	0	0	0	0	0	-1	-1	0	0	0	0					1	-1											$= 0$
1	0	0	0	0	0	0	0	0	0	0	0					0	-1											≤ 0
0	0	0	0	0	0	0	0	0	0	0	0					0	1											≤ 1400
0	0	0	1	0	0	0	0	-1	-1	0	0							1	-1									$= 0$
0	0	0	1	0	0	0	0	0	0	0	0							0	-1									≤ 0
0	0	0	0	0	0	0	0	0	0	0	0							0	1									≤ 1200
0	1	0	0	0	0	1	0	0	0	-1	0									1	-1							$= 0$
0	1	0	0	0	0	1	0	0	0	0	0									0	-1							≤ 0
0	0	0	0	0	0	0	0	0	0	0	0									0	1							≤ 2000
0	0	0	0	1	0	0	0	1	0	0	-1											1	-1					$= 0$
0	0	0	0	1	0	0	0	1	0	0	0											0	-1					≤ 0
0	0	0	0	0	0	0	0	0	0	0	0											0	1					≤ 1500
0	0	1	0	0	0	0	1	0	0	1	0													1	-1			$= 0$
0	0	0	0	0	0	0	0	0	0	0	0													0	1			≤ 1600
0	0	0	0	0	1	0	0	0	1	0	1															1	-1	$= 0$
0	0	0	0	0	0	0	0	0	0	0	0															0	1	≤ 100

23.6-1.

(a)

$$\begin{array}{ll} \text{maximize} & 20x_1 + 30x_2 + 25x_3 \\ \text{subject to} & 3x_1 + 2x_2 + x_3 \leq 29 \\ & 2x_1 + 4x_2 + 2x_3 \leq 48 \\ & x_1 + 3x_2 + 5x_3 \leq 57 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(b) Let x_{21}, x_{22}, x_{23} be the values of x_2 when $29, 30, 31$ are observed respectively for b_1 and x_{31}, x_{32}, $\ldots$, x_{39} be the values of x_3 when the values for (b_1, b_2) are $(29, 48)$, $(29, 50)$, $(29, 52)$, $(30, 48)$, $(30, 50)$, $(30, 52)$, $(31, 48)$, $(31, 50)$, $(31, 52)$ respectively.

$$\begin{array}{ll} \text{maximize} & 20x_1 + \left(\frac{1}{4}\right)\left\{30x_{21} + 25\left[\left(\frac{1}{4}\right)x_{31} + \left(\frac{1}{2}\right)x_{32} + \left(\frac{1}{4}\right)x_{33}\right]\right\} \\ & + \left(\frac{1}{2}\right)\left\{30x_{22} + 25\left[\left(\frac{1}{4}\right)x_{34} + \left(\frac{1}{2}\right)x_{35} + \left(\frac{1}{4}\right)x_{36}\right]\right\} \\ & + \left(\frac{1}{4}\right)\left\{30x_{23} + 25\left[\left(\frac{1}{4}\right)x_{37} + \left(\frac{1}{2}\right)x_{38} + \left(\frac{1}{4}\right)x_{39}\right]\right\} \end{array}$$

$$\begin{array}{ll} \text{subject to} & 3x_1 + 2x_{21} + x_{31} \leq 29 \\ & 3x_1 + 2x_{21} + x_{32} \leq 29 \\ & 3x_1 + 2x_{21} + x_{33} \leq 29 \\ & 3x_1 + 2x_{22} + x_{34} \leq 30 \\ & 3x_1 + 2x_{22} + x_{35} \leq 30 \\ & 3x_1 + 2x_{22} + x_{36} \leq 30 \\ & 3x_1 + 2x_{23} + x_{37} \leq 31 \\ & 3x_1 + 2x_{23} + x_{38} \leq 31 \\ & 3x_1 + 2x_{23} + x_{39} \leq 31 \\ & \\ & 2x_1 + 4x_{21} + 2x_{31} \leq 48 \\ & 2x_1 + 4x_{21} + 2x_{32} \leq 50 \\ & 2x_1 + 4x_{21} + 2x_{33} \leq 52 \\ & 2x_1 + 4x_{22} + 2x_{34} \leq 48 \\ & 2x_1 + 4x_{22} + 2x_{35} \leq 50 \\ & 2x_1 + 4x_{22} + 2x_{36} \leq 52 \\ & 2x_1 + 4x_{23} + 2x_{37} \leq 48 \\ & 2x_1 + 4x_{23} + 2x_{38} \leq 50 \\ & 2x_1 + 4x_{23} + 2x_{39} \leq 52 \\ & \\ & x_1 + 3x_{21} + 5x_{32} \leq 57 \\ & x_1 + 3x_{21} + 5x_{33} \leq 57 \\ & x_1 + 3x_{22} + 5x_{34} \leq 57 \\ & x_1 + 3x_{22} + 5x_{35} \leq 57 \\ & x_1 + 3x_{22} + 5x_{36} \leq 57 \\ & x_1 + 3x_{23} + 5x_{37} \leq 57 \\ & x_1 + 3x_{23} + 5x_{38} \leq 57 \\ & x_1 + 3x_{23} + 5x_{39} \leq 57 \\ & \\ & x_1 \geq 0, x_{2i} \geq 0, x_{3j} \geq 0, \text{ for } i = 1, 2, 3 \text{ and } j = 1, \ldots, 9 \end{array}$$

23.7-1.

(a) $\mathrm{P}\left\{7+\frac{44}{3}+\frac{19}{3}\leq b_1\right\}=\mathrm{P}\{-2\leq z\}=0.9772$

$\mathrm{P}\left\{\frac{14}{3}+\frac{88}{3}+\frac{38}{3}\leq b_2\right\}=\mathrm{P}\{-5/3\leq z\}=0.9515$

$\mathrm{P}\left\{\frac{7}{3}+22+\frac{95}{3}\leq b_3\right\}=\mathrm{P}\{-4/3\leq z\}=0.9082$

$\mathrm{P}\{\text{all constraints are satisfied}\}=0.9772\times 0.9515\times 0.9082=0.8450$

The solution is feasible.

(b) maximize $20x_1+30x_2+25x_3$

subject to
$$\begin{aligned}3x_1+2x_2+x_3&\leq 28.04\\2x_1+4x_2+2x_3&\leq 46.71\\x_1+3x_2+5x_3&\leq 56.16\\x_1,x_2,x_3&\geq 0\end{aligned}$$

(c) maximize $20y_1-30y_2-25y_3$

subject to
$$\begin{aligned}3y_1-2y_2-y_3&\leq -26.96\\2y_1-4y_2-2y_3&\leq -63.29\\y_1-3y_2-5y_3&\leq -90.15\\y_1,y_2,y_3&\geq 0\end{aligned}$$

$$\mu_1=30-2\left[\left(\frac{1}{4}\right)30\right]-1\left[\left(\frac{1}{2}\right)30+\left(\frac{1}{2}\right)50\right]=-25$$

$$\mu_2=50-4\left[\left(\frac{1}{4}\right)30\right]-2\left[\left(\frac{1}{2}\right)30+\left(\frac{1}{2}\right)50\right]=-60$$

$$\mu_3=60-3\left[\left(\frac{1}{4}\right)30\right]-5\left[\left(\frac{1}{2}\right)30+\left(\frac{1}{2}\right)50\right]=-\frac{330}{4}$$

$$\sigma_1^2=\left(\frac{1}{2}\right)^2\cdot 4+\left(1-\frac{1}{2}-\frac{1}{2}\right)^2\cdot 1=1$$

$$\sigma_2^2=\left[4\left(\frac{1}{4}\right)+2\left(\frac{1}{2}\right)\right]^2\cdot 1+\left[1-2\left(\frac{1}{2}\right)\right]^2\cdot 4=4$$

$$\sigma_3^2=\left[3\left(\frac{1}{4}\right)+5\left(\frac{1}{2}\right)\right]^2\cdot 1+\left[5\left(\frac{1}{2}\right)\right]^2\cdot 4=\frac{569}{16}$$

23.7-2.

(a) $b_i-\sum_{j=1}^n a_{ij}x_j$ has a normal distribution with mean $\mathrm{E}(b_i)-\sum_{j=1}^n x_j\mathrm{E}(a_{ij})$ and variance $\sigma^2(b_i)+\sum_{j=1}^n x_j^2\sigma^2(a_{ij})$. Hence,

$$\mathrm{P}\left\{0\leq b_i-\sum_{j=1}^n a_{ij}x_j\right\}=\mathrm{P}\left\{\frac{-\mathrm{E}(b_i)+\sum_{j=1}^n x_j\mathrm{E}(a_{ij})}{\left[\sigma^2(b_i)+\sum_{j=1}^n x_j^2\sigma^2(a_{ij})\right]^{1/2}}\leq z\right\}\geq\alpha_i$$

if and only if

$$-\mathrm{E}(b_i)+\sum_{j=1}^n x_j\mathrm{E}(a_{ij})\leq K_{\alpha_i}\left[\sigma^2(b_i)+\sum_{j=1}^n x_j^2\sigma^2(a_{ij})\right]^{1/2}.$$

(b) $b_i - \sum_{j=1}^{n} a_{ij} x_j = b_i - \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} b_k d_{jk}$ has a normal distribution with mean $\mathrm{E}(b_i) - \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} d_{jk} \mathrm{E}(b_k)$ and variance $\sigma^2(b_i) + \sum_{j=1}^{n} a_{ij}^2 \sum_{k=1}^{m} d_{jk}^2 \sigma^2(b_k)$. Hence,

$$\mathrm{P}\left\{0 \leq b_i - \sum_{j=1}^{n} a_{ij} x_j\right\} = \mathrm{P}\left\{ \frac{-\mathrm{E}(b_i) + \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} d_{jk} \mathrm{E}(b_k)}{\left[\sigma^2(b_i) + \sum_{j=1}^{n} a_{ij}^2 \sum_{k=1}^{m} d_{jk}^2 \sigma^2(b_k)\right]^{1/2}} \leq z \right\} \geq \alpha_i$$

if and only if

$$-\mathrm{E}(b_i) + \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} d_{jk} \mathrm{E}(b_k) \leq K_{\alpha_i} \left[\sigma^2(b_i) + \sum_{j=1}^{n} a_{ij}^2 \sum_{k=1}^{m} d_{jk}^2 \sigma^2(b_k)\right]^{1/2}.$$

CHAPTER 24: PROBABILITY THEORY

24.1.

(a) The six colored sides: red, white, blue, green, yellow, and violet.

(b) $P\{X=0\} = P\{X=1\} = P\{X=2\} = 1/3$

(c) $E(Y) = E(X+1)^2 = \sum_{k=0}^{2}(k+1)^2 P\{X=k\} = 4\frac{2}{3}$

24.2.

(a) $P_{X_1}(i) = \begin{cases} P\{w_1 \cup w_2\} = P\{w_1\} + P\{w_2\} = 1/3 + 1/5 = 8/15 & \text{if } i = 1 \\ P\{w_3\} = 3/10 & \text{if } i = 4 \\ P\{w_4\} = 1/6 & \text{if } i = 5 \\ 0 & \text{else} \end{cases}$

(b) $E(X_1) = 1 \cdot \frac{8}{15} + 4 \cdot \frac{3}{10} + 5 \cdot \frac{1}{6} = 2\frac{17}{30}$

(c) $P_{X_1+X_2}(i) = \begin{cases} P\{w_1 \cup w_2\} = P\{w_1\} + P\{w_2\} = 1/3 + 1/5 = 8/15 & \text{if } i = 2 \\ P\{w_3\} = 3/10 & \text{if } i = 5 \\ P\{w_4\} = 1/6 & \text{if } i = 10 \\ 0 & \text{else} \end{cases}$

(d) $$E(X_1 + X_2) = 2 \cdot \frac{8}{15} + 5 \cdot \frac{3}{10} + 10 \cdot \frac{1}{6} = 4\frac{7}{30}$$

$$E(X_2) = 1 \cdot \left(\frac{1}{3} + \frac{1}{5} + \frac{3}{10}\right) + 5 \cdot \frac{1}{6} = 1\frac{2}{3}$$

or $$E(X_2) = E(X_1 + X_2) - E(X_1)$$

(e) $F_{X_1X_2}(b_1, b_2) = \begin{cases} 0 & \text{for } b_1 < 1 \text{ or } b_2 < 1 \\ 8/15 & \text{for } 1 \le b_1 < 4 \text{ and } 1 \le b_2 < \infty \\ 5/6 & \text{for } 4 \le b_1 < 5 \text{ and } 1 \le b_2 < \infty \\ 5/6 & \text{for } 4 \le b_1 < \infty \text{ and } 1 \le b_2 < 5 \\ 1 & \text{for } 5 \le b_1 \text{ and } 5 \le b_2 \end{cases}$

(f)

$$\rho = \frac{E[X_1 - E(X_1)][X_2 - E(X_2)]}{\sqrt{E[X_1 - E(X_1)]^2 E[X_2 - E(X_2)]^2}}$$

Since $E(X_1) = 77/30$, $E(X_1^2) = 285/30$, $E(X_2) = 50/30$, $E(X_2^2) = 150/30$ and $E(X_1X_2) = 177/30$, $\rho \simeq 0.64$.

(g) $E(2X_1 - 3X_2) = 2E(X_1) - 3E(X_2) = 2/15$

24.3.

(a)	(b)	(c)
GG	4	$1/4$
GM	3	$1/6$
GB	2	$1/12$
MG	3	$1/6$
MM	2	$1/9$
MB	1	$1/18$
BG	2	$1/12$
BM	1	$1/18$
BB	0	$1/36$

(d) $X \in \{0, 1, 2, 3, 4\}$

$$\begin{aligned} &P\{X = 0\} = 1/36,\\ &P\{X = 1\} = 1/18 + 1/18 = 1/9,\\ &P\{X = 2\} = 1/12 + 1/9 + 1/12 = 5/18,\\ &P\{X = 3\} = 1/6 + 1/6 = 1/3,\\ &P\{X = 4\} = 1/4,\\ &P\{X = k\} = 0 \text{ for } k \notin \{0, 1, 2, 3, 4\}. \end{aligned}$$

(e) $E(X) = 0 \cdot 1/36 + 1 \cdot 1/9 + 2 \cdot 5/18 + 3 \cdot 1/3 + 4 \cdot 1/4 = 2\frac{2}{3}$

24.4.

(a) $1 = \int_0^1 f_X(y)dy = \int_0^\theta \theta dy + \int_\theta^1 K dy = \theta^2 + K - K\theta$, so $K = \frac{(1-\theta)^2}{(1-\theta)} = 1 + \theta$

(b)

$$F_X(b) = \begin{cases} 0 & \text{if } b < 0 \\ \int_0^b \theta dy = \theta b & \text{if } 0 \le b < \theta \\ \theta^2 + \int_\theta^b (1+\theta) dy = \theta^2 + (1+\theta)b - (1+\theta)\theta = b + \theta b - \theta & \text{if } \theta \le b < 1 \\ 1 & \text{if } 1 \le b \end{cases}$$

(c) $E(X) = \int_0^\theta y\theta dy + \int_\theta^1 y(1+\theta)dy = (1 + \theta - \theta^2)/2$

(d) No, a counterexample is obtained by choosing $0 \le a \le \theta = 1/3$. In that case,

$$\begin{aligned} P\{X - 1/3 < a\} &= P\{X < a + 1/3\} = F_X(a + 1/3)\\ &= (a + 1/3) + (1/3)(a + 1/3) - 1/3 = (4/3)a + 1/9\\ P\{-(X - 1/3) < a\} &= P\{X > -a + 1/3\} = 1 - F_X(-a + 1/3)\\ &= 1 - (1/3)(-a + 1/3) = (1/3)a + 8/9, \end{aligned}$$

so the equality does not hold.

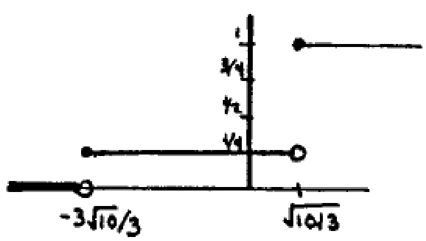
24.5.

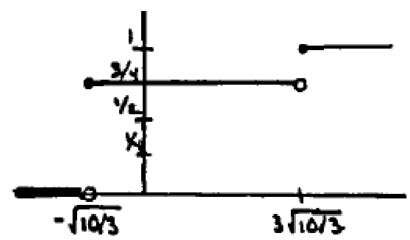
(a) $E(X) = \frac{1}{4}x_1 + \frac{3}{4}x_2 = 0 \ \Rightarrow \ x_1 = -3x_2$

$\text{var}(X) = E(X^2) - [E(X)]^2 = E(X^2) = \frac{1}{4}x_1^2 + \frac{3}{4}x_2^2 = 10$

$$\Rightarrow \ \frac{1}{4}(-3x_2)^2 + \frac{3}{4}x_2^2 = 3x_2^2 = 10 \ \Rightarrow \ \begin{cases} x_1 = -3\sqrt{10/3} \text{ and } x_2 = \sqrt{10/3} \\ x_1 = 3\sqrt{10/3} \text{ and } x_2 = -\sqrt{10/3} \end{cases}$$

(b) Depending on x_1 and x_2, the CDF can be represented as either one of the following two graphs





24.6.

(a) $P\{X \geq 250\} = 1 - P\{X < 250\} = 1 - \int_0^{250} f_X(y)dy = 1 - \int_{100}^{250} \frac{100}{y^2} dy$

$$= 1 - \left(-\frac{100}{y}\right)_{100}^{250} = 1 + 2/5 - 1 = 2/5$$

(b) $E(X) = \int_0^{\infty} y f_X(y)dy = \int_{100}^{\infty} \frac{100}{y} dy = 100(\ln \infty - \ln 100) = \infty$

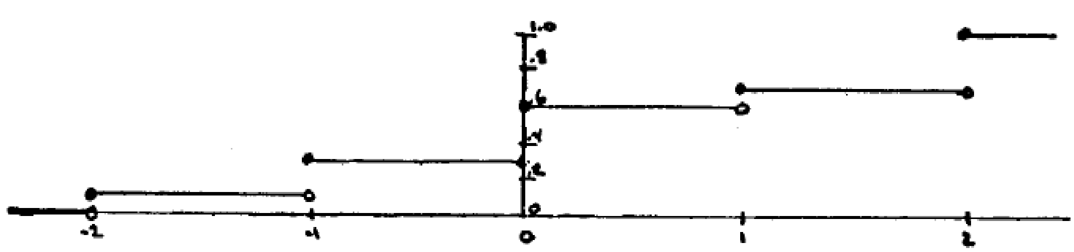
24.7.

(a) $\begin{cases} P\{-1 < X < 2\} = P\{X = 0\} + P\{X = 1\} = 0.4 \\ P\{X = 0\} = 0.3 \\ P\{|X| \leq 1\} = P\{X = -1\} + P\{X = 0\} + P\{X = 1\} = 0.6 \\ P\{X \geq 2\} = P\{X = 2\} = P\{X = -1\} + P\{X = 1\} \\ P\{X = -2\} + P\{X = -1\} + P\{X = 0\} + P\{X = 1\} + P\{X = 2\} = 1 \end{cases}$

Solving this system of equations gives:

k	-2	-1	0	1	2
$P\{X = k\}$	0.1	0.2	0.3	0.1	0.3

(b)



(c) $E(X) = 0.1 \cdot (-2) + 0.2 \cdot (-1) + 0.3 \cdot (0) + 0.1 \cdot (1) + 0.3 \cdot (2) = 0.3$

24.8.

(a) $\int_{-1}^{1} K(1-y^2)dy = K\left(y - \frac{y^3}{3}\right)_{-1}^{1} = \frac{4K}{3} = 1 \;\Rightarrow\; K = \frac{3}{4}$

(b)

$$F_X(b) = \begin{cases} 0 & \text{if } b < -1 \\ \int_{-1}^{b} K(1-y^2)dy = \frac{3}{4}\left(y - \frac{y^3}{3}\right)_{-1}^{b} = \frac{3}{4}(b+1) - \frac{1}{4}(b^3+1) & \text{if } -1 \geq b < 1 \\ 1 & \text{if } 1 \geq b \end{cases}$$

(c) $E(2X-1) = 2E(X) - 1 = 2\left(\int_{-1}^{1} y\frac{3}{4}(1-y^2)dy\right) - 1 = \frac{3}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)_{-1}^{1} - 1 = -1$

Note that $E(X) = 0$.

(d) $\text{var}(X) = E(X^2) - [E(X)]^2 = E(X^2) = \int_{-1}^{1} y^2 \frac{3}{4}(1-y^2)dy = 1/5$

(e) From the Central Limit Theorem, $\overline{X}$ is approximately normal with mean $E(X)$ and variance var(X), equivalently $\frac{\overline{X}-E(X)}{\sqrt{\text{var}(X)/n}} \sim \text{N}(0,1)$ and hence

$$P\{\overline{X} > 0\} = P\left\{\frac{\overline{X}-E(X)}{\sqrt{\text{var}(X)/n}} > \frac{-E(X)}{\sqrt{\text{var}(X)/n}}\right\} = P\{\text{N}(0,1) > 0\} = 0.5$$

24.9.

(a) $1 = \int_0^{1000} \frac{a}{1000}\left(1 - \frac{y}{1000}\right)dy = \frac{a}{1000}\left(y - \frac{y^2}{2000}\right)_0^{1000} = \frac{a}{2} \;\Rightarrow\; a = 2$

(b) $E(X) = \int_0^{1000} y\frac{2}{1000}\left(1 - \frac{y}{1000}\right)dy = \frac{1}{500}\left(\frac{y^2}{2} - \frac{y^3}{3000}\right)_0^{1000} = 333\frac{1}{3}$

(c) $$F_X(b) = \begin{cases} 0 & \text{if } b < 0 \\ \int_0^{b} \frac{2}{1000}\left(1 - \frac{y}{1000}\right)dy = \frac{1}{500}\left(y - \frac{y^2}{2000}\right)_0^{b} = \frac{b}{500} - \frac{b^2}{10^6} & \text{if } 0 \leq b < 1000 \\ 1 & \text{if } 1000 \leq b \end{cases}$$

(d) $$F_Z(b) = F_X(b/3) = \begin{cases} 0 & \text{if } b < 0 \\ \frac{b}{1500} - \frac{b^2}{9 \cdot 10^6} & \text{if } 0 \leq b < 3000 \\ 1 & \text{if } 3000 \leq b \end{cases}$$

24.10.

(a) $P\{X \geq 25\} = 1 - P\{X \leq 24\} = 1 - 0.473 = 0.527$

$P\{X = 20\} = P\{X \leq 20\} - P\{X \leq 19\} = 0.185 - 0.134 = 0.051$

(b) $P\{\text{shortage}\} = P\{X > 35\} = 1 - P\{X \leq 35\} = 1 - 0.978 = 0.022$

24.11.

(a) $E(X) = \sum_{n=1}^{\infty} 2^n (1/2)^n = \sum_{n=1}^{\infty} 1 = \infty$

Hence, player B should pay ∞ to player A so that the game is fair. Otherwise, the game can never be made fair.

(b) Since the mean is infinite and $E(X^2) \geq [E(X)]^2 = \infty$, the variance is $\infty - \infty$, so not well-defined.

(c) $P\{X \leq 8\} = P\{X = 2\} + P\{X = 4\} + P\{X = 8\} = 1/2 + 1/4 + 1/8 = 7/8$

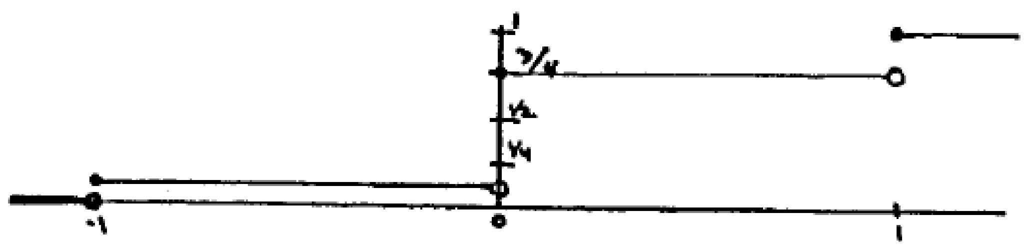
24.12.

(a) $1 = P\{D = -1\} + P\{D = 0\} + P\{D = 1\} = 1/8 + 5/8 + c/8 = 6/8 + c/8$

Solving this equation for c gives $c = 2$.

(b) $E\left(e^{D^2}\right) = \frac{1}{8} \cdot e + \frac{5}{8} \cdot 1 + \frac{2}{8} \cdot e = \frac{1}{8}(5 + 3e)$

(c)



24.13.

(a) Let X_i denote the volume of bottle i for $i = 1, 2, 3$ and $Z = X_1 + X_2 + X_3$.

$$E(Z) = E(X_1) + E(X_2) + E(X_3) = 3 \cdot 15 = 45$$

$$\text{var}(Z) = \text{var}(X_1) + \text{var}(X_2) + \text{var}(X_3) = 3 \cdot (0.08)^2 = 0.0192$$

$$\sigma_Z = \sqrt{\text{var}(Z)} = 0.139$$

(b) $Z \sim \text{N}(45, 0.0192)$

$$P\{Z \geq 45.2\} = P\left\{\frac{Z-45}{0.139} \geq \frac{45.2-45}{0.139}\right\} = P\{\text{N}(0,1) \geq 1.44\} = 0.075$$

24.14.

(a) $F_X(b) = \begin{cases} 0 & \text{if } b < 0 \\ \int_0^b 6y(1-y)dy = 6\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^b = 3b^2 - 2b^3 & \text{if } 0 \leq b < 1 \\ 1 & \text{if } 1 \leq b \end{cases}$

(b) $E(X) = \int_0^1 y6y(1-y)dy = 6\left(\frac{y^3}{3} - \frac{y^4}{4}\right)_0^1 = 0.5$

$$\text{var}(X) = E(X^2) - [E(X)]^2 = \int_0^1 y^2 6y(1-y)dy - 0.25$$

$$= 6\left(\frac{y^4}{4} - \frac{y^5}{5}\right)_0^1 - 0.25 = 0.05$$

(c) $P\{X > 0.5\} = 1 - P\{X \le 0.5\} = 1 - (3 \cdot 0.5^2 - 2 \cdot 0.5^3) = 0.5$

(d) $E\left(\frac{X_1+X_2+X_3+X_4+X_5+X_6}{6}\right) = \frac{1}{6} \cdot 6 \cdot E(X_1) = 0.5$

(e) $\text{var}\left(\frac{X_1+X_2+X_3+X_4+X_5+X_6}{6}\right) = \frac{1}{36} \cdot 6 \cdot \text{var}(X_1) = 1/120$

24.15.

(a) Let X_1 and X_2 be the voltage of battery 1 and 2 respectively, and $Z = X_1 + X_2$. Since

$$X_1 \sim \text{N}\left(1\tfrac{1}{2}, 0.0625\right) \text{ and } X_2 \sim \text{N}\left(1\tfrac{1}{2}, 0.0625\right),\ Z \sim \text{N}(3, 0.125).$$

$$P\{\text{failure}\} = P\{Z < 2.75\} + P\{Z > 3.25\} = 2 \cdot P\{Z > 3.25\}$$

$$= 2 \cdot P\left\{\text{N}(0,1) > \frac{3.25-3}{\sqrt{0.125}}\right\} = 2 \cdot P\{\text{N}(0,1) > 0.707\} = 0.48$$

The second equality is a result of the symmetry of normal distribution.

(b) Chebyshev's Inequality states $P\{|X-\mu| \ge K\sigma\} \le 1/K^2$. Hence, the probability $P\{Z < 2.75\} + P\{Z > 3.25\} = P\{|X - \mu| \ge 0.25\} \le 1/(0.25/\sigma)^2$ and since $\sigma \simeq 0.354$, the upper bound is $1/(0.706)^2$. This value exceeds 1, so it is not a useful bound on the probability.

24.16.

$P\left\{1000 \cdot \frac{1}{5000} \cdot |\overline{X} - \mu| \le 15\right\} = 0.90 \Leftrightarrow P\{|\overline{X} - \mu| \le 75\} = 0.90$

$\Leftrightarrow P\{|\overline{X} - \mu| > 75\} = 0.10 \Leftrightarrow P\{\overline{X} - \mu > 75\} = 0.05$

$\Leftrightarrow P\left\{\frac{|\overline{X}-\mu|}{\sigma_{\overline{X}}} > \frac{75}{\sigma_{\overline{X}}}\right\} = 0.05 \Leftrightarrow P\left\{\text{N}(0,1) > \frac{75}{\sigma_{\overline{X}}}\right\} = 0.05$

$\Leftrightarrow\ \frac{75}{\sigma_{\overline{X}}} = 1.645\ \Leftrightarrow\ \sigma_{\overline{X}} = 45.6$ or $\sigma^2_{\overline{X}} \simeq 2079$

Since $\sigma^2_{\overline{X}} = \sigma^2_X/n$, $2079 = 40000/n\ \Rightarrow n = 19.24$. Hence, choosing $n \ge 20$ is sufficient.

24.17.

(a) $f_{X_1}(s) = \int_{-\infty}^{\infty} f_{X_1,X_2}(s,t)dt$

Let $\mu = \frac{s-\mu_{X_1}}{\sigma_{X_1}}$ and $\nu = \frac{t-\mu_{X_2}}{\sigma_{X_2}}$ so that $dt = \sigma_{X_2} dv$.

$$f_{X_1}(s) = \frac{1}{2\pi\sigma_{X_1}\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{ \left(\frac{-1}{2(1-\rho^2)} \right) (\mu^2 - 2\rho\mu\nu + \nu^2) \right\} dv$$

$$= \frac{1}{2\pi\sigma_{X_1}\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{ \left(\frac{-1}{2(1-\rho^2)} \right) (\nu^2 - 2\rho\mu\nu + \rho^2\mu^2 - \rho^2\mu^2 + \mu^2) \right\} dv$$

Now let $z = \frac{\nu-\rho\mu}{\sqrt{1-\rho^2}}$ so that $dv = \sqrt{1-\rho^2}dz$.

$$f_{X_1}(s) = \frac{\exp(-\mu^2/2)}{2\pi\sigma_{X_1}} \int_{-\infty}^{\infty} \exp(-z^2/2)dz = \frac{\exp(-\mu^2/2)}{2\pi\sigma_{X_1}} \cdot \sqrt{2\pi} = \frac{1}{\sqrt{2\pi}\sigma_{X_1}} \exp\left[-\frac{1}{2} \left(\frac{s-\mu_{X_1}}{\sigma_{X_1}} \right)^2 \right]$$

Hence, $X_1 \sim \mathrm{N}(\mu_{X_1}, \sigma_{X_1}^2)$ and the same analysis leads to the conclusion $X_2 \sim \mathrm{N}(\mu_{X_2}, \sigma_{X_2}^2)$.

(b) $\mathrm{Corr}(X_1, X_2) = \frac{E[X_1 - E(X_1)][X_2 - E(X_2)]}{\sigma_{X_1}\sigma_{X_2}}$

$$= \frac{1}{\sigma_{X_1}\sigma_{X_2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (s - \mu_{X_1})(t - \mu_{X_2}) f_{X_1,X_2}(s,t) ds dt$$

Let $\mu = \frac{s-\mu_{X_1}}{\sigma_{X_1}}$ and $\nu = \frac{t-\mu_{X_2}}{\sigma_{X_2}}$.

$$\mathrm{Corr}(X_1, X_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu\nu \exp\left\{ \left(\frac{-1}{2(1-\rho^2)} \right) (\mu^2 - 2\rho\mu\nu + \nu^2) \right\} d\mu d\nu$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} d\mu\ \mu e^{-\mu^2/2} \int_{-\infty}^{\infty} d\nu\ \nu \exp\left\{ \left(\frac{-1}{2(1-\rho^2)} \right) (\nu - \rho\mu)^2 \right\}$$

Now let $z = \frac{\nu-\rho\mu}{\sqrt{1-\rho^2}}$.

$$\mathrm{Corr}(X_1, X_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} d\mu\ \mu e^{-\mu^2/2} [0 + \rho\mu\sqrt{1-\rho^2}\sqrt{2\pi}]$$

$$= \frac{\rho}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\mu\ \mu e^{-\mu^2/2} = \rho$$

(c) See part (a).

(d) Let $\mu = \frac{x_1-\mu_{X_1}}{\sigma_{X_1}}$ and $\nu = \frac{x_2-\mu_{X_2}}{\sigma_{X_2}}$.

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} = \frac{\left(\frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}} \right) \exp\left\{ \left(\frac{-1}{2(1-\rho^2)} \right) (\mu^2 - 2\rho\mu\nu + \nu^2) \right\}}{\left(\frac{1}{\sqrt{2\pi}\sigma_{X_2}} \right) e^{-\nu^2/2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_{X_1}\sqrt{1-\rho^2}} \exp\left\{ \left(-\frac{1}{2} \right) \left[\frac{x_2 - \mu_{X_1} - \rho\frac{\sigma_{X_1}}{\sigma_{X_2}}(x_2 - \mu_{X_2})}{\sigma_{X_1}\sqrt{1-\rho^2}} \right] \right\}$$

24.18.

(a) $1 = \int_{100}^{150}\int_{50}^{100} c\,ds\,dt = 2500c \;\Rightarrow\; c = 1/2500$

(b)

$$F_{X_1X_2}(b_1,b_2) = \begin{cases} 0 & \text{for } b_1 < 100 \text{ or } b_2 < 50 \\ \int_{100}^{b_1}\int_{50}^{b_2}\frac{1}{2500}ds\,dt = \frac{1}{2500}(b_1-100)(b_2-50) & \text{for } 100 \le b_1 < 150 \text{ and } 50 \le b_2 < 100 \\ \int_{100}^{150}\int_{50}^{b_2}\frac{1}{2500}ds\,dt = \frac{(b_2-50)}{50} & \text{for } 150 \le b_1 \text{ and } 50 \le b_2 < 100 \\ \int_{100}^{b_1}\int_{50}^{100}\frac{1}{2500}ds\,dt = \frac{(b_1-100)}{50} & \text{for } 100 \le b_1 < 150 \text{ and } 100 \le b_2 \\ 1 & \text{for } 150 \le b_1 \text{ and } 100 \le b_2 \end{cases}$$

$$F_{X_1}(b_1) = \begin{cases} 0 & \text{for } b_1 < 100 \\ \int_{100}^{b_1}\int_{50}^{100}\frac{1}{2500}ds\,dt = \frac{(b_1-100)}{2500} & \text{for } 100 \le b_1 < 150 \\ 1 & \text{for } 150 \le b_1 \end{cases}$$

$$F_{X_2}(b_2) = \begin{cases} 0 & \text{for } b_2 < 50 \\ \int_{100}^{150}\int_{50}^{b_2}\frac{1}{2500}ds\,dt = \frac{(b_2-50)}{2500} & \text{for } 50 \le b_2 < 100 \\ 1 & \text{for } 100 \le b_2 \end{cases}$$

(c) $f_{X_1}(s) = 1/50$ for $100 \le s < 150$

$f_{X_2|X_1=s}(t) = \frac{f_{X_1,X_2}(s,t)}{f_{X_1}(s)} = \frac{1/2500}{1/50} = \frac{1}{50}$ for $100 \le s < 150$ and $f_{X_2|X_1=s}(t) = 0$ else.

24.19.

(a) $$P_{X_1}(0) = \sum_{k=0}^{2} P_{X_1,X_2}(0,k) = 1/2$$

$$P_{X_1}(1) = 1 - P_{X_1}(0) = 1/2$$

$$P_{X_2}(0) = \sum_{k=0}^{1} P_{X_1,X_2}(k,0) = 1/8$$

$$P_{X_2}(1) = \sum_{k=0}^{1} P_{X_1,X_2}(k,1) = 3/8$$

$$P_{X_2}(2) = 1 - P_{X_2}(0) - P_{X_2}(1) = 1/2$$

(b) $$P_{X_1|X_2=1}(0) = \frac{P_{X_1,X_2}(0,1)}{P_{X_2}(1)} = \frac{1/4}{3/8} = \frac{2}{3}$$

$$P_{X_1|X_2=1}(1) = \frac{P_{X_1,X_2}(1,1)}{P_{X_2}(1)} = \frac{1/8}{3/8} = \frac{1}{3}$$

(c) No, consider $P_{X_1|X_2=1}(0) = 2/3 \neq 1/2 = P_{X_1}(0)$.

(d) $E(X_1) = 1/2$ and $\mathrm{var}(X_1) = 1/4$

$E(X_2) = 11/8$ and $\mathrm{var}(X_2) = 31/64$

(e) $P_{X_1+X_2}(0) = 1/8$

$P_{X_1+X_2}(1) = 1/4 + 0 = 1/4$

$P_{X_1+X_2}(2) = 1/8 + 1/8 = 1/4$

$P_{X_1+X_2}(3) = 3/8$

24.20.

(a) $P\{F\} = P\{F \cap \Omega\} = P\{F \cap (E_1 \cup E_2 \cup \ldots \cup E_m)\} = P\{\bigcup_{i=1}^{m}(F \cap E_i)\}$

$= \sum_{i=1}^{m} P\{F \cap E_i\}$ since $P\{E_i \cap E_j\} = 0$ for $i \neq j$

$= \sum_{i=1}^{m} P\{F \mid E_i\} P\{E_i\}$ since $P\{F \mid E_i\} = \frac{P\{F \cap E_i\}}{P\{E_i\}}$

(b) $P\{E_i \mid F\} = \frac{P\{E_i \cap F\}}{P\{F\}} = \frac{P\{E_i \cap F\}}{\sum_{i=1}^{m} P\{F \mid E_i\} P\{E_i\}} = \frac{P\{F \mid E_i\} P\{E_i\}}{\sum_{i=1}^{m} P\{F \mid E_i\} P\{E_i\}}$

CHAPTER 25: RELIABILITY

25.1-1.

The minimal paths for the system are X_1X_2 and X_1X_3. Hence,

$$\begin{aligned}\phi(X_1,X_2,X_3) &= \max[X_1X_2, X_1X_3] = X_1 \max[X_2,X_3] \\ &= X_1[1-(1-X_2)(1-X_3)].\end{aligned}$$

25.1-2.

The minimal paths for the system are $X_1X_2X_3$ and $X_1X_2X_4$. Hence,

$$\begin{aligned}\phi(X_1,X_2,X_3,X_4) &= \max[X_1X_2X_3, X_1X_2X_4] = X_1X_2 \max[X_3,X_4] \\ &= X_1X_2[1-(1-X_3)(1-X_4)].\end{aligned}$$

25.2-1.

Note that throughout this chapter we assume that the component reliabilities are independent.

$$R(p_1,p_2,p_3) = \mathrm{E}[\phi(X_1,X_2,X_3)] = p_1[1-(1-p_2)(1-p_3)]$$

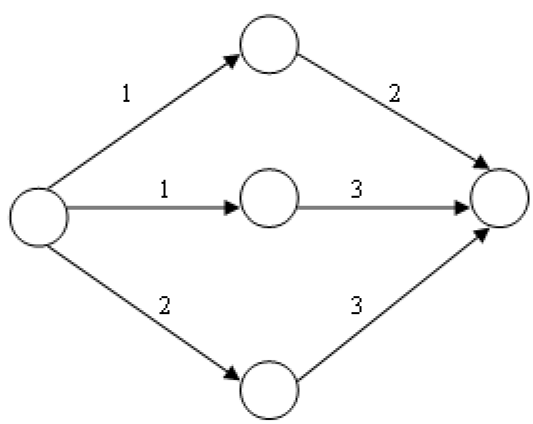
25.2-2.

$$R(p_1,p_2,p_3,p_4) = \mathrm{E}[\phi(X_1,X_2,X_3,X_4)] = p_1p_2[1-(1-p_3)(1-p_4)]$$

25.3-1.

(a) Yes, $k=2$, $n=3$.

(b)



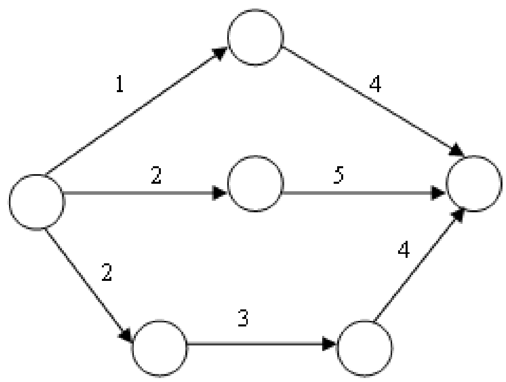
(c) $\phi(X_1,X_2,X_3) = 1-(1-X_1X_2)(1-X_1X_3)(1-X_2X_3)$

$$= X_1^2X_2X_3 + X_1X_2^2X_3 + X_1X_2X_3^2 - X_1X_2 - X_1X_3 - X_1^2X_2^2X_3^2$$

(d) $R(p_1,p_2,p_3) = 1-(1-p_1p_2)(1-p_1p_3)(1-p_2p_3)$

25.3-2.

(a)



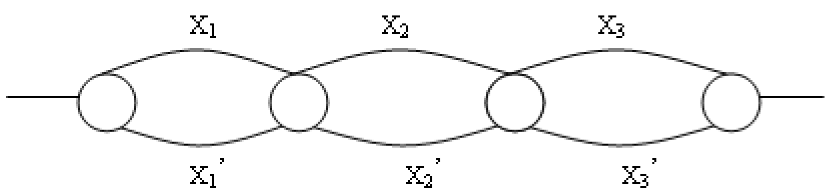
(b) $\phi(X_1, X_2, X_3, X_4, X_5) = 1 - (1 - X_1 X_4)(1 - X_2 X_5)(1 - X_2 X_3 X_4)$

(c) $R(t) = 1 - (1 - R_1(t)R_4(t))(1 - R_2(t)R_5(t))(1 - R_2(t)R_3(t)R_4(t))$

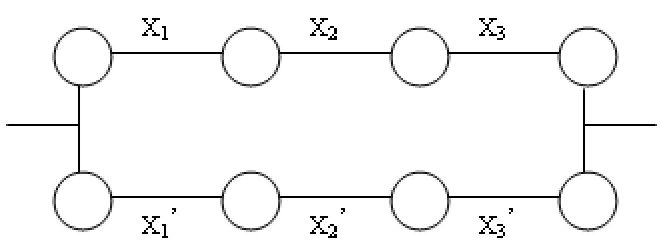
25.3-3.

Let X_i and X_i' denote the two units of type $i = 1, 2, 3$. Then, the two systems to be compared can be represented as follows.

System A



System B



$\phi_A(X_1, X_2, X_3, X_1', X_2', X_3') = [\max(X_1, X_1')][\max(X_2, X_2')][\max(X_3, X_3')]$

$\phi_B(X_1, X_2, X_3, X_1', X_2', X_3') = \max(X_1 X_2 X_3, X_1' X_2' X_3')$

$[\max(X_1, X_1')][\max(X_2, X_2')][\max(X_3, X_3')] \geq X_1 X_2 X_3$

$[\max(X_1, X_1')][\max(X_2, X_2')][\max(X_3, X_3')] \geq X_1' X_2' X_3'$

Hence, $\phi_A(X, X') \geq \max(X_1 X_2 X_3, X_1' X_2' X_3') = \phi_B(X, X'')$ and system A is more reliable than system B.

25.4-1.

(a) Minimal paths: X_1X_3 and X_2X_4

Minimal cuts: X_1X_2, X_1X_4, X_2X_3 and X_3X_4

(b) From the minimal path representation:

$\phi(X_1, X_2, X_3, X_4) = \max[X_1X_3, X_2X_4] = 1 - (1 - X_1X_3)(1 - X_2X_4)$

$R(p_1, p_2, p_3, p_4) = 1 - (1 - p_1p_3)(1 - p_2p_4)$.

If $p_i = p = 0.90$ for all i, $R(p) = 0.9639$.

(c) Upper bound $= 1 - (1 - p_1p_3)(1 - p_2p_4)$

Lower bound $= (1 - q_1q_2)(1 - q_1q_4)(1 - q_2q_3)(1 - q_3q_4)$

where $q_i = 1 - p_i$. If $p_i = p = 0.90$ for all i, then the upper bound is 0.9639 and the lower bound is 0.96060.

25.4-2.

(a) Minimal paths: X_1X_5, $X_1X_3X_4$, $X_2X_3X_5$ and X_2X_4

Minimal cuts: X_1X_2, $X_1X_3X_4$, $X_2X_3X_5$ and X_4X_5

(b) $R(p_1, p_2, p_3, p_4, p_5)$

$$
\begin{aligned}
&= \mathrm{P}\{(X_1X_5 = 1) \cup (X_1X_3X_4 = 1) \cup (X_2X_3X_5 = 1) \cup (X_2X_4 = 1)\} \\
&= \mathrm{P}(X_1X_5 = 1) + \mathrm{P}(X_1X_3X_4 = 1) + \mathrm{P}(X_2X_3X_5 = 1) + \mathrm{P}(X_2X_4 = 1) \\
&- \mathrm{P}(X_1X_3X_4X_5 = 1) - \mathrm{P}(X_1X_2X_3X_5 = 1) - \mathrm{P}(X_1X_2X_4X_5 = 1) \\
&- \mathrm{P}(X_1X_2X_3X_4X_5 = 1) - \mathrm{P}(X_1X_2X_3X_4 = 1) - \mathrm{P}(X_2X_3X_4X_5 = 1) \\
&+ \mathrm{P}(X_1X_2X_3X_4X_5 = 1) + \mathrm{P}(X_1X_2X_3X_4X_5 = 1) + \mathrm{P}(X_1X_2X_3X_4X_5 = 1) \\
&+ \mathrm{P}(X_1X_2X_3X_4X_5 = 1) - \mathrm{P}(X_1X_2X_3X_4X_5 = 1) \\
&= p_1p_5 + p_1p_3p_4 + p_2p_3p_5 + p_2p_4 - p_1p_3p_4p_5 - p_1p_2p_3p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4 \\
&- p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5
\end{aligned}
$$

If $p_i = p = 0.90$ for all i, $R(p) = 0.97848$.

(c) Upper bound $= 1 - (1 - p_1p_5)(1 - p_1p_3p_4)(1 - p_2p_3p_5)(1 - p_2p_4)$

Lower bound $= (1 - q_1q_2)(1 - q_1q_3q_4)(1 - q_2q_3q_5)(1 - q_4q_5)$

where $q_i = 1 - p_i$. If $p_i = p = 0.90$ for all i, then the upper bound is 0.99735 and the lower bound is 0.97814.

25.4-3.

(a) Minimal paths: X_1X_2 and X_2X_3

Minimal cuts: X_1X_3 and X_2

(b) From the minimal path representation:

$\phi(X_1, X_2, X_3) = \max[X_1X_2, X_2X_3] = X_2[1 - (1 - X_1)(1 - X_3)]$

$R(p_1, p_2, p_3) = p_2[1 - (1 - p_1)(1 - p_3)] = p_1p_2 + p_2p_3 - p_1p_2p_3$.

If $p_i = p = 0.90$ for all i, $R(p) = 0.891$.

(c) Upper bound $= 1 - (1 - p_1 p_2)(1 - p_2 p_3)$

Lower bound $= (1 - q_1 q_3)(1 - q_2)$

where $q_i = 1 - p_i$. If $p_i = p = 0.90$ for all i, then the upper bound is 0.9639 and the lower bound is 0.891.

25.4-4.

(a) Minimal paths: $X_1 X_5$, $X_1 X_3 X_6$, $X_2 X_6$ and $X_2 X_4 X_5$

Minimal cuts: $X_1 X_2$, $X_1 X_4 X_6$, $X_2 X_3 X_5$ and $X_5 X_6$

(b) $R(p_1, p_2, p_3, p_4, p_5, p_6)$

$= \text{P}\{(X_1 X_5 = 1) \cup (X_1 X_3 X_6 = 1) \cup (X_2 X_6 = 1) \cup (X_2 X_4 X_5 = 1)\}$

$= p_1 p_5 + p_1 p_3 p_6 + p_2 p_6 + p_2 p_4 p_5 - p_1 p_3 p_5 p_6 - p_1 p_2 p_5 p_6 - p_1 p_2 p_4 p_5 - p_1 p_2 p_3 p_6$

$- p_2 p_4 p_5 p_6 + p_1 p_2 p_3 p_5 p_6 + p_1 p_2 p_4 p_5 p_6$

If $p_i = p$ for all i, $R(p) = 2p^2 + 2p^3 - 5p^4 + 2p^5$ and if $p = 0.9$, then $R(p) = 0.97848$..

(c) Upper bound $= 1 - (1 - p_1 p_5)(1 - p_1 p_3 p_6)(1 - p_2 p_6)(1 - p_2 p_4 p_5)$

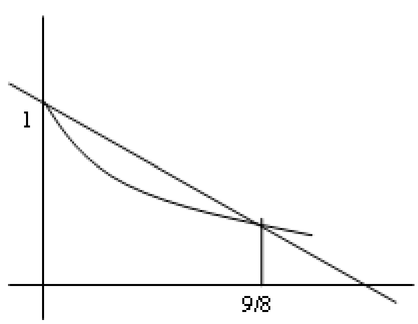
Lower bound $= (1 - q_1 q_2)(1 - q_1 q_4 q_6)(1 - q_2 q_3 q_5)(1 - q_5 q_6)$

where $q_i = 1 - p_i$. If $p_i = p = 0.90$ for all i, then the upper bound is 0.99735 and the lower bound is 0.97814.

25-5.1.

(a) $R(t) \geq e^{-t/\mu}$ for $t \leq \mu \Rightarrow R(1/4) \geq e^{-(1/4)/0.6} \approx 0.659$, so $0.659 \leq R(1/4) \leq 1$.

(b) $R(t) \leq e^{-wt}$ for $t > \mu$ where $1 - \mu w = e^{-wt}$, so we need to find w such that $e^{-w} = 1 - 0.6w$.



Hence, $w \approx 9/8$ and $0 \leq R(t) \leq e^{-9/8} \approx 0.325$.

25-5.2.

$$f(t) = \frac{\beta}{\eta} t^{\beta-1} e^{-t^\beta/\eta} \text{ and } R(t) = e^{-t^\beta/\eta}, \text{ so } r(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\eta} t^{\beta-1},$$

which is nondecreasing if $\beta \geq 1$, nonincreasing if $\beta \leq 1$. Therefore, the Weibull distribution is IFR for $\beta \geq 1$ and DFR for $\beta \leq 1$.

25-5.3.

$$R(t) = \text{P}\{T_1 > t \text{ and } T_2 > t\} = e^{-\frac{t}{\theta_1}} e^{-\frac{t}{\theta_2}} = e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)},$$

so the failure rate of the system is exponentially distributed with parameter $(1/\theta_1)+(1/\theta_2)$ and as noted in Section 25.5, the exponential distribution is both IFR and DFR.

25.5-4.

Let X_i denote the failure time of component i and X the failure time of the system. Also let $\lambda_i = 1/\mu_i$. Then

$$F(t) = \text{P}\{X \le t\} = \text{P}\{X_1 \le t,\, X_2 \le t\} = (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t}),$$

$$r(t) = \frac{f(t)}{1-F(t)} = \frac{\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}}{e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}}.$$

Note that $r(0) = 0$.

$$\begin{aligned} \frac{dr(t)}{dt} &= \frac{\lambda_1^2 e^{-(\lambda_1+2\lambda_2)t} + \lambda_2^2 e^{-(2\lambda_1+\lambda_2)t} - (\lambda_1-\lambda_2)^2 e^{-(\lambda_1+\lambda_2)t}}{[e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}]^2} \\ &= \frac{e^{-(\lambda_1+\lambda_2)t}[\lambda_1^2 e^{-\lambda_2 t} + \lambda_2^2 e^{-\lambda_1 t} - (\lambda_1-\lambda_2)^2]}{[e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}]^2} \end{aligned}$$

Let $K(t) = \lambda_1^2 e^{-\lambda_2 t} + \lambda_2^2 e^{-\lambda_1 t} - (\lambda_1 - \lambda_2)^2$ and note that:

$K(0) = 2\lambda_1\lambda_2 > 0$,

$K(\infty) = -(\lambda_1 - \lambda_2)^2 < 0$, since $\lambda_1 \ne \lambda_2$ and

$\frac{dK(t)}{dt} = -\lambda_1^2\lambda_2 e^{-\lambda_2 t} - \lambda_1\lambda_2^2 e^{-\lambda_1 t} < 0.$

Hence, $K(t)$ is a strictly decreasing function of t. It is positive at $t = 0$ and negative as t tends to ∞. These together with the continuity imply that $K(t) = 0$ has a unique solution. Now, suppose $K(t_0) = 0$ for some $0 < t_0 < \infty$.

$$K(t)\begin{cases} > 0 & \text{for } t < t_0 \\ = 0 & \text{for } t = t_0 \\ < 0 & \text{for } t > t_0 \end{cases} \qquad \frac{dr(t)}{dt}\begin{cases} > 0 & \text{for } t < t_0 \\ = 0 & \text{for } t = t_0 \\ < 0 & \text{for } t > t_0 \end{cases}$$

Then, $r(t)$ is increasing for $t \le t_0$ and decreasing for $t \ge t_0$. Thus, the system can be IFR if and only if $t_0 = \infty$. But since $K(\infty) = -(\lambda_1 - \lambda_2)^2$, this can occur if and only if $\lambda_1 = \lambda_2$, which contradicts the assumption that $\mu_1 \ne \mu_2$.

25.5-5.

Each component has an exponential failure time. The exponential distribution is IFR and hence the time to failure distribution of each component is IFRA, so the system of Problem 25.5-4 is composed of two independent IFRA components. The last paragraph of Section 25.5 states the result that the time to failure distribution of the system is IFRA.

CHAPTER 26: THE APPLICATION OF QUEUEING THEORY

26.2-1.

	Service Costs	Waiting Costs
(a)	Salaries of checkers, cost of cash registers	Lost profit from lost business
(b)	Salaries of firemen, cost of fire trucks	Cost of destruction due to waiting
(c)	Salaries of toll takers, cost of constructing toll lane	Cost of waiting for commuters
(d)	Salaries of repairpersons, cost of tools	Lost profit from lost business
(e)	Salaries of longshoremen, cost of equipment	Lost profit from ships not loaded or unloaded
(f)	Salary of an operator as a function of their experience	Lost profit/productivity from unused machines
(g)	Salaries of operators, cost of equipment	Lost profit/productivity from waiting materials
(h)	Salaries of plumbers, cost of tools	Lost profit from lost business
(i)	Salaries of employees, cost of equipment	Lost profit from lost business
(j)	Salaries of typists, cost of typewriters	Lost profit from unfinished jobs

26.3-1.

$s = 1, \lambda = 2, \mu = 4 \Rightarrow \rho = 0.5 \Rightarrow P_n = 0.5^{n+1}$ and $f_n(t) = 2e^{-2t}$

The answers in (a) and (b) are based on the following identities.

(i) $\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$ if $|x| < 1$

(ii) $\sum_{n=0}^{\infty} n^2x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$ if $|x| < 1$

(iii) $\int_0^b xe^{-\alpha x}dx = \frac{1}{\alpha^2}(1 - e^{-\alpha b} - \alpha b e^{-\alpha b}) \Rightarrow \int_0^\infty xe^{-\alpha x}dx = \frac{1}{\alpha^2}$

(iv) $\int_0^\infty x^3e^{-\alpha x}dx = \frac{6}{\alpha^4}$

(a)
$$\begin{aligned}\text{E(WC)} &= \sum_{n=0}^{\infty}(10n + 2n^2)\text{P}_n = 10\sum_{n=0}^{\infty} n0.5^{n+1} + 2\sum_{n=0}^{\infty} n^2 0.5^{n+1} \\ &= 5\sum_{n=0}^{\infty} n0.5^n + \sum_{n=0}^{\infty} n^2 0.5^n = 5\left(\frac{0.5}{\left(1-0.5\right)^2}\right) + \left(\frac{2\cdot 0.5^2}{\left(1-0.5\right)^3} + \frac{0.5}{\left(1-0.5\right)^2}\right) = 16\end{aligned}$$

(b)
$$\begin{aligned}\text{E(WC)} &= \lambda \text{E}[h(\mathcal{W})] = 2\int_0^\infty (25w + w^3)(2e^{-2w})dw \\ &= 100\int_0^\infty we^{-2w}dw + 4\int_0^\infty w^3e^{-2w}dw = 100 \cdot \frac{1}{2^2} + 4 \cdot \frac{6}{2^4} = 26.5\end{aligned}$$

26.3-2.

The answers in (a) and (b) are based on the following identities.

(i) $\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$ if $|x| < 1$

(ii) $\sum_{n=0}^{\infty} n^2x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$ if $|x| < 1$

(iii) $\sum_{n=0}^{\infty} n^3x^n = \frac{6x^3}{(1-x)^4} + \frac{6x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$ if $|x| < 1$

(iv) $\int_0^b xe^{-\alpha x}dx = \frac{1}{\alpha^2}(1 - e^{-\alpha b} - \alpha b e^{-\alpha b}) \Rightarrow \int_0^\infty xe^{-\alpha x}dx = \frac{1}{\alpha^2}$

(v) $\int_b^\infty x^2 e^{-\alpha x}dx = \frac{1}{\alpha^3}(2+2\alpha b+\alpha^2 b^2)e^{-\alpha b}$

(a)
$$\begin{aligned}\text{E(WC)} &= 10\sum_{n=0}^{2} n0.5^{n+1} + \sum_{n=3}^{5} 6n^2 0.5^{n+1} + \sum_{n=6}^{\infty} n^3 0.5^{n+1}\\ &= 10\cdot\tfrac{1}{4} + 20\cdot\tfrac{1}{8} + 54\cdot\tfrac{1}{16} + 96\cdot\tfrac{1}{32} + 150\cdot\tfrac{1}{64} + \sum_{n=6}^{\infty} n^3 0.5^{n+1}\\ &= 20 + \tfrac{419}{128} = 23.273\end{aligned}$$

(b)
$$\begin{aligned}\text{E(WC)} &= 2\int_0^1 w2e^{-2w}dw + 2\int_1^\infty w^2 2e^{-2w}dw\\ &= 4\left[\tfrac{1}{2^2}(1-e^{-2}-2e^{-2})\right] + 4\left[\tfrac{1}{2^3}(2+4+4)e^{-2}\right]\\ &= 1-3e^{-2}+5e^{-2} = 1.271\end{aligned}$$

26.4-1.

$\lambda = 4$, $\mu = 5$, $C_S = 20$

$$g(N) = \begin{cases} 0 & \text{for } N=0\\ 120 & \text{for } N=1\\ 120+180(N-1) & \text{for } N\geq 2\end{cases}$$

$$\begin{aligned}\text{E(WC)} &= \sum_{n=0}^{\infty} g(n)\text{P}_n = 120\sum_{n=1}^{\infty}\text{P}_n + 180\sum_{n=2}^{\infty} n\text{P}_n - 180\sum_{n=2}^{\infty}\text{P}_n\\ &= 120(1-\text{P}_0) + 180(\text{L}-\text{P}_1) - 180(1-\text{P}_0-\text{P}_1) = 60\text{P}_0 + 180\text{L} - 60\end{aligned}$$

s	$\rho = 4/55$	P_0	L	E(WC)	E(SC)	E(TC)
1	0.8	0.20	4.0	672.00	20.0	692.00
2	0.4	0.43	0.95	136.80	40.0	176.80
3	0.267	0.45	0.82	114.60	60.0	174.60
4	0.2	0.44	0.80	110.40	80.0	190.40

Hence, $s^* = 3$ and E(TC) $=$ \$ 174.60 per hour.

26.4-2.

(a) Model 2 with s $= 1$ fixed, A $= \{30, 40\}$, $\lambda = 20$,

$$f(\mu) = \begin{cases} 4 & \text{for } \mu = 30\\ 12 & \text{for } \mu = 40\end{cases}$$

We need to choose between a slow server consisting of only the cashier and a fast one consisting of the cashier and a box boy.

(b) E(WC) $= \lambda\text{E}[h(\mathcal{W})] = \lambda\text{E}[(0.08)\mathcal{W}] = \lambda(0.08)\text{W} = 0.08\text{L} = 0.08\frac{\lambda}{\mu-\lambda}$

μ	$f(\mu)$	E(WC)	E(TC)
30	4	0.16	4.16
40	12	0.08	12.08

Hence, the status quo should be maintained.

26.4-3.

(a) $L = 1.5 \Rightarrow W = \frac{L}{\lambda} = \frac{1.5}{0.2} = 7.5 \Rightarrow W_q = W - \frac{1}{\mu} = 7.5 - \frac{1}{0.167} = 1.5$

$\Rightarrow L_q = \lambda W_q = 0.2(1.5) = 0.3$

(b)

Template for M/D/1 Queueing Model

	Data			Results
λ =	0.2	(mean arrival rate)	L =	1.05
μ =	0.333333	(mean service rate)	L_q =	0.45
s =	1	(# servers)		
			W =	5.25
			W_q =	2.25

(c) TC(Alternative 1) $= \$70 + (\$100)(L) = \$220$
TC(Alternative 2) $= \$100 + (\$100)(L) = \$205$

Alternative 2 should be chosen.

26.4-4.

(a)

Template for the M/G/1 Queueing Model

	Data			Results
λ =	0.05	(mean arrival rate)	L =	3.000
1/μ =	15	(expected service time)	L_q =	2.250
σ=	15	(standard deviation)		
s =	1	(# servers)	W =	60.000
			W_q =	45.000

(b)

Template for the M/G/1 Queueing Model

	Data			Results
λ =	0.05	(mean arrival rate)	L =	2.963
1/μ =	16	(expected service time)	L_q =	2.163
σ=	9.486833	(standard deviation)		
s =	1	(# servers)	W =	59.250
			W_q =	43.250

(c) The new proposal shows that they will be slightly better off if they switch to the new queueing system.

(d) TC(Status quo) $= \$40 + (L_q)(\$20) = \$85$/hour

TC(Proposal) $= \$40 + (L_q)(\$20) = \$83$/hour

26.4-5.

(a) $L = 2 \Rightarrow W = \frac{L}{\lambda} = \frac{2}{0.3} = 6.67 \Rightarrow W_q = W - \frac{1}{\mu} = 6.67 - \frac{1}{0.2} = 1.67$

$\Rightarrow L_q = \lambda W_q = 0.3(1.67) = 0.5$

(b)

Template for the M/G/1 Queueing Model

	Data			Results
λ =	0.3	(mean arrival rate)	L =	5.587
$1/\mu$ =	3	(expected service time)	L_q =	4.687
σ=	1.19	(standard deviation)		
s =	1	(# servers)	W =	18.624
			W_q =	15.624

(c) TC(Alternative 1) $= \$3000 + (\$150)(L) = \$3,300$
TC(Alternative 2) $= \$2750 + (\$150)(L) = \$3,589$

Alternative 1 should be chosen.

26.4-6.

For the status quo, the system has Poisson arrivals with $\lambda = 15$, exponential service time with $\mu = 15$, $s = 1$ and the capacity of the waiting room is $K = 4$. There is a waiting cost of $6W_q$ for each customer due to loss of good will and also a waiting cost of \$45 per hour when the system is full (i.e., when there are four cars in the system) due to loss of potential customers.

$$\text{E(TC)} = \text{E(WC)} = \lambda 6 W_q + 45P_4 = 6L_q + 45P_4$$

$$\rho = \lambda/\mu = 1 \Rightarrow P_n = \tfrac{1}{K+1} = \tfrac{1}{5} \text{ for } n = 0,1,2,3,4$$

$$L = \sum_{n=1}^{K} nP_n = \tfrac{1}{5}(1+2+3+4) = 2$$

$$L_q = L - (1 - P_0) = 2 - \tfrac{4}{5} = \tfrac{6}{5}$$

$$\text{E(TC)} = 6 \cdot \tfrac{6}{5} + 45 \cdot \tfrac{1}{5} = \$16.20 \text{ per hour}$$

For Proposal 1, the system has Poisson arrivals with $\lambda = 15$, exponential service time with $\mu = 20$ and $s = 1$. In addition to the waiting cost of $6L_q$ due to loss of good will, there is an expected waiting cost of \$2 per customer that waits longer than half an hour before his car is ready. The expected value of this additional waiting cost is given by:

$$2\lambda P\{\mathcal{W} > 0.5\} = 2\lambda e^{-\mu(1-\rho)/2} = 30e^{-2.5} = 2.46.$$

$$L_q = \tfrac{\lambda^2}{\mu(\mu-\lambda)} = \tfrac{225}{20\cdot 5} = 2.25$$

$$\text{E(TC)} = 3 + 6 \cdot 2.25 + 2.46 = \$18.96 \text{ per hour,}$$

where \$3 is the capitalized cost of the new equipment.

For Proposal 2, the system has Poisson arrivals with $\lambda = 15$, Erlang service time with $\mu = 30, k = 2$ and $s = 1$. The only waiting cost is $6L_q$ due to loss of good will.

$$L_q = \left(\tfrac{k+1}{2k}\right)\left(\tfrac{\lambda^2}{\mu(\mu-\lambda)}\right) = \tfrac{3}{4} \cdot \tfrac{225}{30\cdot 15} = 0.375$$

$$\text{E(TC)} = 10 + 2.25 = \$12.25 \text{ per hour}$$

Hence, Proposal 2 should be adopted.

26.4-7.

(a) The customers are trucks to be loaded or unloaded and the servers are crews. The system currently has one server.

(b)

Template for the M/M/s Queueing Model

	Data			Results
$\lambda =$	1	(mean arrival rate)	$L =$	0.333333333
$\mu =$	4	(mean service rate)	$L_q =$	0.083333333
$s =$	1	(# servers)		
			$W =$	0.333333333
$Pr(W > t) =$	0.049787		$W_q =$	0.083333333
when $t =$	1			
			$\rho =$	0.25
$Prob(W_q > t) =$	0.012447			
when $t =$	1		n	P_n
			0	0.75

(c)

Template for the M/M/s Queueing Model

	Data			Results
$\lambda =$	1	(mean arrival rate)	$L =$	0.5
$\mu =$	3	(mean service rate)	$L_q =$	0.166666667
$s =$	1	(# servers)		
			$W =$	0.5
$Pr(W > t) =$	0.135335		$W_q =$	0.166666667
when $t =$	1			
			$\rho =$	0.333333333
$Prob(W_q > t) =$	0.045112			
when $t =$	1		n	P_n
			0	0.666666667

(d)

Template for the M/M/s Queueing Model

	Data			Results
$\lambda =$	1	(mean arrival rate)	$L =$	1
$\mu =$	2	(mean service rate)	$L_q =$	0.5
$s =$	1	(# servers)		
			$W =$	1
$Pr(W > t) =$	0.367879		$W_q =$	0.5
when $t =$	1			
			$\rho =$	0.5
$Prob(W_q > t) =$	0.18394			
when $t =$	1		n	P_n
			0	0.5

(e) A one person team should not be considered since that would lead to a utilization factor of $\rho = 1$, which is not permitted in this model.

(f) - (g)

$$\begin{aligned}
&\text{TC}(m) = (\$20)(m) + (\$30)(L_q)\\
&\text{TC}(4) = (\$20)(4) + (\$30)(0.0833) = \$82.50\text{/hour}\\
&\text{TC}(3) = (\$20)(3) + (\$30)(0.167) = \$65\text{/hour}\\
&\text{TC}(2) = (\$20)(2) + (\$30)(0.5) = \$55\text{/hour}
\end{aligned}$$

A crew of 2 people will minimize the expected total cost per hour.

(h)

s	$\mu_s = \sqrt{s}$	$L = \frac{\lambda}{\mu_s - \lambda}$	E(WC) = 15L	E(SC) = 10s	E(TC)
1	1.000	∞	∞	10	∞
2	1.414	2.414	36.21	20	56.21
3	1.732	1.366	20.49	30	50.49
4	2.000	1.000	15.00	40	55.00
5	2.236	0.809	13.75	50	63.75

Since clearly E(SC) > 50.49 for s ≥ 6, it follows that s$^* = 3$.

26.4-8.

$\lambda = 4$, $\mu = 6n$, E$(N) = \lambda/(\mu - \lambda) = 4/(6n - 4)$

Hourly cost $c(n) = 18n + 20\text{E}(N) = 18n + \frac{80}{6n-4}$

One can easily check that $c(n)$ is convex in n. When n is restricted to be integer, $c(n)$ attains its minimum at $n = 2$, so two leaders would minimize the expected hourly cost.

26.4-9.

$\lambda = 3$, E(T) $= (\mu - 3)^{-1}$

Expected cost $c(\mu) = 5\mu + 60\text{E(T)} \cdot \lambda = 5\mu + 180(\mu - 3)^{-1}$

$c'(\mu) = 5 - 180(\mu - 3)^{-2}$

The derivative is zero at $\mu = 9$ and $c(\mu)$ is convex in μ, so $c(\mu)$ attains its minimum at $\mu = 9$. Equivalently, an hourly wage of \$45 minimizes the expected total cost.

26.4-10.

(a) $\lambda = 0.5$, $s = 1$

Recall: $\rho = \frac{\lambda}{s\mu}$, $\text{P}_0 = 1 - \rho$, $\text{P}_n = (1 - \rho)\rho^n$

$$\text{L} = \frac{\lambda}{\mu - \lambda},\ \text{L}_\text{q} = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$\text{P}(\mathcal{W} > t) = e^{-\mu(1-\rho)t},\ \text{P}(\mathcal{W}_\text{q} > t) = \rho e^{-\mu(1-\rho)t}$$

$$\text{W} = \frac{1}{\mu - \lambda},\ \text{W}_\text{q} = \frac{\lambda}{\mu(\mu - \lambda)}$$

$\underline{\mu = 2}$: $\rho = 0.25$, $\text{P}_0 = 0.75$, $\text{P}_n = 0.75 \cdot 0.25^n$
L $= 1/3$, $\text{L}_\text{q} = 0.083$
$\text{P}(\mathcal{W} > s) = 0.000553$, $\text{P}(\mathcal{W}_\text{q} > s) = 0.000138$
W $= 0.67$, $\text{W}_\text{q} = 0.17$

$\underline{\mu = 1}$: $\rho = 0.5$, $P_0 = 0.5$, $P_n = 0.5^{n+1}$
$L = 1$, $L_q = 0.5$
$P(\mathcal{W} > s) = 0.082$, $P(\mathcal{W}_q > s) = 0.041$
$W = 2$, $W_q = 1$

$\underline{\mu = 2/3}$: $\rho = 0.75$, $P_0 = 0.25$, $P_n = 0.25 \cdot 0.75^n$
$L = 3$, $L_q = 2.25$
$P(\mathcal{W} > s) = 0.435$, $P(\mathcal{W}_q > s) = 0.326$
$W = 6$, $W_q = 4.5$

(b) TC(mean $= 0.5) = 1.60 + 0.8(1/3) = 1.87$
TC(mean $= 1) = 0.40 + 0.8(1) = 1.20$
TC(mean $= 1.5) = 0.20 + 0.8(3) = 2.60$

Hence, $\mu^* = 1$.

26.4-11.

Given that $s = 1$, from the optimality of a single server result,

$$\text{E(TC)} = C_r\mu + C_w\text{L} = C_r\mu + C_w\left(\frac{\lambda}{\mu-\lambda}\right)$$

$$\frac{d\text{E(TC)}}{d\mu} = C_r - C_w\left(\frac{\lambda}{(\mu-\lambda)^2}\right) = 0 \Rightarrow \mu = \lambda + \sqrt{\lambda C_w/C_r}$$

$$\frac{d^2\text{E(TC)}}{d\mu^2} = 2C_w\left(\frac{\lambda}{(\mu-\lambda)^3}\right) > 0 \text{ for all } \mu > \lambda.$$

Assuming $C_w > 0$ and $C_w \neq 0$, E(TC) is strictly convex in μ and $\mu = \lambda + \sqrt{\lambda C_w/C_r}$ is the unique minimizer.

26.4-12.

$$\text{E(TC)} = D\mu + \frac{\lambda C}{(\mu-\lambda)^2}$$

$$\frac{d\text{E(TC)}}{d\mu} = D - \frac{2\lambda C}{(\mu-\lambda)^3} = 0 \Rightarrow \mu = \lambda + \sqrt[3]{2\lambda C/D}$$

$$\frac{d^2\text{E(TC)}}{d\mu^2} = \frac{6\lambda C}{(\mu-\lambda)^4} > 0 \text{ for all } C > 0,$$

so E(TC) is strictly convex in μ and $\mu = \lambda + \sqrt[3]{2\lambda C/D}$ is the unique minimizer.

26.4-13.

(a) The original design would give a smaller expected number of customers in the system because of the pooling effect of multiple servers.

(b) The original design is an M/M/2 queue where $\lambda = 5$ and $\mu = 6$. Running ProMod, we find $L = 1.1$ from Figure 17.7. The alternative design consists of two M/M/1 queues with $L = 2\lambda/(\mu - \lambda) = 10$. This result agrees with the claim in (a).

26.4-14.

(a) Part (a) of Problem 17.6-31 is a special case of Model 3, in which $s = 1$ is fixed and the goal is to determine the mean arrival rate λ, or equivalently the number of machines assigned to one operator.

(b) (i) The resulting system is an M/M/s queue with finite calling population, whose size equals the total number of machines. The associated decision problem fits Model 1, with s being unknown.

(ii) The resulting system is a collection of independent M/M/1 queues with finite calling populations. The appropriate decision model is a combination of Model 2 and Model 3, since the goal is to determine μ, depending on the number of operators assigned, and λ, depending on the number of machines assigned. In this case, $s = 1$ is fixed.

(iii) This system does not fit any of the models described in section 26.4.

Each of the proposed alternatives allows resource (operator) sharing to some extent in contrast to the original proposal. Since in the original proposal, the operators would be idle most of the time, it is reasonable to expect that allowing interaction will result in an increase of the production rate obtained with the same number of operators. As a consequence of this, the number of operators needed to achieve a given production rate will decrease. Then, the question is what could prevent this from happening. In alternatives (i) and (iii), the travel time, which is not considered in the preceding argument, may pose a problem. The idle time could turn into travel time rather than service time. Moreover, in alternative (iii), the service rate of a group of n workers can be smaller than n times the individual service rate, since they will not be working together regularly. This is not the case in alternative (ii), where the members of a crew do work together regularly; even then, the service rate of a crew of n operators may be strictly less than n times the individual rate.

26.4-15.

From Table 17.3:

	$s=1$	$s=2$
$W_1 - \frac{1}{\mu}$	0.024	0.00037
$W_2 - \frac{1}{\mu}$	0.154	0.00793
$W_3 - \frac{1}{\mu}$	1.033	0.06542

Note that $\lambda_1 = 0.2$, $\lambda_2 = 0.6$ and $\lambda_3 = 1.2$.

s	E(WC)				E(SC)	E(TC)
	critical	serious	stable	total		
1	480.00	92.40	12.40	584.80	40.00	624.80
2	7.40	4.76	0.79	12.95	80.00	92.95

Hiring two doctors incurs less cost.

26.5-1.

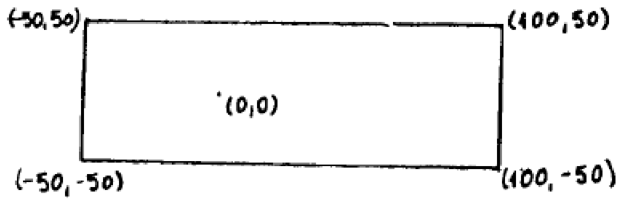
$a = b = c = d = 300$ and $v = 3$ miles/hour $= 264$ feet/min

$$\mathrm{E(T)} = \frac{1}{264}\left[\frac{(300)^2+(300)^2}{(300+300)} + \frac{(300)^2+(300)^2}{(300+300)}\right] = 2.27 \text{ minutes}$$

26.5-2.

$\mu = 30$, $s = 1$, $\lambda_p = 24$, $C_f = 20$, $C_s = 15$, $C_t = 25$

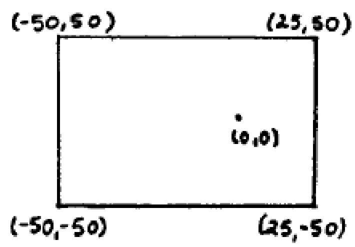
<u>$n = 1$:</u> $\lambda = \lambda_p/n = 24$, $a = b = d = 50$ and $c = 100$



$$2\text{E(T)} = \frac{1}{5{,}000}\left(\frac{50^2+100^2}{50+100} + \frac{50^2+50^2}{50+50}\right) = 0.0267 \text{ hours}$$

$$\text{L} = \frac{\lambda}{\mu-\lambda} = 4$$

<u>$n = 2$:</u> $\lambda = \lambda_p/n = 12$, $a = b = d = 50$ and $c = 25$ by relabeling symmetric areas:



$$\text{E(T)} = \frac{1}{5{,}000}\left(\frac{50^2+25^2}{50+25} + \frac{50^2+50^2}{50+50}\right) = 0.0183 \text{ hours}$$

$$\text{L} = \frac{\lambda}{\mu-\lambda} = \frac{2}{3}$$

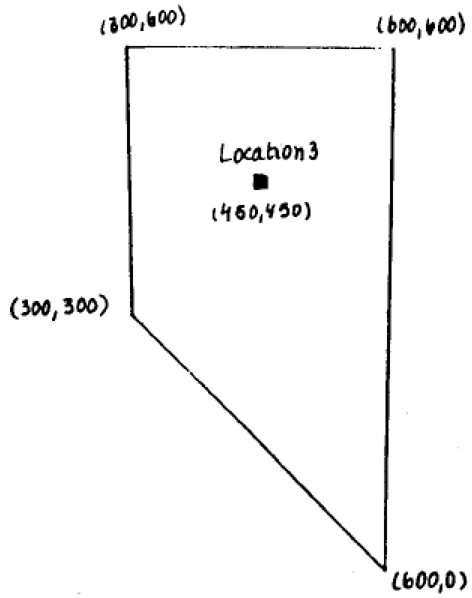
$\text{E(TC)} = n[(C_f + C_s) + C_t\text{L} + \lambda C_t\text{E(T)}]$

n	λ	E(T)	L	$C_f + C_s$	C_tL	λC_tE(T)	E(TC)
1	24	0.0267	4	35	100	16	151
2	12	0.0183	2/3	35	50/3	5.5	114.33

So, there should be two facilities.

26.5-3.

The first step is to relabel Location 3 as the origin $(0, 0)$ for an (x, y) coordinate system by subtracting 450 from all coordinates shown in the following figure.



The probability density function of X is obtained by using the height of the area assigned to the tool crib at Location 3 for each possible value of $X = x$ and then dividing by the size of the area, as given in figure 1-(a) below. This then yields the uniform distribution of $|X|$ shown in 1-(b).

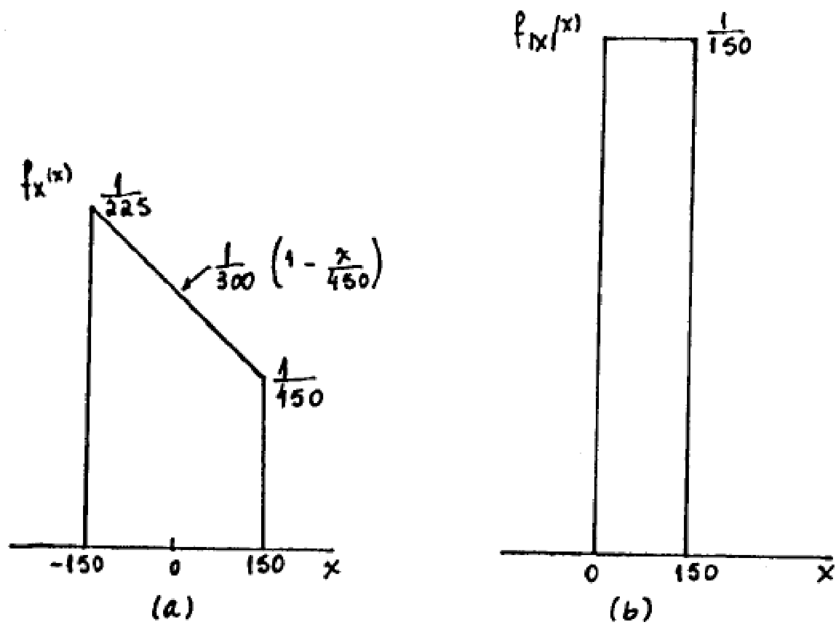


Figure 1 - Probability density functions of (a) X and (b) $|X|$

Thus, $\text{E}(|X|) = \frac{1}{150}\int_0^{150} x dx = 75.$

The probability density function of Y is obtained by using the width of the area assigned to tool crib at Location 3 for each possible value of $Y = y$ and then dividing by the size of the area, as given in figure 2-(a). This then leads to the probability density function of $|Y|$ shown in 2-(b).

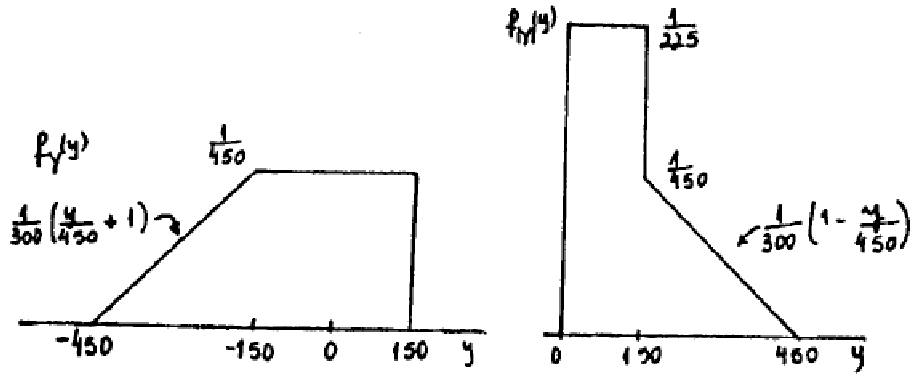


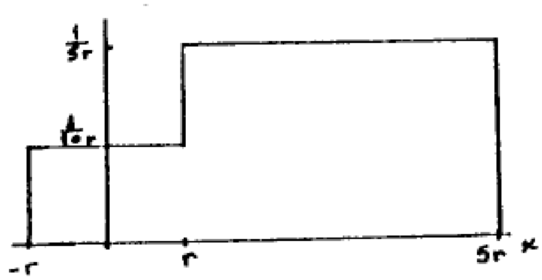
Figure 2 - Probability density functions of (a) Y and (b) $|Y|$

Thus, $\mathrm{E}(|Y|) = \frac{1}{225}\int_0^{150} y dy + \frac{1}{300}\int_{150}^{450}\left(1 - \frac{y}{450}\right) y dy = 133\frac{1}{3}$.

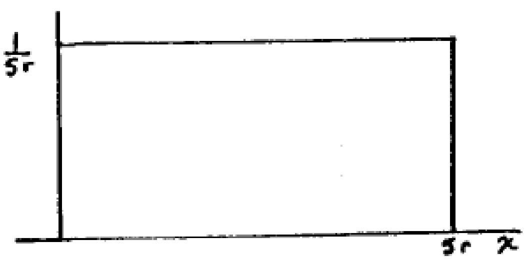
$\mathrm{E(T)} = \frac{2}{v}[\mathrm{E}(|X|) + \mathrm{E}(|Y|)] = \frac{2}{15{,}000}\left(75 + 133\frac{1}{3}\right) = 0.0278$ hr

26.5-4.

(a) Total area $= (2r)^2 + (4r)^2 = 20r^2$

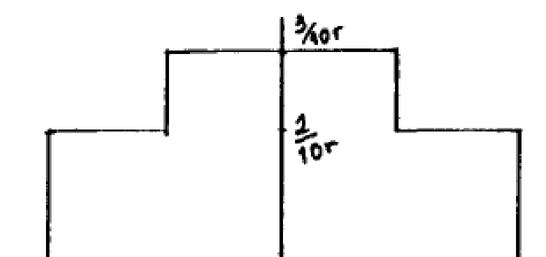


Probability density of X

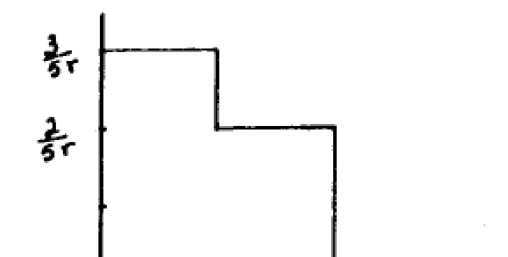


Probability density of $|X|$

$\mathrm{E}(|X|) = \int_0^{5r} \frac{1}{5r} x dx = 2.5r$



Probability density of Y

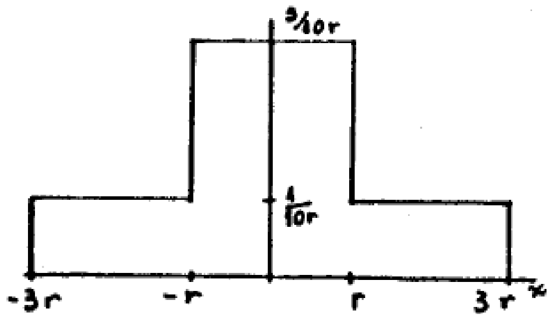


Probability density of $|Y|$

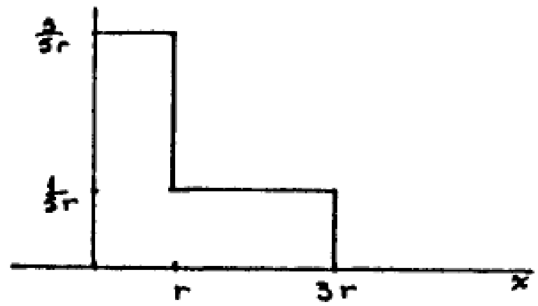
$\mathrm{E}(|Y|) = \int_0^{r} \frac{3}{5r} y dy + \int_r^{2r} \frac{2}{5r} y dy = 0.9r$

$\mathrm{E(T)} = \frac{2}{v}(2.5 + 0.9)r = \frac{6.8r}{v}$

(b) The area is symmetric about $(0,0)$, so $\mathrm{E}(|X|)=\mathrm{E}(|Y|)$ and the total area is $5(2r)^2=20r^2$.



Probability density of X

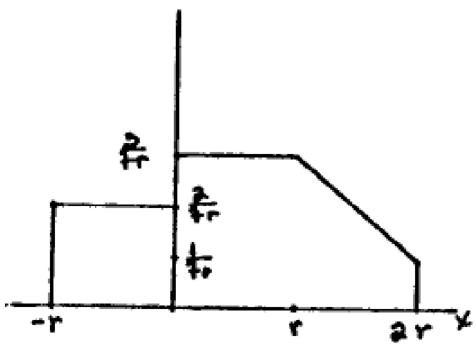


Probability density of $|X|$

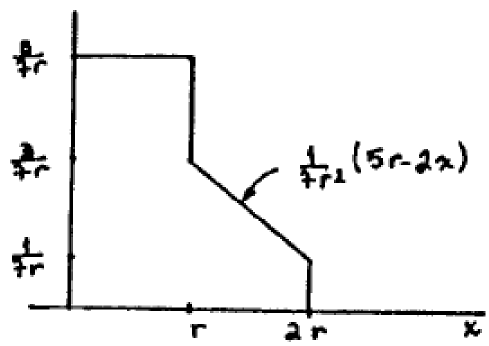
$\mathrm{E}(|X|)=\int_0^r \frac{3}{5r}x dx+\int_r^{3r}\frac{1}{5r}x dx=1.1r$

$\mathrm{E(T)}=\frac{2}{v}(1.1+1.1)r=\frac{4.4r}{v}$

(c) Total area $=2(2r^2+r^2+0.5r^2)=7r^2$

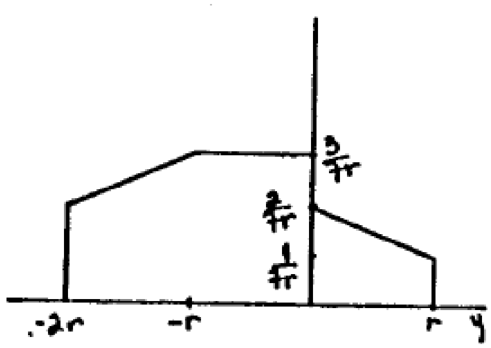


Probability density of X

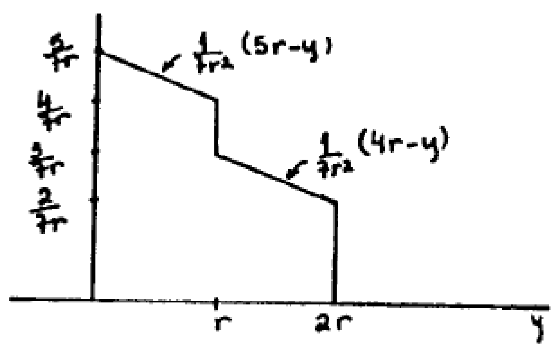


Probability density of $|X|$

$\mathrm{E}(|X|)=\int_0^r \frac{5}{7r}x dx+\int_r^{2r}\frac{1}{7r^2}(5r-2x)x dx=\frac{16}{21}r$



Probability density of Y

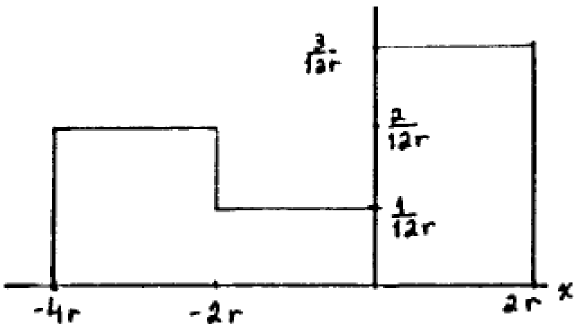


Probability density of $|Y|$

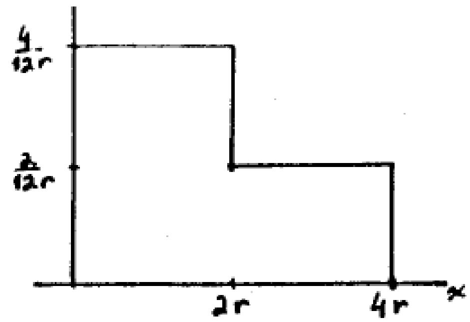
$\mathrm{E}(|Y|)=\frac{1}{7r^2}\bigg(\int_0^r(5r-y)y dy+\int_r^{2r}(4r-y)y dy=\frac{5}{6}r$

$\mathrm{E(T)}=\frac{2}{v}\left(\frac{16}{21}+\frac{5}{6}\right)r=\frac{3.19r}{v}$

(d) Total area $= 6(4r^2) = 24r^2$

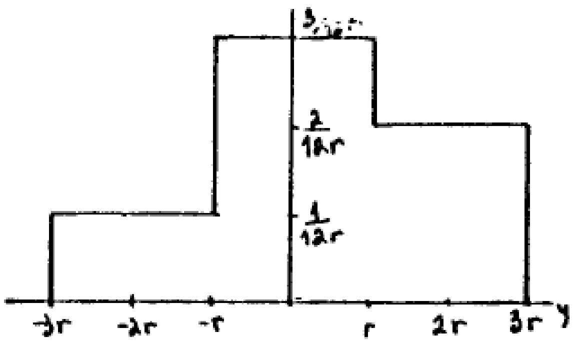


Probability density of X

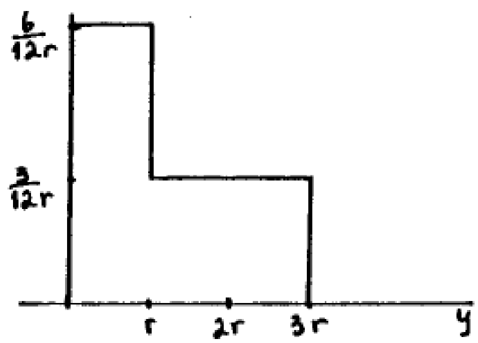


Probability density of $|X|$

$$\mathrm{E}(|X|) = \int_0^{2r} \frac{4}{12r} x dx + \int_{2r}^{4r} \frac{2}{12r} x dx = \frac{5}{3} r$$



Probability density of Y



Probability density of $|Y|$

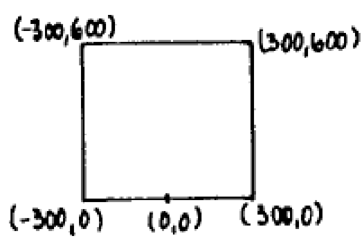
$$\mathrm{E}(|Y|) = \int_0^{r} \frac{6}{12r} y dy + \int_{r}^{3r} \frac{3}{12r} y dy = \frac{5}{4} r$$

$$\mathrm{E(T)} = \frac{2}{v}\left(\frac{5}{3} + \frac{5}{4}\right) r = \frac{5.83r}{v}$$

26.5-5.

Given $C_f = 10$, $C_m = 15$, $C_t = 40$, $\lambda_p = 90$, $v = 20,000$ feet/hour, the expected loading time is $1/20$ hours. For unloading, $\mu_m = 30m$ where m is the crew size.

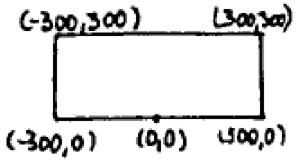
<u>$n = 1$:</u> $a = c = 300$, $b = 0$, $d = 600$



$$\mathrm{E(T)} = \frac{1}{20{,}000}\left[\frac{(300)^2+(300)^2}{(300+300)} + \frac{(600)^2}{600}\right] = 0.045 \text{ hours}$$

$$\mathrm{L} = \frac{\lambda}{\mu_m - \lambda} = \frac{3}{m-3}$$

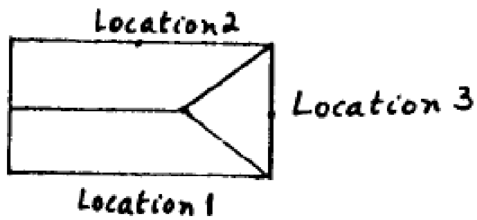
$\underline{n=2}$: $a=c=300$, $b=0$, $d=300$



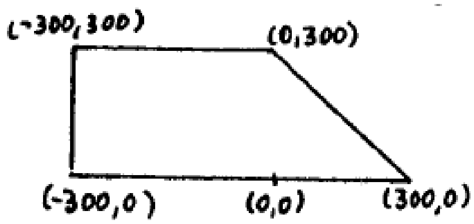
$$\text{E(T)} = \frac{1}{20{,}000}\left[\frac{(300)^2+(300)^2}{(300+300)} + \frac{(300)^2}{300}\right] = 0.030 \text{ hours}$$

$$\text{L} = \frac{\lambda}{\mu_m-\lambda} = \frac{3}{2m-3} \text{ since } \lambda = \frac{\lambda_p}{n} = 45$$

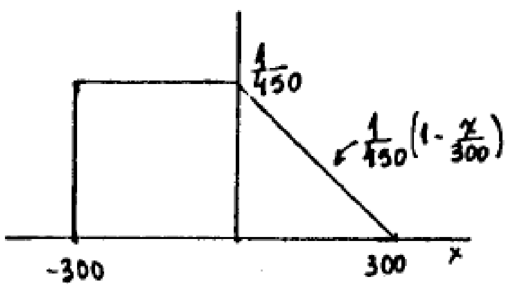
$\underline{n=3}$: The facilities would be located as follows:



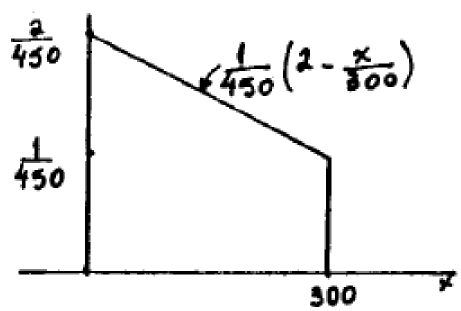
Consider Locations 1 and 2, which are symmetric. Each can be labeled as:



with a total area of $135,000$.

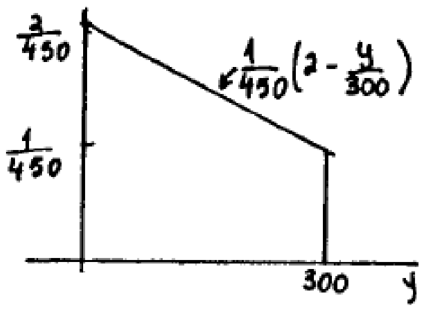


Probability density of X



Probability density of $|X|$

$$\text{E}(|X|) = \int_0^{300} \frac{1}{450}\left(2-\frac{x}{300}\right)x\,dx = \frac{400}{3}$$



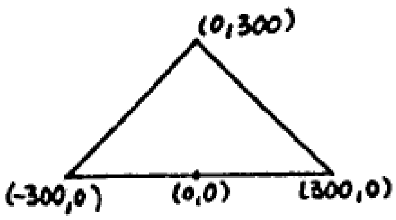
Probability density of $Y = |Y|$

$\mathrm{E}(|Y|) = \mathrm{E}(|X|) = \frac{400}{3}$

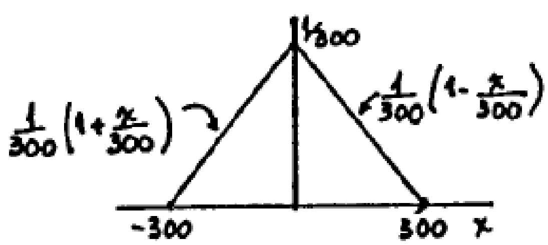
$\mathrm{E(T)} = \frac{2}{20{,}000}\left(\frac{400}{3} + \frac{400}{3}\right) = \frac{4}{150} = 0.0267$

$\mathrm{L} = \frac{135/4}{30m - 135/4} = \frac{9}{8m-9}$ since $\lambda = \frac{135}{4}$

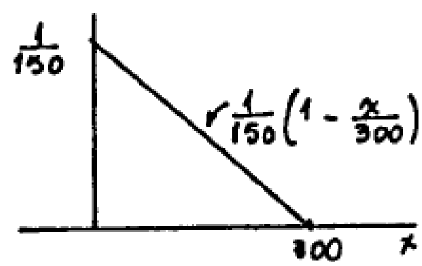
Now consider Location 3. The area would be labeled as follows:



with a total area of $90,000$.

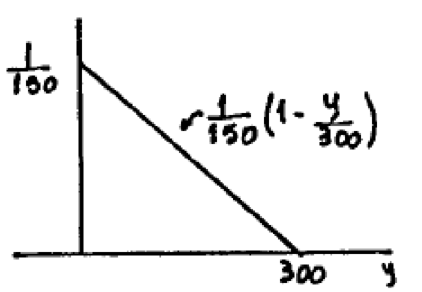


Probability density of X



Probability density of $|X|$

$\mathrm{E}(|X|) = \int_0^{300} \frac{1}{150}\left(1 - \frac{x}{300}\right) x dx = 100$



Probability density of $Y = |Y|$

$\mathrm{E}(|Y|) = \mathrm{E}(|X|) = 100$

$\mathrm{E(T)} = \frac{2}{20{,}000}(100 + 100) = 0.020$ hours

$\text{L} = \frac{45/2}{30m - 45/2} = \frac{3}{4m-3}$

<u>$n = 4$:</u> The areas served by the four facilities would be identical to that of Location 3 for $n = 3$, so E(T) $= 4/200 = 0.020$ hours and L $= 3/(4m - 3)$.

n	E(T) in hours	L
1	0.045	$\frac{3}{m-3}$
2	0.030	$\frac{3}{2m-3}$
3 L1,L2	0.0267	$\frac{9}{8m-9}$
L3	0.020	$\frac{3}{4m-3}$
4	0.020	$\frac{3}{4m-3}$

where L1, L2, and L3 represent Locations 1, 2 and 3 respectively.

If $n = 1$, E(TC) $= (C_f + mC_m) + C_t\text{L} + \lambda C_t\text{E(T)} + \lambda C_t/20$ where $\lambda = 90$.

m	L	E(T)	$C_f + mC_m$	C_tL	λC_tE(T)	$\lambda C_t/20$	E(TC)
4	3	0.045	70	120	162	180	532.00
5	1.5	0.045	85	60	162	180	487.00
6	1	0.045	100	40	162	180	482.00
7	0.75	0.045	115	30	162	180	487.00

For $n = 1$, the minimum cost per hour is \$482 with $m = 6$.

If $n = 2$, E(TC) $= 2[(C_f + mC_m) + C_t\text{L} + \lambda C_t\text{E(T)} + \lambda C_t/20]$ where $\lambda = 45$.

m	L	E(T)	$C_f + mC_m$	C_tL	λC_tE(T)	$\lambda C_t/20$	E(TC)
2	3	0.030	40	120	54	90	608.00
3	1	0.030	55	40	54	90	478.00
4	0.6	0.030	70	24	54	90	476.00
5	$3/7$	0.030	85	17.14	54	90	492.29

For $n = 2$, the minimum cost per hour is \$476 with $m = 4$.

If $n = 3$, at Locations 1 and 2 where $\lambda = 135/4$:

m	L	E(T)	$C_f + mC_m$	C_tL	λC_tE(T)	$\lambda C_t/20$	E(TC)
2	$9/7$	0.0267	40	51.43	36	67.5	194.93
3	$3/5$	0.0267	55	24	36	67.5	182.50
4	$9/23$	0.0267	70	15.65	36	67.5	189.15

At Location 3 where $\lambda = 22.5$:

m	L	E(T)	$C_f + mC_m$	C_tL	λC_tE(T)	$\lambda C_t/20$	E(TC)
1	3	0.020	25	120	18	45	208.00
2	$3/5$	0.020	40	24	18	45	127.00
3	$1/3$	0.020	55	13.33	18	45	131.33

So, for $n = 3$, the minimum cost per hour is $2(182.50) + 127 = 492$ with $m = 3$ at Locations 1 and 2, and $m = 2$ at Location 3.

If $n = 4$, since all areas served are symmetric and each one is same as Location 3 of the case with $n = 3$, the minimum cost per hour is $4(127) = 508$ with $m = 2$.

The following table summarizes these results.

n	m	E(TC)
1	6	482
2	4 at both locations	476
3	3 at Locations 1 and 2	492
	2 at Location 3	
4	2 at all locations	508

Therefore, the best is to have two facilities with a crew size of 4.

CHAPTER 27: FORECASTING

27.1-1.

Answers will vary.

27.1-2.

Answers will vary.

27.4-1.

(a)

$$F_6 = x_5 = 39$$

(b)

$$F_6 = \frac{\sum_{t=1}^{5} x_t}{5} = \frac{5+17+29+41+39}{5} = 26$$

(c)

$$F_6 = \frac{\sum_{t=3}^{5} x_t}{3} = \frac{29+41+39}{3} = 36$$

(d) The demand seems to be rising, so the average forecasting method may be inappropriate, since it uses older, out of data.

27.4-2.

(a)

$$F_6 = x_5 = 13$$

(b)

$$F_6 = \frac{\sum_{t=1}^{5} x_t}{5} = \frac{15+18+12+17+13}{5} = 15$$

(c)

$$F_6 = \frac{\sum_{t=3}^{5} x_t}{3} = \frac{12+17+13}{3} = 14$$

(d) The averaging method seems to be the best, since all five months of data are relevant in determining the forecast of sales for the next month.

27.4-3.

$$F_{t+1} = \frac{1977-1945}{4} + 2083 = 2091$$

27.4-4

$$F_{t+1} = \frac{793-805}{3} + 782 = 778$$

27.4-5.

$$F_{t+1} = \frac{1532-1632}{10} + 1551 = 1541$$

27.4-6.

$$F_{t+1} = \alpha x_t + (1-\alpha)F_t$$

$$F_{t+1}(0.1) = (0.1)(792) + (1-0.1)(782) = 783$$

$$F_{t+1}(0.3) = (0.3)(792) + (1-0.3)(782) = 785$$

$$F_{t+1}(0.5) = (0.5)(792) + (1-0.5)(782) = 787$$

27.4-7.

$$F_{t+1} = \alpha x_t + (1-\alpha)F_t$$
$$F_{t+1}(0.1) = (0.1)(1973) + (1-0.1)(2083) = 2072$$
$$F_{t+1}(0.3) = (0.3)(1973) + (1-0.3)(2083) = 2050$$
$$F_{t+1}(0.5) = (0.5)(1973) + (1-0.5)(2083) = 2028$$

27.4-8.

$$\alpha = 0 \Rightarrow F_{t+1} = F_t = \ldots = F_1$$

The forecast remains equal to the best initial guess for the variable and never changes.

$$\alpha = 1 \Rightarrow F_{t+1} = x_t$$

The forecast always equals the current value of the variable.

27.4-9.

(a) $F_{t+1} = \alpha x_t + (1-\alpha)F_t \Rightarrow x_t = \frac{1}{\alpha}[F_{t+1} - (1-\alpha)F_t] = 2F_{t+1} - F_t$

$\Rightarrow$ Actual demand in April: $2(390) - 380 = 400$

Actual demand in May: $2(380) - 390 = 370$

(b) $F_{\text{Feb}} = 0.5x_{\text{Jan}} + 0.5F_{\text{Jan}}$

$F_{\text{March}} = 0.5x_{\text{Feb}} + 0.5F_{\text{Feb}} = 0.5x_{\text{Feb}} + 0.25x_{\text{Jan}} + 0.25F_{\text{Jan}}$

$x'_{\text{Jan}} = x_{\text{Jan}} + 32$, $x'_{\text{Feb}} = x_{\text{Feb}}$, $F'_{\text{Jan}} = F_{\text{Jan}}$

$\Rightarrow F'_{\text{March}} = F_{\text{March}} + (0.25)(32) = 408$

	Jan	Feb	March	April	May	June
Forecast			408	384	392	381
Actual	400		360	400	370	

27.5-1.

(a)

Quarter	Call Volume	Seasonal Factor
1	6809	$\frac{6809}{7027} = 0.97$
2	6465	$\frac{6465}{7027} = 0.92$
3	6569	$\frac{6569}{7027} = 0.93$
4	8266	$\frac{8266}{7027} = 1.18$

(b)

Quarter	Seasonal Factor	Actual Call Volume	Seasonally Adjusted Call Volume
1	0.97	7257	$\frac{7257}{0.97} = 7481$
2	0.92	7064	$\frac{7064}{0.92} = 7678$
3	0.93	7784	$\frac{7784}{0.93} = 8370$
4	1.18	8724	$\frac{8724}{1.18} = 7393$

(c)

Quarter	Two-year Average	Seasonal Factor
1	7033	$\frac{7033}{7367} = 0.95$
2	6765	$\frac{6765}{7367} = 0.92$
3	7177	$\frac{7177}{7367} = 0.97$
4	8495	$\frac{8495}{7367} = 1.15$

(d)

Quarter	Seasonal Factor	Actual Call Volume	Seasonally Adjusted Call Volume
1	0.95	6992	$\frac{6992}{0.95} = 7360$
2	0.92	6822	$\frac{6822}{0.92} = 7415$
3	0.97	7949	$\frac{7949}{0.97} = 8195$
4	1.15	9650	$\frac{9650}{1.15} = 8391$

27.5-2.

(a)

Quarter	Unemployment Rate	Seasonal Factor
1	0.062	$\frac{0.062}{0.063} = 0.98$
2	0.060	$\frac{0.060}{0.063} = 0.95$
3	0.075	$\frac{0.075}{0.063} = 1.19$
4	0.055	$\frac{0.055}{0.063} = 0.87$

(b)

Quarter	Seasonal Factor	Act. Unemploy. Rate	Seasonally Adj. Unemploy. Rate
1	0.98	0.078	$\frac{0.078}{0.98} = 0.080$
2	0.95	0.074	$\frac{0.074}{0.95} = 0.078$
3	1.19	0.087	$\frac{0.087}{1.19} = 0.073$
4	0.87	0.061	$\frac{0.061}{0.87} = 0.070$

This progression indicates that the state's economy is improving with the unemployment rate decreasing from 8% to 7% (seasonally adjusted) over the four quarters.

27.5-3.

(a)

Quarter	Three-year Average	Seasonal Factor
1	21	$\frac{21}{25} = 0.84$
2	23	$\frac{23}{25} = 0.92$
3	30	$\frac{30}{25} = 1.2$
4	26	$\frac{26}{25} = 1.04$

(b) Seasonally adjusted value: $\frac{28}{1.04} = 27 \Rightarrow$ forecast: $(27)(0.84) = 23$

(c) Quarter 1: seasonally adjusted value: $23/0.84 = 27 \Rightarrow$ forecast:$(27)(0.84) = 23$
Quarter 2: seasonally adjusted value: $25/0.92 = 27 \Rightarrow$ forecast:$(27)(1.20) = 33$
Quarter 3: seasonally adjusted value: $33/1.20 = 27 \Rightarrow$ forecast:$(27)(1.04) = 28$

(d)

Quarter	Seasonal Factor	Avg. House Sales	Seasonally Adjusted Forecast
1	0.84	25	$(25)(0.84) = 21$
2	0.92	25	$(25)(0.92) = 23$
3	1.20	25	$(25)(1.20) = 30$
4	1.04	25	$(25)(1.04) = 26$

27.5-4.

(a) - (b) - (c) - (d) $\alpha = 0.1, \gamma = 0.2$

Year	Quarter	Sales	I	F	S
2000	1	6900	0.965		
	2	6700	0.937		
	3	7900	1.105		
	4	7100	0.993		7150
2001	1	8200	0.997	6900	7285
	2	7000	0.941	6826	7303
	3	7300	1.086	8069	7234
	4	7500	1.001	7183	7266
2002	1	9400	1.049	7245	7482
	2	9200	0.992	7043	7711
	3	9800	1.119	8372	7842
	4	9900	1.047	7849	8047
2003	1	11400	1.113	8442	8329
	2	10000	1.029	8260	8505
	3	9400	1.116	9513	8495
	4	8400	1.036	8892	8448
2004	1	8800		9402	8394
	2	7600		8633	8293
	3	7500		9256	8136
	4			8431	

(e) There is a seasonal effect: $1 \underset{\text{down}}{\rightarrow} 2$, and it is incorporated by the parameter I.

(f) There is a substantial error in these estimates, the constant level assumption is not good enough with $\alpha = 0.1$ and $\gamma = 0.2$.

27.6-1.

$$F_{t+1} = \alpha x_t + (1-\alpha)F_t + \beta[\alpha(x_t - x_{t-1}) + (1-\alpha)(F_t - F_{t-1})] + (1-\beta)T_{t+1}$$

$$F_1 = x_0 + T_1 = 3900 + 700 = 4600$$

$$F_2 = (0.25)(4600) + (0.75)(4600) + (0.25)[(0.25)(700) + (0.75)(700)] + (0.75)(700)$$
$$= 5300$$

$$F_3 = (0.25)(5300) + (0.75)(5300) + (0.25)[(0.25)(700) + (0.75)(700)] + (0.75)(700)$$
$$= 6000$$

27.6-2.

$$F_{t+1} = \alpha x_t + (1-\alpha)F_t + \beta[\alpha(x_t - x_{t-1}) + (1-\alpha)(F_t - F_{t-1})] + (1-\beta)T_{t+1}$$

$$F_{t+1} = (0.2)(550) + (0.8)(540) + (0.3)[(0.2)(15) + (0.8)(10)] + (0.7)(10) = 552$$

27.6-3.

$$F_{t+1} = \alpha x_t + (1-\alpha)F_t + \beta[\alpha(x_t - x_{t-1}) + (1-\alpha)(F_t - F_{t-1})] + (1-\beta)T_{t+1}$$

$$F_{t+1} = (0.1)(4395) + (0.9)(4975) + (0.2)[(0.1)(280) + (0.9)(255)] + (0.8)(240)$$

$$= 5215$$

27.6-4.

Time Period	True Value	Latest Trend	Estimated Trend	Exponential Smoothing Forecast	Forecasting Error			
							Smoothing Constants	
1	15		5.00	15	0		α =	0.2
2	21	5.00	5.00	20	1		β =	0.2
3	24	5.20	5.04	25	1			
4	32	4.79	4.99	30	2		Initial Estimates	
5	37	5.39	5.07	35	2		Average =	10
6	41	5.38	5.13	41	0		Trend =	5
7	40	5.15	5.14	46	6			
8	47	3.93	4.89	50	3		Mean Absolute Deviation	
9	51	4.35	4.79	54	3		MAD =	2.3
10	53	4.19	4.67	58	5			
11		3.66	4.46	62			Mean Square Error	
							MSE =	8.8

Forecast for next production yield: 62%

27.7-1.

(a) Best $\alpha = 0.25$, forecast 50

Time Period	True Value	
1	51	
2	48	
3	52	
4	49	
5	53	
6	49	
7	48	Best α
8	51	0.25
9	50	
10	49	
11	50	

(b) Best $\alpha = 0.114$, forecast 51

Time Period	True Value	
1	52	
2	50	
3	53	
4	51	
5	52	
6	48	
7	52	Best α
8	53	0.114
9	49	
10	52	
11	51	

(c) Best $\alpha = 0.268$, forecast 54

Time Period	True Value	
1	50	
2	52	
3	51	
4	55	
5	53	
6	56	
7	52	Best α
8	55	0.268
9	54	
10	53	
11	54	

27.7-2.

(a) Best $\alpha = 0.637$, best $\beta = 0.488$, forecast 77

Time Period	True Value	
1	52	
2	55	
3	55	
4	58	
5	59	
6	63	
7	64	Best α
8	66	0.637
9	67	
10	72	Best β
11	73	0.488
12	74	
13	77	

(b) Best $\alpha = 0.84$, best $\beta = 0.582$, forecast 74

Time Period	True Value	
1	52	
2	55	
3	59	
4	61	
5	66	
6	69	
7	71	Best α
8	72	0.84
9	73	
10	74	Best β
11	73	0.582
12	74	
13	74	

(c) Best $\alpha = 0.904$, best $\beta = 0.999$, forecast 79

Time Period	True Value	
1	52	
2	53	
3	51	
4	50	
5	48	
6	47	
7	49	Best α
8	52	0.904
9	57	
10	62	Best β
11	69	0.999
12	74	
13	79	

27.7-3.

The best method is exponential smoothing with trend, using $\alpha = 0.317$ and $\beta = 0.999$.

Time Period	True Value	
1	382	
2	405	
3	398	Best Method
4	421	Exp. Smoothing
5	426	with Trend
6	415	
7	443	Best α
8	451	0.317
9	446	
10	464	Best β
11	473	0.999

27.8-1.

Quarter	Forecast	True Value	Error
1	327	345	18
2	332	317	15
3	328	336	8
4	330	311	19

$\text{MAD} = \frac{\text{sum of forecasting errors}}{\text{number of forecasts}} = \frac{18+15+8+19}{4} = 15$

$\text{MSE} = \frac{\text{sum of squares of forecasting errors}}{\text{number of forecasts}} = 243.5$

27.8-2.

(a) <u>Method 1:</u> MAD $= \frac{258+499+560+809+609}{5} = 547$

<u>Method 2:</u> MAD $= \frac{374+471+293+906+396}{5} = 488$

(b) <u>Method 1:</u> MSE $= 330,905$

<u>Method 2:</u> MSE $= 285,044$

(c) She can use the older data to calculate more forecasting errors and compare MSE and MAD for a longer time span. This may make her feel more comfortable with her decision.

27.8-3.

(a) $F_{t+1} = \alpha x_t + (1-\alpha)F_t$

$$F_1 = x_0 = 5000$$

$$F_2 = (0.25)(4600) + (1-0.25)(5000) = 4900$$

$$F_3 = (0.25)(5300) + (1-0.25)(4900) = 5000$$

(b) MAD $= \frac{400+400+1000}{3} = 600$

(c) MSE $= \frac{400^2+400^2+1000^2}{3} = 440,000$

(d) $F_{t+1} = \; = (0.25)(6000) + (1-0.25)(5000) = 5250$

27.8-4.

(a) Since sales are relatively stable, the averaging method would be appropriate for forecasting future sales. This method uses a larger sample size than the last-value method, which should make it more accurate and since the older data is still relevant, it should not be excluded, as would be the case in the moving-average method.

(b) Last-Value Method

Time Period	True Value	Last-Value Forecast	Forecasting Error
1	23		
2	24	23	1
3	22	24	2
4	28	22	6
5	22	28	6
6	27	22	5
7	20	27	7
8	26	20	6
9	21	26	5
10	29	21	8
11	23	29	6
12	28	23	5
13		28	

Mean Absolute Deviation	
MAD =	5.2

Mean Square Error	
MSE =	30.6

(c) Averaging Method

Time Period	True Value	Averaging Forecast	Forecasting Error
1	23		
2	24	23	1
3	22	24	2
4	28	23	5
5	22	24	2
6	27	24	3
7	20	24	4
8	26	24	2
9	21	24	3
10	29	24	5
11	23	24	1
12	28	24	4
13		24	

Mean Absolute Deviation	
MAD =	3.0
Mean Square Error	
MSE =	11.1

(d) Moving-Average Method $(n = 3)$

Time Period	True Value	Moving Average Forecast	Forecasting Error
1	23		
2	24		
3	22		
4	28	23	5
5	22	25	3
6	27	24	3
7	20	26	6
8	26	23	3
9	21	24	3
10	29	22	7
11	23	25	2
12	28	24	4
13		27	

Number of previous periods to consider	
n=	3
Mean Absolute Deviation	
MAD =	3.9
Mean Square Error	
MSE =	17.4

(e) Considering the MAD values $(5.2, 3.0, 3.9)$, the averaging method is the best.

(f) Considering the MSE values $(30.6, 11.1, 17.4)$, the averaging method is the best.

(g) Unless there is a reason to believe that sales will not continue to be relatively stable, the averaging method should be the most accurate in the future as well.

27.8-5.

Ben Swanson should choose 0.1 for the smoothing constant.

Smoothing Constant	MAD	MSE
0.1	2.70	9.44
0.2	2.82	10.24
0.3	2.97	11.20
0.4	3.13	12.35
0.5	3.32	13.75

27.8-6.

(a) Answers will vary. The averaging or the moving-average methods seem to do a better job than the last-value method.

(b) For the last-value method, a change in April affects only the forecast of May. For the averaging method, it affects all forecasts after April and for the moving-average method, it affects the forecasts for May, June and July.

(c) Answers will vary. The averaging and the moving-average methods seem to do slightly better than the last-value method.

(d) Answers will vary. The averaging and the moving-average methods seem to do slightly better than the last-value method.

27.8-7.

(a) Since the sales level is shifting significantly from month to month and there is no consistent trend, the last-value method seems to be appropriate. The averaging method will not do as well because it places too much weight on the old data. The moving-average method will be better than the averaging method, but it will lag any short-term trends. The exponential smoothing method will also lag trends by placing too much weight on the old data. Exponential smoothing with trend will likely not to do well because the trend is not consistent.

(b) Last-Value Method

Time Period	True Value	Last-Value Forecast	Forecasting Error
1	126		
2	137	126	11
3	142	137	5
4	150	142	8
5	153	150	3
6	154	153	1
7	148	154	6
8	145	148	3
9	147	145	2
10	151	147	4
11	159	151	8
12	166	159	7
13		166	

Mean Absolute Deviation	
MAD =	5.3
Mean Square Error	
MSE =	36.2

Averaging Method

Time Period	True Value	Averaging Forecast	Forecasting Error
1	126		
2	137	126	11
3	142	132	11
4	150	135	15
5	153	139	14
6	154	142	12
7	148	144	4
8	145	144	1
9	147	144	3
10	151	145	6
11	159	145	14
12	166	147	19
13		148	

Mean Absolute Deviation	
MAD =	10.0
Mean Square Error	
MSE =	131.4

Moving-Average Method

Time Period	True Value	Moving Average Forecast	Forecasting Error
1	126		
2	137		
3	142		
4	150	135	15
5	153	143	10
6	154	148	6
7	148	152	4
8	145	152	7
9	147	149	2
10	151	147	4
11	159	148	11
12	166	152	14
13		159	

Number of previous periods to consider	
n=	3
Mean Absolute Deviation	
MAD =	8.1
Mean Square Error	
MSE =	84.3

Comparing MAD values $(5.3, 10.0, 8.1)$ and MSE values $(36.2, 131.4, 84.3)$, the last-value method is the best.

(c) Using the template for exponential smoothing with an initial estimate of 120, the following forecast errors are obtained for various values of the smoothing constant α.

α	MAD	MSE
0.1	18.5	382.7
0.2	13.0	210.2
0.3	10.1	139.7
0.4	8.7	104.2
0.5	8.0	82.9

Considering both MAD and MSE, a high value of the smoothing constant seems to be appropriate.

(d) Using the template for exponential smoothing with trend using an initial estimate of 120 for the average value and 10 for the trend, the following forecast errors are obtained for various values of the smoothing constants α and β.

α	β	MAD	MSE
0.1	0.1	25.4	919.6
0.1	0.3	21.2	634.1
0.1	0.5	17.7	450.6
0.3	0.1	13.5	261.9
0.3	0.3	9.8	144.1
0.3	0.5	8.8	111.5
0.5	0.1	8.4	116.1
0.5	0.3	7.0	72.2
0.5	0.5	6.5	61.1

Considering both MAD and MSE, high values of the smoothing constants seem to be appropriate.

(e) The management should use the last-value method to forecast sales. Using this method, the forecast for January of the new year is 166. Exponential smoothing with trend using high smoothing constants, e.g., $\alpha = \beta = 0.5$, also works well. With this method, the forecast for January of the new year is 165.

27.8-8.

(a) Answers will vary. The last-value method seems to be the best. Exponential smoothing with trend is a close second.

(b) For the last-value method, a change in April affects only the forecast for May. For the averaging method, exponential smoothing with or without trend, it affects all forecasts after April. For the moving-average method, it affects the forecasts for May, June, and July.

(c) Answers will vary. The last-value method and exponential smoothing seem to do better than the others.

(d) Answers will vary. The last-value method and exponential smoothing seem to do better than the others.

27.8-9.

(a)

α	MAD
0.1	1.51
0.2	1.62
0.3	1.73
0.4	1.84
0.5	1.95

Choose $\alpha = 0.1$.

(b)

α	MAD
0.1	1.84
0.2	1.88
0.3	1.92
0.4	2.00
0.5	2.10

Choose $\alpha = 0.1$.

(c)

α	MAD
0.1	2.82
0.2	2.54
0.3	2.26
0.4	2.06
0.5	1.90

Choose $\alpha = 0.5$.

27.8-10.

(a)

β	MAD
0.1	0.740
0.2	0.749
0.3	0.759
0.4	0.770
0.5	0.782

Choose $\beta = 0.1$.

(b)

β	MAD
0.1	2.61
0.2	2.76
0.3	2.87
0.4	2.99
0.5	3.05

Choose $\beta = 0.1$.

(c)

β	MAD
0.1	5.66
0.2	6.02
0.3	6.23
0.4	6.36
0.5	6.54

Choose $\beta = 0.1$.

27.8-11.

(a) The time series is not stable enough for the moving-average method.

(b)

Time Period	True Value	Moving Average Forecast	Forecasting Error
1	382		
2	405		
3	398		
4	421	395	26
5	426	408	18
6	415	415	0
7	443	421	22
8	451	428	23
9	446	436	10
10	464	447	17
11		454	

Number of previous periods to consider	
n=	3
Mean Absolute Deviation	
MAD =	16.6
Mean Square Error	
MSE =	346.0

(c)

Time Period	True Value	Exponential Smoothing Forecast	Forecasting Error
1	382	380	2
2	405	381	24
3	398	393	5
4	421	396	26
5	426	408	18
6	415	417	2
7	443	416	27
8	451	430	21
9	446	440	6
10	464	443	21
11		454	

Smoothing Constant	
α =	0.5
Initial Estimate	
Average =	380
Mean Absolute Deviation	
MAD =	15
Mean Square Error	
MSE =	323

(d)

Time Period	True Value	Latest Trend	Estimated Trend	Exponential Smoothing Forecast	Forecasting Error
1	382		10.00	380	2
2	405	10.50	10.13	391	14
3	398	13.72	11.02	405	7
4	421	9.21	10.57	414	7
5	426	12.32	11.01	427	1
6	415	10.82	10.96	438	23
7	443	5.33	9.55	441	2
8	451	9.94	9.65	451	0
9	446	9.53	9.62	461	15
10	464	5.87	8.68	466	2
11		8.20	8.56	474	

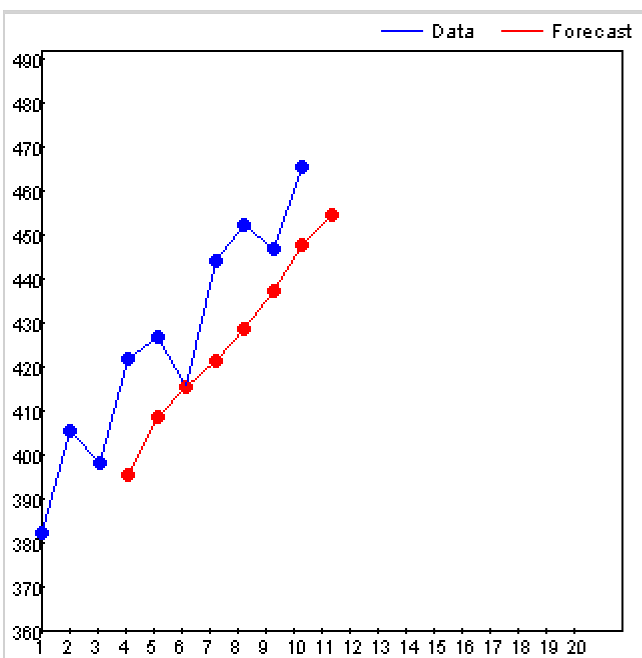
Smoothing Constants	
α =	0.25
β =	0.25
Initial Estimates	
Average =	370
Trend =	10
Mean Absolute Deviation	
MAD =	7.3
Mean Square Error	
MSE =	105.1

(e) Exponential smoothing with a trend is recommended, since it offers the smallest MAD.

27.8-12.

```
Forecasting -- Moving-Average Method
MAD = 16.62          MSE = 345.95
Fetch average of last 3 values
```

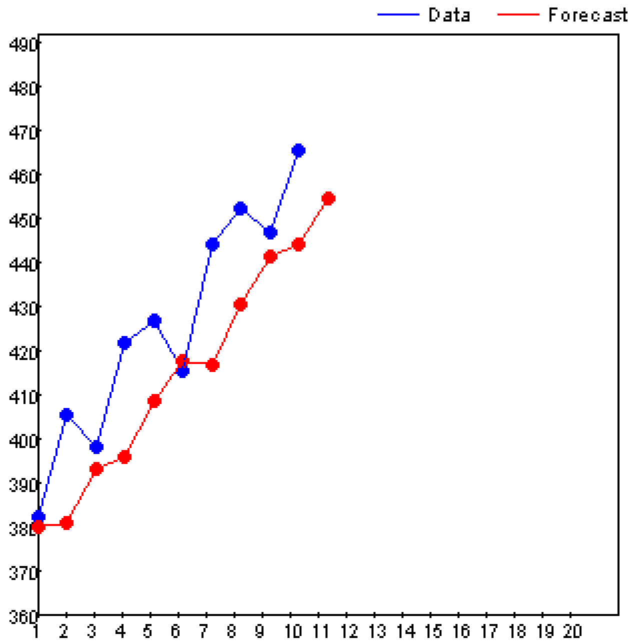
Period	Data	Forecast	Error
1	382		
2	405		
3	398		
4	421	395	26
5	426	408	18
6	415	415	0
7	443	420.67	22.33
8	451	428	23
9	446	436.33	9.67
10	464	446.67	17.33
11	0	453.67	



Moving-Average Method: The forecasts typically lie below the demands.

```
Forecasting -- Exponential Smoothing Method
MAD = 15.14         MSE = 322.97
alpha = 0.5
```

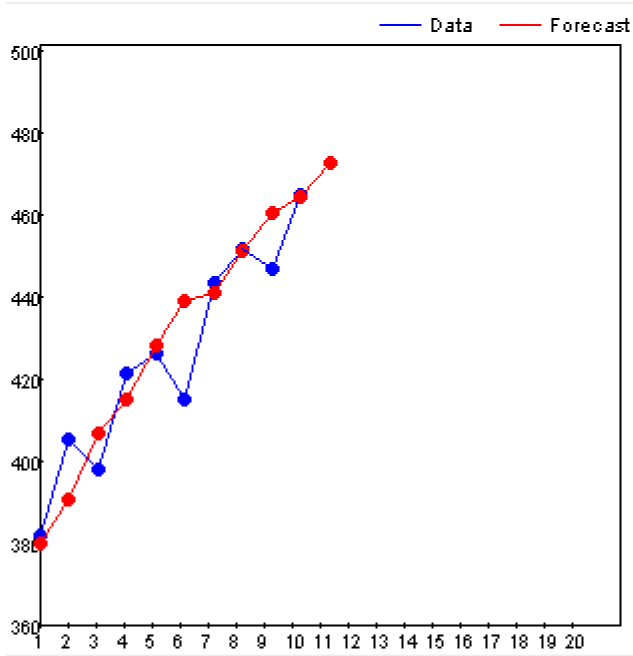
Period	Data	Forecast	Error
1	382	380	2
2	405	381	24
3	398	393	5
4	421	395.5	25.5
5	426	408.25	17.75
6	415	417.12	2.12
7	443	416.06	26.94
8	451	429.53	21.47
9	446	440.27	5.73
10	464	443.13	20.87
11	0	453.57	



Exponential Smoothing: The forecasts typically lie below the demands.

```
Forecasting -- Exponential Smoothing with Trend
MAD = 7.36          MSE = 106.82
alpha = 0.3         beta = 0.3
```

Period	Data	Forecast	Error
1	382	380	2
2	405	390.78	14.22
3	398	406.51	8.51
4	421	414.65	6.35
5	426	427.82	1.82
6	415	438.38	23.38
7	443	440.36	2.64
8	451	450.39	0.61
9	446	459.86	13.86
10	464	463.75	0.25
11	0	471.89	



Exponential Smoothing with Trend: The forecasts are at about the same level as demands (perhaps slightly above). This indicates that exponential smoothing with trend is the best method to use hereafter.

27.8-13.

(a)

Year	Quarter	True Value
1	1	25
1	2	47
1	3	68
1	4	42
2	1	27
2	2	46
2	3	72
2	4	39
3	1	24
3	2	49
3	3	70
3	4	44

Type of Seasonality	
Quarterly	
	Estimate for
Quarter	Seasonal Factor
1	0.5497
2	1.0271
3	1.5190
4	0.9042

(b) Forecast: 27 acre-feet

	Type of Seasonality
	Quarterly
Quarter	Seasonal Factor
1	0.550
2	1.027
3	1.519
4	0.904

Mean Absolute Deviation	
MAD =	2.4
Mean Square Error	
MSE =	8

			Seasonally	Seasonally		
		True	Adjusted	Adjusted	Actual	Forecasting
Year	Quarter	Value	Value	Forecast	Forecast	Error
1	1	25	45			
1	2	47	46	45	47	0
1	3	68	45	46	70	2
1	4	42	46	45	40	2
2	1	27	49	46	26	1
2	2	46	45	49	50	4
2	3	72	47	45	68	4
2	4	39	43	47	43	4
3	1	24	44	43	24	0
3	2	49	48	44	45	4
3	3	70	46	48	72	2
3	4	44	49	46	42	2
4	1			49	27	

(c) Winter: $(49)(0.55) = 27$, Spring: $(49)(1.03) = 50$,

Summer: $(49)(1.52) = 74$, Fall: $(49)(0.9) = 44$

(d) Forecast: 25 acre-feet

			Seasonally	Seasonally		
		True	Adjusted	Adjusted	Actual	Forecasting
Year	Quarter	Value	Value	Forecast	Forecast	Error
1	1	25	45			
1	2	47	46	45	47	0
1	3	68	45	46	69	1
1	4	42	46	45	41	1
2	1	27	49	46	25	2
2	2	46	45	46	48	2
2	3	72	47	46	70	2
2	4	39	43	46	42	3
3	1	24	44	46	25	1
3	2	49	48	46	47	2
3	3	70	46	46	70	0
3	4	44	49	46	41	3
4	1			46	25	

	Type of Seasonality
	Quarterly
Quarter	Seasonal Factor
1	0.550
2	1.027
3	1.519
4	0.904

Mean Absolute Deviation	
MAD =	1.57
Mean Square Error	
MSE =	3.07

(e) Forecast: 26 acre-feet

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	25	45			
1	2	47	46			
1	3	68	45			
1	4	42	46			
2	1	27	49	46	25	2
2	2	46	45	47	48	2
2	3	72	47	46	70	2
2	4	39	43	47	42	3
3	1	24	44	46	25	1
3	2	49	48	45	46	3
3	3	70	46	45	69	1
3	4	44	49	45	41	3
4	1			47	26	

Number of previous periods to consider	
n =	4
	Type of Seasonality
	Quarterly
Quarter	Seasonal Factor
1	0.550
2	1.027
3	1.519
4	0.904

Mean Absolute Deviation	
MAD =	2.2
Mean Square Error	
MSE =	5.5

(f) Forecast: 25 acre-feet

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	25	45	46	25	0
1	2	47	46	46	47	0
1	3	68	45	46	70	2
1	4	42	46	46	41	1
2	1	27	49	46	25	2
2	2	46	45	46	47	1
2	3	72	47	46	70	2
2	4	39	43	46	42	3
3	1	24	44	46	25	1
3	2	49	48	46	47	2
3	3	70	46	46	70	0
3	4	44	49	46	41	3
4	1			46	25	

Smoothing Constant	
α =	0.1
Initial Estimate	
Average =	46
	Type of Seasonality
	Quarterly
Quarter	**Seasonal Factor**
1	0.550
2	1.027
3	1.519
4	0.904

Mean Absolute Deviation	
MAD =	1.4
Mean Square Error	
MSE =	2.7

(g) Exponential smoothing results in the lowest MAD value, 1.4.

(h) Exponential smoothing gives the lowest MSE value, 2.7.

27.8-14.

(a)

Year	Quarter	True Value
1	1	23
1	2	22
1	3	31
1	4	26
2	1	19
2	2	21
2	3	27
2	4	24
3	1	21
3	2	26
3	3	32
3	4	28

Type of Seasonality	
Quarterly	
Quarter	**Estimate for Seasonal Factor**
1	0.8400
2	0.9200
3	1.2000
4	1.0400

(b)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	23	27			
1	2	22	24	27	25	3
1	3	31	26	24	29	2
1	4	26	25	26	27	1
2	1	19	23	25	21	2
2	2	21	23	23	21	0
2	3	27	23	23	27	0
2	4	24	23	23	23	1
3	1	21	25	23	19	2
3	2	26	28	25	23	3
3	3	32	27	28	34	2
3	4	28	27	27	28	0
4	1			27	23	

	Type of Seasonality
	Quarterly
Quarter	Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

Mean Absolute Deviation	
MAD =	1.5
Mean Square Error	
MSE =	3

(c)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	23	27			
1	2	22	24	27	25	3
1	3	31	26	26	31	0
1	4	26	25	26	27	1
2	1	19	23	26	21	2
2	2	21	23	25	23	2
2	3	27	23	25	30	3
2	4	24	23	24	25	1
3	1	21	25	24	20	1
3	2	26	28	24	22	4
3	3	32	27	25	30	2
3	4	28	27	25	26	2
4	1			25	21	

	Type of Seasonality
	Quarterly
Quarter	Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

Mean Absolute Deviation	
MAD =	1.94
Mean Square Error	
MSE =	4.85

(d)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	23	27			
1	2	22	24			
1	3	31	26			
1	4	26	25			
2	1	19	23	26	21	2
2	2	21	23	24	22	1
2	3	27	23	24	29	2
2	4	24	23	23	24	0
3	1	21	25	23	19	2
3	2	26	28	23	21	5
3	3	32	27	25	30	2
3	4	28	27	26	27	1
4	1			27	22	

Number of previous periods to consider	
n =	4
	Type of Seasonality
	Quarterly
Quarter	Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

Mean Absolute Deviation	
MAD =	2.0
Mean Square Error	
MSE =	5.3

(e)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	23	27	25	21	2
1	2	22	24	26	24	2
1	3	31	26	25	30	1
1	4	26	25	25	26	0
2	1	19	23	25	21	2
2	2	21	23	25	23	2
2	3	27	23	24	29	2
2	4	24	23	24	25	1
3	1	21	25	24	20	1
3	2	26	28	24	22	4
3	3	32	27	25	30	2
3	4	28	27	25	26	2
4	1			26	22	

Smoothing Constant	
α =	0.25
Initial Estimate	
Average =	25
	Type of Seasonality
	Quarterly
Quarter	Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

Mean Absolute Deviation	
MAD =	1.7
Mean Square Error	
MSE =	3.6

(f)

			Seasonally			Seasonally		
		True	Adjusted	Latest	Estimated	Adjusted	Actual	Forecasting
Year	Quarter	Value	Value	Trend	Trend	Forecast	Forecast	Error
1	1	23	27		0	25	21	2
1	2	22	24	1	0	26	24	2
1	3	31	26	0	0	25	30	1
1	4	26	25	0	0	26	27	1
2	1	19	23	0	0	25	21	2
2	2	21	23	-1	0	25	23	2
2	3	27	23	-1	0	24	29	2
2	4	24	23	-1	0	23	24	0
3	1	21	25	0	0	23	19	2
3	2	26	28	0	0	23	21	5
3	3	32	27	1	0	25	29	3
3	4	28	27	1	0	25	26	2
4	1			1	0	26	22	

Smoothing Constant	
α =	0.25
β =	0.25
Initial Estimate	
Average =	25
Trend =	0
	Type of Seasonality
	Quarterly
Quarter	**Seasonal Factor**
1	0.84
2	0.92
3	1.20
4	1.04

Mean Absolute Deviation	
MAD =	2
Mean Square Error	
MSE =	4

(g) Using the last-value method with seasonality $(\text{MAD} = 1.5)$, the forecast for first quarter is 23 houses.

(h) Quarter 2: $(27)(0.92) = 25$, Quarter 3: $(27)(1.2) = 32$, Quarter 4: $(27)(1.04) = 28$

27.8-15.

(a) Last-Value Method with Seasonality

Year	Month	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	68	76			
1	Feb	71	81	76	66	5
1	Mar	66	73	81	73	7
1	Apr	72	77	73	67	5
1	May	77	80	77	74	3
1	June	85	78	80	87	2
1	July	94	80	78	91	3
1	Aug	96	83	80	92	4
1	Sep	80	82	83	81	1
1	Oct	73	80	82	75	2
1	Nov	84	80	80	84	0
1	Dec	89	82	80	86	3
2	Jan			82	74	

Mean Absolute Deviation	
MAD =	3.07
Mean Square Error	
MSE =	12.89

Averaging Method with Seasonality

Year	Month	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	68	76			
1	Feb	71	81	76	66	5
1	Mar	66	73	78	71	5
1	Apr	72	77	76	71	1
1	May	77	80	77	73	4
1	June	85	78	77	84	1
1	July	94	80	77	91	3
1	Aug	96	83	78	89	7
1	Sep	80	82	79	76	4
1	Oct	73	80	79	72	1
1	Nov	84	80	79	83	1
1	Dec	89	82	79	86	3
2	Jan			79	71	

Mean Absolute Deviation	
MAD =	3.12
Mean Square Error	
MSE =	13.07

Moving-Average Method with Seasonality

Year	Month	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	68	76			
1	Feb	71	81			
1	Mar	66	73			
1	Apr	72	77	76	71	1
1	May	77	80	77	74	3
1	June	85	78	77	84	1
1	July	94	80	79	92	2
1	Aug	96	83	80	91	5
1	Sep	80	82	81	78	2
1	Oct	73	80	82	75	2
1	Nov	84	80	82	86	2
1	Dec	89	82	81	87	2
2	Jan			81	73	

Mean Absolute Deviation	
MAD =	2.18
Mean Square Error	
MSE =	5.79

Exponential Smoothing Method with Seasonality

Year	Month	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	68	76	80	72	4
1	Feb	71	81	79	70	1
1	Mar	66	73	79	72	6
1	Apr	72	77	78	73	1
1	May	77	80	78	75	2
1	June	85	78	78	85	0
1	July	94	80	78	92	2
1	Aug	96	83	79	91	5
1	Sep	80	82	80	77	3
1	Oct	73	80	80	73	0
1	Nov	84	80	80	84	0
1	Dec	89	82	80	87	2
2	Jan			81	73	

Mean Absolute Deviation	
MAD =	2.34
Mean Square Error	
MSE =	9.31

Method	MAD	MSE
Last-Value	3.07	12.89
Averaging	3.12	13.07
Moving-Average	2.18	5.79
Exponential Smoothing	2.34	9.31

(b) The moving-average method with seasonality has the lowest MAD value. With this method, the forecast for January is 73 passengers.

27.8-16.

(a)

Method	MAD	MSE
Last-Value	2.46	8.34
Averaging	7.06	74.73
Moving-Average	2.79	9.68
Exp. Smoothing	4.28	25.87

(b) Forecast: 94

Year	Month	True Value	Seasonally Adjusted Value	Latest Trend	Estimated Trend	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	75	83		2	82	74	1
1	Feb	76	86	2	2	84	74	2
1	Mar	81	89	2	2	87	79	2
1	Apr	84	90	3	2	90	83	1
1	May	85	89	2	2	92	88	3
1	June	99	91	2	2	93	102	3
1	July	107	91	2	2	95	111	4
1	Aug	108	94	1	2	96	110	2
1	Sep	94	97	1	2	97	95	1
1	Oct	90	99	2	2	99	90	0
1	Nov	106	101	2	2	101	106	0
1	Dec	110	102	2	2	103	111	1
2	Jan			2	2	104	94	

Mean Absolute Deviation	
MAD =	1.66
Mean Square Error	
MSE =	4.21

MAD and MSE values are lower than those in (a).

(c)

Mean Absolute Deviation	
MAD =	2.74
Mean Square Error	
MSE =	10.44

MAD and MSE values obtained are higher than the ones in (b).

Year	Month	True Value	Seasonally Adjusted Value	Latest Trend	Estimated Trend	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	68	76		0	80	72	4
1	Feb	71	81	-1	0	79	69	2
1	Mar	66	73	0	0	79	72	6
1	Apr	72	77	-1	0	77	72	0
1	May	77	80	0	0	77	74	3
1	June	85	78	0	0	77	84	1
1	July	94	80	0	0	77	90	4
1	Aug	96	83	0	0	78	89	7
1	Sep	80	82	1	0	79	77	3
1	Oct	73	80	1	0	80	73	0
1	Nov	84	80	0	0	80	84	0
1	Dec	89	82	0	0	80	87	2
2	Jan	75	83	1	0	81	73	2
2	Feb	76	86	1	0	82	72	4
2	Mar	81	89	1	1	83	76	5
2	Apr	84	90	2	1	85	79	5
2	May	85	89	2	1	87	84	1
2	June	99	91	1	1	89	97	2
2	July	107	91	1	1	90	106	1
2	Aug	108	94	1	1	92	105	3
2	Sep	94	97	2	1	93	91	3
2	Oct	90	99	2	1	96	87	3
2	Nov	106	101	2	2	98	103	3
2	Dec	110	102	2	2	100	108	2
3	Jan			2	2	102	92	

(d) Exponential smoothing with seasonality and trend (with parameters as in (b) should be used.

(e) The best values for the smoothing constants are $\alpha = \beta = 0.3$ and $\gamma = 0.001$. The forecasts for the coming year are given in the table below.

Month	Forecast
Feb	98.4
Mar	98.5
Apr	105.3
May	110.3
Jun	125.7
Jul	138.1
Aug	141
Sep	120.7
Oct	113.4
Nov	132.6
Dec	139.7

27.8-17.

(a) Based on past sales:

Month	Estimate for Seasonal Factor
Jan	0.808186196
Feb	0.807383628
Mar	0.876404494
Apr	0.921348315
May	1.016051364
Jun	1.105136437
Jul	1.017656501
Aug	1.188603531
Sep	0.805778491
Oct	0.760834671
Nov	1.436597111
Dec	1.256019262

(b) Moving Average with Seasonality

Year	Month	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan					
1	Feb					
1	Mar					
1	Apr					
1	May					
1	June					
1	July					
1	Aug					
1	Sep					
1	Oct	335	440			
1	Nov	594	413			
1	Dec	527	420			
2	Jan	364	450	424	343	21
2	Feb	343	425	428	345	2
2	Mar	391	446	432	378	13
2	Apr	437	474	440	406	31
2	May	458	451	448	456	2
2	June	494	447	457	505	11
2	July	468	460	457	465	3
2	Aug	555	467	453	538	17
2	Sep	387	480	458	369	18
2	Oct	364	478	469	357	7
2	Nov	662	461	475	683	21
2	Dec	581	463	473	594	13
3	Jan			467	378	

Mean Absolute Deviation	
MAD =	13.30
Mean Square Error	
MSE =	249.09

(c) Exponential Smoothing with Seasonality

Year	Month	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	364	450	420	339	25
1	Feb	343	425	426	344	1
1	Mar	391	446	426	373	18
1	Apr	437	474	430	396	41
1	May	458	451	439	446	12
1	June	494	447	441	488	6
1	July	468	460	442	450	18
1	Aug	555	467	446	530	25
1	Sep	387	480	450	363	24
1	Oct	364	478	456	347	17
1	Nov	662	461	461	662	0
1	Dec	581	463	461	579	2
2	Jan			461	373	

Mean Absolute Deviation	
MAD =	15.83
Mean Square Error	
MSE =	384.99

(d) Exponential Smoothing with Seasonality and Trend

Year	Month	True Value	Seasonally Adjusted Value	Latest Trend	Estimated Trend	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	364	450		0	420	339	25
1	Feb	343	425	6	1	427	345	2
1	Mar	391	446	1	1	428	375	16
1	Apr	437	474	5	2	433	399	38
1	May	458	451	10	3	445	452	6
1	June	494	447	5	4	450	497	3
1	July	468	460	3	4	453	461	7
1	Aug	555	467	5	4	458	545	10
1	Sep	387	480	6	4	464	374	13
1	Oct	364	478	7	5	472	359	5
1	Nov	662	461	6	5	479	688	26
1	Dec	581	463	2	4	479	602	21
2	Jan			1	4	480	388	

Mean Absolute Deviation	
MAD =	14.26
Mean Square Error	
MSE =	314.71

(e) The moving-average method results in the best MAD value (13.30) and the best MSE value (249.09).

(f)

Month	Avg. Forecast	Forecasting Error
January	341	23
February	345	2
March	375	16
April	400	37
May	451	7
June	497	3
July	459	9
August	537	18
September	369	18
October	354	10
November	677	15
December	592	12

$\text{MAD} = 14.17$

(g) The moving-average method performed better than the average of all three, so it should be used next year.

(h) The best method is exponential smoothing with seasonality and trend, using $\alpha = \beta = 0.3$ and $\gamma = 0.001$.

Month	Forecast
Jan	389
Feb	394
Mar	430
Apr	453
May	502
Jun	548
Jul	507
Aug	594
Sep	405
Oct	383
Nov	726
Dec	637

27.8-18.

Quarter	Sales	Forecast a)	Squared Error a)	Forecast b)	Squared Error b)	Forecast c)	Squared Error c)	Forecast d)	Squared Error d)
1	6900								
2	6700								
3	7900			6880	1040400	6840	1123600	6500	1960000
4	7100			6982	13924	7158	3364	6846	64516
5	8200	7150	1102500	6994	1454918	7141	1122328	6871	1766082
6	7000	7475	225625	7114	13092	7458	210149	7338	114384
7	7300	7550	62500	7103	38818	7321	437	7275	637
8	7500	7400	10000	7123	142370	7315	34364	7323	31458
9	9400	7500	3610000	7160	5015754	7370	4119934	7432	3872612
10	9200	7800	1960000	7384	3296509	7979	1490434	8256	891432
11	9800	8350	2102500	7566	4991051	8345	2115813	8857	888430
12	9900	8975	855625	7789	4454884	8782	1250390	9543	127179
13	11400	9575	3330625	8000	11557236	9117	5210929	10086	1727552
14	10000	10075	5625	8340	2754386	9802	39173	11034	1068141
15	9400	10275	765625	8506	798647	9861	212940	11184	3182671
16	8400	10175	3150625	8596	38297	9723	1750377	10949	6496332
17	8800	9800	1000000	8576	50119	9326	276795	10255	2116296
18	7600	9150	2402500	8599	997030	9168	2459499	9758	4656936
19	7500	8550	1102500	8499	997327	8698	1434713	8856	1838858
	MSE		1445750		2214986		1344426		1811971

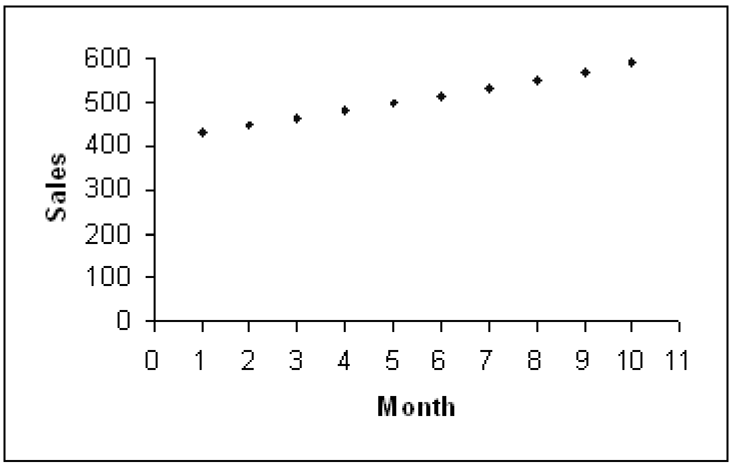
SMALLEST

27.8-19.

Quarter	Sales	Forecast a)	Squared Error a)	Forecast b)	Squared Error b)	Forecast c)	Squared Error c)	Forecast d)	Squared Error d)
1	546								
2	528								
3	530			544	202	541	112	510	400
4	508			543	1210	537	866	500	67
5	647	528	14161	539	11599	523	15277	487	25665
6	594	553	1661	550	1930	560	1124	534	3622
7	665	570	9073	554	12218	631	1149	556	11828
8	630	604	702	566	4158	641	127	603	727
9	736	634	10404	572	26907	655	6642	628	11727
10	724	656	4590	588	18396	679	2030	687	1404
11	813	689	15438	602	44549	732	6496	727	7314
	MSE		8004		13463		3758		6973
							smallest		

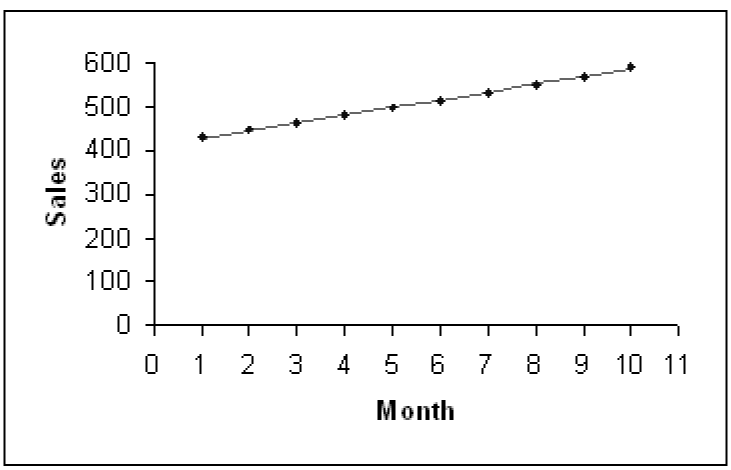
27.10-1.

(a)



(b) $y = 410 + 17.6x$

(c)



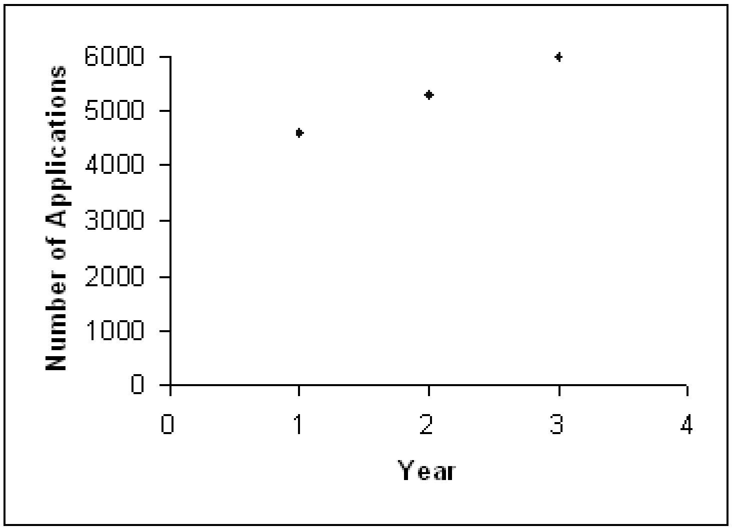
(d) $y = 410 + (17.6)(11) = 604$

(e) $y = 410 + (17.6)(20) = 762$

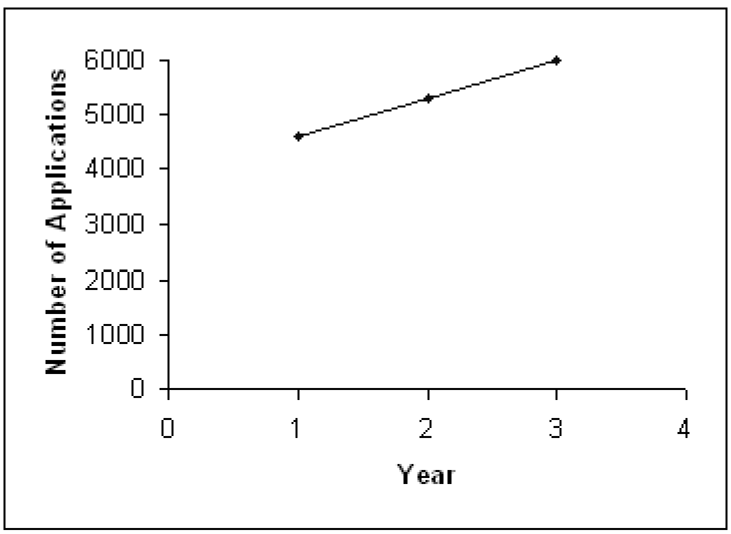
(f) The average growth in monthly sales is 17.6.

27.10-2.

(a)



(b)

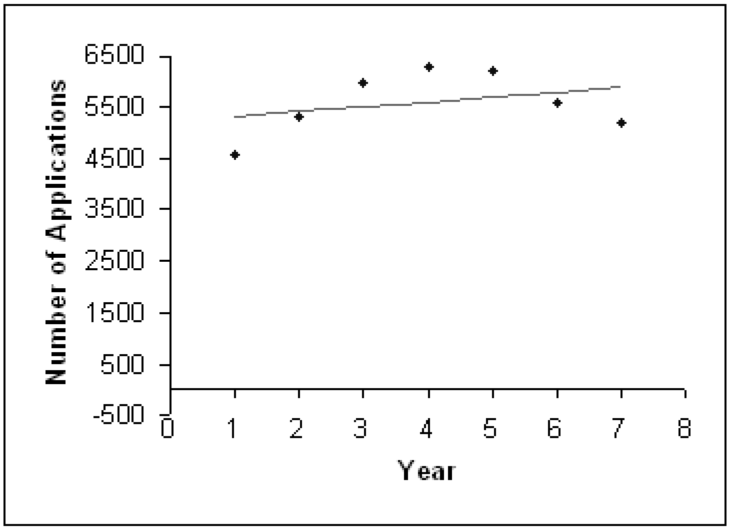


(c) $y = 3900 + 700x$

(d) $y(\text{Year } 4) = 3900 + (700)(4) = 6700$
$y(\text{Year } 5) = 3900 + (700)(5) = 7400$
$y(\text{Year } 6) = 3900 + (700)(6) = 8100$
$y(\text{Year } 7) = 3900 + (700)(7) = 8800$
$y(\text{Year } 8) = 3900 + (700)(8) = 9500$

(e) It does not make sense to use the forecast obtained earlier, 9500. The relationship between the variables has changed and thus the linear regression that was used is no longer appropriate.

(f)



$y = 5228 + 92.9x$
$y = 5228 + (92.9)(8) = 5971$

The linear regression line does not provide a close fit to the data. Consequently, the forecast that it provides for year 8 is not likely to be accurate. It does not make sense to continue to use a linear regression line when changing conditions cause a large shift in the underlying trend in the data.

(g)

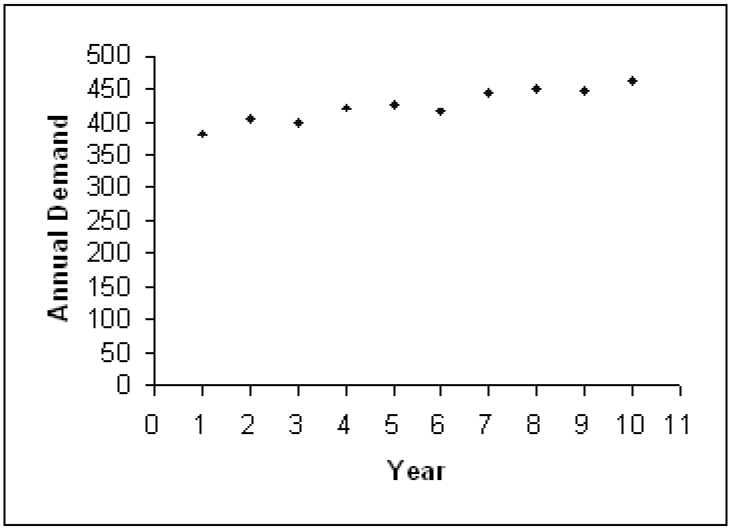
Time Period	True Value	Latest Trend	Estimated Trend	Exponential Smoothing Forecast	Forecasting Error
1	4,600		700.00	4,600	0
2	5,300	700.00	700.00	5,300	0
3	6,000	700.00	700.00	6,000	0
4	6,300	700.00	700.00	6,700	400
5	6,200	500.00	600.00	7,100	900
6	5,600	150.00	375.00	7,025	1,425
7	5,200	-337.50	18.75	6,331	1,131
8		-546.88	-264.06	5,502	

Mean Absolute Deviation	
MAD =	550.9
Mean Square Error	
MSE =	611,478.8

Casual forecasting takes all the data into account, even the data from before changing conditions cause a shift. Exponential smoothing with trend adjusts to shifts in the underlying trend.

27.10-3.

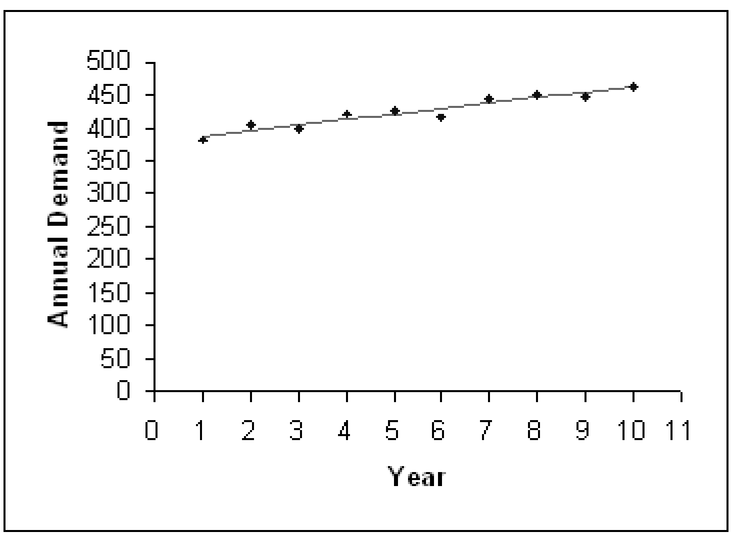
(a)



(b) $y = 380.27 + 8.15x$

Time Period	Independent Variable	Dependent Variable	Estimate	Estimation Error	Square of Error
1	1	382	388	6.42	41
2	2	405	397	8.43	71
3	3	398	405	6.72	45
4	4	421	413	8.13	66
5	5	426	421	4.98	25
6	6	415	429	14.18	201
7	7	443	437	5.67	32
8	8	451	445	5.52	30
9	9	446	454	7.63	58
10	10	464	462	2.22	5

(c)



(d) $y = 380 + (8.15)(11) = 470$

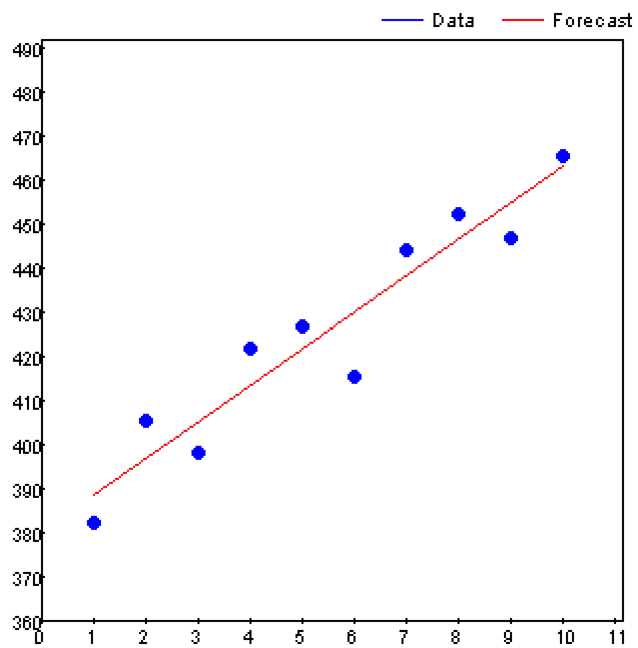
(e) $y = 380 + (8.15)(15) = 503$

(f) The average growth per year is 8.15 tons.

(g)

Forecasting -- Linear Regression Method
MAD = 6.99   MSE = 57.5
a = 380.27   b = 8.15   y = 380.27 + 8.15x

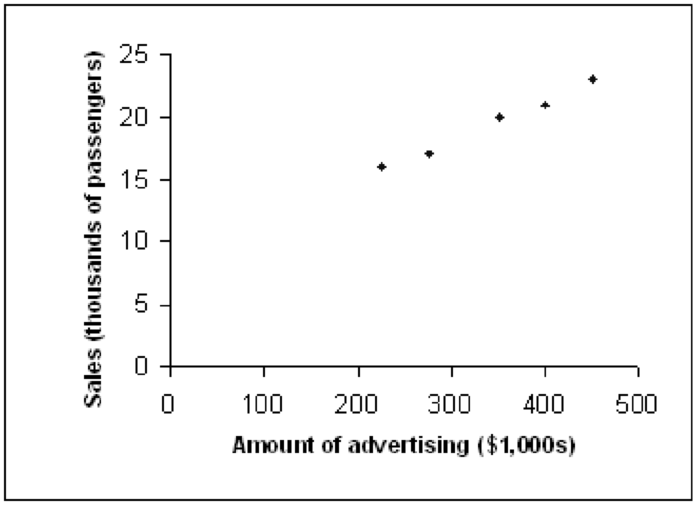
x	y	Forecast	Error
1	382	388.42	6.42
2	405	396.57	8.43
3	398	404.72	6.72
4	421	412.87	8.13
5	426	421.02	4.98
6	415	429.18	14.18
7	443	437.33	5.67
8	451	445.48	5.52
9	446	453.63	7.63
10	464	461.78	2.22



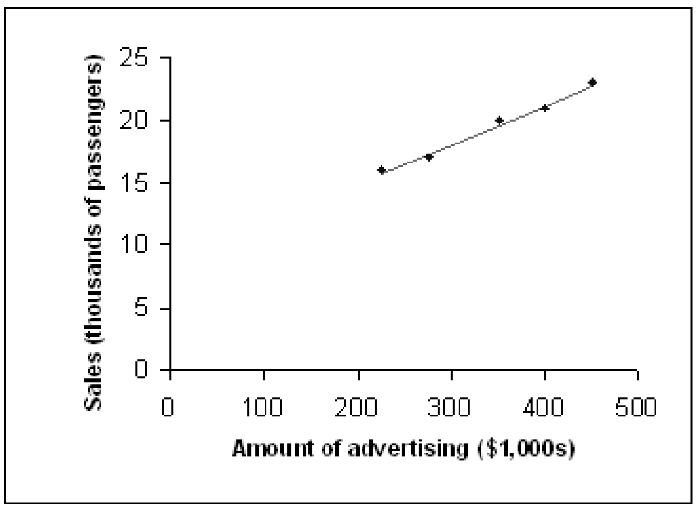
27.10-4.

(a) The amount of advertising is the independent variable and sales is the dependent variable.

(b)



(c) $y = 8.71 + 0.031x$



(d) $y = 8.71 + (0.031)(300) = 18,000$ passengers

(e) $22 = 8.71 + (0.031)(x) \Rightarrow x = \$429,000$

(f) An increase of 31 passengers can be attained.

27.10-5.

(a) If the sales increase from 16 to 19 when the amount of advertising is 225, then the linear regression line shifts below this point. The line actually shifts up, but not as much as the data point has shifted up.

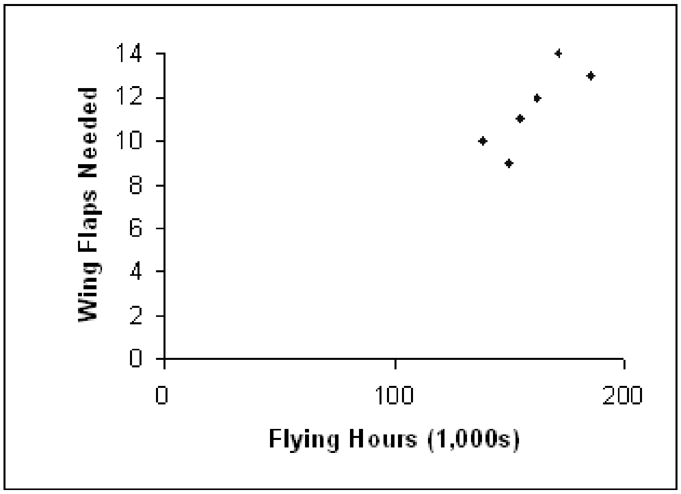
(b) If the sales increase from 23 to 26 when the amount of advertising is 450, then the linear regression line shifts below this point. The line actually shifts up, but not as much as the data point has shifted up.

(c) If the sales increase from 20 to 23 when the amount of advertising is 350, then the linear regression line shifts below this point. The line actually shifts up, but not as much as the data point has shifted up.

27.10-6.

(a) The number of flying hours is the independent variable and the number of wing flaps needed is the dependent variable.

(b)



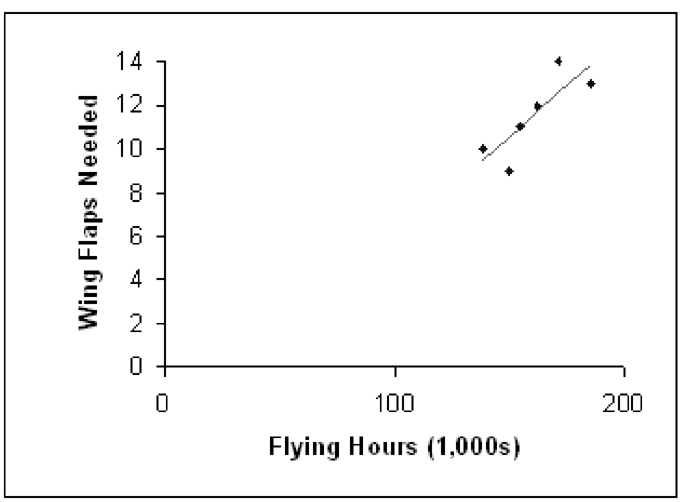
(c) $y = -3.382 + 0.093x$

Time Period	Independent Variable	Dependent Variable	Estimate	Estimation Error	Square of Error
1	162	12	12	0.30	0
2	149	9	10	1.49	2
3	185	13	14	0.84	1
4	171	14	13	1.46	2
5	138	10	9	0.53	0
6	154	11	11	0.04	0

Linear Regression Line	
y = a + bx	
a =	-3.382
b =	0.093

Estimator	
If x =	150
then y=	10.584

(d)



(e) $y = -3.382 + (0.093)(150) = 11$

(f) $y = -3.382 + (0.093)(200) = 15$

27.10-7.

Time Period	Independent Variable	Dependent Variable	Estimate	Estimation Error	Square of Error
1	323	24	22	2.48	6
2	359	23	25	2.02	4
3	396	28	29	0.63	0
4	421	32	31	0.93	1
5	457	34	35	0.57	0
6	472	37	36	0.97	1
7	446	33	34	0.50	0
8	407	30	30	0.30	0
9	374	27	26	0.51	0
10	343	22	23	1.47	2

Linear Regression Line	
y = a + bx	
a =	-9.954
b =	0.097

Joe should use the linear regression line $y = -9.95 + 0.097x$ to develop a forecast for jobs in the future.

27.10-8.

(a) $\widehat{y}(x) = 121.04 - 1.0346x \Rightarrow \widehat{y}(55) = 64.137$

(b) $t_{0.025:5} = 2.571, s_{y|x} = 6.34$

$$\sqrt{1 + \frac{1}{7} + \frac{(x_t-\overline{x})^2}{\sum(x_i-\overline{x})^2}} = 1.0735$$

The 95% prediction interval is $[46.64, 81.64]$.

(c) By interpolation:

$$t_{0.0125:5} = 3.365 - \frac{0.0025}{0.015}(3.365 - 2.571) = 3.233$$

The simultaneous 95% prediction interval is $[42.13, 86.14]$.

(d) By interpolation:

$$c^{**} = 10.722 + \frac{1}{2}(11.150 - 10.722) = 10.936$$

The simultaneous tolerance interval is $[37.1, 91.2]$.

27.10-9.

(a) $\sum_{i=1}^{5} x = 20, \sum_{i=1}^{5} y = 40, \sum_{i=1}^{5} xy = 242, \sum_{i=1}^{5} x^2 = 120$

$$\Rightarrow \widehat{y}(x) = -0.2 + 2.05x \Rightarrow \widehat{y}(10) = 20.3$$

(b) $s_{y|x}^2 = 0.6333, t_{0.025:3} = 3.182$

$$\sqrt{\frac{1}{5} + \frac{(x_t-\overline{x})^2}{\sum(x_i-\overline{x})^2}} = \sqrt{1.1}$$

The 95% prediction interval is $[17.64, 22.9]$.

(c)

$$\sqrt{1+\frac{1}{5}+\frac{(x_t-\overline{x})^2}{\sum(x_i-\overline{x})^2}}=\sqrt{2.1}$$

The 95% prediction interval is $[16.630, 23.970]$.

(d) By interpolation:

$$c^{**}=11.150+\frac{1}{2}(14.953-11.150)=13.0515$$

The simultaneous tolerance interval is $[9.406, 31.194]$.

27.10-10.

(a)

$$k=\frac{\sum\limits_{i=1}^{5}x_iy_i-\left(\sum\limits_{i=1}^{5}x_i\sum\limits_{i=1}^{5}y_i\right)/5}{\sum\limits_{i=1}^{5}x_i^2-\left(\sum\limits_{i=1}^{5}x_i\right)^2/5}=\frac{19.96-0}{10-0}=1.996$$

$$\log g=\frac{\sum\limits_{i=1}^{5}(y_i-kx_i)}{5}=\frac{0.08}{5}=0.016$$

$$\Rightarrow \log r=0.016+1.996\log t$$

$$\log t=3\Rightarrow \log r=0.016+1.996\times 3=6.004$$

The forecast for the distance traveled when $\log\ t=3$ is then $10^{6.004}$, which is approximately one million.

(b)

$\log t$	$\log r$	$\widehat{E}(\log r)$
-2.0	-3.95	$-$
-1.0	-2.12	$-$
0.0	0.08	-3.767
1.0	2.20	-3.382
2.0	3.87	-2.824
3.0	$-$	-2.155

(c)

$\log t$	$\log r$	$\alpha x+(1-\alpha)F$	Trend	$\widehat{E}(\log r)$
-2.0	-3.95	-3.950	1.996	$-$
-1.0	-2.12	-1.971	1.994	$-$
0.0	0.08	0.029	1.995	0.024
1.0	2.20	2.042	1.997	2.024
2.0	3.87	4.022	1.995	4.039
3.0	$-$	$-$	$-$	6.017

27.10-11.

$$Q=\sum_{i=1}^{n}(y_i-bx_i)^2\Rightarrow \frac{dQ}{db}=\sum_{i=1}^{n}-2x_i(y_i-bx_i)=0\Rightarrow B=\frac{\sum\limits_{i=1}^{n}x_iy_i}{\sum\limits_{i=1}^{n}x_i^2}$$

Cases

27 - 1 a) We need to forecast the call volume for each day separately.

1) To obtain the seasonally adjusted call volume for the past 13 weeks, we first have to determine the seasonal factors. Because call volumes follow seasonal patterns within the week, we have to calculate a seasonal factor for Monday, Tuesday, Wednesday, Thursday, and Friday. We calculate the seasonal factors using the following formula:

$$\frac{\text{Average for the Day}}{\text{Overall Average}}$$

The spreadsheet used to calculate the seasonal factors follows.

	Monday	Tuesday	Wednesday	Thursday	Friday	Total
Week 44	1130	851	859	828	726	4394
Week 45	1085	1042	892	840	799	4658
Week 46	1303	1121	1003	1113	1005	5545
Week 47	2652	2825	1841	0	0	7318
Week 48	1949	1507	989	990	1084	6519
Week 49	1260	1134	941	847	714	4896
Week 50	1002	847	922	842	784	4397
Week 51	823	0	0	401	429	1653
Week 52/1	1209	830	0	1082	841	3962
Week 2	1362	1174	967	930	853	5286
Week 3	924	954	1346	904	758	4886
Week 4	886	878	802	945	610	4121
Week 5	910	754	705	729	772	3870
Average	1268.846	1070.538	866.692	803.923	721.154	946.231
Seasonal Factors	**1.341**	**1.131**	**0.916**	**0.850**	**0.762**	

Now that we have found the seasonal factors, we can obtain the seasonally adjusted series. For each day within each of the 13 weeks, we need to calculate the seasonally adjusted daily call volumes using the following formula:

$$\frac{\text{Actual Daily Call Volume}}{\text{Seasonal Factor for the Corresponding Day of the Week}}$$

The spreadsheet for the seasonally adjusted call volumes follows.

	Monday	Tuesday	Wednesday	Thursday	Friday
Week 44	842.69	752.18	937.83	974.57	952.59
Week 45	809.13	921.01	973.86	988.69	1048.37
Week 46	971.70	990.83	1095.05	1310.02	1318.67
Week 47	1977.71	2496.97	2009.95	0.00	0.00
Week 48	1453.45	1332.01	1079.76	1165.25	1422.32
Week 49	939.63	1002.32	1027.36	996.93	936.84
Week 50	747.23	748.65	1006.61	991.05	1028.69
Week 51	613.74	0.00	0.00	471.98	562.89
Week 52/1	901.60	733.62	0.00	1273.53	1103.48
Week 2	1015.70	1037.68	1055.74	1094.63	1119.23
Week 3	689.06	843.22	1469.53	1064.02	994.58
Week 4	660.73	776.05	875.60	1112.28	800.39
Week 5	678.62	666.45	769.70	858.05	1012.95

2) To forecast the call volume for the next week using the last-value forecasting method, we need to use the Last Value with Seasonality template. However, we need to modify this template because it does not provide the option for daily seasonality. The modified spreadsheet follows.

Week	Day	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
44	Mon	1130	843			
44	Tues	851	753	843	952	101
44	Wed	859	938	753	690	169
44	Thurs	828	974	938	797	31
44	Fri	726	955	974	740	14
45	Mon	1085	809	955	1281	196
45	Tues	1042	922	809	914	128
45	Wed	892	974	922	845	47
45	Thurs	840	988	974	828	12
45	Fri	799	1051	988	751	48
46	Mon	1303	972			
46	Tues	1121	992			
46	Wed	1003	1095			
46	Thurs	1113	1309			
46	Fri	1005	1322			
47	Mon	2652	1978			
47	Tues	2825	2500			
47	Wed	1841	2010			
47	Thurs	0	0			
47	Fri	0	0			
48	Mon	1949	1453			
48	Tues	1507	1334			
48	Wed	989	1080			
48	Thurs	990	1165			
48	Fri	1084	1426			
49	Mon	1260	940			
49	Tues	1134	1004			
49	Wed	941	1027			
49	Thurs	847	996			
49	Fri	714	939			
50	Mon	1002	747			
50	Tues	847	750			
50	Wed	922	1007			
50	Thurs	842	991			
50	Fri	784	1032			
51	Mon	823	614			
51	Tues	0	0			
51	Wed	0	0			
51	Thurs	401	472			
51	Fri	429	564			
52/1	Mon	1209	902			
52/1	Tues	830	735			
52/1	Wed	0	0			
52/1	Thurs	1082	1273			
52/1	Fri	841	1107			
2	Mon	1362	1016			
2	Tues	1174	1039			
2	Wed	967	1056			
2	Thurs	930	1094			
2	Fri	853	1122			
3	Mon	924	689			
3	Tues	954	844			
3	Wed	1346	1469			
3	Thurs	904	1064			
3	Fri	758	997			
4	Mon	886	661			
4	Tues	878	777			
4	Wed	802	876			
4	Thurs	945	1112			
4	Fri	610	803			
5	Mon	910	679			
5	Tues	754	667			
5	Wed	705	770			
5	Thurs	729	858			
5	Fri	772	1016			
6	Mon	1362	1016	1016	1362	0
6	Tues	1148	1016	1016	1148	0
6	Wed	931	1016	1016	931	0
6	Thurs	864	1016	1016	864	0
6	Fri	773	1017	1016	773	0

Weekly Demand for the Next We 5077

Type of Seasonalit

Daily

Day	Seasonal Factor
Mon	1.34
Tues	1.13
Wed	0.92
Thurs	0.85
Fri	0.76

The forecasted call volume for the next week is 5077 calls: 1362 calls are received on Monday, 1148 calls are received on Tuesday, 931 calls are received on Wednesday, 864 calls are received on Thursday, and 773 calls are received on Friday.

3) To forecast the call volume for the next week using the averaging forecasting method, we need to use the Averaging with Seasonality template. However, we need to modify this template because it does not provide the option for daily seasonality. The modified spreadsheet follows.

Week	Day	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
44	Mon	1130	843			
44	Tues	851	753			
44	Wed	859	938			
44	Thurs	828	974			
44	Fri	726	955			
45	Mon	1085	809			
45	Tues	1042	922			
45	Wed	892	974			
45	Thurs	840	988			
45	Fri	799	1051			
46	Mon	1303	972			
46	Tues	1121	992			
46	Wed	1003	1095			
46	Thurs	1113	1309			
46	Fri	1005	1322			
47	Mon	2652	1978			
47	Tues	2825	2500			
47	Wed	1841	2010			
47	Thurs	0	0			
47	Fri	0	0			
48	Mon	1949	1453			
48	Tues	1507	1334			
48	Wed	989	1080			
48	Thurs	990	1165			
48	Fri	1084	1426			
49	Mon	1260	940			
49	Tues	1134	1004			
49	Wed	941	1027			
49	Thurs	847	996			
49	Fri	714	939			
50	Mon	1002	747			
50	Tues	847	750			
50	Wed	922	1007			
50	Thurs	842	991			
50	Fri	784	1032			
51	Mon	823	614			
51	Tues	0	0			
51	Wed	0	0			
51	Thurs	401	472			
51	Fri	429	564			
52/1	Mon	1209	902			
52/1	Tues	830	735			
52/1	Wed	0	0			
52/1	Thurs	1082	1273			
52/1	Fri	841	1107			
2	Mon	1362	1016			
2	Tues	1174	1039			
2	Wed	967	1056			
2	Thurs	930	1094			
2	Fri	853	1122			
3	Mon	924	689			
3	Tues	954	844			
3	Wed	1346	1469			
3	Thurs	904	1064			
3	Fri	758	997			
4	Mon	886	661			
4	Tues	878	777			
4	Wed	802	876			
4	Thurs	945	1112			
4	Fri	610	803			
5	Mon	910	679			
5	Tues	754	667			
5	Wed	705	770			
5	Thurs	729	858			
5	Fri	772	1016			
6	Mon	1270	947	947	1270	0
6	Tues	1070	947	947	1070	0
6	Wed	867	947	947	867	0
6	Thurs	805	947	947	805	0
6	Fri	720	947	947	720	0

Type of Seasonality

Daily

Day	Seasonal Factor
Mon	1.34
Tues	1.13
Wed	0.92
Thurs	0.85
Fri	0.76

Weekly Demand for the Next Week 4732

The forecasted call volume for the next week is 4732 calls: 1270 calls are received on Monday, 1070 calls are received on Tuesday, 867 calls are received on Wednesday, 805 calls are received on Thursday, and 720 calls are received on Friday.

4) To forecast the call volume for the next week using the moving-average forecasting method, we need to use the Moving Averaging with Seasonality template. However, we need to modify this template because it does not provide the option for daily seasonality. The modified spreadsheet follows.

Week	Day	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
44	Mon	1130	843			
44	Tues	851	753			
44	Wed	859	938			
44	Thurs	828	974			
44	Fri	726	955			
45	Mon	1085	809			
45	Tues	1042	922			
45	Wed	892	974			
45	Thurs	840	988			
45	Fri	799	1051			
46	Mon	1303	972			
46	Tues	1121	992			
46	Wed	1003	1095			
46	Thurs	1113	1309			
46	Fri	1005	1322			
47	Mon	2652	1978			
47	Tues	2825	2500			
47	Wed	1841	2010			
47	Thurs	0	0			
47	Fri	0	0			
48	Mon	1949	1453			
48	Tues	1507	1334			
48	Wed	989	1080			
48	Thurs	990	1165			
48	Fri	1084	1426			
49	Mon	1260	940			
49	Tues	1134	1004			
49	Wed	941	1027			
49	Thurs	847	996			
49	Fri	714	939			
50	Mon	1002	747			
50	Tues	847	750			
50	Wed	922	1007			
50	Thurs	842	991			
50	Fri	784	1032			
51	Mon	823	614			
51	Tues	0	0			
51	Wed	0	0			
51	Thurs	401	472			
51	Fri	429	564			
52/1	Mon	1209	902			
52/1	Tues	830	735			
52/1	Wed	0	0			
52/1	Thurs	1082	1273			
52/1	Fri	841	1107			
2	Mon	1362	1016			
2	Tues	1174	1039			
2	Wed	967	1056			
2	Thurs	930	1094			
2	Fri	853	1122			
3	Mon	924	689			
3	Tues	954	844			
3	Wed	1346	1469			
3	Thurs	904	1064			
3	Fri	758	997			
4	Mon	886	661			
4	Tues	878	777			
4	Wed	802	876			
4	Thurs	945	1112			
4	Fri	610	803			
5	Mon	910	679			
5	Tues	754	667			
5	Wed	705	770			
5	Thurs	729	858			
5	Fri	772	1016			
6	Mon	1070	798	798	1070	0
6	Tues	928	822	822	928	0
6	Wed	781	853	853	781	0
6	Thurs	739	869	869	739	0
6	Fri	662	871	871	662	0

Number of previous periods to consider

n = 5

Type of Seasonality

Daily

Day	Seasonal Factor
Mon	1.34
Tues	1.13
Wed	0.92
Thurs	0.85
Fri	0.76

Weekly Demand for the Next Week 4180

The forecasted call volume for the next week is 4189 calls: 1070 calls are received on Monday, 928 calls are received on Tuesday, 781 calls are received on Wednesday, 739 calls are received on Thursday, and 662 calls are received on Friday.

5) To forecast the call volume for the next week using the exponential smoothing forecasting method, we need to use the Exponential with Seasonality template. However, we need to modify this template because it does not provide the option for daily seasonality. The modified spreadsheet follows.

Week	Day	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
44	Mon	1130	843			
44	Tues	851	753			
44	Wed	859	938			
44	Thurs	828	974			
44	Fri	726	955			
45	Mon	1085	809			
45	Tues	1042	922			
45	Wed	892	974			
45	Thurs	840	988			
45	Fri	799	1051			
46	Mon	1303	972			
46	Tues	1121	992			
46	Wed	1003	1095			
46	Thurs	1113	1309			
46	Fri	1005	1322			
47	Mon	2652	1978			
47	Tues	2825	2500			
47	Wed	1841	2010			
47	Thurs	0	0			
47	Fri	0	0			
48	Mon	1949	1453			
48	Tues	1507	1334			
48	Wed	989	1080			
48	Thurs	990	1165			
48	Fri	1084	1426			
49	Mon	1260	940			
49	Tues	1134	1004			
49	Wed	941	1027			
49	Thurs	847	996			
49	Fri	714	939			
50	Mon	1002	747			
50	Tues	847	750			
50	Wed	922	1007			
50	Thurs	842	991			
50	Fri	784	1032			
51	Mon	823	614			
51	Tues	0	0			
51	Wed	0	0			
51	Thurs	401	472			
51	Fri	429	564			
52/1	Mon	1209	902			
52/1	Tues	830	735			
52/1	Wed	0	0			
52/1	Thurs	1082	1273			
52/1	Fri	841	1107			
2	Mon	1362	1016			
2	Tues	1174	1039			
2	Wed	967	1056			
2	Thurs	930	1094			
2	Fri	853	1122			
3	Mon	924	689			
3	Tues	954	844			
3	Wed	1346	1469			
3	Thurs	904	1064			
3	Fri	758	997			
4	Mon	886	661			
4	Tues	878	777			
4	Wed	802	876			
4	Thurs	945	1112			
4	Fri	610	803			
5	Mon	910	679			
5	Tues	754	667			
5	Wed	705	770			
5	Thurs	729	858			
5	Fri	772	1016	946		
6	Mon	1278	953	953	1278	0
6	Tues	1077	953	953	1077	0
6	Wed	873	953	953	873	0
6	Thurs	810	953	953	810	0
6	Fri	724	953	953	724	0

Weekly Demand for the Next Week 4763

Smoothing Constant

$\alpha =$ 0.1

Initial Estimate

Average = 946.2307692

Type of Seasonality

Daily

Day	Seasonal Factor
Mon	1.34
Tues	1.13
Wed	0.92
Thurs	0.85
Fri	0.76

The forecasted call volume for the next week is 4763 calls: 1278 calls are received on Monday, 1077 calls are received on Tuesday, 873 calls are received on Wednesday, 810 calls are received on Thursday, and 724 calls are received on Friday.

b) To obtain the mean absolute deviation for each forecasting method, we simply need to subtract the true call volume from the forecasted call volume for each day in the sixth week. We then need to take the absolute value of the five differences. Finally, we need to take the average of these five absolute values to obtain the mean absolute deviation.

The spreadsheet for the calculation of the mean absolute deviation for the last-value forecasting method follows.

LAST VALUE

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	1362	639
6	Tues	677	1148	471
6	Wed	521	931	410
6	Thurs	571	864	293
6	Fri	498	773	275

Mean Absolute Deviation

MAD =	417

This method is the least effective of the four methods because this method depends heavily upon the <u>average</u> seasonality factors. If the average seasonality factors are not the true seasonality factors for week 6, a large error will appear because the average seasonality factors are used to transform the Friday call volume in week 5 to forecasts for <u>all</u> call volumes in week 6. We calculated in part (a) that the call volume for Friday is 0.76 times lower than the overall average call volume. In week 6, however, the call volume for Friday is only 0.83 times lower than the average call volume over the week. Also, we calculated that the call volume for Monday is 1.34 times higher than the overall average call volume. In Week 6, however, the call volume for Monday is only 1.21 times higher than the average call volume over the week. These differences introduce a large error.

The spreadsheet for the calculation of the mean absolute deviation for the averaging forecasting method appears below.

Averaging

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	1270	547
6	Tues	677	1070	393
6	Wed	521	867	346
6	Thurs	571	805	234
6	Fri	498	720	222

Mean Absolute Deviation

MAD =	348

This method is the second-most effective of the four methods. Again, the reason lies in the average seasonality factors. Applying the average seasonality factors to an average call volume yields a much more accurate result than applying average seasonality factors to only one call volume. This method is not the most effective method, however, because the centralized call center experiences not only daily seasonality, but also weekly seasonality. For example, the call volumes in weeks 45 and 46 are much greater than the call volumes in week 6. Therefore, these larger call volumes inflate the average call volume, which in turn inflates the forecasts for Week 6.

The spreadsheet for the calculation of the mean absolute deviation for the moving-average forecasting method appears below.

MOVING AVERAGE

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	1070	347
6	Tues	677	928	251
6	Wed	521	781	260
6	Thurs	571	739	168
6	Fri	498	662	164

Mean Absolute Deviation

MAD =	238

This method is the most effective of the four methods because this method only uses the average week 5 call volume to forecast the call volumes for week 6. Again, applying the average seasonality factors to an average call volume yields a much more accurate result than applying average seasonality factors to only one call volume. Also, the average call volume used in this method is not overly inflated since it is an average of the week 5 call volumes, which are closer to the week 6 call volumes than any other of the 13 weeks.

The spreadsheet for the calculation of the mean absolute deviation for exponential forecasting method follows.

EXPONENTIAL SMOOTHING

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	1278	555
6	Tues	677	1077	400
6	Wed	521	873	352
6	Thurs	571	810	239
6	Fri	498	724	226

Mean Absolute Deviation

MAD =	355

This method is almost as effective as the averaging forecasting method because the smoothing constant used is 0.1. Therefore, the call volumes from earlier weeks are still weighted in calculating the call volume average. This method is a little less effective than the averaging forecasting method because the smoothing constant causes less weight to be placed on the call volumes in weeks 44 and 45. These call volumes are lower than volumes in weeks 46, 48, and 49, however, and they help lower the already inflated average call volume.

c) This problem is simply a linear regression problem.

1) To find a mathematical relationship, we use the Linear Regression template. The decentralized case volumes are the independent variables, and the centralized case volumes are the dependent variables. Substituting the case volume data, we obtain the following spreadsheet:

Week	Independent Variable	Dependent Variable	Estimate	Estimation Error	Square of Error
44	612	2052	2038	13.84	192
45	721	2170	2121	49.45	2445
46	693	2779	2099	679.61	461872
47	540	2334	1984	350.27	122690
48	1386	2514	2623	109.26	11938
49	577	1713	2012	298.70	89221
50	405	1927	1882	45.32	2054
51	441	1167	1909	741.89	550400
52/1	655	1549	2071	521.66	272132
2	572	2126	2008	118.08	13943
3	475	2337	1935	402.41	161932
4	530	1916	1976	60.17	3620
5	595	2098	2025	72.69	5284

Linear Regression Line
y = a + bx

a =	1575.516778
b =	0.755947559

Estimator

If x =	613
then y=	2038.912632

2) To forecast the week 6 call volume for the centralized call center, we simply input the week 6 decentralized case volume for the value of x in the Estimator section of the Linear Regression Spreadsheet. The value of y then represents the week 6 centralized case volume. We multiply this value of y by 1.5 to obtain the week 6 centralized call volume. The calculations in Excel appear below.

Estimator

If x =	613
then y=	2038.912632

Calls = 1.5 * y
3058.368948

We then break this weekly call volume into daily call volume. We do this conversion by dividing the weekly call volume by the sum of the seasonal factors calculated in part (a) and then multiplying this weekly call volume by the appropriate seasonal factor to find the call volume for each of the five days of the week. The spreadsheet showing these calculations follows:

Seasonal Factors	**1.341**	**1.131**	**0.916**	**0.850**	**0.762**

Week 6 Call Volume	**3058**
Sum of Seasonal Factors	**5.000**
Converted Week 6 Call Volume	**611.6**
Call Volume for:	
Monday	**820.1237**
Tuesday	**691.9468**
Wednesday	**560.19**
Thursday	**519.6189**
Friday	**466.1206**
Week 6 Call Volume	**3058**

The forecasted call volume for week 6 is 3058 calls: 820 calls are received on Monday, 692 calls are received on Tuesday, 560 calls are received on Wednesday, 520 calls are received on Thursday, and 466 calls are received on Friday.

3) To calculate the mean absolute deviation, we need to subtract the true call volume from the forecasted call volume for each day in the sixth week. We then need to take the absolute value of the five differences. Finally, we need to take the average of these five absolute values to obtain the mean absolute deviation.

The spreadsheet for the calculation of the mean absolute deviation follows.

CAUSAL FORECASTING

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	820	97
6	Tues	677	692	15
6	Wed	521	560	39
6	Thurs	571	520	51
6	Fri	498	466	32

Mean Absolute Deviation

MAD =	47

This forecasting method is by far the most effective method. The centralized center performs the same services and serves the same population as the decentralized center. Therefore, the call volume trends are the same. Once we have a factor to scale the decentralized call volumes to the centralized call volumes, we have a very effective forecasting method.

d) We would definitely recommend using the causal forecasting method implemented in part (c) because it yields the lowest error. The causal method shows us that the call volume trends remain relatively the same year after year. We had to convert between case volumes and call volumes in part (c), however, and such a conversion introduces error. For example, what if a case generates a higher or lower number of calls? We therefore recommend that call volume data be meticulously recorded as the centralized center continues its operation. Once one year's worth of call volumes have been collected, the causal forecasting model should be updated. The model should be updated to use the historical centralized call volume data instead of the historical decentralized case volume data.

CHAPTER 28: EXAMPLES OF PERFORMING SIMULATIONS ON SPREADSHEETS WITH CRYSTAL BALL

28.1.

(a) Answers will vary. A typical set of 5 runs: 46.49, 45.98, 45.76, 45.99, and 46.74.

(b) Answers will vary. A typical set of 5 runs: 46.13, 46.15, 46.42, 46.14, and 46.27.

(c) The mean completion times in (b) should be more consistent.

28.2.

(a) Triangular Distribution (Min = 293.51, Likeliest = 503.00, Max = 599.50)

(b) Min Extreme Distribution (Likeliest = 492.26, Scale = 56.34)

28.3.

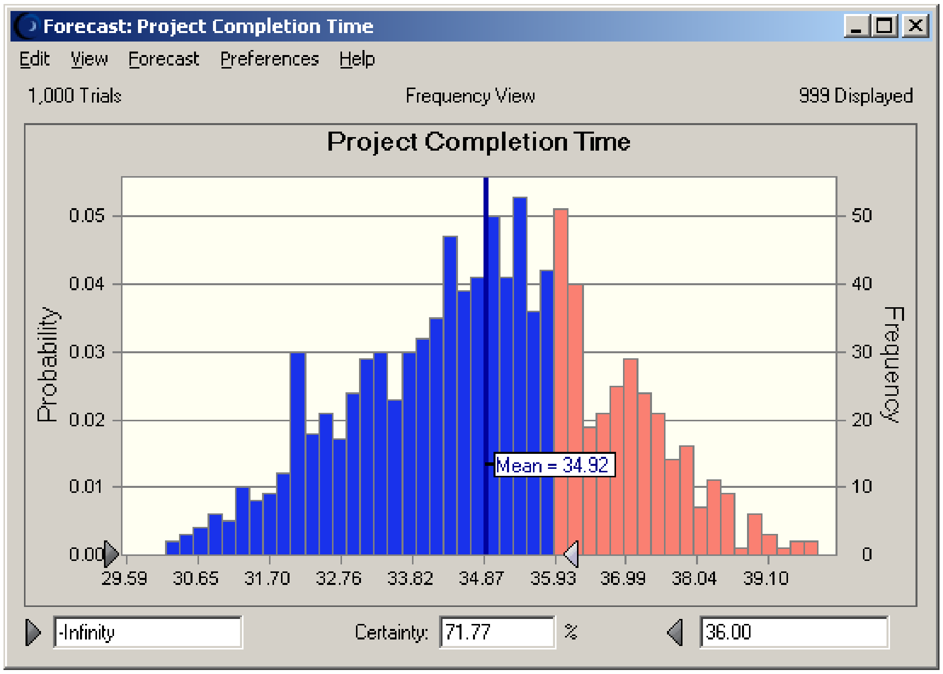
(a) Uniform Distribution (Min = 299.27, Max = 498.73)

(b) Lognormal Distribution (Mean = 390.84, Standard Deviation = 59.91)

28.4.

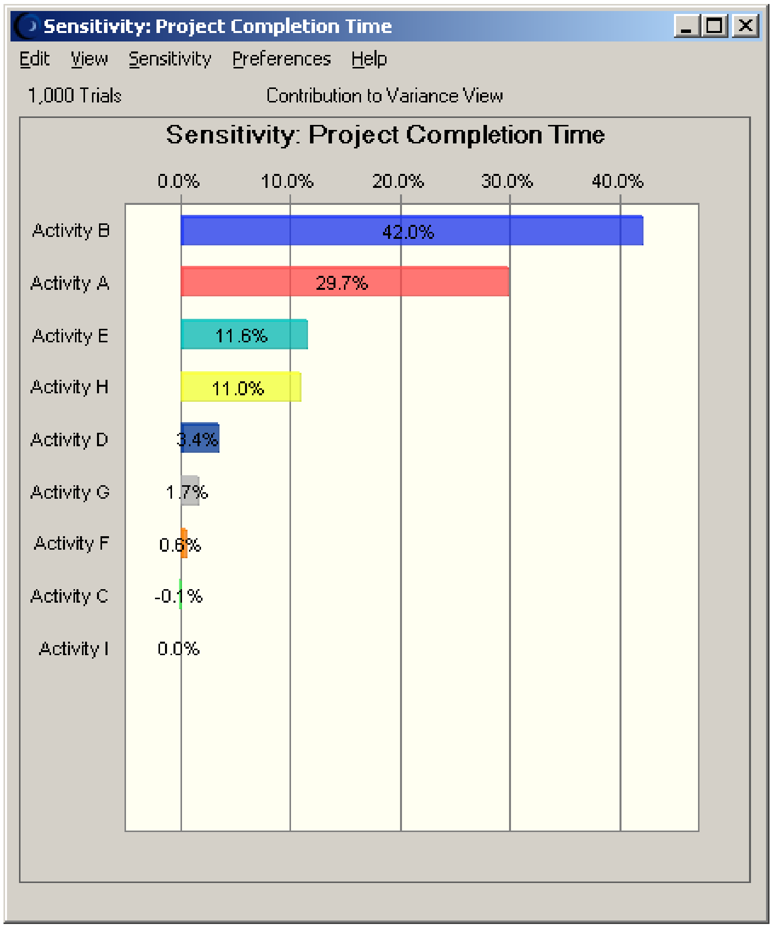
	A	B	C	D	E	F	G	H	I
1							(all times in months)		
2							Start	Activity	Finish
3	Activity	Predecessor	Distribution	Parameters			Time	Time	Time
4	A Secure funding	—	Normal (mean, st. dev.)	6	1		0.0	6	6.0
5	B Design Building	A	Uniform (min, max)	6	10		6.0	8	14.0
6	C Site Preparation	A	Triangular (min, most likely,	1.5	2	2.5	6.0	2	8.0
7	D Foundation	B, C	Triangular (min, most likely,	1.5	2	3	14.0	2.1666667	16.2
8	E Framing	D	Triangular (min, most likely,	3	4	6	16.2	4.3333333	20.5
9	F Electrical	E	Triangular (min, most likely,	2	3	5	20.5	3.3333333	23.8
10	G Plumbing	E	Triangular (min, most likely,	3	4	5	20.5	4	24.5
11	H Walls and Roof	F, G	Triangular (min, most likely,	4	5	7	24.5	5.3333333	29.8
12	I Finish Work	H	Triangular (min, most likely,	5	6	7	29.8	6	35.8
13	J Landscaping	H	Fixed (5)				29.8	5	34.8
14									
15					Project Completion Time				34.8

(a) The mean project completion time is approximately 35 months.



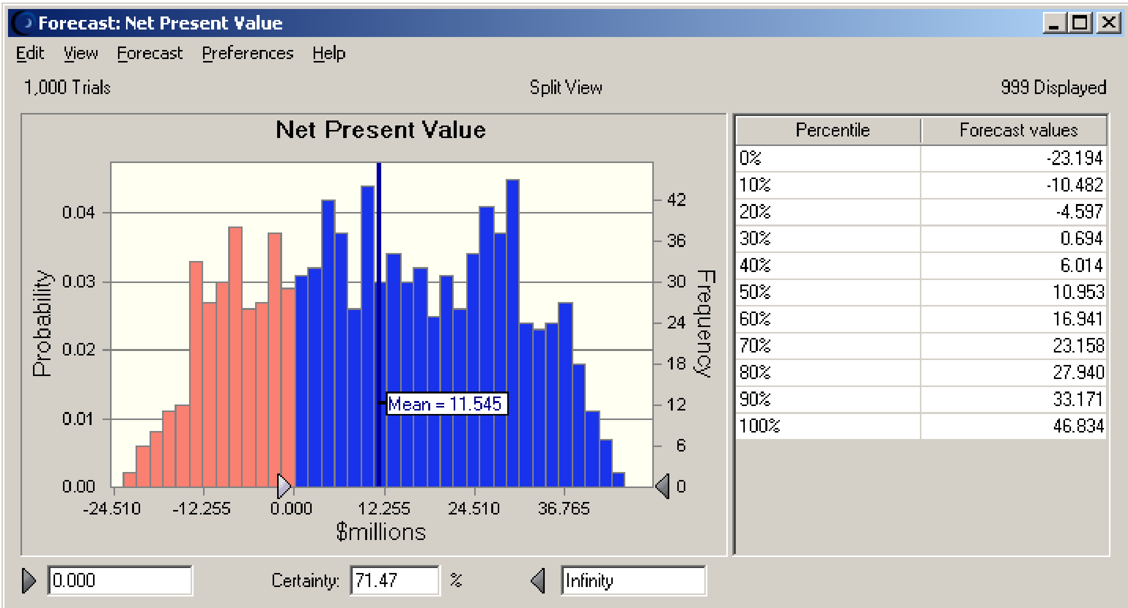
(b) The probability that the project completion time will be less than 36 months is approximately 71.8%.

(c) Activity A and Activity B have the largest impact on the variability of the project completion time.

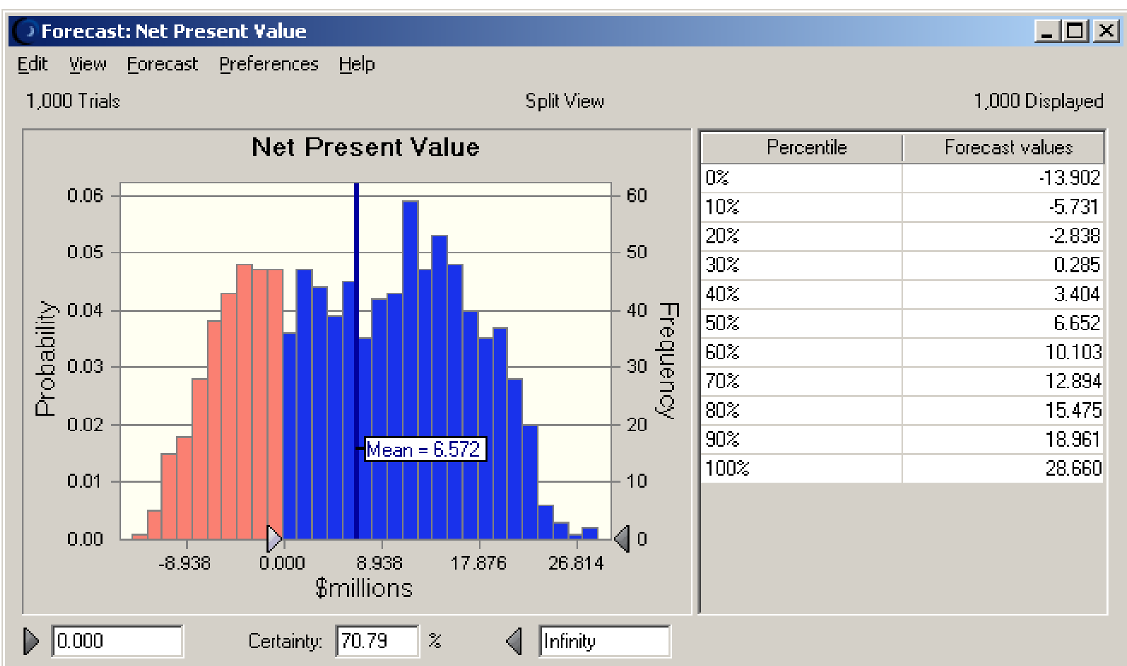


28.5.

(a) Option 2: Hotel Project Only



(b) Option 3: Shopping Center Project Only



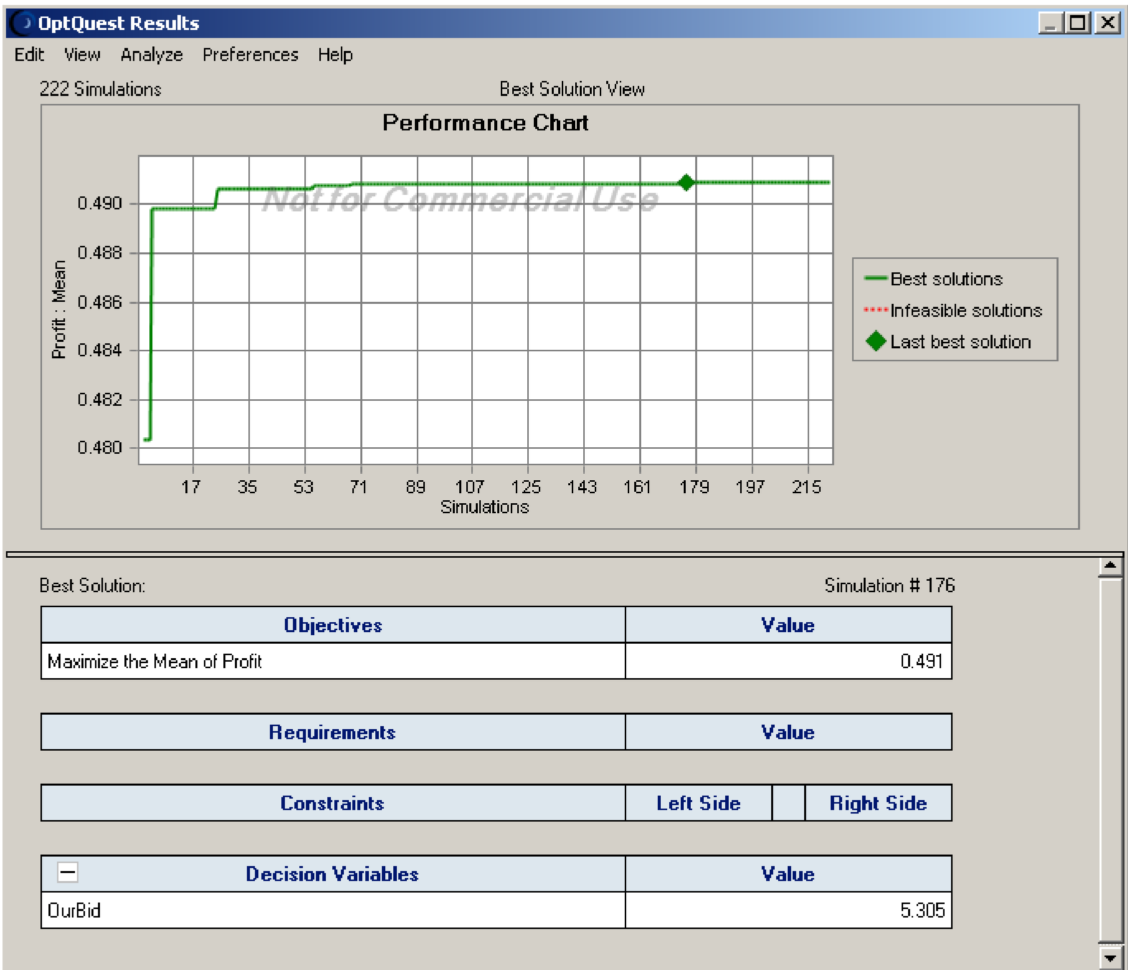
(c) Option 1 appears to be the best. It has the highest expected NPV, $18 million whereas Option 2 has an expected NPV less than $12 million and Option 3 has an expected NPV less than $7 million. Moreover, there is less chance of losing money if one chooses Option 1. This probability is less than 20% for Option 1 while for the other options, it exceeds 25%.

28.6.

(a) A bid of approximately $5.3 million maximizes the mean profit.

OurBid (5.200)	OurBid (5.250)	OurBid (5.300)	OurBid (5.350)	OurBid (5.400)	OurBid (5.450)	OurBid (5.500)	OurBid (5.550)	OurBid (5.600)
0.473	0.486	0.489	0.488	0.480	0.463	0.392	0.311	0.241

(b) The optimal bid is approximately $5.305 million, as found by OptQuest.

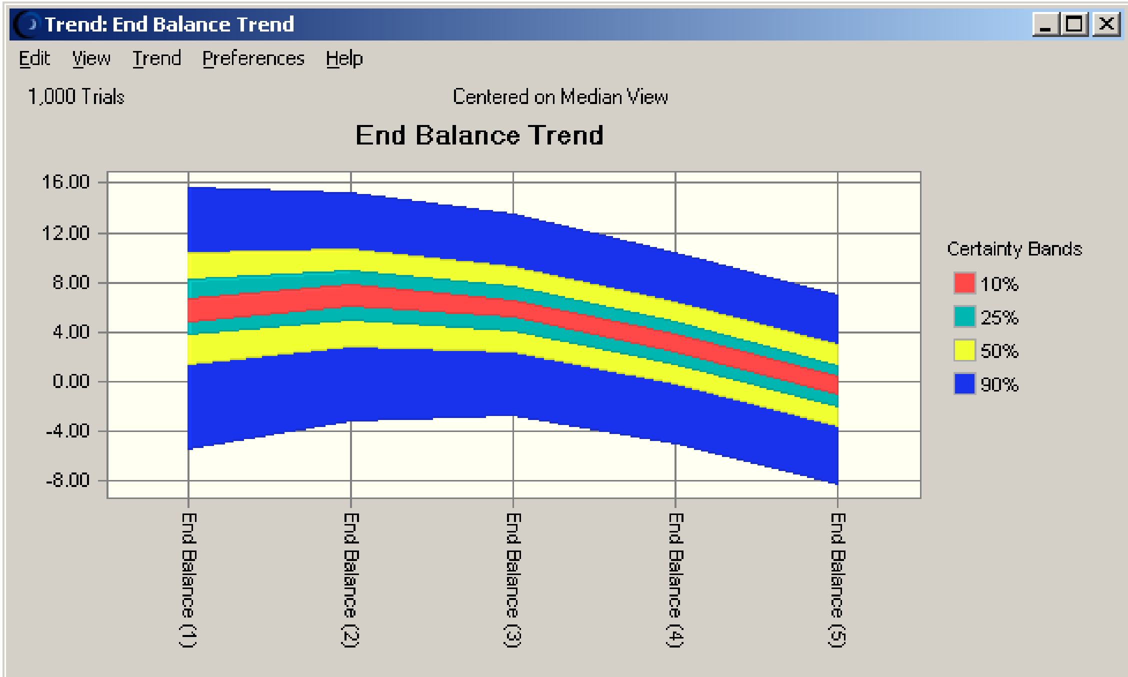


28.7.

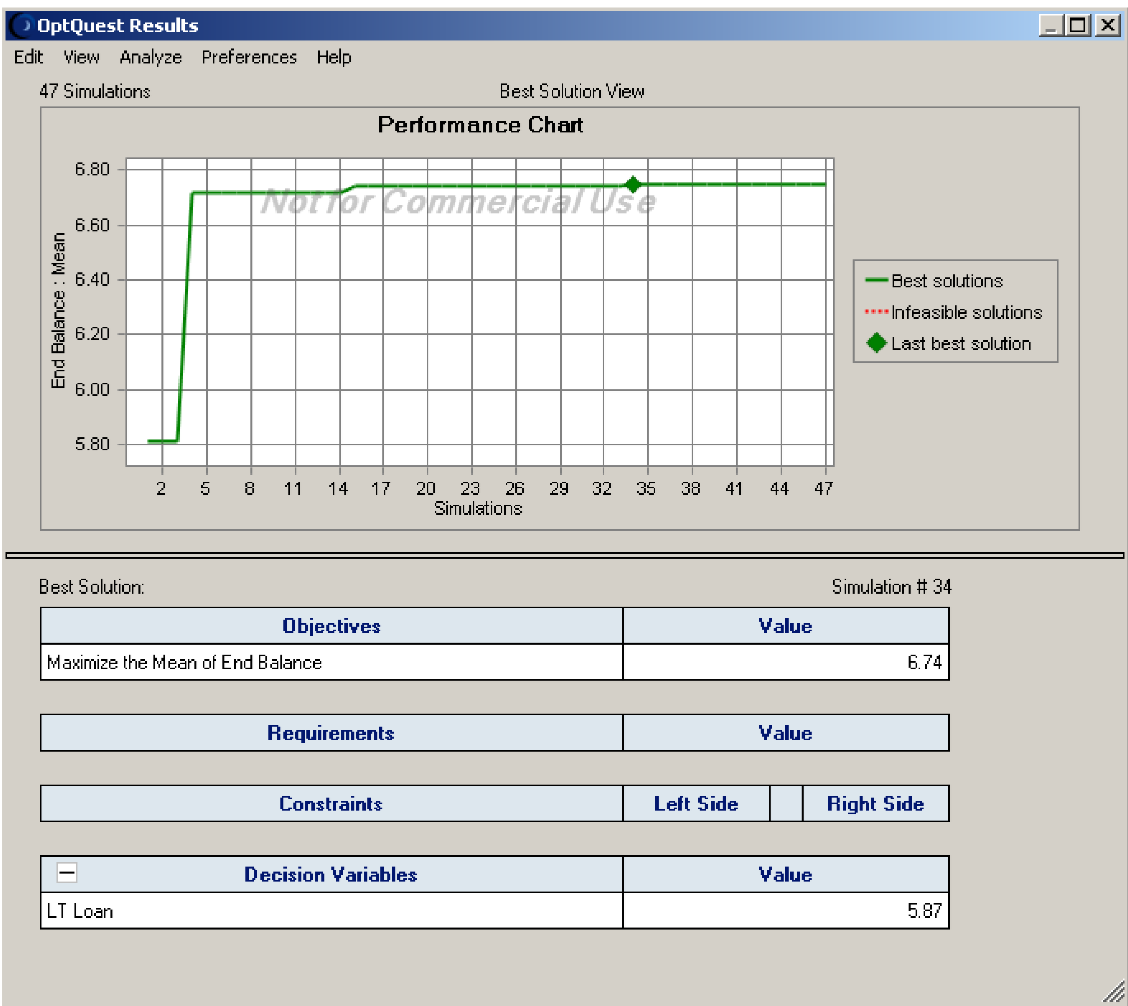
(a) A long-term loan of approximately $5 million maximizes Everglade's mean ending balance.

LT Loan (0.00)	LT Loan (5.00)	LT Loan (10.00)	LT Loan (15.00)	LT Loan (20.00)
5.73	6.72	5.82	3.07	-0.33

(b)



(c) The optimal long-term loan is approximately $5.87 million, as found by OptQuest.

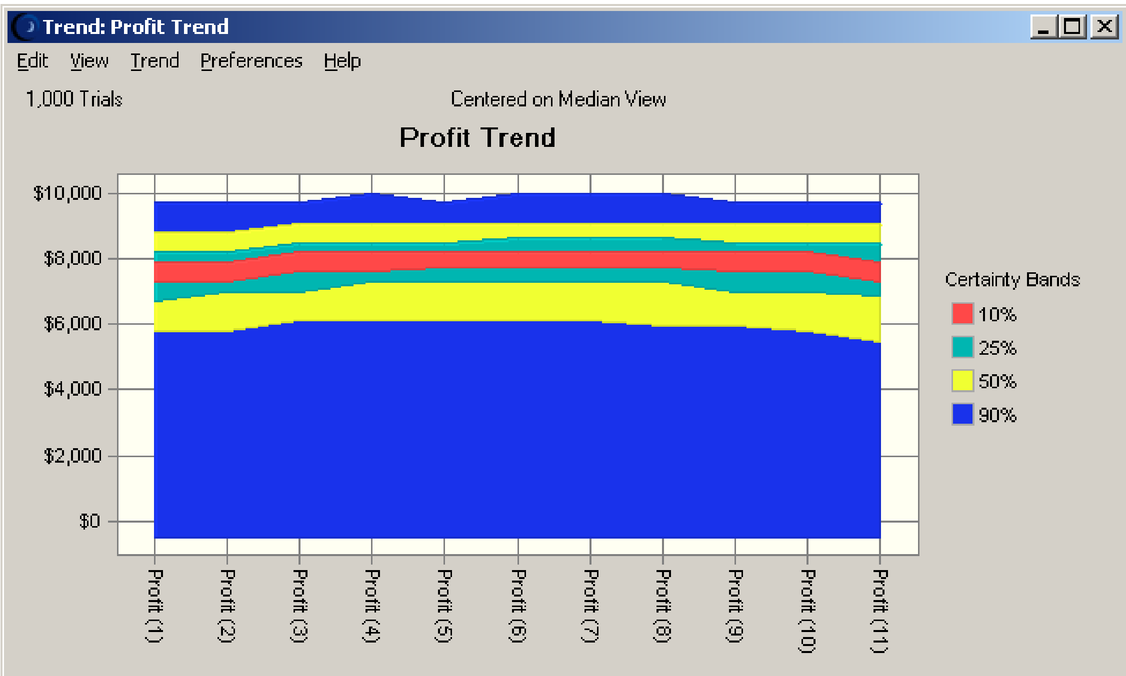


28.8.

(a) Accepting approximately 185 reservations maximizes the mean profit.

ReservationsToAccept (180)	ReservationsToAccept (181)	ReservationsToAccept (182)	ReservationsToAccept (183)	ReservationsToAccept (184)	ReservationsToAccept (185)	ReservationsToAccept (186)	ReservationsToAccept (187)	ReservationsToAccept (188)	ReservationsToAccept (189)	ReservationsToAccept (190)
$6,613	$6,719	$6,803	$6,869	$6,908	$6,926	$6,924	$6,894	$6,841	$6,777	$6,693

(b)



(c) The optimal number of reservations to accept is approximately 185, as found by OptQuest.